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Cup-one algebras and 1-minimal models

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Previously we introduced the notion of binomial cup-one algebras, which are differential graded algebras endowed with Steenrod \cup_1 -products and compatible binomial operations. In this paper we show that binomial cup-one algebras capture homotopy 1-type. In particular, given such an R -dga, (A, d_A) , defined over the ring $R = \mathbb{Z}$ or \mathbb{F}_p (for p a prime), with $H^0(A) = R$ and with $H^1(A)$ a finitely generated, free R -module, we show that A admits a functorially defined 1-minimal model, $\rho : (\mathcal{M}(A), d) \rightarrow (A, d_A)$, which is unique up to isomorphism. Furthermore, we associate to this model a pronilpotent group, whose continuous cohomology is isomorphic to that of $\mathcal{M}(A)$. These constructions, which refine classical notions from rational homotopy theory, allow us to distinguish spaces with isomorphic torsion-free integral cohomology rings. Moreover, we show that there is an equivalence of categories between isomorphism classes of finitely generated, torsion-free-nilpotent groups and isomorphism classes of finitely generated 1-minimal models over the integers.

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1 Introduction

1.1 Overview

Previously we combined properties of the Steenrod cup-one products of cochains and binomial rings in the cochain complex of a space to define the algebraic categories of binomial cup-one differential

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graded algebras over the integers and over \mathbb{F}_p for p a prime. We construct here the 1-minimal model \mathcal{M} for such a binomial, cup-one dga (A, d) , and prove its key properties; namely, that it is a free binomial cup-one dga unique up to isomorphism, and that it determines a pronilpotent group, also unique up to isomorphism, whose continuous cohomology is isomorphic to that of \mathcal{M} .

Since the cochains of a space X with coefficients in the ring $R = \mathbb{Z}$ or \mathbb{F}_p are binomial \cup_1 -dgas, it follows that invariants of the 1-minimal model for $A = C^*(X; R)$ give homotopy-type invariants of X . As an application, we define in Section 12 one such invariant, called *n-step equivalence*, and exhibit a family of spaces that can be distinguished using the integral 1-minimal model, where the corresponding approach in rational homotopy theory fails to distinguish those spaces. Moreover, we show in Section 13 that there is an equivalence of categories between isomorphism classes of finitely generated, torsion-free-nilpotent groups and isomorphism classes of finitely generated 1-minimal models over the integers.

The reader familiar with rational homotopy theory [10; 11; 16], will note that the 1-minimal models defined here over \mathbb{Z} and over \mathbb{F}_p can be viewed as analogs of Sullivan’s 1-minimal model over the rationals. Integral homotopy theory, as initiated by Ekedahl [8] and Kadeishvili [22], deals with E_∞ -algebras rather than commutative differential graded algebras that are used in rational homotopy theory. In [28], Mandell showed that two nilpotent, finite-type spaces are weakly equivalent if and only if their singular cochain complexes are quasi-isomorphic as E_∞ -algebras. From this perspective, the 1-minimal model constructed in this paper expresses precisely the E_∞ -data that matter for the fundamental group.

In other recent work of interest, Petersen [31] develops a framework for studying the compactly supported cohomology of configuration spaces using a forgetful functor from E_∞ -algebras to twisted cdgas; Richter and Sagave [37] use diagrams of chain complexes to model E_∞ -dgas by strictly commutative objects; Horel [21] constructs a fully faithful functor from finite-type nilpotent spaces to cosimplicial binomial rings; Flynn-Connolly [12] explores the relationship between commutative algebras and E_∞ -algebras in characteristic p ; and Gadish [15] connects the first layer of the E_∞ -algebra structure on the cochain complex of a space to “letter-braiding” invariants of the fundamental group.

1.2 Steenrod \cup_i -products and binomial cup-one dgas

Let $A = (C^*(X; R), d)$ be the cellular cochain complex of a Δ -set X , with coefficients in a commutative ring R . Then A is, in fact, a differential graded R -algebra, with multiplication given by the cup-product. In [39], Steenrod introduced a sequence of operations, $\cup_i : A^p \otimes_R A^q \rightarrow A^{p+q-i}$, starting with $\cup_0 = \cup$, the usual cup-product. We are mainly interested here in the additional structure on the cochain algebra provided by the \cup_1 -product, which is tied to the differential and the cup product via the Steenrod [39] and Hirsch [20] identities (see Section 4.1).

We defined in [35] several categories of differential graded algebras (over $R = \mathbb{Z}$ or \mathbb{F}_p) with extra structure, coming from either the cup-one products, or from a binomial ring structure, or both, bound together by suitable compatibility conditions. In Sections 4.3 and 4.4 we recall these notions, with some mild modifications to better fit the present context. A *cup-one differential graded algebra* is an R -dga (A, d) with a cup-one product map $\cup_1 : A^1 \otimes_R A^1 \rightarrow A^1$ that gives $R \oplus A^1$ the structure of a

commutative ring and satisfies the Hirsch identity, as well as the “ \cup_1 - d formula,”

$$(1-1) \quad d(a \cup_1 b) = -a \cup b - b \cup a + da \cup_1 b + db \cup_1 a - da \circ db,$$

for all $a, b \in A^1$ with da, db equal to sums of cup products, where the map \circ is bilinear and satisfies $(a_1 \cup a_2) \circ (b_1 \cup b_2) = (a_1 \cup_1 b_1) \cup (a_2 \cup_1 b_2)$. The significance of (1-1) is that it expresses the differentials of cup-one products of elements in A^1 in terms of the differentials of the factors and \cup_1 -products of elements in A^1 , as opposed to Steenrod’s formula, which involves \cup_1 -products of elements in A^2 with elements in A^1 . Moreover, as shown in [35], if $A = C^*(X; R)$ is a cochain algebra, Steenrod’s formula restricts to (1-1) for elements $a, b \in A^1$ with da and db equal to sums of cup products.

In Section 5, we add extra structure to these algebras. A commutative ring A is called a *binomial ring* if A is torsion-free as a \mathbb{Z} -module, and has the property that the elements $\binom{a}{n} := a(a-1)\cdots(a-n+1)/n!$ lie in A for every $a \in A$ and every $n > 0$. An analogous notion holds for \mathbb{F}_p -algebras. These objects come equipped with maps $\zeta_n : A \rightarrow A, a \mapsto \binom{a}{n}$, defined for all $n > 0$ over \mathbb{Z} , and only for $n < p$ over \mathbb{F}_p . Now consider a cup-one dga (A, d) over $R = \mathbb{Z}$ or \mathbb{F}_p . Such an object is called a *binomial cup-one algebra* if A^0 , with multiplication $A^0 \otimes_R A^0 \rightarrow A^0$ given by the cup-product, is a binomial R -algebra, and the R -submodule $R \oplus A^1 \subset A^{\leq 1}$, with multiplication $A^1 \otimes_R A^1 \rightarrow A^1$ given by the cup-one product, is an R -binomial algebra.

Our main motivating example is the cochain algebra of a space. In Theorem 5.12 we show that, for any Δ -set X , the cellular cochain algebra $C = (C^*(X; R), d)$ is a binomial cup-one dga, with maps $\cup_1 : C^1 \otimes_R C^1 \rightarrow C^1$ given by $(a \cup_1 b)(e) = a(e) \cdot b(e)$ for all 1-simplices e and with the \circ map equal to Steenrod’s \cup_2 -product, with $\circ = \cup_2 : C^2 \otimes_R C^2 \rightarrow C^2$ given by $(v \circ w)(s) = v(s) \cdot w(s)$, for all 2-simplices s , and with binomial maps given by $\zeta_n(a)(e) = \binom{a(e)}{n}$ when $R = \mathbb{Z}$ and analogously for $R = \mathbb{F}_p$.

1.3 Free binomial cup-one dgas

These structures allow us to define in Section 6 the *free binomial cup-one graded algebra*, $T = T_R^*(X)$, on a set X . When $R = \mathbb{Z}$, the starting point is the ring $\text{Int}(\mathbb{Z}^X) = \{q \in \mathbb{Q}[X] \mid q(\mathbb{Z}^X) \subseteq \mathbb{Z}\}$ of integrally valued polynomials with variables in X . This is a binomial ring, generated by the polynomials $\binom{X}{\mathbf{n}} = \prod_{x \in X} \binom{x}{n_x}$ with $\mathbf{n}_x \in \mathbb{Z}_{\geq 0}$. We define T to be the tensor algebra on \mathfrak{m}_X , the maximal ideal at 0 in $\text{Int}(\mathbb{Z}^X)$. When $R = \mathbb{F}_p$, an analogous definition applies, with suitable modifications. In either case, we have R -linear maps $\cup_1 : T^1 \otimes T^1 \rightarrow T^1$, given by $a \cup_1 b = ab$ and $\circ : T^2 \otimes T^2 \rightarrow T^2$ given by $(a_1 \otimes a_2) \circ (b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2)$. Using the binomial structure on T^1 , we show that the map sending each $x \in X$ to 0 extends to a linear map $d_0 : T^1 \rightarrow T^2$. In turn, d_0 extends to a differential on the whole tensor algebra by the graded Leibniz rule, and $(T_R(X), d_0)$ is then a binomial cup-one dga. The key result that allows us to construct the differential d_0 is Theorem 6.6, which reads as follows.

Theorem 1.1 *Let $d : T_R(X) \rightarrow T_R(X)$ be a degree-one map satisfying the \cup_1 - d formula and the Leibniz rule. Then $d^2(x) = 0$ for all $x \in X$ if and only if $d^2(u) = 0$ for all $u \in T_R(X)$, in which case $(T_R(X), d)$ is a binomial cup-one dga.*

Taking $d(x) = 0$ for all $x \in X$ yields the differential d_0 from above. As shown in [35], this differential is compatible with the binomial structure on $\mathbb{T}_R(X)$; more precisely,

$$(1-2) \quad d_0(\zeta_n(x)) = -\sum_{\ell=1}^{n-1} \zeta_\ell(x) \otimes \zeta_{n-\ell}(x),$$

for all $x \in X$, and for all $n \geq 1$ when $R = \mathbb{Z}$ and for $1 \leq n \leq p - 1$ when $R = \mathbb{F}_p$. As an application of the methods developed here, we give in Theorem 7.10 a quicker, more conceptual proof of this result.

1.4 Differentials defined by admissible maps

A key thread in our paper involves the correspondence (described in Section 2) between a magma, that is, a set M with a binary operation $\mu : M \times M \rightarrow M$, and a certain 2-dimensional Δ -set, $\Delta^{(2)}(M)$. In the case when μ is associative, that is, (M, μ) is a semigroup, this 2-complex extends to an infinite-dimensional cell complex $\Delta(M)$, whose n -simplices are given by ordered n -tuples of elements in M . Moreover, if (M, μ) is a group, then $\Delta(M)$ is the cell complex of the bar construction applied to M ; that is, an Eilenberg–MacLane classifying space for M .

Properties of the cellular cochain algebras $(C^*(\Delta(M); R), d_\Delta)$ are used in Section 7 to derive properties of differential graded algebras in our category of binomial \cup_1 -dgas, as follows. Given a set X and a set map $\tau : X \rightarrow \mathbb{T}_R^2(X)$, we start by defining a binary operation, $\mu_\tau : M \times M \rightarrow M$, on the R -module $M = M(X, R)$ of all functions from X to the ring $R = \mathbb{Z}$ or \mathbb{F}_p . Letting $\Delta^{(2)}(M_\tau)$ be the 2-dimensional Δ -set associated to the magma $M_\tau = (M, \mu_\tau)$, we define a degree-preserving, R -linear map, $\psi = \psi_{X, \mu} : \mathbb{T}_R^{\leq 2}(X) \rightarrow C^*(\Delta^{(2)}(M_\tau); R)$. This map sends $1 \in \mathbb{T}_R^0(X) = R$ to the unit 0-cochain; a polynomial $q \in \mathbb{T}_R^1(X) = \mathfrak{m}_X$ to the 1-cochain whose value on a 1-simplex $\mathbf{a} : X \rightarrow R$ is $q(\mathbf{a})$; and a tensor $q \otimes q' \in \mathbb{T}_R^2(X)$ to the 2-cochain whose value on a 2-simplex $(\mathbf{a}, \mathbf{a}')$ is $q(\mathbf{a}) \cdot q'(\mathbf{a}')$. We then show in Lemma 7.1 that the map ψ is a monomorphism which commutes with cup products, cup-one products, and the \circ maps. Using the embedding ψ , we show in Theorem 7.3 that there is a unique extension of the map $\tau : X \rightarrow \mathbb{T}_R^2(X)$ to an R -linear map $d_\tau : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X)$ that satisfies the Leibniz rule and the \cup_1 - d formula.

To make further headway, we focus on the case when M_τ is a semigroup (we then say τ is *admissible*), and consider the associated cell complex, $\Delta(M_\tau)$. We then show in Theorem 7.6 that the map ψ extends uniquely to an inclusion $\psi : (\mathbb{T}_R(X), d_\tau) \hookrightarrow C^*(\Delta(M_\tau); R)$ that satisfies $\psi \circ d_\tau = d_\Delta \circ \psi$, from which it follows that d_τ^2 is the zero map. These results can be summarized as follows.

Theorem 1.2 *If $\tau : X \rightarrow \mathbb{T}_R^2(X)$ is admissible, then $d_\tau^2 \equiv 0$ and the map $\mathbb{T}^1(X) \rightarrow C^1(\Delta(M_\tau); R)$ given by $q \mapsto (\mathbf{a} \mapsto q(\mathbf{a}))$ extends uniquely to a monomorphism*

$$\psi_X : (\mathbb{T}_R(X), d_\tau) \hookrightarrow (C^*(\Delta(M_\tau); R), d_\Delta)$$

of binomial cup-one dgas.

1.5 Hirsch extensions

In Section 8 we continue laying out the groundwork for the construction of 1-minimal models for \cup_1 -dgas. The first step in the construction is a free binomial \cup_1 -dga of the form $(\mathbb{T}_R(X), d_0)$ with X a finite set of n elements. In this case, the map $\tau : X \rightarrow \mathbb{T}_R^2(X)$ is the zero map, and the corresponding R -module, $M = M_\tau$, is isomorphic to R^n . Using a spectral sequence argument, we prove in Theorem 8.9 that the map $\psi_X : (\mathbb{T}_R(X), d_0) \rightarrow C^*(B(R^n); R)$ induces an isomorphism on cohomology.

As in rational homotopy theory, Hirsch extensions of free binomial \cup_1 -dgas are the basic building blocks for constructing 1-minimal models. An inclusion $i : (\mathbb{T}_R(X), d) \rightarrow (\mathbb{T}_R(X \cup Y), \bar{d})$ is called a *Hirsch extension* if $\bar{d}(y)$ is a cocycle in $\mathbb{T}_R^2(X)$ for all $y \in Y$. As shown in Theorem 8.2, there is a bijection between maps of sets from Y to cocycles in $\mathbb{T}_R^2(X)$ and Hirsch extensions of this sort.

Assume now that $X = \bigcup_{i \geq 1} X_i$, with each X_i a finite set and $X_1 \neq \emptyset$. An R -dga $\mathbb{T} = (\mathbb{T}_R(X), d)$ is called a *colimit of Hirsch extensions* if the differential d restricts to differentials d_n on $\mathbb{T}_R(X^n)$ such that $d_1(x) = 0$ for all $x \in X_1$ and each dga $(\mathbb{T}_R(X^{n+1}), d_{n+1})$ is a Hirsch extension of $(\mathbb{T}_R(X^n), d_n)$. To such a colimit of Hirsch extensions, \mathbb{T} , we associate in Lemma 8.11 a pronilpotent group, $G_\mathbb{T}$, together with a \cup_1 -dga map, $\psi_\mathbb{T} : \mathbb{T} \rightarrow C^*(B(G_\mathbb{T}); R)$, inducing an isomorphism on H^1 . As shown in Theorem 8.13, if $\psi_\mathbb{T}$ is a quasi-isomorphism and $0 \rightarrow F \rightarrow \bar{G} \xrightarrow{\pi} G \rightarrow 1$ is a central extension of groups with F a finitely generated, free R -module, then there is a Hirsch extension $i : \mathbb{T} \hookrightarrow \bar{\mathbb{T}}$ such that $\bar{G} = G_{\bar{\mathbb{T}}}$, the map $\psi_{\bar{\mathbb{T}}}$ is also a quasi-isomorphism, and $B(\pi)^* \circ \psi_\mathbb{T} = \psi_{\bar{\mathbb{T}}} \circ i$.

In [36], we build on this correspondence between colimits of Hirsch extensions and sequences of central extensions of groups. For the cochain algebra $A = C^*(Y; R)$ of a path-connected space Y , we will describe a concrete relationship between the group $G_\mathbb{T}$ and the fundamental group $\pi_1(Y)$, where \mathbb{T} is a colimit of Hirsch extensions together with a 1-quasi-isomorphism $\rho : \mathbb{T} \rightarrow C^*(Y; R)$.

1.6 1-minimal models

In Sections 9 and 10 (which form the core of this work), we develop these ideas into a theory of 1-minimal models over a ring R equal to \mathbb{Z} or \mathbb{F}_p . A key technical tool is provided by the following lifting criterion (Theorem 9.4): Let $f : A \rightarrow A'$ be a surjective 1-quasi-isomorphism between binomial cup-one R -dgas and let $\varphi : \mathbb{T} \rightarrow A'$ be a morphism from a colimit of Hirsch extensions to A' ; there is then a morphism $\hat{\varphi} : \mathbb{T} \rightarrow A$ such that $f \circ \hat{\varphi} = \varphi$.

Now let (A, d) be a binomial cup-one R -dga. A *1-minimal model* for A is a colimit of Hirsch extensions $\mathcal{M}_n = (\mathbb{T}_R(X^n), d_n)$, together with morphisms $\rho_n : \mathcal{M}_n \rightarrow A$ compatible with the Hirsch extensions of \mathcal{M}_n into \mathcal{M}_{n+1} . Additionally, the map $H^1(\rho_1) : H^1(\mathcal{M}_1) \rightarrow H^1(A)$ is required to be an isomorphism; in particular, X_1 corresponds to a basis for $H^1(A)$. For $n \geq 1$, the set X_{n+1} is a basis for the free submodule $\ker(H^2(\rho_n)) \subset H^2(\mathcal{M}_n)$ given by the cohomology classes of the 2-cocycles $d_{n+1}(x)$ with $x \in X_{n+1}$.

In Theorems 9.8 and 10.3 we show that every binomial cup-one dga admits (under some mild finiteness assumptions) a 1-minimal model, unique up to isomorphism. These results may be summarized as follows.

Theorem 1.3 *Let (A, d_A) be a binomial cup-one dga over $R = \mathbb{Z}$ or \mathbb{F}_p , with p a prime. Assume $H^0(A) = R$ and $H^1(A)$ is a finitely generated, free R -module. Then,*

- (1) *There is a 1-minimal model, $\mathcal{M} = (\mathbb{T}_R(X), d)$, for A , and a structural morphism, $\rho : \mathcal{M} \rightarrow A$, that is a 1-quasi-isomorphism.*
- (2) *Given 1-minimal models, $\rho : \mathcal{M} \rightarrow A$ and $\rho' : \mathcal{M}' \rightarrow A$, there is an isomorphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ and a dga homotopy $\Phi : \mathcal{M} \rightarrow A \otimes_R C^*([0, 1]; R)$ from ρ to $\rho' \circ f$.*

In the case when (A, d_A) admits an augmentation, that is, a dga morphism $\varepsilon : A \rightarrow R$, the isomorphism f from above is unique (in the category of augmented dgas). More precisely, we prove in Theorems 9.10 and 10.8 that A has an augmented 1-minimal model, \mathcal{M} , such that the structural morphism is an augmentation-preserving 1-quasi-isomorphism. Moreover, given augmented 1-minimal models, $\rho : \mathcal{M} \rightarrow A$ and $\rho' : \mathcal{M}' \rightarrow A$, there is a *unique* augmentation-preserving isomorphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ such that ρ is augmentation-preserving homotopic to $\rho' \circ f$.

1.7 Compatibility of integer and rational 1-minimal models

In Section 11 we show that the integer 1-minimal model of a space Y tensored with the rationals is weakly equivalent as a dga to the 1-minimal model for Y in rational homotopy theory.

The algebra of polynomial forms with rational coefficients on a standard simplex was used by Sullivan in [42] to define the algebra $A_{\text{PL}}(Y)$ of compatible polynomial forms on the singular simplices of a space Y ; this algebra is a commutative dga over the rationals. The properties of the 1-minimal model of a cdga over \mathbb{Q} are analogous to — and in fact are the motivation for — the properties we use to define the 1-minimal model for a binomial cup-one dga over \mathbb{Z} or \mathbb{F}_p .

In addition to the Sullivan algebra $A_{\text{PL}}(Y)$ and the singular cochain algebra $C^*(Y; \mathbb{Q})$, there is a dga over the rationals $\text{CA}(Y)$ with the property proved in [10] that for topological spaces Y , there are natural quasi-isomorphisms $C^*(Y; \mathbb{Q}) \rightarrow \text{CA}(Y) \leftarrow A_{\text{PL}}(Y)$. Consequently, $A_{\text{PL}}(Y)$ is weakly equivalent (as a dga) to $C^*(Y; \mathbb{Q})$. The following result, Theorem 11.4, shows that weak equivalence extends to 1-minimal models.

Theorem 1.4 *Let Y be a connected topological space with $H^1(Y; \mathbb{Z})$ finitely generated. Then the 1-minimal model for $C^*(Y; \mathbb{Z})$ tensored with the rationals is weakly equivalent as a dga to the 1-minimal model in rational homotopy theory for $A_{\text{PL}}(Y)$.*

As we shall see next, although the 1-minimal model for $C^*(Y; \mathbb{Z})$ is weakly equivalent over \mathbb{Q} to the (rational) 1-minimal model for $A_{\text{PL}}(Y)$, the integral version does contain more refined information than its rational version.

1.8 n -step equivalence and triple Massey products

Given a positive integer n , we define in Section 12 the relation of *n -step equivalence* on the set of augmented binomial cup-one dgas (A, d) over \mathbb{Z} for which $H^0(A) = \mathbb{Z}$, $H^1(A)$ is finitely generated and torsion-free, and $H^2(A)$ is finitely generated. We then construct an infinite family of spaces that can be

distinguished using the 1-minimal model over \mathbb{Z} , though the same approach in rational homotopy theory fails to distinguish among the spaces in the family.

The definition of n -step equivalence is motivated as follows. If a morphism $\varphi : A \rightarrow A'$ induces isomorphisms of cohomology groups in degrees up to 2, then for each $n \geq 1$ there is an isomorphism of the n -th step in the respective 1-minimal models, $f_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$, such that this diagram commutes:

$$(1-3) \quad \begin{array}{ccc} H^2(\mathcal{M}_n) & \xrightarrow{H^2(f_n)} & H^2(\mathcal{M}'_n) \\ H^2(\rho_n) \downarrow & & \downarrow H^2(\rho'_n) \\ H^2(A) & \xrightarrow{H^2(\varphi)} & H^2(A') \end{array}$$

The horizontal arrows in (1-3) are isomorphisms. We say that A and A' are n -step equivalent if there are isomorphisms $f_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$ and $e_n : H^2(A) \rightarrow H^2(A')$ such that the diagram (1-3) commutes with $H^2(\varphi)$ replaced by e_n .

If A and A' are n -step equivalent, then the cokernels of the homomorphisms $H^2(\rho_n)$ and $H^2(\rho'_n)$ are isomorphic, and hence have isomorphic torsion subgroups. Given a space X with n -th step in the 1-minimal model given by (\mathcal{M}_n, ρ_n) , we define $\kappa_n(X) = \text{Tors}(\text{coker } H^2(\rho_n))$. The following result (proved in Theorem 12.4) relates the invariant $\kappa_n(X)$ of the n -step equivalence class of $C^*(X; \mathbb{Z})$ to the fundamental group of X .

Theorem 1.5 *Let X and X' be two connected Δ -complexes with first and second integral cohomology groups finitely generated.*

- (1) *If $\pi_1(X) \cong \pi_1(X')$, then $\kappa_n(X) \cong \kappa'_n(X)$ for all $n \geq 1$.*
- (2) *If $\kappa_n(X) \not\cong \kappa'_n(X)$ for some $n \geq 1$, then the cochain algebras $C^*(X; \mathbb{Z})$ and $C^*(X'; \mathbb{Z})$ are not n -step equivalent.*

We apply this result to an infinite family of links in the three-sphere, $\{L(n)\}_{n \geq 1}$, the first term of which is the well-known Borromean rings. Set $X(n)$ equal to the complement of $L(n)$ in S^3 . In Proposition 12.5, we show that $\kappa_2(X(n)) = \mathbb{Z}_n \oplus \mathbb{Z}_n$, so by part (2) of Theorem 1.5, $X(n)$ and $X(m)$ are not 2-step equivalent for $n \neq m$. We also show that the Sullivan algebras $A_{\text{pl}}(X(n))$ and $A_{\text{pl}}(X(m))$ are 2-step equivalent for all $n, m \geq 1$.

In the proof of Proposition 12.5, the cokernel of $H^2(\rho_2)$ is given by triple Massey products of cohomology classes in $H^1(X(n); \mathbb{Z})$. This framework provides the context for defining restricted Massey products, which is a particular case of a more general construction that will be developed in [33]. The theory of generalized Massey products continues the program initiated in [34] and is being developed more fully in [33].

1.9 1-minimal models for nilmanifolds

Finally, in Section 13 we establish a correspondence between finitely generated, torsion-free nilpotent groups (for short, \mathcal{T} -groups) and finite colimits of Hirsch extensions over \mathbb{Z} . The main result (Theorem 13.1) may be summarized as follows.

Theorem 1.6 *There is a bijection between \mathcal{T} -groups and finite colimits of Hirsch extensions which preserves cohomology algebras and induces an equivalence of categories between isomorphism classes of \mathcal{T} -groups and augmentation-preserving, isomorphism classes of finitely generated, 1-minimal models.*

Every \mathcal{T} -group G can be realized as the fundamental group of a nilmanifold M , which is a classifying space for G . In [25], Lambe and Priddy sharpened a classical result of Nomizu [30], which identifies the cohomology algebra of M over \mathbb{R} with the Lie algebra cohomology of the corresponding nilpotent real Lie algebra. They associated to G a Lie algebra $L(G)$ defined over the subring of \mathbb{Q} generated by the coefficients of the Hall polynomials of G , and showed that $H^*(G; S) \cong H^*(L(G); S)$, for a certain subring $S \subset \mathbb{Q}$.

One can ask whether this isomorphism holds over the (possibly smaller) ring R . For instance, Kuzmin and Semenov showed in [23] that $H^*(G; \mathbb{Z}) \cong H^*(L(G); \mathbb{Z})$ when G is a free nilpotent group of class 2. It still appears to be an open question when this is the case in general. The approach we take is illustrated by two examples in Section 13.2, where we apply our theory of integral 1-minimal models to the general question of whether the group and Lie algebra cohomologies of a \mathcal{T} -group are isomorphic with integer coefficients. We will further develop this approach in [36].

2 Delta-sets, magmas, and cochain algebras

2.1 Δ -sets and Δ -complexes

We start the section by reviewing the notion of a Δ -complex, in the sense of Rourke and Sanderson [38]; see also Hatcher [19] and Friedman [14]. We will view such a complex as the geometric realization of the corresponding Δ -set; see [14].

An (abstract) n -simplex Δ^n is simply a finite ordered set, $(0, 1, \dots, n)$. The face maps $d_i : \Delta^n \rightarrow \Delta^{n-1}$, given by omitting the i -th element in the set, satisfy $d_i d_j = d_{j-1} d_i$ whenever $0 \leq i < j \leq n$. The geometric realization of the simplex, $|\Delta^n|$, is the convex hull of $n+1$ affinely independent vectors in \mathbb{R}^{n+1} , endowed with the subspace topology; the face maps induce continuous maps, $d_i : |\Delta^n| \rightarrow |\Delta|^{n-1}$.

More generally, a Δ -set consists of a sequence of sets $X = \{X_n\}_{n \geq 0}$ and maps $d_i : X_n \rightarrow X_{n-1}$ for each $0 \leq i \leq n$ such that $d_i d_j = d_{j-1} d_i$ whenever $i < j$. This is the generalization of the notion of ordered (abstract) simplicial complex, where the sets X_n are the sets of n -simplices and the maps d_i are the face maps. We refer to $X^{(n)} = \{X_i\}_{i=0}^n$ as the n -skeleton of the Δ -set, and say that $X^{(n)}$ has dimension (at most) n .

The geometric realization of a Δ -set X is the topological space $|X| = \coprod_{n \geq 0} X_n \times |\Delta^n| / \sim$, where \sim is the equivalence relation generated by $(x, d^i(p)) \sim (d_i(x), p)$ for $x \in X_{n+1}$, $p \in |\Delta^n|$, and $0 \leq i \leq n$, where $d^i : |\Delta^n| \rightarrow |\Delta^{n+1}|$ is the inclusion of the i -th face. Such a space is called a Δ -complex, and can be viewed either as a special kind of CW-complex, or a generalized simplicial complex.

The assignment $X \mapsto |X|$ is functorial: if $f : X \rightarrow Y$ is a map of Δ -sets (i.e., f is a family of maps $f_n : X_n \rightarrow Y_n$ commuting with the face maps), there is an obvious realization, $|f| : |X| \rightarrow |Y|$, and this is a (continuous) map of Δ -complexes.

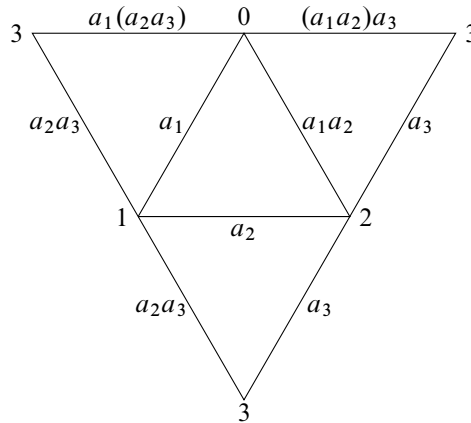


Figure 1: Some simplices in $\Delta^{(2)}(M)$.

The chain complex of a Δ -set X , denoted by $C_*(X; \mathbb{Z})$, coincides with the simplicial chain complex of its geometric realization: for each $n \geq 0$, the chain group $C_n(X)$ is the free abelian group on X_n , while the boundary maps $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ are the linear maps given by $\partial_n = \sum_i (-1)^i d_i$. If B is an abelian group, the chain complex of X with coefficients in B is defined as $C_*(X; B) = C_*(X; \mathbb{Z}) \otimes B$. The cochain complex $C^*(X; B)$ is defined by setting $C^n(X; B) = \text{Hom}(C_n(X), B)$ and dualizing the differentials. We denote by $H_*(X; B)$ and $H^*(X; B)$, respectively, the homology groups of these complexes.

2.2 From binary operations to Δ -sets

Let $M = (M, \mu)$ be a magma, that is, a set M equipped with a binary operation, $\mu : M \times M \rightarrow M$, commonly written as $(a_1, a_2) \mapsto a_1 a_2$. These data determine a 2-dimensional Δ -set, denoted by $\Delta^{(2)}(M)$, whose geometric realization can be described as follows. There is a single vertex; each element $a \in M$ gives a 1-simplex, and to each ordered pair of 1-simplices, a_1 and a_2 , we assign a 2-simplex, (a_1, a_2) , with front face a_1 , back face a_2 , and third face equal to $a_1 a_2$.

This construction is functorial, in the following sense. Suppose $h : (M, \mu) \rightarrow (M', \mu')$ is a morphism of magmas, that is, $\mu'(h(a_1), h(a_2)) = h(\mu(a_1, a_2))$ for all $a_1, a_2 \in M$. Then h determines in a straightforward manner a simplicial map between the respective Δ -complexes, $\Delta(h) : \Delta^{(2)}(M', \mu') \rightarrow \Delta^{(2)}(M, \mu)$, so that $\Delta(h \circ g) = \Delta(h) \circ \Delta(g)$.

Now suppose $M = (M, \mu)$ is a semigroup, that is, the operation μ on the magma M is associative. Then, as indicated in Figure 1, the 2-dimensional Δ -complex corresponding to $\Delta^{(2)}(M)$ extends to a 3-dimensional Δ -set, whose 3-simplices are ordered triples, (a_1, a_2, a_3) , of elements in A . A routine computation shows that this construction can be pushed through in all dimensions; the upshot is summarized in the next lemma.

Lemma 2.1 *Let M be a semigroup, and let $\Delta^{(2)}(M)$ be the 2-dimensional Δ -set determined by the underlying magma. Then $\Delta^{(2)}(M)$ is the 2-skeleton of a Δ -set, $\Delta(M)$, whose n -simplices are given by ordered n -tuples of elements in M .*

Given a semigroup M , we define a Δ -set $S(M)$, as follows. We let $S_n(M)$ equal to the set of all functions, f , from the 1-simplices of the abstract n -simplex Δ^n to M with the property that $f(i, \ell) = f(i, j) \cdot f(j, \ell)$ for all $0 \leq i < j < \ell \leq k$. The face maps $d_i : S_n(M) \rightarrow S_{n-1}(M)$ are given by the restriction of f to the faces of Δ^n . It is readily seen that the Δ -set $S(M)$ coincides with $\Delta(M)$.

Remark 2.2 Of particular importance is the case when M is a *monoid*, that is, a semigroup with multiplication $\mu : M \times M \rightarrow M$ and two-sided identity e . Then $\Delta(M)$ is the bar construction on M : the corresponding Δ -complex, $B(M) = |\Delta(M)|$, has a single 0-cell, and an n -cell $[g_1 | \dots | g_n]$ for each n -tuple $(g_1, \dots, g_n) \in M^n$. The chain complex $C_*(B(M); \mathbb{Z})$ yields a resolution by free $\mathbb{Z}[M]$ -modules of the group \mathbb{Z} , viewed as a trivial module over the monoid-ring $\mathbb{Z}[M]$. Finally, if $M = G$ is a group, then $B(G)$ is an Eilenberg–MacLane classifying space $K(G, 1)$; see [26, Chapter 10] and also [3; 5; 7].

2.3 Cocycles and Δ -complexes

Let $M = (M, \mu)$ be a magma, with multiplication $\mu : M \times M \rightarrow M$ written as $\mu(a_1, a_2) = a_1 a_2$, and let B be an abelian group, together with a map of sets $\nu : M \times M \rightarrow B$, written $(a_1, a_2) \mapsto a_1 * a_2$. Defining a binary operation on $M \times B$ by

$$(2-1) \quad (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 + b_2 + a_1 * a_2)$$

for all $a_i \in M$ and $b_i \in B$ turns the set $M \times B$ into a magma which we call the *extension of (M, μ) by ν* . It is readily seen that this binary operation is associative if and only if

$$(2-2) \quad a_2 * a_3 + a_1 * (a_2 a_3) = a_1 * a_2 + (a_1 a_2) * a_3$$

for all $a_i \in M$. If (M, μ) has a two-sided identity, e , and if $(a, b_1) \cdot (e, b_2) = (e, b_2) \cdot (a, b_1)$ and $(e, b_1) \cdot (e, b_2) = (e, b_1 + b_2)$ for all $a \in M$ and $b_1, b_2 \in B$, then the extension is called a *central extension*.

From the correspondence between 2-simplices in $|\Delta(M)|$ and ordered pairs of elements in M , it follows that ν may be viewed as an element in $C^2(\Delta(M); B)$. If the magma (M, μ) has a two-sided identity e , we say that ν is *normalized* if $\nu(a, e) = \nu(e, a) = 0$ for all $a \in M$. In the case when (M, μ) is a monoid (that is, the operation μ is associative and has a two-sided identity e), a cochain $\xi \in C^k(\Delta(M); B)$ is called *normalized* if $\xi(a_1, \dots, a_k) = 0$ whenever $a_i = e$ for some i .

The following lemma gives conditions on (M, μ) and ν for the extension of a semigroup to be a semigroup, the extension of a monoid to be a monoid, and for the extension of a group to be a group. Note that a monoid is a semigroup with identity.

Lemma 2.3 *Given a magma (M, μ) and a map $\nu : M \times M \rightarrow B$, the extension $E = (M \times B, \cdot)$ of M by the abelian group B as defined above has the following properties.*

- (1) *Suppose (M, μ) has a two-sided identity e and ν is a normalized cochain. Then E is a central extension.*
- (2) *Suppose (M, μ) is a semigroup. Then (E, \cdot) is a semigroup if and only if ν is a cocycle in $C^2(\Delta(M); B)$.*

(3) Suppose (M, μ) is a monoid and ν is a normalized cocycle in $Z^2(\Delta(M); B)$. Then (E, \cdot) is a monoid.

(4) Suppose (M, μ) is a group and ν is a normalized cocycle in $Z^2(\Delta(M); B)$. Then (E, \cdot) is a group.

Proof Part (1) follows at once from the definitions.

Part (2): Let us view the map $\nu : M \times M \rightarrow B$ as a B -valued 2-cochain on $\Delta(M)$. Using Figure 1 to identify a triple $(a_1, a_2, a_3) \in M^3$ with the standard 3-simplex on vertices $0, \dots, 3$ and then to identify ordered pairs of elements in M with 2-simplices, we get

$$\begin{aligned} \delta\nu(a_1, a_2, a_3) &= \nu([1, 2, 3]) - \nu([0, 2, 3]) + \nu([0, 1, 3]) - \nu([0, 1, 2]) \\ &= \nu(a_2, a_3) - \nu(a_1 a_2, a_3) + \nu(a_1, a_2 a_3) - \nu(a_1, a_2) \\ &= a_2 * a_3 - (a_1 a_2) * a_3 + a_1 * (a_2 a_3) - a_1 * a_2. \end{aligned}$$

Hence, (2-2) is satisfied precisely when ν is a cocycle.

Part (3): We know from part (2) that $E = (M \times B, \cdot)$ is a semigroup. Moreover, if e is a 2-sided identity for M , a routine computation shows that $(e, 0)$ is a 2-sided identity for E .

Part (4): It follows from parts (2) and (3) that E is a monoid. A routine computation shows $(a, b) \in M \times B$ has $(a^{-1}, -b - a * a^{-1})$ as a 2-sided inverse. \square

3 Differential graded algebras and homotopies

3.1 Differential graded algebras

Throughout this section, we work over a fixed coefficient ring R , assumed to be commutative and with unit 1. We start with some basic definitions.

Definition 3.1 A *graded algebra* over R is an R -algebra A such that the underlying R -module is a direct sum of R -modules, $A = \bigoplus_{i \geq 0} A^i$, and such that the product $A \otimes_R A \rightarrow A$ sends $A^i \otimes_R A^j$ to A^{i+j} .

We refer to the multiplication maps $\cup : A^i \otimes_R A^j \rightarrow A^{i+j}$, given by $\cup(a \otimes b) = a \cup b$ as the *cup-product maps*; we also refer to the elements of A^i as i -cochains. A morphism of graded algebras is a map of R -algebras preserving degrees.

Definition 3.2 A *differential graded algebra* over R (for short, a dga) is a graded R -algebra $A = \bigoplus_{i \geq 0} A^i$ endowed with a degree-1 map, $d : A \rightarrow A$, satisfying $d^2 = 0$ and the graded Leibniz rule,

$$d(a \cup b) = da \cup b + (-1)^{|a|} a \cup db,$$

for all homogenous elements $a, b \in A$, where $|a|$ is the degree of a .

We denote by $[a] \in H^i(A)$ the cohomology class of a cocycle $a \in Z^i(A)$. As usual, the graded R -module $H^*(A)$ inherits an algebra structure from A .

Observe that A^0 is a subring of A and the structure map $R \rightarrow A$ sends the unit $1 \in R$ to the unit of A , which we will also denote by 1 , and which necessarily has degree 0. Consequently, R may be viewed as a subring of A^0 , and the graded pieces A^i may be viewed as A^0 -modules. We say that A is *connected* if the structure map $R \rightarrow A$ maps R isomorphically to A^0 , and we say that A is *graded commutative* (for short, A is a cdga) if $ab = (-1)^{|a||b|}ba$ for all homogeneous elements $a, b \in A$.

A morphism of differential graded R -algebras is an R -linear map $\varphi : A \rightarrow B$ between two dgas which preserves the grading and commutes with the respective differentials and products. The induced map in cohomology, $\varphi^* : H^*(A) \rightarrow H^*(B)$, $[a] \mapsto [\varphi(a)]$, is a morphism of graded R -algebras. The map φ is called a *quasi-isomorphism* if φ^* is an isomorphism. Two dgas are called *weakly equivalent* if there is a zigzag of quasi-isomorphisms connecting one to the other; plainly, this is an equivalence relation among dgas. A dga (A, d) is said to be *formal* if it is weakly equivalent to its cohomology algebra, $H^*(A)$, endowed with the zero differential.

All these notions have partial analogs. Fix an integer $q \geq 1$. A dga map $\varphi : A \rightarrow B$ is a *q -quasi-isomorphism* if the induced homomorphism, $\varphi^* : H^i(A) \rightarrow H^i(B)$, is an isomorphism for $i \leq q$ and a monomorphism for $i = q + 1$. Two dgas are called *q -equivalent* if they may be connected by a zigzag of q -quasi-isomorphisms. Finally, a dga (A, d) is *q -formal* if it is q -equivalent to $(H^*(A), d = 0)$.

Remark 3.3 Suppose $\varphi : A \rightarrow B$ is a surjective q -quasi-isomorphism. Then the long exact sequence in cohomology induced by the exact sequence of cochain complexes $0 \rightarrow \ker(\varphi) \rightarrow A \xrightarrow{\varphi} B \rightarrow 0$ implies that $H^i(\ker(\varphi)) = 0$ for $i \leq q + 1$.

If (A, d_A) and (B, d_B) are two dgas, then the tensor product of the underlying graded R -modules, $A \otimes_R B$, acquires a dga structure, with multiplication and differential given on homogeneous elements by $(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'||b|}aa' \otimes bb'$ and $d_{A \otimes_R B}(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|}a \otimes d_B(b)$. The direct product $A \times B$ also has a natural structure of a dga, with $(a, b) \cdot (a', b') = aa' \otimes bb'$ and $d_{A \times B}(a, b) = (d_A(a), d_B(b))$. If $\varphi : A \rightarrow B$ and $\varphi' : A' \rightarrow B$ are two dga maps, then their *fiber product*, denoted by $A \times_B A'$, is the sub-dga of $A \times B$ consisting of all pairs (a, b) with $\varphi(a) = \varphi'(b)$.

3.2 Cochain algebras

The motivating example for us is the singular cochain algebra $C^*(X; R)$ on a space X , with coefficients in a commutative ring R . This is an R -dga, with differentials given by the usual coboundary maps, and with multiplication given by the cup product. We will be mostly interested in the case when X is a simplicial complex, or, more generally, a Δ -complex (see [14; 19; 38]). We will view such a complex as the geometric realization of the corresponding abstract simplicial complex or Δ -set, respectively, and we will use the simplicial cochain algebra of X , still to be denoted by $C^*(X; R)$. Let us note that the structure map $R \rightarrow C^0(X; R)$ sends an element $r \in R$ to the cochain whose value on every vertex is r .

Example 3.4 Let I be the closed interval $[0, 1]$, viewed as a simplicial complex in the usual way, and let $C = C^*(I; R)$ be its cochain algebra over R . Then $C^0 = R \oplus R$ with generators t_0, t_1 corresponding to the endpoints 0 and 1, and $C^1 = R$ with generator u . The differential $d : C^0 \rightarrow C^1$ is given by

$dt_0 = -u$ and $dt_1 = u$, while the multiplication is given on generators by $t_i t_j = \delta_{ij} t_i$, $t_0 u = u t_1 = u$, and $u t_0 = t_1 u = 0$. The cocycle $t_0 + t_1$ is the unit of C . Furthermore, $H^*(C) = R$, concentrated in degree 0.

Example 3.5 Now let G be a group. Recall from Section 2.2 that the Δ -complex $B(G)$ for the bar construction on G has one 0-cell, one 1-cell $[g]$ for each $g \in G$, and one 2-cell for each ordered pair $[g_1|g_2]$ of elements in G . Thus, the 1-cochains are functions $f : G \rightarrow R$ and the 2-cochains are functions from $G \times G$ to R . The cup product and differential are given by $(f \cup h)([g_1|g_2]) = f(g_1) \cdot h(g_2)$ and $(df)([g_1|g_2]) = f([g_1]) + f([g_2]) - f([g_1 \cdot g_2])$, where \cdot denotes the product in R or G , depending on the context.

Following R. H. Fox [13] and J. H. C. Whitehead [43; 44], we say that two maps of spaces, $f, g : X \rightarrow Y$, are n -homotopic if $f \circ h \simeq g \circ h$, for every map $h : K \rightarrow X$ from a CW-complex K of dimension at most n . A map $f : X \rightarrow Y$ is an n -homotopy equivalence (for some $n \geq 1$) if it admits an n -homotopy inverse. If such a map f exists, we say that X and Y have the same n -homotopy type. Two CW-complexes, X and Y , are said to be of the same n -type if their n -skeleta have the same $(n-1)$ -homotopy type. Any two connected CW-complexes have the same 1-type, and they have the same 2-type if and only if their fundamental groups are isomorphic.

A (cellular) map $f : X \rightarrow Y$ between two CW-complexes induces a morphism of dgas, $f^\# : C^*(Y; R) \rightarrow C^*(X; R)$, between the respective cochain algebras, and thus a morphism, $f^* : H^*(Y; R) \rightarrow H^*(X; R)$, between their cohomology algebras. If f is a homotopy equivalence, then $f^\#$ is a quasi-isomorphism of R -dgas. The next result, which develops ideas from [44], was proved in [35].

Theorem 3.6 [35] *If X and Y are CW-complexes of the same n -type, then the cochain algebras $C^*(X; R)$ and $C^*(Y; R)$ are $(n-1)$ -equivalent. In particular, if $\pi_1(X) \cong \pi_1(Y)$, then $C^*(X; R)$ and $C^*(Y; R)$ are 1-equivalent.*

The $(n-1)$ -equivalence between the n -skeleta of X and Y takes a special form, which we now recall, for it will be needed in the proof of Theorem 12.4. By [44, Theorem 6], there is a homotopy equivalence, f , from $\bar{X}^{(n)} = X^{(n)} \vee \bigvee_{i \in I} S_i^n$ to $\bar{Y}^{(n)} = Y^{(n)} \vee \bigvee_{j \in J} S_j^n$, for some indexing sets I and J . Let $q_X : \bar{X}^{(n)} \rightarrow X^{(n)}$ and $q_Y : \bar{Y}^{(n)} \rightarrow Y^{(n)}$ be the obvious collapse maps, and consider the induced morphisms on cochain algebras,

$$(3-1) \quad C^*(X^{(n)}; R) \xrightarrow{q_X^\#} C^*(\bar{X}^{(n)}; R) \xleftarrow{f^\#} C^*(\bar{Y}^{(n)}; R) \xleftarrow{q_Y^\#} C^*(Y^{(n)}; R).$$

The map $f^\#$ is a quasi-isomorphism, while $q_X^\#$ and $q_Y^\#$ are $(n-1)$ -quasi-isomorphisms; thus, (3-1) is the desired $(n-1)$ -equivalence between $C^*(X; R)$ and $C^*(Y; R)$.

3.3 Homotopy invariance

Let $C^*(I; R)$ be the cochain algebra of the interval I , as described in Example 3.4, and let $\eta_0, \eta_1 : C^*(I; R) \rightarrow R$ denote the R -linear maps induced by restriction to the endpoints of I ; that is to say, $\eta_i(t_j) = \delta_{ij}$ and $\eta_i(u) = 0$.

Definition 3.7 Two dga maps, $\varphi_0, \varphi_1 : A \rightarrow B$, are said to be *homotopic* (denoted by $\varphi_0 \simeq \varphi_1$) if there is a dga map $\Phi : A \rightarrow B \otimes_R C^*(I; R)$ such that the following diagram commutes for $i = 0, 1$:

$$(3-2) \quad \begin{array}{ccc} B \otimes_R C^*(I; R) & \xrightarrow{\text{id}_B \otimes \eta_i} & B \otimes_R R \\ \Phi \uparrow & & \parallel \\ A & \xrightarrow{\varphi_i} & B \end{array}$$

From the commutativity of the diagram (3-2) it follows that a homotopy Φ is given on elements $a \in A^i$ by $\Phi(a) = \varphi_0(a)t_0 + \varphi_1(a)t_1 - c(a)u$, for some $c(a) \in B^{i-1}$. In particular, if $a \in A^0$, then $c(a) = 0$, and so $\Phi(a) = \varphi_0(a)t_0 + \varphi_1(a)t_1$. The next theorem can be proven by a standard argument, in a manner similar to the proof of [10, Proposition 12.8(i)] (see also [18, Remark 5.10.3]).

Theorem 3.8 *Homotopic dga maps induce the same map on cohomology: if $\varphi_0 \simeq \varphi_1$, then $\varphi_0^* = \varphi_1^*$.*

3.4 Augmented dgas and the wedge sum

Let (A, d_A) be a differential graded algebra over a unital commutative ring R . Let us view the ground ring R as a dga concentrated in degree 0 and with differential $d = 0$. An *augmentation* for A , then, is a dga-map, $\varepsilon_A : A \rightarrow R$. We call the triple (A, d_A, ε_A) an *augmented dga*. A morphism in this category is a dga map, $\varphi : (A, d_A) \rightarrow (B, d_B)$, such that $\varepsilon_B \circ \varphi = \varepsilon_A$.

Recall that A is connected if the structure map $\sigma_A : R \rightarrow A^0$ is an isomorphism of rings; in this case we assume the augmentation $\varepsilon_A : A \rightarrow R$ then restricts to an isomorphism from A^0 to R . The composition $\varepsilon_A \circ \sigma_A : R \rightarrow R$ then is an isomorphism of R -algebras, and hence, is the identity map. Thus, if A is connected, it has a unique augmentation map. Moreover, if $\varphi : A \rightarrow B$ is an augmentation-preserving morphism between connected R -dgas, the map $\varphi^0 : A^0 \rightarrow B^0$ may be identified with id_R . In general, though, a dga may admit many augmentations.

If A and B are two augmented dgas, we denote by $A \vee B = A \times_R B$ the fiber product of the augmentation maps $\varepsilon_A : A \rightarrow R$ and $\varepsilon_B : B \rightarrow R$. Note that $(A \vee B)^0$ is the kernel of the map $(\varepsilon_A, -\varepsilon_B) : A^0 \oplus B^0 \rightarrow R$, while $(A \vee B)^i = A^i \oplus B^i$ for $i > 0$.

The motivation for these definitions comes from topology. Let X be a topological space, and let $C^*(X; R)$ be its singular cochain algebra. Choosing a basepoint $x_0 \in X$ yields an augmentation, $\varepsilon_0 : C^*(X; R) \rightarrow R$, which sends a 0-cochain ξ to its evaluation $\xi(x_0) \in R$ and any cochain of higher degree to 0.

Example 3.9 Let $C^*(I; R)$ be the cochain algebra of the unit interval $I = [0, 1]$ as in Example 3.4, and let $x_0 = 0$. Then $\varepsilon_0(t_0) = 1$, while $\varepsilon_0(t_1) = \varepsilon_0(u) = 0$.

If the function $f : (X, x_0) \rightarrow (Y, y_0)$ is a pointed map, then the induced morphism of cochain algebras, $f^* : C^*(Y; R) \rightarrow C^*(X; R)$, preserves the respective augmentations. Finally, if $X \vee Y$ is the wedge sum of two pointed spaces, then $C^*(X \vee Y; R)$ is the subalgebra of $C = C^*(X; R) \times C^*(Y; R)$ equal to C^n

in degrees $n > 0$ and $\{(a, b) \in C^0 : a(x_0) = b(y_0)\}$ in degree 0. Therefore, $C^*(X \vee Y; R)$ is isomorphic to $C^*(X; R) \vee C^*(Y; R)$.

3.5 Augmentation-preserving homotopies

Two augmented dga maps, $\varphi_0, \varphi_1 : A \rightarrow B$, are said to be *augmentation-preserving homotopic* if there is a homotopy $\Phi : A \rightarrow B \otimes_R C^*(I; R)$ between them such that

$$(3-3) \quad \begin{array}{ccc} A & \xrightarrow{\Phi} & B \otimes_R C^*(I; R) \\ \downarrow \varepsilon_A & & \downarrow \varepsilon_B \otimes \text{id} \\ R & \longrightarrow & R \otimes_R C^*(I; R) \\ & \searrow & \parallel \\ & & C^*(I; R) \end{array}$$

commutes, where the diagonal map is the structure map for the R -algebra $C^*(I; R)$, which sends 1 to $t_0 + t_1$. As noted in Section 3.3, the homotopy Φ is given on $a \in A^i$ by $\Phi(a) = \varphi_0(a)t_0 + \varphi_1(a)t_1 - c(a)u$, for some $c(a) \in B^{i-1}$. The commutativity of (3-3) implies that $\varepsilon_B(c(a)) = 0$. When both A and B are connected and A is generated in degree 1, augmentation-preserving homotopies take a very special form, which we describe in the proof of the next lemma.

Lemma 3.10 *Let A and B be augmented R -dgas such that A and B are connected and A is generated as a graded R -algebra by A^1 . Let $\varphi_0, \varphi_1 : A \rightarrow B$ be augmentation-preserving morphisms. If there is an augmentation-preserving homotopy between φ_0 and φ_1 , then $\varphi_0 = \varphi_1$.*

Proof Let $\Phi : A \rightarrow B \otimes_R C^*(I; R)$ be an augmentation-preserving homotopy between φ_0 and φ_1 . We claim that

$$(3-4) \quad \Phi(a) = \varphi_0(a)t_0 + \varphi_1(a)t_1 \quad \text{for all } a \in A.$$

To prove this claim, let $a \in A^i$ and write as before $\Phi(a) = \varphi_0(a)t_0 + \varphi_1(a)t_1 - c(a)u$, for some $c(a) \in B^{i-1}$. When $a \in A^0$, we necessarily have $c(a) = 0$. When $a \in A^1$, we have that $c(a) \in B^0$ and $\varepsilon_B(c(a)) = 0$; since B is connected, it follows that $c(a) = 0$. Therefore, (3-4) holds for all $a \in A^0 \oplus A^1$. Now recall that $t_i t_j = \delta_{ij} t_i$. Since φ_0, φ_1 , and Φ are all maps of graded algebras, A is generated in degree 1, and (3-4) holds for all $a \in A^1$, it follows that (3-4) holds for all $a \in A$.

Using (3-4), we have that $(\Phi \circ d_A)(a) = \varphi_0(d_A a)t_0 + \varphi_1(d_A a)t_1$ and

$$\begin{aligned} (d_{B \otimes_R C^*(I; R)} \circ \Phi)(a) &= d_{B \otimes_R C^*(I; R)}(\varphi_0(a)t_0 + \varphi_1(a)t_1) \\ &= \varphi_0(d_A a)t_0 + \varphi_1(d_A a)t_1 - (\varphi_0(a) - \varphi_1(a))u \end{aligned}$$

for every $a \in A$. Since Φ is a map of dgas, it follows that $\varphi_0(a) - \varphi_1(a) = 0$ for all $a \in A$. □

4 The Steenrod \cup_i -products

4.1 The \cup_i operations

We now enrich the notion of a differential graded algebra with extra structure, motivated by properties of the cochain algebra of a space, as laid out in the foundational paper of Steenrod [39], and further developed by Hirsch in [20].

Let X be a Δ -complex, and let $A = (C^*(X; R), d)$ be its cochain algebra with coefficients in a commutative ring R , with multiplication given by the cup product $\cup : A^p \otimes_R A^q \rightarrow A^{p+q}$. This R -dga comes endowed with R -linear maps, $\cup_i : A^p \otimes_R A^q \rightarrow A^{p+q-i}$, which coincide with the usual cup product when $i = 0$, vanish if either $p < i$ or $q < i$, and satisfy

$$(4-1) \quad d(a \cup_i b) = (-1)^{|a|+|b|-i} a \cup_{i-1} b + (-1)^{|a||b|+|a|+|b|} b \cup_{i-1} a + da \cup_i b + (-1)^{|a|} a \cup_i db,$$

$$(4-2) \quad (a \cup b) \cup_1 c = a \cup (b \cup_1 c) + (-1)^{|b|(|c|-1)} (a \cup_1 c) \cup b$$

for all homogeneous elements $a, b, c \in A$. We shall refer to (4-1) as the ‘‘Steenrod identities’’ and to (4-2), with the cup product to the left of the cup-one product, as the ‘‘Hirsch identity’’.

Steenrod’s \cup_i operations enjoy the following naturality property. Suppose $f : X \rightarrow Y$ is a map of Δ -complexes which preserves the ordering of the vertices of simplices. Then, by [39, Theorem 3.1], the induced map on cochains, $f^* : C^*(Y; R) \rightarrow C^*(X; R)$, is a morphism of differential graded algebras that commutes with the \cup_i products.

Steenrod’s \cup_i products also occur in the theory of noncommutative differential forms, as developed by M. Karoubi, N. Battikh, and A. Abbassi. We refer to [35] for an overview of this subject.

4.2 Cup and cup-one operations on 1-cochains

Henceforth, we will focus on the aforementioned operations on cochains in low degrees. Let (A, d) be an R -dga and assume we have an R -linear map $\cup_1 : A^1 \otimes_R A^1 \rightarrow A^1$ that satisfies the Steenrod identity,

$$(4-3) \quad d(a \cup_1 b) = -a \cup b - b \cup a + da \cup_1 b - a \cup_1 db,$$

for all $a, b \in A^1$. In particular, if $a, b \in Z^1(A)$ are 1-cocycles, we then have

$$(4-4) \quad d(a \cup_1 b) = -(a \cup b + b \cup a).$$

Under these assumptions, the operation $\cup_1 : Z^1(A) \otimes_R Z^1(A) \rightarrow A^2$ provides an explicit witness for the noncommutativity of the multiplication map $\cup : Z^1(A) \otimes_R Z^1(A) \rightarrow Z^2(A)$ and shows that $uv = -vu$ for elements $u, v \in H^1(A)$.

Now let X be a Δ -complex and let $C^*(X; R)$ be its cochain algebra with coefficients in a commutative ring R . By [39, Theorem 2.1], $u \cup_1 v = 0$ if either u or v is a 0-cochain. Formulas for computing the cup

products and cup-one products for 1-cochains $u, v \in C^1(X; R)$ are as follows:

$$(4-5) \quad (u \cup v)(s) = u(e_1) \cdot v(e_2), \quad (u \cup_1 v)(e) = u(e) \cdot v(e),$$

where in the first formula s is a 2-simplex with front face e_1 and back face e_2 , while in the second formula e is a 1-simplex, and \cdot denotes the product in R . In particular, the \cup_1 -product on $C^1(X; R)$ is both associative and commutative, and thus defines an R -algebra structure on $C^{\leq 1}(X; R)$.

Example 4.1 For the cochain algebra $C = C^*(I; R)$ from Example 3.4, the cup-one product $C^1 \otimes_R C^1 \rightarrow C^1$ is given by $u \cup_1 u = u$.

Example 4.2 Let G be a group, and let $C^*(B(G); R)$ be the cochain algebra of the bar construction on G , as described in Example 3.5. The \cup_1 -product on $C^1(B(G); R)$ is given by $(f \cup_1 h)([g]) = f(g) \cdot h(g)$.

4.3 Graded algebras with cup-one products

Recall from Section 3.1 that the structure map $R \rightarrow A$ sends the unit $1 \in R$ to the unit of A (which belongs to A^0); thus, R may be viewed as a subring of A^0 . Furthermore, the restriction of the map $\cup : A^0 \otimes A^1 \rightarrow A^1$ to $R \otimes_R A^1 \rightarrow A^1$ may be identified with the identity of A^1 , while the restriction of the map $\cup : A^0 \otimes_R A^0 \rightarrow A^0$ to $R \otimes_R R \rightarrow R$ may be identified with the identity of R .

Given a graded R -algebra A , we let $D^2(A)$ denote the *decomposables* in A^2 , that is, the image of the cup-product map $A^1 \otimes_R A^1 \rightarrow A^2$, or, the R -submodule of A^2 spanned by all elements of the form $a \cup b$ with $a, b \in A^1$.

Definition 4.3 A *graded R -algebra with cup-one products* is a graded R -algebra A equipped with cup-one product maps, $\cup_1 : A^1 \otimes_R A^1 \rightarrow A^1$ and $\cup_1 : D^2(A) \otimes_R A^1 \rightarrow A^2$, such that:

- (i) The R -module $R \oplus A^1$, with multiplication defined by the maps $R \otimes_R R \rightarrow R$ and $R \otimes_R A^1 \rightarrow A^1$ mentioned above, together with the map $\cup_1 : A^1 \otimes_R A^1 \rightarrow A^1$ is a commutative, graded R -algebra, with identity $1 \in R$.
- (ii) The \cup - and \cup_1 -maps satisfy the Hirsch identity

$$(4-6) \quad (a \cup b) \cup_1 c = a \cup (b \cup_1 c) + (a \cup_1 c) \cup b$$

for all $a, b, c \in A^1$.

A morphism of graded algebras with cup-one products is a map $\varphi : A \rightarrow B$ between two such objects which is a map of graded algebras such that $\varphi(a_1 \cup_1 a_2) = \varphi(a_1) \cup_1 \varphi(a_2)$ for all $a_1, a_2 \in A^1$.

Lemma 4.4 Let A and B be two graded R -algebras with cup-one products. Then the tensor product $A \otimes_R B$ is again a graded algebra with cup-one products.

Proof First, we extend the \cup_1 -products on A^1 and B^1 to a \cup_1 -product on

$$(A \otimes_R B)^1 = (A^1 \otimes_R B^0) \oplus (A^0 \otimes_R B^1)$$

by setting

$$(4-7) \quad \begin{aligned} (a_1 \otimes b_0) \cup_1 (a'_1 \otimes b'_0) &= (a_1 \cup_1 a'_1) \otimes b_0 b'_0, \\ (a_0 \otimes b_1) \cup_1 (a'_0 \otimes b'_1) &= a_0 a'_0 \otimes (b_1 \cup_1 b'_1), \\ (a_1 \otimes b_0) \cup_1 (a'_0 \otimes b'_1) &= (a_0 \otimes b_1) \cup_1 (a'_1 \otimes b'_0) = 0 \end{aligned}$$

for all $a_i, a'_i \in A^i$ and $b_i, b'_i \in B^i$ ($i = 0, 1$) and extending linearly to $(A \otimes_R B)^1$. Since the \cup_1 -product on A^1 and B^1 and the multiplication on A^0 and B^0 are all commutative, it follows that the \cup_1 -product on $(A \otimes_R B)^1$ is also commutative.

Next, note that $(A \otimes_R B)^2 = (A^2 \otimes_R B^0) \oplus (A^1 \otimes_R B^1) \oplus (A^0 \otimes_R B^2)$ and the R -submodule $D^2(A \otimes_R B)$ is generated by all possible cup-products of elements from $A^1 \otimes_R B^0$ and $A^0 \otimes_R B^1$. We now extend the cup-one product maps $\cup_1 : D^2(A) \otimes_R A^1 \rightarrow A^2$ and $\cup_1 : D^2(B) \otimes_R B^1 \rightarrow B^2$ to a map $\cup_1 : D^2(A \otimes_R B) \otimes_R (A \otimes_R B)^1 \rightarrow (A \otimes_R B)^2$ by setting

$$(4-8) \quad \begin{aligned} ((a_1 \otimes b_0) \cup (a'_1 \otimes b'_0)) \cup_1 (a''_1 \otimes b''_0) &= ((a_1 \cup_1 a'_1) \cup_1 a''_1) \otimes b_0 b'_0 b''_0, \\ ((a_0 \otimes b_1) \cup (a'_0 \otimes b'_1)) \cup_1 (a''_0 \otimes b''_1) &= a_0 a'_0 a''_0 \otimes ((b_1 \cup_1 b'_1) \cup_1 b''_1), \\ ((a_1 \otimes b_0) \cup (a_0 \otimes b_1)) \cup_1 (a'_0 \otimes b'_1) &= a_1 a_0 a'_0 \otimes b_0 (b_1 \cup_1 b'_1), \\ ((a_0 \otimes b_1) \cup (a_1 \otimes b_0)) \cup_1 (a'_1 \otimes b'_0) &= a_0 (a_1 \cup_1 a'_1) \otimes b_1 b_0 b'_0 \end{aligned}$$

for $a_i, a'_i, a''_i \in A^i$ and $b_i, b'_i, b''_i \in B^i$ ($i = 0, 1$), with the other 4 types of products equal to 0, and then extending linearly to $D^2(A \otimes_R B) \otimes_R (A \otimes_R B)^1$. To verify that this operation is well defined, suppose $(a_1 \otimes b_0) \cup (a'_1 \otimes b'_0) = (\bar{a}_1 \otimes \bar{b}_0) \cup (\bar{a}'_1 \otimes \bar{b}'_0)$. Then $a_1 \cup_1 a'_1 = \bar{a}_1 \cup_1 \bar{a}'_1$ and $b_0 b'_0 = \bar{b}_0 \bar{b}'_0$, and it follows that the operation is well defined on products such as those in the first line of (4-8). The other cases follow similarly.

Since the Hirsch identity (4-6) holds for both A and B , it also holds for $A \otimes_R B$; for instance,

$$\begin{aligned} (a_1 \otimes b_0 \cup a'_1 \otimes b'_0) \cup_1 (a''_1 \otimes b''_0) &= ((a_1 \cup_1 a'_1) \cup_1 a''_1) \otimes b_0 b'_0 b''_0 \\ &= ((a_1 \cup_1 a'_1) \cup_1 a''_1 + a_1 \cup_1 (a'_1 \cup_1 a''_1)) \otimes b_0 b'_0 b''_0 \\ &= (a_1 \otimes b_0 \cup_1 a''_1 \otimes b''_0) \cup_1 a'_1 \otimes b'_1 + a_1 \otimes b_0 \cup_1 (a'_1 \otimes b'_0 \cup_1 a''_1 \otimes b''_0), \end{aligned}$$

and similarly for the other types of \cup - and \cup_1 -products. \square

4.4 Cup-one differential graded algebras

In this section we make a definition that will play an important role in our investigation. We begin with some motivation. Note that if a dga is generated by a set of elements $\{x_i\}_{i \in J}$ in degree 1, then the Leibniz

rule gives a formula for the differential of any product of the x_i as a sum of cup products of the x_i and dx_i . Hence, the differential on the algebra is completely determined by the differentials of the generators x_i . This raises the question of whether there is a formula for the differential of cup-one products of the generators x_i that allows one to write $d(x_i \cup_1 x_j)$ as a sum of cup products of 1-cochains. If so, then it follows that if a dga is generated by elements x_i in degree one and by iterated cup-one products of the x_i , then the differential on the algebra is completely determined by the differentials of the x_i .

The next definition answers this question by giving as part of hypothesis (iv) a formula for the differential of a cup-one product of 1-cochains that, along with the Hirsch identity, allows one to write the differential of cup-one products of 1-cochains as a sum of cup products. This definition is a slight modification of a notion introduced in [35], better adapted to the current context by including the additional hypothesis (iii).

Definition 4.5 A differential graded R -algebra (A, d) is called a *cup-one differential graded algebra* if the following conditions hold.

- (i) A is a graded R -algebra with cup-one products.
- (ii) There is an R -linear map $\circ : D^2(A) \otimes_R D^2(A) \rightarrow D^2(A)$ such that

$$(4-9) \quad (u \cup v) \circ (w \cup z) = (u \cup_1 w) \cup (v \cup_1 z)$$

for all $u, v, w, z \in A^1$.

- (iii) The differential d and the \cup and \cup_1 products satisfy

$$(4-10) \quad a \cup_1 dc = a \cup c - c \cup a$$

for all $a \in A^1$ and $c \in A^0$.

- (iv) The differential d satisfies the “ \cup_1 - d formula,”

$$(4-11) \quad d(a \cup_1 b) = -a \cup b - b \cup a + da \cup_1 b + db \cup_1 a - da \circ db,$$

for all $a, b \in A^1$ with $da, db \in D^2(A)$.

Remark 4.6 Formula (4-11) comes from Steenrod’s definition of the \cup_i products in a cochain algebra $A = C^*(X; R)$, as follows. By (4-3), we have that $d(a \cup_1 b) = -a \cup b - b \cup a + da \cup_1 b - a \cup_1 db$ for all $a, b \in A^1$. If da is decomposable, then $da \cup_1 b$ can be written as a sum of cup products using the Hirsch formula (4-6). This leaves the problem of writing $a \cup_1 db$ as a sum of cup products. By a direct computation using Steenrod’s definition of the cup-one product $\cup_1 : A^1 \otimes_R A^2 \rightarrow A^2$, it follows that

$$(4-12) \quad a \cup_1 (b_1 \cup b_2) = da \circ (b_1 \cup b_2) - (b_1 \cup b_2) \cup_1 a,$$

where we assume da is decomposable and \circ is given by (4-9). This then gives the \cup_1 - d formula, (4-11).

Our motivation for Definition 4.5 arises from the cochain algebras of Δ -complexes. As shown in [35, Theorem 4.4], such algebras are indeed \cup_1 -algebras. We briefly review this result, with the necessary modifications for our context here.

Theorem 4.7 [35] *Let X be a nonempty Δ -complex, and let R be a unital commutative ring. The cochain algebra $(C^*(X; R), \delta)$ is a cup-one dga.*

Proof As we saw in Sections 3.2 and 4.2, the cellular cochain algebra $C = (C^*(X; R), \delta)$ is a graded algebra with cup-one products. Moreover, it is a differential graded algebra, and the Steenrod identities (4-1) hold in full generality. Setting $(c_1 \circ c_2)(s) = c_1(s) \cdot c_2(s)$ for any 2-cochains c_1, c_2 and any 2-simplex s defines an R -linear map $\circ : C^2 \otimes_R C^2 \rightarrow C^2$. It follows straight from the definitions of the \cup - and \cup_1 -products that

$$(4-13) \quad (u \cup v)(s) \cdot (w \cup z)(s) = ((u \cup_1 w) \cup (v \cup_1 z))(s),$$

for all 1-cochains u, v, w, z . Thus, the restriction of the \circ -map to decomposable elements yields a map, $\circ : D^2(C) \otimes_R D^2(C) \rightarrow D^2(C)$, which clearly obeys (4-9). It is now straightforward to verify that the simplicial differential d satisfies (4-11).

It remains to check that (4-10) holds. Given a 1-cochain u , a 0-cochain c , and a 1-simplex e with endpoints v_0 and v_1 , we have

$$(4-14) \quad \begin{aligned} (u \cup_1 \delta c)(e) &= u(e) \cdot (\delta c)(e) = u(e) \cdot c(v_1 - v_0) \\ &= u(e) \cdot c(v_1) - c(v_0) \cdot u(e) = (u \cup c)(e) - (c \cup u)(e), \end{aligned}$$

and this completes the proof. □

Remark 4.8 Comparing the definition of the map $\circ : C^2 \otimes_R C^2 \rightarrow C^2$ given in the above proof to that of Steenrod's map $\cup_2 : C^2 \otimes_R C^2 \rightarrow C^2$, we readily see that these two maps coincide. Moreover, as shown in Remark 4.6, in this case the \cup_1 - d formula (4-11) is a consequence of Steenrod's formula (4-3).

4.5 Tensor products of \cup_1 -dgas

We conclude this section with a result showing that the category of \cup_1 -dgas is closed under taking tensor products.

Proposition 4.9 *If (A, d_A) and (B, d_B) are cup-one differential graded algebras, then the tensor product $(A \otimes_R B, d_{A \otimes B})$ is again a cup-one differential graded algebra.*

Proof By Lemma 4.4, $A \otimes_R B$ is a graded algebra with cup-one products. The \circ operations on $D^2(A)$ and $D^2(B)$ extend to a binary operation, $\circ : D^2(A \otimes_R B) \otimes_R D^2(A \otimes_R B) \rightarrow D^2(A \otimes_R B)$, by letting $[(a_1 \otimes b_0) \cup (a'_1 \otimes b'_0)] \circ [(a''_1 \otimes b''_0) \cup (a'''_1 \otimes b'''_0)]$ equal to $[(a_1 \cup a'_1) \circ (a''_1 \cup a'''_1)] \otimes b_0 b'_0 b''_0 b'''_0$, and so on. To verify that this operation is well defined, suppose $(a_1 \otimes b_0) \cup (a'_1 \otimes b'_0) = (\bar{a}_1 \otimes \bar{b}_0) \cup (\bar{a}'_1 \otimes \bar{b}'_0)$ and $(a''_1 \otimes b''_0) \cup (a'''_1 \otimes b'''_0) = (\bar{a}''_1 \otimes \bar{b}''_0) \cup (\bar{a}'''_1 \otimes \bar{b}'''_0)$. Then $a_1 \cup a'_1 = \bar{a}_1 \cup \bar{a}'_1$ and $a''_1 \cup a'''_1 = \bar{a}''_1 \cup \bar{a}'''_1$, and likewise, $b_0 b'_0 = \bar{b}_0 \bar{b}'_0$ and $b''_0 b'''_0 = \bar{b}''_0 \bar{b}'''_0$. The claim follows.

Using (4-7), it is readily checked that (4-9) holds for $(A \otimes_R B, d_{A \otimes B})$. Next, we verify (4-10):

$$\begin{aligned}
 (4-15) \quad (a_1 \otimes b_0) \cup_1 d_{A \otimes B}(a'_0 \otimes b'_0) &= (a_1 \otimes b_0) \cup_1 (d_A(a'_0)b'_0 + a'_0 d_B(b'_0)) \\
 &= (a_1 \cup_1 d_A(a'_0)) \otimes b_0 b'_0 \\
 &= (a_1 \cup a'_0 - a'_0 \cup a_1) \otimes b_0 b'_0 \\
 &= (a_1 \otimes b_0) \cup (a'_0 \otimes b'_0) - (a'_0 \otimes b'_0) \cup (a_1 \otimes b_0),
 \end{aligned}$$

and similarly for the other cases.

The last step is to verify that (4-11) holds for the tensor product of A and B . First let $a \in A^1$ such that $d_A(a) \in D^2(A)$, and let $b \in B^0$. It is readily seen that $d_{A \otimes B}(a \otimes b) \in D^2(A \otimes_R B)$; for instance, if $d_A(a) = u \cup v$ for some $u, v \in A^1$, then

$$(4-16) \quad d_{A \otimes B}(a \otimes b) = d_A a \otimes b - a \otimes d_B b = (u \otimes 1) \cup (v \otimes b) - (a \otimes 1) \cup (1 \otimes d_B b).$$

Now also let a' be an element in A^1 such that $d_A(a') \in D^2(A)$, and let $b' \in B^0$. Using (4-9) and (4-7), we find that

$$\begin{aligned}
 (4-17) \quad (d_A(a) \otimes b) \circ (a' \otimes d_B(b')) &= ((u \cup v) \otimes b) \circ ((a' \otimes 1) \cup (1 \otimes d_B(b'))) \\
 &= (u \cup_1 (a' \otimes b)) \cup ((v \otimes 1) \cup_1 (1 \otimes d_B(b'))) \\
 &= ((u \cup_1 a') \otimes b) \cup 0 \\
 &= 0,
 \end{aligned}$$

and similarly $(a \otimes d_B(b)) \circ (d_A(a') \otimes b') = 0$. Furthermore, using (4-9) and (4-10), we get

$$\begin{aligned}
 (4-18) \quad (a \otimes d_B(b)) \circ (a' \otimes d_B(b')) &= ((a \otimes 1) \cup (1 \otimes d_B(b))) \circ ((a' \otimes 1) \cup (1 \otimes d_B(b'))) \\
 &= ((a \otimes 1) \cup_1 (a' \otimes 1)) \cup ((1 \otimes d_B(b)) \cup_1 (1 \otimes d_B(b'))) \\
 &= (a \cup_1 a') \otimes (d_B(b') \cup_1 d_B(b)) \\
 &= (a \cup_1 a') \otimes (d_B(b')b - b d_B(b')).
 \end{aligned}$$

Finally, using (4-11) for A as well as (4-17) and (4-18) we find that

$$\begin{aligned}
 d_{A \otimes B}[(a \otimes b) \cup_1 (a' \otimes b')] &= -(a \otimes b) \cup (a' \otimes b') - (a' \otimes b') \cup (a \otimes b) + d_{A \otimes B}(a \otimes b) \cup_1 (a' \otimes b') \\
 &\quad + d_{A \otimes B}(a' \otimes b') \cup_1 (a \otimes b) - d_{A \otimes B}(a \otimes b) \circ d_{A \otimes B}(a' \otimes b').
 \end{aligned}$$

This shows that (4-11) holds for elements in $A \otimes_R B$ of the form $(a \otimes b) \cup_1 (a' \otimes b')$ with $|a| = |a'| = 1$ and $|b| = |b'| = 0$. The case $|a| = |b'| = 1, |b| = |a'| = 0$ follows using similar computations to show that in this case the right side of (4-11) equals zero. The case $|a| = |a'| = 0, |b| = |b'| = 1$ follows by the same computations as in the first case with the elements in A and B interchanged. \square

5 Binomial cup-one differential graded algebras

In this section we begin by reviewing the definition and basic properties of binomial rings and then define \mathbb{F}_p -binomial algebras for p a prime. This leads to the definition of binomial cup-one differential graded algebras over the ring $R = \mathbb{Z}$ or \mathbb{F}_p . A consequence of including the binomial algebra structure is that it then follows that the cohomology of the free binomial cup-one differential graded algebra on a single generator in degree 1 is isomorphic to the cohomology ring $H^*(K(R, 1); R)$ of the Eilenberg–MacLane space $K(R, 1)$ with $R = \mathbb{Z}$ or \mathbb{F}_p (see Theorem 8.9).

5.1 Binomial rings and R -valued polynomials

Following P. Hall [17] and J. Elliott [9], we say that a commutative ring A is a *binomial ring* if A is torsion-free (as a \mathbb{Z} -module) and the element

$$(5-1) \quad \binom{a}{n} := a(a-1)\cdots(a-n+1)/n! \in A \otimes_{\mathbb{Z}} \mathbb{Q}$$

lies in A for every $a \in A$ and every $n > 0$. Therefore we have maps $\zeta_n : A \rightarrow A$, $a \mapsto \binom{a}{n}$ for all $n \in \mathbb{N}$, with the convention that $\zeta_0(a) = 1$ for all $a \in A$.

Let

$$(x)_n := x(x-1)\cdots(x-n+1) \in \mathbb{Z}[x]$$

be the “falling factorial” polynomial. Writing $(x)_n = \sum_{k=0}^n s(n, k)x^k$, the coefficients $s(n, k)$ of this polynomial are the Stirling numbers of the first kind. Now note that numerator of the fraction in (5-1) is obtained by evaluating the polynomial $(x)_n$ at the value $x = a$, and so we may also write $\zeta_n(a) = (a)_n/n!$.

The next lemma follows straight from the definitions.

Lemma 5.1 *Let R be a binomial ring and let $M = R^X$ be a free R -module on a set X . Then the dual module, $M^\vee = \text{Hom}_R(M, R)$, is a binomial ring with product defined by $f \cdot g(x) = f(x) \cdot g(x)$ for all $x \in X$ and with maps $\zeta_n : M^\vee \rightarrow M^\vee$ given by $\zeta_n(f)(x) = \zeta_n(f(x))$ for all $f \in M^\vee$ and $x \in X$.*

Now suppose the binomial ring R is an integral domain, and let $K = \text{Frac}(R)$ be its field of fractions. Let $K[X]$ be the ring of polynomials in a set of formal variables X , with coefficients in K . Following [4; 9], we define the *ring of R -valued polynomials* (in the variables X and with coefficients in K) as the subring

$$(5-2) \quad \text{Int}(R^X) := \{p \in K[X] \mid p(R^X) \subseteq R\}.$$

If, moreover, the domain R has characteristic 0 (that is, R is torsion-free as an abelian group), then $\text{Int}(R^X)$ is a binomial ring, generated by the polynomials $\binom{X}{n} := \prod_{x \in X} \binom{x}{n_x}$, for all multi-indices $n = (n_x)_{x \in X} \in \bigoplus_X \mathbb{Z}_{\geq 0}$, and $\text{Int}(R^X)$ satisfies a type of universality property which makes it into the *free binomial ring* on variables in X (see Corollary 5.3). As a consequence (at least when $R = \mathbb{Z}$), any binomial ring is a quotient of $\text{Int}(R^X)$, for some set X .

Let $I : X \rightarrow \mathbb{Z}_{\geq 0}$ be a function whose support, $\text{supp}(I) := \{x \in X \mid I(x) \neq 0\}$, is a finite set. Given a binomial domain R with field of fractions K , we associate to such a function the polynomial $\zeta_I \in K[X]$

given by

$$(5-3) \quad \zeta_I := \prod_{x \in X} \zeta_{I(x)}.$$

This possibly infinite product is well defined, since the support of I is finite and since $\zeta_0(x) = 1$ for all $x \in X$. Clearly, ζ_I is an R -valued polynomial in $\text{Int}(R^X)$. That is to say, given an element $\mathbf{a} = \sum_{x \in X} \mathbf{a}_x x \in R^X$ with $\mathbf{a}_x \in R$ and $\mathbf{a}_x = 0$ for all but finitely many $x \in X$, the evaluation $\zeta_I(\mathbf{a}) = \prod_{x \in X} \zeta_{I(x)}(\mathbf{a}_x)$ is an element of R .

5.2 A basis for integer-valued polynomials

We restrict now to the case when $R = \mathbb{Z}$. The next theorem provides a useful \mathbb{Z} -basis for the ring $\text{Int}(\mathbb{Z}^X)$ of integer-valued polynomials; for a proof, we refer to [4, Proposition XI.I.12] and [9, Lemma 2.2].

Theorem 5.2 [4; 9] *The \mathbb{Z} -module $\text{Int}(\mathbb{Z}^X)$ is free, with basis consisting of all polynomials of the form ζ_I with $I : X \rightarrow \mathbb{Z}_{\geq 0}$ a function with finite support.*

Alternatively, one may take as a basis for $\text{Int}(\mathbb{Z}^X)$ all polynomials $\zeta_I(x)$ with $\text{supp}(I) = x$, a finite set of variables $x \subset X$, together with the constant polynomial ζ_0 . We emphasize that in the products $\zeta_I(x)$, there is no repetition allowed among the variables comprising the set x . For instance, the product $\zeta_m \zeta_n$ is not part of the aforementioned \mathbb{Z} -basis; rather, it may be expressed as a linear combination of the binomials $\zeta_m, \dots, \zeta_{m+n}$. On the other hand, if I and J have disjoint supports, we have that $\zeta_I \cdot \zeta_J = \zeta_{I+J}$, and this polynomial is again part of the aforementioned basis for $\text{Int}(\mathbb{Z}^X)$.

Corollary 5.3 [9; 35] *Let X be a set, let A be a binomial ring, and let $\phi : X \rightarrow A$ be a map of sets. There is then a unique extension of ϕ to a map $\tilde{\phi} : \text{Int}(\mathbb{Z}^X) \rightarrow A$ of binomial rings.*

Corollary 5.4 [9; 35] *Let R_1 and R_2 be binomial rings. Then the tensor product $R_1 \otimes_{\mathbb{Z}} R_2$, with product $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$, is a binomial ring.*

5.3 \mathbb{F}_p -binomial algebras

Fix a prime p , and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Let A be a commutative \mathbb{F}_p -algebra; we will assume that the structure map $\mathbb{F}_p \rightarrow A$ which sends $1 \in \mathbb{F}_p$ to the identity $1 \in A$ is injective. The binomial operations $\zeta_n(a) = (a)_n/n!$ with $a \in A$ are defined for $1 \leq n \leq p-1$, since $n!$ is then a unit in \mathbb{F}_p .

Example 5.5 Let $A = C^*(X; \mathbb{F}_p)$ be the cochain algebra over \mathbb{F}_p of a Δ -complex X . For a cochain $a \in A^1$, we have that $(a)_p = 0$, where the product is the \cup_1 -product on A^1 . To see this, let e be any 1-simplex in X ; then the elements $a(e), a(e) - 1, \dots, a(e) - p + 1$ are distinct elements in \mathbb{F}_p . Since there are p of these elements, one of the elements must be 0 and the property follows.

This motivates the following definition.

Definition 5.6 Let A be a commutative \mathbb{F}_p -algebra. We say that A is a \mathbb{F}_p -binomial algebra if $(a)_p = 0$, for all $a \in A$.

Clearly, this condition is equivalent to $(a)_n = 0$ for all integers $n \geq p$ and all $a \in A$. In $\mathbb{F}_p[x]$ we have $(x)_p = x^p - x$. Indeed, both polynomials are monic, of degree p , and both have the same set of p distinct roots, namely $0, 1, \dots, p-1$. Therefore, a commutative \mathbb{F}_p -algebra A is a \mathbb{F}_p -binomial algebra if and only if $a^p = a$, for all $a \in A$; that is, A is a p -Boolean algebra, in the sense of Kriz [24].

The next step is to derive properties of binomials in a \mathbb{F}_p -binomial algebra analogous to those for a binomial ring over \mathbb{Z} . We start by defining the analog of $\text{Int}(\mathbb{Z}^X)$.

Given a set X , we will denote by $\text{Int}(\mathbb{F}_p^X)$ the quotient of the free binomial algebra $\text{Int}(\mathbb{Z}^X)$ by the ideal generated by the elements $\zeta_n(x)$ for $x \in X$ and $n \geq p$, tensored with \mathbb{F}_p . The next result shows that, modulo the constant terms, $\text{Int}(\mathbb{F}_p^X)$ has \mathbb{F}_p -basis given by products of the elements $\zeta_i(x)$ for $0 < i < p$ and $x \in X$. Recall from (5-3) that, for a finitely supported function $I : X \rightarrow \mathbb{Z}_{\geq 0}$, we write $\zeta_I = \prod_{x \in X} \zeta_{I(x)}$.

Lemma 5.7 [35] *The ring $\text{Int}(\mathbb{F}_p^X)$ is a \mathbb{F}_p -binomial algebra, with \mathbb{F}_p -basis given by the \mathbb{F}_p -valued polynomials $\zeta_I(x)$ with $I : X \rightarrow \{0, \dots, p-1\}$.*

Theorem 5.8 [35] *Let A be a \mathbb{F}_p -binomial algebra. There is then a bijection between maps of \mathbb{F}_p -binomial algebras from $\text{Int}(\mathbb{F}_p^X)$ to A and set maps from X to A .*

Lemma 5.9 *Let A and B be \mathbb{F}_p -binomial algebras. Then the tensor product $A \otimes_{\mathbb{F}_p} B$, with product $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$, is a \mathbb{F}_p -binomial algebra.*

Proof Let $a \in A$ and $b \in B$, so that $a^p = a$ and $b^p = b$. Then $(a \otimes b)^p = a^p \otimes b^p = a \otimes b$. \square

5.4 Binomial cup-one differential graded algebras

Following [35], we combine the notions of cup-one algebras and binomial algebras into a single package. Henceforth we will assume that our ground ring R is equal to either \mathbb{Z} or \mathbb{F}_p for some prime p .

Definition 5.10 A differential graded algebra (A, d) over $R = \mathbb{Z}$ or \mathbb{F}_p is called a *binomial cup-one algebra* if

- (i) A is a cup-one algebra,
- (ii) A^0 , with multiplication $A^0 \otimes_R A^0 \rightarrow A^0$ given by the cup-product, is an R -binomial algebra,
- (iii) the R -algebra $R \oplus A^1$ from Definition 4.3, part (i) is an R -binomial algebra.

A morphism of binomial cup-one R -algebras is a map $\varphi : A \rightarrow B$ between two such objects which is a map of cup-one R -algebras that satisfies $\varphi(\zeta_n(a)) = \zeta_n(\varphi(a))$, for all $n \geq 1$ and all $a \in A^1$.

Proposition 5.11 *Let (A, d_A) and (B, d_B) be binomial cup-one algebras over $R = \mathbb{Z}$ or \mathbb{F}_p . Then the tensor product $(A \otimes_R B, d_{A \otimes B})$ of the underlying dgas is again a binomial cup-one algebra.*

Proof Recall that $(A \otimes_R B)^1 = (A^1 \otimes_R B^0) \oplus (A^0 \otimes_R B^1)$. The claim follows at once from Proposition 4.9, Corollary 5.4, and Lemma 5.9. \square

5.5 Binomial operations on cochains

In this section, we show that cochain complexes with coefficients in a binomial algebra are binomial cup-one algebras and give examples. The next result builds on [35].

Theorem 5.12 *For any nonempty Δ -complex X , the cochain algebra $C^*(X; R)$, where $R = \mathbb{Z}$ or \mathbb{F}_p , is a binomial \cup_1 -dga.*

Proof First assume $R = \mathbb{Z}$. The cochain algebra $C^*(X; \mathbb{Z})$ is then a cup-one dga by Theorem 4.7, and the claim follows from Lemma 5.1.

Now assume $R = \mathbb{F}_p$. We define maps $\zeta_n^X : C^1(X; \mathbb{F}_p) \rightarrow C^1(X; \mathbb{F}_p)$ for $1 \leq n \leq p-1$, by setting $\zeta_n^X(f)(e) := (f(e))_n/n!$ for each 1-cochain $f \in C^1(X; \mathbb{F}_p) = \text{Hom}(C_1(X; \mathbb{F}_p), \mathbb{F}_p)$ and each 1-simplex e in X . As noted in Example 5.5, we have that $(f)_p = 0$. With this structure, it is readily verified that $C^*(X; \mathbb{F}_p)$ is a \mathbb{F}_p -binomial \cup_1 -dga. \square

This theorem together with Proposition 5.11 yield the following corollary.

Corollary 5.13 *Let A be a binomial \cup_1 -dga over $R = \mathbb{Z}$ or \mathbb{F}_p . Then the tensor product $A \otimes_R C^*(I; R)$ is again a binomial \cup_1 -dga.*

It is readily seen that the ζ -maps enjoy the following naturality property: If $h : X \rightarrow Y$ is a map of Δ -complexes, then each ζ_n commutes with the pullback of cochains, that is,

$$h^* \circ \zeta_n^Y = \zeta_n^X \circ h^* : C^*(Y; R) \rightarrow C^*(X; R).$$

Note that in the case $R = \mathbb{Z}$, the evaluation $\zeta_n(f)(e)$ is simply the binomial coefficient $\binom{f(e)}{n}$, for all $f \in \text{Hom}(C_1(X; R), R)$ and all $e \in C_1(X; R)$.

Example 5.14 For the cochain algebra $C = C^*(I; \mathbb{Z})$ from Example 3.4, the ζ_n -maps are given by $\zeta_n(ku) = \binom{k}{n}u$. In particular, $\zeta_n(u) = 0$ for $n \geq 2$.

Example 5.15 Let G be a group, and let $C^*(B(G); R)$ be the cochain algebra of the bar construction on G , as described in Example 3.5, with coefficient ring R a binomial algebra. Then the ζ_n maps on $C^1(B(G); R)$ are given by $\zeta_n(f)([g]) = \binom{f(g)}{n}$.

6 Free binomial \cup_1 -differential graded algebras

In this section, we define $\mathbb{T}(X)$, the free binomial graded cup-one algebra over the rings $R = \mathbb{Z}$ and $R = \mathbb{F}_p$ generated by a set X . Given a map $d : \mathbb{T}(X) \rightarrow \mathbb{T}(X)$ that satisfies the \cup_1 - d formula and the Leibniz rule, we show that $d^2 \equiv 0$ if and only if $d^2(x) = 0$ for all $x \in X$. In particular, setting $d_0(x) = 0$ for all $x \in X$ yields the dga $(\mathbb{T}(X), d_0)$, which we call the free binomial \cup_1 -dga generated by X .

6.1 The free binomial cup-one graded algebra

Let $R = \mathbb{Z}$ or \mathbb{F}_p . Given a set X , we let $\mathfrak{m}_{X,R}$ denote the R -submodule of $\text{Int}(R^X)$ consisting of polynomials with zero constant term.

Definition 6.1 [35] The free binomial \cup_1 -graded algebra over R on a set X , denoted by $\mathbb{T} = \mathbb{T}_R(X)$, is the tensor algebra on the free R -module $\mathfrak{m}_{X,R}$; that is,

$$(6-1) \quad \mathbb{T}_R^*(X) = T^*(\mathfrak{m}_{X,R}).$$

By construction, $\mathbb{T}_R^0(X) = R$ and $\mathbb{T}_R^1(X) = \mathfrak{m}_{X,R}$, and so $\mathbb{T}_R^{\leq 1}(X) = \mathbb{T}^0 \oplus \mathbb{T}^1$ is isomorphic to the free binomial algebra $\text{Int}(R^X)$. By Theorem 5.2 and Lemma 5.7, respectively, \mathbb{T}^1 is a free R -module, with basis consisting of all R -valued polynomials of the form ζ_I , where $I : X \rightarrow \mathbb{Z}_{\geq 0}$ has finite, nonempty support when $R = \mathbb{Z}$, and $I : X \rightarrow \{0, 1, \dots, p-1\}$ when $R = \mathbb{F}_p$, excluding the constant-0 function. Furthermore, the R -module \mathbb{T}^1 comes endowed with a cup-one product map, $\cup_1 : \mathbb{T}^1 \otimes \mathbb{T}^1 \rightarrow \mathbb{T}^1$, given by $a \cup_1 b = ab$. By analogy with the classical Hirsch formula for cochain algebras, we use this cup-one product to define a linear map $\mathbb{T}^2 \otimes \mathbb{T}^1 \rightarrow \mathbb{T}^2$ by

$$(6-2) \quad (a \otimes b) \otimes c \mapsto ac \otimes b + a \otimes bc.$$

Recall that the \cup_1 - d formula (4-11) involves an operation (denoted by \circ) between degree-2 elements. For this to work, we include the linear map $\circ : \mathbb{T}^2 \otimes \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined on basis elements by

$$(6-3) \quad (a_1 \otimes a_2) \circ (b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2).$$

With this structure, $\mathbb{T}(X)$ is a graded R -algebra with cup-one products, in the sense of Definition 4.3.

The assignment $X \rightsquigarrow \mathbb{T}_R(X)$ is functorial: a map of sets, $h : X \rightarrow X'$, extends to a map of polynomials from $\text{Int}(R^X)$ to $\text{Int}(R^{X'})$ that restricts to an R -linear map $\mathfrak{m}_{X,R} \rightarrow \mathfrak{m}_{X',R}$ which then extends to a map between tensor algebras, $\mathbb{T}(h) : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X')$. Clearly, $\mathbb{T}(h)$ is a morphism of graded algebras that preserves \cup_1 -products; moreover, $\mathbb{T}(h \circ g) = \mathbb{T}(h) \circ \mathbb{T}(g)$.

In the following, the ring R will be equal to either \mathbb{Z} or \mathbb{F}_p . A graded R -algebra A with cup-one products such that the augmented algebra $R \oplus A^1$ is a binomial algebra is called a *binomial graded R -algebra with cup-one products*. In this category, the free binomial graded R -algebra $\mathbb{T}_R(X)$ enjoys the following universality property.

Lemma 6.2 Let A be a binomial graded R -algebra with cup-one products, let X be a set, and let $\phi : X \rightarrow A^1$ be a map of sets. There is then a unique extension of ϕ to a map $f : \mathbb{T}_R(X) \rightarrow A$ of binomial graded R -algebras with cup-one products.

Proof From [35, Lemmas 7.4 and 8.13], it follows that there is a unique extension of ϕ to a degree-preserving map $f^{>0} : \mathbb{T}_R^{>0}(X) \rightarrow A^{>0}$ which commutes with cup products, cup-one products, and the ζ maps. Let $\sigma_{\mathbb{T}} : R \rightarrow \mathbb{T}_R^0(X)$ and $\sigma_A : R \rightarrow A^0$ be the structure maps for \mathbb{T} and A ; respectively, and define $f^0 : \mathbb{T}_R^0(X) \rightarrow A^0$ to be the composition $\sigma_A \circ \sigma_{\mathbb{T}}^{-1}$. Then the resulting map $f : \mathbb{T}_R(X) \rightarrow A$ is a morphism of binomial graded R -algebras. \square

6.2 Maps from free binomial cup-one dgas

Consider now a differential $d : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X)$ making $(\mathbb{T}_R(X), d)$ into a binomial cup-one dga, and let (A, d_A) be an arbitrary binomial cup-one dga over R . The next lemma gives a handy criterion for

deciding whether a map $f : \mathbb{T}_R(X) \rightarrow A$ between the underlying binomial graded algebras with cup-one products is a morphism of binomial cup-one dgas.

Lemma 6.3 *A map $f : (\mathbb{T}_R(X), d) \rightarrow (A, d_A)$ of binomial graded R -algebras with cup-one products commutes with the differentials if and only if $d_A f(x) = f(dx)$ for all $x \in X$.*

Proof Recall from Section 6.1 that $\mathbb{T}_R^1(X)$ is the free R -module with basis consisting of all R -valued polynomials of the form ζ_I , where $I : X \rightarrow \mathbb{Z}_{\geq 0}$ has finite, nonempty support when $R = \mathbb{Z}$ and $I : X \rightarrow \{0, \dots, p-1\}$ has nonempty support when $R = \mathbb{F}_p$. The claim follows from formulas (4-9) and (4-11); formula (6-9) expressing $\zeta_{n+1}(x)$ in terms of $\zeta_n(x)$; and induction on n . \square

Under a connectivity assumption on $H^*(A)$, we may improve on the conclusion of Lemma 6.2, as follows.

Lemma 6.4 *Let (A, d_A) be a binomial cup-one R -dga with $H^0(A) \cong R$, let $(\mathbb{T}_R(X), d)$ be a free binomial cup-one dga, and let $\phi : X \rightarrow A^1$ be a map of sets. There is then a unique extension of ϕ to a map $f : \mathbb{T}_R(X) \rightarrow A$ of binomial graded R -algebras with cup-one products such that the function $H^0(f) : H^0(\mathbb{T}_R(X)) \rightarrow H^0(A)$ is an isomorphism.*

Proof Let $f : \mathbb{T}_R(X) \rightarrow A$ be the extension of ϕ constructed in Lemma 6.2 and let ε denote the isomorphism from $H^0(A)$ to R . Since $H^0(A) = \ker d_A : A^0 \rightarrow A^1$, it follows that the image of the structure map σ_A is contained in $H^0(A) \subseteq A^0$. Thus, the composition $\varepsilon \circ \sigma_A$ is an isomorphism of rings from R to R , and so equals id_R . It then follows that f^0 is the unique R -linear map from $\mathbb{T}^0(X)$ to A^0 that commutes with the structure maps; that is, $f^0 \circ \sigma_{\mathbb{T}} = \sigma_A$. \square

6.3 Differentials on $\mathbb{T}_R(X)$

As before, let $\mathbb{T} = \mathbb{T}_R(X)$ be a free binomial \cup_1 -graded R -algebra on a set X . In this section we show that if a map $d : \mathbb{T} \rightarrow \mathbb{T}$ satisfies the \cup_1 - d formula and the Leibniz rule, then $d^2(u) = 0$ for all $u \in \mathbb{T}$ if and only if $d^2(x) = 0$ for all $x \in X$. For that, we define additional \cup_1 and \circ maps in \mathbb{T} , as follows. First, we define a linear map $\cup_1 : \mathbb{T}^3 \otimes \mathbb{T}^1 \rightarrow \mathbb{T}^3$ by

$$(6-4) \quad (u_1 \cup u_2 \cup u_3) \cup_1 v = (u_1 \cup_1 v) \cup u_2 \cup u_3 + u_1 \cup (u_2 \cup_1 v) \cup u_3 + u_1 \cup u_2 \cup (u_3 \cup_1 v)$$

and a map $\cup_1 : \mathbb{T}^2 \otimes \mathbb{T}^2 \rightarrow \mathbb{T}^3$ by

$$(6-5) \quad \begin{aligned} (a_1 \cup a_2) \cup_1 (b_1 \cup b_2) &= -a_1 \cup (b_1 \cup_1 a_2) \cup b_2 - a_1 \cup b_1 \cup (b_2 \cup_1 a_2) \\ &\quad + \sum_j a_1 \cup (a_{2,1,j} \cup_1 b_1) \cup (a_{2,2,j} \cup_1 b_2) \\ &\quad + (b_1 \cup_1 a_1) \cup b_2 \cup a_2 + b_1 \cup (b_2 \cup_1 a_1) \cup a_2 \\ &\quad - \sum_i (a_{1,1,i} \cup_1 b_1) \cup (a_{1,2,i} \cup_1 b_2) \cup a_2, \end{aligned}$$

where $da_1 = \sum_i a_{1,1,i} \cup a_{1,2,i}$ and $da_2 = \sum_j a_{2,1,j} \cup a_{2,2,j}$. Next, we define a map $\circ : T^2 \otimes T^3 \rightarrow T^3$ by

$$(6-6) \quad (a_1 \cup a_2) \circ (v_1 \cup v_2 \cup v_3) = (a_1 \cup_1 v_1) \cup (a_2 \cup_1 v_2) \cup v_3 \\ + (a_1 \cup_1 v_1) \cup v_2 \cup (a_2 \cup_1 v_3) \\ + v_1 \cup (a_1 \cup_1 v_2) \cup (a_2 \cup_1 v_3) \\ - \sum_i (a_{1,1,i} \cup_1 v_1) \cup (a_{1,2,i} \cup_1 v_2) \cup (a_2 \cup_1 v_3),$$

where $da_1 = \sum_i a_{1,1,i} \cup a_{1,2,i}$. Finally, we define a map $\circ : T^3 \otimes T^2 \rightarrow T^3$ by

$$(6-7) \quad (u_1 \cup u_2 \cup u_3) \circ (b_1 \cup b_2) = u_1 \cup (u_2 \cup_1 b_1) \cup (u_3 \cup_1 b_2) \\ + (u_1 \cup_1 b_1) \cup (u_2 \cup_1 b_2) \cup u_3 \\ + (u_1 \cup_1 b_1) \cup u_2 \cup (u_3 \cup_1 b_2) \\ - \sum_k (u_1 \cup_1 b_1) \cup (u_2 \cup_1 b_{2,1,k}) \cup (u_3 \cup_1 b_{2,2,k}),$$

where $db_2 = \sum_k b_{2,1,k} \cup b_{2,2,k}$.

The proof of the following two equations is a straightforward, though computationally intensive, verification using the definitions of the \cup_1 - d formula along with the \cup_1 and \circ maps in T :

$$(6-8) \quad d(da \cup_1 b) = da \cup b - b \cup da + da \cup_1 db + d^2 a \cup_1 b, \\ d(da \circ db) = da \cup_1 db + db \cup_1 da + d^2 a \circ db + da \circ d^2 b.$$

Note that these equations are analogous to (4-1) for $i = 1, 2$ and $|a| + |b| - i \leq 3$, with the \cup_2 map replaced by the \circ map.

Lemma 6.5 *If $d : T \rightarrow T$ is a degree-one map satisfying the \cup_1 - d formula and the Leibniz rule, and if a, b are elements in T^1 with $d^2 a = d^2 b = 0$, then $d^2(a \cup_1 b) = 0$.*

Proof By the \cup_1 - d formula (4-11), we have that

$$d(a \cup_1 b) = -a \cup b - b \cup a + da \cup_1 b + db \cup_1 a - da \circ db.$$

From (6-8), it follows that

$$d^2(a \cup_1 b) = d^2 a \cup_1 b + d^2 b \cup_1 a - d^2 a \circ db - da \circ d^2 b,$$

and this proves the claim, since $d^2 a = d^2 b = 0$. □

Theorem 6.6 *Let $d : T_R(X) \rightarrow T_R(X)$ be a degree-one map satisfying the \cup_1 - d formula and the Leibniz rule. Then $d^2(x) = 0$ for all $x \in X$ if and only if $d^2(u) = 0$ for all $u \in T_R(X)$, in which case $(T_R(X), d)$ is a binomial \cup_1 -dga.*

Proof Let $x \in X$. Recall that the binomial ζ -functions satisfy

$$(6-9) \quad \zeta_{n+1}(x) = \frac{\zeta_n(x) \cup_1 x - n \zeta_n(x)}{n+1},$$

where $n + 1 \leq p - 1$ for $R = \mathbb{F}_p$. Using this formula and Lemma 6.5, induction on n shows that $d^2 \zeta_n(x) = 0$ for all $n \geq 1$ in the case $R = \mathbb{Z}$ and for all $n \leq p - 2$ in the case $R = \mathbb{F}_p$.

Making use of this fact and of Lemma 6.5 once again, induction on the number of elements in the support of I shows that $d^2(\zeta_I(x)) = 0$ for all I . The claim now follows by the Leibniz rule. \square

As a corollary, we recover a result from [35], which gives the free binomial graded algebra $\mathbb{T}_R(X)$ the structure of a binomial \cup_1 -dga structure, with differential d_0 vanishing on all the generators $x \in X$.

Corollary 6.7 [35] *For any set X , the algebra $\mathbb{T}_R(X)$ is a binomial \cup_1 -dga, with differential d_0 satisfying $d_0(x) = 0$ for all $x \in X$.*

7 Differentials defined by admissible maps

In this section, we define embeddings of the free binomial R -algebra $\mathbb{T}_R(X)$ (or its truncation in degree 2) into a suitable cochain algebra. Using these embeddings, we show in Theorem 7.3 that there is a bijection between degree-one linear maps $d : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X)$ that satisfy the \cup_1 - d formula and the Leibniz rule and maps of sets $\tau : X \rightarrow \mathbb{T}_R^2(X)$. Theorem 7.6 gives a sufficient condition on τ , called admissibility, for d^2 to be the zero map.

7.1 Embedding $\mathbb{T}_R^{\leq 2}(X)$ into a cochain algebra

Recall that the ring R equals \mathbb{Z} or \mathbb{F}_p with p a prime. Given a set X , we let $M(X, R)$ be the set of all functions $\mathbf{a} : X \rightarrow R$. This is an abelian group under pointwise addition, with neutral element the zero function, denoted by $\mathbb{0}$. Furthermore, to every set map $f : X \rightarrow Y$ we assign (in a functorial way) the R -linear map $f^\vee : M(Y, R) \rightarrow M(X, R)$ given by $f^\vee(\mathbf{b})(x) = \mathbf{b}(f(x))$, for $\mathbf{b} : Y \rightarrow R$ and $x \in X$.

Now let $\mu : M \times M \rightarrow M$ be an arbitrary binary operation on $M = M(X, R)$. By the construction from Section 2.2, the magma (M, μ) determines a 2-dimensional Δ -set, $\Delta^{(2)}(M, \mu)$. Let $C_\mu(X) = C_\mu(X; R)$ denote the cochain complex (over R) of this Δ -set. Next, we define a degree-preserving, R -linear map,

$$(7-1) \quad \psi = \psi_{X, \mu} : \mathbb{T}_R^{\leq 2}(X) \rightarrow C_\mu(X),$$

as follows. First define a map $\psi : \mathbb{T}_R^0(X) \rightarrow C_\mu^0(X)$ by sending $1 \in \mathbb{T}_R^0(X) = R$ to the unit cochain $1 \in C_\mu^0(X)$. For each polynomial $p \in \mathbb{T}_R^1(X) = \mathfrak{m}_X$, we set $\psi(p) \in C_\mu^1(X)$ equal to the 1-cochain whose value on a 1-simplex \mathbf{a} is $p(\mathbf{a})$. Finally, we set $\psi(p \otimes q) \in C_\mu^2(X)$ equal to the 2-cochain whose value on a 2-simplex $(\mathbf{a}, \mathbf{a}')$ is $p(\mathbf{a}) \cdot q(\mathbf{a}')$. If the zero function $\mathbb{0}$ is a two-sided identity in (M, μ) , it is readily seen that the image of ψ is contained in the normalized cochains on $\Delta^{(2)}(M, \mu)$.

Lemma 7.1 *With notation as above, the map $\psi = \psi_{X, \mu} : \mathbb{T}_R^{\leq 2}(X) \rightarrow C_\mu(X)$ is a monomorphism that commutes with cup products, cup-one products, and the \circ map.*

Proof The proof of the lemma follows in general outline the proof of the first part of [35, Theorem 7.2]; since the context here is somewhat different, we provide full details.

It follows directly from the definitions that the map ψ commutes with cup products, cup-one products, and the \circ map, so it suffices to show that ψ is a monomorphism. To prove this, first suppose that $\psi(p) = 0$, for some $p \in \mathfrak{m}_X$. Then for all functions $\mathbf{a} : X \rightarrow R$, we have that $p(\mathbf{a}) = 0$, and so p is the zero polynomial. Therefore, $\psi : \mathbb{T}_R^1(X) \rightarrow C_\mu^1(X)$ is a monomorphism. Now suppose $\psi(\sum \alpha_{I,J} \zeta_I \otimes \zeta_J) = 0$, for some $\alpha_{I,J} \in R$ and each of I, J not identically 0. Then $\sum \alpha_{I,J} \zeta_I(\mathbf{a}) \cdot \zeta_J(\mathbf{a}') = 0$, for all functions $\mathbf{a}, \mathbf{a}' : X \rightarrow R$. Let X' be another (disjoint) copy of X , and for each J , let $J' : X' \rightarrow \mathbb{Z}_{\geq 0}$ be the corresponding indexing function. Viewing each $\zeta_{J'}$ as a polynomial in $\text{Int}(R^{X'})$, it follows that $\sum \alpha_{I,J} \zeta_I \cdot \zeta_{J'} \in \text{Int}(R^{X \sqcup X'})$ is the zero polynomial. For each pair I and J of indexing functions, the functions I and J' have disjoint supports; hence, $\zeta_I \cdot \zeta_{J'} = \zeta_K$, where $K|_X = I$ and $K|_{X'} = J'$. Since these polynomials are elements in a basis for $\text{Int}(R^{X \sqcup X'})$, it follows that each $\alpha_{I,J}$ is equal to 0, thus showing that $\psi : \mathbb{T}_R^2(X) \rightarrow C_\mu^2(X)$ is a monomorphism. \square

The map $\psi = \psi_{X,\mu}$ constructed above enjoys a naturality property that we now proceed to describe. Let $h : X \rightarrow X'$ be a map of sets. By the discussion in Section 6.1, we have an induced morphism, $\mathbb{T}(h) : \mathbb{T}_R^{\leq 2}(X) \rightarrow \mathbb{T}_R^{\leq 2}(X')$. Now let μ' be a binary operation on $M' = M(X', R)$. Composition with h defines a map $h^* : M' \rightarrow M$. Suppose this map is a morphism of magmas, that is, $\mu'(\mathbf{a} \circ h, \mathbf{a}' \circ h) = h(\mu(\mathbf{a}, \mathbf{a}'))$ for all $\mathbf{a}, \mathbf{a}' : X \rightarrow R$. Then, as noted in Section 2.2, h^* yields a simplicial map between the respective Δ -complexes, $\Delta(h^*) : \Delta^{(2)}(M', \mu') \rightarrow \Delta^{(2)}(M, \mu)$, which in turn induces a morphism between the corresponding cochain algebras, $\Delta(h^*)^* : C_\mu(X; R) \rightarrow C_{\mu'}(X'; R)$. The next lemma now follows straight from the definitions.

Lemma 7.2 *Let $h : X \rightarrow X'$ be a set map and suppose $h^* : (M', \mu') \rightarrow (M, \mu)$ is a magma map. Then the following diagram commutes:*

$$(7-2) \quad \begin{array}{ccc} \mathbb{T}_R^{\leq 2}(X) & \xrightarrow{\psi_{X,\mu}} & C_\mu(X; R) \\ \downarrow \mathbb{T}(h) & & \downarrow \Delta(h^*)^* \\ \mathbb{T}_R^{\leq 2}(X') & \xrightarrow{\psi_{X',\mu'}} & C_{\mu'}(X'; R) \end{array}$$

In particular, if $h : X \rightarrow X'$ is injective, $\mu' : M' \times M' \rightarrow M'$ is a binary operation on M' , and μ is the restriction of μ' to $M \times M$, then clearly $h^* : (M', \mu') \rightarrow (M, \mu)$ is a magma map, and thus $\psi_{X',\mu'} \circ \mathbb{T}(h) = \Delta(h^*)^* \circ \psi_{X,\mu}$.

7.2 From 2-tensors to simplicial complexes

We now refine the above construction, in a more specialized setting. Consider a set map $\tau : X \rightarrow \mathbb{T}_R^2(X)$. Recall from (6-1) that $\mathbb{T}_R(X)$ is the tensor algebra on the maximal ideal $\mathfrak{m}_{X,R} \subset \text{Int}(R^X)$. Thus, for each $x \in X$, the 2-tensor $\tau(x) \in \mathbb{T}_R^2(X)$ may be written as

$$(7-3) \quad \tau(x) = \sum_{i=1}^{s_x} p_{x,i} \otimes q_{x,i},$$

for some polynomials $p_{x,i}, q_{x,i} \in \text{Int}(R^X)$ with $p_{x,i}(0) = q_{x,i}(0) = 0$.

As before, let $M = M(X, R)$ be the set of all functions $\mathbf{a} : X \rightarrow R$, with R -module structure given by pointwise addition. We define an operation, $\mu_\tau : M \times M \rightarrow M$, by setting

$$(7-4) \quad \mu_\tau(\mathbf{a}, \mathbf{a}') = \mathbf{a} + \mathbf{a}' - \sum_{i=1}^{s_x} p_{x,i}(\mathbf{a}) \cdot q_{x,i}(\mathbf{a}').$$

The pair $M_\tau := (M, \mu_\tau)$ is a unital magma, with unit the zero function $\mathbb{0}$; indeed, (7-4) implies that $\mu_\tau(\mathbf{a}, \mathbb{0}) = \mu_\tau(\mathbb{0}, \mathbf{a}) = \mathbf{a}$, for all \mathbf{a} . In the particular case when τ itself is the zero function (that is, $\tau(x) = 0$ in $\mathbb{T}_R^2(X)$ for all $x \in X$), the corresponding magma is just the aforementioned abelian group M . In general, though, the operation μ_τ is not associative, and so M_τ need not be a (unital) semigroup (also known as monoid).

We denote by $\Delta^{(2)}(M_\tau)$ the 2-dimensional Δ -set associated to the magma M_τ by the constructions from Sections 2.2 and 7.1, and we let $C_\tau(X) := (C^\bullet(\Delta^{(2)}(M_\tau)), d_\Delta)$ denote the simplicial cochain algebra associated to the Δ -set $\Delta^{(2)}(M_\tau)$.

7.3 The endomorphism d_τ of $\mathbb{T}_R(X)$

Our next objective is to define a degree-1 endomorphism $d_\tau : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X)$ of the free binomial algebra on X that extends the map τ and satisfies some desirable properties. We achieve this by embedding $\mathbb{T}_R^{\leq 2}(X)$ into the cochain algebra $C_\tau(X)$ defined above.

Theorem 7.3 *Given a map of sets, $\tau : X \rightarrow \mathbb{T}_R^2(X)$, there exists a unique degree-1 linear map, $d_\tau : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X)$, such that $d_\tau(x) = \tau(x)$ for all $x \in X$ and both the \cup_1 - d formula and the graded Leibniz rule are satisfied.*

Proof Let $\psi = \psi_{X,\tau} : \mathbb{T}_R^{\leq 2}(X) \hookrightarrow C_\tau(X)$ be the monomorphism from Lemma 7.1. First note that from the formula for the coboundary of a 1-cochain in a Δ -complex, we have

$$(7-5) \quad \begin{aligned} d_\Delta \psi(x)(\mathbf{a}, \mathbf{a}') &= \mathbf{a}(x) + \mathbf{a}'(x) - (\mathbf{a}(x) + \mathbf{a}'(x) - \tau(\mathbf{a}, \mathbf{a}')(x)) \\ &= \tau(\mathbf{a}, \mathbf{a}')(x) \\ &= \sum_i (\varphi(p_{x,i})(\mathbf{a}) \cdot \varphi(q_{x,i})(\mathbf{a}')) \\ &= \sum_i (\varphi(p_{x,i}(x) \cup \varphi(q_{x,i}(x)))(\mathbf{a}, \mathbf{a}')) \\ &= \sum_i (\varphi(p_{i,x}(x)) \otimes \varphi(q_{x,i}(x)))(\mathbf{a}, \mathbf{a}')) \\ &= (\varphi\tau(x))(\mathbf{a}, \mathbf{a}'). \end{aligned}$$

Since this holds for all pairs $(\mathbf{a}, \mathbf{a}') \in M \times M$, it follows that $d_\Delta(\psi(x)) = \psi(\tau(x))$.

The next step is to show that the differential $d_\Delta : C_\tau^\bullet(X) \rightarrow C_\tau^{\bullet+1}(X)$ leaves invariant the subgroup $\psi(\mathbb{T}_R^{\leq 2}(X)) \subset C_\tau(X)$. Let $p = p(x)$ be a polynomial in $\mathbb{T}_R^1(X)$, where $\mathbf{x} \subset X$ denotes the subset of

variables appearing in the monomials comprising p . Given 1-chains \mathbf{a}, \mathbf{a}' , we have

$$(7-6) \quad d_{\Delta} p(\mathbf{a}, \mathbf{a}') = p(\mathbf{a}) + p(\mathbf{a}') - p(\mathbf{a} + \mathbf{a}' - \tau(\mathbf{a}, \mathbf{a}')).$$

By Theorem 5.2, we may write

$$(7-7) \quad d_{\Delta} p(\mathbf{x}, \mathbf{x}') = \sum c_{I_i, J_i} \cdot \zeta_{I_i}(\mathbf{x}) \otimes \zeta_{J_i}(\mathbf{x}'),$$

where the c_{I_i, J_i} are constants in R . From (7-6) it follows that the polynomial $d_{\Delta} p(0, \mathbf{x}') = \sum c_{0, J_i} \cdot \zeta_{J_i}(\mathbf{x}')$ vanishes for all values of \mathbf{x}' ; thus all coefficients c_{0, J_i} vanish. A similar argument shows that $c_{I_i, 0} = 0$ for all i . Therefore, $d_{\Delta} p$ is a sum of products of polynomials in $\text{Int}(R^{\mathbf{X}})$ and polynomials in $\text{Int}(R^{\mathbf{X}'})$ with zero constant term in each factor; that is, $d_{\Delta} p$ is in the image of $\mathbb{T}_R^{\leq 2}(\mathbf{X})$ under the map ψ . This completes the proof that $\psi(\mathbb{T}_R^{\leq 2}(\mathbf{X}))$ is closed under d_{Δ} .

Now set $d_{\tau} : \mathbb{T}_R^{\leq 2}(\mathbf{X}) \rightarrow \mathbb{T}_R^{\leq 2}(\mathbf{X})$ equal to the restriction of d_{Δ} to the invariant subgroup $\psi(\mathbb{T}_R^{\leq 2}(\mathbf{X}))$. Since the differential d_{Δ} satisfies the \cup_1 - d formula (4-11), it follows that d_{τ} also satisfies this formula. Finally, we extend d_{τ} to the whole free cup-one algebra $\mathbb{T} = \mathbb{T}_R(\mathbf{X})$ using the graded Leibniz rule.

The final step is to show that the map $d_{\tau} : \mathbb{T} \rightarrow \mathbb{T}$ defined above is the unique degree-1 linear map for which $d_{\tau}(x) = \tau(x)$ for all $x \in \mathbf{X}$, and the \cup_1 - d formula and the graded Leibniz rule are satisfied. Let $d : \mathbb{T}^1 \rightarrow \mathbb{T}^2$ be any map that satisfies the \cup_1 - d formula with $d(x) = d_{\tau}(x)$ for all $x \in \mathbf{X}$. It suffices to show that $d(p) = d_{\tau}(p)$ for all $p \in \mathbb{T}^1$. Since both d and d_{τ} satisfy the \cup_1 - d formula, it follows that

$$(7-8) \quad d(p \cup_1 q) = d_{\tau}(p \cup_1 q) \quad \text{if } d(p) = d_{\tau}(p) \text{ and } d(q) = d_{\tau}(q).$$

Then from (7-8) and induction on i using $\zeta_{i+1}(x) = (\zeta_i(x) \cup_1 x - i\zeta_i(x))/(i + 1)$, it follows that $d(\zeta_i(x)) = d_{\tau}(\zeta_i(x))$ for all $x \in \mathbf{X}$ and all $i \geq 1$ in the case $R = \mathbb{Z}$ and $1 \leq i \leq p - 2$ in the case $R = \mathbb{F}_p$.

It then follows using (7-8) and induction on the length of $\text{supp}(I)$ that $d(\zeta_I(\mathbf{x})) = d_{\tau}(\zeta_I(\mathbf{x}))$ for all I and \mathbf{x} . Since the polynomials $\zeta_I(\mathbf{x})$ form a basis for \mathbb{T}^1 , the proof of uniqueness is complete. \square

Remark 7.4 Let $\mu : M \times M \rightarrow M$ be an operation on the set $M = M(\mathbf{X}, R)$ of all functions from \mathbf{X} to R and let $\psi : \mathbb{T}_R^{\leq 2}(\mathbf{X}) \hookrightarrow C_{\mu}(\mathbf{X})$ be the monomorphism in Lemma 7.1. It can be shown that the differential $d_{\Delta} : C_{\mu}(\mathbf{X}) \rightarrow C_{\mu}(\mathbf{X})$ leaves invariant the subgroup $\psi(\mathbb{T}_R^{\leq 2}(\mathbf{X})) \subset C_{\mu}(\mathbf{X})$ if and only if $\mu = \mu_{\tau}$, for some function $\tau : \mathbf{X} \rightarrow \mathbb{T}^2(\mathbf{X})$.

7.4 The differential d_{τ} associated to an admissible map τ

Recall that a map $\tau : \mathbf{X} \rightarrow \mathbb{T}_R^2(\mathbf{X})$ given by (7-3) determines a binary operation, $\mu_{\tau} : M \times M \rightarrow M$, given by (7-4). The case when the corresponding unital magma, $M_{\tau} = (M, \mu_{\tau})$, is a monoid is particularly interesting.

Definition 7.5 A map of sets $\tau : \mathbf{X} \rightarrow \mathbb{T}_R^2(\mathbf{X})$ is said to be *admissible* if the corresponding binary operation, $\mu_{\tau} : M \times M \rightarrow M$, is associative, or, equivalently, the magma M_{τ} is a monoid.

Theorem 7.6 *If the map $\tau : \mathbf{X} \rightarrow \mathbb{T}_R^2(\mathbf{X})$ is admissible, then the map $\psi = \psi_{\mathbf{X}, \tau} : \mathbb{T}_R(\mathbf{X}) \rightarrow C^*(\Delta(M_{\tau}))$ is a monomorphism and $d_{\tau}^2 \equiv 0$.*

Proof Let $\Delta_\tau = \Delta(M_\tau)$ be the Δ -set associated to the monoid M_τ . Note that the 2-skeleton of Δ_τ is the previously defined Δ -set $\Delta^{(2)}(M_\tau)$. The arguments in the proofs of Lemma 7.1 and Theorem 7.3 generalize as follows to show that the map $\psi_{\leq 2} : \mathbb{T}_R^{\leq 2}(X) \rightarrow C^{\leq 2}(\Delta_\tau^{(2)})$ extends to a monomorphism $\psi : \mathbb{T}_R(X) \rightarrow C^*(\Delta_\tau)$ with $d_\Delta \circ \psi = \psi \circ d_\tau$.

Set $\psi : \mathbb{T}_R(X) \rightarrow C^*(\Delta_\tau)$ equal to the unique map of algebras that restricts to $\psi_{\leq 2}$ on elements of degree less than or equal to 2.

The next step is to show that ψ is a monomorphism. Let $p_1(\mathbf{x}) \otimes \cdots \otimes p_n(\mathbf{x})$ be a basis element in $\mathbb{T}_R^n(X)$ and let X_1, \dots, X_n be disjoint copies of X . Set $e : \mathbb{T}_R^n(X) \rightarrow \text{Int}(R^{X_1 \cup \cdots \cup X_n})$ equal to the map that sends $p_1(\mathbf{x}) \otimes \cdots \otimes p_n(\mathbf{x})$ to the product of polynomials $p_1(x_1) \cdots p_n(x_n)$, where $p_i(x_i)$ denotes the polynomial $p_i(\mathbf{x})$ with the variables $x \in X$ replaced by the corresponding variables $x_i \in X_i$. Since e is a bijection on basis elements, it follows that e is a bijection. If $\psi(\sum_i p_{1,i}(\mathbf{x}) \otimes \cdots \otimes p_{n,i}(\mathbf{x}))$ is the zero element in $C^n(\Delta_\tau)$, then $e(\sum_i p_{1,i}(\mathbf{a}_1) \otimes \cdots \otimes p_{n,i}(\mathbf{a}_n))$ is zero for all maps $\mathbf{a}_i : X_i \rightarrow R$, and hence, is the zero polynomial. The result that ψ is a monomorphism now follows since e is a monomorphism.

Since $d_\Delta \circ \psi = \psi \circ d_\tau : \mathbb{T}_R^1(X) \rightarrow C^2(\Delta_\tau)$ by Theorem 7.3, and since $\mathbb{T}_R(X)$ is generated by products of elements in $\mathbb{T}_R^1(X)$, it follows that $d_\Delta \circ \psi = \psi \circ d_\tau : \mathbb{T}_R^i(X) \rightarrow C^{i+1}(\Delta_\tau)$ for all $i \geq 1$. Then since $d_\Delta^2 \equiv 0$ and ψ is a monomorphism, it follows that $d_\tau^2 \equiv 0$. \square

Remark 7.7 It can be shown that the homomorphism $\psi : \mathbb{T}_R(X) \rightarrow C^\bullet(\Delta_\tau)$ sends the \cup_1 and \circ products in $\mathbb{T}_R(X)$ given in Section 6.3; respectively, to the \cup_1 and \cup_2 product maps in $C^\bullet(\Delta_\tau)$.

7.5 The differentials of the ζ -maps

We consider now in more detail the simplest possible case of Theorem 7.3; namely, the case when $\tau = 0$. To begin, we recall the following result, which is proved in [35, Theorems 7.5 and 8.14].

Theorem 7.8 Let (A, d_A) be a \cup_1 -dga over $R = \mathbb{Z}$ or \mathbb{F}_p , let X be a set, and let $f : \mathbb{T}_R(X) \rightarrow A$ be a morphism of graded R -algebras with cup-one products. Then $f : (\mathbb{T}_R(X), d_0) \rightarrow (A, d_A)$ is a map of \cup_1 -dgas if and only if $d_A \circ f(x) = f \circ d_0(x)$ for all $x \in X$.

Lemma 6.4 and the above theorem have the following immediate corollary.

Corollary 7.9 If (A, d_A) is a binomial cup-one R -dga with $H^0(A) = R$, then there is a bijection between binomial \cup_1 -dga maps from $(\mathbb{T}_R(X), d_0)$ to (A, d_A) and maps of sets from X to $Z^1(A)$.

The following theorem (which will be used in the proof of Lemma 8.5) gives an explicit formula for the differential $d_0 : \mathbb{T}_R^1(X) \rightarrow \mathbb{T}_R^2(X)$. This result recovers Theorem 6.11 from [35], proved there by other methods. Since the current proof is shorter, more transparent, and illustrates the strength of the techniques developed here, we provide full details.

Theorem 7.10 Let X be a set, let $\tau : X \rightarrow \mathbb{T}_R^2(X)$ be the zero map, and let d_0 be the corresponding differential on $\mathbb{T}_R(X)$, given by $d_0(x) = 0$ for all $x \in X$. Then we have

$$(7-9) \quad d_0(\zeta_k(x)) = -\sum_{\ell=1}^{k-1} \zeta_\ell(x) \otimes \zeta_{k-\ell}(x),$$

for all $x \in X$ and for $k \geq 1$ in the case $R = \mathbb{Z}$ and for $1 \leq k \leq p-1$ in the case $R = \mathbb{F}_p$. More generally,

$$(7-10) \quad d_0(\zeta_I(\mathbf{x})) = - \sum_{\substack{I_1+I_2=I \\ I_j \neq \emptyset}} \zeta_{I_1}(\mathbf{x}) \otimes \zeta_{I_2}(\mathbf{x}),$$

where in the case $R = \mathbb{F}_p$ we have $k \leq p-1$ and each of the indices in I is less than or equal to $p-1$.

Proof Since τ is the zero map, the magma $M = M_\tau$ is a monoid (in fact, an abelian group), and hence, τ is admissible. By Theorem 7.6, we have that $d_0 = d_\tau$ is a differential on $\mathbb{T}_R(X)$, and hence, $\mathbb{T} = (\mathbb{T}_R(X), d_0)$ is a binomial \cup_1 -dga.

To prove (7-9) and (7-10) recall that in a binomial algebra with elements a, b , we have

$$(7-11) \quad \zeta_k(a+b) = \sum_{i+j=k} \zeta_i(a)\zeta_j(b),$$

for $k \geq 1$ in the case $R = \mathbb{Z}$ and for $1 \leq k \leq p-1$ in the case $R = \mathbb{F}_p$. Set $C_\tau(X) = (C^\bullet(\Delta_\tau(X)), d_\Delta)$, and let $\psi : \mathbb{T}^{\leq 2} \hookrightarrow C_\tau^{\leq 2}(X)$ be the injective map of binomial \cup_1 -dgas defined in the proof of Theorem 7.3, so that $\psi(\zeta_k(x)) = \zeta_k(\mathbf{a}(x))$ for all $\mathbf{a} \in C_\tau^1(X)$ and all $k \geq 1$. Then,

$$\begin{aligned} \psi(d_\tau \zeta_k(x))(\mathbf{a}, \mathbf{a}') &= d_\Delta \psi(\zeta_k(x))(\mathbf{a}, \mathbf{a}') \\ &= \mathbf{a}(x) + \mathbf{a}'(x) - \zeta_k(\mathbf{a}(x) + \mathbf{a}'(x)) && \text{(by (7-6))} \\ &= \mathbf{a}(x) + \mathbf{a}'(x) - \mathbf{a}(x) - \mathbf{a}'(x) - \sum_{\ell=1}^{k-1} \zeta_\ell(\mathbf{a}(x)) \cdot \zeta_{k-\ell}(\mathbf{a}'(x)) && \text{(by (7-11))} \\ &= - \sum_{\ell=1}^{k-1} \zeta_\ell(\mathbf{a}(x)) \cdot \zeta_{k-\ell}(\mathbf{a}'(x)) \\ &= - \sum_{\ell=1}^{k-1} [\psi(\zeta_\ell(x)) \cup \psi(\zeta_{k-\ell}(x))](\mathbf{a}, \mathbf{a}') \\ &= - \sum_{\ell=1}^{k-1} \psi[\zeta_\ell(x) \otimes \zeta_{k-\ell}(x)](\mathbf{a}, \mathbf{a}'). \end{aligned}$$

Since this equality holds for all 2-simplices $(\mathbf{a}, \mathbf{a}')$ in $\Delta_\tau(X)$, (7-9) now follows. Equation (7-10) follows similarly, by applying (7-11) to products of the form $\zeta_{i_1}(a_1 + b_1)\zeta_{i_2}(a_2 + b_2) \cdots \zeta_{i_n}(a_n + b_n)$. \square

Corollary 7.11 *Let A be a binomial \cup_1 -dga over R . Then for $a \in Z^1(A)$ we have*

$$(7-12) \quad d_A(\zeta_k(a)) = - \sum_{\ell=1}^{k-1} \zeta_\ell(a) \otimes \zeta_{k-\ell}(a),$$

for all $k \geq 1$ in the case $R = \mathbb{Z}$ and for $1 \leq k \leq p-1$ in the case $R = \mathbb{F}_p$.

Proof By Corollary 7.9, an element $a \in Z^1(A)$ corresponds to a map of binomial \cup_1 -dgas from $(\mathbb{T}_R(\{x\}), d_0)$ to (A, d_A) which sends x to a , and the result follows from Theorem 7.10. \square

7.6 Homotopies between binomial cup-one dga maps

As we saw in Theorem 3.8, homotopic dga maps induce the same homomorphism on cohomology. The next lemma provides a partial converse to this theorem, in the context of binomial cup-one algebras.

Lemma 7.12 *Let (A, d_A) be a binomial cup-one dga over $R = \mathbb{Z}$ or \mathbb{F}_p such that $H^0(A) = R$ and $H^1(A)$ is a finitely generated, free R -module. Suppose $\varphi_0, \varphi_1 : (\mathbb{T}_R(X), d_0) \rightarrow (A, d_A)$ are morphisms of binomial \cup_1 -dgas such that $H^1(\varphi_0) = H^1(\varphi_1)$. Then $\varphi_0 \simeq \varphi_1$.*

Proof We construct a homotopy $\Phi : \mathbb{T}_R(X) \rightarrow A \otimes_R C^*(I; R)$ between φ_0 and φ_1 , as follows. For each $x \in X$, set

$$(7-13) \quad \Phi(x) = \varphi_0(x)t_0 + \varphi_1(x)t_1 - c(x)u,$$

where $c(x)$ is an element in A^0 such that $d_A(c(x)) = \varphi_0(x) - \varphi_1(x)$; such an element exists by our assumption that $H^1(\varphi_0)([x]) = H^1(\varphi_1)([x])$. It is readily verified that $\Phi(x)$ is a 1-cocycle in $A \otimes_R C^*(I; R)$:

$$(7-14) \quad d_{A \otimes_R C^*(I; R)}(\Phi(x)) = \varphi_0(x)u - \varphi_1(x)u - (\varphi_0(x) - \varphi_1(x))u = 0.$$

It now follows from Corollary 7.9 that the set map $\Phi : X \rightarrow Z^1(A \otimes_R C^*(I; R))$ extends to a map of binomial \cup_1 -dgas, $\Phi : \mathbb{T}_R(X) \rightarrow A \otimes_R C^*(I; R)$. By construction, this map is a homotopy between φ_0 and φ_1 . \square

8 Hirsch extensions

In this section, we consider Hirsch extensions of $(\mathbb{T}_R(X), d)$, the free binomial graded algebra with cup-one products on a set X equipped with a differential $d : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X)$ making it into a \cup_1 -dga. Furthermore, for $R = \mathbb{Z}$ or \mathbb{F}_p , we show that the map $\psi_{X, R} : (\mathbb{T}_R(X), d_0) \rightarrow C^*(B(R^n); R)$ induces an isomorphism of cohomology rings.

8.1 Hirsch extensions of $\mathbb{T}_R(X)$

The following definition is motivated by the notion of Hirsch extension in the context of commutative dgas over fields of characteristic 0. Let $(\mathbb{T}_R(X), d)$ be as above.

Definition 8.1 Let Y be a set. An inclusion $i : (\mathbb{T}_R(X), d) \rightarrow (\mathbb{T}_R(X \cup Y), \bar{d})$ of binomial \cup_1 -dgas is called a *Hirsch extension* (over R) if $\bar{d}(y)$ is a cocycle in $(\mathbb{T}_R^2(X), d)$ for all $y \in Y$. If $Y = \{y\}$ is a singleton, we call such an extension an *elementary Hirsch extension*.

Theorem 8.2 *Let $(\mathbb{T}_R(X), d)$ be a free binomial \cup_1 -dga on a set X .*

- (1) *For every set Y , there is a bijection between Hirsch extensions $(\mathbb{T}_R(X), d) \rightarrow (\mathbb{T}_R(X \cup Y), \bar{d})$ and maps of sets $\tau_Y : Y \rightarrow Z(\mathbb{T}_R^2(X))$.*
- (2) *If $\tau = d|_X$ is admissible, then $\bar{\tau} = \bar{d}|_{X \cup Y}$ is admissible.*

Proof Given a Hirsch extension $(\mathbb{T}_R(X), d) \hookrightarrow (\mathbb{T}_R(X \cup Y), \bar{d})$, the restriction of \bar{d} to Y gives a map $\tau_Y = \bar{d}|_Y : Y \rightarrow Z(\mathbb{T}_R^2(X))$. In the opposite direction, assume that the map $\tau_Y : Y \rightarrow Z(\mathbb{T}_R^2(X))$ is given. Set $\bar{\tau} : X \cup Y \rightarrow \mathbb{T}_R^2(X \cup Y)$ equal to the map given by $\bar{\tau}|_X = d|_X$ and $\bar{\tau}|_Y = \tau|_Y$. By Theorem 7.3, the map $\bar{\tau}$ determines an extension of d to a map $\bar{d} = d_{\bar{\tau}} : \mathbb{T}_R(X \cup Y) \rightarrow \mathbb{T}_R(X \cup Y)$ satisfying the \cup_1 - d formula and the Leibniz rule with $d_{\bar{\tau}}|_{X \cup Y} = \bar{\tau}$. Since $\tau_Y(y)$ is a cocycle for all $y \in Y$, it follows from Theorem 6.6 that $\bar{d}^2(u) = 0$ for all $u \in \mathbb{T}_R(X \cup Y)$, and the proof of claim (1) is complete.

Recall from Definition 7.5 that τ is admissible precisely when the corresponding magma, M_τ , is a monoid. It follows from the above proof that $M_{\bar{\tau}}$ is the extension of M_τ by the R -module $M(Y, R)$ of functions from Y to R given by the normalized cocycle $\nu \in Z^2(\Delta(M_\tau); M(Y, R))$, where for $y \in Y$, we have $\tau(y) = \sum_{i=1}^{s_y} p_{y,i} \otimes q_{y,i}$ and $\nu(\mathbf{a}, \mathbf{a}')(y) = \sum_{i=1}^{s_y} p_{y,i}(\mathbf{a}) \cdot q_{y,i}(\mathbf{a}')$.

Claim (2) now follows from Lemma 2.3, part (3). \square

Given a map $\tau : Y \rightarrow Z(\mathbb{T}_R^2(X), d)$, denote by $[\tau]$ the map from Y to $H^2(\mathbb{T}_R(X), d)$ that sends each element $y \in Y$ to the cohomology class of $\tau(y)$.

Definition 8.3 Given maps τ and τ' from a set Y to $Z(\mathbb{T}_R^2(X), d)$, the corresponding Hirsch extensions, $(\mathbb{T}_R(X \cup Y), \bar{d})$ and $(\mathbb{T}_R(X \cup Y), \bar{d}')$, are called *equivalent Hirsch extensions* if $[\tau] = [\tau']$.

Lemma 8.4 If $(\mathbb{T}_R(X \cup Y), \bar{d})$ and $(\mathbb{T}_R(X \cup Y), \bar{d}')$ are equivalent Hirsch extensions, then they are isomorphic.

Proof First recall from Lemmas 6.3 and 6.4 that a map $f : (\mathbb{T}_R(X), d) \rightarrow (A, d_A)$ of binomial graded R -algebras with cup-one products commutes with the differentials if and only if $d_A f(x) = f(dx)$ for all $x \in X$; moreover, if $H^0(A) = R$ and f induces an isomorphism on H^0 , then f is determined by its restriction to the set X .

From the definition of equivalent Hirsch extensions it follows that for each $y \in Y$ there is an element $c_1(y) \in \mathbb{T}_R^1(X)$ with $\tau'(y) = \tau(y) + dc_1(y)$. Define a linear map $f : (\mathbb{T}_R(X \cup Y), \bar{d}) \rightarrow (\mathbb{T}_R(X \cup Y), \bar{d}')$ by setting $f(u) = u$ for $u \in \mathbb{T}_R(X)$ and $f(y) = y - c_1(y)$ for $y \in Y$. Then

$$\begin{aligned} \bar{d}'(f(y)) &= \bar{d}'(y - c_1(y)) = \bar{d}'(y) - dc_1(y) = \tau'(y) - dc_1(y) \\ &= (\tau(y) + dc_1(y)) - dc_1(y) = \tau(y) = f(\bar{d}(y)), \end{aligned}$$

and so f commutes with the differentials. Similarly, a linear map $g : (\mathbb{T}_R(X \cup Y), \bar{d}') \rightarrow (\mathbb{T}_R(X \cup Y), \bar{d})$ is defined by setting $g(u) = u$ for $u \in \mathbb{T}_R(X)$ and $g(y) = y + c_1(y)$ for $y \in Y$. Then g commutes with the differentials, and the result follows since f and g are inverses of each other. \square

8.2 A spectral sequence

We now set up a cohomological spectral sequence that will prove useful for our purposes.

Lemma 8.5 Let $(\mathbb{T}_R(X), d)$ be a free binomial \cup_1 -dga on a set X . Given an elementary Hirsch extension $(\mathbb{T}_R(X), d) \rightarrow (\mathbb{T}_R(X \cup \{y\}), \bar{d})$, there is a spectral sequence, $(E_r^{p,q}, d_r)$, converging to $H^*(\mathbb{T}_R(X \cup \{y\}), \bar{d})$ with $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ and $E_2^{p,q} \cong H^p(\mathbb{T}_R(X), d) \otimes H^q(\mathbb{T}_R(\{y\}), d_0)$, where $d_0(y) = 0$.

Proof Denote $\mathbb{T}_R(X \cup \{y\})$ by \mathbb{T} . A basis for the free R -module \mathbb{T}^1 is given by elements of the form $\zeta_I(x_1, \dots, x_\ell)\zeta_k(y)$, where $I = (i_1, \dots, i_\ell)$ and $\zeta_I(x_1, \dots, x_\ell) = \zeta_{i_1}(x_1) \cdots \zeta_{i_\ell}(x_\ell) \in \mathbb{T}_R(X)$, with the x_j distinct elements in X . If $k = 0$, then $\zeta_I(x_1, \dots, x_\ell)\zeta_k(y)$ denotes $\zeta_I(x_1, \dots, x_\ell)$; and if $I = (0, \dots, 0)$, then $\zeta_I(x_1, \dots, x_\ell)\zeta_k(y)$ denotes $\zeta_k(y)$, where in the case $R = \mathbb{F}_p$, we have that $1 \leq i_j \leq p - 1$.

Define a bigrading on \mathbb{T} by setting $D^{p,q}$ equal to the summand of \mathbb{T}^{p+q} generated by the tensor products $u_1 \otimes \cdots \otimes u_{p+q}$ of basis elements in \mathbb{T}^1 for which exactly p of the factors have $I \neq (0, \dots, 0)$. We claim that the differential d restricts to maps

$$(8-1) \quad d : D^{0,1} \rightarrow D^{2,0} \oplus D^{0,2} \quad \text{and} \quad d : D^{1,0} \rightarrow D^{2,0} \oplus D^{1,1}.$$

The claim follows by induction using the \cup_1 - d formula, the Hirsch identity, and

$$\zeta_{n+1}(y) = \frac{\zeta_n(y)y - n\zeta_n(y)}{n + 1}.$$

The group $D^{0,1}$ is free abelian, with basis $\{\zeta_i(y)\}$, and the induction is on i with base case $i = 1$. The group $D^{1,0}$ has basis $\{\zeta_I(x_1, \dots, x_\ell)\zeta_i(y)\}$, where $I \neq \{0\}$, and the induction is on i with base case $i = 0$.

From (8-1), it follows that $F^\ell(\mathbb{T}) := \bigoplus_{p \geq \ell, q \geq 0} D^{p,q}$ defines a decreasing filtration,

$$\mathbb{T} = F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots,$$

of subcomplexes. A direct computation shows that in the resulting spectral sequence the E_2 term is given by $E_2^{p,q} = H^p(\mathbb{T}_R(X), d) \otimes H^q(\mathbb{T}_R(y), d_0)$, where $d_0(y) = 0$. □

8.3 The cohomology of $(\mathbb{T}_R(X), d_0)$

We are now in a position to compute the cohomology algebra of the free binomial graded cup-one algebra $\mathbb{T}_R(X)$, endowed with the differential $d_\tau = d_0$ corresponding to the admissible function $\tau : X \rightarrow \mathbb{T}_R^2(X)$ given by $\tau(x) = 0$ for all $x \in X$. We first assume $R = \mathbb{Z}$, in which case we write $\mathbb{T}(X) := \mathbb{T}_R(X)$.

Proposition 8.6 *Given a finite set X , there is a natural isomorphism*

$$(8-2) \quad \kappa_X : H^*(\mathbb{T}(X)) \xrightarrow{\cong} \bigwedge^*(X)$$

between the cohomology algebra of the dga $(\mathbb{T}(X), d_0)$ and the exterior algebra on the free abelian group \mathbb{Z}^X .

Proof We establish the existence of the isomorphism κ_X is by induction on k , the size of X . For the base case $k = 1$, write $\mathbb{T} = \mathbb{T}(X)$, and define two subcomplexes, \mathbb{T}_0 and \mathbb{T}_1 , as follows. Set $\mathbb{T}_0^0 = \mathbb{Z}$, $\mathbb{T}_0^1 = \mathbb{Z}$ with generator x , and $\mathbb{T}_0^i = 0$ for $i \geq 2$. Furthermore, set \mathbb{T}_1^1 equal to the submodule of $\mathbb{T}^1(\{x\})$ generated by the elements of the form $\zeta_k(x)$ with $k \geq 2$, and set $\mathbb{T}_1^j = \mathbb{T}^j(\{x\})$ for $j \geq 2$. It is now readily verified that $\mathbb{T} = \mathbb{T}_0 \oplus \mathbb{T}_1$.

Clearly, $T_0 = \bigwedge(x)$, with zero differential; thus, $H^*(T_0) = \bigwedge(x)$. Denote $\zeta_k(x)$ by ζ_k , and define homomorphisms $h_\ell : T_1^\ell \rightarrow T_1^{\ell-1}$ by

$$(8-3) \quad h_\ell(\zeta_{i_1} \otimes \zeta_{i_2} \otimes \cdots \otimes \zeta_{i_\ell}) = \begin{cases} -\zeta_{i_2+1} \otimes \cdots \otimes \zeta_{i_\ell} & \text{if } i_1 = 1, \\ 0 & \text{if } i_1 > 1. \end{cases}$$

By direct computation using (7-10), it follows that $d_0 \circ h_\ell + h_{\ell+1} \circ d_0 = \text{id}_{T_1}$. Hence, the cohomology of T_1 is zero, and we conclude that the cohomology of $(T(\{x\}), d_0)$ is the exterior algebra with generator x .

For the inductive step, we assume the result holds for $X_k = \{x_1, \dots, x_k\}$ and show the result then holds for $X_{k+1} = \{x_1, \dots, x_k, x_{k+1}\}$. We use the spectral sequence in Lemma 8.5 with $X = X_k$ and $y = x_{k+1}$; applying the base case with $T = T(x_{k+1})$, we have by induction that the E_2 term in the spectral sequence of the form $E_2 = \bigwedge(x_1, \dots, x_k) \otimes \bigwedge(x_{k+1})$. Since $d_0 x_{k+1} = 0$ in T , it follows that the spectral sequence collapses, from which we obtain an isomorphism of graded \mathbb{Z} -modules, $H^*(T(X), d_0) \cong \bigwedge(x_1, \dots, x_{k+1})$.

For $x \in X$, set $[x]$ equal to the cohomology class of x in $H^1(T(X), d_0)$; note that $d_0 \zeta_2(x) = -x \cup x$, and so $[x] \cup [x] = 0$ in $H^*(T(X))$. Moreover, for x_1, x_2 distinct elements in X , we have

$$d_0(x_1 \cup_1 x_2) = -x_1 \cup x_2 - x_2 \cup x_1,$$

so $[x_1] \cup [x_2] = -[x_2] \cup [x_1]$. It follows that the isomorphism κ_X of graded \mathbb{Z} -modules between $H^*(T(X), d_0)$ and the exterior algebra is a map of graded algebras.

To prove the naturality of the isomorphism κ_X , let $h : X \rightarrow Y$ be a map of sets, let $\bigwedge(h) : \bigwedge(X) \rightarrow \bigwedge(Y)$ be its extension to exterior algebras, and let $T(h) : T(X) \rightarrow T(Y)$ be its extension to free binomial graded algebras with cup-one products, as in Section 6.1. It is readily verified that $\kappa_Y \circ T(h) = \bigwedge(h) \circ \kappa_X$. \square

Corollary 8.7 *If X is a set with n elements, then the map $\psi_X : (T(X), d_0) \rightarrow C^*(B(\mathbb{Z}^n); \mathbb{Z})$ induces an isomorphism of cohomology rings.*

Proof The proof is by induction. To prove the result in the case $n = 1$, let $X = \{x\}$. The morphism ψ in Theorem 7.6 maps $T(\{x\})$ to $C^*(\Delta(M(\{x\})); \mathbb{Z}) = C^*(B(\mathbb{Z}); \mathbb{Z})$. Note that $C^1(B(\mathbb{Z}); \mathbb{Z})$ is the free abelian group of maps of sets from \mathbb{Z} to \mathbb{Z} and $\psi(x)$ is the identity map from \mathbb{Z} to \mathbb{Z} . The identity map of \mathbb{Z} is a generator for $H^1(\mathbb{Z})$ and it now follows from Proposition 8.6 that the map $H^*(\psi)$ is an isomorphism.

For the inductive step, write $X^n = \{x_1, \dots, x_n\}$ and $X = X^n \cup \{x_{n+1}\}$, and consider the morphism $\psi = \psi_X : (T(X), d_0) \rightarrow C^*(B(\mathbb{Z}^n \oplus \mathbb{Z}); \mathbb{Z})$ from Theorem 7.6. Assume by induction that the restriction of ψ to a map $T(X^n) \rightarrow C^*(B(\mathbb{Z}^n); \mathbb{Z})$ induces an isomorphism on cohomology. By the case $n = 1$ above, the restriction of ψ to the map from $(T(\{x_{n+1}\}), d_0)$ to $C^*(B(\mathbb{Z}); \mathbb{Z})$ induces an isomorphism on cohomology. Thus the map of E_2 terms from the spectral sequence of the Hirsch extension $T(X^n) \rightarrow T(X)$ to the spectral sequence of the central extension $\mathbb{Z} \rightarrow \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}^n$ is an isomorphism. Since both spectral sequences collapse, it follows that $H^*(\psi)$ is an isomorphism. \square

The next step is to consider the case $R = \mathbb{F}_p$. Recall that for $R = \mathbb{F}_2$, the cohomology ring $H^*(B(R); R)$ is the polynomial algebra $R[x]$ on a generator x in degree 1, and for $R = \mathbb{F}_p$ with p odd, $H^*(B(R); R)$ is the free commutative algebra $\bigwedge(x) \otimes_R R[y]$ with one generator x in degree 1 and one generator y in degree 2, with the relation $x^2 = 0$.

Proposition 8.8 *For $R = \mathbb{F}_p$, the map $\psi : (\mathbb{T}_R(\{x\}), d_0) \rightarrow C^*(B(R); R)$ induces an isomorphism $\psi^* : H^*(\mathbb{T}_R(\{x\}), d_0) \xrightarrow{\cong} H^*(B(R); R)$.*

Proof Consider first the case $R = \mathbb{F}_2$. In this case $\mathbb{T}_R(\{x\})$ is equal to $\mathbb{F}_2[x]$, the polynomial algebra on a single generator x . Hence, the differential d_0 is identically zero and we have that $H^*(\mathbb{T}_R(\{x\}), d_0) = \mathbb{F}_2[x]$. To see that ψ induces an isomorphism on cohomology, note that the 1-chain, $[1]$, in the chain complex of the bar construction on \mathbb{F}_2 is a generator of $H_1(B(\mathbb{F}_2); \mathbb{F}_2)$ and $\psi(x)([1]) = 1 \in \mathbb{F}_2$ so $\psi(x)$ is a generator of $H^1(B(\mathbb{F}_2); \mathbb{F}_2)$. The result follows, since ψ^* is an isomorphism in degree 1 and both its source and target are polynomial algebras on a single generator in degree 1.

Now consider the case $R = \mathbb{F}_p$ with $p \geq 3$. Denoting $\zeta_i(x)$ by ζ_i , the R -vector space $\mathbb{T} = \mathbb{T}_R(\{x\})$ has basis consisting of all elements of the form $\zeta_{i_1} \otimes \zeta_{i_2} \otimes \cdots \otimes \zeta_{i_\ell}$, where each $1 \leq i_j \leq p-1$. Set D equal to the graded R -submodule of \mathbb{T} generated by the basis elements ζ_i for $2 \leq i \leq p-1$ and $\zeta_{i_1} \otimes \zeta_{i_2} \otimes \cdots \otimes \zeta_{i_\ell}$ for $(i_1, i_2) \neq (1, p-1)$. Clearly, D is closed under the differential d_0 ; moreover, the cochain homotopy used over \mathbb{Z} restricts to D . Hence, $H^*(D, d_0) = 0$, and it follows that $H^*(\mathbb{T}, d_0) \cong H^*(\mathbb{T}/D, d_0)$.

Write x and ζ_i for the images of those elements from \mathbb{T} in \mathbb{T}/D , and note that both x and $\zeta_1 \otimes \zeta_{p-1}$ are cocycles in \mathbb{T}/D . The \mathbb{F}_p -algebra \mathbb{T}/D is generated in degree 1 by $[x]$ and in degree 2 by $[\zeta_1 \otimes \zeta_{p-1}]$. It follows that $H^1(\mathbb{T}/D; \mathbb{F}_p) \cong \mathbb{F}_p$, with generator $[x]$, and $H^2(\mathbb{T}/D; \mathbb{F}_p) \cong \mathbb{F}_p$, with generator $[\zeta_1 \otimes \zeta_{p-1}]$. Now note that $(\mathbb{T}/D)^{i+2}$ is isomorphic to $\zeta_1 \otimes \zeta_{p-1} \otimes \mathbb{T}^i$ for $i \geq 1$. The degree-2 map $\mathbb{T}^* \rightarrow (\mathbb{T}/D)^{*+2}$ given by $\alpha \mapsto \zeta_1 \otimes \zeta_{p-1} \otimes \alpha$ for $\alpha \in \mathbb{T}^i$ is an isomorphism of graded R -algebras. Since $\zeta_1 \otimes \zeta_{p-1}$ is a cocycle in \mathbb{T}/D , it follows from the graded Leibniz rule that this map commutes with the differentials, and hence induces an isomorphism on cohomology. This gives $H^i(\mathbb{T}) \cong H^i(\mathbb{T}/D) \cong H^{i+2}(\mathbb{T})$ for $i \geq 1$, and it follows that $H^i(\mathbb{T}/D) \cong \mathbb{F}_p$ for $i \geq 1$. The generators of these groups are

$$\begin{aligned} &\zeta_1 \otimes \zeta_{p-1} \otimes \zeta_1 \otimes \zeta_{p-1} \otimes \cdots \otimes \zeta_1 \otimes \zeta_{p-1} && \text{if } i \text{ is even,} \\ &\zeta_1 \otimes \zeta_{p-1} \otimes \zeta_1 \otimes \zeta_{p-1} \otimes \cdots \otimes \zeta_1 \otimes \zeta_{p-1} \otimes \zeta_1 && \text{if } i \text{ is odd.} \end{aligned}$$

The next step is to show that ψ induces isomorphisms $\psi^i : H^i(\mathbb{T}(\{x\}), d_0) \rightarrow H^i(B(\mathbb{F}_p); \mathbb{F}_p)$ in degrees $i = 1$ and 2. The 1-chain $[1]$ is a generator of $H_1(B(\mathbb{F}_p); \mathbb{F}_p) = \mathbb{F}_p$ and $\psi(x)([1]) = 1$, so ψ^1 is an isomorphism. Note that the cocycle $c = \sum_{i=1}^p \zeta_i(x) \cup \zeta_{k-i}(x) \in \mathbb{T}$ projects to the cocycle $\zeta_1 \otimes \zeta_{p-1} \in \mathbb{T}/D$. Moreover, the homology class of the 2-cycle $g = \sum_{i=1}^{p-1} [i|1]$ is a generator of $H_2(B(\mathbb{F}_p); \mathbb{F}_p)$. Since $\psi(c)(g) = 1$, we conclude that ψ^2 is also an isomorphism.

Finally, since $H^*(B(\mathbb{F}_p); \mathbb{F}_p)$ is generated in degrees 1 and 2 and ψ^* is an isomorphism in those degrees, it follows that ψ^* is an epimorphism. Since $H^i(\mathbb{T}(\{x\}), d_0)$ and $H^i(B(\mathbb{F}_p); \mathbb{F}_p)$ are both isomorphic to \mathbb{F}_p for each $i \geq 0$, it follows that ψ^* is an isomorphism. □

The next theorem is a synthesis of the preceding results.

Theorem 8.9 *If X is a finite set with n elements and if $R = \mathbb{Z}$ or \mathbb{F}_p , then the dga morphism $\psi_{X,R} : (\mathbb{T}_R(X), d_0) \rightarrow C^*(B(R^n); R)$ induces an isomorphism of cohomology rings.*

Proof For $R = \mathbb{Z}$ the result is Corollary 8.7. For $R = \mathbb{F}_p$ and $n = 1$, the result is Proposition 8.8. For $n > 1$, the result follows by induction on n , using the property that the spectral sequence of the Hirsch extension $(\mathbb{T}_R(X), d_0) \rightarrow (\mathbb{T}_R(X \cup \{y\}), d_0)$ collapses with $E_2 = E_\infty$. \square

8.4 Colimits of Hirsch extensions

We now consider a type of free binomial \cup_1 -dgas that arise as unions (or, colimits) of certain sequences of Hirsch extensions. These objects will play an important role for the rest of this paper.

Definition 8.10 A free binomial \cup_1 -dga $(\mathbb{T}_R(X), d)$ is called a *colimit of Hirsch extensions* if the following conditions hold.

- (1) $X = \bigcup_{i \geq 1} X_i$ with each set X_i finite.
- (2) For $X^n = \bigcup_{i=1}^n X_i$ and $n \geq 1$, the differential d on $\mathbb{T}_R(X)$ restricts to a differential d_n on $\mathbb{T}_R(X^n)$.
- (3) Each map $i_n : (\mathbb{T}_R(X^n), d_n) \rightarrow (\mathbb{T}_R(X^{n+1}), d_{n+1})$ is a Hirsch extension.
- (4) $X_1 \neq \emptyset$, and $d_1(x) = 0$ for all $x \in X_1$.

A *morphism of colimits of Hirsch extensions* is a map of binomial \cup_1 -dgas as above, $f : (\mathbb{T}_R(X), d) \rightarrow (\mathbb{T}_R(X'), d')$, with the property that for each $n \geq 1$, it restricts to a morphism $f_n : \mathbb{T}_R(X^n) \rightarrow \mathbb{T}_R(X'^n)$. Note that these morphisms are compatible with the respective colimits; that is, the diagram below commutes for each $n \geq 1$:

$$(8-4) \quad \begin{array}{ccc} \mathbb{T}_R(X^{n+1}) & \xrightarrow{f_{n+1}} & \mathbb{T}_R(X'^{n+1}) \\ \uparrow i_n & & \uparrow i'_n \\ \mathbb{T}_R(X^n) & \xrightarrow{f_n} & \mathbb{T}_R(X'^n) \end{array}$$

8.5 The group associated to a colimit of Hirsch extensions

We now associate in a functorial way to each colimit of Hirsch extensions a pronilpotent group.

Lemma 8.11 *Let $\mathbb{T} = (\mathbb{T}_R(X), d)$ be a colimit of Hirsch extensions $\mathbb{T}_n = (\mathbb{T}_R(X^n), d_n)$.*

- (1) *There is a pronilpotent group $G_{\mathbb{T}}$ and a \cup_1 -dga map $\psi_{\mathbb{T}} : \mathbb{T} \rightarrow C^*(B(G_{\mathbb{T}}); R)$.*
- (2) *If X is finite, then $G_{\mathbb{T}}$ is a nilpotent group and $\psi_{\mathbb{T}}$ is a quasi-isomorphism. Moreover, if $R = \mathbb{Z}$, then $G_{\mathbb{T}}$ is torsion-free.*
- (3) *In general, $\psi_{\mathbb{T}}$ induces a quasi-isomorphism from \mathbb{T} to $C^*_{\text{cts}}(B(G_{\mathbb{T}}); R) = \text{colim}_{n \rightarrow \infty} C^*(B(G_{\mathbb{T}_n}); R)$, the continuous cochains on $B(G_{\mathbb{T}})$.*
- (4) *Every morphism of colimits of Hirsch extensions, $f : \mathbb{T} \rightarrow \mathbb{T}'$, induces (in a functorial way) a group homomorphism, $\tilde{f} : G_{\mathbb{T}'} \rightarrow G_{\mathbb{T}}$.*

Proof (1) We start by defining for each $n \geq 1$ a finitely generated nilpotent group, $G_n = G_{T_n}$, corresponding to the free \cup_1 -dgas $T_n = (T_R(X^n), d_n)$, as well as a \cup_1 -dga map $\psi_n : T_R(X^n) \rightarrow C^*(B(G_n); R)$ inducing an isomorphism on cohomology. This is done inductively, as follows.

First let $G_1 = M(X_1, R)$. As noted in Section 7.1, this is a free R -module with basis X_1 ; we view it now as a finitely generated abelian group. By Theorem 8.9, there is a quasi-isomorphism $\psi_1 : (T_R(X_1), d_0) \rightarrow C^*(B(G_1); R)$. Assume now that a finitely generated nilpotent group G_n has been constructed, together with a \cup_1 -dga quasi-isomorphism $\psi_n : (T_R(X^n), d_n) \rightarrow C^*(B(G_n); R)$ inducing an isomorphism on H^1 . By Theorem 8.2, the differential d_{n+1} on $T(X_{n+1})$ restricts to an admissible map $\tau_n : X_{n+1} \rightarrow Z(T_R^2(X^n))$. The composition $\psi_n \circ \tau_n$, then, defines a cocycle in $Z^2(B(G_n); M(X_{n+1}, R))$; let

$$(8-5) \quad 0 \longrightarrow M(X_{n+1}, R) \longrightarrow G_{n+1} \xrightarrow{q_n} G_n \longrightarrow 1$$

be the corresponding central extension. Since G_n is a group, Lemma 2.3 insures that G_{n+1} is also a group; by construction, this is again a finitely generated nilpotent group (torsion-free if $R = \mathbb{Z}$). Since the map τ_n is admissible, Theorem 8.2 insures that the map $\tau_{n+1} = d_{n+1}|_{X_{n+1}}$ is also admissible. Since X_{n+1} is finite, the Hirsch extension $T_R(X_n) \hookrightarrow T_R(X_{n+1})$ can be realized as a sequence of elementary Hirsch extensions. The inductive assumption together with Lemma 8.5 and Theorem 8.9 then show that ψ_{n+1} is a quasi-isomorphism. This completes the construction of the G_n and the argument that ψ_n is a quasi-isomorphism.

Let $G_T = \varprojlim G_n$ be the limit of the inverse system of groups $\{G_n, q_n\}_{n \geq 1}$ and $\psi_T : T \rightarrow C^*(B(G_T); R)$ be the colimit of the directed system of maps $\psi_n : (T_R(X^n), d_n) \rightarrow C^*(B(G_n); R)$. By construction, both G_T and ψ_T satisfy the claimed properties. The underlying magma of G_T is $(M(X, R), \mu_\tau)$, where $\tau : X \rightarrow Z(T_R^2(X))$ is the colimit of the maps τ_n , while ψ_T coincides with $\psi_X : T(X) \rightarrow C^*(\Delta(M_\tau); R)$.

(2) If X is finite, then $X = X^n$ for some $n \geq 1$, and the claimed properties for G_T and ψ_T follow from the above proof.

(3) The fact that the map $T \rightarrow C_{\text{cts}}^*(B(G_T); R)$ induced by ψ_T is a quasi-isomorphism (when X is not necessarily finite) follows from part (2).

(4) Let $f : T \rightarrow T'$ be a morphism of colimits Hirsch extensions, so that, for each $n \geq 1$, the diagram (8-4) commutes and $d'_n \circ f_n = f_{n+1} \circ d_n$. Setting $\tilde{i}_n := q_n$, we define inductively homomorphisms $\tilde{f}_n : G'_n \rightarrow G_n$ which satisfy $\tilde{f}_n \circ \tilde{i}'_n = \tilde{i}_n \circ \tilde{f}_{n+1}$, as follows. We first let $\tilde{f}_1 : G'_1 \rightarrow G_1$ be equal to the R -linear map $f_1^\vee : M(X'_1, R) \rightarrow M(X_1, R)$ from Section 7.1. Assuming that \tilde{f}_n has been defined, we let $(f_{n+1}|_{X_{n+1}})^\vee : M(X'_{n+1}, R) \rightarrow M(X_{n+1}, R)$ be equal to the $\text{Hom}(-, R)$ -dual of the set map $f_{n+1}|_{X_{n+1}} : X_{n+1} \rightarrow X'_{n+1}$. The fact that f_n and f_{n+1} are compatible dga-maps implies that the homomorphisms \tilde{f}_n and $(f_{n+1}|_{X_{n+1}})^\vee$ are compatible with the k -invariants of the extensions (8-5), and thus define a homomorphism $\tilde{f}_{n+1} : G'_{n+1} \rightarrow G_{n+1}$ with the claimed property. Passing to the limit yields a homomorphism $\tilde{f} : G_{T'} \rightarrow G_T$. Finally, if $g : T' \rightarrow T''$ is another morphism of colimits of Hirsch extensions, it is readily verified that $\tilde{f} \circ \tilde{g} = \tilde{g} \circ f$. \square

Remark 8.12 In [1; 2], Bousfield and Kan associated to every space X and commutative ring R an R -completion, $R_\infty(X)$. This is an R -space (i.e., its homology groups in positive degrees are R -modules) which comes equipped with a structure map, $\kappa_X : X \rightarrow R_\infty X$, with the following property: Given a map $f : X \rightarrow Y$, there is an induced map, $R_\infty f : R_\infty X \rightarrow R_\infty Y$, such that $R_\infty f \circ \kappa_X = \kappa_Y \circ f$; moreover, f is an R -homology equivalence if and only if $R_\infty f$ is a weak homotopy equivalence. The Bousfield–Kan completion also works at the level of groups, sending in a functorial way a group G to its R -completion, \widehat{G}_R . In particular, if $R = \mathbb{Z}$, then $\widehat{G}_\mathbb{Z} = \varprojlim_n G / \Gamma_n(G)$ is the pronilpotent completion of G , built from the lower central series $\{\Gamma_n(G)\}_{n \geq 1}$. It is shown in [36] that if \mathbb{T} is the 1-minimal model of a path-connected space X , then $G_\mathbb{T}$ is the pro-torsion-free-nilpotent completion of $G = \pi_1(X)$ if $R = \mathbb{Z}$ and $G_\mathbb{T} = \widehat{G}_{\mathbb{F}_p}$ if $R = \mathbb{F}_p$, and thus, an R -free pronilpotent group.

The next lemma describes another type of functoriality property of the above construction, related to maps between classifying spaces of R -free pronilpotent groups.

Theorem 8.13 *Let $\mathbb{T} = (\mathbb{T}_R(X), d)$ be a colimit of Hirsch extensions, let $G = G_\mathbb{T}$ be the corresponding pronilpotent group, and assume the map $\psi_\mathbb{T} : \mathbb{T}_R(X) \rightarrow C^*(BG; R)$ is a quasi-isomorphism. Let also $0 \rightarrow F \rightarrow \overline{G} \rightarrow G \rightarrow 1$ be a central extension of groups, with F a finitely generated, free R -module, and let $\pi : B\overline{G} \rightarrow BG$ be the corresponding fibration.*

- (1) *There is a Hirsch extension $i : \mathbb{T} \hookrightarrow \overline{\mathbb{T}} = (\mathbb{T}_R(X \cup Y), \bar{d})$ such that $\overline{G} = G_{\overline{\mathbb{T}}}$.*
- (2) *The diagram below commutes:*

$$(8-6) \quad \begin{array}{ccc} \overline{\mathbb{T}} & \xrightarrow{\psi_{\overline{\mathbb{T}}}} & C^*(B\overline{G}; R) \\ \uparrow i & & \uparrow \pi^* \\ \mathbb{T} & \xrightarrow{\psi_\mathbb{T}} & C^*(BG; R) \end{array}$$

- (3) *The map $\psi_{\overline{\mathbb{T}}}$ is a quasi-isomorphism.*

Proof It suffices to prove the case where $F = R$. Let $c \in \mathbb{T}_R^2(X)$ be a cocycle such that the cohomology class of $\psi_X(c)$ in $H^2(BG; R)$ corresponds to the central extension. Set $\overline{\mathbb{T}} = (\mathbb{T}_R(X \cup \{y\}), \bar{d})$ equal to the Hirsch extension with $\bar{d}(y) = c$ and set $\overline{G} = G_{\overline{\mathbb{T}}}$. This gives the commutative diagram (8-6).

The map $\psi_{\overline{\mathbb{T}}}$ induces a map of spectral sequences from the spectral sequence of the Hirsch extension to the spectral sequence of the fibration. The map of the terms $E_2^{p,q}$ is

$$(8-7) \quad H^p(\mathbb{T}_R(X)) \otimes H^q(\mathbb{T}_R(\{y\})) \xrightarrow{\psi_\mathbb{T} \otimes f} H^p(BG; R) \otimes H^q(BR; R),$$

where $H^q(\mathbb{T}(\{y\}))$ denotes the cohomology computed with $dy = 0$ and f denotes the map on cohomology induced by the map $\psi_{\mathbb{T}(\{y\})} : \mathbb{T}_R(\{y\}) \rightarrow C^*(BR; R)$. The map $\psi_\mathbb{T}$ is an isomorphism by assumption, while f has been shown to be an isomorphism in Theorem 8.9. Hence, the map of E_2 terms is an isomorphism and it follows that $\psi_{\overline{\mathbb{T}}}$ is a quasi-isomorphism. □

Corollary 8.14 *The assignment $\mathbb{T} = (\mathbb{T}_R(X), d) \rightsquigarrow G_{\mathbb{T}}$ establishes an equivalence of categories between colimits of Hirsch extensions over R and R -free pronilpotent groups, with the property that $H^*(\mathbb{T}_R(X)) \cong H_{\text{cts}}^*(G_{\mathbb{T}}; R)$.*

Proof By part (1) of Lemma 8.11 it follows that if $(\mathbb{T}_R(X), d)$ is a colimit of Hirsch extensions over R , then the corresponding group $G_{\mathbb{T}}$ is an R -free pronilpotent group.

Conversely, if $G = \varprojlim G_n$ is an R -free pronilpotent group, by Theorem 8.13 and induction on n , $\mathbb{T} = (\mathbb{T}_R(X), d)$ is a colimit of Hirsch extensions. Also, by Lemma 8.11(3), $H_{\text{cts}}^*(G_{\mathbb{T}}; R) \cong H^*(\mathbb{T})$. \square

9 The existence of 1-minimal models

In this section, we define 1-minimal models and show that every binomial \cup_1 -dga A over the ring $R = \mathbb{Z}$ or $R = \mathbb{F}_p$ admits a 1-minimal model, provided that $H^0(A) = R$ and $H^1(A)$ is a finitely generated, free R -module.

9.1 Solving an extension problem

We start by setting up an extension problem in the category of binomial \cup_1 -dgas and give an if and only if criterion to solve it.

Definition 9.1 Let $f : (\mathbb{T}_R(X), d) \rightarrow (A, d_A)$ be a morphism of binomial \cup_1 -dgas, and let the map $i : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X \cup Y)$ be a Hirsch extension of $\mathbb{T}_R(X)$. A morphism $\bar{f} : \mathbb{T}_R(X \cup Y) \rightarrow A$ is an *extension of f* if the following diagram commutes:

$$(9-1) \quad \begin{array}{ccc} & & \mathbb{T}_R(X \cup Y) \\ & \nearrow \bar{f} & \uparrow i \\ A & \xleftarrow{f} & \mathbb{T}_R(X) \end{array}$$

Theorem 9.2 *Given a morphism of binomial \cup_1 -dgas, $f : (\mathbb{T}_R(X), d) \rightarrow (A, d_A)$, and a Hirsch extension, $i : (\mathbb{T}_R(X), d) \rightarrow (\mathbb{T}_R(X \cup Y), \bar{d})$, there is a bijection between the set of extensions $\bar{f} : \mathbb{T}_R(X \cup Y) \rightarrow A$ of f and the set of functions $\sigma : Y \rightarrow A^1$ such that $d_A(\sigma(y)) = f(dy)$ for all $y \in Y$.*

Proof First note that there is an extension \bar{f} of f if and only if $[f(dy)] = 0$ for all $y \in Y$. Thus, the set of such extensions is empty if and only if the set $\{\sigma : Y \rightarrow A^1 : d_A(\sigma(y)) = f(dy), \forall y \in Y\}$ is empty, and we are done in this case.

So assume there is an extension $\bar{f} : \mathbb{T}_R(X \cup Y) \rightarrow A$. The corresponding map $\sigma : Y \rightarrow A^1$ is then given by $\sigma(y) = \bar{f}(y)$ for $y \in Y$. The condition that $d_A(\sigma(y)) = f(dy)$ follows from the assumption that \bar{f} is a map of dgas.

In the opposite direction, assume $\sigma : Y \rightarrow A^1$ is a map of sets with $d_A(\sigma(y)) = f(dy)$ for all $y \in Y$. Then by Lemma 6.2, $y \mapsto \sigma(y)$ extends f uniquely to a map $\bar{f} : \mathbb{T}_R(X \cup Y) \rightarrow A$ of binomial graded R -algebras with cup-one products. Since $d_A(\sigma(y)) = f(dy)$, it then follows from Theorem 7.8 that this extension commutes with the differentials d and d_A , and so \bar{f} is a morphism of binomial \cup_1 -dgas. \square

Lemma 9.3 *With notation as above, let $\bar{f} : \mathbb{T}_R(X \cup Y) \rightarrow A$ be an extension of $f : \mathbb{T}_R(X) \rightarrow A$ with $Y = \{y\}$, where \bar{d} denotes the differential on $\mathbb{T}_R(X \cup Y)$ and d denotes its restriction to $\mathbb{T}_R(X)$. Assume both $H^1(A)$ and $\ker(H^2(f))$ are finitely generated, free R -modules, and that the cohomology classes of the cocycles $dy, c_1, c_2, \dots, c_\ell$ in $\mathbb{T}_R(X)$ form a basis for $\ker(H^2(f))$.*

- (1) *The inclusion $i : \mathbb{T}_R(X) \rightarrow \mathbb{T}_R(X \cup \{y\})$ induces an isomorphism on H^1 .*
- (2) *The set $\{[i(c_1)], \dots, [i(c_\ell)]\}$ is a basis for $\text{im}(H^2(i)) \cap \ker(H^2(\bar{f}))$.*
- (3) *The kernel of $H^2(\bar{f})$ is a finitely generated, free R -module.*

Proof Consider the spectral sequence from Lemma 8.5 associated with the elementary Hirsch extension $(\mathbb{T}_R(X), d) \hookrightarrow (\mathbb{T}_R(X \cup \{y\}), \bar{d})$. We then have $E_2^{0,1} = R$ with generator y , and $\bar{d}y \in \ker(H^2(f))$. Hence, by assumption, $ndy \neq 0$ for all $n \in R, n \neq 0$. It follows that $E_3^{0,1} = 0$. Therefore, the induced homomorphism $H^1(f) : H^1(\mathbb{T}_R(X)) \rightarrow H^1(\mathbb{T}_R(X \cup \{y\}))$ is an isomorphism, and the proof of claim (1) is complete.

To prove claim (2) in the case $R = \mathbb{F}_p$, note that since \mathbb{F}_p is a field, we can write $H^2(\mathbb{T}_R(X))$ as a direct sum $\text{span}([dy], [c_1], \dots, [c_\ell]) \oplus \bar{B}$, with $H^2(f)$ restricted to \bar{B} a monomorphism. Then $\text{im}(H^2(i)) = E_\infty^{2,0} = \text{span}([c_1], \dots, [c_\ell]) \oplus \bar{B}$, and the result follows since $H^2(\bar{f}) \circ H^2(i) = H^2(f)$.

To prove claim (2) in the case $R = \mathbb{Z}$, note that since $E_2^{0,2} = 0$, the terms $E_\infty^{2,0}$ and $E_\infty^{1,1}$ give an exact sequence,

$$(9-2) \quad 0 \longrightarrow E_\infty^{2,0} = H^2(\mathbb{T}_R(X))/[\bar{d}y] \longrightarrow H^2(\mathbb{T}_R(X \cup \{y\})) \longrightarrow E_\infty^{1,1} \longrightarrow 0,$$

and claim (2) follows at once.

For $R = \mathbb{F}_p$, claim (3) follows since in this case every submodule of a finitely generated, free R -module is a finitely generated, free R -module.

To prove claim (3) in the case $R = \mathbb{Z}$, let $\{[dy], [c_1], \dots, [c_\ell]\}$ be a basis for $\ker(H^2(f))$, and let $[c_2, \dots, c_\ell]$ denote the submodule of $H^2(\mathbb{T}_R(X \cup \{y\}))$ generated by the elements $[i(c_2)], \dots, [i(c_\ell)]$. Then since $E_\infty^{1,1}$ is finitely generated and torsion-free, the sequence (9-2) is split exact and the kernel of the map from $H^2(\mathbb{T}_R(X \cup \{y\}))/[c_2, \dots, c_\ell]$ to $H^2(A)$ is the submodule K of $E_\infty^{1,1}$ consisting of all elements k for which there is an element $\alpha \in \text{im}(H^2(i))/[c_2, \dots, c_\ell]$ with $H^2(\bar{f})(k + \alpha) = 0$. It follows that $\ker(H^2(\bar{f})) \cong [c_2, \dots, c_\ell] \oplus K$. This establishes claim (3) in the case $R = \mathbb{Z}$. \square

9.2 A lifting criterion

The next theorem corresponds to an analogous rational homotopy result from [10, Lemma 12.4].

Theorem 9.4 *Let (A, d_A) and $(A', d_{A'})$ be binomial cup-one R -dgas over $R = \mathbb{Z}$ or \mathbb{F}_p , let $f : A \rightarrow A'$ be a surjective 1-quasi-isomorphism, and let φ be a morphism from a colimit of Hirsch extensions, $(\mathbb{T}_R(X), d)$, to $(A', d_{A'})$. There is then a lift of φ through f ; that is, a morphism $\hat{\varphi} : \mathbb{T}_R(X) \rightarrow A$ such*

that the following diagram commutes:

$$\begin{array}{ccc}
 & & A \\
 & \nearrow \widehat{\varphi} & \downarrow f \\
 \mathbb{T}_R(X) & \xrightarrow{\varphi} & A'
 \end{array}$$

Proof As in Definition 8.10, let $\{(\mathbb{T}_R(X^n), d_n)\}_{n \geq 1}$ be the sequence of binomial \cup_1 -dgas whose colimit is $(\mathbb{T}_R(X), d)$. Set $\varphi_n : \mathbb{T}_R(X^n) \rightarrow A'$ equal to the restriction of φ to $\mathbb{T}_R(X^n)$. It suffices to show that for each $n \geq 1$, there is a lift $\widehat{\varphi}_n$ of φ_n through f .

The argument proceeds by induction. For $n = 1$, we have that $d_1(x) = 0$ for all $x \in X_1$. Thus, by Lemmas 6.2 and 6.3 it suffices to show that for each $x \in X_1$ there is a cocycle a_x in A with $f(a_x) = \varphi(x)$. To do this, given $x \in X_1$, let b_x be an element in A^1 with $f(b_x) = \varphi(x)$. Then $d_A(b_x)$ is a cocycle in $\ker(f)$. By Remark 3.3, we have that $H^2(\ker(f)) = 0$. Hence, there is an element $c_x \in \ker(f)$ with $d_A(c_x) = d_A(b_x)$. Then $a_x = b_x - c_x$ is a cocycle in A with $f(a_x) = \varphi(x)$. This completes the argument for the case $n = 1$.

Now assume there is a lifting $\widehat{\varphi}_n$ of φ_n through f . In order to show that $\widehat{\varphi}_n$ extends to a lifting $\widehat{\varphi}_{n+1}$ through f , it suffices by Lemmas 6.2 and 6.3 to show that for each $x \in X_{n+1}$ there is an element a_x in A with $f(a_x) = \varphi(x)$ and $\widehat{\varphi}_n(dx) = d_A(a_x)$. Given $x \in X_{n+1}$, let b_x be an element in A with $f(b_x) = \varphi(x)$. Then

$$\begin{aligned}
 f(\widehat{\varphi}_n(dx) - d_A(b_x)) &= \varphi(dx) - f(d_A(b_x)) = \varphi(dx) - d_{A'}(f(b_x)) \\
 &= \varphi(dx) - d_{A'}(\varphi(x)) = 0,
 \end{aligned}$$

and so $\widehat{\varphi}_n(dx) - d_A(b_x) \in \ker(f)$. We have that $\widehat{\varphi}_n(dx) - d_A(b_x)$ is a cocycle in $\ker(f)$ and $H^2(\ker(f)) = 0$; therefore, there is an element $c_x \in \ker(f)$ with $d_A(c_x) = \widehat{\varphi}_n(dx) - d_A(b_x)$. Setting $a_x = b_x - c_x$, we have that $f(a_x) = \varphi(x)$ and $\widehat{\varphi}_n(dx) = d_A(a_x)$. □

9.3 1-minimal models

Colimits of Hirsch extensions lead to the notion of 1-minimal model, which is central to the study done in this paper. Let (A, d_A) be a binomial \cup_1 -dga over $R = \mathbb{Z}$ or \mathbb{F}_p such that $H^0(A) = R$ and $H^1(A)$ is a finitely generated, free R -module.

Definition 9.5 A 1-minimal model for A is a free binomial \cup_1 -dga $\mathcal{M} = (\mathbb{T}_R(X), d)$ which arises as the colimit of a sequence of Hirsch extensions, $\mathcal{M}_n = (\mathbb{T}_R(X^n), d_n)$, together with morphisms $\rho_n : \mathcal{M}_n \rightarrow A$ such that, for each $n \geq 1$, the diagram

$$(9-3) \quad \begin{array}{ccc}
 & & \mathcal{M}_{n+1} \\
 & \swarrow \rho_{n+1} & \uparrow i_n \\
 A & \xleftarrow{\rho_n} & \mathcal{M}_n
 \end{array}$$

is a commutative diagram of binomial \cup_1 -dgas and the following conditions are satisfied:

- (1) The maps $H^i(\rho_1) : H^i(\mathcal{M}_1) \rightarrow H^i(A)$ are isomorphisms for $i = 0$ and $i = 1$.
- (2) The submodule $\ker(H^2(\rho_n)) \subset H^2(\mathcal{M}_n)$ is a free R -module with basis given by the cohomology classes of the cocycles $\{d_{n+1}(x) \mid x \in X_{n+1}\} \subset Z^2(\mathbb{T}_R(X^n))$.

Set $\mathcal{M} := \bigcup_n \mathcal{M}_n$. Since all diagrams of type (9-3) commute, there is a morphism of \cup_1 -dgas, $\rho : (\mathcal{M}, d) \rightarrow (A, d_A)$, whose restriction to \mathcal{M}_n coincides with ρ_n for all $n \geq 1$. We will oftentimes refer to $\rho : \mathcal{M} \rightarrow A$, or simply to \mathcal{M} as being a 1-minimal model for A ; when needed, we will refer to the map $\rho : \mathcal{M} \rightarrow A$ as the structural morphism for \mathcal{M} . If the inclusion maps $i_n : \mathcal{M}_n \rightarrow \mathcal{M}$ satisfy properties (1) and (2), then we simply say that \mathcal{M} is a 1-minimal model. If $\mathcal{M} = \mathbb{T}(X)$ is a 1-minimal model with X a finite set, then we say that \mathcal{M} is a finitely generated 1-minimal model.

From the definition of colimit of Hirsch extensions we have that $\mathcal{M}_1 = (\mathbb{T}_R(X_1), d_0)$. Also note that from part (2) of Theorem 8.2 it follows that the map $d|_{X^n}$ is admissible for each $n \geq 1$.

Lemma 9.6 *Let $\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n$ be a 1-minimal model for A . Then, for all $n \geq 1$,*

- (1) $Z^1(\mathcal{M}_n) = H^1(\mathcal{M}_n)$,
- (2) *the inclusion $\mathcal{M}_1 \hookrightarrow \mathcal{M}_n$ induces an isomorphism, $H^1(\mathcal{M}_1) \xrightarrow{\cong} H^1(\mathcal{M}_n)$.*

Proof Part (1) follows since $d : (\mathcal{M}_n)^0 \rightarrow (\mathcal{M}_n)^1$ is the zero map for all n . Part (2) follows from Lemma 9.3, part (1). \square

The next corollary follows at once from the lemma.

Corollary 9.7 *If \mathcal{M} is a 1-minimal model for A , then, for all $n \geq 1$,*

- (1) $H^1(\mathcal{M}_n) \cong M(X_1, R)$, *the free R -module with basis given by the elements in X_1 ,*
- (2) $Z^1(\mathcal{M}_{n+1}) = Z^1(\mathcal{M}_n)$.

9.4 Existence of 1-minimal models

We are now in a position to state and prove the main result of this section.

Theorem 9.8 *Let A be a binomial \cup_1 -dga over $R = \mathbb{Z}$ or \mathbb{F}_p , with $H^0(A) = R$ and $H^1(A)$ a finitely generated, free R -module. There is then a 1-minimal model, \mathcal{M} , and a structural morphism, $\rho : \mathcal{M} \rightarrow A$, which is a 1-quasi-isomorphism.*

Proof Since \mathcal{M} is connected, we can define $\rho^0 : \mathcal{M}^0 \rightarrow A^0$ to be the composition of the inverse of the structure map from R to \mathcal{M}^0 followed by the structure map for A .

Now let u_1, \dots, u_k be cocycles in A^1 whose cohomology classes give a basis for $H^1(A)$. Let $X_1 = \{x_1, \dots, x_k\}$ and set $\mathcal{M}_1 = (\mathbb{T}_R(X_1), d_{\tau_1})$, where $\tau_1 = 0$, that is, $d_{\tau_1}(x_i) = 0$ for all i . In view of Corollary 7.9, we may define a morphism $\rho_1 : \mathcal{M}_1 \rightarrow A$ by setting $\rho_1(x_i) = u_i$ for $1 \leq i \leq k$ such that the induced map on H^0 is an isomorphism. By construction, the map $H^1(\rho_1)$ is also an isomorphism.

Assume by induction that an extension $(\mathcal{M}_n, d_{\tau_n}) = (\mathbb{T}_R(X_1 \cup \cdots \cup X_n), d_{\tau_n})$ has been constructed, along with a map $\rho_n : \mathcal{M}_n \rightarrow A$ inducing isomorphisms on H^0 and H^1 and such that the kernel of ρ_n is a finitely generated, free R -module. Then by repeated applications of Corollary 7.9, Theorem 8.2, and Lemma 9.3, it follows that there is a finite set X_{n+1} , an extension $\mathcal{M}_{n+1} = \mathbb{T}_R(X_1 \cup \cdots \cup X_{n+1})$ with differential $d_{\tau_{n+1}}$, and an extension ρ_{n+1} of ρ_n such that ρ_{n+1} induces isomorphisms on H^0 and H^1 , the kernel of $H^2(\rho_{n+1})$ is a finitely generated, free R -module, and the restriction of $H^2(\rho_{n+1})$ to the image of \mathcal{M}_n in \mathcal{M}_{n+1} is a monomorphism.

If for some n the map $H^2(\rho_n)$ is a monomorphism, then set $\mathcal{M} = \mathcal{M}_n$; if not, then set $\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n$. It then follows that \mathcal{M} is a 1-minimal model for A . Its structural morphism, $\rho : \mathcal{M} \rightarrow A$, is defined to be the direct limit of the morphisms $\rho_n : \mathcal{M}_n \rightarrow A$; that is, $\rho|_{\mathcal{M}_n} = \rho_n$, for all $n \geq 1$. By construction, the map $H^i(\rho) : H^i(\mathcal{M}) \rightarrow H^i(A)$ is an isomorphism for $i = 0$ and 1 and a monomorphism for $i = 2$. Therefore, ρ is a 1-quasi-isomorphism. \square

The theorem has an immediate corollary in the case when A is a cochain algebra of a space.

Corollary 9.9 *Let X be a connected Δ -complex, and assume $H^1(X; R)$ is a finitely generated module over $R = \mathbb{Z}$ or \mathbb{F}_p . There is then a 1-minimal model, $\rho : \mathcal{M} \rightarrow A$, for the cochain algebra $A = C^*(X; R)$.*

9.5 Augmented 1-minimal models

When the binomial \cup_1 -dga A admits an augmentation, the above theorem can be enhanced, accordingly.

Theorem 9.10 *Let A be binomial \cup_1 -dga such that there is an augmentation $\varepsilon_A : A \rightarrow R$ which induces an isomorphism from $H^0(A)$ to R and such that $H^1(A)$ is a finitely generated, free R -module. There is then an augmented 1-minimal model, \mathcal{M} , such that the structural morphism, $\rho : \mathcal{M} \rightarrow A$, is an augmentation-preserving 1-quasi-isomorphism.*

Proof By Theorem 9.8, the binomial \cup_1 -dga A has a 1-minimal model, $\rho : \mathcal{M} \rightarrow A$, which is a 1-quasi-isomorphism. Since the tensor algebra $\mathcal{M} = \mathbb{T}_R(X)$ is connected, it admits a canonical augmentation, $\varepsilon_{\mathcal{M}} : \mathcal{M} \rightarrow R$, which sends $\mathcal{M}^{>0}$ to 0 and identifies \mathcal{M}^0 with R .

Since both ε_A and ρ are dga maps, their composite, $\varepsilon_A \circ \rho : \mathcal{M} \rightarrow R$ is again a dga map. Owing to our hypothesis on ε_A , the map from $R = H^0(\mathcal{M})$ to R induced by the composition is an isomorphism of rings from R to R and so equals the identity of R . It follows that $\varepsilon_A \circ \rho$ is an augmentation for \mathcal{M} . By the uniqueness of augmentation maps for connected dgas, we have that $\varepsilon_A \circ \rho = \varepsilon_{\mathcal{M}}$. \square

Recall from Section 3.4 that a choice of basepoint x_0 for a space X yields an augmentation map, $\varepsilon_0 : C^*(X; R) \rightarrow R$.

Corollary 9.11 *Let (X, x_0) be a connected, pointed Δ -complex, and assume $H^1(X; R)$ is a finitely generated module over $R = \mathbb{Z}$ or \mathbb{F}_p . There is then an augmented 1-minimal model, \mathcal{M} , for the cochain algebra $C^*(X; R)$ and a structural morphism, $\rho : \mathcal{M} \rightarrow A$, which is a 1-quasi-isomorphism preserving augmentations, that is, $\varepsilon_0 \circ \rho = \varepsilon_{\mathcal{M}}$.*

10 Uniqueness and functoriality of 1-minimal models

10.1 Maps between 1-minimal models

Let (A, d) and (A', d') be binomial \cup_1 -dgas over the ring $R = \mathbb{Z}$ or \mathbb{F}_p , with $H^0 = R$ and H^1 a finitely generated, free R -module. By Theorem 9.8, there are 1-minimal models $\rho : \mathcal{M} \rightarrow A$ and $\rho' : \mathcal{M}' \rightarrow A'$, in the sense of Definition 9.5. We then have $\mathcal{M} = \bigcup_{n \geq 1} \mathcal{M}_n$, with inclusion maps $i_n : \mathcal{M}_n \hookrightarrow \mathcal{M}_{n+1}$ which are Hirsch extensions, and morphisms $\rho_n : \mathcal{M}_n \rightarrow A$ such that $\rho_n = \rho|_{\mathcal{M}_n}$ and $\rho_{n+1} \circ i_n = \rho_n$ for all $n \geq 1$, and similarly for A' and \mathcal{M}' .

Let $\varphi : A \rightarrow A'$ be a map of binomial \cup_1 -dgas. A map $f : \mathcal{M} \rightarrow \mathcal{M}'$ between the corresponding 1-minimal models is said to be a *morphism compatible with φ* if f is a map of binomial \cup_1 -dgas such that $\rho' \circ f = \varphi \circ \rho$. That is, we have a sequence of morphisms, $f_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$, such the following diagrams commute, for all $n \geq 1$:

$$(10-1) \quad \begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{f_{n+1}} & \mathcal{M}'_{n+1} \\ \uparrow i_n & & \uparrow i'_n \\ \mathcal{M}_n & \xrightarrow{f_n} & \mathcal{M}'_n \end{array}$$

$$(10-2) \quad \begin{array}{ccc} \mathcal{M}_n & \xrightarrow{f_n} & \mathcal{M}'_n \\ \downarrow \rho_n & & \downarrow \rho'_n \\ A & \xrightarrow{\varphi} & A' \end{array}$$

The commutativity of the diagrams (10-1) means that the morphisms $f_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$ between the n -th stages of the respective colimits of Hirsch extensions are compatible, in the sense delineated in Section 8.4.

A weaker notion is that of a morphism *compatible up to homotopy*; that is, a morphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ preserving dga and binomial structures and such that $\rho' \circ f \simeq \varphi \circ \rho$. In this case, the diagram (10-1) commutes for all $n \geq 1$ and the diagram (10-2) commutes only up to homotopy, in the sense that there are homotopies $\Phi_n : \mathcal{M}_n \rightarrow A' \otimes_R C^*(I; R)$ between $\varphi \circ \rho_n$ and $\rho'_n \circ f_n$ for all $n \geq 1$, with the restriction of Φ_{n+1} to \mathcal{M}_n equal to Φ_n for all $n \geq 1$. Since by Theorem 3.8 homotopic maps induce the same map on cohomology, we still get commuting diagrams in cohomology

$$(10-3) \quad \begin{array}{ccc} H^i(\mathcal{M}_n) & \xrightarrow{H^i(f_n)} & H^i(\mathcal{M}'_n) \\ \downarrow H^i(\rho_n) & & \downarrow H^i(\rho'_n) \\ H^i(A) & \xrightarrow{H^i(\varphi)} & H^i(A') \end{array}$$

for all $i \geq 0$ and all $n \geq 1$.

10.2 Extending dga maps to 1-minimal models

Our next goal is to show that, given a morphism $\varphi : A \rightarrow A'$, there is a morphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ compatible with φ up to homotopy. We start with a lemma.

Lemma 10.1 *Let $\varphi : A \rightarrow A'$ be a morphism of binomial \cup_1 -dgas as above. Let $\rho : \mathcal{M} \rightarrow A$ and $\rho' : \mathcal{M}' \rightarrow A$ be 1-minimal models for (A, d) and (A', d') , respectively, and let n be a positive integer. Assume there is a morphism $f_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$ such that this diagram commutes:*

$$(10-4) \quad \begin{array}{ccc} H^2(\mathcal{M}_n) & \xrightarrow{H^2(f_n)} & H^2(\mathcal{M}'_n) \\ H^2(\rho_n) \downarrow & & \downarrow H^2(\rho'_n) \\ H^2(A) & \xrightarrow{H^2(\varphi)} & H^2(A') \end{array}$$

Then:

- (1) There is a morphism $f_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ such that diagram (10-1) commutes.
- (2) There is a bijection between morphisms f_{n+1} as above and maps of sets from X_{n+1} to $H^1(\mathcal{M}'_n)$.
- (3) If f_n is an isomorphism and $H^2(\varphi)$ is a monomorphism, then f_{n+1} is also an isomorphism.

Proof Set $K_n = \ker H^2(\rho_n)$ and $K'_n = \ker H^2(\rho'_n)$. To prove parts (1) and (2), there are three cases to consider:

- (a) $K_n = 0$,
- (b) $K_n \neq 0$, $K'_n = 0$, and
- (c) $K_n \neq 0$, $K'_n \neq 0$.

(a) If $K_n = 0$, then $\mathcal{M}_n = \mathcal{M}_{n+1}$ and the results in both parts follow, since $f_{n+1} = i'_n \circ f_n$ is the unique map for which the diagram (10-1) commutes.

(b) Next consider the case where $K_n \neq 0$ and $K'_n = 0$. In this case, $\mathcal{M}_{n+1} = \mathbb{T}(X_n \cup X_{n+1})$ with $X_{n+1} \neq \emptyset$, and $\mathcal{M}'_{n+1} = \mathcal{M}'_n$. For $x \in X_{n+1}$, it follows from the commutativity of the diagram (10-4) that the cocycle $f_n(d_{\mathcal{M}_n}(x))$ is cohomologous to 0 in $H^2(\mathcal{M}'_n)$. Thus, for each $x \in X_{n+1}$, we can choose an element, $f_{n+1}(x) \in \mathcal{M}'_n$ with $d_{\mathcal{M}'_n}(f_{n+1}(x)) = f_n(d_{\mathcal{M}_n}(x))$. Then by Theorem 9.2 the map of sets $X_{n+1} \rightarrow \mathcal{M}'_n$ given by $x \mapsto f_{n+1}(x)$ extends uniquely to a morphism $f_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ of binomial \cup_1 -dgas. This completes the proof of part (1) in this case.

Note that if $\hat{f}_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ is a morphism with $\hat{f}_{n+1} \circ i_n = f_n$, then

$$d_{\mathcal{M}'_{n+1}}(\hat{f}_{n+1}(x)) - d_{\mathcal{M}'_{n+1}}(f_{n+1}(x)) = f_n(d_{\mathcal{M}_n}(x)) - f_n(d_{\mathcal{M}_n}(x)) = 0.$$

Hence, the map of sets $X_{n+1} \rightarrow Z^1(\mathcal{M}'_n)$ given by $x \mapsto c_x := f_{n+1}(x) - \hat{f}_{n+1}(x)$ is a bijection, in this case, between morphisms \hat{f}_{n+1} with $\hat{f}_{n+1} \circ i_n = f_n$ and maps of sets from X_{n+1} to $Z^1(\mathcal{M}'_n)$. From part (1) of Lemma 9.6, we have that $Z^1(\mathcal{M}_n) = H^1(\mathcal{M}_n)$. This completes the proof of part (2) in case (b).

(c) Finally, consider the case where $K_n \neq 0$ and $K'_n \neq 0$. Since the diagram (10-4) commutes, the map $H^2(f_n)$ restricts to a homomorphism $k_n : K_n \rightarrow K'_n$ which fits into the commuting diagram

$$(10-5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K_n & \longrightarrow & H^2(\mathcal{M}_n) & \xrightarrow{H^2(\rho_n)} & H^2(A) \\ & & \downarrow k_n & & \downarrow H^2(f_n) & & \downarrow H^2(\varphi) \\ 0 & \longrightarrow & K'_n & \longrightarrow & H^2(\mathcal{M}'_n) & \xrightarrow{H^2(\rho'_n)} & H^2(A') \end{array}$$

Note that by assumption both K_n and K'_n are nonzero, finitely generated, free R -modules. Now let $\mathcal{M}_{n+1} = \mathbb{T}(X^n \cup X_{n+1})$ and $\mathcal{M}'_{n+1} = \mathbb{T}(X'^m \cup X'_{n+1})$. From hypothesis (2) in the definition of a 1-minimal for A , it follows that the composition of $d : \mathbb{T}^1(X_{n+1}) \rightarrow Z^2(\mathcal{M}_n)$ followed by the projection of a cocycle to its cohomology class gives an isomorphism $\mathbb{T}^1(X_{n+1}) \xrightarrow{\cong} K_n$. Similarly, the differential from $\mathbb{T}^1(X'_{n+1})$ to \mathcal{M}'_{n+1} gives an isomorphism $\mathbb{T}^1(X'_{n+1}) \xrightarrow{\cong} K'_n$. Therefore, k_n gives a homomorphism $\mathbb{T}^1(X_{n+1}) \rightarrow \mathbb{T}^1(X'_{n+1})$. By Theorem 7.8, this homomorphism extends uniquely to a map $f_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ of binomial \cup_1 -dgas such that the diagram (10-1) commutes. This completes the proof of part (1) in this case.

Now fix a choice for f_{n+1} and let $\hat{f}_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ be any morphism such that the diagram (10-1) commutes. Let x be any element in X_{n+1} . Since \hat{f}_{n+1} commutes with the differentials and $dx \in \mathcal{M}_n$, we have $f_n \circ d(x) = d' \circ \hat{f}_{n+1}(x)$ and similarly $f_n \circ d(x) = d' \circ \hat{f}_{n+1}(x)$. Hence,

$$(10-6) \quad d' f_{n+1}(x) - d' \hat{f}_{n+1}(x) = f_n(dx) - f_n(dx) = 0,$$

and it follows that $f_{n+1}(x) - \hat{f}_{n+1}(x)$ is a cocycle in $(\mathcal{M}'_n)^1$. Therefore, for each $x \in X_{n+1}$ there is a cocycle $c(x) \in (\mathcal{M}'_n)^1$ such that

$$(10-7) \quad \hat{f}_{n+1}(x) = f_{n+1}(x) + c(x).$$

By Lemma 9.6, part (1) we have that $Z^1(\mathcal{M}'_n) = H^1(\mathcal{M}'_n)$. This shows that a choice for f_{n+1} gives a map that sends isomorphisms \hat{f}_{n+1} such that the diagram (10-1) commutes to maps from X_{n+1} to $H^1(\mathcal{M}'_n)$. This completes the proof of part (2) in the last case.

We now turn to part (3). Suppose that f_n is an isomorphism and $H^2(\varphi)$ is a monomorphism. Then $H^2(f_n)$ is also an isomorphism, and so chasing diagram (10-5) shows that $H^2(f_n)$ restricts to an isomorphism $k_n : K_n \xrightarrow{\cong} K'_n$. Since the differentials $d : \mathbb{T}^1(X_{n+1}) \rightarrow Z^2(\mathcal{M}_n)$ and $d' : \mathbb{T}^1(X'_{n+1}) \rightarrow Z^2(\mathcal{M}'_n)$ are monomorphisms, it follows that k_n gives an isomorphism $\mathbb{T}^1(X_{n+1}) \xrightarrow{\cong} \mathbb{T}^1(X'_{n+1})$. By Theorem 7.8, this isomorphism extends uniquely to a morphism $f_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ of binomial \cup_1 -dgas such that the diagram (10-1) commutes.

We claim that if \hat{f}_{n+1} is any morphism from \mathcal{M}_{n+1} to \mathcal{M}'_{n+1} such that the diagram (10-1) commutes, then \hat{f}_{n+1} is in fact, an isomorphism. To prove the claim, first note that since f_{n+1} restricts to an isomorphism from $\mathbb{T}^1(X_{n+1})$ to $\mathbb{T}^1(X'_{n+1})$, it follows that $f_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ is an isomorphism. By (10-7) it follows that f_{n+1} and \hat{f}_{n+1} induce the same map of R -modules from $\mathcal{M}_{n+1}/\mathcal{M}_n$ to $\mathcal{M}'_{n+1}/\mathcal{M}'_n$.

Consider the following commutative diagram of exact sequences of R -modules:

$$(10-8) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_n & \xrightarrow{i_n} & \mathcal{M}_{n+1} & \xrightarrow{q_n} & \mathcal{M}_{n+1}/\mathcal{M}_n & \longrightarrow & 0 \\ & & \downarrow f_n & & \downarrow \hat{f}_{n+1} & & \downarrow [f_{n+1}] = [\hat{f}_{n+1}] & & \\ 0 & \longrightarrow & \mathcal{M}'_n & \xrightarrow{i'_n} & \mathcal{M}'_{n+1} & \xrightarrow{q'_n} & \mathcal{M}'_{n+1}/\mathcal{M}'_n & \longrightarrow & 0 \end{array}$$

By assumption, f_n is an isomorphism; moreover, $[\hat{f}_{n+1}]$ is an isomorphism, since $[f_{n+1}]$ is an isomorphism. The claim now follows from the five lemma. \square

10.3 Lifting homotopies to 1-minimal models

The next step is to show that homotopies between maps of binomial \cup_1 -dgas lift to the respective 1-minimal models.

Lemma 10.2 (homotopy lifting lemma) *Let (A, d_A) and $(A', d_{A'})$ be binomial cup-one dgas over $R = \mathbb{Z}$ or \mathbb{F}_p with $H^0 = R$ and H^1 finitely generated, free R -modules. Let $\rho : \mathcal{M}(A) \rightarrow A$ and $\rho' : \mathcal{M}(A') \rightarrow A'$ be 1-minimal models, and let $\varphi : A \rightarrow A'$ be a morphism. Suppose for a given $n \geq 1$ there is a morphism $f_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$ and a homotopy $\Phi_n : \mathcal{M}_n \rightarrow A' \otimes_R C^*(I; R)$ between $\varphi \circ \rho_n$ and $\rho'_n \circ f_n$.*

- (1) *There is a unique morphism $f_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ such that $f_{n+1} \circ i_n = i'_n \circ f_n$ and such that there is a homotopy $\Phi_{n+1} : \mathcal{M}_{n+1} \rightarrow A' \otimes_R C^*(I; R)$ between $\varphi \circ \rho_{n+1}$ and $\rho'_{n+1} \circ f_{n+1}$ with $\Phi_{n+1}|_{\mathcal{M}_n} = \Phi_n$.*
- (2) *If in addition f_n is an isomorphism and $H^2(\varphi)$ is a monomorphism, then f_{n+1} is an isomorphism.*

Proof As before, let $\mathcal{M}_{n+1} = \mathbb{T}(X^n \cup X_{n+1})$ and let $\mathcal{M}'_{n+1} = \mathbb{T}(X'^n \cup X'_{n+1})$, with corresponding maps ρ and ρ' as pictured in

$$(10-9) \quad \begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{f_{n+1}} & \mathcal{M}'_{n+1} \\ \uparrow i_n & & \uparrow i'_n \\ \mathcal{M}_n & \xrightarrow{f_n} & \mathcal{M}'_n \\ \downarrow \rho_n & & \downarrow \rho'_n \\ A & \xrightarrow{\varphi} & A' \end{array}$$

(Curved arrows labeled ρ_{n+1} and ρ'_{n+1} connect \mathcal{M}_{n+1} to A and \mathcal{M}'_{n+1} to A' respectively.)

Let d_{n+1} denote the differential on \mathcal{M}_{n+1} , and note that for $x \in X_{n+1}$, we have $d_{n+1}(x) \in \mathcal{M}_n$. By Theorem 7.8, it suffices to define for each $x \in X_{n+1}$ an element $\Phi_{n+1}(x) \in A' \otimes_R C^*(I; R)$ such that $\Phi_{n+1} \circ d_{n+1}(x) = d_{A' \otimes_R C^*(I; R)} \circ \Phi_{n+1}(x)$. Let $x \in X_{n+1}$. Then $d_{n+1}(x)$ is a cocycle in \mathcal{M}_n , and hence, $\Phi_n(d_{n+1}(x))$ is a cocycle in $A' \otimes_R C^*(I; R)$. We can assume this cocycle has the form

$$(10-10) \quad \Phi_n(d_{n+1}(x)) = \varphi(\rho_n(d_{n+1}x))t_0 + \rho'_n(f_n(d_{n+1}x))t_1 + c_1(x)u,$$

with $c_1(x) \in (A')^1$.

The condition that $\Phi_n(d_{n+1}(x))$ is a cocycle then leads to an equation for $d_{A'}(c_1(x))$, as follows. Recall that if u_0, u_1, c are homogeneous elements in A' with $|u_0| = |u_1| = |c| + 1$, then $u = u_0t_0 + u_1t_1 + cu$ is a homogeneous element in $A' \otimes_R C^*(I; R)$ with

$$(10-11) \quad d_{A' \otimes C^*(I; R)}(u_0t_0 + u_1t_1 + cu) = d_{A'}(u_0)t_0 + d_{A'}(u_1)t_1 + ((-1)^{|u_0|+1}u_0 + (-1)^{|u_1|}u_1 + d_{A'}(c))u.$$

In particular, if $u_0t_0 + u_1t_1 + cu$ is a cocycle in $A' \otimes_R C^*(I; R)$, then u_0 and u_1 are cocycles in A' and $d_{A'}(c) = (-1)^{|u_0|}u_0 + (-1)^{|u_1|+1}u_1$. Since $\Phi_n(d_{n+1}x)$ is a cocycle in $A' \otimes_R C^*(I; R)$, we have that

$$(10-12) \quad d_{A'}(c_1(x)) = \varphi(\rho_n(d_{n+1}x)) - \rho'_n(f_n(d_{n+1}x)).$$

Now by Lemma 10.1 there are morphisms $f_{n+1} : \mathcal{M}_{n+1} \rightarrow \mathcal{M}'_{n+1}$ such that the diagram (10-1) commutes. Choose such a map f_{n+1} . The map f_{n+1} then determines a map from X_{n+1} to $H^1(\mathcal{M}'_n)$, as follows. For each $x \in X_{n+1}$, we have

$$(10-13) \quad d_{A'}(\varphi(\rho_{n+1}(x))) = \varphi(\rho_n(d_{n+1}x)) \quad \text{and} \quad d_{A'}(\rho'_{n+1}(f_{n+1}(x))) = \rho'_n(f_n(d_{n+1}x)),$$

and it follows from (10-12) and (10-13) that the element $z(x) := c_1(x) - \varphi(\rho_{n+1}(x)) + \rho'_{n+1}(f_{n+1}(x))$ is a cocycle in A' . Thus, we have a map of sets $X_{n+1} \rightarrow H^1(A')$ given by $x \mapsto [z(x)]$, where $[w]$ is the cohomology class of a cocycle w . By Definition 9.5 and Lemma 9.6, this map corresponds uniquely to a map of sets from X_{n+1} to $H^1(\mathcal{M}'_n)$.

By Lemma 10.1 we can assume that f_{n+1} has been chosen so that for each $x \in X_{n+1}$, we have that $[z(x)] = 0$. It then follows that for each x there is an element $c_0(x) \in (A')^0$ with

$$(10-14) \quad d_{A'}(c_0(x)) = z(x) = c_1(x) - \varphi(\rho_{n+1}(x)) + \rho'_{n+1}(f_{n+1}(x)).$$

Now set

$$(10-15) \quad \Phi_{n+1}(x) := \varphi(\rho_{n+1}(x))t_0 + \rho'_{n+1}(f_{n+1}(x))t_1 + c_0(x)u.$$

The final step is to show that with this choice of Φ_{n+1} , it follows that

$$\Phi_n(d_{n+1}(x)) = d_{A' \otimes C^*(I)}(\Phi_{n+1}(x))$$

for all $x \in X_{n+1}$. Using (10-15), (10-14), and (10-10), we have that

$$(10-16) \quad \begin{aligned} d_{A' \otimes C^*(I)}(\Phi_{n+1}(x)) &= d_{A' \otimes C^*(I)}(\varphi(\rho_{n+1}(x))t_0 + \rho'_{n+1}(f_{n+1}(x))t_1 + c_0(x)u) \\ &= \varphi(\rho_n(d_{n+1}x))t_0 + \rho'_n(f_n(d_{n+1}x))t_1 \\ &\quad + [\varphi(\rho_{n+1}(x)) - \rho'_{n+1}(f_{n+1}(x)) + d_{A'}(c_0(x))]u \\ &= \varphi(\rho_n(d_{n+1}x))t_0 + \rho'_n(f_n(d_{n+1}x))t_1 + c_1(x)u \\ &= \Phi_n(d_{n+1}x), \end{aligned}$$

and the proof is complete. □

10.4 Homotopy functoriality of 1-minimal models

We are now in a position to show that 1-minimal models are unique up to homotopy.

Theorem 10.3 *Let (A, d) and (A', d') be binomial cup-one dgas over $R = \mathbb{Z}$ or \mathbb{F}_p such that $H^0(A)$ and $H^0(A')$ are isomorphic to R and $H^1(A)$ and $H^1(A')$ are finitely generated, free R -modules. Let $\rho : \mathcal{M}(A) \rightarrow A$ and $\rho' : \mathcal{M}(A') \rightarrow A'$ be 1-minimal models, and let $\varphi : A \rightarrow A'$ be a morphism.*

(1) *There is a morphism $\widehat{\varphi} : \mathcal{M}(A) \rightarrow \mathcal{M}(A')$ compatible with the respective colimit structures (that is, $\widehat{\varphi}_{n+1} \circ i_n = i'_n \circ \widehat{\varphi}_n$ for all n), and there is a homotopy $\Phi : \mathcal{M}(A) \rightarrow A' \otimes_R C^*(I; R)$ between $\varphi \circ \rho$ and $\rho' \circ \widehat{\varphi}$ (also preserving colimit structures), so that the diagram below commutes up to homotopy:*

$$(10-17) \quad \begin{array}{ccc} \mathcal{M}(A) & \overset{\widehat{\varphi}}{\dashrightarrow} & \mathcal{M}(A') \\ \downarrow \rho & & \downarrow \rho' \\ A & \xrightarrow{\varphi} & A' \end{array}$$

(2) *If φ is a 1-quasi-isomorphism, then $\widehat{\varphi}$ is an isomorphism.*

Proof To prove part (1), first set $\mathcal{M}_n = \mathcal{M}_n(A)$ and $\mathcal{M}'_n = \mathcal{M}_n(A')$. We need to construct isomorphisms $\widehat{\varphi}_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$ and homotopies $\Phi_n : \mathcal{M}_n \rightarrow A \otimes_R C^*(I; R)$ between ρ_n and $\rho'_n \circ \widehat{\varphi}_n$ such that $\widehat{\varphi}_{n+1} \circ i_n = i'_n \circ \widehat{\varphi}_n$ and $\Phi_{n+1}|_{\mathcal{M}_n} = \Phi_n$. The proof is by induction on n .

The base case is to show that there is an isomorphism $\widehat{\varphi}_1 : \mathcal{M}_1 \rightarrow \mathcal{M}'_1$ and a homotopy $\Phi_1 : \mathcal{M}_1 \rightarrow A \otimes_R C^*(I; R)$ between ρ_1 and $\rho'_1 \circ \widehat{\varphi}_1$. Since $H^1(\rho_1) = H^1(\rho'_1 \circ \widehat{\varphi}_1)$, the claim follows from Lemma 7.12. The induction step now follows from the homotopy lifting lemma (Lemma 10.2).

To prove part (2), assume that φ is a 1-quasi-isomorphism; that is, the map $H^1(\varphi) : H^1(A) \rightarrow H^1(A')$ is an isomorphism and $H^2(\varphi) : H^2(A) \rightarrow H^2(A')$ is a monomorphism. Now, the map $H^1(\varphi)$ determines a map $\widehat{\varphi}_1 : \mathcal{M}_1 \rightarrow \mathcal{M}'_1$, which must also be an isomorphism. Since $H^2(\varphi)$ is a monomorphism, Lemma 10.1, part (3) insures that $\widehat{\varphi}_1$ lifts to compatible isomorphisms, $\widehat{\varphi}_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$, for all $n \geq 1$. It follows that the family of maps $\{\widehat{\varphi}_n\}_{n \geq 1}$ defines the desired isomorphism $\widehat{\varphi} : \mathcal{M}(A) \rightarrow \mathcal{M}(A')$. □

Taking $A = A'$ in the above theorem, we obtain the following corollary.

Corollary 10.4 *Let A be a binomial cup-one R -dga as above, and let (\mathcal{M}, ρ) and (\mathcal{M}', ρ') be any two 1-minimal models for A . Then there is an isomorphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\rho' \circ f$ is homotopic to ρ .*

10.5 1-minimal models and homotopies

The next lemma corresponds to an analogous result in [16, Corollary 11.4]; see also [10, Proposition 12.7].

Lemma 10.5 *Homotopy is an equivalence relation on the set of morphisms from a colimit of Hirsch extensions, \mathcal{M} , to a binomial cup-one dga, A .*

Proof Clearly, \simeq is reflexive. To show symmetry, let $\Phi : \mathcal{M} \rightarrow A \otimes_R C^*(I; R)$ be a homotopy from φ_0 to φ_1 , given on elements $a \in M^i$ by $\Phi(a) = \varphi_0(a)t_0 + \varphi_1(a)t_1 + c(a)u$, for some $c(a) \in A^{i-1}$; then the map $\bar{\Phi}$ given by $\bar{\Phi}(a) = \varphi_1(a)t_0 + \varphi_0(a)t_1 + c(a)u$ is a homotopy from φ_1 to φ_0 .

It remains to show \simeq is transitive. With Φ as above, let I' be another copy of the interval, let t'_0, t'_1, u' be the corresponding generators of $C^*(I'; R)$, and let $\Phi' : \mathcal{M} \rightarrow A \otimes_R C^*(I'; R)$ be a homotopy from φ_1 to φ_2 . Finally, let $C^*(I; R) \vee C^*(I'; R)$ be the fiber product corresponding to the augmentations $\varepsilon : C^*(I; R) \rightarrow R$ and $\varepsilon' : C^*(I'; R) \rightarrow R$ given by $\varepsilon(t_1) = \varepsilon'(t'_0) = 1$ and $\varepsilon(t_0) = \varepsilon'(t'_1) = 0$. With this setup, we define a map

$$(10-18) \quad \Psi : \mathcal{M} \longrightarrow A \otimes_R (C^*(I; R) \vee C^*(I'; R))$$

by setting $\Psi(a) = (\Phi(a), \Phi'(a))$. Now let Δ be a triangle with oriented edges $e_1 = I$, $e_2 = I'$, and $e_3 = I''$. The inclusions of the edges in the triangle induce epimorphisms $q_j : C^*(\Delta; R) \twoheadrightarrow C^*(e_j; R)$, which give a surjection $f : C^*(\Delta; R) \twoheadrightarrow C^*(I; R) \vee C^*(I'; R)$. By Theorem 9.4, the morphism Ψ lifts through f to a morphism $\widehat{\Psi} : \mathcal{M} \rightarrow A \otimes_R C^*(\Delta; R)$. The map $q_3 \circ \widehat{\Psi} : \mathcal{M} \rightarrow A \otimes_R C^*(I''; R)$, then, is the desired homotopy from φ_0 to φ_2 . \square

We will write $[\mathcal{M}, A]$ for the set of homotopy classes of morphisms $\varphi : \mathcal{M} \rightarrow A$. Given a morphism $\xi : A \rightarrow A'$, composition with ξ defines a function, $\xi_* : [\mathcal{M}, A] \rightarrow [\mathcal{M}, A']$. Similar notions hold for augmented dgas and augmentation-preserving morphisms between them. The next lemma corresponds to an analogous result in [16, Theorem 11.5]; see also [10, Proposition 12.9].

Lemma 10.6 *Let $(\mathbb{T}_R(X), d)$ be a colimit of Hirsch extensions and assume A and A' are binomial \cup_1 -dgas with augmentations, ε_A and $\varepsilon_{A'}$, that induce isomorphisms from H^0 to R . Assume further that there is an augmentation-preserving 1-quasi-isomorphism, $\xi : A \rightarrow A'$. Then the induced map $\xi_* : [\mathbb{T}_R(X), A] \rightarrow [\mathbb{T}_R(X), A']$ of equivalence classes of augmentation-preserving homotopies of augmentation-preserving maps is injective.*

Proof Let $f, g : \mathbb{T}_R(X) \rightarrow A$ be augmentation-preserving morphisms and let $H : \mathbb{T}_R(X) \rightarrow A' \otimes_R C^*(I; R)$ be an augmentation-preserving homotopy between $\xi \circ f$ and $\xi \circ g$. We will show that H lifts to an augmentation-preserving homotopy \widehat{H} between f and g .

Write $X = \{x_1, x_2, \dots\}$ with $X^n = \{x_1, \dots, x_n\}$. Set $\mathcal{M}_n = (\mathbb{T}_R(X^n), d_n)$ and let f_n, g_n and H_n denote the restrictions of f, g , and H to \mathcal{M}_n . Then x_1 is a cocycle in \mathcal{M}_1 , and we have

$$(10-19) \quad H(x_1) = \xi \circ f(x_1)t_0 + \xi \circ g(x_1)t_1 + c(x_1)u$$

with

$$(10-20) \quad dc(x_1) = \xi \circ f(x_1) - \xi \circ g(x_1).$$

Since $H^1(\xi) : H^1(A) \rightarrow H^1(A')$ is a monomorphism, it follows that there is an element $\hat{c}(x_1) \in \ker(\varepsilon_A)$ with

$$(10-21) \quad d\hat{c}(x_1) = f(x_1) - g(x_1) = f_1(x_1) - g_1(x_1).$$

Since $f_1(x_1)t_0 + g_1(x_1)t_1 + \hat{c}(x_1)u$ is a cocycle in $A \otimes_R C^1(I; R)$ and $\hat{c}(x_1) \in \ker(\varepsilon_A)$ with ε_A a binomial subalgebra of A^0 , it follows that the map $x_1 \mapsto f_1(x_1)t_0 + g_1(x_1)t_1 + \hat{c}(x_1)u$ extends uniquely to an augmentation-preserving homotopy \widehat{H}_1 between f_1 and g_1 .

The next step is to show that \widehat{H}_1 is a lifting of H_1 . Since $\hat{c}(x_1) \in \ker(\varepsilon_A)$ and ξ is augmentation-preserving, it follows that $\xi \circ \hat{c}(x_1) \in \ker(\varepsilon_{A'})$. From (10-20) and (10-21), we have that the elements $\xi \circ \hat{c}(x_1)$ and $c(x_1)$ have the same coboundary. Since $\varepsilon_{A'}$ induces an isomorphism from $H^0(A')$ to R , it follows that two elements in $\ker(\varepsilon_{A'})$ with the same coboundary are equal to each other; hence $\xi \circ \hat{c}(x_1) = c(x_1)$ and \widehat{H}_1 is a lifting of H_1 .

Now assume by induction that the homotopy H_n lifts to a homotopy \widehat{H}_n between f_n and g_n and show that \widehat{H}_n then extends to a lifting \widehat{H}_{n+1} of H_{n+1} . Note that dx_{n+1} is a cocycle in $\mathbb{T}(X^n)$. Hence $\widehat{H}_n(dx_{n+1}) = f_n(dx_{n+1})t_0 + g_n(dx_{n+1})t_1 + \hat{c}(dx_{n+1})u$ is a cocycle in $A \otimes_R C^*(I; R)$, and it follows that $d\hat{c}(dx_{n+1}) = f_n(dx_{n+1}) - g_n(dx_{n+1})$.

The obstruction to extending \widehat{H}_n to a homotopy \widehat{H}_{n+1} is finding an element $\hat{c}(x_{n+1}) \in A^0$ such that the map $x_{n+1} \mapsto \widehat{H}_{n+1}(x_{n+1}) = f(x_{n+1})t_0 + g(x_{n+1})t_1 + \hat{c}(x_{n+1})u$ commutes with the coboundary map; that is, $d\hat{c}(x_{n+1}) = g(x_{n+1}) - f(x_{n+1}) + \hat{c}(dx_{n+1})$. Since \widehat{H}_n is a lifting of H_n , we have that $H_{n+1}(x_{n+1})$ has the form $\xi \circ f(x_{n+1})t_0 + \xi \circ g(x_{n+1})t_1 + c(x_{n+1})u$, where

$$dc(x_{n+1}) = \xi \circ g(x_{n+1}) - \xi \circ f(x_{n+1}) + \xi \circ \hat{c}(dx_{n+1}).$$

In particular, the cocycle $\xi \circ g(x_{n+1}) - \xi \circ f(x_{n+1}) + \xi \circ \hat{c}(dx_{n+1})$ in A' is cohomologous to zero, and then since $\xi: H^1(A) \rightarrow H^1(A')$ is a monomorphism, it follows that the cocycle $g(x_{n+1}) - f(x_{n+1}) + \hat{c}(dx_{n+1})$ is cohomologous to zero in A . Thus, there is an element $\hat{c}(x_{n+1})$ in $\ker(\varepsilon_A)$ with

$$d\hat{c}(x_{n+1}) = g(x_{n+1}) - f(x_{n+1}) + \hat{c}(dx_{n+1}),$$

and hence, \widehat{H}_n extends to a homotopy \widehat{H}_{n+1} .

The final step is to see that \widehat{H}_{n+1} is a lifting of H_{n+1} . Since $\hat{c}(x_{n+1}) \in \ker(\varepsilon_A)$ and ξ is augmentation-preserving, it follows that $\xi \circ \hat{c}(x_{n+1}) \in \ker(\varepsilon_{A'})$. The elements $\xi \circ \hat{c}(x_{n+1})$ and $c(x_{n+1})$ have the same coboundary and are both elements in $\ker(\varepsilon_{A'})$; hence they are equal to each other given that the augmentation $\varepsilon_{A'}$ induces an isomorphism from $H^0(A')$ to R . This completes the argument that \widehat{H}_{n+1} is a lifting of H_{n+1} . \square

Remark 10.7 If $A = C^*(X; R)$ with X a path-connected Δ -complex, then there is an augmentation $\varepsilon_A: A \rightarrow R$ inducing an isomorphism from $H^0(A)$ to R .

10.6 1-minimal models of augmented binomial dgas

Let $R = \mathbb{Z}$ or \mathbb{F}_p . For binomial cup-one R -dgas (A, d_A) that come equipped with an augmentation, $\varepsilon_A: A \rightarrow R$, that induces an isomorphism from $H^0(A)$ to R , and for which $H^1(A)$ is a finitely generated, free R -module, Theorem 10.3 may be refined. Recall from Theorem 9.10 that any such dga admits an augmented 1-minimal model, $\rho: \mathcal{M} \rightarrow A$.

Theorem 10.8 *Let (A, d) and (A', d') be augmented binomial cup-one dgas as above. Let $\rho : \mathcal{M} \rightarrow A$ and $\rho' : \mathcal{M}' \rightarrow A'$ be augmented 1-minimal models, and let $\varphi : A \rightarrow A'$ be an augmentation-preserving morphism. There is then a unique augmentation-preserving morphism $\widehat{\varphi} : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\varphi \circ \rho$ is augmentation-preserving homotopic to $\rho' \circ \widehat{\varphi}$.*

Proof By Theorem 10.3, there is an isomorphism $\widehat{\varphi} : \mathcal{M} \rightarrow \mathcal{M}'$ and a homotopy $\Phi : \mathcal{M} \rightarrow A \otimes_R C^*(I; R)$ between $\varphi \circ \rho$ and $\rho' \circ \widehat{\varphi}$. Let $\widetilde{\varphi} : \mathcal{M} \rightarrow \mathcal{M}'$ be another such isomorphism. Since $\rho' \circ \widehat{\varphi}$ and $\rho' \circ \widetilde{\varphi}$ are both homotopic to $\varphi \circ \rho$, it follows from Lemma 10.5 that $\rho' \circ \widehat{\varphi}$ and $\rho' \circ \widetilde{\varphi}$ are homotopic to each other. Then from Lemma 10.6, it follows that $\widehat{\varphi}$ and $\widetilde{\varphi}$ are homotopic morphisms from \mathcal{M} to \mathcal{M}' .

Since the proofs of Theorem 10.3 and Lemma 10.5 apply as well to augmentation-preserving homotopies, it follows that $\widehat{\varphi}$ and $\widetilde{\varphi}$ are homotopic by an augmentation-preserving homotopy. It then follows from Lemma 3.10 that $\widehat{\varphi} = \widetilde{\varphi}$. □

The following uniqueness result follows from Theorem 10.8 by taking $A = A'$.

Corollary 10.9 *Let A be an augmented binomial cup-one R -dga as above, and let (\mathcal{M}, ρ) and (\mathcal{M}', ρ') be any two augmented 1-minimal models for A . Then there is a unique augmentation-preserving isomorphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\rho' \circ f$ is augmentation-preserving homotopic to ρ .*

11 Compatibility of integral and rational 1-minimal models

In this section we define the Postnikov 1-tower of a connected space Y with $H^1(Y; R)$ finitely generated and show that the steps \mathcal{M}_n in the 1-minimal model of Y are quasi-isomorphic to the cochains of the corresponding spaces Y_n in the Postnikov system. We then use this result to show that the integral 1-minimal model of Y tensored with \mathbb{Q} is weakly equivalent as a dga to the 1-minimal model of Y defined in rational homotopy theory.

11.1 Postnikov 1-towers and 1-minimal models

A connected space Y with $H^1(Y; R)$ finitely generated determines a tower of spaces with compatible maps from Y to the tower, as pictured here:

$$(11-1) \quad \begin{array}{ccc} & & \cdots \\ & & \downarrow \\ & & Y_3 \\ & \nearrow h_3 & \downarrow \pi_2 \\ Y & & Y_2 \\ & \nearrow h_2 & \downarrow \pi_1 \\ Y & \xrightarrow{h_1} & Y_1 \end{array}$$

The spaces Y_n and maps h_n are defined by induction as follows. Set Y_1 equal to the Eilenberg–MacLane space $K(H^1(Y; R), 1)$ and let h_1 be a map inducing an isomorphism from $H^1(Y_1; R)$ to $H^1(Y; R)$.

Assume Y_n and $h_n : Y \rightarrow Y_n$ have been defined. Let P_n be the kernel of $H^2(h_n)$. Using the argument in the proof of property (3) of Lemma 9.3, we can assume by induction that P_n is a finitely generated, free R -module. Set $Q_n = \text{Hom}_R(P_n, R)$. Then the set of homotopy classes of maps from Y_n to $K(Q_n; 2)$ can be identified with the set of R -linear maps from P_n to $H^2(Y_n; R)$. Choose a map $f_n : Y_n \rightarrow K(Q_n; 2)$ corresponding to the map of P_n to the kernel of $H^2(h_n)$. Set Y_{n+1} equal to the pullback over f_n of the path space fibration on $K(Q_n; 2)$, and let h_{n+1} be a lifting of h_n . We call the resulting tower the *Postnikov 1-tower* of the space Y . It will be shown in [36] that the R -modules Q_n are successive quotients in a suitable descending series of normal subgroups of $\pi_1(Y)$.

Lemma 11.1 *Let Y be a connected space with $H^1(Y; R)$ finitely generated. Let $\{Y_n\}_{n \geq 1}$ be a Postnikov 1-tower for Y , as in diagram (11-1). Then there is a colimit of Hirsch extensions \mathcal{M} and quasi-isomorphisms $\psi_n : \mathcal{M}_n \rightarrow C^*(Y_n; R)$ such that \mathcal{M} with structure maps $\rho_n : \mathcal{M}_n \rightarrow C^*(Y; R)$ given by $\rho_n = h_n^* \circ \psi_n$ is a 1-minimal model for $C^*(Y; R)$.*

Proof Note that $H^2(K(H^1(Y; R), 1); R)$ is a finitely generated free R -module. Then by induction, using the argument in the proof of property (3) of Lemma 9.3, it follows that the kernel of the map $H^2(h_n) : H^2(Y_n; R) \rightarrow H^2(Y; R)$ is also a finitely generated, free R -module. The existence of the colimit of Hirsch extensions \mathcal{M} and quasi-isomorphisms $\psi_n : \mathcal{M}_n \rightarrow C^*(Y_n; R)$ such that \mathcal{M} with structure maps $\rho_n = h_n^* \circ \psi_n$ is a 1-minimal model for $C^*(Y; R)$ now follows from Theorem 8.13. \square

11.2 Polynomial differential forms

We now briefly review a construction in rational homotopy theory due to Sullivan [42]. For each integer $n \geq 0$, set

$$(11-2) \quad (A_{\text{PL}})_n = \frac{\Lambda(t_0, \dots, t_n, y_0, \dots, y_n)}{(\sum t_i - 1, \sum y_j)},$$

where $\Lambda(t_0, \dots, t_n, y_0, \dots, y_n)$ denotes the free commutative algebra over \mathbb{Q} generated by elements t_i of degree zero and elements y_j of degree one, and define a differential d on $(A_{\text{PL}})_n$ by setting $dt_i = y_i$ and $dy_j = 0$. Given a space Y , an element $u \in A_{\text{PL}}^p(Y)$ is a rule that associates to each singular n -simplex σ of Y an element $u(\sigma) \in (A_{\text{PL}})_n^p$ compatible with the face and degeneracy maps $\partial_i : (A_{\text{PL}})_{n+1} \rightarrow (A_{\text{PL}})_n$ and $s_j : (A_{\text{PL}})_n \rightarrow (A_{\text{PL}})_{n+1}$ as given in [10, p. 122]. Then $A_{\text{PL}}(Y)$ is a commutative differential graded algebra over the rationals and the assignment $Y \rightsquigarrow A_{\text{PL}}(Y)$ is functorial.

In addition to the Sullivan algebra $A_{\text{PL}}(Y)$ and the cochain algebra $C^*(Y; \mathbb{Q})$, there is a differential graded algebra over the rationals, $\text{CA}(Y)$, with the following property.

Theorem 11.2 [10, Corollary 10.10] *For topological spaces Y there are natural quasi-isomorphisms*

$$C^*(Y; \mathbb{Q}) \longrightarrow \text{CA}(Y) \longleftarrow A_{\text{PL}}(Y).$$

Consequently, $A_{\text{PL}}(Y)$ is weakly equivalent (as a dga) to $C^*(Y; \mathbb{Q})$.

11.3 Rational 1-minimal models

Let $(\Lambda(X), d)$ be the free commutative algebra over \mathbb{Q} generated by the elements in a set X , equipped with a differential d . We say that a dga $(\Lambda(X) \otimes_{\mathbb{Q}} \Lambda(Y), \bar{d})$ is a *Hirsch extension* of $(\Lambda(X), d)$ if $\bar{d}x = dx$ for all $x \in X$ and $\bar{d}y$ is a cocycle in $\Lambda(X)$ for all $y \in Y$.

Now let Y be a topological space. A sequence $\mathcal{M}_{n,\mathbb{Q}}(Y) = (\Lambda(X_1 \cup \dots \cup X_n), d_n)$ of Hirsch extensions with $n \geq 1$ together with maps

$$(11-3) \quad \begin{array}{ccc} & & \mathcal{M}_{n+1,\mathbb{Q}}(Y) \\ & \swarrow \rho_{n+1} & \uparrow i_n \\ A_{\text{PL}}(Y) & \xleftarrow{\rho_n} & \mathcal{M}_{n,\mathbb{Q}}(Y) \end{array}$$

is a *rational 1-minimal model* for Y if the following conditions are satisfied:

- (1) $d_1(x) = 0$ for all $x \in X_1$.
- (2) The map $\rho_1^1 : H^1(\mathcal{M}_{1,\mathbb{Q}}) \rightarrow H^1(A_{\text{PL}}(Y))$ is an isomorphism.
- (3) Under the assignment $x \mapsto d_{n+1}(x)$, the set X_{n+1} corresponds to a basis for $\ker(\rho_n^2) \subset H^2(\mathcal{M}_{n,\mathbb{Q}})$ given by the cohomology classes of the cocycles $\{d_{n+1}(x) \mid x \in X_{n+1}\} \subset Z^2(\mathcal{M}_{n,\mathbb{Q}})$.

As shown by Sullivan [42] (see also [10; 16; 40; 41]), every connected space Y admits a rational 1-minimal model, unique up to an isomorphism of cdgas.

The following gives the definition of n -step equivalence in rational homotopy theory.

Definition 11.3 Given commutative dgas (A, d_A) and $(A', d_{A'})$ over \mathbb{Q} with 1-minimal models $(\mathcal{M}_{\mathbb{Q}}, \rho)$ and $(\mathcal{M}'_{\mathbb{Q}}, \rho')$; respectively, and an integer $n \geq 1$, we say that A and A' are *n -step equivalent over \mathbb{Q}* if there are isomorphisms $f_n : \mathcal{M}_{n,\mathbb{Q}} \rightarrow \mathcal{M}'_{n,\mathbb{Q}}$ and $e_n : H^2(A) \rightarrow H^2(A')$ such that this diagram commutes:

$$(11-4) \quad \begin{array}{ccc} H^2(\mathcal{M}_{n,\mathbb{Q}}) & \xrightarrow{H^2(f_n)} & H^2(\mathcal{M}'_{n,\mathbb{Q}}) \\ H^2(\rho_n) \downarrow & & \downarrow H^2(\rho'_n) \\ H^2(A) & \xrightarrow{e_n} & H^2(A') \end{array}$$

11.4 Relating the integer and rational 1-minimal models

We are now in a position to state and prove the main result of this section.

Theorem 11.4 *Let Y be a connected topological space with $H^1(Y; \mathbb{Z})$ finitely generated. Then the 1-minimal model for $C^*(Y; \mathbb{Z})$ tensored with \mathbb{Q} is weakly equivalent as a differential graded algebra to the 1-minimal model in rational homotopy theory for $A_{\text{PL}}(Y)$.*

Proof Let $\{Y_n\}_{n \geq 1}$ be a Postnikov 1-tower for Y , and let $\mathcal{M} = \{\mathcal{M}_{n,\mathbb{Z}}(Y), \rho_{n,\mathbb{Z}}\}_{n \geq 1}$ be an integral 1-minimal model for Y , with quasi-isomorphisms $\psi_n : \mathcal{M}_{n,\mathbb{Z}}(Y) \rightarrow C^*(Y_n; \mathbb{Z})$ as in Lemma 11.1. The proof is to show by induction that for the rational 1-minimal model $\{\mathcal{M}_{n,\mathbb{Q}}(Y), \rho_{n,\mathbb{Q}}\}_{n \geq 1}$ for $A_{\text{PL}}(Y)$,

there are 1-quasi-isomorphisms $e_n : \mathcal{M}_{n,\mathbb{Q}}(Y) \rightarrow A_{\text{PL}}(Y_n)$ with $\rho_{n,\mathbb{Q}} = f_n^* \circ e_n$. The result then follows from the natural equivalences between $A_{\text{PL}}(Y_n)$ and $C^*(Y_n; \mathbb{Q})$ and between $A_{\text{PL}}(Y)$ and $C^*(Y; \mathbb{Q})$.

Assume that for the sets X_i ($i \geq 1$), we have $\mathcal{M}_{n,\mathbb{Z}}(Y) = T(X_1 \cup \dots \cup X_n)$. In order to prove the base case, set $\mathcal{M}_{1,\mathbb{Q}} = \Lambda(X_1)$ and define $e_1 : \Lambda(X_1) \rightarrow A_{\text{PL}}(Y_1)$ by setting $e_1(x)$ equal to a cocycle in $A_{\text{PL}}(Y_1)$ whose cohomology class corresponds to $\psi_1(x)$ under the weak equivalence between $C^*(Y_1; \mathbb{Q})$ and $A_{\text{PL}}(Y_1)$. Then e_1 is a 1-quasi-isomorphism and $\rho_{1,\mathbb{Q}} = f_1^* \circ e_1$ induces an isomorphism from $H^1(\mathcal{M}_{1,\mathbb{Q}}(Y))$ to $H^1(A_{\text{PL}}(Y))$.

To prove the inductive step, assume $\mathcal{M}_{n,\mathbb{Q}}(Y)$ and a 1-quasi-isomorphism e_n have been constructed. Consider the diagram below. Note that the diagram of solid arrows commutes and each horizontal arrow represents a 1-quasi-isomorphism:

$$\begin{array}{ccccccc}
 \mathcal{M}_{n+1,\mathbb{Z}}(Y) \otimes \mathbb{Q} & \xrightarrow{\psi_{n+1}} & C^*(Y_{n+1}; \mathbb{Q}) & \longrightarrow & \text{CA}(Y_{n+1}) & \longleftarrow & A_{\text{PL}}(Y_{n+1}) \xleftarrow{e_{n+1}} \mathcal{M}_{n+1,\mathbb{Q}}(Y) \\
 \uparrow & & \uparrow & & \uparrow & & p_n \uparrow & & i_n \uparrow \\
 \mathcal{M}_{n,\mathbb{Z}}(Y) \otimes \mathbb{Q} & \xrightarrow{\psi_n} & C^*(Y_n; \mathbb{Q}) & \longrightarrow & \text{CA}(Y_n) & \longleftarrow & A_{\text{PL}}(Y_n) \xleftarrow{e_n} \mathcal{M}_{n,\mathbb{Q}}(Y) \\
 & \searrow \rho_{n,\mathbb{Z}} & \downarrow & & \downarrow & & \downarrow f_n & & \swarrow \rho_{n,\mathbb{Q}} \\
 & & C^*(Y; \mathbb{Q}) & \longrightarrow & \text{CA}(Y) & \longleftarrow & A_{\text{PL}}(Y)
 \end{array}$$

Define $\mathcal{M}_{n+1,\mathbb{Q}}(Y)$ and e_{n+1} as follows. Set $\mathcal{M}_{n+1,\mathbb{Q}}(Y)$ equal to the Hirsch extension

$$(\mathcal{M}_{n,\mathbb{Q}}(Y) \otimes \Lambda(X_{n+1}), d_{n,\mathbb{Q}}),$$

where for each $x \in X_{n+1}$, its differential $d_{n,\mathbb{Q}}(x)$ is equal to a cocycle in $\mathcal{M}_{n,\mathbb{Q}}(Y)$ whose cohomology class corresponds to the cohomology class of $d_{n,\mathbb{Z}}(x) \in \mathcal{M}_{n,\mathbb{Z}}(Y) \otimes \mathbb{Q}$ under the 1-equivalence between $\mathcal{M}_{n,\mathbb{Z}}(Y) \otimes \mathbb{Q}$ and $\mathcal{M}_{n,\mathbb{Q}}(Y)$.

Given $x \in X_{n+1}$, set $e_{n+1}(x)$ equal to a cochain in $A_{\text{PL}}(Y_{n+1})$ whose coboundary equals $e_n(d_{n,\mathbb{Q}}(x))$. Then $p_n \circ e_n = e_{n+1} \circ i_n$. As in the proof of Theorem 8.13, the map e_{n+1} gives a map of the spectral sequence of the Hirsch extension i_n to the spectral sequence of the fibration p_n inducing an isomorphism of E_2 terms. Thus, e_{n+1} is a quasi-isomorphism. \square

12 Distinguishing homotopy types via 1-minimal models

In this section we use the 1-minimal model of a binomial \cup_1 -dga over $R = \mathbb{Z}$ or \mathbb{F}_p to define n -step equivalence for $n \geq 1$. We show in Theorem 12.4 that if X and X' have isomorphic fundamental groups, then $C^*(X; \mathbb{Z})$ and $C^*(X'; \mathbb{Z})$ are n -step equivalent for all $n \geq 1$, and in Proposition 12.5 we give an example of a family of spaces that can be distinguished using n -step equivalence with $R = \mathbb{Z}$, where the corresponding approach in rational homotopy theory fails to distinguish the spaces.

12.1 n -step equivalence

Let (A, d_A) be a binomial \cup_1 -dga over the ring $R = \mathbb{Z}$ or $R = \mathbb{F}_p$. We will assume throughout this section that $H^0(A) = R$ and $H^1(A)$ is a finitely generated, free R -module. By Theorem 9.8, there is

a 1-minimal model, (\mathcal{M}, d) , which comes equipped with a structure map, $\rho : \mathcal{M} \rightarrow A$, that induces an isomorphism on H^1 and a monomorphism on H^2 . Furthermore, by Theorem 10.3, any morphism $\varphi : A \rightarrow A'$ between two such binomial \cup_1 -dgas lifts to a morphism $\widehat{\varphi} : \mathcal{M} \rightarrow \mathcal{M}'$ which is compatible with the respective colimit structures and with the structure maps (up to homotopy). The next result extracts further information from these data.

Proposition 12.1 *Let A and A' be two binomial \cup_1 -dgas as above, with 1-minimal models $\rho : \mathcal{M} \rightarrow A$ and $\rho' : \mathcal{M}' \rightarrow A'$. Let $\varphi : A \rightarrow A'$ be a morphism, and let $\widehat{\varphi} : \mathcal{M} \rightarrow \mathcal{M}'$ be a lift of φ .*

- (1) *The map $H^2(\varphi)$ induces homomorphisms $\bar{\varphi}_n : \text{coker}(H^2(\rho_n)) \rightarrow \text{coker}(H^2(\rho'_n))$, for all $n \geq 1$.*
- (2) *If φ is a 1-quasi-isomorphism, then all the homomorphisms $\bar{\varphi}_n$ are injective.*

Proof Since the morphism $\widehat{\varphi} : \mathcal{M} \rightarrow \mathcal{M}'$ is compatible with the colimit structures, it restricts to R -linear maps $\widehat{\varphi}_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$. Since $\rho' \circ \widehat{\varphi}$ is homotopic to $\varphi \circ \rho$ via a homotopy which is also compatible with the colimit structures, we have that $H^2(\rho') \circ H^2(\widehat{\varphi}) = H^2(\varphi) \circ H^2(\rho)$ and similarly for the maps at level $n \geq 1$. We thus obtain for each $n \geq 1$ a commuting diagram in the category of R -modules,

$$(12-1) \quad \begin{array}{ccc} H^2(\mathcal{M}_n) & \xrightarrow{H^2(\widehat{\varphi}_n)} & H^2(\mathcal{M}'_n) \\ H^2(\rho_n) \downarrow & & \downarrow H^2(\rho'_n) \\ H^2(A) & \xrightarrow{H^2(\varphi)} & H^2(A') \\ \downarrow & & \downarrow \\ \text{coker}(H^2(\rho_n)) & \xrightarrow{\bar{\varphi}_n} & \text{coker}(H^2(\rho'_n)) \end{array}$$

where, by definition, $\bar{\varphi}_n$ is the R -linear map induced by $H^2(\varphi)$ on cokernels.

Now suppose $\varphi : A \rightarrow A'$ is a 1-quasi-isomorphism. Then the map $H^2(\varphi) : H^2(A) \rightarrow H^2(A')$ is injective; moreover, by Theorem 10.3, the map $H^2(\widehat{\varphi})$ is an isomorphism. A straightforward diagram chase with (12-1) then shows that the map $\bar{\varphi}_n$ is injective. □

Next we define an equivalence relation such that $\text{coker } H^2(\rho_n)$ is an invariant for each $n \geq 1$.

Definition 12.2 Given binomial cup-one dgas A, A' with 1-minimal models (\mathcal{M}, ρ) and (\mathcal{M}', ρ') ; respectively, and an integer $n \geq 1$ we say that A and A' are n -step equivalent if there are isomorphisms $f_n : \mathcal{M}_n \rightarrow \mathcal{M}'_n$ and $e_n : H^2(A) \rightarrow H^2(A')$ such that the diagram below commutes:

$$(12-2) \quad \begin{array}{ccc} H^2(\mathcal{M}_n) & \xrightarrow{H^2(f_n)} & H^2(\mathcal{M}'_n) \\ H^2(\rho_n) \downarrow & & \downarrow H^2(\rho'_n) \\ H^2(A) & \xrightarrow{e_n} & H^2(A') \end{array}$$

If A and A' are q -equivalent for some $q \geq 2$, then A and A' are n -step equivalent for all $n \geq 1$.

For the rest of this paper, we will assume $R = \mathbb{Z}$ and that $H^2(A)$ and $H^2(A')$ are finitely generated. In this case, the cokernels of the maps $H^2(\rho_n)$ and $H^2(\rho'_n)$ are also finitely generated. Given a 1-minimal model $\rho : \mathcal{M} \rightarrow A$ over \mathbb{Z} , we define for each $n \geq 1$ a finite abelian group $\kappa_n(A)$ by

$$(12-3) \quad \kappa_n(A) := \text{Tors}(\text{coker}(H^2(\rho_n) : H^2(\mathcal{M}_n) \rightarrow H^2(A))).$$

That is, $\kappa_n(A)$ is the torsion subgroup of the finitely generated abelian group $\text{coker}(H^2(\rho_n))$.

From Proposition 12.1 it follows that the groups $\text{coker } H^2(\rho_n)$ and hence $\kappa_n(A)$ are independent of the choice of 1-minimal model for A used in constructing them. The considerations above yield the following proposition.

Proposition 12.3 *Let A and A' be binomial \cup_1 -dgas over \mathbb{Z} , with $H^0 = \mathbb{Z}$, H^1 finitely generated and torsion-free, and H^2 finitely generated. Let (\mathcal{M}, ρ) and (\mathcal{M}', ρ') be 1-minimal models over \mathbb{Z} for A and A' , respectively. Then, if A and A' are n -step equivalent, then $\text{coker } H^2(\rho_n)$ is isomorphic to $\text{coker } H^2(\rho'_n)$ and $\kappa_n(A)$ is isomorphic to $\kappa_n(A')$*

Proof The result follows from the definition of n -step equivalence by a diagram chase using the diagram (12-2) expanded, as in diagram (12-1), to include the cokernels. □

12.2 Distinguishing homotopy 2-types

Making use of the above setup, we obtain in this section an invariant of 2-type for spaces based on a cohomological comparison between their integral cochain algebra and the corresponding 1-minimal model.

In this section, all our spaces are assumed to be connected Δ -complexes X with finitely generated cohomology groups $H^1(X; \mathbb{Z})$ and $H^2(X; \mathbb{Z})$ (for instance, presentation 2-complexes for finitely presented groups). Let $C^*(X; \mathbb{Z})$ be the simplicial cochain algebra of such a space, viewed as a \cup_1 -algebra over \mathbb{Z} . We consider the sequence of finite abelian groups $\kappa_n(X) := \kappa_n(C^*(X; \mathbb{Z}))$, for $n \geq 1$. The next result shows that these groups depend only on the homotopy 2-type of X , or, equivalently, on the isomorphism type of its fundamental group.

Theorem 12.4 *Let X and X' be two Δ -complexes as above.*

- (1) *If $\pi_1(X) \cong \pi_1(X')$, then $\kappa_n(X) \cong \kappa'_n(X')$ for all $n \geq 1$.*
- (2) *If $\kappa_n(X) \not\cong \kappa'_n(X')$ for some $n \geq 1$, then the cochain algebras $C^*(X; \mathbb{Z})$ and $C^*(X'; \mathbb{Z})$ are not n -step equivalent.*

Proof Let $\rho : \mathcal{M} \rightarrow C^*(X; \mathbb{Z})$ and $\rho' : \mathcal{M}' \rightarrow C^*(X'; \mathbb{Z})$ be 1-minimal models for the respective cochain algebras. We begin by proving property (1) in the case where X and X' are 2-dimensional. By assumption, $\pi_1(X) \cong \pi_1(X')$, that is, X and X' have the same 2-type. As in the proof of Theorem 3.6, it follows that there is a homotopy equivalence, $f : \bar{X} \rightarrow \bar{X}'$, where $\bar{X} = X \vee \bigvee_{i \in I} S_i^2$ and $\bar{X}' = X' \vee \bigvee_{j \in J} S_j^2$, for some indexing sets I and J . We let $q : \bar{X} \rightarrow X$ and $q' : \bar{X}' \rightarrow X'$ be the maps that collapse the wedges of 2-spheres to the basepoint, and we let $\bar{\rho} : \bar{\mathcal{M}} \rightarrow C^*(\bar{X}; \mathbb{Z})$ and $\bar{\rho}' : \bar{\mathcal{M}}' \rightarrow C^*(\bar{X}'; \mathbb{Z})$ be 1-minimal models for the respective cochain algebras.

By Theorem 10.3, the induced maps on cochain algebras, f^\sharp , q^\sharp , and q'^\sharp , lift to maps between 1-minimal models. For each $n \geq 1$, this leads to the following diagram, which commutes up to homotopy:

$$(12-4) \quad \begin{array}{ccccccc} \mathcal{M}_n & \xrightarrow[\simeq]{\widehat{q}^\sharp} & \overline{\mathcal{M}}_n & \xleftarrow[\simeq]{\widehat{f}^\sharp} & \overline{\mathcal{M}}'_n & \xleftarrow[\simeq]{\widehat{q}'^\sharp} & \mathcal{M}'_n \\ \downarrow \rho & & \downarrow \bar{\rho}_n & & \downarrow \bar{\rho}'_n & & \downarrow \rho'_n \\ C^*(X; \mathbb{Z}) & \xrightarrow{q^\sharp} & C^*(\overline{X}; \mathbb{Z}) & \xleftarrow[\simeq]{f^\sharp} & C^*(\overline{X}'; \mathbb{Z}) & \xleftarrow{q'^\sharp} & C^*(X'; \mathbb{Z}) \end{array}$$

Since all the maps on the bottom row are 1-quasi-isomorphisms, the maps on the top row are isomorphisms, again by Theorem 10.3. Applying the $H^2(-)$ functor to (12-4), we obtain the commuting diagram

$$(12-5) \quad \begin{array}{ccccccc} H^2(\mathcal{M}_n) & \xrightarrow[\simeq]{H^2(\widehat{q}^\sharp)} & H^2(\overline{\mathcal{M}}_n) & \xleftarrow[\simeq]{H^2(\widehat{f}^\sharp)} & H^2(\overline{\mathcal{M}}'_n) & \xleftarrow[\simeq]{H^2(\widehat{q}'^\sharp)} & H^2(\mathcal{M}'_n) \\ \downarrow H^2(\rho_n) & & \downarrow H^2(\bar{\rho}_n) & & \downarrow H^2(\bar{\rho}'_n) & & \downarrow H^2(\rho'_n) \\ H^2(X; \mathbb{Z}) & \xrightarrow{H^2(q)} & H^2(\overline{X}; \mathbb{Z}) & \xleftarrow[\simeq]{H^2(f)} & H^2(\overline{X}'; \mathbb{Z}) & \xleftarrow{H^2(q')} & H^2(X'; \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{coker}(H^2(\rho_n)) & \hookrightarrow & \text{coker}(H^2(\bar{\rho}_n)) & \xleftarrow[\simeq]{} & \text{coker}(H^2(\bar{\rho}'_n)) & \hookrightarrow & \text{coker}(H^2(\rho'_n)) \end{array}$$

From the way the collapse map q is defined, the induced homomorphism $H^2(q)$ may be identified with the natural inclusion $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}) \oplus \mathbb{Z}^I$; thus the induced map on cokernels restricts to an isomorphism on torsion subgroups. Similar considerations apply to the map $H^2(q')$, while of course $H^2(f)$ is an isomorphism. It follows that $\text{Tors}(\text{coker}(H^2(\bar{\rho}_n))) \cong \text{Tors}(\text{coker}(H^2(\bar{\rho}'_n)))$. This completes the proof of property (1) in the case where X and X' are 2-dimensional.

Now let X be an arbitrary Δ -complex satisfying our assumptions. We let $(C^*(X; \mathbb{Z}), \delta^*)$ be its simplicial cochain complex, $i : X^{(2)} \hookrightarrow X$ the inclusion of the 2-skeleton into X , and $\rho^{(2)} : \mathcal{M}^{(2)} \rightarrow C^*(X^{(2)}; \mathbb{Z})$ a 1-minimal model for $X^{(2)}$. The induced homomorphism $H^2(i) : H^2(X; \mathbb{Z}) \rightarrow H^2(X^{(2)}; \mathbb{Z})$ may be identified with the natural inclusion $H^2(X; \mathbb{Z}) \hookrightarrow H^2(X; \mathbb{Z}) \oplus \text{im}(\delta^2)$. Since $\text{im}(\delta^2) \subset C^3(X; \mathbb{Z})$ is a free abelian group, the map induced by $H^2(i)$ on cokernels, $\text{coker}(H^2(\rho_n)) \rightarrow \text{coker}(H^2(\rho_n^{(2)}))$, restricts to an isomorphism on torsion subgroups, and the proof of part (1) is complete.

Part (2) follows at once from Proposition 12.3. □

12.3 Integral versus rational 1-minimal models

In this final section we give a family of links in the 3-sphere that can be distinguished from each other using the integral 1-minimal model, but are not distinguished from each other using the corresponding approach in rational homotopy theory. We begin with a quick review of triple Massey products and the cohomology of link complements.

We refer to [29] for the general definition and properties of Massey products. Here we restrict our attention to triple Massey products of elements in the first cohomology. Given a dga (A, d) and elements

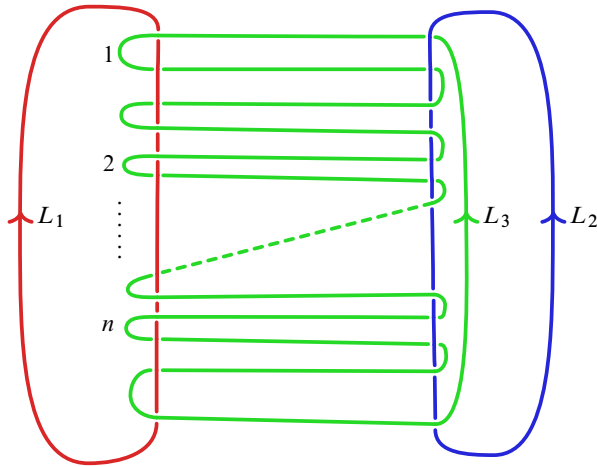


Figure 2: Generalized Borromean rings.

u_1, u_2, u_3 in $H^1(A)$ with $u_1u_2 = u_2u_3 = 0$, let c_i be cocycles with cohomology classes $[c_i] = u_i$, and let $c_{i,j}$ for $i < j$ be elements in A^1 with $dc_{i,j} = c_ic_j$. Then $c_1c_{2,3} + c_{1,2}c_3$ is a 2-cocycle; let $\langle u_1, u_2, u_3 \rangle$ denote the subset of $H^2(A)$ consisting of the set of cohomology classes $[c_1c_{2,3} + c_{1,2}c_3]$ of this type. It follows that the difference between any two elements in $\langle u_1, u_2, u_3 \rangle$ is an element in $u_1 \cup H^1(A) + H^1(A) \cup u_3$.

Now let $L = \bigcup_{i=1}^n L_i$ be an n -component link in the 3-sphere, with complement $X_L = S^3 \setminus L$. This space has the homotopy type of a finite, 2-dimensional CW-complex; moreover, $H^1(X_L; \mathbb{Z}) = \mathbb{Z}^n$, with basis elements corresponding by Alexander duality to generators of $H_1(L_i; \mathbb{Z})$, and with $H^2(X_L; \mathbb{Z}) = \mathbb{Z}^{n-1}$ generated by the Lefschetz duals γ_{ij} to paths from L_i to L_j . A formula for the Massey products of elements in the first cohomology of a CW-complex in terms of the Magnus coefficients of the attaching maps of the 2-cells is given in [32], along with a proof that Massey products in the complement of a link and Milnor's $\bar{\mu}$ -invariants of the link determine each other.

Consider now the following family of links in the 3-sphere. For each $n \geq 1$, let $L(n)$ be the link pictured in Figure 2, where the pattern in the middle repeats n times. Note that $L(1)$ is the well-known Borromean rings, and that the three components of $L(n)$ have pairwise linking numbers equal to 0, for all $n \geq 1$. We denote by $X(n) = X_{L(n)}$ the complement of $L(n)$ in S^3 , and we let $\rho(n) : \mathcal{M}(n) \rightarrow A(n)$ be a 1-minimal model for the cochain algebra $A(n) := C^*(X(n); \mathbb{Z})$.

Proposition 12.5 *For the links $L(n)$ with complement $X(n)$ described above, we have:*

- (1) *The cokernel of $H^2(\rho_2(n))$ is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$, and so $\kappa_2(X(n)) = \mathbb{Z}_n \oplus \mathbb{Z}_n$.*
- (2) *The cochain algebras $C^*(X(m); \mathbb{Z})$ and $C^*(X(n); \mathbb{Z})$ are not 2-step equivalent for $m \neq n$.*
- (3) *The Sullivan algebras $A_{\text{pl}}(X(m))$ and $A_{\text{pl}}(X(n))$ are 2-step equivalent for all positive integers m and n .*

Proof (1) To compute the cokernel of $H^2(\rho_2(n))$, we proceed as follows. First note that $\mathcal{M}_1(n)$ equals $(\mathbb{T}(\{x_1, x_2, x_3\}), d_0)$, with $\rho_1(n)(x_i)$ equal to a cocycle whose cohomology class u_i is Alexander dual to the generator of $H_1(L_i; \mathbb{Z})$ given by the orientation of L_i .

Since all linking numbers for $L(n)$ are zero, we have that all cup products of elements in $H^1(X(n); \mathbb{Z})$ are zero, and it follows that

$$\mathcal{M}_2(n) = (\mathbb{T}(\{x_1, x_2, x_3, x_{1,2}, x_{1,3}, x_{2,3}\}), d_2(n)),$$

where $d_2(n)(x_{i,j}) = x_i \otimes x_j$ and $\rho_2(n)$ is any extension of $\rho_1(n)$. In the spectral sequence of the Hirsch extension $\mathcal{M}_1(n) \hookrightarrow \mathcal{M}_2(n)$, it follows from Theorem 8.9 that the E_2 term is the tensor product of two exterior algebras,

$$(12-6) \quad E_2 = \wedge([x_1], [x_2], [x_3]) \otimes_{\mathbb{Z}} \wedge([x_{1,2}], [x_{2,3}], [x_{1,3}]),$$

where $[x]$ denotes the image of x in E_2 . The differential $d_2 : E_2 \rightarrow E_2$ is given by $d_2[x_i] = 0$ and $d_2[x_{i,j}] = [x_i x_j]$. Then by direct computation of the $E_{\infty}^{p,q}$ terms with $p + q = 2$, it follows that $H^2(\mathcal{M}_2(n)) = \mathbb{Z}^8$, with basis given by the triple Massey products

$$(12-7) \quad \begin{aligned} &\langle [x_1], [x_1], [x_2] \rangle, \quad \langle [x_1], [x_2], [x_2] \rangle, \quad \langle [x_1], [x_1], [x_3] \rangle, \quad \langle [x_1], [x_3], [x_3] \rangle, \\ &\langle [x_2], [x_2], [x_3] \rangle, \quad \langle [x_2], [x_3], [x_3] \rangle, \quad \langle [x_1], [x_2], [x_3] \rangle, \quad \langle [x_1], [x_3], [x_2] \rangle. \end{aligned}$$

The correspondence between Massey products and elements in E_{∞} is indicated as follows. An element in $E_{\infty}^{1,1}$ such as $[x_1] \otimes [x_{1,2}]$ is represented by the element in the set $\langle [x_1], [x_1], [x_2] \rangle$ given by the cohomology class of the cocycle $x_1 \otimes x_{1,2} - \zeta_2(x_1) \otimes x_2$ in $\mathcal{M}_2(n)$ and $\langle [x_1], [x_2], [x_3] \rangle$ is taken to be the cohomology class of the cocycle $x_1 \otimes x_{2,3} + x_{1,2} \otimes x_3$, which corresponds to the element $[x_1] \otimes [x_{2,3}] - [x_3] \otimes [x_{1,2}]$ in $E_{\infty}^{1,1}$.

We now determine the cokernel of $H^2(\rho_2(n))$. By the naturality of Massey products, $H^2(\rho_2(n))$ sends a Massey product $\langle [x_i], [x_j], [x_k] \rangle$ to $\langle u_i, u_j, u_k \rangle$. All cup products of elements in $H^1(X(n); \mathbb{Z})$ are zero, so each triple Massey product contains only one element. Since each 2-component sublink of L is isotopic to the unlink, it follows that each of the first six Massey products from (12-7) maps to zero. From the computations in [32, Example 3, p. 46], we have that $\langle u_1, u_2, u_3 \rangle = -n\gamma_{1,3}$ and $\langle u_1, u_3, u_2 \rangle = n\gamma_{1,2}$. Since $\{\gamma_{1,3}, \gamma_{1,2}\}$ is a basis for $H^2(X(n); \mathbb{Z})$, the argument that $\text{coker } H^2(\rho_2(n)) = \mathbb{Z}_n \oplus \mathbb{Z}_n$ is complete.

Part (2) follows at once from part (1) and Theorem 12.4.

(3) We now show that if the binomial cup-one dga $C^*(X(n); \mathbb{Z})$ is replaced by the cdga $B(n) = A_{\text{PL}}(X(n))$ and the integral 1-minimal model \mathcal{M} is replaced with the Sullivan rational 1-minimal model $\mathcal{M}_{\mathbb{Q}} \simeq \mathcal{M} \otimes \mathbb{Q}$ from [42], then $B(m)$ and $B(n)$ are 2-step equivalent for all positive integers m and n .

Let $\rho_{\mathbb{Q}}(n) : (\mathcal{M}_{\mathbb{Q}}, d) \rightarrow B(n)$ be the rational 1-minimal model for $B(n)$. Then $\mathcal{M}_{2,\mathbb{Q}}$ is the exterior algebra over \mathbb{Q} given by $\wedge(y_1, y_2, y_3, y_{1,2}, y_{1,3}, y_{2,3})$, with differential given by $dy_i = 0$ and $dy_{i,j} = y_i \wedge y_j$. Let $v_i \in H^1(B(n))$ be the elements that correspond to the elements $u_i \in H^1(X(n); \mathbb{Q})$, and let $\omega_{i,j} \in H^2(B(n))$ be the elements that correspond to the elements $\gamma_{i,j} \in H^2(X(n); \mathbb{Q})$ via the

zigzag of quasi-isomorphisms

$$(12-8) \quad C^*(X(n); \mathbb{Q}) \longrightarrow \text{CA}(X(n)) \longleftarrow A_{\text{PL}}(X(n)) = B(n).$$

We can assume that $\rho_{1, \mathbb{Q}}(y_i) = v_i$ for $i \in \{1, 2, 3\}$. Since the maps in (12-8) are dga maps inducing isomorphisms on cohomology, it follows from the computation of Massey products in $C^*(X(n); \mathbb{Q})$ that in $H^2(B(n))$ we have $\langle v_1, v_2, v_3 \rangle = -n\omega_{1,3}$ and $\langle v_1, v_3, v_2 \rangle = n\omega_{1,2}$, while all triple products of the form $\langle v_i, v_j, v_k \rangle$ with $\{i, j, k\}$ a proper subset of $\{1, 2, 3\}$ are zero. Thus, $H^2(\rho_2)\langle [y_1], [y_2], [y_3] \rangle = -n\omega_{1,3}$ and $H^2(\rho_2)\langle [y_1], [y_3], [y_2] \rangle = n\omega_{1,2}$, while $H^2(\rho_2)\langle [y_i], [y_j], [y_k] \rangle = 0$ if $\{i, j, k\} \subsetneq \{1, 2, 3\}$.

Now given positive integers n and m , we define a homomorphism $e_{n,m} : H^2(B(n)) \rightarrow H^2(B(m))$ by $\omega_{i,j} \mapsto \frac{m}{n} \cdot \omega_{i,j}$ for $i = 1$ and $j \in \{2, 3\}$. Then $e_{n,m}$ is an isomorphism and this diagram commutes:

$$(12-9) \quad \begin{array}{ccc} & H^2(\mathcal{M}_2, \mathbb{Q}) & \\ H^2(\rho_{2, \mathbb{Q}}(n)) \swarrow & & \searrow H^2(\rho_{2, \mathbb{Q}}(m)) \\ H^2(B(n)) & \xrightarrow{e_{n,m}} & H^2(B(m)) \end{array}$$

The argument that over the rationals the link complements $X(m)$ and $X(n)$ are 2-step equivalent for all $m, n \geq 1$ is complete. (We do not know whether the two link complements have the same rational homotopy type.) □

13 Torsion-free nilpotent groups and integral 1-minimal models

13.1 Nilmanifolds, torsion-free nilpotent groups, and Hall polynomials

We start by reviewing some well-known facts about nilmanifolds and finitely generated, torsion-free nilpotent groups, for short, \mathcal{T} -groups. We refer to [6; 17; 25; 27; 41] for more details.

Given a \mathcal{T} -group G , there is a refinement of the upper central series, $G = G_1 > \dots > G_n > G_{n+1} = 1$, with each subgroup G_i a normal subgroup of G_{i+1} , and each quotient G_i/G_{i+1} an infinite cyclic group. The integer n is an invariant of the group, called the *length of G* . From the well-known correspondence [3] between central extensions of a group G by an abelian group B and elements in $H^2(G; B)$, it follows that every \mathcal{T} -group G can be viewed as a multiplication defined on the set \mathbb{Z}^n , where n is the length of G , with the multiplication given by the cocycles representing the iterated central extensions.

More precisely, we can choose a *Malcev basis* $\{u_1, \dots, u_n\}$ for G , which satisfies $G_i = \langle G_{i+1}, u_i \rangle$. Consequently, each element $u \in G$ can be written uniquely as $u_1^{a_1} u_2^{a_2} \dots u_n^{a_n}$ for some $a_i \in \mathbb{Z}$. Using this basis, the group G , as a set, can be identified with \mathbb{Z}^n via the map sending $u_1^{a_1} \dots u_n^{a_n}$ to $a = (a_1, \dots, a_n)$. The multiplication in G then takes the form

$$(13-1) \quad u_1^{a_1} \dots u_n^{a_n} \cdot u_1^{b_1} \dots u_n^{b_n} = u_1^{\rho_1(a,b)} \dots u_n^{\rho_n(a,b)},$$

where each map $\rho_i : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ is given by an integrally valued rational polynomial. This procedure identifies the group G with the group (\mathbb{Z}^n, ρ) , with multiplication the map $\rho = (\rho_1, \dots, \rho_n) : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ given by

$$(13-2) \quad \rho_i(x, y) = x_i + y_i + \tau_i(x_1, \dots, x_{i-1}, y_1, \dots, y_{i-1}),$$

where each τ_i is a linear combination of integrally valued rational polynomials of the form

$$\zeta_I(x_1, \dots, x_{i-1})\zeta_J(y_1, \dots, y_{i-1})$$

with both I and J nonzero. The polynomials τ_i are called the *Hall polynomials* of the group G ; see [17].

The group $G = (\mathbb{Z}^n, \rho)$ is a discrete subgroup of the real Lie group $G \otimes \mathbb{R}$. The quotient space, $M = (G \otimes \mathbb{R})/G$, is a compact, connected, orientable manifold, called a *nilmanifold*. By construction, the fundamental group $\pi_1(M)$ is isomorphic to G . Moreover, the nilmanifold M is an Eilenberg–MacLane space of type $K(G, 1)$.

Let R be the subring of \mathbb{Q} generated by the coefficients of ρ . As shown by Lambe and Priddy in [25], one can associate with the group G a Lie algebra over R , by setting $L(G) = (R^n, [\cdot, \cdot])$, with Lie bracket given by $[x, y] = \sigma(x, y) - \sigma(y, x)$, where $\sigma = (\sigma_1, \dots, \sigma_n)$ is the quadratic part of ρ . The real Lie algebra $\mathcal{L}(G) = L(G) \otimes_R \mathbb{R}$, then, coincides with the classical Lie algebra of $G \otimes \mathbb{R}$. Moreover, as shown by Nomizu in [30], the Chevalley–Eilenberg complex $\bigwedge^* \mathcal{L}(G)^\vee$ is a (commutative) minimal model over \mathbb{R} for the nilmanifold M , and thus $H^*(M; \mathbb{R}) \cong H^*(\mathcal{L}(G); \mathbb{R})$.

In [25], Lambe and Priddy strengthened Nomizu’s result, by showing that $H^*(G; S) \cong H^*(L(G); S)$, where S is a subring of \mathbb{Q} obtained from \mathbb{Z} by adjoining fractions of the form $1/n$, for n large enough (depending on G). One can ask whether this isomorphism holds over the (possibly smaller) ring \mathbb{Z} . In [23], Kuzmin and Semenov showed that $H^*(G; \mathbb{Z}) \cong H^*(L(G); \mathbb{Z})$ when G is a free nilpotent group of class 2. It still appears to be an open question when this is the case in general.

13.2 An equivalence between nilmanifolds and finitely generated 1-minimal models

In this section we show that there is a one-to-one correspondence between nilmanifolds (or, equivalently, \mathcal{T} -groups) and finite colimits of Hirsch extensions that leads to an equivalence between the category of isomorphism classes of \mathcal{T} -groups and isomorphism classes of finitely generated 1-minimal models. We start with a correspondence between Hall polynomials and differentials of generators in a colimit of Hirsch extensions over the ring $R = \mathbb{Z}$.

Given a finite, ordered set $X = \{x_1, \dots, x_n\}$, write $X^i = \{x_1, \dots, x_i\}$ and $Y^i = \{y_1, \dots, y_i\}$. Let $T(X^{i-1}, Y^{i-1})$ denote the submodule of $T^1(X \cup Y)$ consisting of linear combinations of polynomials of the form $\zeta_I(x_1, \dots, x_{i-1})\zeta_J(y_1, \dots, y_{i-1})$ with both I and J nonzero. There is then a bijection between maps $\rho : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ satisfying (13-2) where each τ_i is a linear combination of polynomials $\zeta_I(x_1, \dots, x_{i-1})\zeta_J(y_1, \dots, y_{i-1})$ with both I and J nonzero and elements τ_i with $\tau_1 = \tau_2 = 0$ and $\tau_i \in T(X^{i-1}, Y^{i-1})$ for $3 \leq i \leq n$.

From the basis theorem for $T^1(X)$ and the fact that $T^2(X) = T^1(X) \otimes T^1(X)$, it follows that the map $e : T(X^{i-1}, Y^{i-1}) \rightarrow T^1(X^{i-1}) \otimes T^1(X^{i-1})$ given by

$$(13-3) \quad e(\zeta_I(x_1, \dots, x_n)\zeta_J(y_1, \dots, y_n)) = -\zeta_I(x_1, \dots, x_n) \otimes \zeta_J(y_1, \dots, y_n)$$

is a bijection. Given $\tau = (\tau_1, \dots, \tau_n)$ with $\tau_i \in T(X^{i-1}, Y^{i-1})$, define a map $\rho_\tau : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ by

$$(13-4) \quad (\rho_\tau)_i(x, y) = x_i + y_i + \tau_i(x_1, \dots, x_{i-1}, y_1, \dots, y_{i-1})$$

and set $d_\tau : T(X) \rightarrow T(X)$ equal to the unique degree-one map that satisfies the \cup_1 - d formula and the graded Leibniz rule with $d_\tau(x_i) = -e \circ \tau_i$.

Theorem 13.1 Given $\tau = (\tau_1, \dots, \tau_n)$ as above, $G = (\mathbb{Z}^n, \rho_\tau)$ is a \mathcal{T} -group if and only if $T = (T(X), d_\tau)$ is a colimit of Hirsch extensions, in which case, $H^*(G; \mathbb{Z}) \cong H^*(T)$.

Moreover, this bijection between \mathcal{T} -groups and finite colimits of Hirsch extensions induces an equivalence of categories between isomorphism classes of \mathcal{T} -groups and augmentation-preserving isomorphism classes of finitely generated 1-minimal models.

Proof The first statement follows from the discussion above and Lemma 8.11.

The second statement follows from Corollaries 8.14 and 10.9, using the fact that the classifying space BG is connected, and thus $C^*(BG; \mathbb{Z})$ has a canonical augmentation. \square

13.3 On the integral cohomology of nilmanifolds

We conclude this section with two examples (extracted from [25]) which illustrate the computation of the integral 1-minimal model of a \mathcal{T} -group, and its use in the determination of the cohomology ring of the corresponding nilmanifold.

Example 13.2 For a fixed integer $k > 0$, let $G(k)$ be the group of upper triangular matrices

$$(13-5) \quad \begin{pmatrix} 1 & a_1 & a_{1,2}/k \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix},$$

with $a_1, a_2, a_{1,2} \in \mathbb{Z}$. The multiplication in $G(k)$ has the form

$$(a_1, a_2, a_{1,2}) \cdot (a'_1, a'_2, a'_{1,2}) = (a_1 + a'_1, a_2 + a'_2, a_{1,2} + a'_{1,2} + ka_1a_2).$$

Then $G(k)$ is a \mathcal{T} -group and hence from Theorem 13.1 it follows that $(T(\{x_1, x_2, x_{12}\}), d)$ with $d(x_1) = d(x_2) = 0$ and $d(x_{12}) = -kx_1 \otimes x_2$ is a colimit of Hirsch extensions. Moreover,

$$H^*(G(k); \mathbb{Z}) \cong H^*(T(\{x_1, x_2, x_{12}\})) = \wedge(x_1, x_2, x_{12}, x_1x_{12}, x_2x_{12})/(kx_1x_2).$$

Now take $X_1 = \{x_1, x_2\}$ and $X_2 = \{x_{12}\}$ and let i_1 be the inclusion of $T(X_1)$ in $T(X)$. Since $\ker(H^2(i_1))$ is the free \mathbb{Z} -module generated by the cohomology class of the cocycle $-kx_1 \otimes x_2$, it follows that $(T(X), d)$ is a 1-minimal model for $G(k)$. Furthermore, the Lie algebra $L(G)$ is defined over \mathbb{Z} , and, as noted in [25], $H^*(G(k); \mathbb{Z}) \cong H^*(L(G(k)); \mathbb{Z})$.

Example 13.3 Let k and ℓ be positive integers, and let $G(k, \ell)$ be the group of upper triangular matrices

$$(13-6) \quad \begin{pmatrix} 1 & a_1 & a_{1,2}/k & a_{13}/k\ell \\ 0 & 1 & a_2 & a_{2,3}/\ell \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $a_1, a_2, a_3, a_{1,2}, a_{2,3}, a_{13} \in \mathbb{Z}$. The group $G(k, \ell)$ is a \mathcal{T} -group, with multiplication map ρ given by

$$\begin{aligned} \rho_i(a, a') &= a_i + a'_i \quad \text{for } i = 1, 2, 3; \\ \rho_{12}(a, a') &= a_{12} + a'_{12} + ka_1a'_2; \\ \rho_{23}(a, a') &= a_{23} + a'_{23} + \ell a_2a'_3; \\ \rho_{13}(a, a') &= a_{13} + a'_{13} + ka_1a'_{23} + \ell a_{12}a'_3. \end{aligned}$$

Applying Theorem 13.1, we find that

$$\mathbb{T} = (\mathbb{T}(\{x_1, x_2, x_3, x_{12}, x_{23}, x_{13}\}), d)$$

with

$$\begin{aligned} d(x_1) &= d(x_2) = d(x_3) = 0, \\ d(x_{12}) &= -kx_1 \otimes x_2, \\ d(x_{23}) &= -\ell x_2 \otimes x_3, \\ d(x_{13}) &= -kx_1 \otimes x_{23} - \ell x_{12} \otimes x_3 \end{aligned}$$

is a colimit of Hirsch extensions, whose cohomology is isomorphic to that of $G(k, \ell)$.

Take $\mathbf{X}_1 = \{x_1, x_2, x_3\}$, $\mathbf{X}_2 = \{x_{12}, x_{23}\}$, and $\mathbf{X}_3 = \{x_{13}\}$. Let i_1 be the inclusion of $\mathbb{T}(\mathbf{X}_1)$ in $\mathbb{T}(\mathbf{X})$, and let i_2 be the inclusion of $\mathbb{T}(\mathbf{X}_1 \cup \mathbf{X}_2)$ in $\mathbb{T}(\mathbf{X})$. Since $\ker(H^2(i_1))$ is the free \mathbb{Z} -module generated by the cohomology classes of the cocycles $-kx_1 \otimes x_2$ and $-\ell x_2 \otimes x_3$ and $\ker(H^2(i_2))$ is the free \mathbb{Z} -module generated by the cohomology class of the cocycle $-kx_1 \otimes x_{23} - \ell x_{12} \otimes x_3$, it follows that $(\mathbb{T}(\mathbf{X}), d)$ is a 1-minimal model for $G(k, \ell)$. It is shown in [25] that $H^*(G(k, \ell); S) \cong H^*(L(G(k, \ell)); S)$, provided the ring S contains the set $\{1/n : 2 \leq n \leq 10\}$. Using our integral 1-minimal model, we can sharpen their result, and show that the two cohomology groups are in fact isomorphic over \mathbb{Z} .

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
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