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Algebraicity in monochromatic homotopy theory

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Using Patchkoria and Pstrągowski's version of Franke's algebraicity theorem, we prove that the category of $K_p(n)$-local spectra is exotically equivalent to the category of derived I_n-complete periodic comodules over the Adams Hopf algebroid (E_*, E_*E) for large primes. This gives a finite prime result analogous to the asymptotic algebraicity for $\mathrm{Sp}_{K_p(n)}$ of Barthel, Schlank, and Stapleton.

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1 Introduction

The central idea in chromatic homotopy theory is to study the symmetric monoidal stable ∞-category of spectra, Sp, via its smaller building blocks. These are the categories $\mathrm{Sp}_{n,p}$ and $\mathrm{Sp}_{K_p(n)}$ of $E_{n,p}$-local and $K_p(n)$-local spectra, where $E = E_{n,p}$ is Morava E-theory, and $K_p(n)$ is Morava K-theory; see, for example, [26]. These categories depend on a prime p and an integer n, called the height. For a fixed height n, increasing the prime p makes both categories behave more algebraically. This manifests itself, for example, in the E-Adams spectral sequence of signature

$$E_2^{s,t}(L_n\mathbb{S}) = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*) \Rightarrow \pi_{t-s} L_n\mathbb{S}$$

computing the homotopy groups of the E-local sphere. By the smash product theorem of Ravenel, see [40, 7.5.6], this spectral sequence has a horizontal vanishing line at a finite page. If $p > n+1$, this vanishing line appears already on the second page, where the information is completely described by the homological algebra of Comod_{E_*E} — the Grothendieck abelian category of comodules over the Hopf algebroid (E_*, E_*E).

Increasing the prime p correspondingly increases the distance between objects appearing in the E-Adams spectral sequence. When $2p-2$ exceeds n^2+n, there is no longer room for any differentials, and the spectral sequence in fact collapses to an isomorphism

$$\pi_* L_n \mathbb{S} \cong \mathrm{Ext}^{*,*}_{E_*E}(E_*, E_*),$$

for degree reasons. In other words, the homotopy groups are completely algebraic in this range.

A natural question to ask is whether this collapse is a feature solely of the E-Adams spectral sequence or if it is a feature of the category $\mathrm{Sp}_{n,p}$. More precisely, is the entire category of E-local spectra algebraic, in the sense that it is equivalent to a derived category of an abelian category, whenever $2p-2 > n^2+n$? What about the category of $K_p(n)$-local spectra?

At height $n=0$, both categories $\mathrm{Sp}_{n,p}$ and $\mathrm{Sp}_{K_p(n)}$ are the category of rational spectra $\mathrm{Sp}_{\mathbb{Q}}$, which can be seen to be equivalent to the derived ∞-category of rational vector spaces, but at positive heights $n>0$, there can never be an equivalence of ∞-categories $\mathrm{Sp}_{n,p} \simeq \mathrm{D}(\mathcal{A})$ or $\mathrm{Sp}_{K_p(n)} \simeq \mathrm{D}(\mathcal{A})$.

However, in [16] Bousfield showed that for $p>2$ and $n=1$, that there is an equivalence of homotopy categories

$$h\mathrm{Sp}_{1,p} \simeq h\mathrm{Fr}_{1,p},$$

where $\mathrm{Fr}_{n,p}$ is a certain derived ∞-category of twisted comodules over (E_*, E_*E). As this cannot be lifted to an equivalence of ∞-categories, it is sometimes referred to as an *exotic* equivalence.

Franke expanded upon this in [19] by conjecturing — and attempting to prove — that for $2p-2 > n^2+n$ there should be an equivalence of homotopy categories

$$h\mathrm{Sp}_{n,p} \simeq h\mathrm{Fr}_{n,p}.$$

Unfortunately, a subtle error was discovered in the proof by Patchkoria in [35], but the result was recovered in [37] with a slightly worse bound: $2p-2 > 2n^2+2n$. Pstrągowski also proved that this equivalence gets "stronger" the larger the prime, where we not only get an equivalence of categories but an equivalence of k-categories

$$h_k\mathrm{Sp}_{n,p} \simeq h_k\mathrm{Fr}_{n,p},$$

for $k = 2p-2-2n^2-2n$. Here $h_k\mathcal{C}$ denotes taking the homotopy k-category, given by $(k-1)$-truncating the mapping spaces in $\mathcal{C}$. At $k=1$, this gives the classical situation of taking the homotopy category $h\mathcal{C}$. Using and developing a more general machinery, Pstrągowski and Patchkoria proved in [36] that the above equivalence holds in Franke's conjectured bound, $2p-2 > n^2+n$.

The current belief is that these bounds are optimal. We know this to be true at the prime 2, as Roitzheim proved in [41] that the category $\mathrm{Sp}_{1,2}$ is *rigid*, in the sense that any equivalence of homotopy categories $h\mathrm{Sp}_{1,2} \simeq h\mathcal{C}$ lifts to an equivalence $\mathrm{Sp}_{1,2} \simeq \mathcal{C}$. The $K_p(n)$-local analogue of Roitzheim's result also holds, as Ishak proved in [29] that $\mathrm{Sp}_{K_2(1)}$ is rigid as well. Hence, exotic equivalences for $\mathrm{Sp}_{n,p}$ or $\mathrm{Sp}_{K_p(n)}$ can only exist at primes that are large compared to the height.

The above results imply that increasing the prime p decreases how destructive the k-truncation of the mapping spaces needs to be. In the limit $p \to \infty$, we might expect that there is no need to truncate at all, giving an equivalence of ∞-categories. But, there needs to be an appropriate notion of what "going to the infinite prime" should be. In [11], the authors use a notion of ultraproducts over a nonprincipal ultrafilter of primes, $\mathcal{F}$, to formalize this limiting process. They use this to prove the existence of a symmetric monoidal equivalence of ∞-categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{n,p} \simeq \prod_{\mathcal{F}} \mathrm{Fr}_{n,p}.$$

Expanding on their work, Barthel, Schlank, and Stapleton proved in [12] a $K_p(n)$-local version of the above result. More precisely, they show that there is a symmetric monoidal equivalence of ∞-categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{K_p(n)} \simeq \prod_{\mathcal{F}} \widehat{\mathrm{Fr}}_{n,p},$$

where the right-hand side consists of derived complete twisted comodules for the naturally occurring Landweber ideal $I_n \subseteq E_*$.

Statement of results

We can summarize the most general of the above algebraicity results in the table

	$p < \infty$	$p \to \infty$
$\mathrm{Sp}_{n,p}$	[36]	[11]
$\mathrm{Sp}_{K_p(n)}$		[12]

A natural question arises: is there a finite prime exotic algebraicity for $\mathrm{Sp}_{K_p(n)}$? The goal of this paper is to give an affirmative answer. More precisely, we prove the following.

Theorem A (Theorem 4.15) *Let p be a prime and $n \in \mathbb{N}$. If $k = 2p-2-n^2-n > 0$, then there is an equivalence of k-categories*

$$h_k\mathrm{Sp}_{K_p(n)} \simeq h_k\widehat{\mathrm{Fr}}_{n,p}.$$

In other words, $K_p(n)$-local spectra are exotically algebraic at large primes.

The available tools for proving such a statement require an abelian category with enough injective objects admitting lifts to a stable ∞-category. In lack of such a well-behaved abelian approximation for $\mathrm{Sp}_{K_p(n)}$, we take inspiration from [12] and instead use the dual category $\mathcal{M}_{n,p}$ of monochromatic spectra, which we show has the needed properties. Theorem A will then follow from the following result.

Theorem B (Theorem 4.13) *Let p be a prime and $n \in \mathbb{N}$. If $k = 2p-2-n^2-n > 0$, then there is an equivalence of k-categories $h_k\mathcal{M}_{n,p} \simeq h_k\mathrm{Fr}_{n,p}^{I_n\text{-tors}}$.*

In order to prove Theorem B, we first prove the analogous statement for monochromatic E-modules.

Theorem C (Theorem 4.5) *Let p be a prime and $n \in \mathbb{N}$. If $k = 2p-2-n > 0$, then there is an equivalence of k-categories $h_k\mathrm{Mod}_E^{I_n\text{-tors}} \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n\text{-tors}})$.*

Overview of the paper

Section 1 introduces local duality, and the proposed exotic algebraic model using periodic chain complexes of torsion comodules. Section 3 focuses on Franke's algebraicity theorem. Most of the new results of the paper are presented in Sections 4.1 and 4.2, where we prove Theorems A, B and C. In Section 4.2 we prove that Barr–Beck adjunctions interact well with local duality, which is used to prove that periodization, torsion and taking the derived category all commute.

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2 The algebraic model

The goal of this section is to set up the necessary background material that will be used throughout the paper. We use these to construct convenient algebraic approximations of categories arising from chromatic homotopy theory.

Some conventions

We freely use the language of ∞-categories, as developed by Joyal [31] and Lurie [32; 33]. Even though we are dealing with both classical 1-categories and ∞-categories in this paper, we will sometimes refer to them both as *categories*, hoping that the prefix is clear from the context.

We denote by $\mathrm{Pr}_{\mathrm{st}}^L$ the ∞-category of presentable stable ∞-categories and colimit preserving functors. Together with the Lurie tensor product, it is a symmetric monoidal ∞-category. The category of algebras $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$ is then the category of presentable stable ∞-categories with a symmetric monoidal structure commuting with colimits separately in each variable.

Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$. A localization is a functor $f\colon \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint i. We denote the composite by $L = i \circ f$. The adjoint i identifies $\mathcal{D}$ with a full subcategory of $\mathcal{C}$, which we denote by $\mathcal{C}_L$. We then view L as a functor $L\colon \mathcal{C} \to \mathcal{C}_L$, that is left adjoint to the inclusion, and by abuse of notation also call these localizations.

2.1 Local duality

The theory of abstract local duality, proved in [25] and generalized to the ∞-categorical setting in [8], will be important for the entire paper. In particular, it will allow us to translate Theorem B into Theorem A.

Definition 2.1 A pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ is compactly generated by dualizable objects, and $\mathcal{K}$ is a subset of compact objects, is called a *local duality context*.

Construction 2.2 Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We define $\mathcal{C}^{\mathcal{K}\text{-tors}}$ to be the localizing tensor ideal generated by $\mathcal{K}$, denoted by $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$. Further we define $\mathcal{C}^{\mathcal{K}\text{-loc}}$ to be the left orthogonal complement $(\mathcal{C}^{\mathcal{K}\text{-tors}})^{\perp}$, ie, the full subcategory consisting of objects $C \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(T, C) \simeq 0$ for all $T \in \mathcal{C}^{\mathcal{K}\text{-tors}}$. Similarly, define $\mathcal{C}^{\mathcal{K}\text{-comp}}$ to be the left-orthogonal complement of $\mathcal{C}^{\mathcal{K}\text{-loc}}$, ie $\mathcal{C}^{\mathcal{K}\text{-comp}} = (\mathcal{C}^{\mathcal{K}\text{-loc}})^{\perp}$. These full subcategories are, respectively, called the $\mathcal{K}$-torsion, $\mathcal{K}$-local and $\mathcal{K}$-complete objects in $\mathcal{C}$. We have inclusions into $\mathcal{C}$, denoted by $i_{\mathcal{K}\text{-tors}}$, $i_{\mathcal{K}\text{-loc}}$ and $i_{\mathcal{K}\text{-comp}}$, respectively.

By the adjoint functor theorem, [32, 5.5.2.9], the inclusions $i_{\mathcal{K}\text{-loc}}$ and $i_{\mathcal{K}\text{-comp}}$ have left adjoints $L_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$, respectively, while $i_{\mathcal{K}\text{-tors}}$ and $i_{\mathcal{K}\text{-loc}}$ have right adjoints $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$, respectively. These are then, by definition, localizations and colocalizations. Since the torsion, local and complete objects are ideals, these localizations and colocalizations are compatible with the symmetric monoidal structure of $\mathcal{C}$, in the sense of [33, 2.2.1.7]. In particular, by [33, 2.2.1.9] we get unique induced symmetric monoidal structures such that $L_{\mathcal{K}}$, $\Lambda_{\mathcal{K}}$, $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$ are symmetric monoidal functors.

For any $X \in \mathcal{C}$, these functors assemble into two cofiber sequences:

$$\Gamma_{\mathcal{K}} X \to X \to L_{\mathcal{K}} X \quad \text{and} \quad V_{\mathcal{K}} X \to X \to \Lambda_{\mathcal{K}} X.$$

Also, these functors only depend on the localizing subcategory $\mathcal{C}^{\mathcal{K}\text{-tors}}$, not on the particular choice of generators $\mathcal{K}$. Thus, when the set $\mathcal{K}$ is clear from the context, we often omit it as a subscript when writing the functors.

The next theorem is a slightly restricted version of the abstract local duality theorem of [8, 2.21; 25, 3.3.5].

Theorem 2.3 *Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. Then*

(1) *the functors Γ and L are* ***smashing****, meaning that there are natural equivalences $\Gamma X \simeq X \otimes \Gamma \mathbb{1}$ and $LX \simeq X \otimes L\mathbb{1}$,*

(2) *the functors* Λ *and* V *are* ***cosmashing****, meaning there are natural equivalences* $\Lambda X \simeq \underline{\mathrm{Hom}}(\Gamma\mathbb{1}, X)$ *and* $VX \simeq \underline{\mathrm{Hom}}(L\mathbb{1}, X)$*, and*

(3) *the functors* $\Gamma\colon \mathcal{C}^{\mathcal{K}\text{-comp}} \to \mathcal{C}^{\mathcal{K}\text{-tors}}$ *and* $\Lambda\colon \mathcal{C}^{\mathcal{K}\text{-tors}} \to \mathcal{C}^{\mathcal{K}\text{-comp}}$ *are mutually inverse symmetric monoidal equivalences of categories.*

This can be summarized by the following diagram of adjoints:

$$\begin{tikzcd}
 & \mathcal{C}^{\mathcal{K}\text{-loc}} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathcal{C} \arrow[u, shift left=2, "L"'] \arrow[u, shift right=2, "V"] \arrow[dl, shift left] \arrow[dr, shift left, "\Lambda"] & \\
\mathcal{C}^{\mathcal{K}\text{-tors}} \arrow[ur, shift left, "\Gamma"'] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathcal{C}^{\mathcal{K}\text{-comp}} \arrow[ul, shift left]
\end{tikzcd}$$

Remark 2.4 Theorem 2.3 implies, in particular, that the symmetric monoidal structure induced by the localization L and the colocalization Γ is just the symmetric monoidal structure on $\mathcal{C}$ restricted to the full subcategories. This is not the case for $\mathcal{C}^{\mathcal{K}\text{-comp}}$, where the symmetric monoidal structure is given by $\Lambda(-\otimes_{\mathcal{C}}-)$. The functor V also induces a symmetric monoidal structure on $\mathcal{C}^{\mathcal{K}\text{-loc}}$, but this coincides with the one induced by L, due to their associated endofunctors on $\mathcal{C}$ defining an adjoint symmetric monoidal monad-comonad pair. We will not need or focus on the functor V, hence it will be omitted from the local duality diagrams for the rest of the paper.

We have two main examples of interest for this paper.

Example 2.5 Let (A, Ψ) be an Adams-type Hopf algebroid, for example, the Hopf algebroid (R_*, R_*R) for an Adams-type ring spectrum R — see [24; 39, A.1] for details. Denote by $\mathrm{D}(\Psi)$ the derived ∞-category associated to the symmetric monoidal Grothendieck abelian category Comod_Ψ. This is defined using the model structure from [4]. If $I \subseteq A$ is a finitely generated invariant regular ideal, then $(\mathrm{D}(\Psi), A/I)$ is a local duality context, with associated local duality diagram

$$\begin{tikzcd}
 & \mathrm{D}(\Psi)^{I\text{-loc}} \arrow[d, shift left] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{D}(\Psi) \arrow[u, shift left, "L_I^\Psi"] \arrow[dl, shift left] \arrow[dr, shift left, "\Delta_I^\Psi"] & \\
\mathrm{D}(\Psi)^{I\text{-tors}} \arrow[ur, shift left, "\Gamma_I^\Psi"'] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathrm{D}(\Psi)^{I\text{-comp}} \arrow[ul, shift left]
\end{tikzcd}$$

In Section 2.2 we compare $\mathrm{D}(\Psi)^{I\text{-tors}}$ to a more concrete category: the derived category of I-power torsion comodules.

The following example comes from chromatic homotopy theory. For a good introduction, see [5].

Example 2.6 Let E denote Morava E-theory at prime p and height n. If $F(n)$ is a finite type-n spectrum, then the pair $(\mathrm{Sp}_{n,p}, L_n F(n))$ is a local duality context. The corresponding diagram can be recognized as

$$\begin{tikzcd}
 & \mathrm{Sp}_{n-1,p} \arrow[d, shift left, "L_{n-1}"'] \arrow[ddr, dashed, bend left] & \\
 & \mathrm{Sp}_{n,p} \arrow[u, shift left, hook] \arrow[dl, shift left, "M_{n,p}"] \arrow[dr, shift left, "L_{K_p(n)}"] & \\
\mathcal{M}_{n,p} \arrow[ur, shift left, hook] \arrow[rr, "\simeq"] \arrow[uur, dashed, bend left] & & \mathrm{Sp}_{K_p(n)} \arrow[ul, shift left, hook]
\end{tikzcd}$$

where $\mathcal{M}_{n,p}$ is the height-n monochromatic category and $\mathrm{Sp}_{K_p(n)}$ is the category of spectra localized at height-n Morava K-theory $K_p(n)$. The functor L_{n-1} is the Bousfield localization at E_{n-1}, while $L_{K_p(n)}$ is the Bousfield localization at $K_p(n)$; see [15]. The local duality then exhibits the classical equivalence $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{K_p(n)}$; see [26, 6.19].

Remark 2.7 There is also a version of this local duality diagram for modules over E, see [21, 4.2, 5.1], or alternatively [8, 3.7] for a version more similar to the above. This gives equivalences

$$\mathcal{M}_{n,p}\mathrm{Mod}_E \simeq \mathrm{Mod}_E^{I_n\text{-tors}} \simeq \mathrm{Mod}_E^{I_n\text{-comp}} \simeq L_{K_p(n)}\mathrm{Mod}_E,$$

where I_n is the Landweber ideal $(p, v_1, \dots, v_{n-1}) \subseteq E_*$.

2.2 The periodic derived torsion category

In this section we identify the category $\mathrm{D}(\Psi)^{I\text{-tors}}$ — as obtained in Example 2.5 — as the derived category of I-power torsion comodules. We also modify the category to exhibit some needed periodicity.

Definition 2.8 Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. The *I-power torsion* of a comodule M is defined as

$$T_I^\Psi M = \{x \in M \mid I^k x = 0 \text{ for some } k \in \mathbb{N}\}.$$

We say a comodule M is *I-torsion* if the natural map $T_I^\Psi M \to M$ is an equivalence.

Remark 2.9 One can similarly define I-power torsion A-modules. If (A, Ψ) is an Adams Hopf algebroid, a Ψ-comodule M is I-power torsion if and only if its underlying module is I-power torsion; see [8, 5.7].

Remark 2.10 By [8, 5.10] the full subcategory of I-torsion comodules, which we denote by $\mathrm{Comod}_\Psi^{I\text{-tors}}$, is a Grothendieck abelian category. It also inherits a symmetric monoidal structure from Comod_Ψ.

The following technical lemma will be needed later.

Lemma 2.11 *Let (A, Ψ) be an Adams Hopf algebroid, where A is noetherian and $I \subseteq A$ a regular invariant ideal. Then $\mathrm{Comod}_\Psi^{I\text{-tors}}$ is generated under filtered colimits by the compact I-power torsion comodules.*

Proof By [9, 3.4] $\mathrm{Comod}_\Psi^{I\text{-tors}}$ is generated by the set

$$\mathrm{Tors}_\Psi^{\mathrm{fp}} := \{G \otimes A/I^k \mid G \in \mathrm{Comod}_\Psi^{\mathrm{fp}}, k \geqslant 1\},$$

where $\mathrm{Comod}_\Psi^{\mathrm{fp}}$ is the full subcategory of dualizable Ψ-comodules. Since I is finitely generated and regular, A/I^k is finitely presented as an A-module, hence it is compact in Comod_Ψ by [24, 1.4.2], and in $\mathrm{Comod}_\Psi^{I\text{-tors}}$ as colimits are computed in Comod_Ψ. As A is compact, being finitely generated is equivalent to being finitely presented. The tensor product of finitely generated modules is finitely generated, hence any element in $\mathrm{Tors}_\Psi^{\mathrm{fp}}$ is compact. □

Remark 2.12 The assumption that the ring A is noetherian can most likely be removed, but it makes no difference to the results in this paper.

Notation 2.13 Since $\mathrm{Comod}_\Psi^{I\text{-tors}}$ is Grothendieck abelian we have an associated derived stable ∞-category $\mathrm{D}(\mathrm{Comod}_\Psi^{I\text{-tors}})$ which we denote simply by $\mathrm{D}(\Psi^{I\text{-tors}})$.

We can now compare the torsion category obtained from local duality and the derived category of I-power torsion comodules.

Lemma 2.14 [9, 3.7(2)] *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. There is an equivalence of categories*

$$\mathrm{D}(\Psi)^{I\text{-tors}} \simeq \mathrm{D}(\Psi^{I\text{-tors}}).$$

Furthermore, an object $M \in \mathrm{D}(\Psi)$ is I-torsion if and only if the homology groups H_*M are I-power torsion Ψ-comodules.

In order to state both the general algebraicity machinery of [36] and our results, we need the respective derived categories to exhibit the periodic nature of the spectra we are interested in. This is done via the periodic derived category. There are several ways to constructing this, but we follow [19] in spirit, using periodic chain complexes.

Definition 2.15 Let $\mathcal{A}$ be an abelian category with a local grading, ie, an auto-equivalence $T : \mathcal{A} \to \mathcal{A}$, and denote by $[1]$ the shift functor on the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ in $\mathcal{A}$. A chain complex $C \in \mathrm{Ch}(\mathcal{A})$ is called *periodic* if there is an isomorphism $\phi : C[1] \to TC$. The full subcategory of periodic chain complexes is denoted by $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A})$.

Definition 2.16 The forgetful functor $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ has a left adjoint P, called the *periodization*.

Definition 2.17 Let $\mathcal{A}$ be a locally graded abelian category. Then the *periodic derived category* of $\mathcal{A}$, denoted by $\mathrm{D}^{\mathrm{per}}(\mathcal{A})$, is the ∞-category obtained by localizing $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A})$ at the quasi-isomorphism. It is in fact stable by [36, 7.8].

Remark 2.18 If $\mathcal{A}$ is a symmetric monoidal category, then $P\mathbb{1}$ is a commutative ring object called the *periodic unit*. By [4, 2.3] the category of periodic chain complexes $\mathrm{Ch}^{\mathrm{per}}(\mathcal{A})$ is equivalent to $\mathrm{Mod}_{P\mathbb{1}}(\mathrm{Ch}(\mathcal{A}))$. This descends also to the derived categories, giving an equivalence

$$\mathrm{D}^{\mathrm{per}}(\mathcal{A}) \simeq \mathrm{Mod}_{P\mathbb{1}}(\mathrm{D}(\mathcal{A}));$$

see, for example, [37, 3.7].

We will also need local duality for the periodic derived category associated to a Hopf algebroid.

Construction 2.19 Let (A,Ψ) be an Adams-type (graded) Hopf algebroid. Then the shift functor $[1]\colon \mathrm{Comod}_\Psi \to \mathrm{Comod}_\Psi$ defined by $(TM)_k = M_{k-1}$ is a local grading on Comod_Ψ. Denote the corresponding periodic derived category by $\mathrm{D}^{\mathrm{per}}(\Psi)$. The pair $(\mathrm{D}^{\mathrm{per}}(\Psi), P(A/I))$ is a local duality context with associated local duality diagram

$$\begin{tikzcd}
& \mathrm{D}^{\mathrm{per}}(\Psi)^{I\text{-loc}} \arrow[d, shift left, hook] \arrow[ddr, dashed, bend left] & \\
& \mathrm{D}^{\mathrm{per}}(\Psi) \arrow[u, shift left, "L_I^\Psi"] \arrow[dl, shift left, "\Gamma_I^\Psi"] \arrow[dr, shift left, "\Lambda_I^\Psi"] & \\
\mathrm{D}^{\mathrm{per}}(\Psi)^{I\text{-tors}} \arrow[ur, shift left, hook] \arrow[uur, dashed, bend left] \arrow[rr, "\simeq"] & & \mathrm{D}^{\mathrm{per}}(\Psi)^{I\text{-comp}} \arrow[ul, shift left, hook]
\end{tikzcd}$$

The functors in the diagram are induced by the functors from Example 2.5. In fact, there is a diagram

$$\begin{tikzcd}
\mathrm{D}(\Psi)^{I\text{-tors}} \arrow[r, shift left] \arrow[d, shift right, "P"'] & \mathrm{D}(\Psi) \arrow[l, shift left, "\Gamma_I^\Psi"] \arrow[r, shift left, "L_I^\Psi"] \arrow[d, shift right, "P"'] & \mathrm{D}(\Psi)^{I\text{-loc}} \arrow[l, shift left] \arrow[d, shift right, "P"'] \\
\mathrm{D}^{\mathrm{per}}(\Psi)^{I\text{-tors}} \arrow[r, shift left] \arrow[u, shift right] & \mathrm{D}^{\mathrm{per}}(\Psi) \arrow[l, shift left, "\Gamma_I^\Psi"] \arrow[r, shift left, "L_I^\Psi"] \arrow[u, shift right] & \mathrm{D}^{\mathrm{per}}(\Psi)^{I\text{-loc}} \arrow[l, shift left] \arrow[u, shift right]
\end{tikzcd}$$

that is commutative in all possible directions. Here the unmarked horizontal arrows are the respective fully faithful inclusions.

Remark 2.20 In the specific case of $(A,\Psi) = (E_0, E_0E)$ and $I \subseteq E_0$ the Landweber ideal I_n, the above construction is [12, 3.12].

There is now some ambiguity to take care of for our category of interest $\mathrm{D}^{\mathrm{per}}(\Psi)^{I\text{-tors}}$. In the picture above, we do mean that we take I-torsion objects in $\mathrm{D}^{\mathrm{per}}(\Psi)$, ie, $[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I\text{-tors}}$, but we could also take the periodization of the category $\mathrm{D}(\Psi^{I\text{-tors}})$ as our model. Luckily, there is no choice, as they are equivalent. This can be thought of as the periodic version of Lemma 2.14.

Theorem 2.21 *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a finitely generated invariant regular ideal. Then there is an equivalence of stable ∞-categories*

$$[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I\text{-tors}} \simeq \mathrm{D}^{\mathrm{per}}(\Psi^{I\text{-tors}}).$$

The proof of this uses the fact that Barr–Beck adjunctions commute with local duality. Proving this here disrupts the flow of the paper, so we defer it to Section 4.2.

Proof As Comod_Ψ is symmetric monoidal we have by Remark 2.18 an equivalence

$$\mathrm{D}^{\mathrm{per}}(\Psi) \simeq \mathrm{Mod}_{P\mathbb{1}}(\mathrm{D}(\Psi)),$$

coming from the periodicity Barr–Beck adjunction. By Theorem A.6 this induces a Barr–Beck adjunction on the torsion subcategories, which gives an equivalence

$$[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I\text{-tors}} \simeq \mathrm{Mod}_{\Gamma_I^\Psi(P\mathbb{1})}(\mathrm{D}(\Psi)^{I\text{-tors}}).$$

Since Γ_I^Ψ is a smashing colocalization, and P is given by tensoring with $P(\mathbb{1})$, they do in fact commute. By Lemma 2.14 we have $\mathrm{D}(\Psi)^{I\text{-tors}} \simeq \mathrm{D}(\Psi^{I\text{-tors}})$, hence the above equivalence can be rewritten as

$$[\mathrm{D}^{\mathrm{per}}(\Psi)]^{I\text{-tors}} \simeq \mathrm{Mod}_{P(\Gamma_I^\Psi\mathbb{1})}(\mathrm{D}(\Psi^{I\text{-tors}})).$$

Now, also $\mathrm{Comod}_\Psi^{I\text{-tors}}$ is symmetric monoidal, so Remark 2.18 gives an equivalence

$$\mathrm{D}^{\mathrm{per}}(\Psi^{I\text{-tors}}) \simeq \mathrm{Mod}_{P(\Gamma_I^\Psi\mathbb{1})}(\mathrm{D}(\Psi^{I\text{-tors}})). \qquad \square$$

3 Exotic algebraic models

We now have two sets of local duality diagrams, one coming from chromatic homotopy theory, see Example 2.6, and one from the homological algebra of Adams Hopf algebroids, see Example 2.5. We can also pass between these duality theories, by using homology theories. In particular, if we let $E = E_n$ be height-n Morava E-theory at a prime p, then we have the E-homology functor $E_*\colon \mathrm{Sp}_{n,p} \to \mathrm{Comod}_{E_*E}$ converting between homotopy theory and algebra. We can, in some sense, say that E_* approximates homotopical information by algebraic information.

The goal of this section is to set up an abstract framework for studying how good such approximations are. The version we recall below was developed in [36], taking inspiration from [19; 38].

3.1 Adapted homology theories

Adapted homology theories are particularly well-behaved homology theories that have associated Adams-type spectral sequences giving computational benefits over other homology theories.

Definition 3.1 Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category and $\mathcal{A}$ an abelian category with a local grading [1]. A functor $H\colon \mathcal{C} \to \mathcal{A}$ is called a *conservative homology theory* if:

(1) H is additive.

(2) For a cofiber sequence $X \to Y \to Z$ in $\mathcal{C}$, the sequence $HX \to HY \to HZ$ is exact in $\mathcal{A}$.

(3) There is a natural isomorphism $H(\Sigma X) \cong (HX)[1]$ for any $X \in \mathcal{C}$.

(4) H reflects isomorphisms.

Remark 3.2 The first two axioms make H a homological functor, the third makes H into a locally graded functor, ie, a functor that preserves the local grading, and the last makes it a conservative functor.

Example 3.3 Let R be a ring spectrum. Then the functor $\pi_*\colon \mathrm{Mod}_R \to \mathrm{Mod}_{R_*}$ defined as $\pi_* M = [\mathbb{S}, M]_*$ is a conservative homology theory.

Example 3.4 Let R be a ring spectrum. The functor $R_*(-)\colon \mathrm{Sp} \to \mathrm{Mod}_R$, defined as the composition

$$\mathrm{Sp} \xrightarrow{R\otimes(-)} \mathrm{Mod}_R \xrightarrow{\pi_*} \mathrm{Mod}_{R_*},$$

is a homology theory. If R is of Adams type, then $R_*(-)$ naturally lands in the subcategory Comod_{R_*R}. If we restrict the domain of R_* to the category of R-local spectra, then it is a conservative homology theory. For the rest of the paper we will use R_* to denote the restricted conservative homology theory $R_*\colon \mathrm{Sp}_R \to \mathrm{Comod}_{R_*R}$.

Remark 3.5 Recall that we are really interested in the category $\mathrm{Sp}_{K_p(n)}$ of $K_p(n)$-local spectra. The spectrum $K_p(n)$ is a field object in Sp, and its homotopy groups $\pi_* K_p(n)$ are graded fields. Hence the homology theory $K_p(n)_*\colon \mathrm{Sp}_{K_p(n)} \to \mathrm{Mod}_{K_p(n)_*}$ is too simple to exhibit the algebraicity properties that we want. As $K_p(n)$ is Adams type, $K_p(n)_*(-)$ factors through Comod_{K_*K}, but this category is very complicated. In particular, it does not have finite cohomological dimension, a feature we will need later. We learnt the argument for why this is the case from [10]. Having finite cohomological dimension would imply that the $K_p(n)$-Adams spectral sequence has a horizontal vanishing line at a finite page. The groups in this spectral sequence are all torsion, hence this would imply that, for example, the homotopy groups of the $K_p(n)$-local sphere is a finite filtration of torsion groups. In particular there could be no rational homotopy groups. But, by [13] the rational homotopy groups of the $K_p(n)$-local sphere are highly nontrivial, meaning that the original assumption that Comod_{K_*K} has finite cohomological dimension must be wrong.

There is, however, a version of E_*-homology on $\mathrm{Sp}_{K_p(n)}$, defined by sending a $K(n)$-local spectrum X to $E^\vee_*(X) := \pi_* L_{K_p(n)}(E \otimes X)$. The functor does land in a category of comodules, specifically over the L-complete Hopf algebroid $(E_*, E^\vee_* E)$; see [1, 5.3]. However, the category $\mathrm{Comod}_{E^\vee_* E}$ is not abelian. This is the reason for instead using the monochromatic category $\mathcal{M}_{n,p}$ and the category of I_n-power torsion comodules, as these inherit nicer homological properties we can exploit.

Definition 3.6 Let $H\colon \mathcal{C} \to \mathcal{A}$ be a homology theory and J an injective object in $\mathcal{A}$. An object $\bar{J} \in \mathcal{C}$ is said to be an *injective lift* of J if it represents the functor

$$\mathrm{Hom}_{\mathcal{A}}(H(-), J)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$$

in the homotopy category $h\mathcal{C}$, ie $\mathrm{Hom}_{\mathcal{A}}(H(-), J) \cong [-, \bar{J}]$. We call $\bar{J}$ a *faithful lift* if the map $H(\bar{J}) \to J$ coming from the identity on $\bar{J}$ is an equivalence.

Definition 3.7 A homology theory $H\colon \mathcal{C} \to \mathcal{A}$ is said to be *adapted* if $\mathcal{A}$ has enough injective objects, and for any injective $J \in \mathcal{A}$ there is a faithful lift $\bar{J} \in \mathcal{C}$.

Example 3.8 We return to our two guiding examples $\pi_*\colon \mathrm{Mod}_R \to \mathrm{Mod}_{R_*}$ and $R_*\colon \mathrm{Sp}_R \to \mathrm{Comod}_{R_*R}$, where R is an Adams-type ring spectrum. Both functors are conservative adapted homology theories, with faithful lifts provided by Brown representability; see [36, 8.2] and [36, 8.13], respectively.

Remark 3.9 The definition of an adapted homology theory H states that for any injective $J \in \mathcal{A}$, there is some object $\bar{J} \in \mathcal{C}$ together with an equivalence $[X, \bar{J}] \simeq \mathrm{Hom}_{\mathcal{A}}(HX, J)$ induced by H. Because $\mathcal{A}$ has enough injective objects, we can use these equivalences to approximate homotopy classes of maps by repeatedly mapping into injectives. This gives precisely an associated Adams spectral sequence for the homology theory H. In fact, Patchkoria and Pstrągowski proved that there is a bijection between adapted homology theories and Adams spectral sequences; see [36, 3.24, 3.25]. The construction of the Adams spectral sequence associated to an adapted homology theory $H\colon \mathcal{C} \to \mathcal{A}$ is given in [36, 2.24], or alternatively as a totalization spectral sequence in [36, 2.27].

In our particular interest $R = E_n$, the associated adapted homology theories π_* and E_* are even nicer than a general adapted homology theory. This is because the category of comodules is particularly simple.

Definition 3.10 Let $\mathcal{A}$ be a locally graded abelian category with enough injective objects. Then the *cohomological dimension* of $\mathcal{A}$ is the smallest integer d such that $\mathrm{Ext}^{s,t}_{\mathcal{A}}(-,-) \cong 0$ for all $s > d$.

Example 3.11 Let n be an integer, p a prime such that $p > n+1$ and $E = E_n$ Morava E-theory at height n. Then by [37, 2.5] the category Comod_{E_*E} has cohomological dimension n^2+n.

For certain Adams-type ring spectra R we get decompositions of the category Comod_{R_*R} into periodic families of subcategories. Such decompositions allows for the construction of partial inverses to the associated homology theories.

Construction 3.12 Let R be an Adams-type ring spectrum such that π_*R is concentrated in degrees divisible by some positive number $q+1$, ie, $\pi_m R = 0$ for all $m \neq 0 \bmod q+1$. Any comodule M in the category Comod_{R_*R} splits uniquely into a direct sum of subcomodules $\bigoplus_{\phi \in \mathbb{Z}/q+1} M_\phi$ such that M_ϕ is concentrated in degrees divisible by ϕ. Such a splitting induces a decomposition of the full subcategory of injective objects

$$\mathrm{Comod}^{\mathrm{inj}}_{R_*R} \simeq \mathrm{Comod}^{\mathrm{inj}}_{R_*R,0} \times \mathrm{Comod}^{\mathrm{inj}}_{R_*R,1} \times \cdots \times \mathrm{Comod}^{\mathrm{inj}}_{R_*R,q},$$

where the category $\mathrm{Comod}^{\mathrm{inj}}_{R_*R,\phi}$ denotes the full subcategory spanned by injective comodules concentrated in degrees divisible by ϕ.

Let $h_k\mathcal{C}$ denote the homotopy k-category of $\mathcal{C}$, obtained by $(k+1)$-truncating all the mapping spaces in $\mathcal{C}$. The lift associated with each injective via the Adapted homology theory R_* allows us to construct a partial inverse to R_*, called the Bousfield functor β^{inj} in [36]. It is a functor $\beta^{\mathrm{inj}}\colon \mathrm{Comod}^{\mathrm{inj}}_{R_*R} \to h_{q+1}\mathrm{Sp}^{\mathrm{inj}}_R$, where the latter category is the homotopy $(q+1)$-category of the full subcategory of Sp_R containing all spectra X such that R_*X is injective.

In order to mimic this behavior for a general adapted homology theory, Franke introduced the notion of a splitting of an abelian category.

Definition 3.13 [19] Let $\mathcal{A}$ be an abelian category with a local grading [1]. A *splitting* of $\mathcal{A}$ of order $q+1$ is a collection of Serre subcategories $\mathcal{A}_\phi \subseteq \mathcal{A}$ indexed by $\phi \in \mathbb{Z}/(q+1)$ satisfying

(1) $[k]\mathcal{A}_n \subseteq \mathcal{A}_{n+k \bmod (q+1)}$ for any $k \in \mathbb{Z}$, and

(2) the functor $\prod_\phi \mathcal{A}_\phi \to \mathcal{A}$, defined by $(a_\phi) \mapsto \oplus_\phi a_\phi$, is an equivalence of categories.

Example 3.14 As we saw above in Construction 3.12, the category of comodules over an Adams Hopf algebroid (R_*, R_*R), where R_* is concentrated in degrees divisible by $q+1$, has a splitting of order $q+1$. This, then, also holds for the discrete Hopf algebroid (R_*, R_*), giving the module category Mod_{R_*} a splitting of order $q+1$ as well.

Example 3.15 In the case $R = E(1)$ this has been written out in detail in [4, Section 4]. The Serre subcategories are all copies of the category of p-local abelian groups together with Adams operations ψ^k for $k \neq 0$ in $\mathbb{Z}_{(p)}$. The shift leaves the underlying module unchanged, but changes the Adams operation.

Definition 3.16 We will say that objects $A \in \mathcal{A}_\phi$ are of *pure weight* ϕ.

Remark 3.17 Just as for Comod_{R_*R}, a splitting of order $q+1$ of a locally graded abelian category $\mathcal{A}$ is enough to define, for any adapted homology theory $H\colon \mathcal{C} \to \mathcal{A}$, a partial inverse Bousfield functor β^{inj}; see [36, Section 7.2].

3.2 Exotic homology theories

In order to make some statements about exotic equivalences a bit simpler, we introduce the concept of exotic adapted homology theories. This is not the way similar results are phrased in [36], but the notation serves as a shorthand for the criteria that they use.

Definition 3.18 Let $H:\mathcal{C}\to\mathcal{A}$ be a homology theory. We say H is k*-exotic* if H is adapted, conservative, $\mathcal{A}$ has finite cohomological dimension d and a splitting of order $q+1$ such that $k=q+1-d>0$.

The remarkable thing about a k-exotic homology theory $H:\mathcal{C}\to\mathcal{A}$ is that it forces the stable ∞-category $\mathcal{C}$ to be approximately algebraic. Intuitively: as the order of the splitting is greater than the cohomological dimension, the H-Adams spectral sequence is very sparse and well behaved. There is a partial inverse of H via the Bousfield functor $\beta:\mathcal{A}^{\mathrm{inj}}\to h_{q+1}\mathcal{C}^{\mathrm{inj}}$, which forces a certain subcategory of a categorified deformation of H to be equivalent to both $h_k\mathcal{C}$ and $h_k\mathrm{D}^{\mathrm{per}}(\mathcal{A})$. This is the contents of Franke's algebraicity theorem.

Theorem 3.19 [36, 7.56] *Let $H:\mathcal{C}\to\mathcal{A}$ be a k-exotic homology theory. Then there is an equivalence of homotopy k-categories $h_k\mathcal{C}\simeq h_k\mathrm{D}^{\mathrm{per}}(\mathcal{A})$.*

There are several interesting examples of homology theories satisfying Theorem 3.19; see Section 8 in [36]. We highlight again our two guiding examples but focus specifically on certain Morava E-theories.

Example 3.20 [36, 8.7] Let p be a prime, n be a nonnegative integer, and E a height-n Morava E-theory concentrated in degrees divisible by $2p-2$, for example, Johnson–Wilson theory $E(n)$. If $k=2p-2-n>0$, then the functor $\pi_*:\mathrm{Mod}_E\to\mathrm{Mod}_{E_*}$ is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Mod}_E\simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}).$$

Notation 3.21 For the following example and the rest of the paper, we follow the notation of [3; 11; 12] and denote the category $\mathrm{D}^{\mathrm{per}}(\mathrm{Comod}_{E_*E})$ by $\mathrm{Fr}_{n,p}$.

Example 3.22 [36, 8.13] Let p be a prime, n be a nonnegative integer, and E any height-n Morava E-theory. If $k=2p-2-n^2-n>0$, then the functor $E_*:\mathrm{Sp}_{n,p}\to\mathrm{Comod}_{E_*E}$ is a k-exotic homology theory, giving an equivalence

$$h_k\mathrm{Sp}_{n,p}\simeq h_k\mathrm{Fr}_{n,p}.$$

Remark 3.23 As noted in [11, 5.29], this equivalence is strictly exotic for all $n\geqslant 1$ and primes p. In other words, it can never be made into an equivalence of stable ∞-categories. In particular, the mapping spectra in $\mathrm{Fr}_{n,p}$ are $H\mathbb{Z}$-linear, while the mapping spectra in $\mathrm{Sp}_{n,p}$ are only $H\mathbb{Z}$-linear for $n=0$.

Definition 3.24 Let $H\colon \mathcal{C} \to \mathcal{A}$ be a k-exotic homology theory. The category $\mathrm{D}^{\mathrm{per}}(\mathcal{A})$ is called an *exotic algebraic model* of $\mathcal{C}$ if the equivalence $h_k\mathcal{C} \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathcal{A})$ can not be enhanced to an equivalence of ∞-categories $\mathcal{C} \simeq \mathrm{D}^{\mathrm{per}}(\mathcal{A})$.

Remark 3.25 The notion of being exotically algebraic is part of a complex hierarchy of algebraicity levels; see [30] for a great exposé.

Remark 3.26 The existence of an exotic algebraic model for a stable ∞-category $\mathcal{C}$ implies that the category is not rigid. This means, in particular, that there cannot exist a k-exotic homology theory with source Sp or $\mathrm{Sp}_{(p)}$ as these are all rigid for all primes; see [42; 43; 44]. The same holds for $\mathrm{Sp}_{1,2}$, as this is rigid by [41], and similarly for $\mathrm{Sp}_{K_2(1)}$ by [29]. This shows that being k-exotic is quite a strong requirement.

4 Algebraicity for monochromatic categories

We are now ready to prove our main results. We start by proving Theorem C, which we will later use to prove Theorem B. The main result, Theorem A will then follow by using certain local duality arguments.

4.1 Monochromatic modules

For the rest of this section, we assume that E is the height-n Johnson–Wilson theory $E(n)$. This is an $\mathbb{E}_1$-ring spectrum concentrated in degrees divisible by $2p-2$, with coefficient ring

$$\pi_*E(n) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}],$$

where $|v_i| = 2p^i - 2$. The goal of this section is to prove Theorem C, which we do in three steps. First we show that the functor $\pi_*\colon \mathrm{Mod}_E^{I_n\text{-tors}} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ is a conservative adapted homology theory. We then show that $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ has finite cohomological dimension, and lastly that it admits a splitting.

The following lemma is the I_n-power torsion version [6, 3.14], and the proof is similar.

Lemma 4.1 *If M is an E-module, then $M \in \mathrm{Mod}_E^{I_n\text{-tors}}$ if and only if $\pi_*M \in \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$.*

Proof Let $X \in \mathrm{Mod}_E^{I_n\text{-tors}}$. By [8, 3.19] there is a strongly convergent spectral sequence of $E(n)_*$-modules with signature

$$E_2^{s,t} = (H_{I_n}^{-s}\pi_*X)_t \Longrightarrow \pi_{s+t}M_nX,$$

where $H_{I_n}^{-s}$ denotes local cohomology. By [17, 2.1.3(ii)] the E_2-page consist of only I_n-power torsion modules. As $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ is abelian, it is closed under quotients and subobjects, as the higher pages are created from the E_2-page using quotients and subobjects, they must also consist of only I_n-power torsion modules. In particular, the E_∞-page is all I_n-power torsion. By Grothendieck's vanishing theorem, see, for example, [17, 6.1.2], $H_{I_n}^s(-) \cong 0$ for $s > n$, hence the abutment of the spectral sequence π_*M_nX is a finite filtration of I_n-power torsion E_*-modules, and is therefore itself an I_n-power torsion module. Since X was assumed to be monochromatic, ie $X \in \mathrm{Mod}_E^{I_n\text{-tors}}$, we have $\pi_* M_n X \cong \pi_* X$, and thus $\pi_* X \in \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$.

Assume now $X \in \mathrm{Mod}_E$ such that its homotopy groups are I_n-power torsion. Monochromatization gives a map $\phi\colon M_n X \to X$, and as $\pi_* M_n X$ is I_n-power torsion this map factors on homotopy groups as

$$\pi_* M_n X \to H^0_{I_n} \pi_* X \to \pi_* X,$$

where the first map is the edge morphism in the above-mentioned spectral sequence. As $\pi_* X$ was assumed to be I_n-power torsion we have $\pi_* X \cong H^0_{I_n} \pi_* X$, and $H^s_{I_n} \pi_* X \cong 0$ for $s > 0$. Hence the spectral sequence collapses to give the isomorphism $\pi_* M_n X \cong H^0_{I_n} \pi_* X$, which shows that $\pi_* \phi$ is an isomorphism. As π_* is conservative ϕ was already an isomorphism, hence $X \in \mathrm{Mod}_E^{I_n\text{-tors}}$. □

Lemma 4.2 *For any prime p and nonnegative integer n, the functor*

$$\pi_*\colon \mathrm{Mod}_E^{I_n\text{-tors}} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$$

is a conservative adapted homology theory.

Proof The functor $\pi_*\colon \mathrm{Mod}_E \to \mathrm{Mod}_{E_*}$ is a conservative adapted homology theory. By Lemma 4.1 its restriction to $\mathrm{Mod}_E^{I_n\text{-tors}}$ lands in $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$, hence automatically $\pi_*\colon \mathrm{Mod}_E^{I_n\text{-tors}} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ is a conservative homology theory.

Let J be an injective I_n-power torsion E_*-module. We can embed $J \to Q$ into an injective E_*-module Q, as Mod_{E_*} has enough injectives. After applying the torsion functor $T_{I_n}^{E_*}$ this map has a section, as $J \cong T_{I_n}^{E_*} J$ is injective. In particular, any injective J is a retract of $T_{I_n}^{E_*} Q$ for some injective E_*-module Q, hence we can assume J to be of that form. By [17, 2.1.4] any such $J = T_{I_n}^{E_*} Q$ is injective as an object of Mod_{E_*}.

Now, as π_* is adapted on Mod_E we can chose a faithful injective lift $\bar{J}$ of J to Mod_E, and since $\bar{J}$ was assumed to have I_n-torsion homotopy groups we know by Lemma 4.1 that $\bar{J}$ is an object of $\mathrm{Mod}_E^{I_n\text{-tors}}$. In particular, we have faithful lifts for any injective in $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$, which means that $\pi_*\colon \mathrm{Mod}_E^{I_n\text{-tors}} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ is adapted. □

Let C^{I_n} denote the I_n-adic completion functor on Mod_{E_*}. It is neither left nor right exact; see [26, Appendix A]. As E_* is an integral domain, the higher right derived functors vanish by [20, 5.1]. For $i \geqslant 0$ we denote the i^{th} left derived functor of C^{I_n} by $L_i^{I_n}$. For any $M \in \mathrm{Mod}_E$ there is a natural map $L_0^{I_n} M \to C^{I_n} M$. It is always an epimorphism, but usually not an isomorphism.

Lemma 4.3 *For any prime p and nonnegative integer n, the category $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ has cohomological dimension n.*

Proof The category Mod_{E_*} has cohomological dimension n, and that Ext-groups in $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ are computed in Mod_{E_*}. By [17, 2.1.4], this implies that the cohomological dimension of $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ cannot be greater than n, so it remains to prove that it is exactly n. We prove this by computing an $\mathrm{Ext}^n_{E_*}$ group that is nonzero.

By [26, A.2(d)] we have $L_0^{I_n} M \cong \mathrm{Ext}^n_{E_*}(H^n_{I_n}(E_*), M)$ for any E_*module M. In other words, the derived completion of an E_*-module is the n^{th} derived functor of maps from the I_n-local cohomology of E_* into M. Choosing $M = E_*/I_n$ we get

$$L_0^{I_n}(E_*/I_n) \cong \mathrm{Ext}^n_{E_*}(H^n_{I_n}(E_*), E_*/I_n).$$

As any bounded I_n-torsion E_*-module is I_n-adically complete we have, as remarked in [7, 1.4], an isomorphism $L_0^{I_n}(E_*/I_n) \cong E_*/I_n$. The local cohomology of E_* is also I_n-torsion, in particular $H^n_{I_n} E_* = E_*/I_n^\infty$. Hence we have

$$\mathrm{Ext}^n_{E_*}(E_*/I_n^\infty, E_*/I_n) \cong E_*/I_n \ncong 0,$$

showing that there are two I_n-power torsion E_*-modules with nontrivial n^{th} Ext. □

Lemma 4.4 *For any prime p and nonnegative integer n, the category $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ has a splitting of order $2p-2$.*

Proof By [36, 8.1] the category Mod_{E_*} has a splitting of order $2p-2$. We define the pure weight ϕ component of $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$, denoted by $\mathrm{Mod}_{E_*,\phi}^{I_n\text{-tors}}$, to be the essential image of $T_{I_n}^{E_*} : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ restricted to the pure weight ϕ component $\mathrm{Mod}_{E_*,\phi}$. We claim that this defines a splitting of order $2p-2$ on $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$.

As $\mathrm{Mod}_{E_*,\phi}$ is a Serre subcategory, and being I_n-power torsion is a property closed under subobjects, quotients, and extensions, also $\mathrm{Mod}_{E_*,\phi}^{I_n\text{-tors}}$ is a Serre subcategory. As E_* is concentrated in degrees divisible by $2p-2$ every I_n-power torsion module decomposes into its pure weight components. This also gives a decomposition of $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$. The shift functor on I_n-power torsion modules simply shifts the underlying module, hence shift-invariance follows from the shift-invariance on Mod_{E_*}. □

We can now summarize the above discussion with the first of our main results.

Theorem 4.5 (Theorem C) *Let p be a prime and n a nonnegative integer. If $k = 2p-2-n > 0$, then the functor*

$$\pi_* : \mathrm{Mod}_E^{I_n\text{-tors}} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$$

is a k-exotic homology theory, giving an equivalence

$$h_k \mathrm{Mod}_E^{I_n\text{-tors}} \simeq h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n\text{-tors}}).$$

In particular, monochromatic E-modules are exotically algebraic at large primes.

Proof By Lemma 4.3 the cohomological dimension of the category $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ is n, and by Lemma 4.4 we have a splitting on $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ of order $2p-2$. Hence, by Lemma 4.2 the functor

$$\pi_*\colon \mathrm{Mod}_E^{I_n\text{-tors}} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$$

is a k-exotic homology theory for $k = 2p-2-n > 0$, which gives an equivalence

$$h_k\mathrm{Mod}_E^{I_n\text{-tors}} \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n\text{-tors}})$$

by Theorem 3.19. □

We can also phrase this dually in terms of $K_p(n)$-local E-modules.

Corollary 4.6 *Let p be a prime, n a positive integer and $K_p(n)$ be height-n Morava K-theory at the prime p. If $k = 2p-2-n > 0$, then we have a k-exotic algebraic equivalence*

$$h_k L_{K_p(n)}\mathrm{Mod}_E \simeq h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n\text{-comp}}.$$

In particular, $K_p(n)$-local E-modules are exotically algebraic at large primes.

Proof The equivalence is constructed from the equivalences obtained from Remark 2.7, Theorems 4.5 and 2.21 and Construction 2.19. In particular, we have

$$\begin{aligned} h_k\mathrm{Mod}_E^{I_n\text{-comp}} &\overset{2.7}{\simeq} h_k\mathrm{Mod}_E^{I_n\text{-tors}} \\ &\overset{4.5}{\simeq} h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n\text{-tors}}) \\ &\overset{2.21}{\simeq} h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n\text{-tors}} \\ &\overset{2.19}{\simeq} h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n\text{-comp}}, \end{aligned}$$

where we have used that an equivalence of ∞-categories induces an equivalence on homotopy k-categories. □

Now, let HE_* be the Eilenberg–MacLane spectrum of E_*. By Schwede and Shipleys's derived Morita theory, see [33, 7.1.1.16], there is a symmetric monoidal equivalence of categories $\mathrm{D}(E_*) \simeq \mathrm{Mod}_{HE_*}$, and we can form a local duality diagram for Mod_{HE_*} corresponding to Example 2.5 for the discrete Hopf algebroid (E_*, E_*). By arguments similar to Lemmas 4.1 and 4.2 one can show that the homotopy groups functor $\pi_*\colon \mathrm{Mod}_{HE_*} \to \mathrm{Mod}_{E_*}$ restricts to a conservative adapted homology theory

$$\pi_*\colon \mathrm{Mod}_{HE_*}^{I_n\text{-tors}} \to \mathrm{Mod}_{E_*}^{I_n\text{-tors}}.$$

In the same range as Theorem 4.5 this is also k-exotic. We can then combine the algebraicity for $\mathrm{Mod}_E^{I_n\text{-tors}}$ and Mod_{HE_*} to get the following statement.

Corollary 4.7 *Let p be a prime and n a nonnegative integer. If $k = 2p-2-n > 0$, then there is an exotic equivalence $h_k\mathrm{Mod}_E^{I_n\text{-tors}} \simeq h_k\mathrm{Mod}_{HE_*}^{I_n\text{-tors}}$.*

4.2 Monochromatic spectra

Having proven that monochromatic E-modules are algebraic at large primes, we now turn to the larger category of all monochromatic spectra $\mathcal{M}_{n,p}$ with the same goal. The strategy is exactly the same as in Section 4.1: we first prove that the conservative adapted homology theory $E_*\colon \mathrm{Sp}_{n,p} \to \mathrm{Comod}_{E_*E}$ restricts to a conservative adapted homology theory on $\mathcal{M}_{n,p}$, before proving that $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ has a splitting and finite cohomological dimension. This will prove Theorem B, which we then convert into a proof of Theorem A, as in Corollary 4.6.

In this section the choice of v_n-periodic Landweber exact ring spectrum E does not matter, as the categories $\mathrm{Sp}_{n,p}$ and Comod_{E_*E} are equivalent for all such spectra — see [23, 1.12] and [27, 4.2], respectively. However, to make the interaction with Section 4.1 as simple as possible we will continue to use the height-n Johnson–Wilson spectrum $E(n)$.

Lemma 4.8 *If X is a E-local spectrum, then $X \in \mathcal{M}_{n,p}$ if and only if $E_*X \in \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$.*

Proof Assume first that $X \in \mathcal{M}_{n,p}$. We have $E \otimes X \in \mathrm{Mod}_E^{I_n\text{-tors}}$ as

$$E \otimes X \simeq E \otimes M_n X \simeq M_n(E \otimes X),$$

where the last equivalence follows from M_n being smashing. In particular, the restricted functor $E_*\colon \mathcal{M}_{n,p} \to \mathrm{Comod}_{E_*E}$ factors through $\mathrm{Mod}_E^{I_n\text{-tors}}$. By Lemma 4.1 and Remark 2.9 this means that E_*X is an I_n-power torsion E_*E-comodule.

For the converse, assume that we have $X \in \mathrm{Sp}_{n,p}$ such that $E_*X \in \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$. Using the monochromatization functor we obtain a comparison map $M_n X \to X$, which induces a map on E-modules $E \otimes M_n X \to E \otimes X$. This map is an isomorphism on homotopy groups, as E_*X was assumed to be I_n-power torsion. As E_* is conservative on $\mathrm{Sp}_{n,p}$, the original comparison map $M_n X \to X$ was an isomorphism, meaning that $X \in \mathcal{M}_{n,p}$. □

Lemma 4.9 *For any prime p and nonnegative integer n, the functor*

$$E_*\colon \mathcal{M}_{n,p} \to \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$$

is a conservative adapted homology theory.

Proof First note that the image of the functor $E_*\colon \mathrm{Sp}_{n,p} \to \mathrm{Comod}_{E_*E}$ restricted to $\mathcal{M}_{n,p}$ is contained in $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ by Lemma 4.8. The functor $E_*\colon \mathcal{M}_{n,p} \to \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ is then automatically a conservative homology theory. The category $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ has enough injectives as it is Grothendieck by Remark 2.10. Hence, it only remains to prove that we have faithful lifts for all injective objects.

Let J be an injective in $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$. As in the proof of Lemma 4.2 we can assume that J has the form $J = T_{I_n}^{E_*E} P$ for some injective E_*E-comodule P, as being torsion is a property of the underlying module. By [28, 2.1(c)] any injective E_*E-comodule is a retract of $E_*E \otimes_{E_*} Q$ for some injective E_*-module Q. Hence, we can further assume that J has the form $J = T_{I_n}^{E_*E}(E_*E \otimes_{E_*} Q)$.

From [8, 5.7] it follows that there is a commutative diagram of adjoint functors

$$\begin{tikzcd} \mathrm{Comod}_{E_*E} \arrow[r, shift left, "\varepsilon_*"] \arrow[d, shift left, "T^{E_*E}_{I_n}"] & \mathrm{Mod}_{E_*} \arrow[l, shift left, "\varepsilon^*"] \arrow[d, shift left, "T^{E_*}_{I_n}"] \\ \mathrm{Comod}^{I_n\text{-tors}}_{E_*E} \arrow[r, shift left, "\varepsilon_*"] \arrow[u, shift left] & \mathrm{Mod}^{I_n\text{-tors}}_{E_*} \arrow[l, shift left, "\varepsilon^*"] \arrow[u, shift left] \end{tikzcd}$$

where $\varepsilon_* \dashv \varepsilon^*$ is the forgetful-cofree adjunction. In particular, the functor ε^* is given by $E_*E \otimes_{E_*} (-)$. To justify the notation in the bottom row, let us prove that the cofree functor on I_n-power torsion modules is also given by $E_*E \otimes_{E_*} (-)$. In order to do this we prove that for an I_n-power torsion E_*-module M, that $T^{E_*E}_{I_n}(E_*E \otimes_{E_*} M) \cong E_*E \otimes_{E_*} M$.

By [8, 5.5] there is an isomorphism

$$T^{E_*E}_{I_n}(E_*E \otimes_{E_*} M) \cong \mathrm{colim}_k\, \underline{\mathrm{Hom}}_{E_*E}(E_*/I_n^k, E_*E \otimes_{E_*} M),$$

which by [8, 4.4] gives

$$\mathrm{colim}_k\, \underline{\mathrm{Hom}}_{E_*E}(E_*/I_n^k, E_*E \otimes_{E_*} M) \cong \mathrm{colim}_k (E_*E \otimes_{E_*} \mathrm{Hom}_{E_*}(E_*/I_n^k, M)).$$

As the tensor product $- \otimes_{E_*} -$ commutes with filtered colimits separately in each variable, and M was assumed to be I_n-power torsion, the right hand side is $E_*E \otimes_{E_*} M$.

Now, choosing the injective Q in the top right corner and going through the square gives an isomorphism $T^{E_*E}_{I_n}(E_*E \otimes_{E_*} Q) \cong E_*E \otimes_{E_*} T^{E_*}_{I_n} Q$. By [17, 2.1.4] we know that $T^{E_*}_{I_n} Q$ is an injective E_*-module, and by [28, 2.1(a)] the cofree comodule $E_*E \otimes_{E_*} T^{E_*}_{I_n} Q$ is an injective E_*E-comodule. Hence, $J = T^{E_*E}_{I_n}(E_*E \otimes_{E_*} Q)$ is injective also as an object in Comod_{E_*E}.

Finally, as E_* has faithful injective lifts from Comod_{E_*E} to $\mathrm{Sp}_{n,p}$, there exists a lift $\bar{J}$ such that $[X, \bar{J}] \simeq \mathrm{Hom}_{E_*E}(E_*X, J)$ and $E_*\bar{J} \simeq J$. By Lemma 4.8 we know that $\bar{J} \in \mathcal{M}_{n,p}$, as J was assumed to be I_n-power torsion, hence we have found our faithful injective lift. □

Lemma 4.10 *Let p be a prime and n a nonnegative integer. If $p-1 \nmid n$, then the category $\mathrm{Comod}^{I_n\text{-tors}}_{E_*E}$ has cohomological dimension n^2+n.*

Proof The proof follows [37, 2.5] closely, which is itself a modern reformulation of [19, 3.4.3.9]. As in Lemma 4.3 we note that also Ext-groups in $\mathrm{Comod}^{I_n\text{-tors}}_{E_*E}$ are computed in Comod_{E_*E}. We start by defining *good targets* to be I_n-power torsion comodules N such that $\mathrm{Ext}^{s,t}_{E_*E}(E_*/I_n, N) = 0$ for all $s > n^2+n$ and *good sources* to be I_n-power torsion comodules M such that $\mathrm{Ext}^{s,t}_{E_*E}(M, N) = 0$ for all $s > n^2+n$ and I_n-torsion comodules N.

By the Landweber filtration theorem, see, for example, [27, 5.7], we know that any finitely presented comodule M has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{s-1} \subseteq M_s = M,$$

where $M_r/M_{r-1} \cong E_*/I_{j_r}[t_r]$ and $j_r \leqslant n$. When M is I_n-power torsion we get $j_r = n$ for all r, as noted in [27, 4.3]. For primes p not dividing $n+1$ Morava's vanishing theorem, see, for example, [39, 6.2.10], gives us that $\operatorname{Ext}^{s,t}_{E_*E}(E_*, E_*/I_n) = 0$ for all $s > n^2$. As the generators for the ideal I_i form a regular sequence, we get short exact sequences of the form

$$0 \to E_*/I_{i-1} \xrightarrow{v_i} E_*/I_{i-1} \to E_*/I_i \to 0$$

for $0 \leqslant i \leqslant n$. By the induced long exact sequence in Ext-groups, we get that

$$\operatorname{Ext}^{s,t}_{E_*E}(E_*/I_n, E_*/I_n) = 0$$

for $s > n^2 + n$, which by the Landweber filtration implies that any finitely presented I_n-power torsion comodule is a good target.

The comodule E_*/I_n has a finite resolution of E_*E-comodules that are projective as modules over E_*. The Ext-functor out of these projectives can be computed using the cobar complex, see [39, A1.2.12], implying that the functor $\operatorname{Ext}^{s,t}_{E_*E}(E_*/I_n, -)$ commutes with filtered colimits. By Lemma 2.11 any I_n-power torsion comodule is a filtered colimit of finitely presented ones, hence any I_n-power torsion comodule is a good target.

The above argument also proves that E_*/I_n is a good source, which by the Landweber filtration argument implies that any finitely presented I_n-torsion comodule is a good source. Again, by Lemma 2.11, the category $\operatorname{Comod}^{I_n\text{-tors}}_{E_*E}$ is generated under filtered colimits by finitely presented comodules. Hence, we can apply [37, 2.4] to any injective resolution

$$0 \to M \to J_0 \to J_1 \to \cdots$$

to get that $J_{n^2+n} \to \operatorname{Im}(J_{n^2+n} \to J_{n^2+n+1})$ is a split surjection, and that $\operatorname{Im}(J_{n^2+n} \to J_{n^2+n+1})$ is injective. Hence, any injective resolution can be modified to have length n^2+n. $\square$

Remark 4.11 In a previous version of this paper, we claimed that the cohomological dimension was n^2. We want to thank Piotr Pstrągowski for pointing out the gap in the proof. This means that $\operatorname{Comod}^{I_n\text{-tors}}_{E_*E}$ has the same cohomological dimension as the nontorsion category $\operatorname{Comod}_{E_*E}$, as seen in Example 3.11. However, we do obtain something slightly stronger, as our result holds for all $p-1 \nmid n$, while the analogue in $\operatorname{Comod}_{E_*E}$ only holds when $p-1 > n$. In fact, $\operatorname{Comod}_{E_*E}$ does not have finite cohomological dimension when $p-1 \leqslant n$, as noted in [37, 2.6]. This difference happens because we only need the Ext^s-groups out of E_*/I_n to vanish for large s, which is given to us by Morava's vanishing theorem whenever $p-1 \nmid n$. For nontorsion comodules one has to have stronger vanishing results. These can be obtained by using the chromatic spectral sequence, which only gives the vanishing results for $p-1 > n$ instead of for $p-1 \nmid n$.

Lemma 4.12 *For any prime p and nonnegative integer n, the category $\operatorname{Comod}^{I_n\text{-tors}}_{E_*E}$ has a splitting of order $2p-2$.*

Proof As E is concentrated in degrees divisible by $2p-2$, [36, 8.13] shows that Comod_{E_*E} has a splitting of order $2p-2$. The proof of the induced splitting on the I_n-torsion category is then identical to Lemma 4.4. $\square$

We can now summarize the above results with our second main result, which is the monochromatic analogue of Example 3.22.

Theorem 4.13 (Theorem B) *Let p be a prime and n a nonnegative integer. If $k = 2p-2-n^2-n > 0$, then the restricted functor $E_*\colon \mathcal{M}_{n,p} \to \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ is k-exotic. In particular, there is an equivalence*

$$h_k\mathcal{M}_{n,p} \simeq h_k\mathrm{D}^{\mathrm{per}}(E_*E^{I_n\text{-tors}}),$$

meaning that monochromatic homotopy theory is exotically algebraic at large primes.

Proof By Lemma 4.10, the cohomological dimension of $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ is n^2+n and by Lemma 4.12 we have a splitting of order $2p-2$. The restricted functor E_* is then by Lemma 4.9 k-exotic whenever $k = 2p-2-n^2-n > 0$, which by Theorem 3.19 finishes the proof. $\square$

Remark 4.14 By Theorem 2.21 there is an equivalence

$$\mathrm{D}^{\mathrm{per}}(E_*E^{I_n\text{-tors}}) \simeq \mathrm{Fr}_{n,p}^{I_n\text{-tors}}$$

and by Example 2.6 there is an equivalence

$$\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{n,p}^{I_n\text{-tors}}.$$

This means that we can write the equivalence in Theorem 4.13 as

$$h_k\mathrm{Sp}_{n,p}^{I_n\text{-tors}} \simeq h_k\mathrm{Fr}_{n,p}^{I_n\text{-tors}}$$

for $k = 2p-2-n^2-n > 0$. This is more in line with thinking about Theorem 4.13 as "coming from" the chromatic algebraicity of Example 3.22 on localizing ideals. This formulation is perhaps also easier to connect to the limiting case $p \to \infty$ as described using ultraproducts in [12], which can be stated informally as

$$\lim_{p\to\infty} \mathrm{Sp}_{n,p}^{I_n\text{-tors}} \simeq \lim_{p\to\infty} \mathrm{Fr}_{n,p}^{I_n\text{-tors}}.$$

Via Theorem 2.3 we can now obtain the associated exotic algebraicity statement for the category of $K_p(n)$-local spectra.

Theorem 4.15 (Theorem A) *Let p be a prime, n a nonnegative integer and $K_p(n)$ be height-n Morava K-theory at the prime p. If $k = 2p-2-n^2 > 0$, then we have a k-exotic algebraic equivalence*

$$h_k\mathrm{Sp}_{K_p(n)} \simeq h_k\mathrm{Fr}_{n,p}^{I_n\text{-comp}}.$$

In other words, $K_p(n)$-local homotopy theory is algebraic at large primes.

Proof As we did in Corollary 4.6, we construct the equivalence from a sequence of equivalences coming from Theorems 2.3 and 4.13. More precisely we use equivalences coming from Example 2.6, Theorems 4.13 and 2.21 and Construction 2.19, which give

$$\begin{aligned} h_k\mathrm{Sp}_{K_p(n)} &\overset{2.6}{\simeq} h_k\mathcal{M}_{n,p} \\ &\overset{4.13}{\simeq} h_k\mathrm{D}^{\mathrm{per}}(\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}) \\ &\overset{2.21}{\simeq} h_k\mathrm{Fr}_{n,p}^{I_n\text{-tors}} \\ &\overset{2.19}{\simeq} h_k\mathrm{Fr}_{n,p}^{I_n\text{-comp}}, \end{aligned}$$

where we again have used that an equivalence of ∞-categories induces an equivalence on homotopy k-categories. ◻

Remark 4.16 As in Remark 4.14 we can phrase Theorem 4.15 as $h_k\mathrm{Sp}_{n,p}^{I_n\text{-comp}} \simeq h_k\mathrm{Fr}_{n,p}^{I_n\text{-comp}}$.

Appendix Barr–Beck for localizing ideals

In this appendix we prove that the monoidal Barr–Beck theorem — a monoidal version of Lurie's ∞-categorical version of the classical Barr–Beck monadicity theorem; see [33, Section 4.7] — interacts nicely with local duality.

Theorem A.1 [34, 5.29] *Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ and $(F \dashv G)\colon \mathcal{C} \to \mathcal{D}$ be a monoidal adjunction. If in addition*

1. *G is conservative,*
2. *G preserves colimits, and*
3. *the projection formula holds,*

then (F, G) is a monoidally monadic adjunction and the monad GF is equivalent to the monad $G(\mathbb{1}_{\mathcal{D}})\otimes(-)$. In particular this gives a symmetric monoidal equivalence $\mathcal{D} \simeq \mathrm{Mod}_{G(\mathbb{1}_{\mathcal{D}})}(\mathcal{C})$.

Proof By [33, 4.7.3.5] the adjunction is monadic by the first two criteria, giving an equivalence $\mathcal{D} \simeq \mathrm{Mod}_{GF}(\mathcal{C})$. The map of monads $G(\mathbb{1}_{\mathcal{D}}) \otimes (-) \to GF$ given by [18, 3.6] is seen to be an equivalence by applying the projection formula to the unit $\mathbb{1}_{\mathcal{D}}$. ◻

Definition A.2 When the three criteria above hold for a given monoidal adjunction $(F \dashv G)$, we will say that the adjunction satisfies the monoidal Barr–Beck criteria or that it is a *monoidal Barr–Beck adjunction*. We will sometimes omit the prefix monoidal when it is clear from context.

Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We wish to prove that the associated local duality diagram is compatible with Theorem A.1. By modifying [14, 3.7] slightly, we know that any Barr–Beck adjunction induces a Barr–Beck adjunction on $\mathcal{K}$-local and $\mathcal{K}$-complete objects. Hence, it remains only to prove a similar statement for the $\mathcal{K}$-torsion objects.

Definition A.3 Let $(\mathcal{C}, \mathcal{K})$ and $(\mathcal{D}, \mathcal{L})$ be local duality contexts. A *map of local duality contexts* is a symmetric monoidal colimit-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{K}) \subseteq \mathcal{L}$. If, in addition $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$, then we say F is a *strict* map of local duality contexts. A monoidal adjunction $(F \dashv G)\colon \mathcal{C} \to \mathcal{D}$ such that F is a strict map of local duality contexts is called a *local duality adjunction*, sometimes denoted by

$$(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \to (\mathcal{D}, \mathcal{L}).$$

Given a local duality context and an appropriate functor, one can always extend the functor to a strict map of local duality context in the following way.

Construction A.4 Let $(\mathcal{C}, \mathcal{K})$ be a local duality context, $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ and $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal colimit-preserving functor. The image of $\mathcal{K}$ under F generates a localizing ideal $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K}))$ in $\mathcal{D}$, which makes F a map of local duality contexts. We call this the local duality context on $\mathcal{D}$ induced by $\mathcal{C}$ via F.

The following lemma is essentially the "nongeometric" version of [2, 5.11]. The proof is also similar, but as we have phrased it in a different and slightly more general language, we present a full proof.

Lemma A.5 *Let $(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \to (\mathcal{D}, \mathcal{L})$ be a local duality adjunction. Then, the adjunction induces a monoidal adjunction on localizing ideals*

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}).$$

Proof From Remark 2.4 we know that the symmetric monoidal structures on $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ and $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ is simply the symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, restricted to the full subcategories.

Since F is a map of local duality contexts, we have an inclusion $F(\mathcal{K}) \subseteq \mathcal{L}$, which gives inclusions

$$F(\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})) \subseteq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) \subseteq \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}),$$

meaning that the functor F restricts to the torsion objects. In particular we have for any object $X \in \mathcal{C}^{\mathcal{K}\text{-tors}}$ an equivalence $\Gamma_{\mathcal{L}} F(X) \simeq F(X)$. We let $F' = F_{|\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})}$ and define G' to be the composition

$$Loc^{\otimes}_{\mathcal{D}}(\mathcal{L}) \xrightarrow{i_{\mathcal{L}\text{-tors}}} \mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{\Gamma_{\mathcal{K}}} \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}),$$

which is an adjoint to F'. We need to show that F is a symmetric monoidal functor, but, as the inclusions $i_{\mathcal{K}\text{-loc}}$ and $i_{\mathcal{L}\text{-loc}}$ are nonunitally monoidal all that remains to be proven is that F' sends the monoidal unit $\Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}}$ to the monoidal unit $\Gamma_{\mathcal{L}} \mathbb{1}_{\mathcal{D}}$.

The localizing ideals $\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K})$ and $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ are equal to the localizing ideals generated by the respective units, ie

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) = \mathrm{Loc}^{\otimes}_{\mathcal{C}}(\Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}}) \quad \text{and} \quad \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L}) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\Gamma_{\mathcal{L}} \mathbb{1}_{\mathcal{D}}).$$

Since $(F \dashv G)$ is a local duality adjunction we also know that $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\mathcal{K})) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$, which also means $\mathrm{Loc}_{\mathcal{D}}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})) = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$.

Let $\mathcal{G}$ be the full subcategory of $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ where $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ acts as a unit, in other words objects $M \in \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$ such that $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M \simeq M$. In particular, $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ is in $\mathcal{G}$. The category $\mathcal{G}$ is closed under retracts, suspension, and colimits, as well as tensoring with objects in $\mathcal{D}$, as we have

$$F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} (M \otimes_{\mathcal{D}} D) \simeq (F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M) \otimes_{\mathcal{D}} D \simeq M \otimes_{\mathcal{D}} D$$

for any $M \in \mathcal{G}$ and $D \in \mathcal{D}$. Hence, it is a localizing tensor ideal of $\mathcal{D}$, with a symmetric monoidal structure where the unit is $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$. In particular, $\mathcal{G} = \mathrm{Loc}^{\otimes}_{\mathcal{D}}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}))$, which we already know is equivalent to $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$.

Since the ideals are equivalent, and the unit is unique, we must have $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}}\mathbb{1}_{D}$. □

The key feature for us is that such an induced adjunction inherits the property of being a Barr–Beck adjunction, ie, that the right adjoint is conservative, preserves colimits, and has a projection formula. An analogous, but not equivalent, statement was proven in [14, 4.5]. Another related, but not equivalent statement, is Greenlees and Shipley's cellularization principle; see [22].

Theorem A.6 *Let $(F \dashv G)\colon (\mathcal{C}, \mathcal{K}) \to (\mathcal{D}, \mathcal{L})$ be a local duality adjunction. If $(F \dashv G)$ satisfies the Barr–Beck criteria, then the induced monoidal adjunction on localizing ideals*

$$\mathrm{Loc}^{\otimes}_{\mathcal{C}}(\mathcal{K}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$$

constructed in Lemma A.5 also satisfies the Barr–Beck criteria.

Proof We need to prove that G' is conservative and colimit-preserving and that the projection formula holds. The first two will both follow from the following computation, showing that also G' is just the restriction of G to $\mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$.

Let $X \in \mathrm{Loc}^{\otimes}_{\mathcal{D}}(\mathcal{L})$. By definition we have $G'(X) = \Gamma_{\mathcal{K}}G(X)$, where we have omitted the inclusions from the notation for simplicity. Since $\Gamma_{\mathcal{K}}$ is smashing and $(F \dashv G)$ by assumption has a projection formula,

$$\Gamma_{\mathcal{K}}G(X) \simeq G(X) \otimes_{\mathcal{C}} \Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}} \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})).$$

By Lemma A.5 F' is symmetric monoidal, hence $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}$, which acts on X as the monoidal unit. Thus, we can summarize with

$$G'(X) \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})) \simeq G(X \otimes_{\mathcal{D}} \Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}) \simeq G(X),$$

which shows that also G' is the restriction of G.

Now, as G is both conservative and preserves colimits, and colimits in the localizing ideals are computed in $\mathcal{C}$ and $\mathcal{D}$, respectively, then also G' is conservative and colimit-preserving. The projection formula for $(F' \dashv G')$ also automatically follows from the projection formula for $(F \dashv G)$. □

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