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FEDOR VYLEGZHANIN

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We develop a general homological approach to presentations of connected graded associative algebras, and apply it to the loop homology of moment-angle complexes Z_K that correspond to flag simplicial complexes K . For an arbitrary coefficient ring, we describe generators of the Pontryagin algebra $H_*(\Omega Z_K)$ and defining relations between them. We prove that such moment-angle complexes are coformal over \mathbb{Q} , give a necessary condition for rational formality, and compute their homotopy groups in terms of homotopy groups of spheres.

16E30, 16W50, 57S12; 55P35, 55Q52, 55U15, 57T05

1 Introduction

For a simply connected space X and a commutative ring k with unit, the *Pontryagin algebra* $H_*(\Omega X; k)$ is a connected graded associative k -algebra with respect to the Pontryagin product. We study the Pontryagin algebras of *moment-angle complexes* $X = Z_{\mathcal{K}} := (D^2, S^1)^{\mathcal{K}}$ that correspond to simplicial complexes \mathcal{K} . Moment-angle complexes play an important role in toric topology [15], and they have interesting homotopical properties and surprising connections to several topics in algebra and combinatorics [5]. If \mathcal{K} is a simplicial complex on the vertex set $[m] = \{1, \dots, m\}$, there is an effective action of the m -dimensional torus $\mathbb{T}^m = (S^1)^{\times m}$ on $Z_{\mathcal{K}}$. The homotopy quotient $E\mathbb{T}^m \times_{\mathbb{T}^m} Z_{\mathcal{K}}$ (the Borel construction) is known as the *Davis–Januszkiewicz space* $DJ(\mathcal{K})$ and is homotopy equivalent to the polyhedral product $(\mathbb{C}P^\infty, *)^{\mathcal{K}}$; see [15, Theorem 4.3.2].

Panov and Ray [35] reduced the study of corresponding Pontryagin algebras to an algebraic problem. Applying the based loops functor to the homotopy fibration

$$(1) \quad Z_{\mathcal{K}} \rightarrow DJ(\mathcal{K}) \rightarrow B\mathbb{T}^m,$$

they obtained a split fibration of H-spaces $\Omega Z_{\mathcal{K}} \rightarrow \Omega DJ(\mathcal{K}) \rightarrow \mathbb{T}^m$ and thus an extension of cocommutative Hopf algebras

$$k \rightarrow H_*(\Omega Z_{\mathcal{K}}; k) \rightarrow H_*(\Omega DJ(\mathcal{K}); k) \rightarrow \Lambda[u_1, \dots, u_m] \rightarrow k$$

over a field k . For any \mathcal{K} , there is an isomorphism of Hopf algebras $H_*(\Omega DJ(\mathcal{K}); k) \cong \text{Ext}_{k[\mathcal{K}]}(k, k)$ [20; 35] (moreover, this is true for any principal ideal domain k such that $H_*(\Omega DJ(\mathcal{K}); k)$ is a free

\mathbf{k} -module). If \mathcal{K} is a *flag* simplicial complex, this Hopf algebra is known completely: it is isomorphic to the partially commutative algebra

$$\mathbf{k}[\mathcal{K}]^{\dagger} := T(u_1, \dots, u_m) / (u_i^2 = 0, i = 1, \dots, m; u_i u_j + u_j u_i = 0, \{i, j\} \in \mathcal{K}), \quad \deg u_i = 1.$$

Generators u_i are primitive and have degree $(-1, 2e_i)$ with respect to the $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading introduced in [43]. In this case Grbić, Panov, Theriault and Wu [23] found a minimal generating set for the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$, and the author calculated the number of relations in any minimal presentation (by homogeneous generators and relations) of this algebra [43].

The last calculation relies on homological methods developed by Wall [44] and Lemaire [29] for connected graded associative algebras over a field. Namely, multiplicative generators of a connected \mathbf{k} -algebra A correspond to additive generators of the graded \mathbf{k} -module $\mathrm{Tor}_1^A(\mathbf{k}, \mathbf{k})$, and relations correspond to generators of $\mathrm{Tor}_2^A(\mathbf{k}, \mathbf{k})$. In order to study the integer Pontryagin algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbb{Z})$, we generalise these results to the case of *arbitrary* commutative rings \mathbf{k} with unit, and construct explicit presentations of connected \mathbf{k} -algebras using cycles in the bar construction. These results are presented in Appendix A. We hope that they will be useful in other contexts.

Let us give a general description of our approach. Suppose that we are given a connected \mathbf{k} -algebra A which is a free left module over its subalgebra S , $A \simeq S \otimes_{\mathbf{k}} V$. We wish to construct a presentation of S . Theorem A.6 does that, if we know a set of cycles in the bar construction $\overline{B}(S)$, such that their images generate the \mathbf{k} -modules $H_i(\overline{B}(S)) \simeq \mathrm{Tor}_i^S(\mathbf{k}, \mathbf{k})$, $i = 1, 2$. The following algorithm computes such cycles:

- (i) Build a free resolution $(A \otimes M, d)$ of the left A -module \mathbf{k} .
- (ii) Interpret it as a free resolution $(S \otimes V \otimes M, \widehat{d})$ of the left S -module \mathbf{k} . Compute the functor $\mathrm{Tor}^S(\mathbf{k}, \mathbf{k})$ as the homology of the complex $(V \otimes M, \overline{d})$. Find cycles in $(V \otimes M, \overline{d})$ such that their images generate $\mathrm{Tor}_i^S(\mathbf{k}, \mathbf{k})$, $i = 1, 2$.
- (iii) Construct a morphism $\varphi: (S \otimes V \otimes M, \widehat{d}) \rightarrow (B(S), d_B)$ of free resolutions of the left S -module \mathbf{k} , using the contracting homotopy of the bar resolution (see Corollary 2.2). Obtain a morphism of chain complexes $\overline{\varphi}: (V \otimes M, \overline{d}) \rightarrow (\overline{B}(S), d_{\overline{B}})$ that induces an isomorphism on the homology.
- (iv) Applying $\overline{\varphi}$ to the cycles from (ii), obtain the required cycles in $\overline{B}(S)$.

This situation takes place if $\mathbf{k} \rightarrow S \rightarrow A \rightarrow V \rightarrow \mathbf{k}$ is an extension of connected Hopf algebras; see [2; 31, Proposition 4.9]. In that sense, our algorithm has similarities with the *Reidemeister–Schreier algorithm* that constructs a presentation of a subgroup, given a presentation of the whole group. See [30] for another approach to Hopf subalgebras in connected Hopf algebras. It is well known that extensions of Hopf algebras arise in the study of fibrations $F \rightarrow E \rightarrow B$ that have a section after looping (see Appendix B for the proof). For such “ Ω -split” fibrations, the proposed method allows us to study presentations of $H_*(\Omega F; \mathbf{k})$, if the algebras $H_*(\Omega E; \mathbf{k})$ and $H_*(\Omega B; \mathbf{k})$ are known.

Fibrations of this kind are studied by Theriault [41]; see also [9, Proposition 6.1]. (However, these works deal with cases when the algebra $H_*(\Omega F; \mathbf{k})$ is known better than $H_*(\Omega E; \mathbf{k})$.) We consider the case $F = \mathcal{Z}_{\mathcal{K}}$, $E = \text{DJ}(\mathcal{K})$, $B = (\mathbb{C}P^\infty)^m$. The algorithm is also applicable to *partial quotients* of moment-angle complexes [15, Section 4.8] (we will consider their Pontryagin algebras in subsequent publications) and polyhedral products of the form $(PX, \Omega X)^{\mathcal{K}}$ (here we refer to the recent work [16] by Li Cai).

1.1 Main results

We give a presentation of the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ for a flag simplicial complex \mathcal{K} and any ring \mathbf{k} . The presentation is explicit up to a rewriting process described in Algorithm 5.4. For $x \in H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k})$ and a subset $A = \{a_1 < \dots < a_k\} \subset [m]$, define

$$c(A, x) := [u_{a_1}, [u_{a_2}, \dots, [u_{a_k}, x], \dots]] \in H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k}).$$

This element belongs to the subalgebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \subset H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k})$, if $x = u_i$ and $A \neq \emptyset$ (see Corollary 3.10). For every $J \subset [m]$, denote by $\Theta(J)$ the set of all vertices $i \in J$ such that

- vertices i and $\max(J)$ are in different path components of the complex \mathcal{K}_J ;
- i is the smallest vertex in its path component.

Denote by $\tilde{b}_i(X; \mathbf{k})$ the minimal number of elements that generate the \mathbf{k} -module $\tilde{H}_i(X; \mathbf{k})$. Clearly, $|\Theta(J)| = \tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ for any principal ideal domain \mathbf{k} . Consider the $\tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ -element set

$$\{c(J \setminus \{i\}, u_i) : J \subset [m], i \in \Theta(J)\} \subset H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}).$$

We call its elements the *GPTW generators* (after Grbić, Panov, Theriault and Wu).

Theorem 1.1 *Let \mathbf{k} be a commutative ring with unit and \mathcal{K} be a flag simplicial complex without ghost vertices on vertex set $[m]$.*

(i) *For every $J \subset [m]$, choose a set of simplicial 1-cycles*

$$\sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij}^{(\alpha)} [\{i, j\}] \in C_1(\mathcal{K}_J; \mathbf{k})$$

that generate the \mathbf{k} -module $H_1(\mathcal{K}_J; \mathbf{k})$. Then the algebra $H_(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is generated by GPTW generators modulo the relations*

$$\sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus \{i, j\} = A \sqcup B \\ \max(A) > i, \max(B) > j}} \pm [\hat{c}(A, u_i), \hat{c}(B, u_j)] = 0$$

that correspond to the chosen 1-cycles. (Here $\hat{c}(A, u_i)$, $\hat{c}(B, u_j)$ are the elements $c(A, u_i)$, $c(B, u_j)$ of $H_(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ that are arbitrarily expressed through the GPTW generators, and $[x, y]$ is defined as $x \cdot y - (-1)^{|x| \cdot |y|} y \cdot x$.) In particular, $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ admits a $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation by $\sum_{J \subset [m]} \tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ generators and $\sum_{J \subset [m]} \tilde{b}_1(\mathcal{K}_J; \mathbf{k})$ relations.*

(ii) *If \mathbf{k} is a principal ideal domain, then this presentation is minimal: any $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ contains at least $\sum_{J \subset [m]} \tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ generators and at least $\sum_{J \subset [m]} \tilde{b}_1(\mathcal{K}_J; \mathbf{k})$ relations.*

This theorem follows from Theorems 5.1 and 5.6, proven in Section 5. For field coefficients, these results were partially obtained by Grbić, Panov, Theriault, Wu (the minimal set of generators [23, Theorem 4.3]) and the author (number of relations and their degrees [43, Corollary 4.5]). Sometimes the number of relations can be reduced, if we do not require them to be $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous (see Theorem 5.7).

We also present new results on the homotopy of moment-angle complexes that correspond to flag complexes. Using a result of Huang [26], we prove in Corollary 6.7 that in the flag case $\mathcal{Z}_{\mathcal{K}}$ is *coformal* over \mathbb{Q} in the sense of rational homotopy theory. Results of Berglund [10] then give a necessary condition for such moment-angle complexes to be rationally formal (Theorem 6.13). Finally, we improve a recent result of Stanton [39] about the homotopy type of $\Omega \mathcal{Z}_{\mathcal{K}}$ by finding the explicit number of spheres in the product:

Theorem 1.2 *Let \mathcal{K} be a $(d-1)$ -dimensional flag simplicial complex on $[m]$ with no ghost vertices. Then there is a homotopy equivalence*

$$(2) \quad \Omega \mathcal{Z}_{\mathcal{K}} \simeq \prod_{n \geq 3} (\Omega S^n)^{\times D_n},$$

where the numbers $D_n \geq 0$ are determined by

$$(3) \quad - \sum_{J \subset [m]} \bar{\chi}(\mathcal{K}_J) \cdot t^{|J|} = (1+t)^{m-d} h_{\mathcal{K}}(-t) = \prod_{n \geq 3} (1-t^{n-1})^{D_n},$$

where $\bar{\chi}(X) := \chi(X) - 1 = \sum_{i \geq 0} (-1)^i \dim \tilde{H}_i(X)$ is the reduced Euler characteristic and $h_{\mathcal{K}}(t) := \sum_{i=0}^d h_i(\mathcal{K}) \cdot t^i$ is the h -polynomial [15, Definition 2.2.5] of \mathcal{K} . In particular, for every $N \geq 1$ we have an isomorphism

$$(4) \quad \pi_N(\mathcal{Z}_{\mathcal{K}}) \simeq \bigoplus_{n=3}^N \pi_N(S^n)^{\oplus D_n}.$$

This theorem is proved in Section 6. Using (4), it is easy to describe the homotopy groups of corresponding Davis–Januszkiewicz spaces (using the fibration (1)) and partial quotients of moment-angle complexes, including quasitoric manifolds and smooth toric varieties (using similar fibrations; see [15, Proposition 7.3.13; 19, Section 4]).

1.2 Organisation of the paper

Section 2 consists of algebraic preliminaries. We highlight Corollary 2.2 that allows us to construct chain maps into the bar resolution. In Section 3 we recall notions from toric topology and discuss the properties of Pontryagin algebras $H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k})$ and $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$. Main calculations are carried in Section 4. In Section 5 we prove Theorem 1.1 and consider an example. Section 6 contains results about (co)formality and homotopy groups of moment-angle complexes in the flag case. In Appendix A we develop the homological tools for working with presentations of connected graded algebras over a commutative ring. In Appendix B we prove the following folklore fact: split fibrations of loop spaces correspond (by passing to homology) to extensions of Hopf algebras. Appendix C contains commutator identities that are used in Section 4.

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2 Preliminaries: algebra

2.1 Connected graded algebras

Fix a commutative associative ring k with unit. We consider associative k -algebras with unit that are graded by a commutative monoid G (usually $G = \mathbb{Z}$ or $\mathbb{Z}^k \times \mathbb{Z}_{\geq 0}^m$, $k = 0, 1, 2$.) Left A -modules are also G -graded. Elements of $\mathbb{Z}_{\geq 0}^m$ are denoted by $\alpha = (\alpha_1, \dots, \alpha_m) = \sum_{j=1}^m \alpha_j e_j$, $\alpha_j \geq 0$. Subsets $J \subset [m]$ are identified with elements $\sum_{j \in J} e_j \in \mathbb{Z}_{\geq 0}^m$. Define also

$$|\alpha| := \alpha_1 + \dots + \alpha_m, \quad \text{supp } \alpha := \{i \in [m] : \alpha_i > 0\}.$$

Every $\mathbb{Z}^k \times \mathbb{Z}_{\geq 0}^m$ -graded algebra A is considered as \mathbb{Z} -graded with respect to the total grading

$$A_n := \bigoplus_{n=i_1+\dots+i_k+|\alpha|} A_{i_1, \dots, i_k, \alpha}.$$

The graded algebra A is *connected* if $A_{<0} = 0$ and $A_0 = k \cdot 1$. We have the canonical augmentation $\varepsilon: A \rightarrow A_0 = k$ and the augmentation ideal $I(A) := \text{Ker } \varepsilon$. Examples of connected k -algebras are

- the exterior algebra $\Lambda[m] := \Lambda[u_1, \dots, u_m]$, $\text{deg}(u_i) = (-1, 2e_i) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$, with the basis $\{u_I := u_{i_1} \wedge \dots \wedge u_{i_k} : I = \{i_1 < \dots < i_k\}\}$;
- the polynomial algebra $k[m] := k[v_1, \dots, v_m]$, $\text{deg}(v_i) = (0, 2e_i)$, with the basis

$$\left\{ v^\alpha := \prod_{i=1}^m v_i^{\alpha_i} : \alpha \in \mathbb{Z}_{\geq 0}^m \right\};$$

- the tensor algebra $T(x_1, \dots, x_N)$, where x_i are homogeneous elements of arbitrary positive degrees.

For a homogeneous element a , define $\bar{a} := (-1)^{1+\text{deg}(a)} \cdot a$. Clearly, $\overline{a \cdot b} = -\bar{a} \cdot \bar{b}$ and $\bar{\bar{a}} = a$.

Let A be a G -graded algebra. Complexes of A -modules (M, d) are considered as $\mathbb{Z} \times G$ -graded modules with a differential of degree $(-1, 0)$. We use the Koszul sign rule with respect to the total grading: $d(a \cdot m) = (-1)^{\text{deg}(a)} a \cdot d(m) = -\bar{a} \cdot d(m)$. Several formulas from [43] do not follow this rule and are corrected in this paper.

2.2 Bar resolution and bar construction

Let A be a connected k -algebra and $\varepsilon: A \rightarrow k$ be the augmentation. The resulting left A -module k has the *bar resolution*

$$\dots \rightarrow B_2(A) \rightarrow B_1(A) \rightarrow B_0(A) \rightarrow k \rightarrow 0,$$

where $B_n(A) := A \otimes I(A)^{\otimes n}$. An element of the form $a \otimes a_1 \otimes \dots \otimes a_n \in B_n(A)$ has bidegree $(n, \deg(a) + \sum_{i=1}^n \deg(a_i))$ and is traditionally written as $a[a_1 | \dots | a_n]$. The differential d_B has bi-degree $(-1, 0)$ and is given by

$$-d_B(a[a_1 | \dots | a_n]) := \bar{a} \cdot a_1[a_2 | \dots | a_n] + \sum_{i=1}^{n-1} \bar{a}[\bar{a}_1 | \dots | \bar{a}_{i-1} | \bar{a}_i \cdot a_{i+1} | a_{i+2} | \dots | a_n].$$

Consider also the contracting homotopy $s_n: B_n(A) \rightarrow B_{n+1}(A)$,

$$(5) \quad s_n(a[a_1 | \dots | a_n]) := \begin{cases} 0, & a \in A_0 \simeq k; \\ [a|a_1 | \dots | a_n], & \deg(a) > 0; \end{cases} \quad s_{-1}: k \rightarrow B_0(A), \quad 1 \mapsto 1[\cdot].$$

It is easy to show that $s \circ d_B + d_B \circ s = \text{id}$, $d_B^2 = 0$. Hence $(B(A), d_B)$ is a free resolution of the left A -module k , assuming that A is a free k module. In this case, we obtain

$$\text{Tor}_n^A(k, k) \cong H_n[\bar{B}(A), d_{\bar{B}}],$$

where $\bar{B}(A) := k \otimes_A B(A)$ is the *bar construction* of A . We have

$$(6) \quad \begin{aligned} \bar{B}_n(A) &= I(A)^{\otimes n}, \quad \deg([a_1 | \dots | a_n]) = (n, \deg(a_1) + \dots + \deg(a_n)), \quad \deg d_{\bar{B}} = (-1, 0), \\ d_{\bar{B}}([a_1 | \dots | a_n]) &= \sum_{i=1}^{n-1} [\bar{a}_1 | \dots | \bar{a}_{i-1} | \bar{a}_i \cdot a_{i+1} | a_{i+2} | \dots | a_n] \in \bar{B}_{n-1}(A). \end{aligned}$$

In particular, $d_{\bar{B}}([x|y]) = [\bar{x} \cdot y]$ and $d_{\bar{B}}([x|y|z]) = [\bar{x} \cdot y|z] + [\bar{x}|\bar{y} \cdot z]$.

2.3 Chain maps into resolutions with a contracting homotopy

Any map of modules can be extended to a map of their free resolutions. Moreover, this extension can be described in terms of the contracting homotopy for the latter resolution. This recursive construction seems to be known to specialists: its generalisations and applications are discussed in [12]. The author thanks Georgy Chernykh for the reference.

Lemma 2.1 *Let A be an associative k -algebra. Suppose that the commutative diagram of left A -modules and their homomorphisms*

$$\begin{array}{ccccc} C_n & \xrightarrow{\hat{d}_n} & C_{n-1} & \xrightarrow{\hat{d}_{n-1}} & C_{n-2} \\ & & \downarrow \varphi_{n-1} & & \downarrow \varphi_{n-2} \\ B_n & \xrightarrow{d_n} & B_{n-1} & \xrightarrow{d_{n-1}} & B_{n-2} \end{array}$$

satisfy the conditions

- (i) C_n is a free A -module with a basis $\{e_i\}$;
- (ii) $\widehat{d}_{n-1} \circ \widehat{d}_n = 0$;
- (iii) there are k -linear maps $s_{n-1}: B_{n-1} \rightarrow B_n$ and $s_{n-2}: B_{n-2} \rightarrow B_{n-1}$ such that

$$d_n \circ s_{n-1} + s_{n-2} \circ d_{n-1} = \text{id}_{B_{n-1}}.$$

Define an A -linear map $\varphi_n: C_n \rightarrow B_n$ on the basis by

$$\varphi_n(e_i) := s_{n-1}(\varphi_{n-1}(\widehat{d}_n(e_i))) \in B_n.$$

Then $d_n \circ \varphi_n = \varphi_{n-1} \circ \widehat{d}_n$.

Proof Since $d_n \circ \varphi_n$ and $\varphi_{n-1} \circ \widehat{d}_n$ are maps of A -modules, it is sufficient to show that they agree on the basis of C_n . By definition,

$$d_n(\varphi_n(e_i)) = (d_n \circ s_{n-1} \circ \varphi_{n-1} \circ \widehat{d}_n)(e_i).$$

Condition (iii) gives $d \circ s \circ \varphi \circ \widehat{d} = \varphi \circ \widehat{d} - s \circ d \circ \varphi \circ \widehat{d}$. From the commutativity of the diagram and condition (ii) we obtain $s \circ d \circ \varphi \circ \widehat{d} = s \circ \varphi \circ \widehat{d} \circ \widehat{d} = 0$. Hence

$$d_n(\varphi_n(e_i)) = \varphi_{n-1}(\widehat{d}_n(e_i)) - 0. \quad \square$$

Corollary 2.2 Let A be a connected k -algebra, $(A \otimes V_\bullet, \widehat{d}_\bullet)$ be a free resolution of the left A -module k . Let $\bar{\varphi}_0: V_0 \rightarrow k$ be a map of k -modules such that the diagram

$$\begin{array}{ccccc} A \otimes V_1 & \xrightarrow{\widehat{d}_1} & A \otimes V_0 & \xrightarrow{\widehat{d}_0} & k \\ & & \downarrow \text{id} \otimes \bar{\varphi}_0 & & \parallel \text{id} =: \varphi_{-1} \\ B_1(A) & \xrightarrow{d_{B,1}} & A & \xrightarrow{\varepsilon} & k \end{array}$$

commutes. Choose bases $\{e_i^{(n)}\}$ of k -modules V_n , and define A -linear maps $\varphi_n: A \otimes V_n \rightarrow B_n(A)$ recursively as

$$\varphi_0 := \text{id}_A \otimes \bar{\varphi}_0, \quad \varphi_n(a \otimes e_i^{(n)}) := a \cdot s_{n-1}(\varphi_{n-1}(\widehat{d}_n(e_i^{(n)}))),$$

where $s_{n-1}: B_{n-1}(A) \rightarrow B_n(A)$ is the contracting homotopy (5).

Then $\varphi_\bullet: (A \otimes V_\bullet, \widehat{d}) \rightarrow (B_\bullet(A), d_B)$ is a chain map.

Proof Induction on n . For $n = 0$ the identity

$$d_{B,n} \circ \varphi_n = \varphi_{n-1} \circ \widehat{d}_n$$

holds, since the diagram commutes. The inductive step from $n - 1$ to n is supplied by Lemma 2.1. \square

2.4 Hopf algebra extensions and loop homology

If A is a Hopf algebra over k , we denote the comultiplication by $\Delta: A \rightarrow A \otimes A$ and the (co)unit maps by $\eta_A: k \rightarrow A$, $\varepsilon_A: A \rightarrow k$. A graded k -Hopf algebra A is *connected* if $A_{<0} = 0$, $A_0 = k \cdot 1$. The counit is then the standard augmentation $\varepsilon: A \rightarrow A_0 \simeq k$.

Definition 2.3 Let $\iota: A \rightarrow C$, $\pi: C \rightarrow B$ be morphisms of k -Hopf algebras. They form an *extension of Hopf algebras*, or a *short exact sequence of Hopf algebras*

$$k \rightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \rightarrow k,$$

if

- (i) ι is injective;
- (ii) π is surjective;
- (iii) $\pi \circ \iota = \varepsilon$;
- (iv) $\text{Ker } \pi = I(A) \cdot C$;
- (v) $\text{Im } \iota = \{x \in C : ((\text{id}_C \otimes \pi) \circ \Delta)(x) = x \otimes 1\}$.

See [2, Definition 1.2.0, Proposition 1.2.3] for an equivalent and more “symmetrical” definition. Extensions of connected Hopf algebras were studied implicitly in [31, Section 4].

Proposition 2.4 (see [31, Proposition 4.9]) Let $\iota: A \rightarrow C$, $\pi: C \rightarrow B$ be maps of connected k -Hopf algebras. Suppose that a map $\Phi: A \otimes B \rightarrow C$ is an isomorphism of left A -modules and right C -comodules, and suppose that

$$\iota = \Phi \circ (\text{id}_A \otimes \eta_B), \quad \pi \circ \Phi = \varepsilon_A \otimes \text{id}_B.$$

Then $k \rightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \rightarrow k$ is an extension of Hopf algebras. Conversely, for every Hopf algebra extension there is a map Φ with the described properties. \square

Our main example of Hopf algebras are Pontryagin algebras (loop homology) of connected topological spaces. Let k be a commutative ring, Y be a topological space such that $H_*(Y; k)$ is a free k -module. Then $H_*(Y; k)$ is supplied with the cocommutative *cup coproduct* which is dual to the cup product on $H^*(Y; k)$: it is the composition

$$H_*(Y; k) \xrightarrow{\Delta_*} H_*(Y \times Y; k) \xrightarrow[\simeq]{\text{AW}_*} H_*(C_*(Y; k) \otimes C_*(Y; k)) \xleftarrow[\simeq]{\kappa} H_*(Y; k) \otimes H_*(Y; k),$$

where AW is the Alexander–Whitney map and κ is the Künneth isomorphism. If Y is also an H-space, the cup coproduct respects the Pontryagin product

$$m: H_*(Y; k) \otimes H_*(Y; k) \xrightarrow{\times} H_*(Y \times Y; k) \xrightarrow{\mu_*} H_*(Y; k)$$

and hence $H_*(Y; \mathbf{k})$ is a cocommutative Hopf algebra. In particular, $H_*(\Omega X; \mathbf{k})$ is a connected cocommutative \mathbf{k} -Hopf algebra whenever X is a simply connected space such that $H_*(\Omega X; \mathbf{k})$ is free over \mathbf{k} [31, 8.9]. Otherwise κ fails to be an isomorphism, hence the coproduct is not defined and $H_*(\Omega X; \mathbf{k})$ is merely a connected associative \mathbf{k} -algebra with unit.

In Appendix B we describe a situation when a fibration $F \rightarrow E \rightarrow B$ of simply connected spaces gives rise to an extension $\mathbf{k} \rightarrow H_*(\Omega F; \mathbf{k}) \rightarrow H_*(\Omega E; \mathbf{k}) \rightarrow H_*(\Omega B; \mathbf{k}) \rightarrow \mathbf{k}$ of connected Hopf algebras.

3 Preliminaries: toric topology

3.1 Simplicial complexes and polyhedral products

A simplicial complex \mathcal{K} on the vertex set W is a nonempty family of subsets $I \subset W$ that is closed under taking subsets. Elements $I \in \mathcal{K}$ are called faces. We suppose that \mathcal{K} has no ghost vertices, ie $\{i\} \in \mathcal{K}$ for all $i \in W$. Usually $W \subset [m] := \{1, \dots, m\}$. Sometimes by properties of a complex \mathcal{K} we mean properties of its geometrical realisation, of the topological space $|\mathcal{K}| := \bigcup_{I \in \mathcal{K}} \Delta_I \subset \Delta_W$.

For every $J \subset W$, a simplicial complex $\mathcal{K}_J := \{I \in \mathcal{K} : I \subset J\}$ on the vertex set J (a full subcomplex of \mathcal{K}) is defined.

Throughout the text, we write $I \setminus i := I \setminus \{i\}$ for $i \in I$ and $I \sqcup i := I \sqcup \{i\}$ for $i \in W \setminus I$. A subset $I \subset W$ is a missing face of \mathcal{K} if $I \notin \mathcal{K}$, but $I \setminus i \in \mathcal{K}$ for all $i \in I$. A simplicial complex \mathcal{K} is flag if all its missing faces consist of two elements.

For every complex \mathcal{K} on a vertex set $[m]$, the $\mathbb{Z}_{\geq 0}^m$ -graded Stanley–Reisner ring

$$\mathbf{k}[\mathcal{K}] := \mathbf{k}[v_1, \dots, v_m] / \left(\prod_{i \in I} v_i = 0, I \notin \mathcal{K} \right), \quad \deg v_i := 2e_i \in \mathbb{Z}_{\geq 0}^m,$$

is defined. It has a homogeneous basis $\{v^\alpha := \prod_{i=1}^m v_i^{\alpha_i} \mid \text{supp } \alpha \in \mathcal{K}\}$ as a \mathbf{k} -module. The dual \mathbf{k} -module $\mathbf{k}\langle \mathcal{K} \rangle$ is called the Stanley–Reisner coalgebra. It has an additive basis $\{\chi_\alpha \mid \text{supp } \alpha \in \mathcal{K}\}$, $\deg \chi_\alpha = 2\alpha$, and commutative associative comultiplication $\Delta \chi_\alpha := \sum_{\alpha = \beta + \gamma} \chi_\beta \otimes \chi_\gamma$.

Now let \mathcal{K} be a simplicial complex on $[m]$ and $(\underline{X}, \underline{A}) := ((X_1, A_1), \dots, (X_m, A_m))$ be a sequence of pairs of topological spaces. Their polyhedral product $(\underline{X}, \underline{A})^\mathcal{K}$ is the union

$$(\underline{X}, \underline{A})^\mathcal{K} := \bigcup_{I \in \mathcal{K}} (\underline{X}, \underline{A})^I \subset X^m, \quad (\underline{X}, \underline{A})^I = Y_1 \times \dots \times Y_m, \quad Y_j := \begin{cases} X_j, & j \in I; \\ A_j, & j \notin I. \end{cases}$$

The addition of a ghost vertex v to \mathcal{K} replaces the space $(\underline{X}, \underline{A})^\mathcal{K}$ with $(\underline{X}, \underline{A})^\mathcal{K} \times A_v$. Hence in many cases it is sufficient to consider only complexes without ghost vertices.

Define $(X, A)^\mathcal{K} := (\underline{X}, \underline{A})^\mathcal{K}$ if $X_i = X, A_i = A$ for all $i \in [m]$. We consider two special cases of this construction: moment-angle complexes $\mathcal{Z}_\mathcal{K} := (D^2, S^1)^\mathcal{K}$ and Davis–Januszkiewicz spaces

$DJ(\mathcal{K}) := (\mathbb{C}P^\infty, *)^{\mathcal{K}}$. It is well known that $H^*(DJ(\mathcal{K}); \mathbf{k}) \cong \mathbf{k}[\mathcal{K}]$ and $H^*(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \text{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ as graded rings. Moreover,

$$H^n(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) = \bigoplus_{n=-i+2|J|} H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}), \quad H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \widetilde{H}^{|J|-i-1}(\mathcal{K}_J; \mathbf{k}),$$

and the product has a geometric description in terms of maps $\mathcal{K}_I \sqcup J \hookrightarrow \mathcal{K}_I * \mathcal{K}_J$; see [15, Theorem 4.5.8].

3.2 Loop homology as Hopf algebras

Proposition 3.1 [15, Theorem 4.3.2, Section 8.4] *There is a homotopy fibration*

$$\mathcal{Z}_{\mathcal{K}} \rightarrow DJ(\mathcal{K}) \xrightarrow{i} (\mathbb{C}P^\infty)^m$$

of simply connected spaces, where i is the standard inclusion. The map Ωi admits a homotopy section $\sigma: \mathbb{T}^m \rightarrow \Omega DJ(\mathcal{K})$ that corresponds to the choice of generators in $\pi_2(DJ(\mathcal{K})) \cong \mathbb{Z}^m$ and gives rise to a homotopy equivalence $\Omega DJ(\mathcal{K}) \simeq \Omega \mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$. □

The following description of $H_*(\Omega DJ(\mathcal{K}); \mathbf{k})$ was first given in [35, (8.4)] for $\mathbf{k} = \mathbb{Q}$, but the argument is easily generalised to the arbitrary coefficient ring. The main ingredients are integral formality of $DJ(\mathcal{K})$ [34], Adams’ cobar construction (see [18]) and a result of Fröberg [21].

Theorem 3.2 [43, Theorem 1.1] *For any simplicial complex \mathcal{K} with no ghost vertices and any commutative ring \mathbf{k} , we have an isomorphism $H_*(\Omega DJ(\mathcal{K}); \mathbf{k}) \cong \text{Ext}_{\mathbf{k}[\mathcal{K}]}^i(\mathbf{k}, \mathbf{k})$ of graded \mathbf{k} -algebras (with respect to the Pontryagin product and to the Yoneda product). More precisely,*

$$H_n(\Omega DJ(\mathcal{K}); \mathbf{k}) \cong \bigoplus_{-i+2|\alpha|=n} \text{Ext}_{\mathbf{k}[\mathcal{K}]}^i(\mathbf{k}, \mathbf{k})_{2\alpha}.$$

This isomorphism defines the $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading on $H_*(\Omega DJ(\mathcal{K}); \mathbf{k})$. The “diagonal” subalgebra

$$D = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}^m} H_{-|\alpha|, 2\alpha}(\Omega DJ(\mathcal{K}); \mathbf{k}) \subset H_*(\Omega DJ(\mathcal{K}); \mathbf{k})$$

is isomorphic to the algebra

$$\mathbf{k}[\mathcal{K}]^! := T(u_1, \dots, u_m) / (u_i^2 = 0, i = 1, \dots, m; u_i u_j + u_j u_i = 0, \{i, j\} \in \mathcal{K}), \quad \deg u_i = (-1, 2e_i).$$

For a flag \mathcal{K} , the algebra $H_*(\Omega DJ(\mathcal{K}); \mathbf{k})$ coincides with D , and we have $H_*(\Omega DJ(\mathcal{K}); \mathbf{k}) \cong \mathbf{k}[\mathcal{K}]^!$. □

If $H_*(\Omega Y; \mathbf{k})$ is a free \mathbf{k} -module, the cup coproduct is compatible with the Pontryagin product, hence this associative algebra is a cocommutative \mathbf{k} -Hopf algebra. Similarly, if A is a commutative graded \mathbf{k} -algebra such that $\text{Ext}_A(\mathbf{k}, \mathbf{k})$ is a free \mathbf{k} -module, then the shuffle product on the bar construction (see [28, Theorem X.12.2]) induces a commutative coproduct on $\text{Ext}_A(\mathbf{k}, \mathbf{k})$ that is compatible with the Yoneda product. In our case, these coproduct coincide. This follows from a stronger formality result for Davis–Januszkiewicz spaces, the *hga formality* [20, Theorem 1.3].

Proposition 3.3 [20, Proposition 6.5] *Let \mathcal{K} be a simplicial complex with no ghost vertices, and let \mathbf{k} be a principal ideal domain such that $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ is a free \mathbf{k} -module. Then*

$$H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k}) \cong \text{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k}, \mathbf{k})$$

as Hopf algebras.

Outline of the proof Let A be a dga algebra. The *homotopy Gerstenhaber algebra* (hga) structure on A is a multiplication on its bar construction $\overline{B}(A)$ such that $\overline{B}(A)$ becomes a dga bialgebra [20, Section 4]. This structure arises naturally if A is commutative (then the multiplication is the shuffle product) or if $A = C^*(X; \mathbf{k})$ is the dga algebra of cochains of a 1-reduced simplicial set (then the multiplication was essentially constructed by Baues [8, Section 2]). Then $H^*(\Omega X; \mathbf{k}) \cong H^*(\overline{B}(C^*(X; \mathbf{k})))$ as bialgebras. By a result of Franz [20, Theorem 1.3], hga algebras $C^*(\text{DJ}(\mathcal{K}); \mathbf{k})$ and $\mathbf{k}[\mathcal{K}]$ are quasi-isomorphic. The functor \overline{B} preserves quasi-isomorphisms, so $H^*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k}) \cong H^*(\overline{B}(\mathbf{k}[\mathcal{K}]); \mathbf{k}) \cong \text{Tor}^{\mathbf{k}[\mathcal{K}]}(\mathbf{k}, \mathbf{k})$ as bialgebras. Since the Hopf algebra structure on a bialgebra is unique, it is an isomorphism of Hopf algebras. The statement for $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ follows by dualisation. \square

Remark 3.4 The algebra $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ is not always a free \mathbf{k} -module. For example, let \mathcal{K} be a minimal triangulation of $\mathbb{R}P^2$. Then $\mathcal{Z}_{\mathcal{K}}$ is a wedge of $\Sigma^7\mathbb{R}P^2$ and spheres [23, Example 3.3]. We have $\Omega\text{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$, hence $\Omega\Sigma^7\mathbb{R}P^2$ is a retract of $\Omega\text{DJ}(\mathcal{K})$. It follows that $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbb{Z})$ has 2-torsion.

Recall that an element x is called *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$, and a Hopf algebra is *primitively generated* if it is multiplicatively generated by its primitive elements.

Conjecture 3.5 *The Hopf algebra $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ is primitively generated for every simplicial complex \mathcal{K} and every ring \mathbf{k} such that $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ is a free \mathbf{k} -module.*

By deep results of André and Sjödin — see [4, Theorem 10.2.1(5)] — for every field \mathbf{k} the Hopf algebra $\text{Ext}_A(\mathbf{k}, \mathbf{k})$ is the universal enveloping of a Lie algebra (of a 2-restricted Lie algebra, if $\text{char } \mathbf{k} = 2$). In particular, this Hopf algebra is primitively generated. (This also follows from results of Browder [11]; see [33, Theorem 10.4].) Hence **Conjecture 3.5** holds if \mathbf{k} is a field.

Remark 3.6 The Hopf algebra $H_*(\Omega X; \mathbf{k})$ is not always primitively generated, even if X is a suspension. For example, one can take $X = \Sigma\mathbb{C}P^2$, $\mathbf{k} = \mathbb{Z}$ or $\mathbb{Z}/2$ (see [13, Section 4.2]). On the other hand, [25, Theorem B] implies that $H_*(\Omega\Sigma\mathbb{C}P^d; \mathbb{Z}/p)$ is primitively generated for $p > d$.

Now we describe the connection between the loop homology of Davis–Januszkiewicz spaces and of moment-angle complexes in the form of a Hopf algebra extension.

Proposition 3.7 *Let \mathcal{K} be a simplicial complex on $[m]$ and \mathbf{k} be a commutative ring with unit, such that $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is a free \mathbf{k} -module. Then*

$$\mathbf{k} \rightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \xrightarrow{\iota} H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k}) \xrightarrow{p} \Lambda[u_1, \dots, u_m] \rightarrow \mathbf{k}$$

is an extension of connected $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded \mathbf{k} -Hopf algebras. The projection p maps u_i to u_i . Its \mathbf{k} -linear section $\sigma_* : \Lambda[u_1, \dots, u_m] \rightarrow H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ is given by

$$\sigma_*(u_I) = \hat{u}_I := u_{i_1} \cdots u_{i_k}, \quad I = \{i_1 < \cdots < i_k\}.$$

Therefore, $\Phi(a \otimes u_I) := \iota(a) \cdot \hat{u}_I$ defines an isomorphism of left $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ -modules and right $\Lambda[u_1, \dots, u_m]$ -comodules $\Phi : H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[u_1, \dots, u_m] \rightarrow H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$.

Proof By [Theorem B.3](#), the fibration from [Proposition 3.1](#) gives rise to the required Hopf algebra extension. The formula for p follows from functoriality, since the map $\text{DJ}(\mathcal{K}) \hookrightarrow \text{DJ}(\Delta_{[m]}) \cong (\mathbb{C}P^\infty)^m$ is induced by the inclusion $\mathcal{K} \hookrightarrow \Delta_{[m]}$. The formula for σ_* follows from the description of the homotopy section $\sigma : \mathbb{T}^m \simeq \Omega B\mathbb{T}^m = (\Omega\mathbb{C}P^\infty)^{\times m} \rightarrow \Omega\text{DJ}(\mathcal{K})$ as a concatenation of loops, $(\gamma_1, \dots, \gamma_m) \mapsto \gamma_1 \cdots \gamma_m$. The maps p and σ_* respect the $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading, hence the multigrading on $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is well defined. \square

Since ι is injective, we identify elements of $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ with their images in $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$. Let us describe some of these elements. Recall that we define $[a, b] := ab + (-1)^{\deg(a)\deg(b)+1}ba$ and $c(I, x) := [u_{i_1}, [u_{i_2}, \dots, [u_{i_k}, x], \dots]] \in H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ for $I = \{i_1 < \cdots < i_k\}$ and $x \in H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$. In particular, $c(\emptyset, x) := x$ and $c(\{i\}, u_j) = [u_i, u_j] = u_i u_j + u_j u_i$.

Corollary 3.8 *Let $x \in H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ be a primitive element such that $p(x) = 0$. Then $x \in H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$.*

Proof This follows from [Corollary B.4](#) applied to the Hopf algebra extension from [Proposition 3.7](#). \square

Corollary 3.9 *Let $x \in H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ be a primitive element and $I \subset [m]$, $I \neq \emptyset$. Then $c(I, x)$ is an element of $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$.*

Proof Elements $u_1, \dots, u_m \in H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ are primitive for dimension reasons. Primitive elements form a Lie algebra, hence $c(I, x) \in H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ is primitive. We have $p(c(I, x)) = c(I, p(x)) = 0$, since it is a commutator in the commutative algebra $\Lambda[m]$. Then $c(I, x) \in H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ by [Corollary 3.8](#). \square

Corollary 3.10 *Let $j \in [m]$ and $I \subset [m]$, $I \neq \emptyset$. Then $c(I, u_j)$ is an element of $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$.* \square

3.3 The flag case

Let \mathcal{K} be a flag complex with no ghost vertices. By [Theorem 3.2](#), $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k}) \cong \mathbf{k}[\mathcal{K}]^!$ is a free \mathbf{k} -module, hence the Hopf algebra structure on $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ is well defined. Moreover, the connected \mathbf{k} -algebra $\mathbf{k}[\mathcal{K}]^!$ is generated by elements of degree 1. These conditions determine the Hopf algebra structure on $\mathbf{k}[\mathcal{K}]^!$ uniquely: the elements u_1, \dots, u_m are primitive. Therefore, in the flag case [Conjecture 3.5](#) is true for any \mathbf{k} .

The following important result was recently obtained by Stanton.

Theorem 3.11 [39, Corollary 1.5] *Let \mathcal{K} be a flag simplicial complex or a skeleton of a flag complex. Then $\Omega\mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to a finite-type product of spaces of the form S^1 , S^3 , S^7 and ΩS^n for $n \geq 2$, $n \neq 2, 4, 8$. \square*

This gives a short proof of the fact that $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is free over \mathbf{k} .

Proposition 3.12 [23, Corollary 5.2] *If \mathcal{K} is a flag simplicial complex, then $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is a free \mathbf{k} -module of finite type.*

Proof By the Künneth formula (more precisely, by the collapse of the Künneth spectral sequence — see [37, Theorem 10.90]), $H_*(X \times Y; \mathbf{k}) \simeq H_*(X; \mathbf{k}) \otimes H_*(Y; \mathbf{k})$ if $H_*(X; \mathbf{k})$ and $H_*(Y; \mathbf{k})$ are free over \mathbf{k} . Hence $H_*(X \times Y; \mathbf{k})$ is also a free \mathbf{k} -module.

Clearly, $H_*(S^n; \mathbf{k})$ and $H_*(\Omega S^n; \mathbf{k}) \simeq T(a_{n-1})$ are free \mathbf{k} -modules. By [Theorem 3.11](#) and the arguments above, the same holds for $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$. \square

Hence in the flag case we have a Hopf algebra extension

$$\mathbf{k} \rightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \rightarrow H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k}) \rightarrow \Lambda[m] \rightarrow \mathbf{k}$$

from [Proposition 3.7](#) for any \mathbf{k} .

4 Main calculations

In what follows, \mathcal{K} is a flag simplicial complex on the vertex set $[m]$ with no ghost vertices, and \mathbf{k} is a commutative ring with unit. We consider $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded \mathbf{k} -algebras that are connected with respect to the total grading $A_n := \bigoplus_{n=-i+|\alpha|} A_{-i, \alpha}$.

4.1 Resolutions and formulas for differentials

By [\[43, Proposition 4.1\]](#), the left $H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k})$ -module \mathbf{k} has a free resolution $(H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k}) \otimes \mathbf{k}\langle \mathcal{K} \rangle, d)$, $\deg \chi_\alpha := (|\alpha|, -|\alpha|, 2\alpha)$, $\deg(d) = (-1, 0, 0)$, with the differential

$$d(1 \otimes \chi_\alpha) := \sum_{i \in \text{supp}(\alpha)} u_i \otimes \chi_{\alpha - e_i}.$$

The isomorphism of left $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ -modules

$$\Phi: H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \rightarrow H_*(\Omega\text{DJ}(\mathcal{K}); \mathbf{k}), \quad a \otimes u_I \mapsto a \cdot \hat{u}_I,$$

from [Proposition 3.7](#) allows us to consider this resolution as a free resolution $(H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle, \hat{d})$ of the left $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ -module \mathbf{k} . We apply the functor $\mathbf{k} \otimes_{H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})} (-)$ and obtain a chain complex

$(\Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle, \bar{d})$ whose homology is isomorphic to $\text{Tor}^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})$. The differentials \hat{d} and \bar{d} are determined by the commutative diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k}) \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n)} & \xrightarrow{d} & H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k}) \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n-1)} & \longrightarrow & \dots \\
 & & \uparrow \Phi \otimes \text{id} \simeq & & \uparrow \Phi \otimes \text{id} \simeq & & \\
 \dots & \longrightarrow & H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n)} & \xrightarrow{\hat{d}} & H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n-1)} & \longrightarrow & \dots \\
 & & \downarrow \varepsilon \otimes \text{id} \otimes \text{id} & & \downarrow \varepsilon \otimes \text{id} \otimes \text{id} & & \\
 \dots & \longrightarrow & \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n)} & \xrightarrow{\bar{d}} & \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n-1)} & \longrightarrow & \dots
 \end{array}$$

Here $\mathbf{k}\langle \mathcal{K} \rangle_{(n)}$ is a \mathbf{k} -submodule in $\mathbf{k}\langle \mathcal{K} \rangle$ with the basis $\{\chi_\alpha : |\alpha| = n\}$. With different signs, this construction was considered by the author in [43, Section 4]. Now we describe the differential \hat{d} explicitly. For subsets $A, B \subset [m]$ define the Koszul sign $\theta(A, B) := |\{(a, b) \in A \times B : a > b\}|$.

Proposition 4.1 *The differential \hat{d} is given by*

$$(7) \quad \hat{d}(1 \otimes u_I \otimes \chi_\alpha) = \sum_{i \in \text{supp}(\alpha)} (-1)^{|I|} \cdot 1 \otimes (u_I \wedge u_i) \otimes \chi_{\alpha - e_i} + \sum_{i \in \text{supp}(\alpha)} \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |A|} c(A, u_i) \otimes u_B \otimes \chi_{\alpha - e_i}.$$

The differential \bar{d} is given by

$$(8) \quad \bar{d}(u_I \otimes \chi_\alpha) = (-1)^{|I|} \sum_{i \in \text{supp}(\alpha)} (u_I \wedge u_i) \otimes \chi_{\alpha - e_i}.$$

Remark 4.2 We define $\max(\emptyset) := -\infty$, hence A cannot be empty.

Proof of the proposition Recall that $u_j^2 = 0 \in H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k})$. Therefore, by Proposition C.2,

$$\begin{aligned}
 \hat{u}_I \cdot u_i &= 1 \cdot \begin{cases} (-1)^{|I| > i} \hat{u}_{I \sqcup i}, & i \notin I \\ 0, & i \in I \end{cases} + \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |B|} c(A, u_i) \cdot \hat{u}_B \\
 &= \Phi \left(1 \otimes (u_I \wedge u_i) + \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |B|} c(A, u_i) \otimes u_B \right) \in H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k}).
 \end{aligned}$$

(Here $c(A, u_i) \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ by Corollary 3.10.) Define $\Phi_0 = \Phi \otimes \text{id}_{\mathbf{k}\langle \mathcal{K} \rangle}$. Then

$$\begin{aligned}
 &\Phi_0(\hat{d}(1 \otimes u_I \otimes \chi_\alpha)) \\
 &= d(\Phi_0(1 \otimes u_I \otimes \chi_\alpha)) = d(\hat{u}_I \otimes \chi_\alpha) = (-1)^{|I|} \sum_{i \in \text{supp}(\alpha)} \hat{u}_I u_i \otimes \chi_{\alpha - e_i} \\
 &= (-1)^{|I|} \sum_{i \in \text{supp}(\alpha)} \Phi_0 \left(1 \otimes (u_I \wedge u_i) \otimes \chi_{\alpha - e_i} + \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |B|} c(A, u_i) \otimes u_B \otimes \chi_{\alpha - e_i} \right).
 \end{aligned}$$

Applying Φ_0^{-1} , we obtain precisely (7). After the homomorphism $\varepsilon \otimes \text{id} \otimes \text{id}$ it turns into (8), since $\varepsilon(1) = 1$ and $\varepsilon(c(A, u_i)) = 0$ for $A \neq \emptyset$. □

4.2 Computation of Tor-modules

By [43, Theorem 1.2], for a flag \mathcal{K} we have a $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded isomorphism of \mathbf{k} -modules

$$(9) \quad \text{Tor}^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}_*(\mathcal{K}_J; \mathbf{k}), \quad \text{Tor}_n^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J} \cong \widetilde{H}_{n-1}(\mathcal{K}_J; \mathbf{k}).$$

The homology of $\mathcal{Z}_{\mathcal{K}}$ admits a $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading, and for any \mathcal{K} we have a similar additive isomorphism dual to [15, Theorem 4.5.8]:

$$H_*(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}_*(\mathcal{K}_J; \mathbf{k}), \quad H_{n-|J|, 2J}(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \widetilde{H}_{n-1}(\mathcal{K}_J; \mathbf{k}).$$

Hence $\text{Tor}^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k}) \cong H_*(\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ for flag \mathcal{K} . Moreover, both modules are computed as the homology of $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, d)$.

Remark 4.3 In general, if X is simply connected and $H_*(\Omega X; \mathbf{k})$ is free over \mathbf{k} , there is *Milnor–Moore spectral sequence* $E_{p,q}^2 = \text{Tor}_p^{H_*(\Omega X; \mathbf{k})}(\mathbf{k}, \mathbf{k})_q \Rightarrow H_{p+q}(X; \mathbf{k})$. We see that it collapses at E^2 for $X = \mathcal{Z}_{\mathcal{K}}$ if \mathcal{K} is a flag complex. For $\mathbf{k} = \mathbb{Q}$, the collapse is explained by the *coformality* of $\mathcal{Z}_{\mathcal{K}}$; see [Corollary 6.7](#) and the discussion after.

Now we construct a chain map g that induces the isomorphism (9). For any chain complex (C_\bullet, d) of free \mathbf{k} -modules, we have the *dual complex*

$$(C^\bullet, d_{\text{dual}}), \quad C^n := \text{Hom}_{\mathbf{k}}(C_n, \mathbf{k}), \quad d_{\text{dual}}(f) : c \mapsto f(d(c)).$$

Dualisation preserves isomorphisms and chain homotopies. For a simplicial complex \mathcal{K} , the augmented complex of simplicial chains $\widetilde{C}_*(\mathcal{K}; \mathbf{k})$ has the basis $\{[I] : I \in \mathcal{K}\}$, $\text{deg}[I] := |I| + 1$ and the differential

$$d([I]) := \sum_{i \in I} (-1)^{|I| < i|} [I \setminus \{i\}].$$

The dual complex is the augmented complex of simplicial cochains $(\widetilde{C}^*(\mathcal{K}; \mathbf{k}), d_{\text{dual}})$, which has the basis $\{[I]^* : I \in \mathcal{K}\}$ and the differential

$$d_{\text{dual}}([I]^*) = \sum_{\substack{i \notin I \\ I \sqcup i \in \mathcal{K}}} (-1)^{|I| < i|} [I \sqcup \{i\}]^*.$$

Proposition 4.4 For every $J \subset [m]$, consider the map

$$g_J : \widetilde{C}_{*-1}(\mathcal{K}_J; \mathbf{k}) \rightarrow (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle)_{*, -|J|, 2J}, \quad [L] \mapsto \epsilon(L, J) \cdot u_{J \setminus L} \otimes \chi_L,$$

where $\epsilon(L, J) := (-1)^{\sum_{\ell \in L} |J < \ell|}$. Then g_J are chain maps, and the direct sum

$$g : \bigoplus_{J \subset [m]} \widetilde{C}_*(\mathcal{K}_J; \mathbf{k}) \rightarrow (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d})$$

induces an isomorphism on homology. Therefore,

$$H_{n, -|J|, 2J}(\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d}) \cong \widetilde{H}_{n-1}(\mathcal{K}_J; \mathbf{k}), \quad J \subset [m], n \geq 0,$$

all the other graded components of $H_*(\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d})$ being zero.

Since $\text{Tor}^{H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k}) \cong H(\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d})$, this proposition implies (9). The proof is the dualisation of arguments from [15, Section 3.2].

Proof of Proposition 4.4 Consider the dga algebra $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}], d)$ with the differential that is defined on generators by $d(u_i) = v_i$, $d(v_i) = 0$ and with the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading

$$\deg u_i := (0, -1, 2e_i), \quad \deg v_i := (1, -1, 2e_i), \quad \deg d := (1, 0, 0).$$

This complex has the basis $\{u_I v^\alpha : I \subset [m], \alpha \in \mathbb{Z}_{\geq 0}^m, \text{supp}(\alpha) \in \mathcal{K}\}$ and the differentials

$$d(u_I v^\alpha) = \sum_{i \in I} (-1)^{|I_{<i}|} u_{I_{<i}} v_i u_{I_{>i}} v^\alpha = \sum_{i \in I} (-1)^{|I_{<i}|} u_{I \setminus i} v_i v^\alpha.$$

Then the dual complex $(\Lambda[m] \otimes \mathbf{k}[\mathcal{K}])^*$ has the basis $\{(u_I v^\alpha)^* : I \subset [m], \text{supp}(\alpha) \in \mathcal{K}\}$ and the differential

$$d_{\text{dual}}((u_I v^\alpha)^*) = \sum_{\substack{i \in \text{supp}(\alpha) \\ i \notin I}} (-1)^{|I_{<i}|} (u_{I \sqcup i} v^{\alpha - e_i})^*.$$

This formula is similar to (8). We obtain an isomorphism of chain complexes

$$\psi : (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d}) \rightarrow ((\Lambda[m] \otimes \mathbf{k}[\mathcal{K}])^*, d_{\text{dual}}), \quad u_I \otimes \chi_\alpha \mapsto (u_I v^\alpha)^*.$$

Consider the dga algebra $R^*(\mathcal{K}) := (\Lambda[m] \otimes \mathbf{k}[\mathcal{K}]) / (u_i v_i = v_i^2 = 0, i = 1, \dots, m)$. It is well defined, since the ideal $(u_i v_i, v_i^2) \subset \Lambda[m] \otimes \mathbf{k}[\mathcal{K}]$ is d -invariant. The following facts are obtained in the proof of [15, Theorem 3.2.9].

Lemma 4.5 [15, Lemma 3.2.6] *The natural projection $\pi : \Lambda[m] \otimes \mathbf{k}[\mathcal{K}] \rightarrow R^*(\mathcal{K})$ is a chain homotopy equivalence.* □

Lemma 4.6 *We have well-defined chain maps $f_J : \tilde{\mathcal{C}}^*(\mathcal{K}_J; \mathbf{k}) \rightarrow R^*(\mathcal{K})$,*

$$f_J : \tilde{\mathcal{C}}^{n-1}(\mathcal{K}_J; \mathbf{k}) \xrightarrow{\cong} R^{n, -n, 2J}(\mathcal{K}), \quad [L]^* \mapsto \epsilon(L, J) \cdot u_{J \setminus L} v^L, \quad \epsilon(L, J) := (-1)^{\sum_{\ell \in L} |J_{<\ell}|}.$$

The direct sum $f : \bigoplus_{J \subset [m]} \tilde{\mathcal{C}}^(\mathcal{K}_J; \mathbf{k}) \rightarrow R^*(\mathcal{K})$ is an isomorphism of chain complexes.* □

After dualisation, we obtain a chain homotopy equivalence π^* and an isomorphism f^* of chain complexes. It remains to show that the diagram

$$\begin{array}{ccc} \bigoplus_{J \subset [m]} \tilde{\mathcal{C}}^*(\mathcal{K}_J; \mathbf{k}) & \xrightarrow{g} & (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d}) \\ f^* \uparrow \simeq & & \psi \uparrow \simeq \\ (R^*(\mathcal{K}))^* & \xrightarrow[\sim]{\pi^*} & ((\Lambda[m] \otimes \mathbf{k}[\mathcal{K}])^*, d_{\text{dual}}) \end{array}$$

is commutative. Indeed, $f^*((u_{J \setminus L} v^L)^*) = \epsilon(L, J) \cdot [L]$, hence

$$g(f^*((u_{J \setminus L} v^L)^*)) = \epsilon(L, J) \cdot \epsilon(L, J) u_{J \setminus L} \otimes \chi_L = \psi(\pi^*((u_{J \setminus L} v^L)^*)). \quad \square$$

Remark 4.7 In our notation, $\epsilon(L, J) = (-1)^n$, $n = \theta(J \setminus L, L) + |L|(|L| - 1)/2$ for $L \subset J$.

4.3 A chain map to the bar resolution

Theorem 4.8 *The identity map of the left $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ -module \mathbf{k} can be extended to the map of free resolutions $\varphi_{\bullet}: (H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \otimes \mathbf{k}(\mathcal{K}), \hat{d}) \rightarrow (\mathbf{B}_*(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})), d_B)$, given by*

$$\varphi_n(u_I \otimes \chi_{\alpha}) = (-1)^{|I|} \sum_{\substack{\alpha=e_{i_1}+\dots+e_{i_n} \\ I=A_1 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, \forall t \in [n]}} (-1)^{\sum_{1 \leq t_1 < t_2 \leq n} \theta(A_{t_1}, A_{t_2})} [c(A_1, u_{i_1}) | \dots | c(A_n, u_{i_n})].$$

Proof We apply Corollary 2.2 for $\bar{\varphi}_0(u_I) = \varepsilon(u_I)$. It is sufficient to show that

$$\varphi_{n+1}(u_I \otimes \chi_{\alpha}) = s(\varphi_n(\hat{d}(u_I \otimes \chi_{\alpha})))$$

for $|\alpha| = n + 1, n \geq 0$. By (7) and by the $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ -linearity of φ_n , we have

$$\begin{aligned} \varphi_n(\hat{d}(u_I \otimes \chi_{\alpha})) &= \sum_{i \in \text{supp}(\alpha)} (-1)^{|I|} \varphi_n((u_I \wedge u_i) \otimes \chi_{\alpha - e_i}) \\ &\quad + \sum_{i \in \text{supp}(\alpha)} \sum_{\substack{I=A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |A|} c(A, u_i) \varphi_n(u_B \otimes \chi_{\alpha - e_i}). \end{aligned}$$

The map s is trivial on summands of the first sum, since they belong to

$$\bar{\mathbf{B}}(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})) \subset \text{Ker } s \subset \mathbf{B}(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})).$$

Hence we have

$$\begin{aligned} s(\varphi_n(\hat{d}(u_I \otimes \chi_{\alpha}))) &= 0 + \sum_{i \in \text{supp}(\alpha)} \sum_{\substack{I=A \sqcup B \\ \max(A) > i}} \sum_{\substack{\alpha - e_i = e_{i_1} + \dots + e_{i_n} \\ B=A_1 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, \forall t \in [n]}} (-1)^{\zeta} s(c(A, u_i) [c(A_1, u_{i_1}) | \dots | c(A_n, u_{i_n})]) \\ &= \sum_{i \in \text{supp}(\alpha)} \sum_{\substack{I=A \sqcup B \\ \max(A) > i}} \sum_{\substack{\alpha - e_i = e_{i_1} + \dots + e_{i_n} \\ B=A_1 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, \forall t \in [n]}} (-1)^{\zeta} [c(A, u_i) | c(A_1, u_{i_1}) | \dots | c(A_n, u_{i_n})], \end{aligned}$$

where $\zeta = |B| + \theta(A, B) + |A| + \sum_{1 \leq t_1 < t_2 \leq n} \theta(A_{t_1}, A_{t_2})$. Defining $i = i_0, A = A_0$, we obtain

$$\begin{aligned} s(\varphi_n(\hat{d}(u_I \otimes \chi_{\alpha}))) &= \sum_{\substack{\alpha=e_{i_0}+\dots+e_{i_n} \\ I=A_0 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, 0 \leq t \leq n}} (-1)^{\sum_{t=1}^n \theta(A_0, A_t) + |I| + \sum_{1 \leq t_1 < t_2 \leq n} \theta(A_{t_1}, A_{t_2})} [c(A_0, u_{i_0}) | \dots | c(A_n, u_{i_n})]. \end{aligned}$$

The right-hand side equals $\varphi_{n+1}(u_I \otimes \chi_{\alpha})$ up to a shift of indices. □

Theorem 4.9 *Let $J \subset [m]$. Let a class $\alpha \in \text{Tor}_n^{H_*(\Omega \mathcal{Z}_{\mathcal{K}})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J} \cong \tilde{H}_{n-1}(\mathcal{K}_J; \mathbf{k})$ be represented by a cycle*

$$\kappa = \sum_{\substack{I \in \mathcal{K}_J \\ |I|=n}} \lambda_I \cdot [I] \in \tilde{\mathcal{C}}_{n-1}(\mathcal{K}_J; \mathbf{k}).$$

Then the same class is represented by the cycle $\kappa' \in \bar{B}_n(H_*(\Omega Z_{\mathcal{K}}; \mathbf{k}))_{-|J|, 2J}$ in the bar construction,

$$\kappa' := \sum_{\substack{I \in \mathcal{K}_J \\ |I|=n}} \epsilon(I, J) \lambda_I \sum_{\substack{I=\{i_1, \dots, i_n\} \\ J \setminus I = J_1 \sqcup \dots \sqcup J_n \\ \max(J_t) > i_t, \forall t \in [n]}} (-1)^{\sum_{1 \leq t_1 < t_2 \leq n} \theta(J_{i_1}, J_{i_2})} [c(J_1, u_{i_1}) | \dots | c(J_n, u_{i_n})].$$

Proof The map $\tilde{H}_*(\mathcal{K}_J; \mathbf{k}) \rightarrow \text{Tor}_*^{H_*(\Omega Z_{\mathcal{K}})}(\mathbf{k}, \mathbf{k})$ is induced by the composition

$$\bigoplus_{J \subset [m]} \tilde{C}_*(\mathcal{K}_J; \mathbf{k}) \xrightarrow[\sim]{g} (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d}) \xrightarrow[\sim]{\bar{\varphi}} (\bar{B}(H_*(\Omega Z_{\mathcal{K}})), d_{\bar{B}}),$$

of chain maps, where g is defined in Proposition 4.4 and $\bar{\varphi}$ is induced by the chain map φ from Theorem 4.8. We have $\kappa' = \bar{\varphi}(g(\kappa))$ by construction. □

The formulas become simpler for $n = 1, 2$.

Corollary 4.10 *Let $J \subset [m]$. Let a class*

$$\alpha \in \text{Tor}_2^{H_*(\Omega Z_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J} \cong \tilde{H}_1(\mathcal{K}_J)$$

be represented by a cycle

$$\kappa = \sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij} [\{i, j\}] \in \tilde{C}_1(\mathcal{K}_J; \mathbf{k}).$$

Then the same class is represented by the following cycle in the bar construction:

$$\kappa' = \sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J < i| + |J < j|} \lambda_{ij} \sum_{\substack{J \setminus \{i, j\} = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B)} [c(A, u_i) | c(B, u_j)] + (-1)^{\theta(B, A)} [c(B, u_j) | c(A, u_i)]. \quad \square$$

Corollary 4.11 *Let $J \subset [m]$, and let the simplicial complex \mathcal{K}_J have $t + 1$ path components. Let vertices $i_1, \dots, i_t, \max(J)$ be representatives of these components. Then a basis of the \mathbf{k} -module $\text{Tor}_1^{H_*(\Omega Z_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J} \cong \tilde{H}_0(\mathcal{K}_J; \mathbf{k}) \simeq \mathbf{k}^t$ is represented by cycles*

$$[c(J \setminus i_s, u_{i_s})] \in \bar{B}_1(H_*(\Omega Z_{\mathcal{K}}))_{-|J|, 2J}, \quad s = 1, \dots, t.$$

Proof Define $j := \max(J)$. The cycles $\kappa_s = [\{j\}] - [\{i_s\}] \in \tilde{C}_0(\mathcal{K}_J; \mathbf{k})$, $1 \leq s \leq t - 1$, represent a basis in $\tilde{H}_0(\mathcal{K}_J; \mathbf{k})$. By Theorem 4.9, the basis in $\text{Tor}_1^{H_*(\Omega Z_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J}$ is represented by cycles

$$\kappa'_s = 0 \pm [c(J \setminus i_s, u_{i_s})], \quad s = 1, \dots, t.$$

(The summand $[\{j\}]$ in κ_s does not contribute to κ'_s , since the subset $J_1 := J \setminus \{j\}$ does not satisfy the condition $\max(J_1) > j$.) □

5 Generators and relations in the flag case

5.1 Minimal sets of generators

Let M be a finitely generated \mathbf{k} -module. Denote the smallest number of generators by $\text{gen}(M)$, and define $\tilde{b}_i(X; \mathbf{k}) := \text{gen}(\tilde{H}_i(X; \mathbf{k}))$. Since $\tilde{H}_0(X; \mathbf{k})$ is a free module, this number does not depend on \mathbf{k} if \mathbf{k} is a principal ideal domain, so we write $\tilde{b}_0(X; \mathbf{k}) = \tilde{b}_0(X)$. (In fact, $\tilde{b}_0(X) + 1$ is the number of path components in X).

Theorem 5.1 *Let \mathcal{K} be a flag simplicial complex on a vertex set $[m]$ and \mathbf{k} be a commutative ring with unit. For every $J \subset [m]$, choose a $\tilde{b}_0(\mathcal{K}_J)$ -element subset $\Theta(J) \subset J \setminus \{\max(J)\}$ such that $\Theta(J) \sqcup \{\max(J)\}$ contains exactly one vertex from each path component of \mathcal{K}_J . Then $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is multiplicatively generated by the following set of $\sum_{J \subset [m]} \tilde{b}_0(\mathcal{K}_J)$ elements:*

$$\{c(J \setminus i, u_i) : i \in \Theta(J), J \subset [m]\}, \quad c(J \setminus i, u_i) \in H_{-|J|, 2J}(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}).$$

If \mathbf{k} is a principal ideal domain, this set is minimal: any $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation of $H_(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ contains at least $\tilde{b}_0(\mathcal{K}_J)$ generators of degree $(-|J|, 2J)$; any \mathbb{Z} -homogeneous presentation contains at least $\sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ generators of degree n .*

Proof By Corollary 4.11, images of cycles $\{[c(J \setminus i, u_i)] : J \subset [m], i \in \Theta(J)\} \subset \bar{B}_1(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}))$ additively generate the \mathbf{k} -module $\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})$. Hence, by Theorem A.6(i), the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is multiplicatively generated by the elements in question. The lower bounds on the number of generators follow from (9) and from Theorem A.10(ii). \square

Definition 5.2 Let \mathcal{K} be a simplicial complex on $[m]$, and let $J \subset [m]$. Choose $\Theta(J)$ as the set of the smallest vertices in corresponding path components. More precisely, define $\Theta(J)$ as the set of all vertices $i \in J$ such that

- (i) i and $\max(J)$ belong to different path components of the complex \mathcal{K}_J ;
- (ii) i is the smallest vertex (has the smallest number) in its path component.

The corresponding set of generators $\{c(J \setminus i, u_i) : i \in \Theta(J), J \subset [m]\}$ will be called the *GPTW generators*.

Grbić, Panov, Theriault and Wu proved [23, Theorem 4.3] that GPTW generators minimally generate the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ if \mathbf{k} is a field. The minimality was proved using topological methods. Our Theorem 5.1 gives a purely algebraic proof for any ring \mathbf{k} .

5.2 Rewriting of nested commutators

Thus the GPTW generators are indeed multiplicative generators of the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ for any ring \mathbf{k} and any flag complex \mathcal{K} .

Definition 5.3 Let $i \in J \subset [m]$. Express the element $c(J \setminus i, u_i) \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ as a noncommutative polynomial in GPTW generators (this expression may be nonunique). Any such expression will be denoted by $\hat{c}(J \setminus i, u_i)$.

These noncommutative polynomials can be computed recursively, following the proof of [23, Theorem 4.3]. We describe an explicit rewriting process.

Algorithm 5.4 Suppose that expressions $\hat{c}(A \setminus t, u_t)$, $|A| < |J|$, are already computed, and we compute $\hat{c}(J \setminus i, u_i)$. Three cases are possible:

(i) **$i = \max(J)$** Define $j = \max(J \setminus i)$. Then

$$c(J \setminus i, u_i) = c(J \setminus ij, [u_j, u_i]) = c(J \setminus j, u_j).$$

The task is reduced to the case $i \neq \max(J)$.

(ii) **i and $\max(J)$ belong to the same path component of \mathcal{K}_J** The length of the shortest path from i to $\max(J)$ along the edges of \mathcal{K}_J will be called the *rank* of a vertex i . We proceed by induction on the rank. The case of rank zero is discussed above. If rank equals 1, we have $[u_{\max(J)}, u_i] = 0$, so

$$c(J \setminus i, u_i) = c(J \setminus \{i, \max(J)\}, [u_{\max(J)}, u_i]) = 0.$$

Suppose that rank is greater than one, and let $\{i, j\}$ be the first edge in (any) shortest path from i to $\max(J)$. Since $[u_i, u_j] = 0 \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$, the identity (16) expresses $c(J \setminus i, u_i)$ in terms of $c(J \setminus j, u_j)$ (this element has smaller rank) and commutators of smaller degree (expressions for which are already computed).

(iii) **i and $\max(J)$ are in different path components** Let i_0 be the smallest vertex of the component that contains i . The length of the shortest path from i to i_0 will be called the *rank* of a vertex i . If the rank is zero, then $i \in \Theta(J)$, so we can set $\hat{c}(J \setminus i, u_i) := c(J \setminus i, u_i)$. Otherwise we decrease the rank using (16), as in case (ii).

Remark 5.5 A similar argument works more generally: suppose that we have a set $\{x_{J,i} : i \in J \subset [m]\}$ such that, for any $\{i, j\} \in \mathcal{K}_J$, the linear combination $x_{J,i} \pm x_{J,j}$ is a noncommutative polynomial on elements of smaller degree. Then we can express each element $x_{A,t}$ through the ‘‘GPTW elements’’ $\{x_{J,i} : i \in \Theta(J), J \subset [m]\}$ by a similar rewriting process. In our case $x_{J,i} = c(J \setminus i, u_i)$, and the polynomial is given by the last summand in (16).

5.3 Minimal sets of relations

Theorem 5.6 Let \mathcal{K} be a flag simplicial complex on a vertex set $[m]$, and let \mathbf{k} be a commutative ring. For each $J \subset [m]$, choose a collection of simplicial 1-cycles

$$\sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij}^{(\alpha)} \{i, j\} \in \tilde{\mathcal{C}}_1(\mathcal{K}_J; \mathbf{k})$$

that generate the k -module $H_1(\mathcal{K}_J; k)$. Then the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ is presented by GPTW generators $\{c(J \setminus i, u_i) : i \in \Theta(J), J \subset [m]\}$ (see Definition 5.2) modulo the relations

$$(10) \quad \sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J < i| + |J < j|} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus \{i, j\} = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B) + |A|} [\widehat{c}(A, u_i), \widehat{c}(B, u_j)] = 0.$$

In particular, $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ admits a presentation by $\sum_{J \subset [m]} \widetilde{b}_0(\mathcal{K}_J)$ generators modulo $\sum_{J \subset [m]} \widetilde{b}_1(\mathcal{K}_J; k)$ relations: one should take the 1-cycles that correspond to minimal sets of generators.

If k is a principal ideal domain, this presentation is minimal: any $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation contains at least $\widetilde{b}_1(\mathcal{K}_J; k)$ relations of degree $(-|J|, 2J)$ for every $J \subset [m]$.

Proof By Corollary 4.10, our 1-cycles correspond to the elements

$$\sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J < i| + |J < j|} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus ij = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B)} [c(A, u_i) | c(B, u_j)] + (-1)^{\theta(B, A)} [c(B, u_j) | c(A, u_i)]$$

in bar construction, and their images additively generate $\text{Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)$. We apply Theorem A.6(ii) to this situation. (In the notation of this theorem, we take GPTW generators as a_1, \dots, a_N . Their images freely generate $\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}})}(k, k)$, so we can take $R = 0$. We take $\widehat{c}(A, u_i)$ and $\widehat{c}(B, u_j)$ as polynomials $P_{j, \alpha}$ and $Q_{j, \alpha}$.) It follows that $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ is generated by GPTW generators and presented by the relations

$$\sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J < i| + |J < j|} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus ij = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B)} \overline{\widehat{c}(A, u_i)} \widehat{c}(B, u_j) + (-1)^{\theta(B, A)} \overline{\widehat{c}(B, u_j)} \widehat{c}(A, u_i) = 0.$$

Define $x = \widehat{c}(A, u_i)$, $y = \widehat{c}(B, u_j)$. Since $\theta(A, B) + \theta(B, A) \equiv |A| \cdot |B|$, we have

$$\begin{aligned} (-1)^{\theta(A, B)} \overline{x} y + (-1)^{\theta(B, A)} \overline{y} x &= (-1)^{\theta(A, B)} ((-1)^{|A|} x y + (-1)^{|B| + |A| \cdot |B|} y x) \\ &= (-1)^{\theta(A, B) + |A|} (x y - (-1)^{(|A| + 1)(|B| + 1)} y x) = (-1)^{\theta(A, B) + |A|} [x, y]. \end{aligned}$$

Hence the obtained relations coincide with (10). Finally, the lower bound on the number of relations follows from (9) and Theorem A.10(ii). □

Sometimes we can reduce the number of relations if the presentation is not required to be $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous. For example, suppose that for some $I, J \subset [m]$ we have $|I| = |J| = n$, $H_1(\mathcal{K}_I; \mathbb{Z}) = \mathbb{Z}/2$, $H_1(\mathcal{K}_J; \mathbb{Z}) = \mathbb{Z}/3$. Then the graded components of the module $\text{Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbb{Z})}(\mathbb{Z}, \mathbb{Z})$ having multidegrees $(-n, 2I)$ and $(-n, 2J)$ are equal to $\mathbb{Z}/2$ and $\mathbb{Z}/3$. By Theorem A.10, every $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbb{Z})$ should contain relations of these multidegrees. On the other hand, these $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded components contribute $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \simeq \mathbb{Z}/6$ to the \mathbb{Z} -graded component of degree n . Hence we can take just one \mathbb{Z} -homogeneous relation (for example, the sum of these $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous relations). Let us give a general result.

Theorem 5.7 Let \mathcal{K} be a flag simplicial complex and k be a principal ideal domain. Consider all homogeneous presentations of the \mathbb{Z} -graded k -algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$.

- (i) There is a presentation that consists of, for each $n \geq 0$, exactly $\sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ generators and exactly $\text{gen}(\bigoplus_{|J|=n} H_1(\mathcal{K}_J; k))$ relations of degree n . One can take GPTW generators as generators, and take linear combinations of identities from [Theorem 5.6](#), corresponding to minimal generators of the k -module $\bigoplus_{|J|=n} H_1(\mathcal{K}_J; k)$, as relations.
- (ii) For every $n \geq 0$, any presentation contains at least $\sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ generators and at least $\text{gen}(\bigoplus_{|J|=n} H_1(\mathcal{K}_J; k))$ relations of degree n .

Proof By [Theorem A.10](#), $\text{gen Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n$ and $\text{gen Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n + \text{rel Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n$ are precise bounds on the number of generators and relations, respectively, of degree n . By (9), we have $\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n = \bigoplus_{|J|=n} \tilde{H}_0(\mathcal{K}_J; k) \simeq k^{\oplus \sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)}$, $\text{Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k) = \bigoplus_{|J|=n} H_1(\mathcal{K}_J; k)$;

hence $\text{gen Tor}_1 = \sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ and $\text{rel Tor}_1 = 0$. One can take the GPTW generators since the images of corresponding cycles generate Tor_1 by [Corollary 4.11](#). □

5.4 Example: moment-angle complexes for m -cycles

Let \mathcal{K} be the boundary of an m -gon. The corresponding moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is homeomorphic to a connected sum of sphere products, $\mathcal{Z}_{\mathcal{K}} \cong \#_{k=3}^{m-1} (S^k \times S^{m+2-k})^{\#(k-2) \binom{m-2}{k-1}}$, and hence $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ is a one-relator algebra. It was considered in [\[24; 42\]](#). From the point of view of [Theorem 5.6](#), the relation corresponds to the 1-cycle

$$\kappa = [\{1, m\}] - \sum_{i=1}^{m-1} [\{i, i+1\}] \in \tilde{\mathcal{C}}_1(\mathcal{K}; k)$$

and has the form

$$\sum_{\substack{\{2, \dots, m-1\} = A \sqcup B \\ \max(A) > 1 \\ \max(B) > m}} (\dots) - \sum_{i=1}^{m-1} (-1)^{(i-1)+i} \sum_{\substack{[m] \setminus \{i, i+1\} = A \sqcup B \\ \max(A) > i \\ \max(B) > i+1}} (-1)^{\theta(A, B) + |A|} [\hat{c}(A, u_i), \hat{c}(B, u_{i+1})] = 0.$$

The first sum is empty, since $\max(B) \leq m - 1$. Similarly, in the second sum the inner sum is empty for $i = m - 1, m - 2$. The simplified relation is

$$\sum_{i=1}^{m-3} \sum_{\substack{[m] \setminus \{i, i+1\} = A \sqcup B \\ \max(A), \max(B) \geq i+2}} (-1)^{\theta(A, B) + |A|} [\hat{c}(A, u_i), \hat{c}(B, u_{i+1})] = 0.$$

Some summands are immediately zero. For example, if $\max(B) = i + 2$, then

$$c(B, u_{i+1}) = c(B \setminus \{i + 2\}, [u_{i+2}, u_{i+1}]) = 0,$$

so we can take $\hat{c}(B, u_{i+1}) = 0$. Similarly, $c(A, u_1) = 0$ if $i = 1$ and $m \in A$. Other summands can be computed using Algorithm 5.4. We were not able to obtain a closed formula for this relation (as a polynomial of GPTW generators or other minimal generators). However, we at least have an effective algorithm that computes the relation for any given m .

Consider the case $m = 5$. Besides from the partitions $[5] \setminus \{i, i + 1\} = A \sqcup B$ considered above, for $i = 1$ the allowed partitions are $\{3, 4, 5\} = \{3\} \sqcup \{4, 5\} = \{4\} \sqcup \{3, 5\} = \{3, 4\} \sqcup \{5\}$; for $i = 2$ the allowed partitions are $\{1, 4, 5\} = \{4\} \sqcup \{1, 5\} = \{1, 4\} \sqcup \{5\}$. The resulting relation has five summands:

$$\begin{aligned} &(-1)^{\theta(3,45)+1}[\hat{c}(3, u_1), \hat{c}(45, 2)] + (-1)^{\theta(4,35)+1}[\hat{c}(4, u_1), \hat{c}(35, 2)] + (-1)^{\theta(34,5)+2}[\hat{c}(34, u_1), \hat{c}(5, u_2)] \\ &\quad + (-1)^{\theta(4,15)+1}[\hat{c}(4, u_2), \hat{c}(15, u_3)] + (-1)^{\theta(14,5)+2}[\hat{c}(14, u_2), \hat{c}(5, u_3)] = 0. \end{aligned}$$

All commutators, apart from $\hat{c}(14, u_2) = [u_1, [u_4, u_2]] = -[u_2, [u_4, u_1]] = -c(24, u_1)$, already are GPTW generators. We obtain the following identity between the generators:

$$\begin{aligned} &-[[u_3, u_1], [u_4, [u_5, u_2]]] + [[u_4, u_1], [u_3, [u_5, u_2]]] - [[u_5, u_2], [u_3, [u_4, u_1]]] \\ &\quad + [[u_4, u_2], [u_1, [u_5, u_3]]] + [[u_5, u_3], [u_2, [u_4, u_1]]] = 0. \end{aligned}$$

This relation was first obtained by Veryovkin as a result of brute force [42, Theorem 3.2]. For $m = 6$, the analogous relation is initially the sum of $7 + 10 + 4 = 21$ commutators. After computing the elements $\hat{c}(J \setminus i, u_i)$ and changing the set of generators, it can be written as $\sum_{i=1}^{17} [a_i, b_i] = 0$ (see [42, Theorem 4.1]). This agrees with the homeomorphism $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^5)^{\#9} \# (S^4 \times S^4)^{\#8}$.

6 Homotopical properties in the flag case

6.1 Homotopy groups

As in [39], we denote by \mathcal{P} the class of H-spaces which are homotopy equivalent to finite-type products of spheres and loops on simply connected spheres, and by \mathcal{W} the class of topological spaces which are homotopy equivalent to finite-type wedges of simply connected spheres. The author thanks Lewis Stanton for providing a proof of the following lemma.

Lemma 6.1 *Let A_1, \dots, A_m be connected topological spaces, \mathcal{K} be a simplicial complex on $[m]$, and suppose that $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}} \in \mathcal{P}$. Then $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}}$ is homotopy equivalent to a finite-type product of loops on simply connected spheres.*

Proof By [40, Corollary 9.8], $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}} \simeq \prod_{i=1}^m \Omega \Sigma Y_i$ for some spaces Y_i . Since the class \mathcal{P} is closed under retracts [39, Theorem 3.10], $\Omega \Sigma Y_i \in \mathcal{P}$. By repeated use of the homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ and the James splitting $\Sigma \Omega \Sigma X \simeq \bigvee_{n \geq 1} \Sigma X^{\wedge n}$ [27], we have $\Sigma Z \in \mathcal{W}$ for $Z \in \mathcal{P}$. In particular, $\Sigma \Omega \Sigma Y_i \in \mathcal{W}$. On the other hand, ΣY_i is a retract of $\Sigma \Omega \Sigma Y_i$ by the James splitting. The class \mathcal{W} is closed under retracts (see, for example, [1, Lemma 3.1]), so $\Sigma Y_i \in \mathcal{W}$. Now $\Omega \Sigma Y_i$ is homotopy equivalent to a product of loops on spheres by the Hilton–Milnor theorem. It follows that the same holds for $\prod_{i=1}^m \Omega \Sigma Y_i$. □

Proof of Theorem 1.2 Since \mathcal{K} is flag, we have $\Omega\mathcal{Z}_{\mathcal{K}} \in \mathcal{P}$ by Theorem 3.11. Hence $\Omega\mathcal{Z}_{\mathcal{K}} = \Omega(CS^1, S^1)^{\mathcal{K}}$ is a product of loops on spheres by Lemma 6.1. It follows that for some $D_n \geq 0$ we have a homotopy equivalence

$$\Omega\mathcal{Z}_{\mathcal{K}} \simeq \prod_{n \geq 2} (\Omega S^n)^{\times D_n}.$$

The numbers D_n are finite, since $\dim_{\mathbf{k}} H_i(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) < \infty$ for all i . (Here \mathbf{k} is any field.) Also $D_2 = 0$, since $\mathcal{Z}_{\mathcal{K}}$ is 2-connected [15, Proposition 4.3.5]. In order to compute D_n , we calculate $\dim H_i(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ twice. Recall that the Poincaré series $F(V; t)$ of a graded \mathbf{k} -vector space V are the formal power series

$$F(V; t) := \sum_{i \geq 0} \dim_{\mathbf{k}}(V_i) \cdot t^i \in \mathbb{Z}[[t]].$$

We have

$$F(V \oplus W) = F(V; t) + F(W; t) \quad \text{and} \quad F(V \otimes W; t) = F(V; t) \cdot F(W; t).$$

From $F(H_*(\Omega S^k; \mathbf{k}); t) = (1 - t^{k-1})^{-1}$ and the Künneth formula we have

$$F(H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}); t) = \prod_{n \geq 3} (1 - t^{n-1})^{-D_n}.$$

On the other hand, it is known (see [15, Proposition 8.5.4] and [43, Theorem 4.8]) that

$$F(H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k}); t) = \frac{1}{(1+t)^{m-d} \cdot h_{\mathcal{K}}(-t)} = - \frac{1}{\sum_{J \subset [m]} \bar{\chi}(\mathcal{K}_J) t^{|J|}}$$

for a flag complex \mathcal{K} . We obtain the required identity (3). \square

Remark 6.2 In the proof above, the algebra $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is actually $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded. We expect that factors of the product (2) can be considered as “ $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded spheres”, and thus $\pi_*(\Omega\mathcal{Z}_{\mathcal{K}})$ admits a functorial $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading as conjectured in [43, Remark 4.10].

Problem 6.3 Describe the Whitehead bracket in $\pi_*(\mathcal{Z}_{\mathcal{K}})$ in terms of the decomposition (4).

6.2 Rational coformality of moment-angle complexes

Let X be a simply connected space and ΩX be the space of Moore loops. Since ΩX is a strictly associative topological monoid, the chain complex $C_*(\Omega X; \mathbf{k})$ is a dga algebra with respect to the Pontryagin product for any \mathbf{k} . Also, the cochain complex $C^*(X; \mathbf{k})$ is a dga algebra with respect to the Kolmogorov–Alexander product (cup product).

Definition 6.4 A topological space X is *formal* over a ring \mathbf{k} if the dga algebras $H^*(X; \mathbf{k})$ (with zero differential) and $C^*(X; \mathbf{k})$ are quasi-isomorphic (are connected by a zigzag of dga maps which induce isomorphisms on homology).

Definition 6.5 A simply connected space X is *coformal* over a ring \mathbf{k} if the dga algebras $H_*(\Omega X; \mathbf{k})$ (with zero differential) and $C_*(\Omega X; \mathbf{k})$ are quasi-isomorphic.

The notions of formality and coformality (over a field of characteristic zero) arose in rational homotopy theory, and were initially formulated in terms of Sullivan and Quillen models. The rational homotopy type of a formal (coformal) space is fully determined by the algebra $H^*(X; \mathbb{Q})$ (by the algebra $H_*(\Omega X; \mathbb{Q})$). As proved by Saleh [38, Corollary 1.2, 1.4], our definitions are equivalent to the classical ones.

Notbohm and Ray proved [34, Theorem 4.8] that all Davis–Januszkiewicz spaces $\text{DJ}(\mathcal{K})$ are formal over \mathbb{Z} . (In fact, the proof works over any ring \mathbf{k} ; see also [20, Theorem 1.3] for a stronger formality result). Also, $\text{DJ}(\mathcal{K})$ is coformal over \mathbb{Q} if and only if \mathcal{K} is flag [15, Theorem 8.5.6]. First examples of nonformal moment-angle complexes were constructed by Baskakov [6] using Massey products. See [14, Introduction] for a survey of further developments in this area.

The following result of Huang can be used to prove coformality over \mathbb{Q} .

Proposition 6.6 [26, Proposition 5.1] *Let $F \xrightarrow{i} E \rightarrow B$ be a fibration of nilpotent spaces of finite type, such that*

- *the map $i_* : \pi_*(F) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \pi_*(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective;*
- *E is coformal over \mathbb{Q} .*

Then F is coformal over \mathbb{Q} . □

Corollary 6.7 *Let \mathcal{K} be a flag simplicial complex with no ghost vertices. Then $\mathcal{Z}_{\mathcal{K}}$ is coformal over \mathbb{Q} .*

Proof We apply Proposition 6.6 to the fibration $\mathcal{Z}_{\mathcal{K}} \rightarrow \text{DJ}(\mathcal{K}) \rightarrow B\mathbb{T}^m$. By Proposition 3.1 and exact sequence of homotopy groups, $\pi_*(\mathcal{Z}_{\mathcal{K}}) \rightarrow \pi_*(\text{DJ}(\mathcal{K}))$ is injective. The second condition holds by [15, Theorem 8.5.6]. □

It is natural to hope that Huang’s theorem admits the following generalisation.

Conjecture 6.8 *Let $F \rightarrow E \xrightarrow{p} B$ be a fibration of simply connected spaces of finite type, such that*

- *Ωp has a homotopy section;*
- *E is coformal over \mathbf{k} .*

Then F is coformal over \mathbf{k} .

Let X be a simply connected space such that $H_*(\Omega X; \mathbf{k})$ is a free \mathbf{k} -module. The tensor filtration on the bar construction $\bar{B}(C_*(\Omega X; \mathbf{k}))$ gives rise to the Milnor–Moore spectral sequence

$$E_{p,q}^2 = \text{Tor}_p^{H_*(\Omega X; \mathbf{k})}(\mathbf{k}, \mathbf{k})_q \Rightarrow \text{Tor}_{p+q}^{C_*(\Omega X; \mathbf{k})}(\mathbf{k}, \mathbf{k}) \cong H_{p+q}(X; \mathbf{k}).$$

(The last isomorphism is due to Eilenberg and Moore; see [18, Theorem IV]).

The differential Tor is preserved by quasi-isomorphisms. Hence the spectral sequence collapses at E^2 if X is coformal over \mathbf{k} . On the other hand, it collapses for $X = \mathcal{Z}_{\mathcal{K}}$ in the flag case; see (9). This suggests the following conjecture.

Conjecture 6.9 Let \mathcal{K} be a flag simplicial complex. Then the spaces $\text{DJ}(\mathcal{K})$ and $\mathcal{Z}_{\mathcal{K}}$ are coformal over any commutative ring with unit.

6.3 A necessary condition for the rational formality in the flag case

The space X is Koszul if it is both formal and coformal over \mathbb{Q} . Hence $\text{DJ}(\mathcal{K})$ is Koszul if and only if \mathcal{K} is flag. Koszul spaces were introduced by Berglund [10].

Definition 6.10 Let k be a field, $A = \bigoplus_{n \in \mathbb{Z}} A^n$ be a graded k -algebra that admits an additional “weight” grading $A^n = \bigoplus_{j \geq 0} A^{n,(j)}$. The algebra A is Koszul with respect to the weight grading if $\text{Ext}_A^i(k, k)^{n,(j)} = 0$ for all $i \neq j$.

For every Koszul algebra, there is a quadratic dual Koszul algebra $A^!$; see [22]. More explicitly, we set

$$A^! := \text{Ext}_A(k, k), \quad (A^!)^{n,(i)} = \text{Ext}_A^i(k, k)^{-i-n,(i)}.$$

Then it is known that $(A^!)^! \cong A$ as bigraded algebras.

Remark 6.11 In the classical theory of Koszul algebras [22; 36] the \mathbb{Z} -grading $A = \bigoplus_{n \in \mathbb{Z}} A^n$ is absent, and only the weight grading $(A^!)^{(i)} = \text{Ext}_A^i(k, k)^{(i)}$ is considered. Classical results are readily generalised to the graded case.

The following result is due to Berglund. We replace the Koszul Lie algebras with their universal enveloping algebras. Berglund considers a stronger version of the Koszul duality, the duality between Lie algebras and commutative algebras.

Theorem 6.12 [10, Theorems 2 and 3] Let X be a simply connected space of finite type such that X is coformal over \mathbb{Q} . The following conditions are equivalent:

- (a) X is formal over \mathbb{Q} .
- (b) The graded algebra $A = H_*(\Omega X; \mathbb{Q})$ admits a weight grading $A = \bigoplus_{i \geq 0} A^{(i)}$ such that A is Koszul with respect to it.

Moreover, if these conditions are met, then the \mathbb{Z} -graded algebras $A^!$ and $H^{-*}(X; \mathbb{Q})$ are isomorphic: $H^n(X; \mathbb{Q}) \cong \bigoplus_{i \geq 0} (A^!)^{-n,(i)}$. \square

Theorem 6.13 Let \mathcal{K} be a flag simplicial complex on $[m]$ with no ghost vertices, such that $\mathcal{Z}_{\mathcal{K}}$ is rationally formal. Then $\Gamma = H^*(\mathcal{Z}_{\mathcal{K}}; \mathbb{Q})$ is a Koszul algebra with respect to the grading

$$\Gamma^{(i)} := \bigoplus_{J \subset [m]} H^{i-|J|, 2J}(\mathcal{Z}_{\mathcal{K}}; \mathbb{Q}) = \bigoplus_{J \subset [m]} \tilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q}).$$

In particular, Γ is generated by elements in $\tilde{H}^0(\mathcal{K}_J; \mathbb{Q})$ modulo the relations in $\tilde{H}^1(\mathcal{K}_J; \mathbb{Q})$.

Proof By [Theorem 6.12](#), the algebra $A = H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbb{Q})$ is Koszul with respect to a weight grading $A = \bigoplus_{i \geq 0} A^{(i)}$. From [\[43, Theorem 1.2\]](#) we have $\text{Tor}_i^A(\mathbb{Q}, \mathbb{Q})_j = \bigoplus_{|J|=j} \widetilde{H}_{i-1}(\mathcal{K}_J; \mathbb{Q})$. Therefore, $\text{Ext}_A^i(\mathbb{Q}, \mathbb{Q})^j = \bigoplus_{|J|=j} \widetilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q})$. The algebra A is Koszul, hence

$$\text{Ext}_A^i(\mathbb{Q}, \mathbb{Q})^j = \text{Ext}_A^i(\mathbb{Q}, \mathbb{Q})^{j,(i)} = (A^!)^{-i-j,(i)}.$$

Since $(A^!)^* \cong \Gamma^{-*}$ as graded algebras, we obtain a weight grading

$$\Gamma^{i+j,(i)} = \bigoplus_{|J|=j} \widetilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q}), \quad \Gamma^{(i)} = \bigoplus_{J \subset [m]} \widetilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q})$$

such that Γ is Koszul with respect to it. Finally, any Koszul algebra is generated by elements of weight 1 modulo relations of weight 2. □

Conjecture 6.14 *If \mathcal{K} is a flag complex and $H^*(\mathcal{Z}_{\mathcal{K}}; \mathbb{Q})$ is Koszul with respect to the grading from [Theorem 6.13](#), then $\mathcal{Z}_{\mathcal{K}}$ is formal over \mathbb{Q} .*

Appendix A Presentations of connected graded algebras

In this section we prove [Theorems A.1](#) and [A.10](#) that generalise some results of [Wall \[44, Section 7\]](#). We also prove [Theorem A.6](#), which seems to be new. We use the notation from [Section 2](#); some of which is recalled below.

A.1 Conventions

The ring k is assumed to be an arbitrary commutative associative ring with unit. All tensor products are over k .

We consider G -graded k -algebras, where G is a commutative monoid supplied with a homomorphism $G \rightarrow \mathbb{Z}$. It induces a \mathbb{Z} -grading. Such an algebra A is *connected* if it is connected with respect to the \mathbb{Z} -grading, ie $A_{<0} = 0$ and $A_0 = k \cdot 1$. Then the standard augmentation $\varepsilon: A \rightarrow A_0 \cong k$ makes k a left A -module and a right A -module.

Every complex of G -graded modules is considered as a $\mathbb{Z} \times G$ -graded module with a differential of degree $(-1, 0)$. Hence, A -linear differentials satisfy the following version of Leibniz's rule:

$$d(a \cdot x) = (-1)^{\text{deg}(a)} a \cdot d(x) = -\bar{a} \cdot d(x),$$

where $\bar{a} := (-1)^{1+\text{deg}(a)} a$.

A *presentation* of a connected k -algebra A is an isomorphism of the form $A \simeq T(x_1, \dots, x_N)/(r_1, \dots, r_M)$, sometimes written as

$$A \simeq T(x_1, \dots, x_N)/(r_1 = \dots = r_M = 0),$$

where $T(x_1, \dots, x_N)$ is a tensor algebra and $(r_1, \dots, r_M) \subset T(x_1, \dots, x_N)$ is the two-sided ideal generated by the set $\{r_1, \dots, r_M\}$. It is assumed that generators and relations are homogeneous and have positive degree, hence belong to $\text{Ker } \varepsilon$. Note that A is not required to be a free \mathbf{k} -module, and M, N can be infinite of any cardinality.

A.2 Exact sequence of a presentation

Let $T(x_1, \dots, x_N)$ be a tensor algebra generated by homogeneous elements of positive degrees. Every element $w \in T(x_1, \dots, x_N)$ is uniquely represented as a sum

$$w = \varepsilon(w) + \sum_{i=1}^N w_i \cdot x_i, \quad w_i \in T(x_1, \dots, x_N).$$

In the next proposition we use this representation implicitly. For example, we assume that

$$r_j = \varepsilon(r_j) + \sum_{i=1}^N r_{ji} \cdot x_i.$$

Since $r_j \in \text{Ker } \varepsilon$, the first summand is zero.

Proposition A.1 *Let $A = T(x_1, \dots, x_N)/(r_1, \dots, r_M)$ be a presentation of a connected \mathbf{k} -algebra,*

$$\pi : T(x_1, \dots, x_N) \twoheadrightarrow A$$

be the projection. Then the following sequence of graded free left A -modules is exact:

$$A \cdot \{R_1, \dots, R_M\} \xrightarrow{d_2} A \cdot \{X_1, \dots, X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} \mathbf{k} \rightarrow 0,$$

$$d_2(R_j) := - \sum_{i=1}^N \pi(\overline{r_{ji}}) \cdot X_i, \quad d_1(X_i) := x_i.$$

Proof We first prove that the sequence is a chain complex. Indeed, $\varepsilon(d_1(X_i)) = \varepsilon(x_i) = 0$ and

$$d_1(d_2(R_j)) = \sum_{i=1}^N \pi(r_{ji}) \cdot d_1(X_i) = \sum_{i=1}^N \pi(r_{ji})x_i = \pi\left(\sum_{i=1}^N r_{ji}x_i\right) = \pi(r_j) = 0 \in A$$

($r_j \in \text{Ker } \varepsilon$, hence $r_j = \sum_i r_{ji}x_i$). We check the exactness in the term A . Let $y \in \text{Ker } \varepsilon \subset A$. We have $y = \pi(w)$ for some element $w \in T(x_1, \dots, x_N)$ of positive degree, hence

$$y = \pi\left(\sum_{i=1}^N w_i x_i\right) = \sum_{i=1}^N \pi(w_i)x_i = d_1\left(-\sum_{i=1}^N \pi(\overline{w_i}) \cdot X_i\right) \in \text{Im } d_1.$$

We check the exactness in the term $A \cdot \{X_1, \dots, X_N\}$. Suppose that $\sum_{i=1}^N a_i \cdot X_i \in \text{Ker } d_1$, so $\sum_{i=1}^N \overline{a_i} x_i = 0$. We have $a_i = \pi(v_i)$ for some $v_i \in T(x_1, \dots, x_N)$. Then the element $w := \sum_{i=1}^N \overline{v_i} x_i \in T(x_1, \dots, x_N)$ belongs to $\text{Ker } \pi$. This kernel is a two-sided ideal generated by r_j . Hence $w = \sum_{j=1}^M \sum_{\alpha} u_{j,\alpha} r_j w_{j,\alpha}$ for some $u_{j,\alpha}, w_{j,\alpha} \in T(x_1, \dots, x_N)$. We can rewrite it as

$$w = \sum_{j=1}^M \sum_{\alpha} u_{j,\alpha} r_j \varepsilon(w_{j,\alpha}) + \sum_{j=1}^M \sum_{\alpha} \sum_{i=1}^N u_{j,\alpha} r_j w_{j,\alpha,i} x_i = \sum_{i=1}^N \sum_{j=1}^M \sum_{\alpha} (\varepsilon(w_{j,\alpha})u_{j,\alpha} r_{ji} + u_{j,\alpha} r_j w_{j,\alpha,i}) x_i.$$

On the other hand, $w = \sum_{i=1}^N \bar{v}_i x_i$. Such a representation is unique, so we have

$$\bar{v}_i = \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) u_{j,\alpha} r_{ji} + u_{j,\alpha} r_j w_{j,\alpha,i}, \quad i = 1, \dots, N.$$

Applying π to both parts of this identity, we obtain $\bar{a}_i = \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) \pi(u_{j,\alpha}) \pi(r_{ji})$, since $\pi(v_i) = a_i$ and $\pi(r_j) = 0$. Finally,

$$\sum_{i=1}^N a_i \cdot X_i = - \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) \pi(\bar{u}_{j,\alpha}) \pi(\bar{r}_{ji}) \cdot X_i = d_2 \left(- \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) \pi(u_{j,\alpha}) \cdot R_j \right) \in \text{Im } d_2. \quad \square$$

Remark A.2 Proposition A.1 holds for presentations of augmented algebras such that $\varepsilon(x_i) = \varepsilon(r_j) = 0$. The corresponding exact sequence is called the ‘‘Koszul resolution’’ in [3, Section 2].

Corollary A.3 Let $A = T(x_1, \dots, x_N)/(r_1, \dots, r_M)$ be a presentation of a connected graded k -algebra, which is a free k -module. Then the k -module $\text{Tor}_1^A(k, k)$ is additively generated by images of cycles $[x_1], \dots, [x_N] \in \bar{B}_1(A)$.

Proof We extend the exact sequence from Proposition A.1 to a free resolution

$$\dots \rightarrow A \cdot \{X_1, \dots, X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} k \rightarrow 0, \quad d_1(X_i) = x_i,$$

of the left A -module k . Consider the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & A \cdot \{X_1, \dots, X_N\} & \xrightarrow{d_1} & A & \xrightarrow{\varepsilon} & k \longrightarrow 0 \\ & & \downarrow X_i \mapsto [x_i] & & \downarrow a \mapsto a[\cdot] & & \parallel \\ \dots & \longrightarrow & B_1(A) & \xrightarrow{d_{B,1}} & B_0(A) & \xrightarrow{\varepsilon} & k \longrightarrow 0 \end{array}$$

It is commutative, since $d_1(a \otimes X_i) = -\bar{a}x_i$ and $d_{B,1}(a[x_i]) = -\bar{a}x_i[\cdot]$. Hence it can be extended to a map of resolutions (eg using Lemma 2.1). Apply the functor $k \otimes_A (-)$. We obtain a map of chain complexes

$$\begin{array}{ccccccc} \dots & \longrightarrow & k \cdot \{X_1, \dots, X_N\} & \xrightarrow{0} & k & \longrightarrow & 0 \\ & & \downarrow X_i \mapsto [x_i] & & \downarrow a \mapsto a[\cdot] & & \\ \dots & \longrightarrow & \bar{B}_1(A) & \xrightarrow{d_{\bar{B},1}} & \bar{B}_0(A) & \longrightarrow & 0 \end{array}$$

The homology of both complexes equals $\text{Tor}^A(k, k)$, and the induced map in homology is an isomorphism. The elements X_i in the first complex are cycles, and their images generate $\text{Tor}_1^A(k, k)$. \square

Corollary A.4 Let $A = T(x_1, \dots, x_N)$ be the tensor algebra over a ring k , where x_1, \dots, x_N are homogeneous elements of positive degrees. Then $\text{Tor}_1^A(k, k)$ is a free k -module with the basis represented by cycles $[x_1], \dots, [x_N] \in \bar{B}_1(A)$. Moreover, $\text{Tor}_i^A(k, k) = 0$ for $i > 1$.

Proof By Proposition A.1, the sequence

$$0 \rightarrow A \cdot \{X_1, \dots, X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} \mathbf{k} \rightarrow 0, \quad d_1(X_i) = x_i,$$

is exact. As in the proof of Corollary A.3, we obtain a map of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{k} \cdot \{X_1, \dots, X_N\} & \xrightarrow{0} & \mathbf{k} & \longrightarrow & 0 \\ \downarrow & & \downarrow X_i \mapsto [x_i] & & \downarrow a \mapsto a[\cdot] & & \\ \dots & \longrightarrow & \bar{\mathbf{B}}_1(A) & \xrightarrow{d_{\bar{\mathbf{B}},1}} & \bar{\mathbf{B}}_0(A) & \longrightarrow & 0 \end{array}$$

Homology of both complexes is equal to $\text{Tor}^A(\mathbf{k}, \mathbf{k})$, and the induced map in homology is the identity. \square

A.3 A presentation that corresponds to cycles

Recall that $\text{Tor}^A(\mathbf{k}, \mathbf{k}) \cong H(\bar{\mathbf{B}}(A))$ if A is a free \mathbf{k} -module. The following lemma is proved by Lemaire [29, Corollaire 1.2.3] in the case of field coefficients.

Lemma A.5 *Let $f : A \rightarrow C$ be a morphism of connected \mathbf{k} -algebras, where \mathbf{k} is a commutative ring with unit.*

- (i) *Suppose that the map $f_{*,1} : H_1(\bar{\mathbf{B}}(A)) \rightarrow H_1(\bar{\mathbf{B}}(C))$ is surjective. Then $f : A \rightarrow C$ is surjective.*
- (ii) *Suppose that $f_{*,1} : H_1(\bar{\mathbf{B}}(A)) \rightarrow H_1(\bar{\mathbf{B}}(C))$ is bijective, and $f_{*,2} : H_2(\bar{\mathbf{B}}(A)) \rightarrow H_2(\bar{\mathbf{B}}(C))$ is surjective. Then $f : A \rightarrow C$ is an isomorphism.*

We prove by induction that the maps $f_n : A_n \rightarrow C_n$ are surjective (bijective). The base case is the bijection $A_0 \cong \mathbf{k} \cong C_0$. Recall that the bar construction $\bar{\mathbf{B}}(A)$ is the chain complex

$$\begin{aligned} \dots \rightarrow \bar{\mathbf{B}}_3(A) \xrightarrow{d_3} \bar{\mathbf{B}}_2(A) \xrightarrow{d_2} \bar{\mathbf{B}}_1(A) \xrightarrow{0} \mathbf{k} \rightarrow 0, \\ \bar{\mathbf{B}}_k(A) = I(A)^{\otimes k}, \quad d_2(x \otimes y) = \bar{x}y, \quad d_3(x \otimes y \otimes z) = \bar{x}y \otimes z + \bar{x} \otimes \bar{y}z. \end{aligned}$$

We define $f_{\#} : \bar{\mathbf{B}}(A) \rightarrow \bar{\mathbf{B}}(C)$.

Proof of statement (i) Suppose that $f : A_i \rightarrow C_i$ is surjective for $i < n$. Consider the following map of exact sequences:

$$\begin{array}{ccccccc} \bar{\mathbf{B}}_2(A)_n & \xrightarrow{d_2} & \bar{\mathbf{B}}_1(A)_n \cong A_n & \longrightarrow & H_1(\bar{\mathbf{B}}(A))_n & \longrightarrow & 0 \\ \downarrow f_{\#,2} & & \downarrow f & & \downarrow f_{*,1} & & \parallel \\ \bar{\mathbf{B}}_2(C)_n & \xrightarrow{d_2} & \bar{\mathbf{B}}_1(C)_n \cong C_n & \longrightarrow & H_1(\bar{\mathbf{B}}(C))_n & \longrightarrow & 0 \end{array}$$

The map $f_{\#,2}$ is surjective, since it is a direct sum of maps $f \otimes f : A_i \otimes A_j \rightarrow C_i \otimes C_j$ for $i, j < n$, and f is surjective in these degrees by the inductive hypothesis. The surjectivity $f_{*,1}$ is given, and $0 \rightarrow 0$ is injective. Hence $f : A_n \rightarrow C_n$ is surjective by the “first half of five lemma” [37, Proposition 2.72(i)]. \square

Proof of statement (ii) Suppose that $f : A_i \rightarrow C_i$ is bijective for $i < n$. Consider the following map of exact sequences:

$$\begin{array}{ccccccc} \bar{B}_3(A)_n & \xrightarrow{d_3} & \text{Ker } d_2 & \longrightarrow & H_2(\bar{B}(A))_n & \longrightarrow & 0 \\ \downarrow f_{\#,3} & & \downarrow \varphi & & \downarrow f_{*,2} & & \parallel \\ \bar{B}_3(C)_n & \xrightarrow{d_3} & \text{Ker } d_2 & \longrightarrow & H_2(\bar{B}(C))_n & \longrightarrow & 0 \end{array}$$

The map $f_{\#,3}$ is surjective, since it is a direct sum of maps $f \otimes f \otimes f : A_i \otimes A_j \otimes A_k \rightarrow C_i \otimes C_j \otimes C_k$ for $i, j, k < n$, and f is surjective in these degrees. The surjectivity of $f_{*,2}$ is given, and $0 \rightarrow 0$ is injective. Hence φ is surjective by the “first half of five lemma”. Now consider the following map of exact sequences:

$$\begin{array}{ccccccc} \text{Ker } d_{\bar{B},2} & \longrightarrow & \bar{B}_2(A)_n & \xrightarrow{d_2} & \bar{B}_1(A)_n \cong A_n & \longrightarrow & H_1(\bar{B}(A))_n \\ \downarrow \varphi & & \cong \downarrow f_{\#,2} & & \downarrow f & & \downarrow f_{*,1} \\ \text{Ker } d_{\bar{B},2} & \longrightarrow & \bar{B}_2(C)_n & \xrightarrow{d_2} & \bar{B}_1(C)_n \cong C_n & \longrightarrow & H_1(\bar{B}(C))_n \end{array}$$

We proved that φ is surjective. The map $f_{\#,2}$ is bijective by the inductive hypothesis (it is a direct sum of maps $f \otimes f : A_i \otimes A_j \rightarrow C_i \otimes C_j$, $i, j < n$); in particular, it is injective. The injectivity of $f_{*,1}$ is given. Hence the map $f : A_n \rightarrow C_n$ is injective by the “second half of five lemma” [37, Proposition 2.72(ii)]. By (i), this map is also surjective. □

The following theorem allows one to obtain a presentation of a connected k -algebra A , knowing the structure of k -modules $H_1(\bar{B}(A))$ and $H_2(\bar{B}(A))$. In the proof, we do not use the notation $[x|y|z]$ for elements of the bar construction, and write $x \otimes y \otimes z$ instead. Therefore, $[c]$ always denotes the class in $H(\bar{B}(\Gamma))$ represented by a cycle $c \in \bar{B}(\Gamma)$.

We also use the following notation. Let $a_1, \dots, a_N \in A$ be some homogeneous elements of positive degree and $K, L \in T(x_1, \dots, x_N)$ be homogeneous noncommutative polynomials that belong to the augmentation ideal. Then the elements $K(a_1, \dots, a_N), L(a_1, \dots, a_N) \in I(A)$ are defined, and hence we can consider the elements $K(a_1, \dots, a_N) \otimes L(a_1, \dots, a_N) \in I(A) \otimes I(A) = \bar{B}_2(A)$ and

$$d_{\bar{B},2}(K(a_1, \dots, a_N) \otimes L(a_1, \dots, a_N)) = \bar{K}(a_1, \dots, a_N) \cdot L(a_1, \dots, a_N) \in \bar{B}_1(A) = I(A).$$

Theorem A.6 *Let A be a connected algebra over a commutative ring k with unit.*

(i) *Suppose that, for homogeneous elements $a_1, \dots, a_N \in A_{>0}$, the k -module $H_1(\bar{B}(A))$ is additively generated by the classes $[a_1], \dots, [a_N] \in H_1(\bar{B}(A))$. Then A is multiplicatively generated by a_1, \dots, a_N .*

(ii) *Suppose that the k -module $H_1(\bar{B}(A))$ is additively generated by N elements $[a_1], \dots, [a_N]$ modulo R relations*

$$\sum_{i=1}^N \lambda_{ri} [a_i] = 0 \in H_1(\bar{B}(A)), \quad r = 1, \dots, R, \lambda_{ri} \in k.$$

Suppose that homogeneous polynomials $P_{j,\alpha}, Q_{j,\alpha}, K_{r,\beta}, L_{r,\beta} \in T(x_1, \dots, x_N)$ are such that

$$\sum_{i=1}^N \lambda_{ri} \cdot a_i = d_{\bar{B},2} \left(\sum_{\beta} K_{r,\beta}(a_1, \dots, a_N) \otimes L_{r,\beta}(a_1, \dots, a_N) \right) \in I(A), \quad r = 1, \dots, R,$$

and the cycles in bar construction

$$\sum_{\alpha} P_{j,\alpha}(a_j, \dots, a_N) \otimes Q_{j,\alpha}(a_1, \dots, a_N) \in I(A) \otimes I(A), \quad j = 1, \dots, M,$$

generate the k -module $H_2(\bar{B}(A))$. Then the algebra A has a presentation

$$A \cong T(x_1, \dots, x_N) / \left(\sum_{i=1}^N \lambda_{ri} x_i = \sum_{\beta} \bar{K}_{r,\beta} \cdot L_{r,\beta}, r = 1, \dots, R; \sum_{\alpha} \bar{P}_{j,\alpha} \cdot Q_{j,\alpha} = 0, j = 1, \dots, M \right).$$

(Here N, M, R can be infinite of any cardinality.)

Proof of statement (i) Consider the morphism $f : T(x_1, \dots, x_N) \rightarrow A, x_i \mapsto a_i$, of connected algebras. The classes $[a_1], \dots, [a_N]$ generate $H_1(\bar{B}(A))$ and are images of classes $[x_1], \dots, [x_N]$ with respect to the map $f_{*,1} : H_1(\bar{B}(T(x_1, \dots, x_N))) \rightarrow H_1(\bar{B}(A))$. Hence $f_{*,1}$ is surjective. By Lemma A.5(i), f is surjective. □

Proof of statement (ii) Consider the algebra

$$C := T(x_1, \dots, x_N) / \left(\sum_{i=1}^N \lambda_{ri} x_i = \sum_{\beta} \bar{K}_{r,\beta} \cdot L_{r,\beta}, r = 1, \dots, R; \sum_{\alpha} \bar{P}_{j,\alpha} \cdot Q_{j,\alpha} = 0, j = 1, \dots, M \right).$$

The following identities in A are given:

$$\sum_{i=1}^N \lambda_{ri} \cdot a_i = \sum_{\beta} \bar{K}_{r,\beta}(a_1, \dots, a_N) \cdot L_{r,\beta}(a_1, \dots, a_N), \quad 0 = \sum_{\alpha} \bar{P}_{j,\alpha}(a_1, \dots, a_N) \cdot Q_{j,\alpha}(a_1, \dots, a_N).$$

Hence the morphism $f : C \rightarrow A, x_i \mapsto a_i$, is well defined. The induced map $f_{*,1} : H_1(\bar{B}(C)) \rightarrow H_1(\bar{B}(A))$ is surjective, since the elements $[a_i] = f_{*,1}([x_i])$ generate $H_1(\bar{B}(A))$.

We prove that $f_{*,1}$ is injective. Let $\xi \in H_1(\bar{B}(C))$ and $f_{*,1}(\xi) = 0$. By Corollary A.4 and surjectivity of $T(x_1, \dots, x_N) \rightarrow C$, we have $\xi = \sum_{i=1}^N \mu_i \cdot [x_i]$ for some $\mu_i \in k$. Then $0 = f_*(\xi) = \sum_i \mu_i [a_i] \in H_1(\bar{B}(A))$. All linear relations between $[a_1], \dots, [a_N]$ follow from the relations $\sum_i \lambda_{ri} [a_i] = 0$, and therefore $\mu_i = \sum_{r=1}^R c_r \lambda_{ri}$ for some $c_r \in k$. It follows that ξ is represented by the cycle

$$\sum_{i=1}^N \sum_{r=1}^R c_r \lambda_{ri} \cdot x_i = \sum_{r=1}^R c_r \sum_{\beta} \bar{K}_{r,\beta} \cdot L_{r,\beta} = d_{\bar{B},2} \left(\sum_{r=1}^R c_r \sum_{\beta} K_{r,\beta} \otimes L_{r,\beta} \right) \in \bar{B}_1(C).$$

Hence $\xi = 0$. We proved that $f_{*,1}$ is bijective.

The elements $\sum_{\alpha} P_{i,\alpha} \otimes Q_{i,\alpha} \in I(C) \otimes I(C)$ are cycles in $\bar{B}_2(C)$, and their images generate $H_2(\bar{B}(A))$. Hence $f_{*,2} : H_2(\bar{B}(C)) \rightarrow H_2(\bar{B}(A))$ is surjective. Conditions of Lemma A.5(ii) are satisfied, so f is bijective. □

A.4 Bounds on the number of homogeneous generators and relations

Let A be a connected k -algebra. Proposition A.1 gives a lower bound on the number of generators and relations in the homogeneous presentations of A , and Theorem A.6 gives an upper bound. These bounds coincide if k is a principal ideal domain, A is a free k -module, and graded components are finitely generated. We introduce some notation.

Definition A.7 Let M be a finitely generated module over a principal ideal domain k . By the structure theorem of such modules, we have

$$(11) \quad M \simeq k/(d_1) \oplus \cdots \oplus k/(d_r),$$

where $d_1, \dots, d_r \in k$ are noninvertible, and $d_i \mid d_{i+1}$ for all $i = 1, \dots, r - 1$. The number r is determined uniquely, and the elements d_i — uniquely up to a multiplication by an invertible element. Hence,

$$\text{gen } M := r, \quad \text{rel } M := \max\{s : d_s \neq 0\}$$

are well defined. We get a short exact sequence $k^{\text{rel } M} \rightarrow k^{\text{gen } M} \rightarrow M \rightarrow 0$.

Lemma A.8 Let k be a principal ideal domain. Suppose that there exists a short exact sequence $k^A \xrightarrow{f} k^B \rightarrow M \rightarrow 0$ for some $A, B < \infty$. Then $A \geq \text{rel } M$ and $B \geq \text{gen } M$.

Proof We can assume that f is in the Smith normal form, that is, f is represented by a diagonal matrix with diagonal elements d'_1, \dots, d'_s such that $d'_1 \mid d'_2 \mid \cdots \mid d'_s$. Remove all nonzero columns: this preserves cokernel and does not increase A . If d'_i is invertible, remove the i^{th} row and the i^{th} column: this preserves cokernel and diminish A and B by 1. We obtain a diagonal matrix $B' \times A'$ having no zero columns and no invertible elements on diagonal. Hence the cokernel is exactly of the form (11) for $B' = r = \text{gen } M$ and $A' = s = \text{rel } M$. □

Lemma A.9 Let k be a principal ideal domain and $0 \rightarrow k^a \rightarrow k^b \xrightarrow{f} k^c \rightarrow 0$ be an exact sequence of k -modules for some $a, b, c < \infty$. Then $b = a + c$.

Proof We can assume that f is in a Smith normal form. In this basis, f is represented by a diagonal matrix $c \times b$. Since f is surjective, the matrix has no nonzero rows, and all diagonal elements are noninvertible. Hence $\text{Ker } f \simeq k^{b-c}$. We have $k^d \not\cong k^{d'}$ for $d \neq d'$, so $a = b - c$. □

Recall that we consider G -graded algebras that are connected with respect to the \mathbb{Z} -grading given by a map $G \rightarrow \mathbb{Z}$.

Theorem A.10 Let A be a connected associative algebra with unit over a principal ideal domain k . Suppose that k -modules $\text{Tor}_1^A(k, k)_n$ and $\text{Tor}_2^A(k, k)_n$ are finitely generated for all $n \in G$. Then:

- (i) If A is a free k -module, it admits a homogeneous presentation that contains (for every n) precisely $\text{gen } \text{Tor}_1^A(k, k)_n$ generators and $\text{gen } \text{Tor}_2^A(k, k)_n + \text{rel } \text{Tor}_1^A(k, k)_n$ relations of degree n .
 - (ii) If A admits a homogeneous presentation that contains N_n generators and M_n relations of degree n ,
- $$(12) \quad N_n \geq \text{gen } \text{Tor}_1^A(k, k)_n, \quad M_n \geq \text{gen } \text{Tor}_2^A(k, k)_n + \text{rel } \text{Tor}_1^A(k, k)_n.$$

Proof of statement (i) For every n , choose a set of $\text{gen}(\text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n)$ additive generators for the \mathbf{k} -module $\text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n$, a set of $\text{rel}(\text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n)$ linear relations between them, and a set of $\text{gen}(\text{Tor}_2^A(\mathbf{k}, \mathbf{k})_n)$ generators for $\text{Tor}_2^A(\mathbf{k}, \mathbf{k})_n$. These elements are represented by cycles and boundaries in the bar construction. Applying [Theorem A.6](#) to them, we obtain a presentation of required size. \square

Proof of statement (ii) Apply [Proposition A.1](#) and continue the exact sequence to the free resolution of the left A -module \mathbf{k} . It has the form

$$\dots \rightarrow A \otimes \mathbf{k}^M \rightarrow A \otimes \mathbf{k}^N \rightarrow A \xrightarrow{\varepsilon} \mathbf{k} \rightarrow 0.$$

Applying the functor $\mathbf{k} \otimes_A (-)$, we obtain a chain complex of graded \mathbf{k} -modules

$$\dots \rightarrow \mathbf{k}^M \xrightarrow{\partial} \mathbf{k}^N \xrightarrow{0} \mathbf{k} \rightarrow 0,$$

having $\text{Tor}^A(\mathbf{k}, \mathbf{k})$ as homology. Therefore, for some $\partial_n: \mathbf{k}^{M_n} \rightarrow \mathbf{k}^{N_n}$ we have

$$\text{Coker } \partial_n \simeq \text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n, \quad \text{Ker } \partial_n \twoheadrightarrow \text{Tor}_2^A(\mathbf{k}, \mathbf{k})_n.$$

In particular, $\text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n$ is generated by N_n elements, so $N_n \geq \text{gen } \text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n$.

If M_n is infinite, both inequalities [\(12\)](#) are true, since the right side is finite. If M_n is finite, then N_n is finite, since $\text{Coker } \partial_n$ is finitely generated. Thus $\text{Ker } \partial_n \subset \mathbf{k}^{M_n}$, $\text{Im } \partial_n \subset \mathbf{k}^{N_n}$ are submodules of finitely generated free modules, so these modules are free: $\text{Ker } \partial_n \simeq \mathbf{k}^P$, $\text{Im } \partial_n \simeq \mathbf{k}^Q$. We obtain exact sequences

$$\mathbf{k}^P \rightarrow \mathbf{k}^{N_n} \rightarrow \text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n \rightarrow 0, \quad \mathbf{k}^Q \rightarrow \text{Tor}_2^A(\mathbf{k}, \mathbf{k})_n \rightarrow 0, \quad 0 \rightarrow \mathbf{k}^P \rightarrow \mathbf{k}^{M_n} \rightarrow \mathbf{k}^Q \rightarrow 0.$$

Then $N_n \geq \text{gen } \text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n$, $P \geq \text{rel } \text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n$, $Q \geq \text{gen } \text{Tor}_2^A(\mathbf{k}, \mathbf{k})_n$ by [Lemma A.8](#) and $P + Q = M_n$ by [Lemma A.9](#). This proves [\(12\)](#). \square

As a corollary, we obtain a well known result by Wall [\[44, Section 7\]](#):

Corollary A.11 *Let A be a connected associative algebra with unit over a field \mathbf{k} . Then:*

- (i) *A admits a homogeneous presentation that contains (for every n) precisely $\dim_{\mathbf{k}} \text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n$ generators and $\dim_{\mathbf{k}} \text{Tor}_2^A(\mathbf{k}, \mathbf{k})_n$ relations of degree n .*
- (ii) *If A admits a homogeneous presentation that contains N_n generators and M_n relations of degree n , then $N_n \geq \dim_{\mathbf{k}} \text{Tor}_1^A(\mathbf{k}, \mathbf{k})_n$ and $M_n \geq \dim_{\mathbf{k}} \text{Tor}_2^A(\mathbf{k}, \mathbf{k})_n$.* \square

We also obtain a criterion of freeness.

Corollary A.12 [\[32, Proposition 8.5.4\]](#) *Let A be a connected associative algebra with unit over a principal ideal domain \mathbf{k} , which is a free \mathbf{k} -module. The following conditions are equivalent.*

- (a) *A is a free algebra (a tensor algebra on homogeneous generators).*
- (b) *$\text{Tor}_1^A(\mathbf{k}, \mathbf{k})$ is a free \mathbf{k} -module, and $\text{Tor}_2^A(\mathbf{k}, \mathbf{k}) = 0$.*

Proof By Corollary A.4, (a) implies (b). Conversely, suppose that (b) holds. Then

$$\text{rel Tor}_1^A(\mathbf{k}, \mathbf{k}) = \text{gen Tor}_2^A(\mathbf{k}, \mathbf{k}) = 0.$$

By Theorem A.10(i), the algebra A admits a presentation with no relations. Hence A is free. \square

Appendix B Loop homology and extensions of Hopf algebras

Consider a homotopy fibration $F \rightarrow E \xrightarrow{p} B$ of simply connected spaces, such that $\Omega p: \Omega E \rightarrow \Omega B$ admits a homotopy section (ie there is a continuous map $\sigma: \Omega B \rightarrow \Omega E$ that preserves basepoints, and a homotopy $\Omega p \circ \sigma \sim \text{id}_{\Omega B}$). It is well known that then ΩE is homotopy equivalent to $\Omega F \times \Omega B$ (see [7, Proposition A.2; 17, Theorem 5.2]). If \mathbf{k} -homology of these loop spaces is free, we obtain an extension of Hopf algebras $\mathbf{k} \rightarrow H_*(\Omega F; \mathbf{k}) \rightarrow H_*(\Omega E; \mathbf{k}) \rightarrow H_*(\Omega B; \mathbf{k}) \rightarrow \mathbf{k}$. In Theorem B.3 we give a full proof of this folklore result. We consider ordinary loop spaces instead of Moore loop spaces, so that $\Omega(X \times Y) \cong \Omega X \times \Omega Y$ is a strict isomorphism of H-spaces.

We have a natural isomorphism

$$\alpha: \pi_n(A \times B) \xrightarrow{\cong} \pi_n(A) \oplus \pi_n(B), \quad [f] \mapsto [\text{pr}_A \circ f] \oplus [\text{pr}_B \circ f],$$

for any A, B and $n \geq 1$. We denote basepoint inclusion by $\varepsilon: * \rightarrow Y$ and collapse map by $\eta: Y \rightarrow *$.

Lemma B.1 *Let X be a simply connected space and $\mu: \Omega X \times \Omega X \rightarrow \Omega X$ be the composition of loops. Then the following diagram is commutative:*

$$\begin{array}{ccc} \pi_n(\Omega X \times \Omega X) & \xrightarrow{\mu_*} & \pi_n(\Omega X) \\ \alpha \downarrow \cong & \nearrow & \\ \pi_n(\Omega X) \oplus \pi_n(\Omega X) & & \end{array} \quad \begin{array}{l} \\ \\ (x, y) \mapsto x + y \end{array}$$

Proof Let elements $x, y \in \pi_n(\Omega X)$ be represented by maps $f, g: S^n \rightarrow \Omega X$. Consider the element $z = [f \times \eta] + [\eta \times g] \in \pi_n(\Omega X \times \Omega X)$.

The map $\mu \circ (f \times \eta)$ is the composition $S^n \xrightarrow{f} \Omega X \xrightarrow{\text{id} \times \eta} \Omega X \times \Omega X \xrightarrow{\mu} \Omega X$. The composition of two right maps is homotopic to the identity, hence $\mu \circ (f \times \eta) \sim f$. Passing to homotopy groups, we have $\mu_*([f \times \eta]) = x$. Similarly, $\mu_*([\eta \times g]) = y$, hence $\mu_*(z) = x + y$. On the other hand, $\alpha([f \times \eta]) = [\text{pr}_1 \circ (f \times \eta)] \oplus [\text{pr}_2 \circ (f \times \eta)] = [f] \oplus [\eta \varepsilon] = x \oplus 0$. Similarly, $\alpha([\eta \times g]) = 0 \oplus y$, hence $\alpha(z) = x \oplus y$. We obtained $\mu_*(\alpha^{-1}(x \oplus y)) = \mu_*(z) = x + y$, so the diagram commutes. \square

In the following lemma, we say that a diagram commutes if it homotopy commutes.

Lemma B.2 *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of simply connected spaces, and $\sigma: \Omega B \rightarrow \Omega E$ be a homotopy section for Ωp . Consider the composition*

$$f: \Omega F \times \Omega B \xrightarrow{\Omega i \times \sigma} \Omega E \times \Omega E \xrightarrow{\mu} \Omega E.$$

Then:

- (i) f is a weak homotopy equivalence.
- (ii) f respects the inclusion and the projection, that is, the following diagram commutes:

$$\begin{array}{ccccc}
 & & \Omega E & & \\
 & \nearrow \Omega i & \uparrow f & \searrow \Omega p & \\
 \Omega F \times * & \xrightarrow{\text{id} \times \eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon \times \text{id}} & * \times \Omega B
 \end{array}$$

- (iii) f respects the left action of ΩF , that is, the following diagram commutes:

$$\begin{array}{ccc}
 \Omega F \times \Omega F \times \Omega B & \xrightarrow{\Omega i \times f} & \Omega E \times \Omega E \\
 \downarrow \mu \times \text{id} & & \downarrow \mu \\
 \Omega F \times \Omega B & \xrightarrow{f} & \Omega E
 \end{array}$$

- (iv) f respects the right coaction of ΩB , that is, the following diagram commutes:

$$\begin{array}{ccc}
 \Omega F \times \Omega B & \xrightarrow{f} & \Omega E \\
 \downarrow \text{id} \times \Delta & & \downarrow \Delta \\
 \Omega F \times \Omega B \times \Omega B & \xrightarrow{f \times \text{id}} & \Omega E \times \Omega E \\
 & & \downarrow \text{id} \times \Omega p
 \end{array}$$

Proof We have an exact sequence

$$\dots \rightarrow \pi_n(\Omega F) \xrightarrow{(\Omega i)_*} \pi_n(\Omega E) \xrightarrow{(\Omega p)_*} \pi_n(\Omega B) \rightarrow \dots,$$

where the map $(\Omega p)_*$ has a section σ_* . For every $n \geq 1$, we obtain an isomorphism of groups

$$\varphi: \pi_n(\Omega F) \oplus \pi_n(\Omega B) \xrightarrow{\cong} \pi_n(\Omega E), \quad \varphi(x, y) = (\Omega i)_*(x) + \sigma_*(y).$$

(We use that $\pi_1(\Omega X)$ is abelian.) By [Lemma B.1](#) and naturality of

$$\alpha: \pi_n(\Omega F \times \Omega B) \rightarrow \pi_n(\Omega F) \oplus \pi_n(\Omega B)$$

we have

$$\varphi \circ \alpha = (\mu \circ (\Omega i \times \sigma))_* = f_*: \pi_n(\Omega F \times \Omega B) \rightarrow \pi_n(\Omega E).$$

Hence f_* is an isomorphism for all n , so f is a weak homotopy equivalence. Now consider the diagram

$$\begin{array}{ccccc}
 & & \Omega E & \xrightarrow{\Omega p} & \Omega B \\
 & \nearrow \text{id} & \uparrow \mu & & \uparrow \mu \\
 \Omega E \times * & \xrightarrow{\text{id} \times \eta} & \Omega E \times \Omega E & \xrightarrow{\Omega p \times \Omega p} & \Omega B \times \Omega B \\
 \uparrow \Omega i \times \text{id} & & \uparrow \Omega i \times \sigma & & \uparrow \eta \times \text{id} \\
 \Omega F \times * & \xrightarrow{\text{id} \times \eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon \times \text{id}} & * \times \Omega B
 \end{array}$$

The triangle commutes, since η is a homotopy unit in ΩE . The upper right square commutes, since Ωp is a map of H-spaces. The bottom left square commutes, since $\eta_{\Omega E} = \sigma \circ \eta_{\Omega B} : * \rightarrow \Omega E$. Finally, the commutativity of bottom right square is equivalent to the existence of homotopies $\Omega p \circ \Omega i \sim \eta \circ \varepsilon$ and $\Omega p \circ \sigma \sim \text{id}$. The first homotopy exists, since $p \circ i$ is homotopy trivial; the second exists, since σ is a homotopy section for Ωp . Hence the whole diagram is commutative. The right side of the diagram is homotopic to $\text{id} : \Omega B \rightarrow \Omega B$, since η is a homotopy unit in ΩB . We obtain a commutative diagram

$$\begin{array}{ccccc}
 & & \Omega E & \xrightarrow{\Omega p} & \Omega B \\
 & \nearrow \Omega i \times \text{id} & \uparrow f & & \uparrow \text{id} \\
 \Omega F \times * & \xrightarrow{\text{id} \times \eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon \times \text{id}} & * \times \Omega B
 \end{array}$$

that is equivalent to the diagram from (ii). Now consider the diagram

$$\begin{array}{ccccc}
 \Omega F \times \Omega F \times \Omega B & \xrightarrow{\Omega i \times \Omega i \times \sigma} & \Omega E \times \Omega E \times \Omega E & \xrightarrow{\text{id} \times \mu} & \Omega E \times \Omega E \\
 \downarrow \mu \times \text{id} & & \downarrow \mu \times \text{id} & & \downarrow \mu \\
 \Omega F \times \Omega B & \xrightarrow{\Omega i \times \sigma} & \Omega E \times \Omega E & \xrightarrow{\mu} & \Omega E
 \end{array}$$

The left square commutes, since $\Omega i : \Omega F \rightarrow \Omega E$ is a map of H-spaces; the right square commutes, since μ is homotopy associative. The top side of the diagram equals $\Omega i \times (\mu \circ (\Omega i \times \sigma)) = \Omega i \times f$, the bottom side equals f . Hence, it is the diagram from (iii). Finally, consider the diagram

$$\begin{array}{ccccc}
 \Omega F \times \Omega B & \xrightarrow{\Omega i \times \sigma} & \Omega E \times \Omega E & \xrightarrow{\mu} & \Omega E \\
 \downarrow D & & \downarrow D & & \downarrow \Delta \\
 \Omega F \times \Omega B \times \Omega F \times \Omega B & \xrightarrow{\Omega i \times \sigma \times \Omega i \times \sigma} & \Omega E \times \Omega E \times \Omega E \times \Omega E & \xrightarrow{\mu \times \mu} & \Omega E \times \Omega E \\
 \downarrow \text{pr}_{124} & \searrow \Omega i \times \sigma \times \varepsilon \times \text{id} & \downarrow \text{id} \times \text{id} \times \Omega p \times \Omega p & & \downarrow \text{id} \times \Omega p \\
 \Omega F \times \Omega B \times \Omega B & \xrightarrow{\phi} & \Omega E \times \Omega E \times \Omega B \times \Omega B & \xrightarrow{\mu \times \mu} & \Omega E \times \Omega B
 \end{array}$$

where $\Delta(x) := (x, x)$, $D(x, y) := (x, y, x, y)$ and $\phi(f, b_1, b_2) := (\Omega i(f), \sigma(b_1), *, b_2)$. Clearly, the top two squares commute. The bottom right square commutes, since $\text{id} : \Omega E \rightarrow \Omega E$ and $\Omega p : \Omega E \rightarrow \Omega B$ are maps of H-spaces. The upper triangle commutes, since $\Omega p \circ \Omega i \sim \varepsilon$ and $\Omega p \circ \sigma \sim \text{id}$; the bottom triangle commutes by the definition of ϕ . The outer maps in the diagram give the required diagram (iv). \square

In the proof of next theorem we use the Künneth map $\kappa : H_*(X; \mathbf{k}) \otimes H_*(Y; \mathbf{k}) \rightarrow H_*(X \times Y; \mathbf{k})$. It is natural and associative. It is an isomorphism if $H_*(Y; \mathbf{k})$ is a free \mathbf{k} -module.

If X is a simply connected space and $H_*(\Omega X; \mathbf{k})$ is free over \mathbf{k} , this module is a connected \mathbf{k} -Hopf algebra with the standard cup coproduct (see Section 2.4) and the Pontryagin product

$$m : H_*(\Omega X; \mathbf{k}) \otimes H_*(\Omega X; \mathbf{k}) \xrightarrow{\kappa} H_*(\Omega X \times \Omega X; \mathbf{k}) \xrightarrow{\mu_*} H_*(\Omega X; \mathbf{k}).$$

The unit and counit $\mathbf{k} \xrightarrow{\eta} H_*(\Omega X; \mathbf{k}) \xrightarrow{\varepsilon} \mathbf{k}$ are induced by the H-space maps $* \xrightarrow{\eta} \Omega X \xrightarrow{\varepsilon} *$.

Theorem B.3 Let k be an associative ring with unit. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a homotopy fibration of simply connected spaces such that $H_*(\Omega B; k)$ and $H_*(\Omega F; k)$ are free k -modules, and the map Ωp admits a homotopy section $\sigma : \Omega B \rightarrow \Omega E$. Consider the composition

$$\Phi : H_*(\Omega F; k) \otimes H_*(\Omega B; k) \xrightarrow{(\Omega i)_* \otimes \sigma_*} H_*(\Omega E; k) \otimes H_*(\Omega E; k) \xrightarrow{m} H_*(\Omega E; k).$$

Then

- (i) Φ is an isomorphism of k -modules;
- (ii) $(\Omega i)_* = \Phi \circ (\text{id}_{H_*(\Omega F; k)} \otimes \eta_{H_*(\Omega B; k)})$;
- (iii) $(\Omega p)_* \circ \Phi = \varepsilon_{H_*(\Omega F; k)} \otimes \text{id}_{H_*(\Omega B; k)}$;
- (iv) Φ is a morphism of left $H_*(\Omega F; k)$ -modules and right $H_*(\Omega B; k)$ -comodules, where the (co)module structure on $H_*(\Omega E; k)$ is induced by the maps $(\Omega i)_*$ and $(\Omega p)_*$.

In particular, $k \rightarrow H_*(\Omega F; k) \xrightarrow{(\Omega i)_*} H_*(\Omega E; k) \xrightarrow{(\Omega p)_*} H_*(\Omega B; k) \rightarrow k$ is an extension of connected Hopf algebras over k .

Proof We write $H_*(\Omega X)$ instead of $H_*(\Omega X; k)$. Note that σ is continuous, and $\Omega i, \Omega p$ are maps of H-spaces. Hence σ_* is a map of coalgebras, and $(\Omega i)_*, (\Omega p)_*$ are maps of Hopf algebras. By the naturality of Künneth map, the following diagram commutes:

$$\begin{array}{ccccc} H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{(\Omega i)_* \otimes \sigma_*} & H_*(\Omega E) \otimes H_*(\Omega E) & \xrightarrow{m} & H_*(\Omega E) \\ \kappa \downarrow \simeq & & \kappa \downarrow & & \parallel \\ H_*(\Omega F \times \Omega B) & \xrightarrow{(\Omega i \times \sigma)_*} & H_*(\Omega E \times \Omega E) & \xrightarrow{\mu_*} & H_*(\Omega E) \end{array}$$

The top side of diagram equals Φ , and the bottom side equals f_* . Hence Φ is the composition $H_*(\Omega F) \otimes H_*(\Omega B) \xrightarrow{\kappa} H_*(\Omega F \times \Omega B) \xrightarrow{f_*} H_*(\Omega E)$. The left map is bijective by the assumption, the right map is bijective by Lemma B.2(i). Hence Φ is an isomorphism, so (i) is proved. Consider the diagram

$$\begin{array}{ccccc} & & H_*(\Omega E) & & \\ & \nearrow (\Omega i)_* & \uparrow f_* & \searrow (\Omega p)_* & \\ H_*(\Omega F \times *) & \xrightarrow{(\text{id} \times \eta)_*} & H_*(\Omega F \times \Omega B) & \xrightarrow{(\varepsilon \times \text{id})_*} & H_*(* \times \Omega B) \\ \kappa \uparrow \simeq & & \kappa \uparrow & & \kappa \uparrow \simeq \\ H_*(\Omega F) \otimes k & \xrightarrow{\text{id} \otimes \eta} & H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\varepsilon \otimes \text{id}} & k \otimes H_*(\Omega B) \end{array}$$

The top half of the diagram commutes by Lemma B.2(ii), the bottom half commutes by naturality of κ . Since $f_* \circ \kappa = \Phi$, we have a commutative diagram

$$\begin{array}{ccccc} & & H_*(\Omega E) & & \\ & \nearrow (\Omega i)_* & \uparrow \Phi & \searrow (\Omega p)_* & \\ H_*(\Omega F) & \xrightarrow{\text{id} \otimes \eta} & H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\varepsilon \otimes \text{id}} & H_*(\Omega B) \end{array}$$

which proves (ii) and (iii). Now consider the diagram

$$\begin{array}{ccccc}
 H_*(\Omega F) \otimes H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\text{id} \otimes \kappa} & H_*(\Omega F) \otimes H_*(\Omega F \times \Omega B) & \xrightarrow{(\Omega i)_* \otimes f_*} & H_*(\Omega E) \otimes H_*(\Omega E) \\
 \downarrow \kappa \otimes \text{id} & & \downarrow \kappa & & \downarrow \kappa \\
 H_*(\Omega F \times \Omega F) \otimes H_*(\Omega B) & \xrightarrow{\kappa \otimes \text{id}} & H_*(\Omega F \times \Omega F \times \Omega B) & \xrightarrow{(\Omega i \times f)_*} & H_*(\Omega E \times \Omega E) \\
 \downarrow \mu_* \otimes \text{id} & & \downarrow (\mu \times \text{id})_* & & \downarrow \mu_* \\
 H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\kappa} & H_*(\Omega F \times \Omega B) & \xrightarrow{f_*} & H_*(\Omega E)
 \end{array}$$

The bottom right square commutes by Lemma B.2(iii), the other squares commute by naturality of κ . Since $\mu_* \circ \kappa = m: H_*(\Omega X) \otimes H_*(\Omega X) \rightarrow H_*(\Omega X)$, the outer maps in the diagram are

$$\begin{array}{ccc}
 H_*(\Omega F) \otimes H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{(\Omega i)_* \otimes \Phi} & H_*(\Omega E) \otimes H_*(\Omega E) \\
 \downarrow m \otimes \text{id} & & \downarrow m \\
 H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\Phi} & H_*(\Omega E)
 \end{array}$$

Hence Φ is a map of left $H_*(\Omega F)$ -modules. Similarly, by Lemma B.2(iv) and the Künneth isomorphisms we have the commutative diagram

$$\begin{array}{ccc}
 H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\Phi} & H_*(\Omega E) \\
 \downarrow \text{id} \otimes \Delta & & \downarrow \Delta \\
 H_*(\Omega F) \otimes H_*(\Omega B) \otimes H_*(\Omega B) & \xrightarrow{\Phi \otimes \text{id}} & H_*(\Omega E) \otimes H_*(\Omega B) \\
 & & \downarrow \text{id} \otimes (\Omega p)_* \\
 & & H_*(\Omega E) \otimes H_*(\Omega E)
 \end{array}$$

hence Φ is a map of right $H_*(\Omega B)$ -comodules.

Since (i)–(iv) hold, the maps of Hopf algebras $(\Omega i)_*: H_*(\Omega F) \rightarrow H_*(\Omega E)$ and $(\Omega p)_*: H_*(\Omega E) \rightarrow H_*(\Omega B)$ form an extension of Hopf algebras by Proposition 2.4. \square

Recall that an element $x \in A$ of a Hopf algebra is *primitive* if $\Delta x = 1 \otimes x + x \otimes 1$. The set of primitive elements is a Lie subalgebra $PA \subset A$. Every map of Hopf algebras $f: A \rightarrow A'$ induces a map of Lie algebras $Pf := f|_{PA}: PA \rightarrow PA'$.

Corollary B.4 *Suppose that the conditions of Theorem B.3 are met. Let $x \in H_*(\Omega E; \mathbf{k})$ be a primitive element such that $(\Omega p)_*(x) = 0$. Then $x = (\Omega i)_*(y)$ for some $y \in H_*(\Omega F; \mathbf{k})$.*

Proof Since $\mathbf{k} \rightarrow H_*(\Omega F; \mathbf{k}) \rightarrow H_*(\Omega E; \mathbf{k}) \rightarrow H_*(\Omega B; \mathbf{k}) \rightarrow \mathbf{k}$ is a Hopf algebra extension, the sequence $0 \rightarrow PH_*(\Omega F; \mathbf{k}) \rightarrow PH_*(\Omega E; \mathbf{k}) \rightarrow PH_*(\Omega B; \mathbf{k})$ is exact; see [31, Proposition 4.10]. (This also easily follows from definitions). We have

$$x \in \text{Ker}(PH_*(\Omega E; \mathbf{k}) \rightarrow PH_*(\Omega B; \mathbf{k})) = \text{Im}(PH_*(\Omega F; \mathbf{k}) \rightarrow PH_*(\Omega E; \mathbf{k})). \quad \square$$

Appendix C Commutator identities

Fix elements u_1, \dots, u_m of degree 1 in a graded associative algebra Γ . For a subset $I = \{i_1 < \dots < i_k\} \subset [m]$, we define

$$\hat{u}_I := u_{i_1} \cdots u_{i_k}, \quad c(I, x) := [u_{i_1}, [u_{i_2}, [\dots [u_{i_k}, x], \dots]]], \quad x \in \Gamma.$$

We write $A < B$ when $A, B \subset [m]$ and $\max(A) < \min(B)$. If $A < B$, we have $\hat{u}_{A \sqcup B} = \hat{u}_A \cdot \hat{u}_B$ and $c(A \sqcup B, x) = c(A, c(B, x))$. Also, $\hat{u}_\emptyset = 1, c(\emptyset, x) = x$.

Define the Koszul sign by $\theta(A, B) := |\{(a, b) \in A \times B : a > b\}|$. In a graded commutative algebra, we would have $\hat{u}_A \cdot \hat{u}_B = (-1)^{\theta(A, B)} \hat{u}_{A \sqcup B}$ if $A \cap B = \emptyset$. It has the following properties:

- (i) $\theta(A, B) \equiv |A| \cdot |B| + \theta(B, A) \pmod{2}$.
- (ii) If $A_1 \sqcup B_1 < A_2 \sqcup B_2$, then

$$\theta(A_1 \sqcup A_2, B_1 \sqcup B_2) \equiv \theta(A_1, B_1) + \theta(A_2, B_2) + |A_2| \cdot |B_1|.$$

For $I \subset [m], j \in [m]$, we write $I_{<j} = \{i \in I : i < j\}, I_{>j} = \{i \in I : i > j\}$. We also use i as a shortened notation for $\{i\}$.

C.1 Regrouping of monomials

The following formulas can be used to express any monomial on u_1, \dots, u_m as a linear combination of $c_1 \cdots c_s \cdot \hat{u}_B, c_i = c(A_i, u_{j_i}), A_i \neq \emptyset$.

Lemma C.1 *Let $I \subset [m]$, and let $x \in \Gamma$ be homogeneous. Then*

$$(13) \quad \hat{u}_I \cdot x = \sum_{I=A \sqcup B} (-1)^{\theta(A, B) + \deg(x) \cdot |B|} c(A, x) \hat{u}_B.$$

Proof Define $d := \deg(x)$. We induct on $|I|$. The base case $I = \emptyset$ is clear. For the inductive step, let $i = \min(I), I' = I \setminus i$. Then the right-hand side is equal to

$$\begin{aligned} & \sum_{I'=A \sqcup B} (-1)^{\theta(i \sqcup A, B) + d \cdot |B|} c(i \sqcup A, x) \hat{u}_B + \sum_{I'=A \sqcup B} (-1)^{\theta(A, i \sqcup B) + d \cdot |i \sqcup B|} c(A, x) \hat{u}_{i \sqcup B} \\ &= \sum_{I'=A \sqcup B} (-1)^{\theta(A, B) + d \cdot |B|} ([u_i, c(A, x)] + (-1)^{|A| + d} c(A, x) u_i) \cdot \hat{u}_B \\ &= \sum_{I'=A \sqcup B} (-1)^{\theta(A, B) + d \cdot |B|} u_i c(A, x) \cdot \hat{u}_B. \end{aligned}$$

By the inductive hypothesis, this sum is equal to $u_i \cdot \hat{u}_{I'} x = \hat{u}_I \cdot x$. □

Proposition C.2 *Let $I \subset [m], j \in [m]$. Then*

$$\hat{u}_I \cdot u_j = \sum_{\substack{I=A \sqcup B \\ \max(A) > j}} (-1)^{\theta(A, B) + |B|} c(A, u_j) \hat{u}_B + (-1)^{|I_{>j}|} \cdot \left\{ \begin{array}{ll} \hat{u}_{I \sqcup j}, & j \notin I \\ \hat{u}_{I_{<j}} \cdot u_j^2 \cdot \hat{u}_{I_{>j}}, & j \in I \end{array} \right\}.$$

Proof Define $P = I_{\leq j}$, $Q = I_{> j}$. Then $P < Q$, therefore

$$\hat{u}_I = \hat{u}_P \hat{u}_Q, \quad r := \hat{u}_P u_j \hat{u}_Q = \begin{cases} \hat{u}_{I \sqcup \{j\}}, & j \notin I; \\ \hat{u}_{I_{< j}} \cdot u_j^2 \cdot \hat{u}_{I_{> j}}, & j \in I. \end{cases}$$

Apply (13) to $\hat{u}_Q \cdot u_j$, and consider the summand with $A_2 = \emptyset$ separately:

$$\begin{aligned} \hat{u}_I \cdot u_j &= \hat{u}_P \hat{u}_Q u_j = \sum_{Q=A_2 \sqcup B_2} (-1)^{\theta(A_2, B_2) + |B_2|} \hat{u}_P c(A_2, u_j) \hat{u}_{B_2} \\ &= (-1)^{|Q|} \hat{u}_P u_j \hat{u}_Q + \sum_{\substack{Q=A_2 \sqcup B_2 \\ A_2 \neq \emptyset}} (-1)^{\theta(A_2, B_2) + |B_2|} \hat{u}_P c(A_2, u_j) \hat{u}_{B_2}. \end{aligned}$$

Applying (13) to $\hat{u}_P \cdot c(A_2, u_j)$, we obtain the required identity:

$$\begin{aligned} \hat{u}_I \cdot u_j &= (-1)^{|Q|} r + \sum_{P=A_1 \sqcup B_1} \sum_{\substack{Q=A_2 \sqcup B_2 \\ A_2 \neq \emptyset}} (-1)^{\theta(A_1, B_1) + (|A_2| + 1) \cdot |B_1| + \theta(A_2, B_2) + |B_2|} c(A_1, c(A_2, u_j)) \hat{u}_{B_1} \hat{u}_{B_2} \\ &= (-1)^{|Q|} r + \sum_{\substack{P \sqcup Q = A \sqcup B \\ A_{> j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} c(A, u_j) \hat{u}_B. \end{aligned} \quad \square$$

C.2 Identities for nested commutators

In this section Γ can be a Lie superalgebra.

Lemma C.3 For $I \subset [m]$ and homogeneous elements $x, y \in \Gamma$, we have

$$\begin{aligned} (14) \quad c(I, [x, y]) &= \sum_{I=A \sqcup B} (-1)^{\theta(A, B) + \deg(x) \cdot |B|} [c(A, x), c(B, y)] \\ &= [c(I, x), y] + (-1)^{\deg(x) \cdot |I|} [x, c(I, y)] + \sum_{\substack{I=A \sqcup B \\ A, B \neq \emptyset}} (-1)^{\theta(A, B) + \deg(x) \cdot |B|} [c(A, x), c(B, y)]. \end{aligned}$$

Proof The second identity follows from $\theta(\emptyset, I) = \theta(I, \emptyset) = 0$ and $c(\emptyset, x) = x$. Let us prove the first identity by induction on $|I|$. The base case $I = \emptyset$ is clear. For the inductive step, define $i = \min(I)$, $I' = I \setminus i$, $d = \deg(x)$. Then, by the inductive hypothesis,

$$\begin{aligned} c(I, [x, y]) &= [u_i, c(I', [x, y])] = \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', B') + d \cdot |B'|} [u_i, [c(A', x), c(B', y)]] \\ &= \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', B') + d \cdot |B'|} [[u_i, c(A', x)], c(B', y)] \\ &\quad + \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', B') + d \cdot |B'| + d + |A'|} [c(A', x), [u_i, c(B', y)]] \\ &= \sum_{I'=A' \sqcup B'} (-1)^{\theta(i \sqcup A', B') + d \cdot |B'|} [c(i \sqcup A', x), c(B', y)] \\ &\quad + \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', i \sqcup B') + d \cdot |i \sqcup B'|} [c(A', x), c(i \sqcup B', y)] \\ &= \sum_{I=A \sqcup B} (-1)^{\theta(A, B) + d \cdot |B|} [c(A, x), c(B, y)]. \end{aligned} \quad \square$$

Corollary C.4 Let $I \subset [m]$, $I = I'' \sqcup I'$, $I'' < I'$. Let $x, y \in \Gamma$ be homogeneous, and let $\mathcal{A} \subset 2^{I'} \times 2^{I''}$ be a family of pairs of subsets. Then

$$(15) \quad \sum_{\substack{I' = A' \sqcup B' \\ (A', B') \in \mathcal{A}}} (-1)^{\theta(A', B') + |B'|} c(I'', [c(A', x), c(B', y)]) = \sum_{\substack{I = A \sqcup B \\ (A \cap I', B \cap I') \in \mathcal{A}}} (-1)^{\theta(A, B) + |B|} [c(A, x), c(B, y)].$$

Proof This follows from (14) and from identities

$$c(A'', c(A', x)) = c(A'' \sqcup A', x), \quad \theta(A'' \sqcup A', B'' \sqcup B') = \theta(A'', B'') + \theta(A', B') + |A'| \cdot |B''|$$

that are true for $A'', B'' < A', B'$. □

Proposition C.5 Let $J \subset [m]$ and $i, j \in J$ such that $i < j$ and $J_{>j} \neq \emptyset$. Then

$$(16) \quad c(J \setminus ij, [u_i, u_j]) = (-1)^{|J_{>j}|} c(J \setminus i, u_i) - (-1)^{|J_{>i}|} c(J \setminus j, u_j) + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>i}, B_{>j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)].$$

Proof Define $P = J_{<j}$, $Q = J_{>i} \cap J_{<j}$, $R = J_{>j}$. Hence $P < i < Q < j < R$ and $R \neq \emptyset$. The left-hand side is equal to $x := c(P \sqcup Q, c(R, [u_i, u_j]))$. Define also $y := c(P \sqcup Q, [c(R, u_i), u_j])$, $z := c(P \sqcup Q, [u_i, c(R, u_j)])$. Then

$$\begin{aligned} x &= y + (-1)^{|R|} z + \sum_{\substack{R = A' \sqcup B' \\ A', B' \neq \emptyset}} (-1)^{\theta(A', B') + |B'|} c(P \sqcup Q, [c(A', u_i), c(B', u_j)]) \\ &\stackrel{(15)}{=} y + (-1)^{|R|} z + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>j}, B_{>j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)], \\ y &= (-1)^{|R|} c(P \sqcup Q, [u_j, c(R, u_i)]) = (-1)^{|R|} c(J \setminus i, u_i), \\ z &= c(P, c(Q, [u_i, c(R, u_j)])) \stackrel{(14)}{=} c(P, [c(Q, u_i), c(R, u_j)]) + (-1)^{|Q|} \underbrace{c(P, [u_i, c(Q \sqcup R, u_j)])}_{=c(J \setminus j, u_j)} \\ &\quad + \sum_{\substack{Q = A_2 \sqcup B_2 \\ A_2, B_2 \neq \emptyset}} (-1)^{\theta(A_2, B_2) + |B_2|} c(P, [c(A_2, u_i), c(B_2 \sqcup R, u_j)]) \\ &\stackrel{(15)}{=} (-1)^{|R|} \sum_{\substack{J \setminus ij = A \sqcup B \\ B \cap Q = \emptyset \\ A_{>j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)] \\ &\quad + (-1)^{|Q|} c(J \setminus j, u_j) + (-1)^{|R|} \sum_{\substack{J \setminus ij = A \sqcup B \\ B \cap Q \neq \emptyset, Q \\ A_{>j} = \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)]. \end{aligned}$$

Therefore,

$$\begin{aligned} x &= (-1)^{|R|} c(J \setminus i, u_i) + (-1)^{|Q| + |R|} c(J \setminus j, u_j) \\ &\quad + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>i} \neq \emptyset \\ A_{>j} = \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)] + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>j}, B_{>j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)]. \end{aligned}$$

In the first sum the condition $B_{>j} \neq \emptyset$ is always true, since $R = A_{>j} \sqcup B_{>j}$, $A_{>j} = \emptyset$ and $R \neq \emptyset$. In the second sum, $A_{>i} \neq \emptyset$ is always true. Hence the sums can be merged into

$$\sum_{\substack{J \setminus \{j\} = A \sqcup B \\ A_{>i}, B_{>j} \neq \emptyset}} (-1)^{\theta(A,B) + |B|} [c(A, u_i), c(B, u_j)].$$

Using $|R| = |J_{>j}|$ and $|Q| + |R| = |J_{>i}| - 1$, we obtain (16). \square

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Steklov Mathematical Institute of Russian Academy of Sciences
Moscow, Russia

vylegf@gmail.com

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