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Loop homology of moment-angle complexes in the flag case

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We develop a general homological approach to presentations of connected graded associative algebras, and apply it to the loop homology of moment-angle complexes Z_K that correspond to flag simplicial complexes K. For an arbitrary coefficient ring, we describe generators of the Pontryagin algebra $H_*(\Omega Z_K)$ and defining relations between them. We prove that such moment-angle complexes are coformal over $\mathbb{Q}$, give a necessary condition for rational formality, and compute their homotopy groups in terms of homotopy groups of spheres.

16E30, 16W50, 57S12; 55P35, 55Q52, 55U15, 57T05

1 Introduction

For a simply connected space X and a commutative ring $\boldsymbol{k}$ with unit, the *Pontryagin algebra* $H_*(\Omega X;\boldsymbol{k})$ is a connected graded associative $\boldsymbol{k}$-algebra with respect to the Pontryagin product. We study the Pontryagin algebras of *moment-angle complexes* $X=\mathcal{Z}_{\mathcal{K}}:=(D^2,S^1)^{\mathcal{K}}$ that correspond to simplicial complexes $\mathcal{K}$. Moment-angle complexes play an important role in toric topology [15], and they have interesting homotopical properties and surprising connections to several topics in algebra and combinatorics [5]. If $\mathcal{K}$ is a simplicial complex on the vertex set $[m]=\{1,\dots,m\}$, there is an effective action of the m-dimensional torus $\mathbb{T}^m=(S^1)^{\times m}$ on $\mathcal{Z}_{\mathcal{K}}$. The homotopy quotient $E\mathbb{T}^m\times_{\mathbb{T}^m}\mathcal{Z}_{\mathcal{K}}$ (the Borel construction) is known as the *Davis–Januszkiewicz space* $\mathrm{DJ}(\mathcal{K})$ and is homotopy equivalent to the polyhedral product $(\mathbb{C}P^\infty,*)^{\mathcal{K}}$; see [15, Theorem 4.3.2].

Panov and Ray [35] reduced the study of corresponding Pontryagin algebras to an algebraic problem. Applying the based loops functor to the homotopy fibration

$$\mathcal{Z}_{\mathcal{K}}\to\mathrm{DJ}(\mathcal{K})\to B\mathbb{T}^m, \tag{1}$$

they obtained a split fibration of H-spaces $\Omega\mathcal{Z}_{\mathcal{K}}\to\Omega\mathrm{DJ}(\mathcal{K})\to\mathbb{T}^m$ and thus an extension of cocommutative Hopf algebras

$$\boldsymbol{k}\to H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})\to H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})\to\Lambda[u_1,\dots,u_m]\to\boldsymbol{k}$$

over a field $\boldsymbol{k}$. For any $\mathcal{K}$, there is an isomorphism of Hopf algebras $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})\cong\mathrm{Ext}_{\boldsymbol{k}[\mathcal{K}]}(\boldsymbol{k},\boldsymbol{k})$ [20; 35] (moreover, this is true for any principal ideal domain $\boldsymbol{k}$ such that $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ is a free

$\boldsymbol{k}$-module). If $\mathcal{K}$ is a *flag* simplicial complex, this Hopf algebra is known completely: it is isomorphic to the partially commutative algebra

$$\boldsymbol{k}[\mathcal{K}]^! := T(u_1,\dots,u_m)/(u_i^2=0,\ i=1,\dots,m;\ u_iu_j+u_ju_i=0,\ \{i,j\}\in\mathcal{K}),\quad \deg u_i=1.$$

Generators u_i are primitive and have degree $(-1,2e_i)$ with respect to the $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading introduced in [43]. In this case Grbić, Panov, Theriault and Wu [23] found a minimal generating set for the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})$, and the author calculated the number of relations in any minimal presentation (by homogeneous generators and relations) of this algebra [43].

The last calculation relies on homological methods developed by Wall [44] and Lemaire [29] for connected graded associative algebras over a field. Namely, multiplicative generators of a connected $\boldsymbol{k}$-algebra A correspond to additive generators of the graded $\boldsymbol{k}$-module $\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})$, and relations correspond to generators of $\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})$. In order to study the integer Pontryagin algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{Z})$, we generalise these results to the case of *arbitrary* commutative rings $\boldsymbol{k}$ with unit, and construct explicit presentations of connected $\boldsymbol{k}$-algebras using cycles in the bar construction. These results are presented in Appendix A. We hope that they will be useful in other contexts.

Let us give a general description of our approach. Suppose that we are given a connected $\boldsymbol{k}$-algebra A which is a free left module over its subalgebra S, $A\simeq S\otimes_{\boldsymbol{k}} V$. We wish to construct a presentation of S. Theorem A.6 does that, if we know a set of cycles in the bar construction $\overline{\mathrm{B}}(S)$, such that their images generate the $\boldsymbol{k}$-modules $H_i(\overline{\mathrm{B}}(S))\simeq\operatorname{Tor}_i^S(\boldsymbol{k},\boldsymbol{k})$, $i=1,2$. The following algorithm computes such cycles:

(i) Build a free resolution $(A\otimes M,d)$ of the left A-module $\boldsymbol{k}$.

(ii) Interpret it as a free resolution $(S\otimes V\otimes M,\hat{d})$ of the left S-module $\boldsymbol{k}$. Compute the functor $\operatorname{Tor}^S(\boldsymbol{k},\boldsymbol{k})$ as the homology of the complex $(V\otimes M,\bar{d})$. Find cycles in $(V\otimes M,\bar{d})$ such that their images generate $\operatorname{Tor}_i^S(\boldsymbol{k},\boldsymbol{k})$, $i=1,2$.

(iii) Construct a morphism $\varphi\colon(S\otimes V\otimes M,\hat{d})\to(\mathrm{B}(S),d_{\mathrm{B}})$ of free resolutions of the left S-module $\boldsymbol{k}$, using the contracting homotopy of the bar resolution (see Corollary 2.2). Obtain a morphism of chain complexes $\bar{\varphi}\colon(V\otimes M,\bar{d})\to(\overline{\mathrm{B}}(S),d_{\overline{\mathrm{B}}})$ that induces an isomorphism on the homology.

(iv) Applying $\bar{\varphi}$ to the cycles from (ii), obtain the required cycles in $\overline{\mathrm{B}}(S)$.

This situation takes place if $\boldsymbol{k}\to S\to A\to V\to\boldsymbol{k}$ is an extension of connected Hopf algebras; see [2; 31, Proposition 4.9]. In that sense, our algorithm has similarities with the *Reidemeister–Schreier algorithm* that constructs a presentation of a subgroup, given a presentation of the whole group. See [30] for another approach to Hopf subalgebras in connected Hopf algebras. It is well known that extensions of Hopf algebras arise in the study of fibrations $F\to E\to B$ that have a section after looping (see Appendix B for the proof). For such "Ω-split" fibrations, the proposed method allows us to study presentations of $H_*(\Omega F;\boldsymbol{k})$, if the algebras $H_*(\Omega E;\boldsymbol{k})$ and $H_*(\Omega B;\boldsymbol{k})$ are known.

Fibrations of this kind are studied by Theriault [41]; see also [9, Proposition 6.1]. (However, these works deal with cases when the algebra $H_*(\Omega F;\boldsymbol{k})$ is known better than $H_*(\Omega E;\boldsymbol{k})$.) We consider the case $F=\mathcal{Z}_{\mathcal{K}}$, $E=\mathrm{DJ}(\mathcal{K})$, $B=(\mathbb{C}P^\infty)^m$. The algorithm is also applicable to *partial quotients* of moment-angle complexes [15, Section 4.8] (we will consider their Pontryagin algebras in subsequent publications) and polyhedral products of the form $(PX,\Omega X)^{\mathcal{K}}$ (here we refer to the recent work [16] by Li Cai).

1.1 Main results

We give a presentation of the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ for a flag simplicial complex $\mathcal{K}$ and any ring $\boldsymbol{k}$. The presentation is explicit up to a rewriting process described in Algorithm 5.4. For $x\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ and a subset $A=\{a_1<\dots<a_k\}\subset[m]$, define

$$c(A,x):=[u_{a_1},[u_{a_2},\dots,[u_{a_k},x],\dots]]\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k}).$$

This element belongs to the subalgebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})\subset H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$, if $x=u_i$ and $A\neq\varnothing$ (see Corollary 3.10). For every $J\subset[m]$, denote by $\Theta(J)$ the set of all vertices $i\in J$ such that

- vertices i and $\max(J)$ are in different path components of the complex $\mathcal{K}_J$;
- i is the smallest vertex in its path component.

Denote by $\widetilde{b}_i(X;\boldsymbol{k})$ the minimal number of elements that generate the $\boldsymbol{k}$-module $\widetilde{H}_i(X;\boldsymbol{k})$. Clearly, $|\Theta(J)|=\widetilde{b}_0(\mathcal{K}_J;\boldsymbol{k})$ for any principal ideal domain $\boldsymbol{k}$. Consider the $\widetilde{b}_0(\mathcal{K}_J;\boldsymbol{k})$-element set

$$\bigl\{c(J\setminus\{i\},u_i):J\subset[m],\ i\in\Theta(J)\bigr\}\subset H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}).$$

We call its elements the *GPTW generators* (after Grbić, Panov, Theriault and Wu).

Theorem 1.1 *Let $\boldsymbol{k}$ be a commutative ring with unit and $\mathcal{K}$ be a flag simplicial complex without ghost vertices on vertex set $[m]$.*

(i) *For every $J\subset[m]$, choose a set of simplicial 1-cycles*

$$\sum_{\{i<j\}\in\mathcal{K}_J}\lambda_{ij}^{(\alpha)}[\{i,j\}]\in C_1(\mathcal{K}_J;\boldsymbol{k})$$

that generate the $\boldsymbol{k}$-module $H_1(\mathcal{K}_J;\boldsymbol{k})$. Then the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ is generated by GPTW generators modulo the relations

$$\sum_{\{i<j\}\in\mathcal{K}_J}\lambda_{ij}^{(\alpha)}\sum_{\substack{J\setminus\{i,j\}=A\sqcup B\\ \max(A)>i,\max(B)>j}}\pm[\widehat{c}(A,u_i),\widehat{c}(B,u_j)]=0$$

that correspond to the chosen 1-cycles. (Here $\widehat{c}(A,u_i)$, $\widehat{c}(B,u_j)$ are the elements $c(A,u_i)$, $c(B,u_j)$ of $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ that are arbitrarily expressed through the GPTW generators, and $[x,y]$ is defined as $x\cdot y-(-1)^{|x|\cdot|y|}y\cdot x$.) In particular, $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ admits a $\mathbb{Z}\times\mathbb{Z}^m_{\geq0}$-homogeneous presentation by $\sum_{J\subset[m]}\widetilde{b}_0(\mathcal{K}_J;\boldsymbol{k})$ generators and $\sum_{J\subset[m]}\widetilde{b}_1(\mathcal{K}_J;\boldsymbol{k})$ relations.

(ii) *If $\boldsymbol{k}$ is a principal ideal domain, then this presentation is minimal: any $\mathbb{Z}\times\mathbb{Z}^m_{\geq0}$-homogeneous presentation of $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ contains at least $\sum_{J\subset[m]}\widetilde{b}_0(\mathcal{K}_J;\boldsymbol{k})$ generators and at least $\sum_{J\subset[m]}\widetilde{b}_1(\mathcal{K}_J;\boldsymbol{k})$ relations.*

This theorem follows from Theorems 5.1 and 5.6, proven in Section 5. For field coefficients, these results were partially obtained by Grbić, Panov, Theriault, Wu (the minimal set of generators [23, Theorem 4.3]) and the author (number of relations and their degrees [43, Corollary 4.5]). Sometimes the number of relations can be reduced, if we do not require them to be $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous (see Theorem 5.7).

We also present new results on the homotopy of moment-angle complexes that correspond to flag complexes. Using a result of Huang [26], we prove in Corollary 6.7 that in the flag case $\mathcal{Z}_\mathcal{K}$ is *coformal* over $\mathbb{Q}$ in the sense of rational homotopy theory. Results of Berglund [10] then give a necessary condition for such moment-angle complexes to be rationally formal (Theorem 6.13). Finally, we improve a recent result of Stanton [39] about the homotopy type of $\Omega\mathcal{Z}_\mathcal{K}$ by finding the explicit number of spheres in the product:

Theorem 1.2 *Let $\mathcal{K}$ be a $(d-1)$-dimensional flag simplicial complex on $[m]$ with no ghost vertices. Then there is a homotopy equivalence*

$$\Omega\mathcal{Z}_\mathcal{K}\simeq\prod_{n\geq 3}(\Omega S^n)^{\times D_n}, \tag{2}$$

where the numbers $D_n\geq 0$ are determined by

$$-\sum_{J\subset[m]}\overline{\chi}(\mathcal{K}_J)\cdot t^{|J|}=(1+t)^{m-d}h_\mathcal{K}(-t)=\prod_{n\geq 3}(1-t^{n-1})^{D_n}, \tag{3}$$

where $\overline{\chi}(X):=\chi(X)-1=\sum_{i\geq 0}(-1)^i\dim\widetilde{H}_i(X)$ is the reduced Euler characteristic and $h_\mathcal{K}(t):=\sum_{i=0}^d h_i(\mathcal{K})\cdot t^i$ is the h-polynomial [15, Definition 2.2.5] *of $\mathcal{K}$. In particular, for every $N\geq 1$ we have an isomorphism*

$$\pi_N(\mathcal{Z}_\mathcal{K})\simeq\bigoplus_{n=3}^{N}\pi_N(S^n)^{\oplus D_n}. \tag{4}$$

This theorem is proved in Section 6. Using (4), it is easy to describe the homotopy groups of corresponding Davis–Januszkiewicz spaces (using the fibration (1)) and partial quotients of moment-angle complexes, including quasitoric manifolds and smooth toric varieties (using similar fibrations; see [15, Proposition 7.3.13; 19, Section 4]).

1.2 Organisation of the paper

Section 2 consists of algebraic preliminaries. We highlight Corollary 2.2 that allows us to construct chain maps into the bar resolution. In Section 3 we recall notions from toric topology and discuss the properties of Pontryagin algebras $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ and $H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})$. Main calculations are carried in Section 4. In Section 5 we prove Theorem 1.1 and consider an example. Section 6 contains results about (co)formality and homotopy groups of moment-angle complexes in the flag case. In Appendix A we develop the homological tools for working with presentations of connected graded algebras over a commutative ring. In Appendix B we prove the following folklore fact: split fibrations of loop spaces correspond (by passing to homology) to extensions of Hopf algebras. Appendix C contains commutator identities that are used in Section 4.

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2 Preliminaries: algebra

2.1 Connected graded algebras

Fix a commutative associative ring $\boldsymbol{k}$ with unit. We consider associative $\boldsymbol{k}$-algebras with unit that are graded by a commutative monoid G (usually $G = \mathbb{Z}$ or $\mathbb{Z}^k \times \mathbb{Z}^m_{\geq 0}$, $k = 0, 1, 2$.) Left A-modules are also G-graded. Elements of $\mathbb{Z}^m_{\geq 0}$ are denoted by $\alpha = (\alpha_1, \dots, \alpha_m) = \sum_{j=1}^m \alpha_j e_j$, $\alpha_j \geq 0$. Subsets $J \subset [m]$ are identified with elements $\sum_{j \in J} e_j \in \mathbb{Z}^m_{\geq 0}$. Define also

$$|\alpha| := \alpha_1 + \dots + \alpha_m, \quad \operatorname{supp} \alpha := \{i \in [m] : \alpha_i > 0\}.$$

Every $\mathbb{Z}^k \times \mathbb{Z}^m_{\geq 0}$-graded algebra A is considered as $\mathbb{Z}$-graded with respect to the total grading

$$A_n := \bigoplus_{n = i_1 + \dots + i_k + |\alpha|} A_{i_1, \dots, i_k, \alpha}.$$

The graded algebra A is *connected* if $A_{<0} = 0$ and $A_0 = \boldsymbol{k} \cdot 1$. We have the canonical augmentation $\varepsilon \colon A \to A_0 = \boldsymbol{k}$ and the augmentation ideal $I(A) := \operatorname{Ker} \varepsilon$. Examples of connected $\boldsymbol{k}$-algebras are

- the exterior algebra $\Lambda[m] := \Lambda[u_1, \dots, u_m]$, $\deg(u_i) = (-1, 2e_i) \in \mathbb{Z} \times \mathbb{Z}^m_{\geq 0}$, with the basis $\{u_I := u_{i_1} \wedge \dots \wedge u_{i_k} : I = \{i_1 < \dots < i_k\}\}$;
- the polynomial algebra $\boldsymbol{k}[m] := \boldsymbol{k}[v_1, \dots, v_m]$, $\deg(v_i) = (0, 2e_i)$, with the basis
$$\Big\{v^\alpha := \prod_{i=1}^m v_i^{\alpha_i} : \alpha \in \mathbb{Z}^m_{\geq 0}\Big\};$$
- the tensor algebra $T(x_1, \dots, x_N)$, where x_i are homogeneous elements of arbitrary positive degrees.

For a homogeneous element a, define $\overline{a} := (-1)^{1+\deg(a)} \cdot a$. Clearly, $\overline{a \cdot b} = -\overline{a} \cdot \overline{b}$ and $\overline{\overline{a}} = a$.

Let A be a G-graded algebra. Complexes of A-modules (M, d) are considered as $\mathbb{Z} \times G$-graded modules with a differential of degree $(-1, 0)$. We use the Koszul sign rule with respect to the total grading: $d(a \cdot m) = (-1)^{\deg(a)} a \cdot d(m) = -\overline{a} \cdot d(m)$. Several formulas from [43] do not follow this rule and are corrected in this paper.

2.2 Bar resolution and bar construction

Let A be a connected $\boldsymbol{k}$-algebra and $\varepsilon\colon A\to\boldsymbol{k}$ be the augmentation. The resulting left A-module $\boldsymbol{k}$ has the *bar resolution*

$$\cdots\to \mathrm{B}_2(A)\to \mathrm{B}_1(A)\to \mathrm{B}_0(A)\to \boldsymbol{k}\to 0,$$

where $\mathrm{B}_n(A):=A\otimes I(A)^{\otimes n}$. An element of the form $a\otimes a_1\otimes\cdots\otimes a_n\in \mathrm{B}_n(A)$ has bidegree $\bigl(n,\deg(a)+\sum_{i=1}^n\deg(a_i)\bigr)$ and is traditionally written as $a[a_1|\dots|a_n]$. The differential d_{B} has bidegree $(-1,0)$ and is given by

$$-d_{\mathrm{B}}(a[a_1|\dots|a_n]):=\overline{a}\cdot a_1[a_2|\dots|a_n]+\sum_{i=1}^{n-1}\overline{a}[\overline{a}_1|\dots|\overline{a}_{i-1}|\overline{a}_i\cdot a_{i+1}|a_{i+2}|\dots|a_n].$$

Consider also the contracting homotopy $s_n\colon \mathrm{B}_n(A)\to \mathrm{B}_{n+1}(A)$,

$$(5)\qquad s_n(a[a_1|\dots|a_n]):=\begin{cases}0, & a\in A_0\simeq\boldsymbol{k};\\ [a|a_1|\dots|a_n], & \deg(a)>0;\end{cases}\qquad s_{-1}\colon\boldsymbol{k}\to \mathrm{B}_0(A),\quad 1\mapsto 1[\cdot].$$

It is easy to show that $s\circ d_{\mathrm{B}}+d_{\mathrm{B}}\circ s=\mathrm{id}$, $d_{\mathrm{B}}^2=0$. Hence $(\mathrm{B}(A),d_{\mathrm{B}})$ is a free resolution of the left A-module $\boldsymbol{k}$, assuming that A is a free $\boldsymbol{k}$ module. In this case, we obtain

$$\mathrm{Tor}_n^A(\boldsymbol{k},\boldsymbol{k})\cong H_n[\overline{\mathrm{B}}(A),d_{\overline{\mathrm{B}}}],$$

where $\overline{\mathrm{B}}(A):=\boldsymbol{k}\otimes_A \mathrm{B}(A)$ is the *bar construction* of A. We have

$$\overline{\mathrm{B}}_n(A)=I(A)^{\otimes n},\quad \deg([a_1|\dots|a_n])=(n,\deg(a_1)+\cdots+\deg(a_n)),\quad \deg d_{\overline{\mathrm{B}}}=(-1,0),$$

$$(6)\qquad d_{\overline{\mathrm{B}}}([a_1|\dots|a_n])=\sum_{i=1}^{n-1}[\overline{a}_1|\dots|\overline{a}_{i-1}|\overline{a}_i\cdot a_{i+1}|a_{i+2}|\dots|a_n]\in\overline{\mathrm{B}}_{n-1}(A).$$

In particular, $d_{\overline{\mathrm{B}}}([x|y])=[\overline{x}\cdot y]$ and $d_{\overline{\mathrm{B}}}([x|y|z])=[\overline{x}\cdot y|z]+[\overline{x}|\overline{y}\cdot z]$.

2.3 Chain maps into resolutions with a contracting homotopy

Any map of modules can be extended to a map of their free resolutions. Moreover, this extension can be described in terms of the contracting homotopy for the latter resolution. This recursive construction seems to be known to specialists: its generalisations and applications are discussed in [12]. The author thanks Georgy Chernykh for the reference.

Lemma 2.1 *Let A be an associative $\boldsymbol{k}$-algebra. Suppose that the commutative diagram of left A-modules and their homomorphisms*

$$\begin{array}{ccccc} C_n & \xrightarrow{\hat{d}_n} & C_{n-1} & \xrightarrow{\hat{d}_{n-1}} & C_{n-2}\\ & & \downarrow{\scriptstyle\varphi_{n-1}} & & \downarrow{\scriptstyle\varphi_{n-2}}\\ B_n & \xrightarrow[d_n]{} & B_{n-1} & \xrightarrow[d_{n-1}]{} & B_{n-2}\end{array}$$

satisfy the conditions

(i) *C_n is a free A-module with a basis $\{e_i\}$;*

(ii) $\widehat{d}_{n-1} \circ \widehat{d}_n = 0$;

(iii) *there are $\boldsymbol{k}$-linear maps $s_{n-1}\colon B_{n-1} \to B_n$ and $s_{n-2}\colon B_{n-2} \to B_{n-1}$ such that*

$$d_n \circ s_{n-1} + s_{n-2} \circ d_{n-1} = \mathrm{id}_{B_{n-1}}.$$

Define an A-linear map $\varphi_n\colon C_n \to B_n$ on the basis by

$$\varphi_n(e_i) := s_{n-1}\big(\varphi_{n-1}(\widehat{d}_n(e_i))\big) \in B_n.$$

Then $d_n \circ \varphi_n = \varphi_{n-1} \circ \widehat{d}_n$.

Proof Since $d_n \circ \varphi_n$ and $\varphi_{n-1} \circ \widehat{d}_n$ are maps of A-modules, it is sufficient to show that they agree on the basis of C_n. By definition,

$$d_n(\varphi_n(e_i)) = (d_n \circ s_{n-1} \circ \varphi_{n-1} \circ \widehat{d}_n)(e_i).$$

Condition (iii) gives $d \circ s \circ \varphi \circ \widehat{d} = \varphi \circ \widehat{d} - s \circ d \circ \varphi \circ \widehat{d}$. From the commutativity of the diagram and condition (ii) we obtain $s \circ d \circ \varphi \circ \widehat{d} = s \circ \varphi \circ \widehat{d} \circ \widehat{d} = 0$. Hence

$$d_n(\varphi_n(e_i)) = \varphi_{n-1}(\widehat{d}_n(e_i)) - 0. \qquad \square$$

Corollary 2.2 *Let A be a connected $\boldsymbol{k}$-algebra, $(A \otimes V_\bullet, \widehat{d}_\bullet)$ be a free resolution of the left A-module $\boldsymbol{k}$. Let $\bar{\varphi}_0\colon V_0 \to \boldsymbol{k}$ be a map of $\boldsymbol{k}$-modules such that the diagram*

$$\begin{array}{ccccc} A \otimes V_1 & \xrightarrow{\widehat{d}_1} & A \otimes V_0 & \xrightarrow{\widehat{d}_0} & \boldsymbol{k} \\ & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes \bar{\varphi}_0} & & \Big\| {\scriptstyle \mathrm{id} =: \varphi_{-1}} \\ \mathrm{B}_1(A) & \xrightarrow{d_{\mathrm{B},1}} & A & \xrightarrow{\varepsilon} & \boldsymbol{k} \end{array}$$

commutes. Choose bases $\{e_i^{(n)}\}$ of $\boldsymbol{k}$-modules V_n, and define A-linear maps $\varphi_n\colon A \otimes V_n \to \mathrm{B}_n(A)$ recursively as

$$\varphi_0 := \mathrm{id}_A \otimes \bar{\varphi}_0, \quad \varphi_n(a \otimes e_i^{(n)}) := a \cdot s_{n-1}\big(\varphi_{n-1}(\widehat{d}_n(e_i^{(n)}))\big),$$

where $s_{n-1}\colon \mathrm{B}_{n-1}(A) \to \mathrm{B}_n(A)$ is the contracting homotopy (5).

Then $\varphi_\bullet\colon (A \otimes V_\bullet, \widehat{d}) \to (\mathrm{B}_\bullet(A), d_\mathrm{B})$ is a chain map.

Proof Induction on n. For $n = 0$ the identity

$$d_{\mathrm{B},n} \circ \varphi_n = \varphi_{n-1} \circ \widehat{d}_n$$

holds, since the diagram commutes. The inductive step from $n-1$ to n is supplied by Lemma 2.1. $\square$

2.4 Hopf algebra extensions and loop homology

If A is a Hopf algebra over $\boldsymbol{k}$, we denote the comultiplication by $\Delta\colon A \to A \otimes A$ and the (co)unit maps by $\eta_A\colon \boldsymbol{k} \to A$, $\varepsilon_A\colon A \to \boldsymbol{k}$. A graded $\boldsymbol{k}$-Hopf algebra A is *connected* if $A_{<0} = 0$, $A_0 = \boldsymbol{k} \cdot 1$. The counit is then the standard augmentation $\varepsilon\colon A \to A_0 \simeq \boldsymbol{k}$.

Definition 2.3 Let $\iota\colon A \to C$, $\pi\colon C \to B$ be morphisms of $\boldsymbol{k}$-Hopf algebras. They form an *extension of Hopf algebras*, or a *short exact sequence of Hopf algebras*

$$\boldsymbol{k} \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to \boldsymbol{k},$$

if

(i) ι is injective;

(ii) π is surjective;

(iii) $\pi \circ \iota = \varepsilon$;

(iv) $\operatorname{Ker} \pi = I(A) \cdot C$;

(v) $\operatorname{Im} \iota = \{x \in C : ((\mathrm{id}_C \otimes \pi) \circ \Delta)(x) = x \otimes 1\}$.

See [2, Definition 1.2.0, Proposition 1.2.3] for an equivalent and more "symmetrical" definition. Extensions of connected Hopf algebras were studied implicitly in [31, Section 4].

Proposition 2.4 (see [31, Proposition 4.9]) *Let $\iota\colon A \to C$, $\pi\colon C \to B$ be maps of connected $\boldsymbol{k}$-Hopf algebras. Suppose that a map $\Phi\colon A \otimes B \to C$ is an isomorphism of left A-modules and right C-comodules, and suppose that*

$$\iota = \Phi \circ (\mathrm{id}_A \otimes \eta_B), \quad \pi \circ \Phi = \varepsilon_A \otimes \mathrm{id}_B.$$

Then $\boldsymbol{k} \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to \boldsymbol{k}$ is an extension of Hopf algebras. Conversely, for every Hopf algebra extension there is a map Φ with the described properties. □

Our main example of Hopf algebras are Pontryagin algebras (loop homology) of connected topological spaces. Let $\boldsymbol{k}$ be a commutative ring, Y be a topological space such that $H_*(Y;\boldsymbol{k})$ is a free $\boldsymbol{k}$-module. Then $H_*(Y;\boldsymbol{k})$ is supplied with the cocommutative *cup coproduct* which is dual to the cup product on $H^*(Y;\boldsymbol{k})$: it is the composition

$$H_*(Y;\boldsymbol{k}) \xrightarrow{\Delta_*} H_*(Y \times Y;\boldsymbol{k}) \xrightarrow[\simeq]{\mathrm{AW}_*} H_*(C_*(Y;\boldsymbol{k}) \otimes C_*(Y;\boldsymbol{k})) \xleftarrow[\simeq]{\kappa} H_*(Y;\boldsymbol{k}) \otimes H_*(Y;\boldsymbol{k}),$$

where AW is the Alexander–Whitney map and κ is the Künneth isomorphism. If Y is also an H-space, the cup coproduct respects the Pontryagin product

$$m\colon H_*(Y;\boldsymbol{k}) \otimes H_*(Y;\boldsymbol{k}) \xrightarrow{\times} H_*(Y \times Y;\boldsymbol{k}) \xrightarrow{\mu_*} H_*(Y;\boldsymbol{k})$$

and hence $H_*(Y;\boldsymbol{k})$ is a cocommutative Hopf algebra. In particular, $H_*(\Omega X;\boldsymbol{k})$ is a connected cocommutative $\boldsymbol{k}$-Hopf algebra whenever X is a simply connected space such that $H_*(\Omega X;\boldsymbol{k})$ *is free over* $\boldsymbol{k}$ [31, 8.9]. Otherwise κ fails to be an isomorphism, hence the coproduct is not defined and $H_*(\Omega X;\boldsymbol{k})$ is merely a connected associative $\boldsymbol{k}$-algebra with unit.

In Appendix B we describe a situation when a fibration $F\to E\to B$ of simply connected spaces gives rise to an extension $\boldsymbol{k}\to H_*(\Omega F;\boldsymbol{k})\to H_*(\Omega E;\boldsymbol{k})\to H_*(\Omega B;\boldsymbol{k})\to\boldsymbol{k}$ of connected Hopf algebras.

3 Preliminaries: toric topology

3.1 Simplicial complexes and polyhedral products

A *simplicial complex* $\mathcal{K}$ on the vertex set W is a nonempty family of subsets $I\subset W$ that is closed under taking subsets. Elements $I\in\mathcal{K}$ are called *faces*. We suppose that $\mathcal{K}$ *has no ghost vertices*, ie $\{i\}\in\mathcal{K}$ for all $i\in W$. Usually $W\subset[m]:=\{1,\dots,m\}$. Sometimes by properties of a complex $\mathcal{K}$ we mean properties of its *geometrical realisation*, of the topological space $|\mathcal{K}|:=\bigcup_{I\in\mathcal{K}}\Delta_I\subset\Delta_W$.

For every $J\subset W$, a simplicial complex $\mathcal{K}_J:=\{I\in\mathcal{K}:I\subset J\}$ on the vertex set J (a *full subcomplex* of $\mathcal{K}$) is defined.

Throughout the text, we write $I\setminus i:=I\setminus\{i\}$ for $i\in I$ and $I\sqcup i:=I\sqcup\{i\}$ for $i\in W\setminus I$. A subset $I\subset W$ is a *missing face* of $\mathcal{K}$ if $I\notin\mathcal{K}$, but $I\setminus i\in\mathcal{K}$ for all $i\in I$. A simplicial complex $\mathcal{K}$ is *flag* if all its missing faces consist of two elements.

For every complex $\mathcal{K}$ on a vertex set $[m]$, the $\mathbb{Z}^m_{\geq 0}$-graded *Stanley–Reisner ring*

$$\boldsymbol{k}[\mathcal{K}]:=\boldsymbol{k}[v_1,\dots,v_m]/\Big(\prod_{i\in I}v_i=0,\,I\notin\mathcal{K}\Big),\quad \deg v_i:=2e_i\in\mathbb{Z}^m_{\geq 0},$$

is defined. It has a homogeneous basis $\big\{v^\alpha:=\prod_{i=1}^m v_i^{\alpha_i}\mid\operatorname{supp}\alpha\in\mathcal{K}\big\}$ as a $\boldsymbol{k}$-module. The dual $\boldsymbol{k}$-module $\boldsymbol{k}\langle\mathcal{K}\rangle$ is called the *Stanley–Reisner coalgebra*. It has an additive basis $\{\chi_\alpha\mid\operatorname{supp}\alpha\in\mathcal{K}\}$, $\deg\chi_\alpha=2\alpha$, and commutative associative comultiplication $\Delta\chi_\alpha:=\sum_{\alpha=\beta+\gamma}\chi_\beta\otimes\chi_\gamma$.

Now let $\mathcal{K}$ be a simplicial complex on $[m]$ and $(\underline{X},\underline{A}):=((X_1,A_1),\dots,(X_m,A_m))$ be a sequence of pairs of topological spaces. Their *polyhedral product* $(\underline{X},\underline{A})^{\mathcal{K}}$ is the union

$$(\underline{X},\underline{A})^{\mathcal{K}}:=\bigcup_{I\in\mathcal{K}}(\underline{X},\underline{A})^I\subset X^m,\quad (\underline{X},\underline{A})^I=Y_1\times\cdots\times Y_m,\ \ Y_j:=\begin{cases}X_j, & j\in I;\\ A_j, & j\notin I.\end{cases}$$

The addition of a ghost vertex v to $\mathcal{K}$ replaces the space $(\underline{X},\underline{A})^{\mathcal{K}}$ with $(\underline{X},\underline{A})^{\mathcal{K}}\times A_v$. Hence in many cases it is sufficient to consider only complexes without ghost vertices.

Define $(X,A)^{\mathcal{K}}:=(\underline{X},\underline{A})^{\mathcal{K}}$ if $X_i=X$, $A_i=A$ for all $i\in[m]$. We consider two special cases of this construction: *moment-angle complexes* $\mathcal{Z}_{\mathcal{K}}:=(D^2,S^1)^{\mathcal{K}}$ and *Davis–Januszkiewicz spaces*

$\mathrm{DJ}(\mathcal{K}) := (\mathbb{C}P^\infty, *)^{\mathcal{K}}$. It is well known that $H^*(\mathrm{DJ}(\mathcal{K});\boldsymbol{k}) \cong \boldsymbol{k}[\mathcal{K}]$ and $H^*(\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}) \cong \mathrm{Tor}_{\boldsymbol{k}[m]}(\boldsymbol{k}[\mathcal{K}],\boldsymbol{k})$ as graded rings. Moreover,

$$H^n(\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}) = \bigoplus_{n=-i+2|J|} H^{-i,2J}(\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}), \quad H^{-i,2J}(\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}) \cong \widetilde{H}^{|J|-i-1}(\mathcal{K}_J;\boldsymbol{k}),$$

and the product has a geometric description in terms of maps $\mathcal{K}_{I\sqcup J} \hookrightarrow \mathcal{K}_I * \mathcal{K}_J$; see [15, Theorem 4.5.8].

3.2 Loop homology as Hopf algebras

Proposition 3.1 [15, Theorem 4.3.2, Section 8.4] *There is a homotopy fibration*

$$\mathcal{Z}_{\mathcal{K}} \to \mathrm{DJ}(\mathcal{K}) \xrightarrow{i} (\mathbb{C}P^\infty)^m$$

of simply connected spaces, where i is the standard inclusion. The map Ωi admits a homotopy section $\sigma\colon \mathbb{T}^m \to \Omega\mathrm{DJ}(\mathcal{K})$ that corresponds to the choice of generators in $\pi_2(\mathrm{DJ}(\mathcal{K})) \cong \mathbb{Z}^m$ and gives rise to a homotopy equivalence $\Omega\mathrm{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$. □

The following description of $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ was first given in [35, (8.4)] for $\boldsymbol{k} = \mathbb{Q}$, but the argument is easily generalised to the arbitrary coefficient ring. The main ingredients are integral *formality* of $\mathrm{DJ}(\mathcal{K})$ [34], Adams' cobar construction (see [18]) and a result of Fröberg [21].

Theorem 3.2 [43, Theorem 1.1] *For any simplicial complex $\mathcal{K}$ with no ghost vertices and any commutative ring $\boldsymbol{k}$, we have an isomorphism $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k}) \cong \mathrm{Ext}_{\boldsymbol{k}[\mathcal{K}]}(\boldsymbol{k},\boldsymbol{k})$ of graded $\boldsymbol{k}$-algebras (with respect to the Pontryagin product and to the Yoneda product). More precisely,*

$$H_n(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k}) \cong \bigoplus_{-i+2|\alpha|=n} \mathrm{Ext}^i_{\boldsymbol{k}[\mathcal{K}]}(\boldsymbol{k},\boldsymbol{k})_{2\alpha}.$$

This isomorphism defines the $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading on $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$. The "diagonal" subalgebra

$$D = \bigoplus_{\alpha\in\mathbb{Z}^m_{\geq 0}} H_{-|\alpha|,2\alpha}(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k}) \subset H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$$

is isomorphic to the algebra

$$\boldsymbol{k}[\mathcal{K}]^! := T(u_1,\dots,u_m)/(u_i^2=0,\ i=1,\dots,m;\ u_iu_j+u_ju_i=0,\ \{i,j\}\in\mathcal{K}), \quad \deg u_i = (-1,2e_i).$$

For a flag $\mathcal{K}$, the algebra $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ coincides with D, and we have $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k}) \cong \boldsymbol{k}[\mathcal{K}]^!$. □

If $H_*(\Omega Y;\boldsymbol{k})$ is a free $\boldsymbol{k}$-module, the cup coproduct is compatible with the Pontryagin product, hence this associative algebra is a cocommutative $\boldsymbol{k}$-Hopf algebra. Similarly, if A is a commutative graded $\boldsymbol{k}$-algebra such that $\mathrm{Ext}_A(\boldsymbol{k},\boldsymbol{k})$ is a free $\boldsymbol{k}$-module, then the shuffle product on the bar construction (see [28, Theorem X.12.2]) induces a commutative coproduct on $\mathrm{Ext}_A(\boldsymbol{k},\boldsymbol{k})$ that is compatible with the Yoneda product. In our case, these coproduct coincide. This follows from a stronger formality result for Davis–Januszkiewicz spaces, the *hga formality* [20, Theorem 1.3].

Proposition 3.3 [20, Proposition 6.5] *Let $\mathcal{K}$ be a simplicial complex with no ghost vertices, and let $\boldsymbol{k}$ be a principal ideal domain such that $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k})$ is a free $\boldsymbol{k}$-module. Then*

$$H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k}) \cong \mathrm{Ext}_{\boldsymbol{k}[\mathcal{K}]}(\boldsymbol{k}, \boldsymbol{k})$$

as Hopf algebras.

Outline of the proof Let A be a dga algebra. The *homotopy Gerstenhaber algebra* (hga) structure on A is a multiplication on its bar construction $\overline{\mathrm{B}}(A)$ such that $\overline{\mathrm{B}}(A)$ becomes a dga bialgebra [20, Section 4]. This structure arises naturally if A is commutative (then the multiplication is the shuffle product) or if $A = C^*(X; \boldsymbol{k})$ is the dga algebra of cochains of a 1-reduced simplicial set (then the multiplication was essentially constructed by Baues [8, Section 2]). Then $H^*(\Omega X; \boldsymbol{k}) \cong H^*[\overline{\mathrm{B}}(C^*(X; \boldsymbol{k}))]$ as bialgebras. By a result of Franz [20, Theorem 1.3], hga algebras $C^*(\mathrm{DJ}(\mathcal{K}); \boldsymbol{k})$ and $\boldsymbol{k}[\mathcal{K}]$ are quasi-isomorphic. The functor $\overline{\mathrm{B}}$ preserves quasi-isomorphisms, so $H^*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k}) \cong H^*(\overline{\mathrm{B}}(\boldsymbol{k}[\mathcal{K}]); \boldsymbol{k}) \cong \mathrm{Tor}^{\boldsymbol{k}[\mathcal{K}]}(\boldsymbol{k}, \boldsymbol{k})$ as bialgebras. Since the Hopf algebra structure on a bialgebra is unique, it is an isomorphism of Hopf algebras. The statement for $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k})$ follows by dualisation. □

Remark 3.4 The algebra $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k})$ is not always a free $\boldsymbol{k}$-module. For example, let $\mathcal{K}$ be a minimal triangulation of $\mathbb{R}P^2$. Then $\mathcal{Z}_{\mathcal{K}}$ is a wedge of $\Sigma^7 \mathbb{R}P^2$ and spheres [23, Example 3.3]. We have $\Omega \mathrm{DJ}(\mathcal{K}) \simeq \Omega \mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$, hence $\Omega \Sigma^7 \mathbb{R}P^2$ is a retract of $\Omega \mathrm{DJ}(\mathcal{K})$. It follows that $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbb{Z})$ has 2-torsion.

Recall that an element x is called *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$, and a Hopf algebra is *primitively generated* if it is multiplicatively generated by its primitive elements.

Conjecture 3.5 *The Hopf algebra $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k})$ is primitively generated for every simplicial complex $\mathcal{K}$ and every ring $\boldsymbol{k}$ such that $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k})$ is a free $\boldsymbol{k}$-module.*

By deep results of André and Sjödin — see [4, Theorem 10.2.1(5)] — for every field $\boldsymbol{k}$ the Hopf algebra $\mathrm{Ext}_A(\boldsymbol{k}, \boldsymbol{k})$ is the universal enveloping of a Lie algebra (of a 2-restricted Lie algebra, if $\mathrm{char}\, \boldsymbol{k} = 2$). In particular, this Hopf algebra is primitively generated. (This also follows from results of Browder [11]; see [33, Theorem 10.4].) Hence Conjecture 3.5 holds if $\boldsymbol{k}$ is a field.

Remark 3.6 The Hopf algebra $H_*(\Omega X; \boldsymbol{k})$ is not always primitively generated, even if X is a suspension. For example, one can take $X = \Sigma \mathbb{C}P^2$, $\boldsymbol{k} = \mathbb{Z}$ or $\mathbb{Z}/2$ (see [13, Section 4.2]). On the other hand, [25, Theorem B] implies that $H_*(\Omega \Sigma \mathbb{C}P^d; \mathbb{Z}/p)$ is primitively generated for $p > d$.

Now we describe the connection between the loop homology of Davis–Januszkiewicz spaces and of moment-angle complexes in the form of a Hopf algebra extension.

Proposition 3.7 *Let $\mathcal{K}$ be a simplicial complex on $[m]$ and $\boldsymbol{k}$ be a commutative ring with unit, such that $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$ is a free $\boldsymbol{k}$-module. Then*

$$\boldsymbol{k} \to H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \boldsymbol{k}) \xrightarrow{\iota} H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k}) \xrightarrow{p} \Lambda[u_1, \dots, u_m] \to \boldsymbol{k}$$

is an extension of connected $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$*-graded* $\boldsymbol{k}$*-Hopf algebras. The projection* p *maps* u_i *to* u_i*. Its* $\boldsymbol{k}$*-linear section* $\sigma_*:\Lambda[u_1,\dots,u_m]\to H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ *is given by*

$$\sigma_*(u_I)=\widehat{u}_I:=u_{i_1}\cdots u_{i_k},\quad I=\{i_1<\dots<i_k\}.$$

Therefore, $\Phi(a\otimes u_I):=\iota(a)\cdot\widehat{u}_I$ *defines an isomorphism of left* $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$*-modules and right* $\Lambda[u_1,\dots,u_m]$*-comodules* $\Phi\colon H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})\otimes\Lambda[u_1,\dots,u_m]\to H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$*.*

Proof By Theorem B.3, the fibration from Proposition 3.1 gives rise to the required Hopf algebra extension. The formula for p follows from functoriality, since the map $\mathrm{DJ}(\mathcal{K})\hookrightarrow\mathrm{DJ}(\Delta_{[m]})\cong(\mathbb{C}P^\infty)^m$ is induced by the inclusion $\mathcal{K}\hookrightarrow\Delta_{[m]}$. The formula for σ_* follows from the description of the homotopy section $\sigma\colon\mathbb{T}^m\simeq\Omega B\mathbb{T}^m=(\Omega\mathbb{C}P^\infty)^{\times m}\to\Omega\mathrm{DJ}(\mathcal{K})$ as a concatenation of loops, $(\gamma_1,\dots,\gamma_m)\mapsto\gamma_1\cdots\gamma_m$. The maps p and σ_* respect the $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading, hence the multigrading on $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ is well defined. □

Since ι is injective, we identify elements of $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ with their images in $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$. Let us describe some of these elements. Recall that we define $[a,b]:=ab+(-1)^{\deg(a)\deg(b)+1}ba$ and $c(I,x):=[u_{i_1},[u_{i_2},\dots,[u_{i_k},x],\dots]]\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ for $I=\{i_1<\dots<i_k\}$ and $x\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$. In particular, $c(\varnothing,x):=x$ and $c(\{i\},u_j)=[u_i,u_j]=u_iu_j+u_ju_i$.

Corollary 3.8 *Let* $x\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ *be a primitive element such that* $p(x)=0$*. Then* $x\in H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$*.*

Proof This follows from Corollary B.4 applied to the Hopf algebra extension from Proposition 3.7. □

Corollary 3.9 *Let* $x\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ *be a primitive element and* $I\subset[m]$*,* $I\neq\varnothing$*. Then* $c(I,x)$ *is an element of* $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$*.*

Proof Elements $u_1,\dots,u_m\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ are primitive for dimension reasons. Primitive elements form a Lie algebra, hence $c(I,x)\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ is primitive. We have $p(c(I,x))=c(I,p(x))=0$, since it is a commutator in the commutative algebra $\Lambda[m]$. Then $c(I,x)\in H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ by Corollary 3.8. □

Corollary 3.10 *Let* $j\in[m]$ *and* $I\subset[m]$*,* $I\neq\varnothing$*. Then* $c(I,u_j)$ *is an element of* $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$*.* □

3.3 The flag case

Let $\mathcal{K}$ be a flag complex with no ghost vertices. By Theorem 3.2, $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})\cong\boldsymbol{k}[\mathcal{K}]^!$ is a free $\boldsymbol{k}$-module, hence the Hopf algebra structure on $H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$ is well defined. Moreover, the connected $\boldsymbol{k}$-algebra $\boldsymbol{k}[\mathcal{K}]^!$ is generated by elements of degree 1. These conditions determine the Hopf algebra structure on $\boldsymbol{k}[\mathcal{K}]^!$ uniquely: the elements $u_1,\dots,u_m$ are primitive. Therefore, in the flag case Conjecture 3.5 is true for any $\boldsymbol{k}$.

The following important result was recently obtained by Stanton.

Theorem 3.11 [39, Corollary 1.5] *Let $\mathcal{K}$ be a flag simplicial complex or a skeleton of a flag complex. Then $\Omega\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to a finite-type product of spaces of the form S^1, S^3, S^7 and ΩS^n for $n \geq 2$, $n \neq 2, 4, 8$.* □

This gives a short proof of the fact that $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$ is free over $\boldsymbol{k}$.

Proposition 3.12 [23, Corollary 5.2] *If $\mathcal{K}$ is a flag simplicial complex, then $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$ is a free $\boldsymbol{k}$-module of finite type.*

Proof By the Künneth formula (more precisely, by the collapse of the Künneth spectral sequence — see [37, Theorem 10.90]), $H_*(X \times Y; \boldsymbol{k}) \simeq H_*(X; \boldsymbol{k}) \otimes H_*(Y; \boldsymbol{k})$ if $H_*(X; \boldsymbol{k})$ and $H_*(Y; \boldsymbol{k})$ are free over $\boldsymbol{k}$. Hence $H_*(X \times Y; \boldsymbol{k})$ is also a free $\boldsymbol{k}$–module.

Clearly, $H_*(S^n; \boldsymbol{k})$ and $H_*(\Omega S^n; \boldsymbol{k}) \simeq T(a_{n-1})$ are free $\boldsymbol{k}$-modules. By Theorem 3.11 and the arguments above, the same holds for $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$. □

Hence in the flag case we have a Hopf algebra extension

$$\boldsymbol{k} \to H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k}) \to H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k}) \to \Lambda[m] \to \boldsymbol{k}$$

from Proposition 3.7 for any $\boldsymbol{k}$.

4 Main calculations

In what follows, $\mathcal{K}$ is a flag simplicial complex on the vertex set $[m]$ with no ghost vertices, and $\boldsymbol{k}$ is a commutative ring with unit. We consider $\mathbb{Z} \times \mathbb{Z}^m_{\geq 0}$-graded $\boldsymbol{k}$-algebras that are connected with respect to the total grading $A_n := \bigoplus_{n=-i+|\alpha|} A_{-i,\alpha}$.

4.1 Resolutions and formulas for differentials

By [43, Proposition 4.1], the left $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k})$-module $\boldsymbol{k}$ has a free resolution $(H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k}) \otimes \boldsymbol{k}\langle\mathcal{K}\rangle, d)$, $\deg \chi_\alpha := (|\alpha|, -|\alpha|, 2\alpha)$, $\deg(d) = (-1, 0, 0)$, with the differential

$$d(1 \otimes \chi_\alpha) := \sum_{i \in \mathrm{supp}(\alpha)} u_i \otimes \chi_{\alpha - e_i}.$$

The isomorphism of left $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$-modules

$$\Phi\colon H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k}) \otimes \Lambda[m] \to H_*(\Omega \mathrm{DJ}(\mathcal{K}); \boldsymbol{k}), \quad a \otimes u_I \mapsto a \cdot \hat{u}_I,$$

from Proposition 3.7 allows us to consider this resolution as a free resolution $(H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k}) \otimes \Lambda[m] \otimes \boldsymbol{k}\langle\mathcal{K}\rangle, \hat{d})$ of the left $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$-module $\boldsymbol{k}$. We apply the functor $\boldsymbol{k} \otimes_{H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})} (-)$ and obtain a chain complex

$(\Lambda[m]\otimes \boldsymbol{k}\langle\mathcal{K}\rangle, \overline{d})$ whose homology is isomorphic to $\operatorname{Tor}^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})$. The differentials $\widehat{d}$ and $\overline{d}$ are determined by the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_*(\Omega \mathrm{DJ}(\mathcal{K});\boldsymbol{k})\otimes \boldsymbol{k}\langle\mathcal{K}\rangle_{(n)} & \xrightarrow{\ d\ } & H_*(\Omega \mathrm{DJ}(\mathcal{K});\boldsymbol{k})\otimes \boldsymbol{k}\langle\mathcal{K}\rangle_{(n-1)} & \longrightarrow \cdots \\
& \Phi\otimes\mathrm{id}\uparrow\simeq & & \Phi\otimes\mathrm{id}\uparrow\simeq & \\
\cdots \longrightarrow & H_*(\Omega \mathcal{Z}_{\mathcal{K}};\boldsymbol{k})\otimes\Lambda[m]\otimes \boldsymbol{k}\langle\mathcal{K}\rangle_{(n)} & \overset{\widehat{d}}{\dashrightarrow} & H_*(\Omega \mathcal{Z}_{\mathcal{K}};\boldsymbol{k})\otimes\Lambda[m]\otimes \boldsymbol{k}\langle\mathcal{K}\rangle_{(n-1)} & \longrightarrow \cdots \\
& \varepsilon\otimes\mathrm{id}\otimes\mathrm{id}\downarrow & & \varepsilon\otimes\mathrm{id}\otimes\mathrm{id}\downarrow & \\
\cdots \longrightarrow & \Lambda[m]\otimes \boldsymbol{k}\langle\mathcal{K}\rangle_{(n)} & \overset{\overline{d}}{\dashrightarrow} & \Lambda[m]\otimes \boldsymbol{k}\langle\mathcal{K}\rangle_{(n-1)} & \longrightarrow \cdots
\end{array}$$

Here $\boldsymbol{k}\langle\mathcal{K}\rangle_{(n)}$ is a $\boldsymbol{k}$-submodule in $\boldsymbol{k}\langle\mathcal{K}\rangle$ with the basis $\{\chi_\alpha : |\alpha| = n\}$. With different signs, this construction was considered by the author in [43, Section 4]. Now we describe the differential $\widehat{d}$ explicitly. For subsets $A, B \subset [m]$ define the Koszul sign $\theta(A,B) := |\{(a,b)\in A\times B : a > b\}|$.

Proposition 4.1 *The differential $\widehat{d}$ is given by*

$$\widehat{d}(1\otimes u_I\otimes\chi_\alpha) = \sum_{i\in\operatorname{supp}(\alpha)} (-1)^{|I|}\cdot 1\otimes(u_I\wedge u_i)\otimes\chi_{\alpha-e_i} + \sum_{i\in\operatorname{supp}(\alpha)}\ \sum_{\substack{I=A\sqcup B\\ \max(A)>i}} (-1)^{\theta(A,B)+|A|} c(A,u_i)\otimes u_B\otimes\chi_{\alpha-e_i}. \tag{7}$$

The differential $\overline{d}$ is given by

$$\overline{d}(u_I\otimes\chi_\alpha) = (-1)^{|I|}\sum_{i\in\operatorname{supp}(\alpha)} (u_I\wedge u_i)\otimes\chi_{\alpha-e_i}. \tag{8}$$

Remark 4.2 We define $\max(\varnothing) := -\infty$, hence A cannot be empty.

Proof of the proposition Recall that $u_j^2 = 0 \in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k})$. Therefore, by Proposition C.2,

$$\begin{aligned}
\widehat{u}_I\cdot u_i &= 1\cdot\begin{Bmatrix} (-1)^{|I_{>i}|}\widehat{u}_{I\sqcup i}, & i\notin I \\ 0, & i\in I\end{Bmatrix} + \sum_{\substack{I=A\sqcup B\\ \max(A)>i}} (-1)^{\theta(A,B)+|B|} c(A,u_i)\cdot\widehat{u}_B \\
&= \Phi\Big(1\otimes(u_I\wedge u_i) + \sum_{\substack{I=A\sqcup B\\ \max(A)>i}} (-1)^{\theta(A,B)+|B|} c(A,u_i)\otimes u_B\Big) \in H_*(\Omega\mathrm{DJ}(\mathcal{K});\boldsymbol{k}).
\end{aligned}$$

(Here $c(A,u_i)\in H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ by Corollary 3.10.) Define $\Phi_0 = \Phi\otimes\mathrm{id}_{\boldsymbol{k}\langle\mathcal{K}\rangle}$. Then

$$\begin{aligned}
&\Phi_0(\widehat{d}(1\otimes u_I\otimes\chi_\alpha)) \\
&\quad = d(\Phi_0(1\otimes u_I\otimes\chi_\alpha)) = d(\widehat{u}_I\otimes\chi_\alpha) = (-1)^{|I|}\sum_{i\in\operatorname{supp}(\alpha)} \widehat{u}_I u_i\otimes\chi_{\alpha-e_i} \\
&\quad = (-1)^{|I|}\sum_{i\in\operatorname{supp}(\alpha)} \Phi_0\Big(1\otimes(u_I\wedge u_i)\otimes\chi_{\alpha-e_i} + \sum_{\substack{I=A\sqcup B\\ \max(A)>i}} (-1)^{\theta(A,B)+|B|} c(A,u_i)\otimes u_B\otimes\chi_{\alpha-e_i}\Big).
\end{aligned}$$

Applying Φ_0^{-1}, we obtain precisely (7). After the homomorphism $\varepsilon\otimes\mathrm{id}\otimes\mathrm{id}$ it turns into (8), since $\varepsilon(1) = 1$ and $\varepsilon(c(A,u_i)) = 0$ for $A\neq\varnothing$. □

4.2 Computation of Tor-modules

By [43, Theorem 1.2], for a flag $\mathcal{K}$ we have a $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded isomorphism of $\boldsymbol{k}$-modules

$$(9)\qquad \operatorname{Tor}^{H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})\cong\bigoplus_{J\subset[m]}\widetilde{H}_*(\mathcal{K}_J;\boldsymbol{k}),\quad \operatorname{Tor}_n^{H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_{-|J|,2J}\cong\widetilde{H}_{n-1}(\mathcal{K}_J;\boldsymbol{k}).$$

The homology of $\mathcal{Z}_\mathcal{K}$ admits a $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading, and for any $\mathcal{K}$ we have a similar additive isomorphism dual to [15, Theorem 4.5.8]:

$$H_*(\mathcal{Z}_\mathcal{K};\boldsymbol{k})\cong\bigoplus_{J\subset[m]}\widetilde{H}_*(\mathcal{K}_J;\boldsymbol{k}),\quad H_{n-|J|,2J}(\mathcal{Z}_\mathcal{K};\boldsymbol{k})\cong\widetilde{H}_{n-1}(\mathcal{K}_J;\boldsymbol{k}).$$

Hence $\operatorname{Tor}^{H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})\cong H_*(\mathcal{Z}_\mathcal{K};\boldsymbol{k})$ for flag $\mathcal{K}$. Moreover, both modules are computed as the homology of $(\Lambda[u_1,\dots,u_m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,d)$.

Remark 4.3 In general, if X is simply connected and $H_*(\Omega X;\boldsymbol{k})$ is free over $\boldsymbol{k}$, there is *Milnor–Moore spectral sequence* $E^2_{p,q}=\operatorname{Tor}_p^{H_*(\Omega X;\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_q\Rightarrow H_{p+q}(X;\boldsymbol{k})$. We see that it collapses at E^2 for $X=\mathcal{Z}_\mathcal{K}$ if $\mathcal{K}$ is a flag complex. For $\boldsymbol{k}=\mathbb{Q}$, the collapse is explained by the *coformality* of $\mathcal{Z}_\mathcal{K}$; see Corollary 6.7 and the discussion after.

Now we construct a chain map g that induces the isomorphism (9). For any chain complex $(C_\bullet,d)$ of free $\boldsymbol{k}$-modules, we have the *dual complex*

$$(C^\bullet,d_{\mathrm{dual}}),\quad C^n:=\operatorname{Hom}_{\boldsymbol{k}}(C_n,\boldsymbol{k}),\quad d_{\mathrm{dual}}(f)\colon c\mapsto f(d(c)).$$

Dualisation preserves isomorphisms and chain homotopies. For a simplicial complex $\mathcal{K}$, the augmented complex of simplicial chains $\widetilde{C}_*(\mathcal{K};\boldsymbol{k})$ has the basis $\{[I]:I\in\mathcal{K}\}$, $\deg[I]:=|I|+1$ and the differential

$$d([I]):=\sum_{i\in I}(-1)^{|I_{<i}|}[I\setminus\{i\}].$$

The dual complex is the augmented complex of simplicial cochains $(\widetilde{C}^*(\mathcal{K};\boldsymbol{k}),d_{\mathrm{dual}})$, which has the basis $\{[I]^*:I\in\mathcal{K}\}$ and the differential

$$d_{\mathrm{dual}}([I]^*)=\sum_{\substack{i\notin I\\ I\sqcup i\in\mathcal{K}}}(-1)^{|I_{<i}|}[I\sqcup\{i\}]^*.$$

Proposition 4.4 *For every $J\subset[m]$, consider the map*

$$g_J\colon\widetilde{C}_{*-1}(\mathcal{K}_J;\boldsymbol{k})\to(\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle)_{*,-|J|,2J},\quad [L]\mapsto\epsilon(L,J)\cdot u_{J\setminus L}\otimes\chi_L,$$

where $\epsilon(L,J):=(-1)^{\sum_{\ell\in L}|J_{<\ell}|}$. Then g_J are chain maps, and the direct sum

$$g\colon\bigoplus_{J\subset[m]}\widetilde{C}_*(\mathcal{K}_J;\boldsymbol{k})\to(\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\bar{d})$$

induces an isomorphism on homology. Therefore,

$$H_{n,-|J|,2J}(\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\bar{d})\cong\widetilde{H}_{n-1}(\mathcal{K}_J;\boldsymbol{k}),\quad J\subset[m],\ n\geq 0,$$

all the other graded components of $H_*(\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\bar{d})$ being zero.

Since $\operatorname{Tor}^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k}) \cong H(\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\bar{d})$, this proposition implies (9). The proof is the dualisation of arguments from [15, Section 3.2].

Proof of Proposition 4.4 Consider the dga algebra $(\Lambda[u_1,\dots,u_m]\otimes\boldsymbol{k}[\mathcal{K}],d)$ with the differential that is defined on generators by $d(u_i)=v_i$, $d(v_i)=0$ and with the $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading

$$\deg u_i := (0,-1,2e_i),\quad \deg v_i := (1,-1,2e_i),\quad \deg d := (1,0,0).$$

This complex has the basis $\{u_I v^\alpha : I\subset[m],\ \alpha\in\mathbb{Z}^m_{\geq 0},\ \operatorname{supp}(\alpha)\in\mathcal{K}\}$ and the differentials

$$d(u_I v^\alpha)=\sum_{i\in I}(-1)^{|I_{<i}|}u_{I_{<i}}v_i u_{I_{>i}}v^\alpha=\sum_{i\in I}(-1)^{|I_{<i}|}u_{I\setminus i}v_i v^\alpha.$$

Then the dual complex $(\Lambda[m]\otimes\boldsymbol{k}[\mathcal{K}])^*$ has the basis $\{(u_I v^\alpha)^* : I\subset[m],\ \operatorname{supp}(\alpha)\in\mathcal{K}\}$ and the differential

$$d_{\mathrm{dual}}((u_I v^\alpha)^*)=\sum_{\substack{i\in\operatorname{supp}(\alpha)\\ i\notin I}}(-1)^{|I_{<i}|}(u_{I\sqcup i}v^{\alpha-e_i})^*.$$

This formula is similar to (8). We obtain an isomorphism of chain complexes

$$\psi\colon(\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\bar{d})\to((\Lambda[m]\otimes\boldsymbol{k}[\mathcal{K}])^*,d_{\mathrm{dual}}),\quad u_I\otimes\chi_\alpha\mapsto(u_I v^\alpha)^*.$$

Consider the dga algebra $R^*(\mathcal{K}) := (\Lambda[m]\otimes\boldsymbol{k}[\mathcal{K}])/(u_i v_i = v_i^2 = 0,\ i=1,\dots,m)$. It is well defined, since the ideal $(u_i v_i, v_i^2)\subset\Lambda[m]\otimes\boldsymbol{k}[\mathcal{K}]$ is d-invariant. The following facts are obtained in the proof of [15, Theorem 3.2.9].

Lemma 4.5 [15, Lemma 3.2.6] *The natural projection $\pi\colon\Lambda[m]\otimes\boldsymbol{k}[\mathcal{K}]\to R^*(\mathcal{K})$ is a chain homotopy equivalence.* □

Lemma 4.6 *We have well-defined chain maps $f_J\colon\widetilde{C}^*(\mathcal{K}_J;\boldsymbol{k})\to R^*(\mathcal{K})$,*

$$f_J\colon\widetilde{C}^{n-1}(\mathcal{K}_J;\boldsymbol{k})\xrightarrow{\cong}R^{n,-n,2J}(\mathcal{K}),\quad [L]^*\mapsto\epsilon(L,J)\cdot u_{J\setminus L}v^L,\ \ \epsilon(L,J):=(-1)^{\sum_{\ell\in L}|J_{<\ell}|}.$$

The direct sum $f\colon\bigoplus_{J\subset[m]}\widetilde{C}^*(\mathcal{K}_J;\boldsymbol{k})\to R^*(\mathcal{K})$ is an isomorphism of chain complexes. □

After dualisation, we obtain a chain homotopy equivalence π^* and an isomorphism f^* of chain complexes. It remains to show that the diagram

$$\begin{array}{ccc}\bigoplus_{J\subset[m]}\widetilde{C}^*(\mathcal{K}_J;\boldsymbol{k}) & \xrightarrow{\ g\ } & (\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\bar{d})\\ {\scriptstyle f^*}\uparrow{\scriptstyle\simeq} & & {\scriptstyle\psi}\uparrow{\scriptstyle\simeq}\\ (R^*(\mathcal{K}))^* & \xrightarrow[\sim]{\ \pi^*\ } & ((\Lambda[m]\otimes\boldsymbol{k}[\mathcal{K}])^*,d_{\mathrm{dual}})\end{array}$$

is commutative. Indeed, $f^*((u_{J\setminus L}v^L)^*)=\epsilon(L,J)\cdot[L]$, hence

$$g\big(f^*((u_{J\setminus L}v^L)^*)\big)=\epsilon(L,J)\cdot\epsilon(L,J)u_{J\setminus L}\otimes\chi_L=\psi\big(\pi^*((u_{J\setminus L}v^L)^*)\big).\qquad\square$$

Remark 4.7 In our notation, $\epsilon(L,J)=(-1)^n$, $n=\theta(J\setminus L,L)+|L|(|L|-1)/2$ for $L\subset J$.

4.3 A chain map to the bar resolution

Theorem 4.8 *The identity map of the left $H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})$-module $\boldsymbol{k}$ can be extended to the map of free resolutions $\varphi_\bullet\colon (H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})\otimes\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\widehat{d})\to\big(\mathrm{B}_*(H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})),d_{\mathrm{B}}\big)$, given by*

$$\varphi_n(u_I\otimes\chi_\alpha)=(-1)^{|I|}\sum_{\substack{\alpha=e_{i_1}+\dots+e_{i_n}\\ I=A_1\sqcup\dots\sqcup A_n\\ \max(A_t)>i_t,\forall t\in[n]}}(-1)^{\sum_{1\le t_1<t_2\le n}\theta(A_{t_1},A_{t_2})}[c(A_1,u_{i_1})|\dots|c(A_n,u_{i_n})].$$

Proof We apply Corollary 2.2 for $\bar\varphi_0(u_I)=\varepsilon(u_I)$. It is sufficient to show that

$$\varphi_{n+1}(u_I\otimes\chi_\alpha)=s\big(\varphi_n(\widehat{d}(u_I\otimes\chi_\alpha))\big)$$

for $|\alpha|=n+1$, $n\ge 0$. By (7) and by the $H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})$-linearity of φ_n, we have

$$\varphi_n(\widehat{d}(u_I\otimes\chi_\alpha))=\sum_{i\in\operatorname{supp}(\alpha)}(-1)^{|I|}\varphi_n((u_I\wedge u_i)\otimes\chi_{\alpha-e_i})\\ +\sum_{i\in\operatorname{supp}(\alpha)}\sum_{\substack{I=A\sqcup B\\ \max(A)>i}}(-1)^{\theta(A,B)+|A|}c(A,u_i)\varphi_n(u_B\otimes\chi_{\alpha-e_i}).$$

The map s is trivial on summands of the first sum, since they belong to

$$\overline{\mathrm{B}}(H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k}))\subset\operatorname{Ker}s\subset\mathrm{B}(H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})).$$

Hence we have

$$s\big(\varphi_n(\widehat{d}(u_I\otimes\chi_\alpha))\big)=0+\sum_{i\in\operatorname{supp}(\alpha)}\sum_{\substack{I=A\sqcup B\\ \max(A)>i}}\sum_{\substack{\alpha-e_i=e_{i_1}+\dots+e_{i_n}\\ B=A_1\sqcup\dots\sqcup A_n\\ \max(A_t)>i_t,\forall t\in[n]}}(-1)^{\zeta}s\big(c(A,u_i)[c(A_1,u_{i_1})|\dots|c(A_n,u_{i_n})]\big)\\ =\sum_{i\in\operatorname{supp}(\alpha)}\sum_{\substack{I=A\sqcup B\\ \max(A)>i}}\sum_{\substack{\alpha-e_i=e_{i_1}+\dots+e_{i_n}\\ B=A_1\sqcup\dots\sqcup A_n\\ \max(A_t)>i_t,\forall t\in[n]}}(-1)^{\zeta}\big[c(A,u_i)|c(A_1,u_{i_1})|\dots|c(A_n,u_{i_n})\big],$$

where $\zeta=|B|+\theta(A,B)+|A|+\sum_{1\le t_1<t_2\le n}\theta(A_{t_1},A_{t_2})$. Defining $i=i_0$, $A=A_0$, we obtain

$$s\big(\varphi_n(\widehat{d}(u_I\otimes\chi_\alpha))\big)\\ =\sum_{\substack{\alpha=e_{i_0}+\dots+e_{i_n}\\ I=A_0\sqcup\dots\sqcup A_n\\ \max(A_t)>i_t,0\le t\le n}}(-1)^{\sum_{t=1}^{n}\theta(A_0,A_t)+|I|+\sum_{1\le t_1<t_2\le n}\theta(A_{t_1},A_{t_2})}[c(A_0,u_{i_0})|\dots|c(A_n,u_{i_n})].$$

The right-hand side equals $\varphi_{n+1}(u_I\otimes\chi_\alpha)$ up to a shift of indices. □

Theorem 4.9 *Let $J\subset[m]$. Let a class $\alpha\in\operatorname{Tor}_n^{H_*(\Omega\mathcal{Z}_\mathcal{K})}(\boldsymbol{k},\boldsymbol{k})_{-|J|,2J}\cong\widetilde{H}_{n-1}(\mathcal{K}_J;\boldsymbol{k})$ be represented by a cycle*

$$\kappa=\sum_{\substack{I\in\mathcal{K}_J\\ |I|=n}}\lambda_I\cdot[I]\in\widetilde{C}_{n-1}(\mathcal{K}_J;\boldsymbol{k}).$$

Then the same class is represented by the cycle $\kappa' \in \overline{\mathrm{B}}_n(H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}))_{-|J|,2J}$ *in the bar construction*,

$$\kappa' := \sum_{\substack{I\in\mathcal{K}_J\\ |I|=n}} \epsilon(I,J)\lambda_I \sum_{\substack{I=\{i_1,\dots,i_n\}\\ J\setminus I=J_1\sqcup\dots\sqcup J_n\\ \max(J_t)>i_t,\forall t\in[n]}} (-1)^{\sum_{1\le t_1<t_2\le n}\theta(J_{i_1},J_{i_2})}[c(J_1,u_{i_1})|\dots|c(J_n,u_{i_n})].$$

Proof The map $\widetilde{H}_*(\mathcal{K}_J;\boldsymbol{k}) \to \mathrm{Tor}_*^{H_*(\Omega\mathcal{Z}_{\mathcal{K}})}(\boldsymbol{k},\boldsymbol{k})$ is induced by the composition

$$\bigoplus_{J\subset[m]}\widetilde{C}_*(\mathcal{K}_J;\boldsymbol{k}) \xrightarrow[\sim]{g} (\Lambda[m]\otimes\boldsymbol{k}\langle\mathcal{K}\rangle,\bar{d}) \xrightarrow[\sim]{\bar{\varphi}} (\overline{\mathrm{B}}(H_*(\Omega\mathcal{Z}_{\mathcal{K}})),d_{\overline{\mathrm{B}}}),$$

of chain maps, where g is defined in Proposition 4.4 and $\bar{\varphi}$ is induced by the chain map φ from Theorem 4.8. We have $\kappa' = \bar{\varphi}(g(\kappa))$ by construction. □

The formulas become simpler for $n = 1, 2$.

Corollary 4.10 *Let* $J \subset [m]$. *Let a class*

$$\alpha \in \mathrm{Tor}_2^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_{-|J|,2J} \cong \widetilde{H}_1(\mathcal{K}_J)$$

be represented by a cycle

$$\kappa = \sum_{\{i<j\}\in\mathcal{K}_J} \lambda_{ij}[\{i,j\}] \in \widetilde{C}_1(\mathcal{K}_J;\boldsymbol{k}).$$

Then the same class is represented by the following cycle in the bar construction:

$$\kappa' = \sum_{\{i<j\}\in\mathcal{K}_J} (-1)^{|J_{<i}|+|J_{<j}|}\lambda_{ij} \sum_{\substack{J\setminus\{i,j\}=A\sqcup B\\ \max(A)>i\\ \max(B)>j}} (-1)^{\theta(A,B)}[c(A,u_i)|c(B,u_j)] + (-1)^{\theta(B,A)}[c(B,u_j)|c(A,u_i)].$$ □

Corollary 4.11 *Let* $J \subset [m]$, *and let the simplicial complex* $\mathcal{K}_J$ *have* $t+1$ *path components. Let vertices* $i_1,\dots,i_t,\max(J)$ *be representatives of these components. Then a basis of the* $\boldsymbol{k}$*-module* $\mathrm{Tor}_1^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_{-|J|,2J} \cong \widetilde{H}_0(\mathcal{K}_J;\boldsymbol{k}) \simeq \boldsymbol{k}^t$ *is represented by cycles*

$$[c(J\setminus i_s,u_{i_s})] \in \overline{\mathrm{B}}_1(H_*(\Omega\mathcal{Z}_{\mathcal{K}}))_{-|J|,2J}, \quad s=1,\dots,t.$$

Proof Define $j := \max(J)$. The cycles $\kappa_s = [\{j\}] - [\{i_s\}] \in \widetilde{C}_0(\mathcal{K}_J;\boldsymbol{k})$, $1 \le s \le t-1$, represent a basis in $\widetilde{H}_0(\mathcal{K}_J;\boldsymbol{k})$. By Theorem 4.9, the basis in $\mathrm{Tor}_1^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_{-|J|,2J}$ is represented by cycles

$$\kappa'_s = 0 \pm [c(J\setminus i_s,u_{i_s})], \quad s=1,\dots,t.$$

(The summand $[\{j\}]$ in κ_s does not contribute to κ'_s, since the subset $J_1 := J\setminus\{j\}$ does not satisfy the condition $\max(J_1) > j$.) □

5 Generators and relations in the flag case

5.1 Minimal sets of generators

Let M be a finitely generated $\boldsymbol{k}$-module. Denote the smallest number of generators by $\operatorname{gen}(M)$, and define $\widetilde{b}_i(X;\boldsymbol{k}) := \operatorname{gen}(\widetilde{H}_i(X;\boldsymbol{k}))$. Since $\widetilde{H}_0(X;\boldsymbol{k})$ is a free module, this number does not depend on $\boldsymbol{k}$ if $\boldsymbol{k}$ is a principal ideal domain, so we write $\widetilde{b}_0(X;\boldsymbol{k}) = \widetilde{b}_0(X)$. (In fact, $\widetilde{b}_0(X)+1$ is the number of path components in X).

Theorem 5.1 *Let $\mathcal{K}$ be a flag simplicial complex on a vertex set $[m]$ and $\boldsymbol{k}$ be a commutative ring with unit. For every $J \subset [m]$, choose a $\widetilde{b}_0(\mathcal{K}_J)$-element subset $\Theta(J) \subset J \setminus \{\max(J)\}$ such that $\Theta(J) \sqcup \{\max(J)\}$ contains exactly one vertex from each path component of $\mathcal{K}_J$. Then $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ is multiplicatively generated by the following set of $\sum_{J\subset[m]} \widetilde{b}_0(\mathcal{K}_J)$ elements:*

$$\{c(J\setminus i, u_i) : i \in \Theta(J),\ J \subset [m]\}, \quad c(J\setminus i, u_i) \in H_{-|J|,2J}(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}).$$

If $\boldsymbol{k}$ is a principal ideal domain, this set is minimal: any $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous presentation of $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ contains at least $\widetilde{b}_0(\mathcal{K}_J)$ generators of degree $(-|J|, 2J)$; any $\mathbb{Z}$-homogeneous presentation contains at least $\sum_{|J|=n} \widetilde{b}_0(\mathcal{K}_J)$ generators of degree n.

Proof By Corollary 4.11, images of cycles $\{[c(J\setminus i, u_i)] : J \subset [m],\ i \in \Theta(J)\} \subset \overline{\mathrm{B}}_1(H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k}))$ additively generate the $\boldsymbol{k}$-module $\operatorname{Tor}_1^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})$. Hence, by Theorem A.6(i), the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ is multiplicatively generated by the elements in question. The lower bounds on the number of generators follow from (9) and from Theorem A.10(ii). □

Definition 5.2 Let $\mathcal{K}$ be a simplicial complex on $[m]$, and let $J \subset [m]$. Choose $\Theta(J)$ as the set of the smallest vertices in corresponding path components. More precisely, define $\Theta(J)$ as the set of all vertices $i \in J$ such that

(i) i and $\max(J)$ belong to different path components of the complex $\mathcal{K}_J$;

(ii) i is the smallest vertex (has the smallest number) in its path component.

The corresponding set of generators $\{c(J\setminus i, u_i) : i \in \Theta(J),\ J \subset [m]\}$ will be called the *GPTW generators*.

Grbić, Panov, Theriault and Wu proved [23, Theorem 4.3] that GPTW generators minimally generate the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ if $\boldsymbol{k}$ is a field. The minimality was proved using topological methods. Our Theorem 5.1 gives a purely algebraic proof for any ring $\boldsymbol{k}$.

5.2 Rewriting of nested commutators

Thus the GPTW generators are indeed multiplicative generators of the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ for any ring $\boldsymbol{k}$ and any flag complex $\mathcal{K}$.

Definition 5.3 Let $i \in J \subset [m]$. Express the element $c(J \setminus i, u_i) \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$ as a noncommutative polynomial in GPTW generators (this expression may be nonunique). Any such expression will be denoted by $\widehat{c}(J \setminus i, u_i)$.

These noncommutative polynomials can be computed recursively, following the proof of [23, Theorem 4.3]. We describe an explicit rewriting process.

Algorithm 5.4 Suppose that expressions $\widehat{c}(A \setminus t, u_t)$, $|A| < |J|$, are already computed, and we compute $\widehat{c}(J \setminus i, u_i)$. Three cases are possible:

(i) $\boldsymbol{i = \max(J)}$ Define $j = \max(J \setminus i)$. Then

$$c(J \setminus i, u_i) = c(J \setminus ij, [u_j, u_i]) = c(J \setminus j, u_j).$$

The task is reduced to the case $i \neq \max(J)$.

(ii) **$\boldsymbol{i}$ and $\boldsymbol{\max(J)}$ belong to the same path component of $\boldsymbol{\mathcal{K}_J}$** The length of the shortest path from i to $\max(J)$ along the edges of $\mathcal{K}_J$ will be called the *rank* of a vertex i. We proceed by induction on the rank. The case of rank zero is discussed above. If rank equals 1, we have $[u_{\max(J)}, u_i] = 0$, so

$$c(J \setminus i, u_i) = c(J \setminus \{i, \max(J)\}, [u_{\max(J)}, u_i]) = 0.$$

Suppose that rank is greater than one, and let $\{i, j\}$ be the first edge in (any) shortest path from i to $\max(J)$. Since $[u_i, u_j] = 0 \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$, the identity (16) expresses $c(J \setminus i, u_i)$ in terms of $c(J \setminus j, u_j)$ (this element has smaller rank) and commutators of smaller degree (expressions for which are already computed).

(iii) **$\boldsymbol{i}$ and $\boldsymbol{\max(J)}$ are in different path components** Let i_0 be the smallest vertex of the component that contains i. The length of the shortest path from i to i_0 will be called the *rank* of a vertex i. If the rank is zero, then $i \in \Theta(J)$, so we can set $\widehat{c}(J \setminus i, u_i) := c(J \setminus i, u_i)$. Otherwise we decrease the rank using (16), as in case (ii).

Remark 5.5 A similar argument works more generally: suppose that we have a set $\{x_{J,i} : i \in J \subset [m]\}$ such that, for any $\{i, j\} \in \mathcal{K}_J$, the linear combination $x_{J,i} \pm x_{J,j}$ is a noncommutative polynomial on elements of smaller degree. Then we can express each element $x_{A,t}$ through the "GPTW elements" $\{x_{J,i} : i \in \Theta(J),\ J \subset [m]\}$ by a similar rewriting process. In our case $x_{J,i} = c(J \setminus i, u_i)$, and the polynomial is given by the last summand in (16).

5.3 Minimal sets of relations

Theorem 5.6 *Let $\mathcal{K}$ be a flag simplicial complex on a vertex set $[m]$, and let $\boldsymbol{k}$ be a commutative ring. For each $J \subset [m]$, choose a collection of simplicial 1-cycles*

$$\sum_{\{i<j\} \in \mathcal{K}_J} \lambda_{ij}^{(\alpha)} [\{i, j\}] \in \widetilde{C}_1(\mathcal{K}_J; \boldsymbol{k})$$

that generate the $\boldsymbol{k}$-module $H_1(\mathcal{K}_J;\boldsymbol{k})$. Then the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})$ is presented by GPTW generators $\{c(J\setminus i,u_i): i\in\Theta(J),\ J\subset[m]\}$ (see Definition 5.2) modulo the relations

$$\sum_{\{i<j\}\in\mathcal{K}_J}(-1)^{|J_{<i}|+|J_{<j}|}\lambda_{ij}^{(\alpha)}\sum_{\substack{J\setminus\{i,j\}=A\sqcup B\\ \max(A)>i\\ \max(B)>j}}(-1)^{\theta(A,B)+|A|}[\widehat{c}(A,u_i),\widehat{c}(B,u_j)]=0. \tag{10}$$

In particular, $H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})$ admits a presentation by $\sum_{J\subset[m]}\widetilde{b}_0(\mathcal{K}_J)$ generators modulo $\sum_{J\subset[m]}\widetilde{b}_1(\mathcal{K}_J;\boldsymbol{k})$ relations: one should take the 1-cycles that correspond to minimal sets of generators.

If $\boldsymbol{k}$ is a principal ideal domain, this presentation is minimal: any $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous presentation contains at least $\widetilde{b}_1(\mathcal{K}_J;\boldsymbol{k})$ relations of degree $(-|J|,2J)$ for every $J\subset[m]$.

Proof By Corollary 4.10, our 1-cycles correspond to the elements

$$\sum_{\{i<j\}\in\mathcal{K}_J}(-1)^{|J_{<i}|+|J_{<j}|}\lambda_{ij}^{(\alpha)}\sum_{\substack{J\setminus ij=A\sqcup B\\ \max(A)>i\\ \max(B)>j}}(-1)^{\theta(A,B)}[c(A,u_i)|c(B,u_j)]+(-1)^{\theta(B,A)}[c(B,u_j)|c(A,u_i)]$$

in bar construction, and their images additively generate $\operatorname{Tor}_2^{H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})$. We apply Theorem A.6(ii) to this situation. (In the notation of this theorem, we take GPTW generators as $a_1,\dots,a_N$. Their images freely generate $\operatorname{Tor}_1^{H_*(\Omega\mathcal{Z}_\mathcal{K})}(\boldsymbol{k},\boldsymbol{k})$, so we can take $R=0$. We take $\widehat{c}(A,u_i)$ and $\widehat{c}(B,u_j)$ as polynomials $P_{j,\alpha}$ and $Q_{j,\alpha}$.) It follows that $H_*(\Omega\mathcal{Z}_\mathcal{K};\boldsymbol{k})$ is generated by GPTW generators and presented by the relations

$$\sum_{\{i<j\}\in\mathcal{K}_J}(-1)^{|J_{<i}|+|J_{<j}|}\lambda_{ij}^{(\alpha)}\sum_{\substack{J\setminus ij=A\sqcup B\\ \max(A)>i\\ \max(B)>j}}(-1)^{\theta(A,B)}\overline{\widehat{c}(A,u_i)}\widehat{c}(B,u_j)+(-1)^{\theta(B,A)}\overline{\widehat{c}(B,u_j)}\widehat{c}(A,u_i)=0.$$

Define $x=\widehat{c}(A,u_i)$, $y=\widehat{c}(B,u_j)$. Since $\theta(A,B)+\theta(B,A)\equiv|A|\cdot|B|$, we have

$$\begin{aligned}(-1)^{\theta(A,B)}\overline{x}y+(-1)^{\theta(B,A)}\overline{y}x&=(-1)^{\theta(A,B)}((-1)^{|A|}xy+(-1)^{|B|+|A|\cdot|B|}yx)\\&=(-1)^{\theta(A,B)+|A|}(xy-(-1)^{(|A|+1)(|B|+1)}yx)=(-1)^{\theta(A,B)+|A|}[x,y].\end{aligned}$$

Hence the obtained relations coincide with (10). Finally, the lower bound on the number of relations follows from (9) and Theorem A.10(ii). □

Sometimes we can reduce the number of relations if the presentation is not required to be $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous. For example, suppose that for some $I,J\subset[m]$ we have $|I|=|J|=n$, $H_1(\mathcal{K}_I;\mathbb{Z})=\mathbb{Z}/2$, $H_1(\mathcal{K}_J;\mathbb{Z})=\mathbb{Z}/3$. Then the graded components of the module $\operatorname{Tor}_2^{H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{Z})}(\mathbb{Z},\mathbb{Z})$ having multidegrees $(-n,2I)$ and $(-n,2J)$ are equal to $\mathbb{Z}/2$ and $\mathbb{Z}/3$. By Theorem A.10, every $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous presentation of $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{Z})$ should contain relations of these multidegrees. On the other hand, these $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded components contribute $\mathbb{Z}/2\oplus\mathbb{Z}/3\simeq\mathbb{Z}/6$ to the $\mathbb{Z}$-graded component of degree n. Hence we can take just one $\mathbb{Z}$-homogeneous relation (for example, the sum of these $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous relations). Let us give a general result.

Theorem 5.7 *Let $\mathcal{K}$ be a flag simplicial complex and $\boldsymbol{k}$ be a principal ideal domain. Consider all homogeneous presentations of the $\mathbb{Z}$-graded $\boldsymbol{k}$-algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$.*

(i) *There is a presentation that consists of, for each $n\geq 0$, exactly $\sum_{|J|=n}\widetilde{b}_0(\mathcal{K}_J)$ generators and exactly $\operatorname{gen}\bigl(\bigoplus_{|J|=n}H_1(\mathcal{K}_J;\boldsymbol{k})\bigr)$ relations of degree n. One can take GPTW generators as generators, and take linear combinations of identities from Theorem 5.6, corresponding to minimal generators of the $\boldsymbol{k}$-module $\bigoplus_{|J|=n}H_1(\mathcal{K}_J;\boldsymbol{k})$, as relations.*

(ii) *For every $n\geq 0$, any presentation contains at least $\sum_{|J|=n}\widetilde{b}_0(\mathcal{K}_J)$ generators and at least $\operatorname{gen}\bigl(\bigoplus_{|J|=n}H_1(\mathcal{K}_J;\boldsymbol{k})\bigr)$ relations of degree n.*

Proof By Theorem A.10, $\operatorname{gen}\operatorname{Tor}_1^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_n$ and $\operatorname{gen}\operatorname{Tor}_2^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_n+\operatorname{rel}\operatorname{Tor}_1^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_n$ are precise bounds on the number of generators and relations, respectively, of degree n. By (9), we have

$$\operatorname{Tor}_1^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_n=\bigoplus_{|J|=n}\widetilde{H}_0(\mathcal{K}_J;\boldsymbol{k})\simeq\boldsymbol{k}^{\oplus\sum_{|J|=n}\widetilde{b}_0(\mathcal{K}_J)},\quad \operatorname{Tor}_2^{H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})=\bigoplus_{|J|=n}H_1(\mathcal{K}_J;\boldsymbol{k});$$

hence $\operatorname{gen}\operatorname{Tor}_1=\sum_{|J|=n}\widetilde{b}_0(\mathcal{K}_J)$ and $\operatorname{rel}\operatorname{Tor}_1=0$. One can take the GPTW generators since the images of corresponding cycles generate Tor_1 by Corollary 4.11. □

5.4 Example: moment-angle complexes for m-cycles

Let $\mathcal{K}$ be the boundary of an m-gon. The corresponding moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is homeomorphic to a connected sum of sphere products, $\mathcal{Z}_{\mathcal{K}}\cong\#_{k=3}^{m-1}(S^k\times S^{m+2-k})^{\#(k-2)\binom{m-2}{k-1}}$, and hence $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ is a one-relator algebra. It was considered in [24; 42]. From the point of view of Theorem 5.6, the relation corresponds to the 1-cycle

$$\kappa=[\{1,m\}]-\sum_{i=1}^{m-1}[\{i,i+1\}]\in\widetilde{C}_1(\mathcal{K};\boldsymbol{k})$$

and has the form

$$\sum_{\substack{\{2,\dots,m-1\}=A\sqcup B\\ \max(A)>1\\ \max(B)>m}}(\dots)-\sum_{i=1}^{m-1}(-1)^{(i-1)+i}\sum_{\substack{[m]\setminus\{i,i+1\}=A\sqcup B\\ \max(A)>i\\ \max(B)>i+1}}(-1)^{\theta(A,B)+|A|}[\widehat{c}(A,u_i),\widehat{c}(B,u_{i+1})]=0.$$

The first sum is empty, since $\max(B)\leq m-1$. Similarly, in the second sum the inner sum is empty for $i=m-1,m-2$. The simplified relation is

$$\sum_{i=1}^{m-3}\sum_{\substack{[m]\setminus\{i,i+1\}=A\sqcup B\\ \max(A),\max(B)\geq i+2}}(-1)^{\theta(A,B)+|A|}[\widehat{c}(A,u_i),\widehat{c}(B,u_{i+1})]=0.$$

Some summands are immediately zero. For example, if $\max(B)=i+2$, then

$$c(B,u_{i+1})=c(B\setminus\{i+2\},[u_{i+2},u_{i+1}])=0,$$

so we can take $\widehat{c}(B, u_{i+1}) = 0$. Similarly, $c(A, u_1) = 0$ if $i = 1$ and $m \in A$. Other summands can be computed using Algorithm 5.4. We were not able to obtain a closed formula for this relation (as a polynomial of GPTW generators or other minimal generators). However, we at least have an effective algorithm that computes the relation for any given m.

Consider the case $m = 5$. Besides from the partitions $[5] \setminus \{i, i+1\} = A \sqcup B$ considered above, for $i = 1$ the allowed partitions are $\{3,4,5\} = \{3\} \sqcup \{4,5\} = \{4\} \sqcup \{3,5\} = \{3,4\} \sqcup \{5\}$; for $i = 2$ the allowed partitions are $\{1,4,5\} = \{4\} \sqcup \{1,5\} = \{1,4\} \sqcup \{5\}$. The resulting relation has five summands:

$$(-1)^{\theta(3,45)+1}[\widehat{c}(3,u_1),\widehat{c}(45,2)]+(-1)^{\theta(4,35)+1}[\widehat{c}(4,u_1),\widehat{c}(35,2)]+(-1)^{\theta(34,5)+2}[\widehat{c}(34,u_1),\widehat{c}(5,u_2)] \\ +(-1)^{\theta(4,15)+1}[\widehat{c}(4,u_2),\widehat{c}(15,u_3)]+(-1)^{\theta(14,5)+2}[\widehat{c}(14,u_2),\widehat{c}(5,u_3)]=0.$$

All commutators, apart from $\widehat{c}(14, u_2) = [u_1, [u_4, u_2]] = -[u_2, [u_4, u_1]] = -c(24, u_1)$, already are GPTW generators. We obtain the following identity between the generators:

$$-\big[[u_3,u_1],[u_4,[u_5,u_2]]\big]+\big[[u_4,u_1],[u_3,[u_5,u_2]]\big]-\big[[u_5,u_2],[u_3,[u_4,u_1]]\big] \\ +\big[[u_4,u_2],[u_1,[u_5,u_3]]\big]+\big[[u_5,u_3],[u_2,[u_4,u_1]]\big]=0.$$

This relation was first obtained by Veryovkin as a result of brute force [42, Theorem 3.2]. For $m = 6$, the analogous relation is initially the sum of $7 + 10 + 4 = 21$ commutators. After computing the elements $\widehat{c}(J \setminus i, u_i)$ and changing the set of generators, it can be written as $\sum_{i=1}^{17} [a_i, b_i] = 0$ (see [42, Theorem 4.1]). This agrees with the homeomorphism $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^5)^{\# 9} \# (S^4 \times S^4)^{\# 8}$.

6 Homotopical properties in the flag case

6.1 Homotopy groups

As in [39], we denote by $\mathcal{P}$ the class of H-spaces which are homotopy equivalent to finite-type products of spheres and loops on simply connected spheres, and by $\mathcal{W}$ the class of topological spaces which are homotopy equivalent to finite-type wedges of simply connected spheres. The author thanks Lewis Stanton for providing a proof of the following lemma.

Lemma 6.1 *Let $A_1, \dots, A_m$ be connected topological spaces, $\mathcal{K}$ be a simplicial complex on $[m]$, and suppose that $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}} \in \mathcal{P}$. Then $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}}$ is homotopy equivalent to a finite-type product of loops on simply connected spheres.*

Proof By [40, Corollary 9.8], $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}} \simeq \prod_{i=1}^m \Omega\Sigma Y_i$ for some spaces Y_i. Since the class $\mathcal{P}$ is closed under retracts [39, Theorem 3.10], $\Omega\Sigma Y_i \in \mathcal{P}$. By repeated use of the homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ and the James splitting $\Sigma\Omega\Sigma X \simeq \bigvee_{n \geq 1} \Sigma X^{\wedge n}$ [27], we have $\Sigma Z \in \mathcal{W}$ for $Z \in \mathcal{P}$. In particular, $\Sigma\Omega\Sigma Y_i \in \mathcal{W}$. On the other hand, ΣY_i is a retract of $\Sigma\Omega\Sigma Y_i$ by the James splitting. The class $\mathcal{W}$ is closed under retracts (see, for example, [1, Lemma 3.1]), so $\Sigma Y_i \in \mathcal{W}$. Now $\Omega\Sigma Y_i$ is homotopy equivalent to a product of loops on spheres by the Hilton–Milnor theorem. It follows that the same holds for $\prod_{i=1}^m \Omega\Sigma Y_i$. □

Proof of Theorem 1.2 Since $\mathcal{K}$ is flag, we have $\Omega\mathcal{Z}_{\mathcal{K}} \in \mathcal{P}$ by Theorem 3.11. Hence $\Omega\mathcal{Z}_{\mathcal{K}} = \Omega(CS^1, S^1)^{\mathcal{K}}$ is a product of loops on spheres by Lemma 6.1. It follows that for some $D_n \geq 0$ we have a homotopy equivalence

$$\Omega\mathcal{Z}_{\mathcal{K}} \simeq \prod_{n\geq 2} (\Omega S^n)^{\times D_n}.$$

The numbers D_n are finite, since $\dim_{\boldsymbol{k}} H_i(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k}) < \infty$ for all i. (Here $\boldsymbol{k}$ is any field.) Also $D_2 = 0$, since $\mathcal{Z}_{\mathcal{K}}$ is 2-connected [15, Proposition 4.3.5]. In order to compute D_n, we calculate $\dim H_i(\Omega\mathcal{Z}_{\mathcal{K}}; \boldsymbol{k})$ twice. Recall that the *Poincaré series* $F(V;t)$ of a graded $\boldsymbol{k}$-vector space V are the formal power series

$$F(V;t) := \sum_{i\geq 0} \dim_{\boldsymbol{k}}(V_i)\cdot t^i \in \mathbb{Z}[\![t]\!].$$

We have

$$F(V\oplus W) = F(V;t) + F(W;t) \quad \text{and} \quad F(V\otimes W;t) = F(V;t)\cdot F(W;t).$$

From $F(H_*(\Omega S^k;\boldsymbol{k});t) = (1-t^{k-1})^{-1}$ and the Künneth formula we have

$$F(H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k});t) = \prod_{n\geq 3}(1-t^{n-1})^{-D_n}.$$

On the other hand, it is known (see [15, Proposition 8.5.4] and [43, Theorem 4.8]) that

$$F(H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k});t) = \frac{1}{(1+t)^{m-d}\cdot h_{\mathcal{K}}(-t)} = -\frac{1}{\sum_{J\subset[m]}\overline{\chi}(\mathcal{K}_J)t^{|J|}}$$

for a flag complex $\mathcal{K}$. We obtain the required identity (3). □

Remark 6.2 In the proof above, the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\boldsymbol{k})$ is actually $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded. We expect that factors of the product (2) can be considered as "$\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-graded spheres", and thus $\pi_*(\Omega\mathcal{Z}_{\mathcal{K}})$ admits a functorial $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading as conjectured in [43, Remark 4.10].

Problem 6.3 Describe the Whitehead bracket in $\pi_*(\mathcal{Z}_{\mathcal{K}})$ in terms of the decomposition (4).

6.2 Rational coformality of moment-angle complexes

Let X be a simply connected space and ΩX be the space of Moore loops. Since ΩX is a strictly associative topological monoid, the chain complex $C_*(\Omega X;\boldsymbol{k})$ is a dga algebra with respect to the Pontryagin product for any $\boldsymbol{k}$. Also, the cochain complex $C^*(X;\boldsymbol{k})$ is a dga algebra with respect to the Kolmogorov–Alexander product (cup product).

Definition 6.4 A topological space X is *formal* over a ring $\boldsymbol{k}$ if the dga algebras $H^*(X;\boldsymbol{k})$ (with zero differential) and $C^*(X;\boldsymbol{k})$ are quasi-isomorphic (are connected by a zigzag of dga maps which induce isomorphisms on homology).

Definition 6.5 A simply connected space X is *coformal* over a ring $\boldsymbol{k}$ if the dga algebras $H_*(\Omega X;\boldsymbol{k})$ (with zero differential) and $C_*(\Omega X;\boldsymbol{k})$ are quasi-isomorphic.

The notions of formality and coformality (over a field of characteristic zero) arose in rational homotopy theory, and were initially formulated in terms of Sullivan and Quillen models. The rational homotopy type of a formal (coformal) space is fully determined by the algebra $H^*(X;\mathbb{Q})$ (by the algebra $H_*(\Omega X;\mathbb{Q})$). As proved by Saleh [38, Corollary 1.2, 1.4], our definitions are equivalent to the classical ones.

Notbohm and Ray proved [34, Theorem 4.8] that all Davis–Januszkiewicz spaces $\mathrm{DJ}(\mathcal{K})$ are formal over $\mathbb{Z}$. (In fact, the proof works over any ring $\boldsymbol{k}$; see also [20, Theorem 1.3] for a stronger formality result). Also, $\mathrm{DJ}(\mathcal{K})$ is coformal over $\mathbb{Q}$ if and only if $\mathcal{K}$ is flag [15, Theorem 8.5.6]. First examples of nonformal moment-angle complexes were constructed by Baskakov [6] using Massey products. See [14, Introduction] for a survey of further developments in this area.

The following result of Huang can be used to prove coformality over $\mathbb{Q}$.

Proposition 6.6 [26, Proposition 5.1] *Let $F \xrightarrow{i} E \to B$ be a fibration of nilpotent spaces of finite type, such that*

- *the map $i_*\colon \pi_*(F)\otimes_{\mathbb{Z}}\mathbb{Q} \to \pi_*(E)\otimes_{\mathbb{Z}}\mathbb{Q}$ is injective;*
- *E is coformal over $\mathbb{Q}$.*

Then F is coformal over $\mathbb{Q}$. □

Corollary 6.7 *Let $\mathcal{K}$ be a flag simplicial complex with no ghost vertices. Then $\mathcal{Z}_{\mathcal{K}}$ is coformal over $\mathbb{Q}$.*

Proof We apply Proposition 6.6 to the fibration $\mathcal{Z}_{\mathcal{K}} \to \mathrm{DJ}(\mathcal{K}) \to B\mathbb{T}^m$. By Proposition 3.1 and exact sequence of homotopy groups, $\pi_*(\mathcal{Z}_{\mathcal{K}}) \to \pi_*(\mathrm{DJ}(\mathcal{K}))$ is injective. The second condition holds by [15, Theorem 8.5.6]. □

It is natural to hope that Huang's theorem admits the following generalisation.

Conjecture 6.8 *Let $F \to E \xrightarrow{p} B$ be a fibration of simply connected spaces of finite type, such that*

- *Ωp has a homotopy section;*
- *E is coformal over $\boldsymbol{k}$.*

Then F is coformal over $\boldsymbol{k}$.

Let X be a simply connected space such that $H_*(\Omega X;\boldsymbol{k})$ is a free $\boldsymbol{k}$-module. The tensor filtration on the bar construction $\overline{\mathrm{B}}(C_*(\Omega X;\boldsymbol{k}))$ gives rise to the *Milnor–Moore spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}_p^{H_*(\Omega X;\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k})_q \Rightarrow \mathrm{Tor}_{p+q}^{C_*(\Omega X;\boldsymbol{k})}(\boldsymbol{k},\boldsymbol{k}) \cong H_{p+q}(X;\boldsymbol{k}).$$

(The last isomorphism is due to Eilenberg and Moore; see [18, Theorem IV]).

The differential Tor is preserved by quasi-isomorphisms. Hence the spectral sequence collapses at E^2 if X is coformal over $\boldsymbol{k}$. On the other hand, it collapses for $X = \mathcal{Z}_{\mathcal{K}}$ in the flag case; see (9). This suggests the following conjecture.

Conjecture 6.9 *Let $\mathcal{K}$ be a flag simplicial complex. Then the spaces $\mathrm{DJ}(\mathcal{K})$ and $\mathcal{Z}_\mathcal{K}$ are coformal over any commutative ring with unit.*

6.3 A necessary condition for the rational formality in the flag case

The space X is *Koszul* if it is both formal and coformal over $\mathbb{Q}$. Hence $\mathrm{DJ}(\mathcal{K})$ is Koszul if and only if $\mathcal{K}$ is flag. Koszul spaces were introduced by Berglund [10].

Definition 6.10 Let $\boldsymbol{k}$ be a field, $A = \bigoplus_{n\in\mathbb{Z}} A^n$ be a graded $\boldsymbol{k}$-algebra that admits an additional "weight" grading $A^n = \bigoplus_{j\geq 0} A^{n,(j)}$. The algebra A is *Koszul* with respect to the weight grading if $\mathrm{Ext}^i_A(\boldsymbol{k},\boldsymbol{k})^{n,(j)} = 0$ for all $i\neq j$.

For every Koszul algebra, there is a *quadratic dual* Koszul algebra $A^!$; see [22]. More explicitly, we set

$$A^! := \mathrm{Ext}_A(\boldsymbol{k},\boldsymbol{k}), \quad (A^!)^{n,(i)} = \mathrm{Ext}^i_A(\boldsymbol{k},\boldsymbol{k})^{-i-n,(i)}.$$

Then it is known that $(A^!)^! \cong A$ as bigraded algebras.

Remark 6.11 In the classical theory of Koszul algebras [22; 36] the $\mathbb{Z}$-grading $A = \bigoplus_{n\in\mathbb{Z}} A^n$ is absent, and only the weight grading $(A^!)^{(i)} = \mathrm{Ext}^i_A(\boldsymbol{k},\boldsymbol{k})^{(i)}$ is considered. Classical results are readily generalised to the graded case.

The following result is due to Berglund. We replace the Koszul Lie algebras with their universal enveloping algebras. Berglund considers a stronger version of the Koszul duality, the duality between Lie algebras and commutative algebras.

Theorem 6.12 [10, Theorems 2 and 3] *Let X be a simply connected space of finite type such that X is coformal over $\mathbb{Q}$. The following conditions are equivalent*:

(a) *X is formal over $\mathbb{Q}$.*
(b) *The graded algebra $A = H_*(\Omega X;\mathbb{Q})$ admits a weight grading $A = \bigoplus_{i\geq 0} A^{(i)}$ such that A is Koszul with respect to it.*

Moreover, if these conditions are met, then the $\mathbb{Z}$-graded algebras $A^!$ and $H^{-*}(X;\mathbb{Q})$ are isomorphic: $H^n(X;\mathbb{Q}) \cong \bigoplus_{i\geq 0}(A^!)^{-n,(i)}$. □

Theorem 6.13 *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$ with no ghost vertices, such that $\mathcal{Z}_\mathcal{K}$ is rationally formal. Then $\Gamma = H^*(\mathcal{Z}_\mathcal{K};\mathbb{Q})$ is a Koszul algebra with respect to the grading*

$$\Gamma^{(i)} := \bigoplus_{J\subset[m]} H^{i-|J|,2J}(\mathcal{Z}_\mathcal{K};\mathbb{Q}) = \bigoplus_{J\subset[m]} \widetilde{H}^{i-1}(\mathcal{K}_J;\mathbb{Q}).$$

In particular, Γ is generated by elements in $\widetilde{H}^0(\mathcal{K}_J;\mathbb{Q})$ modulo the relations in $\widetilde{H}^1(\mathcal{K}_J;\mathbb{Q})$.

Proof By Theorem 6.12, the algebra $A = H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{Q})$ is Koszul with respect to a weight grading $A=\bigoplus_{i\geq 0} A^{(i)}$. From [43, Theorem 1.2] we have $\mathrm{Tor}_i^A(\mathbb{Q},\mathbb{Q})_j = \bigoplus_{|J|=j}\widetilde{H}_{i-1}(\mathcal{K}_J;\mathbb{Q})$. Therefore, $\mathrm{Ext}_A^i(\mathbb{Q},\mathbb{Q})^j = \bigoplus_{|J|=j}\widetilde{H}^{i-1}(\mathcal{K}_J;\mathbb{Q})$. The algebra A is Koszul, hence

$$\mathrm{Ext}_A^i(\mathbb{Q},\mathbb{Q})^j = \mathrm{Ext}_A^i(\mathbb{Q},\mathbb{Q})^{j,(i)} = (A^!)^{-i-j,(i)}.$$

Since $(A^!)^* \cong \Gamma^{-*}$ as graded algebras, we obtain a weight grading

$$\Gamma^{i+j,(i)} = \bigoplus_{|J|=j}\widetilde{H}^{i-1}(\mathcal{K}_J;\mathbb{Q}),\quad \Gamma^{(i)} = \bigoplus_{J\subset[m]}\widetilde{H}^{i-1}(\mathcal{K}_J;\mathbb{Q})$$

such that Γ is Koszul with respect to it. Finally, any Koszul algebra is generated by elements of weight 1 modulo relations of weight 2. □

Conjecture 6.14 *If $\mathcal{K}$ is a flag complex and $H^*(\mathcal{Z}_\mathcal{K};\mathbb{Q})$ is Koszul with respect to the grading from Theorem 6.13, then $\mathcal{Z}_\mathcal{K}$ is formal over $\mathbb{Q}$.*

Appendix A Presentations of connected graded algebras

In this section we prove Theorems A.1 and A.10 that generalise some results of Wall [44, Section 7]. We also prove Theorem A.6, which seems to be new. We use the notation from Section 2; some of which is recalled below.

A.1 Conventions

The ring $\boldsymbol{k}$ is assumed to be an arbitrary commutative associative ring with unit. All tensor products are over $\boldsymbol{k}$.

We consider G-graded $\boldsymbol{k}$-algebras, where G is a commutative monoid supplied with a homomorphism $G\to\mathbb{Z}$. It induces a $\mathbb{Z}$-grading. Such an algebra A is *connected* if it is connected with respect to the $\mathbb{Z}$-grading, ie $A_{<0}=0$ and $A_0=\boldsymbol{k}\cdot 1$. Then the standard augmentation $\varepsilon\colon A\to A_0\cong\boldsymbol{k}$ makes $\boldsymbol{k}$ a left A-module and a right A-module.

Every complex of G-graded modules is considered as a $\mathbb{Z}\times G$-graded module with a differential of degree $(-1,0)$. Hence, A-linear differentials satisfy the following version of Leibniz's rule:

$$d(a\cdot x) = (-1)^{\deg(a)} a\cdot d(x) = -\overline{a}\cdot d(x),$$

where $\overline{a} := (-1)^{1+\deg(a)}a$.

A *presentation* of a connected $\boldsymbol{k}$-algebra A is an isomorphism of the form $A\simeq T(x_1,\dots,x_N)/(r_1,\dots,r_M)$, sometimes written as

$$A\simeq T(x_1,\dots,x_N)/(r_1=\dots=r_M=0),$$

where $T(x_1,\dots,x_N)$ is a tensor algebra and $(r_1,\dots,r_M)\subset T(x_1,\dots,x_N)$ is the two-sided ideal generated by the set $\{r_1,\dots,r_M\}$. It is assumed that generators and relations are homogeneous and have positive degree, hence belong to $\operatorname{Ker}\varepsilon$. Note that A is not required to be a free $\boldsymbol{k}$-module, and M, N can be infinite of any cardinality.

A.2 Exact sequence of a presentation

Let $T(x_1,\dots,x_N)$ be a tensor algebra generated by homogeneous elements of positive degrees. Every element $w\in T(x_1,\dots,x_N)$ is uniquely represented as a sum

$$w=\varepsilon(w)+\sum_{i=1}^N w_i\cdot x_i,\quad w_i\in T(x_1,\dots,x_N).$$

In the next proposition we use this representation implicitly. For example, we assume that

$$r_j=\varepsilon(r_j)+\sum_{i=1}^N r_{ji}\cdot x_i.$$

Since $r_j\in\operatorname{Ker}\varepsilon$, the first summand is zero.

Proposition A.1 *Let $A=T(x_1,\dots,x_N)/(r_1,\dots,r_M)$ be a presentation of a connected $\boldsymbol{k}$-algebra,*

$$\pi\colon T(x_1,\dots,x_N)\twoheadrightarrow A$$

be the projection. Then the following sequence of graded free left A-modules is exact:

$$A\cdot\{R_1,\dots,R_M\}\xrightarrow{d_2}A\cdot\{X_1,\dots,X_N\}\xrightarrow{d_1}A\xrightarrow{\varepsilon}\boldsymbol{k}\to 0,$$
$$d_2(R_j):=-\sum_{i=1}^N\pi(\overline{r_{ji}})\cdot X_i,\quad d_1(X_i):=x_i.$$

Proof We first prove that the sequence is a chain complex. Indeed, $\varepsilon(d_1(X_i))=\varepsilon(x_i)=0$ and

$$d_1(d_2(R_j))=\sum_{i=1}^N\pi(r_{ji})\cdot d_1(X_i)=\sum_{i=1}^N\pi(r_{ji})x_i=\pi\Big(\sum_{i=1}^N r_{ji}x_i\Big)=\pi(r_j)=0\in A$$

($r_j\in\operatorname{Ker}\varepsilon$, hence $r_j=\sum_i r_{ji}x_i$). We check the exactness in the term A. Let $y\in\operatorname{Ker}\varepsilon\subset A$. We have $y=\pi(w)$ for some element $w\in T(x_1,\dots,x_N)$ of positive degree, hence

$$y=\pi\Big(\sum_{i=1}^N w_ix_i\Big)=\sum_{i=1}^N\pi(w_i)x_i=d_1\Big(-\sum_{i=1}^N\pi(\overline{w}_i)\cdot X_i\Big)\in\operatorname{Im}d_1.$$

We check the exactness in the term $A\cdot\{X_1,\dots,X_N\}$. Suppose that $\sum_{i=1}^N a_i\cdot X_i\in\operatorname{Ker}d_1$, so $\sum_{i=1}^N\overline{a}_ix_i=0$. We have $a_i=\pi(v_i)$ for some $v_i\in T(x_1,\dots,x_N)$. Then the element $w:=\sum_{i=1}^N\overline{v}_ix_i\in T(x_1,\dots,x_N)$ belongs to $\operatorname{Ker}\pi$. This kernel is a two-sided ideal generated by r_j. Hence $w=\sum_{j=1}^M\sum_\alpha u_{j,\alpha}r_jw_{j,\alpha}$ for some $u_{j,\alpha},w_{j,\alpha}\in T(x_1,\dots,x_N)$. We can rewrite it as

$$w=\sum_{j=1}^M\sum_\alpha u_{j,\alpha}r_j\varepsilon(w_{j,\alpha})+\sum_{j=1}^M\sum_\alpha\sum_{i=1}^N u_{j,\alpha}r_jw_{j,\alpha,i}x_i=\sum_{i=1}^N\sum_{j=1}^M\sum_\alpha(\varepsilon(w_{j,\alpha})u_{j,\alpha}r_{ji}+u_{j,\alpha}r_jw_{j,\alpha,i})x_i.$$

On the other hand, $w = \sum_{i=1}^N \overline{v}_i x_i$. Such a representation is unique, so we have

$$\overline{v}_i = \sum_{j=1}^M \sum_\alpha \varepsilon(w_{j,\alpha}) u_{j,\alpha} r_{ji} + u_{j,\alpha} r_j w_{j,\alpha,i}, \quad i = 1, \dots, N.$$

Applying π to both parts of this identity, we obtain $\overline{a}_i = \sum_{j=1}^M \sum_\alpha \varepsilon(w_{j,\alpha})\pi(u_{j,\alpha})\pi(r_{ji})$, since $\pi(v_i) = a_i$ and $\pi(r_j) = 0$. Finally,

$$\sum_{i=1}^N a_i \cdot X_i = -\sum_{j=1}^M \sum_\alpha \varepsilon(w_{j,\alpha})\pi(\overline{u}_{j,\alpha})\pi(\overline{r}_{ji}) \cdot X_i = d_2\Big(-\sum_{j=1}^M \sum_\alpha \varepsilon(w_{j,\alpha})\pi(u_{j,\alpha}) \cdot R_j\Big) \in \operatorname{Im} d_2. \qquad \square$$

Remark A.2 Proposition A.1 holds for presentations of *augmented* algebras such that $\varepsilon(x_i) = \varepsilon(r_j) = 0$. The corresponding exact sequence is called the "Koszul resolution" in [3, Section 2].

Corollary A.3 *Let $A = T(x_1, \dots, x_N)/(r_1, \dots, r_M)$ be a presentation of a connected graded $\boldsymbol{k}$-algebra, which is a free $\boldsymbol{k}$-module. Then the $\boldsymbol{k}$-module $\operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})$ is additively generated by images of cycles $[x_1], \dots, [x_N] \in \overline{\mathrm{B}}_1(A)$.*

Proof We extend the exact sequence from Proposition A.1 to a free resolution

$$\cdots \to A \cdot \{X_1, \dots, X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} \boldsymbol{k} \to 0, \quad d_1(X_i) = x_i,$$

of the left A-module $\boldsymbol{k}$. Consider the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A \cdot \{X_1, \dots, X_N\} & \xrightarrow{d_1} & A & \xrightarrow{\varepsilon} & \boldsymbol{k} & \longrightarrow & 0 \\
 & & \big\downarrow {\scriptstyle X_i \mapsto [x_i]} & & \big\downarrow {\scriptstyle a \mapsto a[\cdot]} & & \big\| & & \\
\cdots & \longrightarrow & \mathrm{B}_1(A) & \xrightarrow{d_{\mathrm{B},1}} & \mathrm{B}_0(A) & \xrightarrow{\varepsilon} & \boldsymbol{k} & \longrightarrow & 0
\end{array}$$

It is commutative, since $d_1(a \otimes X_i) = -\overline{a} x_i$ and $d_{\mathrm{B},1}(a[x_i]) = -\overline{a} x_i [\cdot]$. Hence it can be extended to a map of resolutions (eg using Lemma 2.1). Apply the functor $\boldsymbol{k} \otimes_A (-)$. We obtain a map of chain complexes

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \boldsymbol{k} \cdot \{X_1, \dots, X_N\} & \xrightarrow{0} & \boldsymbol{k} & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow {\scriptstyle X_i \mapsto [x_i]} & & \big\downarrow {\scriptstyle a \mapsto a[\cdot]} & & \\
\cdots & \longrightarrow & \overline{\mathrm{B}}_1(A) & \xrightarrow{d_{\overline{\mathrm{B}},1}} & \overline{\mathrm{B}}_0(A) & \longrightarrow & 0
\end{array}$$

The homology of both complexes equals $\operatorname{Tor}^A(\boldsymbol{k}, \boldsymbol{k})$, and the induced map in homology is an isomorphism. The elements X_i in the first complex are cycles, and their images generate $\operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})$. $\square$

Corollary A.4 *Let $A = T(x_1, \dots, x_N)$ be the tensor algebra over a ring $\boldsymbol{k}$, where $x_1, \dots, x_N$ are homogeneous elements of positive degrees. Then $\operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})$ is a free $\boldsymbol{k}$-module with the basis represented by cycles $[x_1], \dots, [x_N] \in \overline{\mathrm{B}}_1(A)$. Moreover, $\operatorname{Tor}_i^A(\boldsymbol{k}, \boldsymbol{k}) = 0$ for $i > 1$.*

Proof By Proposition A.1, the sequence

$$0 \to A\cdot\{X_1,\dots,X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} \boldsymbol{k} \to 0, \quad d_1(X_i)=x_i,$$

is exact. As in the proof of Corollary A.3, we obtain a map of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \boldsymbol{k}\cdot\{X_1,\dots,X_N\} & \xrightarrow{0} & \boldsymbol{k} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle X_i\mapsto[x_i]} & & \downarrow{\scriptstyle a\mapsto a[\cdot]} & & \\ \cdots & \longrightarrow & \overline{\mathrm{B}}_1(A) & \xrightarrow{d_{\overline{\mathrm{B}},1}} & \overline{\mathrm{B}}_0(A) & \longrightarrow & 0 \end{array}$$

Homology of both complexes is equal to $\mathrm{Tor}^A(\boldsymbol{k},\boldsymbol{k})$, and the induced map in homology is the identity. □

A.3 A presentation that corresponds to cycles

Recall that $\mathrm{Tor}^A(\boldsymbol{k},\boldsymbol{k}) \cong H(\overline{\mathrm{B}}(A))$ if A is a free $\boldsymbol{k}$-module. The following lemma is proved by Lemaire [29, Corollaire 1.2.3] in the case of field coefficients.

Lemma A.5 *Let $f: A \to C$ be a morphism of connected $\boldsymbol{k}$-algebras, where $\boldsymbol{k}$ is a commutative ring with unit.*

(i) *Suppose that the map $f_{*,1}\colon H_1(\overline{\mathrm{B}}(A)) \to H_1(\overline{\mathrm{B}}(C))$ is surjective. Then $f: A\to C$ is surjective.*

(ii) *Suppose that $f_{*,1}\colon H_1(\overline{\mathrm{B}}(A)) \to H_1(\overline{\mathrm{B}}(C))$ is bijective, and $f_{*,2}\colon H_2(\overline{\mathrm{B}}(A)) \to H_2(\overline{\mathrm{B}}(C))$ is surjective. Then $f: A\to C$ is an isomorphism.*

We prove by induction that the maps $f_n\colon A_n \to C_n$ are surjective (bijective). The base case is the bijection $A_0 \cong \boldsymbol{k} \cong C_0$. Recall that the bar construction $\overline{\mathrm{B}}(A)$ is the chain complex

$$\cdots \to \overline{\mathrm{B}}_3(A) \xrightarrow{d_3} \overline{\mathrm{B}}_2(A) \xrightarrow{d_2} \overline{\mathrm{B}}_1(A) \xrightarrow{0} \boldsymbol{k} \to 0,$$
$$\overline{\mathrm{B}}_k(A) = I(A)^{\otimes k}, \quad d_2(x\otimes y)=\bar{x}y, \quad d_3(x\otimes y\otimes z) = \bar{x}y\otimes z + \bar{x}\otimes \bar{y}z.$$

We define $f_\#\colon \overline{\mathrm{B}}(A) \to \overline{\mathrm{B}}(C)$.

Proof of statement (i) Suppose that $f: A_i \to C_i$ is surjective for $i<n$. Consider the following map of exact sequences:

$$\begin{array}{ccccccc} \overline{\mathrm{B}}_2(A)_n & \xrightarrow{d_2} & \overline{\mathrm{B}}_1(A)_n \cong A_n & \longrightarrow & H_1(\overline{\mathrm{B}}(A))_n & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_{\#,2}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{*,1}} & & \| \\ \overline{\mathrm{B}}_2(C)_n & \xrightarrow{d_2} & \overline{\mathrm{B}}_1(C)_n \cong C_n & \longrightarrow & H_1(\overline{\mathrm{B}}(C))_n & \longrightarrow & 0 \end{array}$$

The map $f_{\#,2}$ is surjective, since it is a direct sum of maps $f\otimes f: A_i\otimes A_j \to C_i\otimes C_j$ for $i,j<n$, and f is surjective in these degrees by the inductive hypothesis. The surjectivity $f_{*,1}$ is given, and $0\to 0$ is injective. Hence $f: A_n \to C_n$ is surjective by the "first half of five lemma" [37, Proposition 2.72(i)]. □

Proof of statement (ii) Suppose that $f\colon A_i \to C_i$ is bijective for $i < n$. Consider the following map of exact sequences:

$$\begin{array}{ccccccc}
\overline{\mathrm{B}}_3(A)_n & \xrightarrow{d_3} & \operatorname{Ker} d_2 & \longrightarrow & H_2(\overline{\mathrm{B}}(A))_n & \longrightarrow & 0 \\
\downarrow f_{\#,3} & & \downarrow \varphi & & \downarrow f_{*,2} & & \| \\
\overline{\mathrm{B}}_3(C)_n & \xrightarrow{d_3} & \operatorname{Ker} d_2 & \longrightarrow & H_2(\overline{\mathrm{B}}(C))_n & \longrightarrow & 0
\end{array}$$

The map $f_{\#,3}$ is surjective, since it is a direct sum of maps $f \otimes f \otimes f\colon A_i \otimes A_j \otimes A_k \to C_i \otimes C_j \otimes C_k$ for $i, j, k < n$, and f is surjective in these degrees. The surjectivity of $f_{*,2}$ is given, and $0 \to 0$ is injective. Hence φ is surjective by the "first half of five lemma". Now consider the following map of exact sequences:

$$\begin{array}{ccccccc}
\operatorname{Ker} d_{\overline{\mathrm{B}},2} & \longrightarrow & \overline{\mathrm{B}}_2(A)_n & \xrightarrow{d_2} & \overline{\mathrm{B}}_1(A)_n \cong A_n & \longrightarrow & H_1(\overline{\mathrm{B}}(A))_n \\
\downarrow \varphi & & \simeq \downarrow f_{\#,2} & & \downarrow f & & \downarrow f_{*,1} \\
\operatorname{Ker} d_{\overline{\mathrm{B}},2} & \longrightarrow & \overline{\mathrm{B}}_2(C)_n & \xrightarrow{d_2} & \overline{\mathrm{B}}_1(C)_n \cong C_n & \longrightarrow & H_1(\overline{\mathrm{B}}(C))_n
\end{array}$$

We proved that φ is surjective. The map $f_{\#,2}$ is bijective by the inductive hypothesis (it is a direct sum of maps $f \otimes f\colon A_i \otimes A_j \to C_i \otimes C_j$, $i, j < n$); in particular, it is injective. The injectivity of $f_{*,1}$ is given. Hence the map $f\colon A_n \to C_n$ is injective by the "second half of five lemma" [37, Proposition 2.72(ii)]. By (i), this map is also surjective. □

The following theorem allows one to obtain a presentation of a connected $\boldsymbol{k}$-algebra A, knowing the structure of $\boldsymbol{k}$-modules $H_1(\overline{\mathrm{B}}(A))$ and $H_2(\overline{\mathrm{B}}(A))$. In the proof, we do not use the notation $[x|y|z]$ for elements of the bar construction, and write $x \otimes y \otimes z$ instead. Therefore, $[c]$ always denotes the class in $H(\overline{\mathrm{B}}(\Gamma))$ represented by a cycle $c \in \overline{\mathrm{B}}(\Gamma)$.

We also use the following notation. Let $a_1, \dots, a_N \in A$ be some homogeneous elements of positive degree and $K, L \in T(x_1, \dots, x_N)$ be homogeneous noncommutative polynomials that belong to the augmentation ideal. Then the elements $K(a_1, \dots, a_N), L(a_1, \dots, a_N) \in I(A)$ are defined, and hence we can consider the elements $K(a_1, \dots, a_N) \otimes L(a_1, \dots, a_N) \in I(A) \otimes I(A) = \overline{\mathrm{B}}_2(A)$ and

$$d_{\overline{\mathrm{B}},2}(K(a_1, \dots, a_N) \otimes L(a_1, \dots, a_N)) = \overline{K}(a_1, \dots, a_N) \cdot L(a_1, \dots, a_N) \in \overline{\mathrm{B}}_1(A) = I(A).$$

Theorem A.6 *Let A be a connected algebra over a commutative ring $\boldsymbol{k}$ with unit.*

(i) *Suppose that, for homogeneous elements $a_1, \dots, a_N \in A_{>0}$, the $\boldsymbol{k}$-module $H_1(\overline{\mathrm{B}}(A))$ is additively generated by the classes $[a_1], \dots, [a_N] \in H_1(\overline{\mathrm{B}}(A))$. Then A is multiplicatively generated by $a_1, \dots, a_N$.*

(ii) *Suppose that the $\boldsymbol{k}$-module $H_1(\overline{\mathrm{B}}(A))$ is additively generated by N elements $[a_1], \dots, [a_N]$ modulo R relations*

$$\sum_{i=1}^{N} \lambda_{ri}[a_i] = 0 \in H_1(\overline{\mathrm{B}}(A)), \quad r = 1, \dots, R, \ \lambda_{ri} \in \boldsymbol{k}.$$

Suppose that homogeneous polynomials $P_{j,\alpha}, Q_{j,\alpha}, K_{r,\beta}, L_{r,\beta} \in T(x_1,\dots,x_N)$ are such that

$$\sum_{i=1}^{N} \lambda_{ri} \cdot a_i = d_{\overline{\mathrm{B}},2}\Big(\sum_{\beta} K_{r,\beta}(a_1,\dots,a_N) \otimes L_{r,\beta}(a_1,\dots,a_N)\Big) \in I(A), \quad r = 1,\dots,R,$$

and the cycles in bar construction

$$\sum_{\alpha} P_{j,\alpha}(a_j,\dots,a_N) \otimes Q_{j,\alpha}(a_1,\dots,a_N) \in I(A) \otimes I(A), \quad j = 1,\dots,M,$$

generate the $\boldsymbol{k}$-module $H_2(\overline{\mathrm{B}}(A))$. Then the algebra A has a presentation

$$A \cong T(x_1,\dots,x_N)/\Big(\sum_{i=1}^{N} \lambda_{ri} x_i = \sum_{\beta} \overline{K}_{r,\beta} \cdot L_{r,\beta},\ r = 1,\dots,R;\ \sum_{\alpha} \overline{P}_{j,\alpha} \cdot Q_{j,\alpha} = 0,\ j = 1,\dots,M\Big).$$

(Here N, M, R can be infinite of any cardinality.)

Proof of statement (i) Consider the morphism $f : T(x_1,\dots,x_N) \to A$, $x_i \mapsto a_i$, of connected algebras. The classes $[a_1],\dots,[a_N]$ generate $H_1(\overline{\mathrm{B}}(A))$ and are images of classes $[x_1],\dots,[x_N]$ with respect to the map $f_{*,1} : H_1\big(\overline{\mathrm{B}}(T(x_1,\dots,x_N))\big) \to H_1(\overline{\mathrm{B}}(A))$. Hence $f_{*,1}$ is surjective. By Lemma A.5(i), f is surjective. □

Proof of statement (ii) Consider the algebra

$$C := T(x_1,\dots,x_N)/\Big(\sum_{i=1}^{N} \lambda_{ri} x_i = \sum_{\beta} \overline{K}_{r,\beta} \cdot L_{r,\beta},\ r = 1,\dots,R;\ \sum_{\alpha} \overline{P}_{j,\alpha} \cdot Q_{j,\alpha} = 0,\ j = 1,\dots,M\Big).$$

The following identities in A are given:

$$\sum_{i=1}^{N} \lambda_{ri} \cdot a_i = \sum_{\beta} \overline{K}_{r,\beta}(a_1,\dots,a_N) \cdot L_{r,\beta}(a_1,\dots,a_N), \quad 0 = \sum_{\alpha} \overline{P}_{j,\alpha}(a_1,\dots,a_N) \cdot Q_{j,\alpha}(a_1,\dots,a_N).$$

Hence the morphism $f : C \to A$, $x_i \mapsto a_i$, is well defined. The induced map $f_{*,1} : H_1(\overline{\mathrm{B}}(C)) \to H_1(\overline{\mathrm{B}}(A))$ is surjective, since the elements $[a_i] = f_{*,1}([x_i])$ generate $H_1(\overline{\mathrm{B}}(A))$.

We prove that $f_{*,1}$ is injective. Let $\xi \in H_1(\overline{\mathrm{B}}(C))$ and $f_{*,1}(\xi) = 0$. By Corollary A.4 and surjectivity of $T(x_1,\dots,x_N) \to C$, we have $\xi = \sum_{i=1}^{N} \mu_i \cdot [x_i]$ for some $\mu_i \in \boldsymbol{k}$. Then $0 = f_*(\xi) = \sum_i \mu_i [a_i] \in H_1(\overline{\mathrm{B}}(A))$. All linear relations between $[a_1],\dots,[a_N]$ follow from the relations $\sum_i \lambda_{ri}[a_i] = 0$, and therefore $\mu_i = \sum_{r=1}^{R} c_r \lambda_{ri}$ for some $c_r \in \boldsymbol{k}$. It follows that ξ is represented by the cycle

$$\sum_{i=1}^{N} \sum_{r=1}^{R} c_r \lambda_{ri} \cdot x_i = \sum_{r=1}^{R} c_r \sum_{\beta} \overline{K}_{r,\beta} \cdot L_{r,\beta} = d_{\overline{\mathrm{B}},2}\Big(\sum_{r=1}^{R} c_r \sum_{\beta} K_{r,\beta} \otimes L_{r,\beta}\Big) \in \overline{\mathrm{B}}_1(C).$$

Hence $\xi = 0$. We proved that $f_{*,1}$ is bijective.

The elements $\sum_{\alpha} P_{i,\alpha} \otimes Q_{i,\alpha} \in I(C) \otimes I(C)$ are cycles in $\overline{\mathrm{B}}_2(C)$, and their images generate $H_2(\overline{\mathrm{B}}(A))$. Hence $f_{*,2} : H_2(\overline{\mathrm{B}}(C)) \to H_2(\overline{\mathrm{B}}(A))$ is surjective. Conditions of Lemma A.5(ii) are satisfied, so f is bijective. □

A.4 Bounds on the number of homogeneous generators and relations

Let A be a connected $\boldsymbol{k}$-algebra. Proposition A.1 gives a lower bound on the number of generators and relations in the homogeneous presentations of A, and Theorem A.6 gives an upper bound. These bounds coincide if $\boldsymbol{k}$ is a principal ideal domain, A is a free $\boldsymbol{k}$-module, and graded components are finitely generated. We introduce some notation.

Definition A.7 Let M be a finitely generated module over a principal ideal domain $\boldsymbol{k}$. By the structure theorem of such modules, we have

$$M \simeq \boldsymbol{k}/(d_1) \oplus \cdots \oplus \boldsymbol{k}/(d_r), \tag{11}$$

where $d_1, \dots, d_r \in \boldsymbol{k}$ are noninvertible, and $d_i \mid d_{i+1}$ for all $i = 1, \dots, r-1$. The number r is determined uniquely, and the elements d_i — uniquely up to a multiplication by an invertible element. Hence,

$$\operatorname{gen} M := r, \quad \operatorname{rel} M := \max\{s : d_s \neq 0\}$$

are well defined. We get a short exact sequence $\boldsymbol{k}^{\operatorname{rel} M} \to \boldsymbol{k}^{\operatorname{gen} M} \to M \to 0$.

Lemma A.8 *Let $\boldsymbol{k}$ be a principal ideal domain. Suppose that there exists a short exact sequence $\boldsymbol{k}^A \xrightarrow{f} \boldsymbol{k}^B \to M \to 0$ for some $A, B < \infty$. Then $A \geq \operatorname{rel} M$ and $B \geq \operatorname{gen} M$.*

Proof We can assume that f is in the Smith normal form, that is, f is represented by a diagonal matrix with diagonal elements $d'_1, \dots, d'_s$ such that $d'_1 \mid d'_2 \mid \cdots \mid d'_s$. Remove all nonzero columns: this preserves cokernel and does not increase A. If d'_i is invertible, remove the i^{th} row and the i^{th} column: this preserves cokernel and diminish A and B by 1. We obtain a diagonal matrix $B' \times A'$ having no zero columns and no invertible elements on diagonal. Hence the cokernel is exactly of the form (11) for $B' = r = \operatorname{gen} M$ and $A' = s = \operatorname{rel} M$. □

Lemma A.9 *Let $\boldsymbol{k}$ be a principal ideal domain and $0 \to \boldsymbol{k}^a \to \boldsymbol{k}^b \xrightarrow{f} \boldsymbol{k}^c \to 0$ be an exact sequence of $\boldsymbol{k}$-modules for some $a, b, c < \infty$. Then $b = a + c$.*

Proof We can assume that f is in a Smith normal form. In this basis, f is represented by a diagonal matrix $c \times b$. Since f is surjective, the matrix has no nonzero rows, and all diagonal elements are noninvertible. Hence $\operatorname{Ker} f \simeq \boldsymbol{k}^{b-c}$. We have $\boldsymbol{k}^d \not\simeq \boldsymbol{k}^{d'}$ for $d \neq d'$, so $a = b - c$. □

Recall that we consider G-graded algebras that are connected with respect to the $\mathbb{Z}$-grading given by a map $G \to \mathbb{Z}$.

Theorem A.10 *Let A be a connected associative algebra with unit over a principal ideal domain $\boldsymbol{k}$. Suppose that $\boldsymbol{k}$-modules $\operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})_n$ and $\operatorname{Tor}_2^A(\boldsymbol{k}, \boldsymbol{k})_n$ are finitely generated for all $n \in G$. Then:*

(i) *If A is a free $\boldsymbol{k}$-module, it admits a homogeneous presentation that contains (for every n) precisely $\operatorname{gen} \operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})_n$ generators and $\operatorname{gen} \operatorname{Tor}_2^A(\boldsymbol{k}, \boldsymbol{k})_n + \operatorname{rel} \operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})_n$ relations of degree n.*

(ii) *If A admits a homogeneous presentation that contains N_n generators and M_n relations of degree n,*

$$N_n \geq \operatorname{gen} \operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})_n, \quad M_n \geq \operatorname{gen} \operatorname{Tor}_2^A(\boldsymbol{k}, \boldsymbol{k})_n + \operatorname{rel} \operatorname{Tor}_1^A(\boldsymbol{k}, \boldsymbol{k})_n. \tag{12}$$

Proof of statement (i) For every n, choose a set of $\operatorname{gen}(\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n)$ additive generators for the $\boldsymbol{k}$-module $\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n$, a set of $\operatorname{rel}(\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n)$ linear relations between them, and a set of $\operatorname{gen}(\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})_n)$ generators for $\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})_n$. These elements are represented by cycles and boundaries in the bar construction. Applying Theorem A.6 to them, we obtain a presentation of required size. □

Proof of statement (ii) Apply Proposition A.1 and continue the exact sequence to the free resolution of the left A-module $\boldsymbol{k}$. It has the form

$$\cdots \to A\otimes \boldsymbol{k}^M \to A\otimes \boldsymbol{k}^N \to A \xrightarrow{\varepsilon} \boldsymbol{k} \to 0.$$

Applying the functor $\boldsymbol{k}\otimes_A(-)$, we obtain a chain complex of graded $\boldsymbol{k}$-modules

$$\cdots \to \boldsymbol{k}^M \xrightarrow{\partial} \boldsymbol{k}^N \xrightarrow{0} \boldsymbol{k} \to 0,$$

having $\operatorname{Tor}^A(\boldsymbol{k},\boldsymbol{k})$ as homology. Therefore, for some $\partial_n\colon \boldsymbol{k}^{M_n} \to \boldsymbol{k}^{N_n}$ we have

$$\operatorname{Coker}\partial_n \simeq \operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n, \quad \operatorname{Ker}\partial_n \twoheadrightarrow \operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})_n.$$

In particular, $\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n$ is generated by N_n elements, so $N_n \geq \operatorname{gen}\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n$.

If M_n is infinite, both inequalities (12) are true, since the right side is finite. If M_n is finite, then N_n is finite, since $\operatorname{Coker}\partial_n$ is finitely generated. Thus $\operatorname{Ker}\partial_n \subset \boldsymbol{k}^{M_n}$, $\operatorname{Im}\partial_n \subset \boldsymbol{k}^{N_n}$ are submodules of finitely generated free modules, so these modules are free: $\operatorname{Ker}\partial_n \simeq \boldsymbol{k}^P$, $\operatorname{Im}\partial_n \simeq \boldsymbol{k}^Q$. We obtain exact sequences

$$\boldsymbol{k}^P \to \boldsymbol{k}^{N_n} \to \operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n \to 0, \quad \boldsymbol{k}^Q \to \operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})_n \to 0, \quad 0 \to \boldsymbol{k}^P \to \boldsymbol{k}^{M_n} \to \boldsymbol{k}^Q \to 0.$$

Then $N_n \geq \operatorname{gen}\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n$, $P \geq \operatorname{rel}\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n$, $Q \geq \operatorname{gen}\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})_n$ by Lemma A.8 and $P+Q = M_n$ by Lemma A.9. This proves (12). □

As a corollary, we obtain a well known result by Wall [44, Section 7]:

Corollary A.11 *Let A be a connected associative algebra with unit over a field $\boldsymbol{k}$. Then:*

(i) *A admits a homogeneous presentation that contains (for every n) precisely $\dim_{\boldsymbol{k}}\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n$ generators and $\dim_{\boldsymbol{k}}\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})_n$ relations of degree n.*

(ii) *If A admits a homogeneous presentation that contains N_n generators and M_n relations of degree n, then $N_n \geq \dim_{\boldsymbol{k}}\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})_n$ and $M_n \geq \dim_{\boldsymbol{k}}\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k})_n$.* □

We also obtain a criterion of freeness.

Corollary A.12 [32, Proposition 8.5.4] *Let A be a connected associative algebra with unit over a principal ideal domain $\boldsymbol{k}$, which is a free $\boldsymbol{k}$-module. The following conditions are equivalent.*

(a) *A is a free algebra (a tensor algebra on homogeneous generators).*

(b) *$\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k})$ is a free $\boldsymbol{k}$-module, and $\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k}) = 0$.*

Proof By Corollary A.4, (a) implies (b). Conversely, suppose that (b) holds. Then

$$\operatorname{rel}\operatorname{Tor}_1^A(\boldsymbol{k},\boldsymbol{k}) = \operatorname{gen}\operatorname{Tor}_2^A(\boldsymbol{k},\boldsymbol{k}) = 0.$$

By Theorem A.10(i), the algebra A admits a presentation with no relations. Hence A is free. □

Appendix B Loop homology and extensions of Hopf algebras

Consider a homotopy fibration $F \to E \xrightarrow{p} B$ of simply connected spaces, such that $\Omega p\colon \Omega E \to \Omega B$ admits a homotopy section (ie there is a continuous map $\sigma\colon \Omega B \to \Omega E$ that preserves basepoints, and a homotopy $\Omega p \circ \sigma \sim \mathrm{id}_{\Omega B}$). It is well known that then ΩE is homotopy equivalent to $\Omega F \times \Omega B$ (see [7, Proposition A.2; 17, Theorem 5.2]). If $\boldsymbol{k}$-homology of these loop spaces is free, we obtain an extension of Hopf algebras $\boldsymbol{k} \to H_*(\Omega F;\boldsymbol{k}) \to H_*(\Omega E;\boldsymbol{k}) \to H_*(\Omega B;\boldsymbol{k}) \to \boldsymbol{k}$. In Theorem B.3 we give a full proof of this folklore result. We consider ordinary loop spaces instead of Moore loop spaces, so that $\Omega(X \times Y) \cong \Omega X \times \Omega Y$ is a strict isomorphism of H-spaces.

We have a natural isomorphism

$$\alpha\colon \pi_n(A \times B) \xrightarrow{\cong} \pi_n(A) \oplus \pi_n(B), \quad [f] \mapsto [\mathrm{pr}_A \circ f] \oplus [\mathrm{pr}_B \circ f],$$

for any A, B and $n \geq 1$. We denote basepoint inclusion by $\varepsilon\colon * \to Y$ and collapse map by $\eta\colon Y \to *$.

Lemma B.1 *Let X be a simply connected space and $\mu\colon \Omega X \times \Omega X \to \Omega X$ be the composition of loops. Then the following diagram is commutative*:

$$\begin{array}{ccc} \pi_n(\Omega X \times \Omega X) & \xrightarrow{\mu_*} & \pi_n(\Omega X) \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & \nearrow{\scriptstyle (x,y)\mapsto x+y} & \\ \pi_n(\Omega X) \oplus \pi_n(\Omega X) & & \end{array}$$

Proof Let elements $x, y \in \pi_n(\Omega X)$ be represented by maps $f, g\colon S^n \to \Omega X$. Consider the element $z = [f \times \eta] + [\eta \times g] \in \pi_n(\Omega X \times \Omega X)$.

The map $\mu \circ (f \times \eta)$ is the composition $S^n \xrightarrow{f} \Omega X \xrightarrow{\mathrm{id}\times\eta} \Omega X \times \Omega X \xrightarrow{\mu} \Omega X$. The composition of two right maps is homotopic to the identity, hence $\mu \circ (f \times \eta) \sim f$. Passing to homotopy groups, we have $\mu_*([f \times \eta]) = x$. Similarly, $\mu_*([\eta \times g]) = y$, hence $\mu_*(z) = x + y$. On the other hand, $\alpha([f \times \eta]) = [\mathrm{pr}_1 \circ (f \times \eta)] \oplus [\mathrm{pr}_2 \circ (f \times \eta)] = [f] \oplus [\eta\varepsilon] = x \oplus 0$. Similarly, $\alpha([\eta \times g]) = 0 \oplus y$, hence $\alpha(z) = x \oplus y$. We obtained $\mu_*(\alpha^{-1}(x \oplus y)) = \mu_*(z) = x + y$, so the diagram commutes. □

In the following lemma, we say that a diagram commutes if it homotopy commutes.

Lemma B.2 *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of simply connected spaces*, *and $\sigma\colon \Omega B \to \Omega E$ be a homotopy section for Ωp. Consider the composition*

$$f\colon \Omega F \times \Omega B \xrightarrow{\Omega i \times \sigma} \Omega E \times \Omega E \xrightarrow{\mu} \Omega E.$$

Then:

(i) f *is a weak homotopy equivalence.*

(ii) f *respects the inclusion and the projection*, *that is*, *the following diagram commutes*:

$$\begin{array}{ccccc} & & \Omega E & & \\ & {\scriptstyle \Omega i}\nearrow & \uparrow{\scriptstyle f} & \searrow{\scriptstyle \Omega p} & \\ \Omega F \times * & \xrightarrow[\mathrm{id}\times\eta]{} & \Omega F \times \Omega B & \xrightarrow[\varepsilon\times\mathrm{id}]{} & * \times \Omega B \end{array}$$

(iii) f *respects the left action of* ΩF, *that is*, *the following diagram commutes*:

$$\begin{array}{ccc} \Omega F \times \Omega F \times \Omega B & \xrightarrow{\Omega i \times f} & \Omega E \times \Omega E \\ \downarrow{\scriptstyle \mu\times\mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ \Omega F \times \Omega B & \xrightarrow{f} & \Omega E \end{array}$$

(iv) f *respects the right coaction of* ΩB, *that is*, *the following diagram commutes*:

$$\begin{array}{ccc} \Omega F \times \Omega B & \xrightarrow{f} & \Omega E \\ \Big\downarrow{\scriptstyle \mathrm{id}\times\Delta} & & \downarrow{\scriptstyle \Delta} \\ & & \Omega E \times \Omega E \\ & & \downarrow{\scriptstyle \mathrm{id}\times\Omega p} \\ \Omega F \times \Omega B \times \Omega B & \xrightarrow{f\times\mathrm{id}} & \Omega E \times \Omega B \end{array}$$

Proof We have an exact sequence

$$\cdots \to \pi_n(\Omega F) \xrightarrow{(\Omega i)_*} \pi_n(\Omega E) \xrightarrow{(\Omega p)_*} \pi_n(\Omega B) \to \cdots,$$

where the map $(\Omega p)_*$ has a section σ_*. For every $n \geq 1$, we obtain an isomorphism of groups

$$\varphi\colon \pi_n(\Omega F) \oplus \pi_n(\Omega B) \xrightarrow{\cong} \pi_n(\Omega E), \quad \varphi(x, y) = (\Omega i)_*(x) + \sigma_*(y).$$

(We use that $\pi_1(\Omega X)$ is abelian.) By Lemma B.1 and naturality of

$$\alpha\colon \pi_n(\Omega F \times \Omega B) \to \pi_n(\Omega F) \oplus \pi_n(\Omega B)$$

we have

$$\varphi \circ \alpha = (\mu \circ (\Omega i \times \sigma))_* = f_*\colon \pi_n(\Omega F \times \Omega B) \to \pi_n(\Omega E).$$

Hence f_* is an isomorphism for all n, so f is a weak homotopy equivalence. Now consider the diagram

$$\begin{array}{ccccc} & & \Omega E & \xrightarrow{\Omega p} & \Omega B \\ & {\scriptstyle \mathrm{id}}\nearrow & \uparrow{\scriptstyle \mu} & & \uparrow{\scriptstyle \mu} \\ \Omega E \times * & \xrightarrow{\mathrm{id}\times\eta} & \Omega E \times \Omega E & \xrightarrow{\Omega p\times\Omega p} & \Omega B \times \Omega B \\ \uparrow{\scriptstyle \Omega i\times\mathrm{id}} & & \uparrow{\scriptstyle \Omega i\times\sigma} & & \uparrow{\scriptstyle \eta\times\mathrm{id}} \\ \Omega F \times * & \xrightarrow{\mathrm{id}\times\eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon\times\mathrm{id}} & * \times \Omega B \end{array}$$

The triangle commutes, since η is a homotopy unit in ΩE. The upper right square commutes, since Ωp is a map of H-spaces. The bottom left square commutes, since $\eta_{\Omega E} = \sigma \circ \eta_{\Omega B} \colon * \to \Omega E$. Finally, the commutativity of bottom right square is equivalent to the existence of homotopies $\Omega p \circ \Omega i \sim \eta \circ \varepsilon$ and $\Omega p \circ \sigma \sim \mathrm{id}$. The first homotopy exists, since $p \circ i$ is homotopy trivial; the second exists, since σ is a homotopy section for Ωp. Hence the whole diagram is commutative. The right side of the diagram is homotopic to $\mathrm{id}\colon \Omega B \to \Omega B$, since η is a homotopy unit in ΩB. We obtain a commutative diagram

$$\begin{array}{ccccc}
 & & \Omega E & \xrightarrow{\Omega p} & \Omega B \\
 & \nearrow^{\Omega i \times \mathrm{id}} & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \mathrm{id}} \\
\Omega F \times * & \xrightarrow{\mathrm{id}\times\eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon\times\mathrm{id}} & * \times \Omega B
\end{array}$$

that is equivalent to the diagram from (ii). Now consider the diagram

$$\begin{array}{ccccc}
\Omega F \times \Omega F \times \Omega B & \xrightarrow{\Omega i\times\Omega i\times\sigma} & \Omega E \times \Omega E \times \Omega E & \xrightarrow{\mathrm{id}\times\mu} & \Omega E \times \Omega E \\
\downarrow{\scriptstyle \mu\times\mathrm{id}} & & \downarrow{\scriptstyle \mu\times\mathrm{id}} & & \downarrow{\scriptstyle \mu} \\
\Omega F \times \Omega B & \xrightarrow{\Omega i\times\sigma} & \Omega E \times \Omega E & \xrightarrow{\mu} & \Omega E
\end{array}$$

The left square commutes, since $\Omega i \colon \Omega F \to \Omega E$ is a map of H-spaces; the right square commutes, since μ is homotopy associative. The top side of the diagram equals $\Omega i \times (\mu \circ (\Omega i \times \sigma)) = \Omega i \times f$, the bottom side equals f. Hence, it is the diagram from (iii). Finally, consider the diagram

$$\begin{array}{ccccc}
\Omega F \times \Omega B & \xrightarrow{\Omega i\times\sigma} & \Omega E \times \Omega E & \xrightarrow{\mu} & \Omega E \\
\downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \Delta} \\
\Omega F \times \Omega B \times \Omega F \times \Omega B & \xrightarrow{\Omega i\times\sigma\times\Omega i\times\sigma} & \Omega E \times \Omega E \times \Omega E \times \Omega E & \xrightarrow{\mu\times\mu} & \Omega E \times \Omega E \\
\downarrow{\scriptstyle \mathrm{pr}_{124}} & \searrow^{\Omega i\times\sigma\times\varepsilon\times\mathrm{id}} & \downarrow{\scriptstyle \mathrm{id}\times\mathrm{id}\times\Omega p\times\Omega p} & & \downarrow{\scriptstyle \mathrm{id}\times\Omega p} \\
\Omega F \times \Omega B \times \Omega B & \xrightarrow{\phi} & \Omega E \times \Omega E \times \Omega B \times \Omega B & \xrightarrow{\mu\times\mu} & \Omega E \times \Omega B
\end{array}$$

where $\Delta(x) := (x,x)$, $D(x,y) := (x,y,x,y)$ and $\phi(f,b_1,b_2) := (\Omega i(f), \sigma(b_1), *, b_2)$. Clearly, the top two squares commute. The bottom right square commutes, since $\mathrm{id}\colon \Omega E \to \Omega E$ and $\Omega p\colon \Omega E \to \Omega B$ are maps of H-spaces. The upper triangle commutes, since $\Omega p \circ \Omega i \sim \varepsilon$ and $\Omega p \circ \sigma \sim \mathrm{id}$; the bottom triangle commutes by the definition of ϕ. The outer maps in the diagram give the required diagram (iv). □

In the proof of next theorem we use the Künneth map $\kappa\colon H_*(X;\boldsymbol{k}) \otimes H_*(Y;\boldsymbol{k}) \to H_*(X \times Y;\boldsymbol{k})$. It is natural and associative. It is an isomorphism if $H_*(Y;\boldsymbol{k})$ is a free $\boldsymbol{k}$-module.

If X is a simply connected space and $H_*(\Omega X;\boldsymbol{k})$ is free over $\boldsymbol{k}$, this module is a connected $\boldsymbol{k}$-Hopf algebra with the standard cup coproduct (see Section 2.4) and the Pontryagin product

$$m\colon H_*(\Omega X;\boldsymbol{k}) \otimes H_*(\Omega X;\boldsymbol{k}) \xrightarrow{\kappa} H_*(\Omega X \times \Omega X;\boldsymbol{k}) \xrightarrow{\mu_*} H_*(\Omega X;\boldsymbol{k}).$$

The unit and counit $\boldsymbol{k} \xrightarrow{\eta} H_*(\Omega X;\boldsymbol{k}) \xrightarrow{\varepsilon} \boldsymbol{k}$ are induced by the H-space maps $* \xrightarrow{\eta} \Omega X \xrightarrow{\varepsilon} *$.

Theorem B.3 *Let $\boldsymbol{k}$ be an associative ring with unit. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a homotopy fibration of simply connected spaces such that $H_*(\Omega B;\boldsymbol{k})$ and $H_*(\Omega F;\boldsymbol{k})$ are free $\boldsymbol{k}$-modules, and the map Ωp admits a homotopy section $\sigma\colon \Omega B \to \Omega E$. Consider the composition*

$$\Phi\colon H_*(\Omega F;\boldsymbol{k}) \otimes H_*(\Omega B;\boldsymbol{k}) \xrightarrow{(\Omega i)_* \otimes \sigma_*} H_*(\Omega E;\boldsymbol{k}) \otimes H_*(\Omega E;\boldsymbol{k}) \xrightarrow{m} H_*(\Omega E;\boldsymbol{k}).$$

Then

- (i) Φ *is an isomorphism of $\boldsymbol{k}$-modules*;
- (ii) $(\Omega i)_* = \Phi \circ (\mathrm{id}_{H_*(\Omega F;\boldsymbol{k})} \otimes \eta_{H_*(\Omega B;\boldsymbol{k})})$;
- (iii) $(\Omega p)_* \circ \Phi = \varepsilon_{H_*(\Omega F;\boldsymbol{k})} \otimes \mathrm{id}_{H_*(\Omega B;\boldsymbol{k})}$;
- (iv) Φ *is a morphism of left $H_*(\Omega F;\boldsymbol{k})$-modules and right $H_*(\Omega B;\boldsymbol{k})$-comodules, where the (co)module structure on $H_*(\Omega E;\boldsymbol{k})$ is induced by the maps $(\Omega i)_*$ and $(\Omega p)_*$.*

In particular, $\boldsymbol{k} \to H_*(\Omega F;\boldsymbol{k}) \xrightarrow{(\Omega i)_*} H_*(\Omega E;\boldsymbol{k}) \xrightarrow{(\Omega p)_*} H_*(\Omega B;\boldsymbol{k}) \to \boldsymbol{k}$ is an extension of connected Hopf algebras over $\boldsymbol{k}$.

Proof We write $H_*(\Omega X)$ instead of $H_*(\Omega X;\boldsymbol{k})$. Note that σ is continuous, and Ωi, Ωp are maps of H-spaces. Hence σ_* is a map of coalgebras, and $(\Omega i)_*$, $(\Omega p)_*$ are maps of Hopf algebras. By the naturality of Künneth map, the following diagram commutes:

$$\begin{array}{ccccc}
H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{(\Omega i)_* \otimes \sigma_*} & H_*(\Omega E) \otimes H_*(\Omega E) & \xrightarrow{m} & H_*(\Omega E) \\
{\scriptstyle\kappa}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle\kappa}\big\downarrow & & \big\| \\
H_*(\Omega F \times \Omega B) & \xrightarrow{(\Omega i \times \sigma)_*} & H_*(\Omega E \times \Omega E) & \xrightarrow{\mu_*} & H_*(\Omega E)
\end{array}$$

The top side of diagram equals Φ, and the bottom side equals f_*. Hence Φ is the composition $H_*(\Omega F) \otimes H_*(\Omega B) \xrightarrow{\kappa} H_*(\Omega F \times \Omega B) \xrightarrow{f_*} H_*(\Omega E)$. The left map is bijective by the assumption, the right map is bijective by Lemma B.2(i). Hence Φ is an isomorphism, so (i) is proved. Consider the diagram

$$\begin{array}{ccccc}
 & \nearrow^{(\Omega i)_*} & H_*(\Omega E) & \searrow^{(\Omega p)_*} & \\
 & & {\scriptstyle f_*}\big\uparrow & & \\
H_*(\Omega F \times *) & \xrightarrow{(\mathrm{id} \times \eta)_*} & H_*(\Omega F \times \Omega B) & \xrightarrow{(\varepsilon \times \mathrm{id})_*} & H_*(* \times \Omega B) \\
{\scriptstyle\kappa}\big\uparrow{\scriptstyle\simeq} & & {\scriptstyle\kappa}\big\uparrow & & {\scriptstyle\kappa}\big\uparrow{\scriptstyle\simeq} \\
H_*(\Omega F) \otimes \boldsymbol{k} & \xrightarrow{\mathrm{id} \otimes \eta} & H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\varepsilon \otimes \mathrm{id}} & \boldsymbol{k} \otimes H_*(\Omega B)
\end{array}$$

The top half of the diagram commutes by Lemma B.2(ii), the bottom half commutes by naturality of κ. Since $f_* \circ \kappa = \Phi$, we have a commutative diagram

$$\begin{array}{ccccc}
 & \nearrow^{(\Omega i)_*} & H_*(\Omega E) & \searrow^{(\Omega p)_*} & \\
 & & {\scriptstyle \Phi}\big\uparrow & & \\
H_*(\Omega F) & \xrightarrow{\mathrm{id} \otimes \eta} & H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\varepsilon \otimes \mathrm{id}} & H_*(\Omega B)
\end{array}$$

which proves (ii) and (iii). Now consider the diagram

$$\begin{array}{ccccc}
H_*(\Omega F)\otimes H_*(\Omega F)\otimes H_*(\Omega B) & \xrightarrow{\mathrm{id}\otimes\kappa} & H_*(\Omega F)\otimes H_*(\Omega F\times\Omega B) & \xrightarrow{(\Omega i)_*\otimes f_*} & H_*(\Omega E)\otimes H_*(\Omega E) \\
\downarrow{\scriptstyle \kappa\otimes\mathrm{id}} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa} \\
H_*(\Omega F\times\Omega F)\otimes H_*(\Omega B) & \xrightarrow{\kappa\otimes\mathrm{id}} & H_*(\Omega F\times\Omega F\times\Omega B) & \xrightarrow{(\Omega i\times f)_*} & H_*(\Omega E\times\Omega E) \\
\downarrow{\scriptstyle \mu_*\otimes\mathrm{id}} & & \downarrow{\scriptstyle (\mu\times\mathrm{id})_*} & & \downarrow{\scriptstyle \mu_*} \\
H_*(\Omega F)\otimes H_*(\Omega B) & \xrightarrow{\kappa} & H_*(\Omega F\times\Omega B) & \xrightarrow{f_*} & H_*(\Omega E)
\end{array}$$

The bottom right square commutes by Lemma B.2(iii), the other squares commute by naturality of κ. Since $\mu_*\circ\kappa = m\colon H_*(\Omega X)\otimes H_*(\Omega X)\to H_*(\Omega X)$, the outer maps in the diagram are

$$\begin{array}{ccc}
H_*(\Omega F)\otimes H_*(\Omega F)\otimes H_*(\Omega B) & \xrightarrow{(\Omega i)_*\otimes\Phi} & H_*(\Omega E)\otimes H_*(\Omega E) \\
\downarrow{\scriptstyle m\otimes\mathrm{id}} & & \downarrow{\scriptstyle m} \\
H_*(\Omega F)\otimes H_*(\Omega B) & \xrightarrow{\Phi} & H_*(\Omega E)
\end{array}$$

Hence Φ is a map of left $H_*(\Omega F)$-modules. Similarly, by Lemma B.2(iv) and the Künneth isomorphisms we have the commutative diagram

$$\begin{array}{ccc}
H_*(\Omega F)\otimes H_*(\Omega B) & \xrightarrow{\Phi} & H_*(\Omega E) \\
 & & \downarrow{\scriptstyle \Delta} \\
\downarrow{\scriptstyle \mathrm{id}\otimes\Delta} & & H_*(\Omega E)\otimes H_*(\Omega E) \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes(\Omega p)_*} \\
H_*(\Omega F)\otimes H_*(\Omega B)\otimes H_*(\Omega B) & \xrightarrow{\Phi\otimes\mathrm{id}} & H_*(\Omega E)\otimes H_*(\Omega B)
\end{array}$$

hence Φ is a map of right $H_*(\Omega B)$-comodules.

Since (i)–(iv) hold, the maps of Hopf algebras $(\Omega i)_*\colon H_*(\Omega F)\to H_*(\Omega E)$ and $(\Omega p)_*\colon H_*(\Omega E)\to H_*(\Omega B)$ form an extension of Hopf algebras by Proposition 2.4. □

Recall that an element $x\in A$ of a Hopf algebra is *primitive* if $\Delta x = 1\otimes x + x\otimes 1$. The set of primitive elements is a Lie subalgebra $PA\subset A$. Every map of Hopf algebras $f\colon A\to A'$ induces a map of Lie algebras $Pf := f|_{PA}\colon PA\to PA'$.

Corollary B.4 *Suppose that the conditions of Theorem B.3 are met. Let $x\in H_*(\Omega E;\boldsymbol{k})$ be a primitive element such that $(\Omega p)_*(x)=0$. Then $x=(\Omega i)_*(y)$ for some $y\in H_*(\Omega F;\boldsymbol{k})$.*

Proof Since $\boldsymbol{k}\to H_*(\Omega F;\boldsymbol{k})\to H_*(\Omega E;\boldsymbol{k})\to H_*(\Omega B;\boldsymbol{k})\to\boldsymbol{k}$ is a Hopf algebra extension, the sequence $0\to PH_*(\Omega F;\boldsymbol{k})\to PH_*(\Omega E;\boldsymbol{k})\to PH_*(\Omega B;\boldsymbol{k})$ is exact; see [31, Proposition 4.10]. (This also easily follows from definitions). We have

$$x\in \mathrm{Ker}(PH_*(\Omega E;\boldsymbol{k})\to PH_*(\Omega B;\boldsymbol{k})) = \mathrm{Im}(PH_*(\Omega F;\boldsymbol{k})\to PH_*(\Omega E;\boldsymbol{k})).$$ □

Appendix C Commutator identities

Fix elements $u_1, \dots, u_m$ of degree 1 in a graded associative algebra Γ. For a subset $I=\{i_1<\dots<i_k\}\subset[m]$, we define

$$\widehat{u}_I := u_{i_1}\cdots u_{i_k}, \quad c(I,x) := \big[u_{i_1},[u_{i_2},[\dots[u_{i_k},x],\dots]]\big], \quad x\in\Gamma.$$

We write $A<B$ when $A,B\subset[m]$ and $\max(A)<\min(B)$. If $A<B$, we have $\widehat{u}_{A\sqcup B}=\widehat{u}_A\cdot\widehat{u}_B$ and $c(A\sqcup B,x)=c(A,c(B,x))$. Also, $\widehat{u}_\varnothing=1$, $c(\varnothing,x)=x$.

Define the Koszul sign by $\theta(A,B) := |\{(a,b)\in A\times B : a>b\}|$. In a graded commutative algebra, we would have $\widehat{u}_A\cdot\widehat{u}_B=(-1)^{\theta(A,B)}\widehat{u}_{A\sqcup B}$ if $A\cap B=\varnothing$. It has the following properties:

(i) $\theta(A,B)\equiv|A|\cdot|B|+\theta(B,A) \bmod 2$.

(ii) If $A_1\sqcup B_1<A_2\sqcup B_2$, then

$$\theta(A_1\sqcup A_2, B_1\sqcup B_2)\equiv\theta(A_1,B_1)+\theta(A_2,B_2)+|A_2|\cdot|B_1|.$$

For $I\subset[m]$, $j\in[m]$, we write $I_{<j}=\{i\in I : i<j\}$, $I_{>j}=\{i\in I : i>j\}$. We also use i as a shortened notation for $\{i\}$.

C.1 Regrouping of monomials

The following formulas can be used to express any monomial on $u_1,\dots,u_m$ as a linear combination of $c_1\cdots c_s\cdot\widehat{u}_B$, $c_i=c(A_i,u_{j_i})$, $A_i\neq\varnothing$.

Lemma C.1 *Let $I\subset[m]$, and let $x\in\Gamma$ be homogeneous. Then*

$$\widehat{u}_I\cdot x=\sum_{I=A\sqcup B}(-1)^{\theta(A,B)+\deg(x)\cdot|B|}c(A,x)\widehat{u}_B. \tag{13}$$

Proof Define $d:=\deg(x)$. We induct on $|I|$. The base case $I=\varnothing$ is clear. For the inductive step, let $i=\min(I)$, $I'=I\setminus i$. Then the right-hand side is equal to

$$\begin{aligned}\sum_{I'=A\sqcup B}(-1)^{\theta(i\sqcup A,B)+d\cdot|B|}c(i\sqcup A,x)\widehat{u}_B+\sum_{I'=A\sqcup B}(-1)^{\theta(A,i\sqcup B)+d\cdot|i\sqcup B|}c(A,x)\widehat{u}_{i\sqcup B}&\\ =\sum_{I'=A\sqcup B}(-1)^{\theta(A,B)+d\cdot|B|}\big([u_i,c(A,x)]+(-1)^{|A|+d}c(A,x)u_i\big)\cdot\widehat{u}_B&\\ =\sum_{I'=A\sqcup B}(-1)^{\theta(A,B)+d\cdot|B|}u_ic(A,x)\cdot\widehat{u}_B.&\end{aligned}$$

By the inductive hypothesis, this sum is equal to $u_i\cdot\widehat{u}_{I'}x=\widehat{u}_I\cdot x$. □

Proposition C.2 *Let $I\subset[m]$, $j\in[m]$. Then*

$$\widehat{u}_I\cdot u_j=\sum_{\substack{I=A\sqcup B\\ \max(A)>j}}(-1)^{\theta(A,B)+|B|}c(A,u_j)\widehat{u}_B+(-1)^{|I_{>j}|}\cdot\begin{Bmatrix}\widehat{u}_{I\sqcup j}, & j\notin I\\ \widehat{u}_{I_{<j}}\cdot u_j^2\cdot\widehat{u}_{I_{>j}}, & j\in I\end{Bmatrix}.$$

Proof Define $P = I_{\leq j}$, $Q = I_{>j}$. Then $P < Q$, therefore

$$\widehat{u}_I = \widehat{u}_P \widehat{u}_Q, \quad r := \widehat{u}_P u_j \widehat{u}_Q = \begin{cases} \widehat{u}_{I \sqcup \{j\}}, & j \notin I; \\ \widehat{u}_{I_{<j}} \cdot u_j^2 \cdot \widehat{u}_{I_{>j}}, & j \in I. \end{cases}$$

Apply (13) to $\widehat{u}_Q \cdot u_j$, and consider the summand with $A_2 = \varnothing$ separately:

$$\begin{aligned} \widehat{u}_I \cdot u_j = \widehat{u}_P \widehat{u}_Q u_j &= \sum_{Q = A_2 \sqcup B_2} (-1)^{\theta(A_2,B_2)+|B_2|} \widehat{u}_P c(A_2, u_j) \widehat{u}_{B_2} \\ &= (-1)^{|Q|} \widehat{u}_P u_j \widehat{u}_Q + \sum_{\substack{Q = A_2 \sqcup B_2 \\ A_2 \neq \varnothing}} (-1)^{\theta(A_2,B_2)+|B_2|} \widehat{u}_P c(A_2, u_j) \widehat{u}_{B_2}. \end{aligned}$$

Applying (13) to $\widehat{u}_P \cdot c(A_2, u_j)$, we obtain the required identity:

$$\begin{aligned} \widehat{u}_I \cdot u_j &= (-1)^{|Q|} r + \sum_{P = A_1 \sqcup B_1} \sum_{\substack{Q = A_2 \sqcup B_2 \\ A_2 \neq \varnothing}} (-1)^{\theta(A_1,B_1)+(|A_2|+1)\cdot|B_1|+\theta(A_2,B_2)+|B_2|} c(A_1, c(A_2, u_j)) \widehat{u}_{B_1} \widehat{u}_{B_2} \\ &= (-1)^{|Q|} r + \sum_{\substack{P \sqcup Q = A \sqcup B \\ A_{>j} \neq \varnothing}} (-1)^{\theta(A,B)+|B|} c(A, u_j) \widehat{u}_B. \end{aligned}$$

□

C.2 Identities for nested commutators

In this section Γ can be a Lie superalgebra.

Lemma C.3 *For $I \subset [m]$ and homogeneous elements $x, y \in \Gamma$, we have*

$$\begin{aligned} (14) \qquad c(I, [x,y]) &= \sum_{I = A \sqcup B} (-1)^{\theta(A,B)+\deg(x)\cdot|B|} [c(A,x), c(B,y)] \\ &= [c(I,x), y] + (-1)^{\deg(x)\cdot|I|} [x, c(I,y)] + \sum_{\substack{I = A \sqcup B \\ A,B \neq \varnothing}} (-1)^{\theta(A,B)+\deg(x)\cdot|B|} [c(A,x), c(B,y)]. \end{aligned}$$

Proof The second identity follows from $\theta(\varnothing, I) = \theta(I, \varnothing) = 0$ and $c(\varnothing, x) = x$. Let us prove the first identity by induction on $|I|$. The base case $I = \varnothing$ is clear. For the inductive step, define $i = \min(I)$, $I' = I \setminus i$, $d = \deg(x)$. Then, by the inductive hypothesis,

$$\begin{aligned} c(I, [x,y]) = \big[u_i, c(I', [x,y])\big] &= \sum_{I' = A' \sqcup B'} (-1)^{\theta(A',B')+d\cdot|B'|} \big[u_i, [c(A',x), c(B',y)]\big] \\ &= \sum_{I' = A' \sqcup B'} (-1)^{\theta(A',B')+d\cdot|B'|} \big[[u_i, c(A',x)], c(B',y)\big] \\ &\qquad + \sum_{I' = A' \sqcup B'} (-1)^{\theta(A',B')+d\cdot|B'|+d+|A'|} \big[c(A',x), [u_i, c(B',y)]\big] \\ &= \sum_{I' = A' \sqcup B'} (-1)^{\theta(i \sqcup A',B')+d\cdot|B'|} [c(i \sqcup A', x), c(B',y)] \\ &\qquad + \sum_{I' = A' \sqcup B'} (-1)^{\theta(A', i \sqcup B')+d\cdot|i \sqcup B'|} [c(A',x), c(i \sqcup B', y)] \\ &= \sum_{I = A \sqcup B} (-1)^{\theta(A,B)+d\cdot|B|} [c(A,x), c(B,y)]. \end{aligned}$$

□

Corollary C.4 *Let $I \subset [m]$, $I = I'' \sqcup I'$, $I'' < I'$. Let $x, y \in \Gamma$ be homogeneous, and let $\mathcal{A} \subset 2^{I'} \times 2^{I'}$ be a family of pairs of subsets. Then*

$$\sum_{\substack{I'=A'\sqcup B'\\(A',B')\in\mathcal{A}}} (-1)^{\theta(A',B')+|B'|} c\bigl(I'',[c(A',x),c(B',y)]\bigr) = \sum_{\substack{I=A\sqcup B\\(A\cap I',B\cap I')\in\mathcal{A}}} (-1)^{\theta(A,B)+|B|}[c(A,x),c(B,y)]. \tag{15}$$

Proof This follows from (14) and from identities

$$c(A'', c(A', x)) = c(A'' \sqcup A', x), \quad \theta(A'' \sqcup A', B'' \sqcup B') = \theta(A'', B'') + \theta(A', B') + |A'| \cdot |B''|$$

that are true for $A'', B'' < A', B'$. $\square$

Proposition C.5 *Let $J \subset [m]$ and $i, j \in J$ such that $i < j$ and $J_{>j} \neq \varnothing$. Then*

$$\begin{aligned} c(J \setminus ij, [u_i, u_j]) = (-1)^{|J_{>j}|} c(J \setminus i, u_i) - (-1)^{|J_{>i}|} c(J \setminus j, u_j) \\ + \sum_{\substack{J\setminus ij = A\sqcup B\\ A_{>i}, B_{>j} \neq \varnothing}} (-1)^{\theta(A,B)+|B|}[c(A,u_i), c(B,u_j)]. \end{aligned} \tag{16}$$

Proof Define $P = J_{<j}$, $Q = J_{>i} \cap J_{<j}$, $R = J_{>j}$. Hence $P < i < Q < j < R$ and $R \neq \varnothing$. The left-hand side is equal to $x := c\bigl(P \sqcup Q, c(R, [u_i, u_j])\bigr)$. Define also $y := c(P \sqcup Q, [c(R, u_i), u_j])$, $z := c\bigl(P \sqcup Q, [u_i, c(R, u_j)]\bigr)$. Then

$$\begin{aligned} x &= y + (-1)^{|R|} z + \sum_{\substack{R=A'\sqcup B'\\ A',B'\neq\varnothing}} (-1)^{\theta(A',B')+|B'|} c\bigl(P \sqcup Q, [c(A', u_i), c(B', u_j)]\bigr) \\ &\underset{(15)}{=} y + (-1)^{|R|} z + \sum_{\substack{J\setminus ij=A\sqcup B\\ A_{>j}, B_{>j}\neq\varnothing}} (-1)^{\theta(A,B)+|B|}[c(A,u_i), c(B,u_j)], \end{aligned}$$

$$\begin{aligned} y &= (-1)^{|R|} c\bigl(P \sqcup Q, [u_j, c(R, u_i)]\bigr) = (-1)^{|R|} c(J \setminus i, u_i), \\ z &= c\bigl(P, c\bigl(Q, [u_i, c(R, u_j)]\bigr)\bigr) \underset{(14)}{=} c\bigl(P, [c(Q, u_i), c(R, u_j)]\bigr) + (-1)^{|Q|} \underbrace{c\bigl(P, [u_i, c(Q \sqcup R, u_j)]\bigr)}_{=c(J\setminus j, u_j)} \\ &\qquad + \sum_{\substack{Q=A_2\sqcup B_2\\ A_2,B_2\neq\varnothing}} (-1)^{\theta(A_2,B_2)+|B_2|} c\bigl(P, [c(A_2, u_i), c(B_2 \sqcup R, u_j)]\bigr) \\ &\underset{(15)}{=} (-1)^{|R|} \sum_{\substack{J\setminus ij=A\sqcup B\\ B\cap Q=\varnothing\\ A_{>j}=\varnothing}} (-1)^{\theta(A,B)+|B|}[c(A,u_i), c(B,u_j)] \\ &\qquad + (-1)^{|Q|} c(J \setminus j, u_j) + (-1)^{|R|} \sum_{\substack{J\setminus ij=A\sqcup B\\ B\cap Q\neq\varnothing, Q\\ A_{>j}=\varnothing}} (-1)^{\theta(A,B)+|B|}[c(A,u_i), c(B,u_j)]. \end{aligned}$$

Therefore,

$$\begin{aligned} x &= (-1)^{|R|} c(J \setminus i, u_i) + (-1)^{|Q|+|R|} c(J \setminus j, u_j) \\ &\quad + \sum_{\substack{J\setminus ij=A\sqcup B\\ A_{>i}\neq\varnothing\\ A_{>j}=\varnothing}} (-1)^{\theta(A,B)+|B|}[c(A,u_i), c(B,u_j)] + \sum_{\substack{J\setminus ij=A\sqcup B\\ A_{>j}, B_{>j}\neq\varnothing}} (-1)^{\theta(A,B)+|B|}[c(A,u_i), c(B,u_j)]. \end{aligned}$$

In the first sum the condition $B_{>j} \neq \varnothing$ is always true, since $R = A_{>j} \sqcup B_{>j}$, $A_{>j} = \varnothing$ and $R \neq \varnothing$. In the second sum, $A_{>i} \neq \varnothing$ is always true. Hence the sums can be merged into

$$\sum_{\substack{J\setminus ij = A\sqcup B \\ A_{>i}, B_{>j} \neq \varnothing}} (-1)^{\theta(A,B)+|B|}[c(A,u_i), c(B,u_j)].$$

Using $|R| = |J_{>j}|$ and $|Q + |R| = |J_{>i}| - 1$, we obtain (16). □

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