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We consider numerical semigroups associated with normal weighted homogeneous surface singularities with rational homology sphere links. We say that a semigroup is representable if it can be realized in this way.

We characterize the representable semigroups by proving that they are exactly those semigroups which can be written as a quotient of a flat semigroup. As an intermediate step, we study the class of flat semigroups and show that they can be represented by a special subclass of isolated complete intersection singularities whose defining equations can be given explicitly.

1 Introduction

1.1

One of the most classical problems in the theory of numerical semigroups is the *Diophantine Frobenius problem*, asking the following: given the numerical semigroup $G(a_1, \dots, a_n)$ generated by relatively prime integers $a_1, \dots, a_n$, find the largest integer — called the *Frobenius number* — that is not contained in the semigroup, i.e., it is not representable as a nonnegative integer combination of $a_1, \dots, a_n$.

Several ideas from different areas of mathematics have been studied to find "closed" formulae and algorithms to calculate the Frobenius number. From the point of view of the formulae, although several results for peculiar cases and general bound estimates exist in the literature, the problem is still open in full generality. In our forthcoming discussions, we will only touch a small part of the "classical approaches", nevertheless the interested reader might consult for more details the excellent monograph of Ramírez Alfonsín [17].

1.2

The present article focuses on a relatively new method developed by T. László and A. Némethi in [7]. This is based on a subtle connection with the theory of complex normal surface singularities. Using geometrical and topological techniques of singularity theory, one can solve the Frobenius problem for numerical semigroups which can be related to certain singularities.

In the theory of surface singularities, numerical semigroups appear naturally in many different contexts. In this article, we consider the case of weighted homogeneous normal surface singularities.

Let $(X, 0)$ be a normal weighted homogeneous surface singularity. Being the germ of an affine variety X with a good $\mathbb{C}^*$-action, its affine coordinate ring is $\mathbb{Z}_{\geq 0}$-graded: $R_X = \bigoplus_{\ell \geq 0} R_{X,\ell}$. Then the set

$$\mathcal{S}_{(X,0)} := \{\ell \in \mathbb{Z}_{\geq 0} \mid R_{X,\ell} \neq 0\}$$

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is a numerical semigroup. In fact, by the work of Pinkham [15] we know that if the link M of the singularity — which is a negative definite Seifert 3-manifold — is a rational homology sphere, then $\mathcal{S}_{(X,0)}$ is a topological invariant associated with the link M, or, with the chosen Seifert structure if that is not unique (e.g., in the case of cyclic quotient singularities). In any case, it can be expressed as $\mathcal{S}_{(X,0)} = \{\ell \in \mathbb{Z} \mid N(\ell) \neq 0\}$ where $N(\ell)$ is a quasilinear function defined by the corresponding Seifert invariants. In this note, we will say that a numerical semigroup is *representable* if it can be realized in this way. More details can be found in Sections 2.2 and 2.3.

Using the above interpretation, [7] developed a "closed" formula for the Frobenius number of representable semigroups. Moreover, in that work the authors proved that a special class of representable semigroups appears naturally in the classical theory of numerical semigroups: the semigroups associated with Seifert integral homology 3-spheres are exactly the "strongly flat semigroups" with at least 3 generators, considered by Raczunas and Chrząstowski-Wachtel [16] as special semigroups realizing a sharp upper bound for the Frobenius problem. The case of two generators will be discussed in this article, clarifying completely the class of strongly flat semigroups.

The aforementioned results imposed the following natural questions to the study of representable semigroups (see [7, Section 8]):

Characterization problem *How can the representable semigroups be characterized*?

Representability problem *How big is the set of representable semigroups inside the set of all numerical semigroups*?

1.3

The aim of this work is to answer the characterization problem. In the sequel, we will provide an overview of our study and state our main result.

First of all, given a numerical semigroup $\mathcal{S}$ and $k \in \mathbb{N}^*$ we can consider the quotient numerical semigroup

$$\mathcal{S}/k := \{\ell \in \mathbb{N} : k\ell \in \mathcal{S}\}.$$

We observe that the quotient of a representable semigroup is also representable. Then we study the class of "flat semigroups" that appeared in the classification theme of [16]. It turns out that they are crucial to the characterization problem. First, we prove that they are representable. Moreover, for a given presentation of a flat semigroup we construct a *canonical representative* M whose specialty induces many of the properties of the semigroup. In particular, one can show that every flat semigroup is symmetric and its Frobenius number realizes a sharp upper bound considered by Brauer [2]. Furthermore, they are interesting from singularity theoretic point of view as well, since these semigroups can be associated with a special family of isolated complete intersection singularities whose equations are given explicitly in Section 4.3.

The main theorem of this manuscript is the following.

Characterization theorem *A numerical semigroup is representable if and only if it is a quotient of a flat semigroup.*

The strategy of the proof is as follows. From the previous observations one deduces that a quotient of a flat semigroup is representable. For the converse statement, we fix a representative of a given numerical semigroup. Then, we prove that by perturbing the Seifert invariants in such a way that the associated semigroup does not change, we can always achieve a representative that allows us to construct its associated semigroup as a quotient of a flat semigroup.

It is worth highlighting that the characterization theorem rephrases the representability problem completely via the language of numerical semigroup theory:

"*How big is the set of quotients of flat semigroups in the set of numerical semigroups*?"

Further speculations on this question and the connection with [18; 20] with respect to the quotients of symmetric semigroups can be found at the end of the article.

1.4

The structure of the article is as follows. Section 2 summarizes the necessary preliminaries regarding the Frobenius problem, the "flat" classification of numerical semigroups following [16], weighted homogeneous normal surface singularities and the combinatorics associated with the dual resolution graph of their canonical equivariant resolution (in rational homology sphere case we call them SSR graphs). Furthermore, the last part defines the representable semigroups and collects some already known results about them.

Section 3 contains some new observations about representable semigroups. In the first part, we introduce the sum of SSR graphs (see Section 3.1) and we discuss the representability of semigroups with two generators. Furthermore, in Section 3.3 we construct monoids as bounds for a representable semigroup. This serves the base idea for the study of flat semigroups in Section 4. In this part, we prove that they are representable (Theorem 4.1.3) by constructing a canonical representative, and we study some of their properties. In particular, in Section 4.3 we give explicit equations for a family of isolated complete intersection singularities whose links are the canonical representatives of a flat semigroup.

Finally, Section 5 explains a perturbation process for the Seifert invariants of a representative, and proves the main theorem (Theorem 5.1.6) of this work. Section 5.2 ends the paper by giving important examples and discussing a new reformulation of the representability problem.

2 Preliminaries

2.1 Numerical semigroups and their "flat" classification

2.1.1 The Frobenius problem The Diophantine Frobenius problem aims to find an explicit formula for the greatest integer not representable as a nonnegative linear form of a given system of $d \geq 2$ relatively prime integers $1 \leq a_1 \leq \cdots \leq a_d$. The integer defined in this way is called the *Frobenius number* of the system. In numerical semigroup language, let $G(a_1,\dots,a_d)$ be the numerical semigroup (i.e., submonoid of $\mathbb{N}$ with finite complement) generated by the integers from the above system. Then the Frobenius number is the largest gap of $G(a_1,\dots,a_d)$, for which we will use the standard notation $f_{G(a_1,\dots,a_d)}$.

The problem is still open in full generality, however several formulae for special systems and general bounds exist in the literature. For example, the very first result is the Sylvester formula that expresses $f_{G(a_1,a_2)} = a_1a_2 - a_1 - a_2$.

For the classical approach and different aspects of the problem the interested reader might consult the monograph of Ramírez Alfonsín [17]. Further details regarding the theory of numerical semigroups can be found, e.g., in [1; 19].

In the sequel, we will present some of the general bounds for the Frobenius number which will be important for our purpose.

Brauer [2] considered the upper bound

$$f_{G(a_1,\dots,a_n)} \le T(a_1,\dots,a_n) := \sum_{i=1}^{n}\left(\frac{d_{i-1}}{d_i}-1\right)a_i, \tag{2.1.1}$$

where $d_0 = 0$ and $d_i = \gcd(a_1,\dots,a_i)$ for all $i \ge 1$. Moreover, in [3] the authors characterized those semigroups which satisfy the equality in (2.1.1), namely

$$f = T \iff a_{i+1}/d_{i+1} \in G(a_1/d_i,\dots,a_i/d_i) \text{ for every } 1 \le i \le n-1.$$

We notice that the value of T, as well as the above criterion depends on the order of the generators, and in general only an appropriate permutation gives $f = T$.

Raczunas and Chrząstowski-Wachtel [16] characterized another subclass of semigroups for which $f = T$ holds and T can be expressed independently of the generator permutation. These are the so-called *flat semigroups*. In particular, they considered another upper bound

$$B(a_1,a_2,\dots,a_n) := (n-1)\,\mathrm{lcm}(a_1,\dots,a_n) - \sum_i a_i, \tag{2.1.2}$$

which satisfies $f_{G(a_1,\dots,a_n)} \le T(a_1,\dots,a_n) \le B(a_1,\dots,a_n)$, and characterized the class of semigroups for which $f = T = B$ holds. They are called the *strongly flat semigroups* and form a subclass of flat semigroups.

In the following, we will describe precisely these classes following the discussion in [16].

2.1.2 Flat classification of semigroups [16] Based on the decomposition of the generators, one considers four "shades" of flatness: *strongly flat*, *flat*, *almost flat and nonflat semigroups*.

Let $A = \{a_1,\dots,a_n\}$ be a system of generators of a numerical semigroup $\mathcal{S}$, $\gcd(a_1,\dots,a_n) = 1$. If we consider the numbers $q_i := \gcd(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n)$ and $\widehat{q}_i := \prod_{j\ne i} q_j$ for all $i \in \{1,\dots,n\}$, then we conclude that $\gcd(q_i,q_j) = 1$ for every $i \ne j$. Hence $\widehat{q}_i \mid a_i$ and we can define $\hat{s}_i := a_i/\widehat{q}_i$ (note also that $\gcd(\hat{s}_i,q_i) = 1$). Then the system of generators can be presented in the form

$$A = \{a_1,\dots,a_n\} = \{\hat{s}_1\widehat{q}_1,\dots,\hat{s}_n\widehat{q}_n\}. \tag{2.1.3}$$

Definition 2.1.4 The set A is

- *strongly flat* (**SF**) if one has $\hat{s}_i = 1$ for all i;
- *flat* (**F**) if there exists an i such that $\hat{s}_i = 1$;
- *almost flat* (**AF**) if there exists an i such that $q_i > 1$; and
- *nonflat* (**NF**) if for all i one has $q_i = 1$.

We say that a numerical semigroup $\mathcal{S}$ is strongly flat, flat, almost flat or nonflat if the corresponding condition is satisfied for the minimal set of generators.

Remark 2.1.5 The full semigroup $\mathcal{S} = \mathbb{N}$ and the semigroups minimally generated by two elements are automatically **SF**. On the other hand, we have **SF** $\subset$ **F** $\subset$ **AF**. Moreover, if one of these three conditions is satisfied for a nonminimal set of generators, then it is automatically satisfied for the minimal too. This property does not hold for **NF**.

As we have already mentioned, the strongly flat semigroups are characterized by $f_{\mathcal{S}} = B(a_1, \dots, a_n)$, where $\{a_1, \dots, a_n\}$ is the minimal set of generators of $\mathcal{S}$. Moreover, using the presentation (2.1.3) of the generators and the notation $a := \mathrm{lcm}(a_1, \dots, a_n)$, in this case the Frobenius number can be rewritten as

$$f_{\mathcal{S}} = a\biggl(n - 1 - \sum_i \frac{1}{q_i}\biggr).$$

If $\mathcal{S}$ is flat then $f_{\mathcal{S}} = T(b_1, \dots, b_n)$, where $b_1, \dots, b_n$ is an appropriate permutation of $a_1, \dots, a_n$. However, in this case the Frobenius number can also be expressed in a direct form as

(2.1.6)
$$f_{\mathcal{S}} = \sum_i (q_i - 1)\, a_i - \prod_i q_i;$$

see [16, Theorem 2.5].

2.2 On weighted homogeneous surface singularities

2.2.1 A weighted homogeneous surface singularity $(X, 0)$ is defined as the germ at the origin of an affine surface X with a good and effective $\mathbb{C}^*$-action. This means that its affine coordinate ring is $\mathbb{Z}_{\geq 0}$-graded: $R_X = \bigoplus_{\ell \geq 0} R_{X,\ell}$. (In fact, every finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra corresponds to an affine variety with good $\mathbb{C}^*$-action.)

Let $(X, 0)$ be a normal weighted homogeneous surface singularity. Then $E_0 := (X \setminus \{0\})/\mathbb{C}^*$ is a smooth compact curve. If we denote by T the closure of the graph of the map $X \setminus \{0\} \to E_0$ in $X \times E_0$, then the first projection $T \to X$ is a modification of $(X, 0)$, while the second projection $T \to E_0$ is a Seifert line bundle with zero section E_0. T has at most a finite number of cyclic quotient singularities at the intersection of E_0 with each singular fiber. After resolving these singularities we get a *canonical* (*equivariant*) *resolution* $\pi : \widetilde{X} \to X$. The exceptional divisor $\pi^{-1}(0)$ is a normal crossing divisor and only the central curve E_0 can have self-intersection number -1.

Let Γ be the dual resolution graph of the canonical resolution π. Then, by [14] Γ is a "star-shaped" graph with a central vertex v_0 and $d \geq 0$ legs connected to it. A leg is a chain of vertices which corresponds to the resolution of a cyclic quotient singularity of T. Note that the canonical resolution is good, however if $(X,0)$ is a cyclic quotient singularity (in particular the $\mathbb{C}^*$-action is not unique), it is not necessarily the minimal good resolution. We denote by $\{E_v\}_{v\in\mathcal{V}}$ the irreducible components of $\pi^{-1}(0)$.

The $\mathbb{C}^*$-action induces an S^1-Seifert action on the link M of the singularity. In particular, M is a negative definite Seifert 3-manifold characterized by its normalized Seifert invariants which will be denoted by $Sf = (-b_0, g; (\alpha_i,\omega_i)_{i=1}^d)$. Each leg is determined by an (α_i,ω_i), where $0 < \omega_i < \alpha_i$ and $\gcd(\alpha_i,\omega_i) = 1$. The i-th leg has ν_i vertices, say $v_{i1},\dots,v_{i\nu_i}$ (v_{i1} is connected to the central vertex v_0) with Euler decorations (self-intersection numbers) $-b_{i1},\dots,-b_{i\nu_i}$, which is given by the Hirzebruch–Jung (negative) continued fraction expansion

$$\alpha_i/\omega_i = [b_{i1},\dots,b_{i\nu_i}] = b_{i1} - 1/(b_{i2} - 1/(\cdots - 1/b_{i\nu_i})\cdots) \quad (b_{ij} \geq 2).$$

All these vertices (except v_0) have genus-decorations zero. The central vertex v_0 corresponds to the central genus g curve E_0 with self-intersection number $-b_0$. It is also useful to define ω_i' satisfying

(2.2.1) $$\omega_i\omega_i' \equiv 1 \pmod{\alpha_i}, \quad 0 < \omega_i' < \alpha_i.$$

One knows that $\alpha_i = \det(\Gamma_i)$, the determinant of the i-th leg Γ_i, $\omega_i = \det(\Gamma_i \setminus v_{i1})$ and $\omega_i' = \det(\Gamma_i \setminus v_{i\nu_i})$.

In the sequel we will assume that the link M is a (*Seifert*) *rational homology sphere*, or equivalently, the curve E_0 has $g = 0$. In this case, we will omit the genus from the notation and we will simply write $Sf = (-b_0; (\alpha_i,\omega_i)_{i=1}^d)$.

Furthermore, for simplicity *we say that a graph* Γ *is an SSR graph*, *if it can be realized as the dual resolution graph of the canonical resolution of a normal weighted homogeneous surface singularity with rational homology sphere link.*

2.2.2 Combinatorics and lattices associated with the resolution graph The smooth complex analytic surface $\widetilde{X}$ is the plumbed 4-manifold associated with Γ, with the boundary $\partial\widetilde{X} = M$. We define the lattice L as $H_2(\widetilde{X},\mathbb{Z})$, endowed with the nondegenerate negative definite intersection form $I := (\cdot,\cdot)$. It is freely generated by the (classes of the) exceptional divisors E_v, $v \in \mathcal{V}$, that is, $L = \bigoplus_{v\in\mathcal{V}} \mathbb{Z}\langle E_v\rangle$. The dual lattice $L' := \mathrm{Hom}(H_2(\widetilde{X},\mathbb{Z}),\mathbb{Z})$ can be identified with $H_2(\widetilde{X},M,\mathbb{Z})$. Moreover, one has $L'/L \cong H_1(M,\mathbb{Z})$, which is a finite group and will be denoted by H; see [9; 11].

Since the intersection form is nondegenerate, L' embeds into $L_{\mathbb{Q}} := L \otimes \mathbb{Q}$, and it can be identified with rational cycles $\{l' \in L_{\mathbb{Q}} : (l', L)_{\mathbb{Q}} \subset \mathbb{Z}\}$, where $(\cdot,\cdot)_{\mathbb{Q}}$ denotes the extension of the intersection form to $L_{\mathbb{Q}}$. Hence, in the sequel we regard L' as $\bigoplus_{v\in\mathcal{V}} \mathbb{Z}\langle E_v^*\rangle$, the lattice generated by the (anti)dual cycles $E_v^* \in L_{\mathbb{Q}}$, $v \in \mathcal{V}$, where $(E_u^*, E_v)_{\mathbb{Q}} = -\delta_{u,v}$ (Kronecker delta) for any $u, v \in \mathcal{V}$.

We can consider the anticanonical cycle $Z_K \in L'$ defined by the adjunction formulae $(Z_K, E_v) = e_v + 2$ for all v.

We say that the singularity $(X,0)$, or its topological type, is *numerically Gorenstein* if $Z_K \in L$. Note that the property $Z_K \in L$ is independent of the resolution, since $Z_K \in L$ if and only if the line bundle $\Omega^2_{X\setminus\{0\}}$ of holomorphic 2-forms on $X\setminus\{0\}$ is topologically trivial. $(X,0)$ is called *Gorenstein* if $\Omega^2_{\widetilde{X}}$ (the sheaf of holomorphic 2-forms) is isomorphic to $\mathcal{O}_{\widetilde{X}}(-Z_K)$ (or, equivalently, if the line bundle $\Omega^2_{X\setminus\{0\}}$ is holomorphically trivial). Note that the adjunction formulae imply the identity

(2.2.2)
$$Z_K - E = \sum_{v\in\mathcal{V}} (\delta_v - 2)E_v^*,$$

where we define $E := \sum_{v\in\mathcal{V}} E_v$ and δ_v is the valency of the vertex v.

2.2.3 Some key numerical invariants The orbifold Euler number of the Seifert 3-manifold M is defined as $e := -b_0 + \sum_i \omega_i/\alpha_i$. Then the negative definiteness of the intersection form is equivalent with $e<0$.

Let $\mathfrak{h} := |H|$ be the order of $H = H_1(M,\mathbb{Z}) = L'/L$, and let $\mathfrak{o}$ be the order of the class $[E_0^*]$ (or the generic S^1 Seifert-orbit) in H. Then, writing $\alpha := \mathrm{lcm}(\alpha_1,\dots,\alpha_d)$, one shows that (see, e.g., [13])

(2.2.3)
$$\mathfrak{h} = \alpha_1\cdots\alpha_d|e| \quad \text{and} \quad \mathfrak{o} = \alpha|e|.$$

In particular, if M is an integral homology sphere (called Seifert homology sphere) then necessarily all α_i's are pairwise relatively prime and by (2.2.3) $\alpha|e| = 1$. This gives the Diophantine equation $\big(b_0 - \sum_i \omega_i/\alpha_i\big)\alpha = 1$, which uniquely determines all ω_i and b_0 by the α_i' s. The corresponding Seifert homology sphere is denoted by $\Sigma(\alpha_1,\dots,\alpha_d)$.

Next, we define the combinatorial number

(2.2.4)
$$\gamma := \frac{1}{|e|}\cdot\Big(d-2-\sum_{i=1}^{d}\frac{1}{\alpha_i}\Big)\in\mathbb{Q},$$

which has a central importance regarding properties of weighted homogeneous surface singularities or Seifert rational homology spheres. It has several interpretations: it is the "exponent" of the weighted homogeneous germ $(X,0)$; $-\gamma$ is also called the "log-discrepancy" of E_0; $\mathfrak{o}\gamma$ is usually named as the Goto–Watanabe a-invariant of the universal abelian cover of $(X,0)$, and $e\gamma$ appears as the orbifold Euler characteristic in [13] (see also [12, 3.3.6]).

Nevertheless, the most important interpretation for our purpose will be the following. In an SSR graph the E_0-coefficients of all E_v^* associated with the end-vertices are computed by $-(E_v^*,E_0^*) = 1/(|e|\alpha_v)$ and the E_0-coefficient of E_0^* is $-(E_0^*,E_0^*) = 1/|e|$ (see [10, (11.1)]). Hence, (2.2.2) gives that the E_0-coefficient of Z_K is exactly $\gamma+1$.

For any $i=1,\dots,d$ let us denote by E_i the base element of the i-th end-vertex $v_{i\nu_i}$ and compute the E_i-coefficient of Z_K. Using the identities $(E_i^*,E_j^*) = (e\alpha_i\alpha_j)^{-1}$ for $i\neq j$ and $(E_i^*,E_i^*) = (e\alpha_i^2)^{-1} - \omega_i'/\alpha_i$ if $i=j$, see [10, (11.1)], by (2.2.2) we deduce that

(2.2.5)
$$-(Z_K,E_i^*) = 1 + (\gamma-\omega_i')/\alpha_i.$$

On the other hand, by [8, Lemma 2.2.1] we know

$$E_{v_{ij}}^* = m_{ij}E_i^* - \sum_{j<r\le\nu_i} m_{ijr}E_{v_{ir}},$$

where m_{ij} and m_{ijr} are positive integers. This yields $-(Z_K, E^*_{v_{ij}}) = M_{ij}(Z_K, -E^*_i) + M'_{ij}$ for some $M_{ij}, M'_{ij} \in \mathbb{Z}$, which gives us the following observation.

Lemma 2.2.6 *Γ is numerically Gorenstein if and only if $\gamma \in \mathbb{Z}$ and $\gamma \equiv \omega'_i \pmod{\alpha_i}$ for all $i = 1, \dots, d$.*

Remark 2.2.7 Note that $\gamma|e| = d - 2 - \sum_i 1/\alpha_i$ is negative if and only if $\pi_1(M)$ is finite; see [4]. This can happen only if $d \le 2$, or if $d = 3$ and $\sum_i 1/\alpha_i > 1$. In these cases $(X, 0)$ is a quotient singularity, hence rational. If $(X, 0)$ is not rational, then $\gamma \ge 0$, that is, the E_0-coefficient of Z_K is ≥ 1. In fact, in the latter cases all the coefficients of Z_K are strict positive; see, e.g., [7, 3.2.5]. Moreover, in the numerically Gorenstein nonrational case — when we already know that $\gamma \ge 0$ — by the congruence from Lemma 2.2.6 we get the stronger $\gamma \ge 1$.

2.3 Representable numerical semigroups

2.3.1 Numerical semigroups associated with weighted homogeneous surface singularities [7] We define the set $\mathcal{S}_{(X,0)} := \{\ell \in \mathbb{Z}_{\ge 0} \mid R_{X,\ell} \ne 0\}$. It is a numerical semigroup according to the grading property and it is called the *numerical semigroup associated with* $(X, 0)$.

By [15] one knows that in general the complex structure of $(X, 0)$ is completely recovered by the Seifert invariants and the configuration of points $\{P_i := E_0 \cap E_{i1}\}_{i=1}^d \subset E_0$, where E_0 is the central curve and E_{i1} is the component corresponding to v_{i1} in Γ. Furthermore, the graded ring of the local algebra of the singularity is interpreted by the so-called *Dolgachev–Pinkham–Demazure* formula as

$$R_X = \bigoplus_{\ell \ge 0} R_{X,\ell} = \bigoplus_{\ell \ge 0} H^0(E_0, \mathcal{O}_{E_0}(D^{(\ell)})), \tag{2.3.1}$$

where $D^{(\ell)} := \ell(-E_0|_{E_0}) - \sum_{i=1}^d \lceil \ell\omega_i/\alpha_i \rceil P_i$, $\lceil r \rceil$ denotes the smallest integer greater or equal to r.

In particular, when M is a rational homology sphere, i.e., $E_0 \simeq \mathbb{P}^1$, equation (2.3.1) implies that $\dim(R_{X,\ell}) = \max\{0, 1 + N(\ell)\}$ is topological, where $N(\ell)$ is the quasilinear function

$$N(\ell) := \deg D^{(\ell)} = b_0\ell - \sum_{i=1}^d \left\lceil \frac{\ell\omega_i}{\alpha_i} \right\rceil. \tag{2.3.2}$$

Since $-\lceil x \rceil \le -x$ one obtains $N(\ell) \le |e|\ell$, hence $N(\ell) < 0$ for $\ell < 0$. This means that the semigroup $\mathcal{S}_{(X,0)}$ can be described purely with the Seifert invariants

$$\mathcal{S}_{(X,0)} = \{\ell \in \mathbb{Z} \mid N(\ell) \ge 0\}. \tag{2.3.3}$$

Hence $\mathcal{S}_{(X,0)}$ is either a topological invariant (of M), or an invariant of the Seifert structure in the case of cyclic quotient singularities. Therefore, we will frequently use the notation $\mathcal{S}_M$, or $\mathcal{S}_\Gamma$ as well.

Definition 2.3.4 We say that a numerical semigroup $\mathcal{S}$ is *representable* if it can be realized as $\mathcal{S}_\Gamma$ for some SSR graph Γ. Accordingly, we will say that the corresponding $(X, 0)$, or its link M, or the graph Γ is a *representative* of the numerical semigroup $\mathcal{S}$.

Finally, we list some important properties of the quasilinear function $N(\ell)$ which will be used later in our discussion.

Proposition 2.3.5 [7, Propositions 3.2.11 and 3.2.13; 12] (a) $-(\alpha-1)|e|-d \le N(\ell)-\lceil \ell/\alpha\rceil\alpha|e| \le -1$. *In particular* $\lim_{\ell\to\infty} N(\ell)=\infty$.

(b) *If* $\ell>\gamma$ *then* $h^1(E_0,\mathcal{O}_{E_0}(D^{(\ell)}))=0$, *i.e.,* $N(\ell)\ge -1$.

(c) $N(\alpha)=\alpha\big(b_0-\sum_i \omega_i/\alpha_i\big)=\alpha|e|=\mathfrak{o}>0$.

(d) $N(\ell+\alpha)=N(\ell)+N(\alpha)=N(\ell)+\mathfrak{o}>N(\ell)$ *for any* $\ell\ge 0$.

(e) $N(\ell)\ge 0$ *for any* $\ell>\alpha+\gamma$.

(f) *If the graph is numerically Gorenstein* (*that is*, $Z_K\in L$), *then*

(2.3.6) $$N(\ell)+N(\gamma-\ell)=-2 \quad \textit{for any } \ell\in\mathbb{Z}.$$

2.3.2 The Frobenius number of representable semigroups Let Γ be as in Section 2.2. If Γ satisfies $b_0\ge d$ then a corresponding weighted homogeneous singularity $(X,0)$ supported on this topological type is minimal rational. In this case $\mathcal{S}_\Gamma=\mathbb{N}$. Otherwise, in nontrivial cases, the Frobenius number of $\mathcal{S}_\Gamma$ is expressed by the following result.

Theorem 2.3.7 [7] *If* $b_0<d$ *then one has*

(2.3.8) $$f_{\mathcal{S}_\Gamma}=\gamma+\frac{1}{|e|}-\check{s},$$

where $\check{s}$ *is the* E_0*-coefficient of the unique minimal element of the Lipman cone*

$$\mathcal{S}'_{[Z_K+E_0^*]}:=\{\ell'\in L' \mid (\ell',E_v)\le 0 \text{ for all } v\in\mathcal{V} \text{ and } [\ell']=[Z_K+E_0^*]\},$$

given by the generalized Laufer's algorithm; *see* [7, 3.1.2].

If Γ is numerically Gorenstein (i.e., $Z_K\in L$) and $\mathfrak{o}=1$ then in (2.3.8) the "algorithmic term" $\check{s}$ vanishes and the corresponding semigroup is symmetric, as clarified in the next proposition.

Proposition 2.3.9 *Let* Γ *be a numerically Gorenstein SSR graph that satisfies* $\mathfrak{o}=1$. *Then* $\mathcal{S}_\Gamma$ *is symmetric. Moreover*, *the Frobenius number of* $\mathcal{S}_\Gamma$ *simplifies to*

(2.3.10) $$f_{\mathcal{S}_\Gamma}=\alpha+\gamma.$$

Proof The assumptions imply $1/|e|=\alpha$ and $\check{s}=0$, so (2.3.8) immediately gives the simplified form of $f_{\mathcal{S}_\Gamma}$; see [7, Corollary 3.2.12 or Example 6.2.4(1)].

The proof of the symmetry is analogous to the case of strongly flat semigroups presented in [7, 4.1.1]. For the sake of completeness, we will clarify the details here as well.

One needs to verify that $\ell\in\mathcal{S}_\Gamma$ if and only if $f_{\mathcal{S}_\Gamma}-\ell\notin\mathcal{S}_\Gamma$ for every $\ell\in\mathbb{Z}$. Using the quasilinear function $N(\ell)$ and the expression (2.3.10) of the Frobenius number, this reads as

(2.3.11) $$N(\ell)\ge 0 \quad \text{if and only if} \quad N(\alpha+\gamma-\ell)<0 \qquad \text{for every } \ell\in\mathbb{Z}.$$

Since Γ is numerically Gorenstein, by Proposition 2.3.5(f) we have $N(\ell)+N(\gamma-\ell)=-2$. On the other hand, part (d) of the same proposition gives $N(\alpha+\gamma-\ell)=N(\gamma-\ell)+\mathfrak{o}=N(\gamma-\ell)+1$, hence $N(\ell)+N(\alpha+\gamma-\ell)=-1$. Since $N(\ell)$ and $N(\alpha+\gamma-\ell)$ are integers, we get (2.3.11). □

2.3.3 Representatives of strongly flat semigroups Assume $d\geq 3$ and let $M=\Sigma(\alpha_1,\dots,\alpha_d)$ be a Seifert integral homology sphere. Thus, $\alpha_1,\dots,\alpha_d\geq 2$ are pairwise relatively prime integers and both b_0 and $(\omega_1,\dots,\omega_d)$ are uniquely determined by the Diophantine equation $\alpha\big(b_0-\sum_{i=1}^d \omega_i/\alpha_i\big)=1$. If we consider the integers $a_i:=\alpha/\alpha_i$ then the greatest common divisor of $a_1,\dots,a_{i-1},a_{i+1},\dots,a_d$ is α_i, hence the system $\{a_i\}_{i=1}^d$ generates a strongly flat semigroup $G(a_1,\dots,a_d)$. In fact, in this case $\mathcal{S}_M=G(a_1,\dots,a_d)$.

Theorem 2.3.12 [7] *The strongly flat semigroups with at least three generators are representable. They can be represented by Seifert integral homology spheres.*

In particular, the theorem implies that strongly flat semigroups with $d\geq 3$ generators can be represented by numerically Gorenstein SSR graphs with $d\geq 3$ legs and $\mathfrak{o}=1$. Consequently, they are symmetric and their Frobenius number is expressed by the formula $f_{G(a_1,\dots,a_n)}=\alpha+\gamma$. Its identification with the bound $B(a_1,\dots,a_d)$ can be seen using (2.1.2), (2.2.3) and by noticing that in this case $\operatorname{lcm}(a_1,\dots,a_d)=\alpha$.

Since the representative of a semigroup is not unique (see Remark 3.1.3), we say that the Seifert integral homology sphere is the *canonical representative* of the corresponding strongly flat semigroup.

Remark 2.3.13 For $d\leq 2$ the only possibility is $M=S^3$. Nevertheless we claim that with a well-chosen Seifert structure on S^3 (see Section 3.2) we can represent the strongly flat semigroups with two generators, a class which was excluded from the result of [7].

3 Further observations on representable semigroups

In this section we make some observations and give some bounds for representable semigroups which will be crucial in the forthcoming sections.

3.1 The sum of SSR graphs

Let Γ_1 and Γ_2 be two SSR graphs with Seifert invariants

$$Sf_1=\big(-b_0,(\alpha_i,\omega_i)_{i=1}^m\big) \quad\text{and}\quad Sf_2=\big(-c_0,(\beta_j,w_j)_{j=1}^n\big).$$

We define the sum $\Gamma:=\Gamma_1+\Gamma_2$ of these two graphs as the SSR graph determined by the Seifert invariants

$$Sf=\big(-b_0-c_0,(\alpha_i,\omega_i)_{i=1}^m,(\beta_j,w_j)_{j=1}^n\big).$$

Note that if e_1,e_2 and e are the orbifold Euler numbers of Γ_1,Γ_2 and Γ respectively, then $e=e_1+e_2<0$. Hence, the sum is well defined.

Now, we study this sum from the perspective of the associated numerical semigroups. Denote the quasilinear functions associated with the graphs Γ_1,Γ_2 and Γ by N_1,N_2 and N. Then the above construction yields $N(\ell) = N_1(\ell) + N_2(\ell)$. This provides an upper and a lower bound for $\mathcal{S}_\Gamma$ in terms of $\mathcal{S}_{\Gamma_1}$ and $\mathcal{S}_{\Gamma_2}$, given by the next lemma. Recall that for two sets $A, B \subset \mathbb{N}$ we denote their sum by $A + B := \{a + b : a \in A,\ b \in B\}$. In particular, if A, B are numerical semigroups then $A + B$ is so.

Lemma 3.1.1 *If Γ_1 and Γ_2 are SSR graphs then*

(3.1.2)
$$\mathcal{S}_{\Gamma_1} \cap \mathcal{S}_{\Gamma_2} \subset \mathcal{S}_{\Gamma_1+\Gamma_2} \subset \mathcal{S}_{\Gamma_1} + \mathcal{S}_{\Gamma_2}.$$

Proof Since the quasilinear function associated with $\Gamma_1 + \Gamma_2$ is $N_1 + N_2$, one shows that

$$\mathcal{S}_{\Gamma_1+\Gamma_2} = \{\ell \in \mathbb{N} : N_1(\ell) + N_2(\ell) \geq 0\} \supset \{\ell \in \mathbb{N} : N_1(\ell) \geq 0 \text{ and } N_2(\ell) \geq 0\} = \mathcal{S}_{\Gamma_1} \cap \mathcal{S}_{\Gamma_2}.$$

Furthermore, we can write

$$\mathcal{S}_{\Gamma_1} + \mathcal{S}_{\Gamma_2} = \langle \mathcal{S}_{\Gamma_1} \cup \mathcal{S}_{\Gamma_2} \rangle = \langle \{\ell \in \mathbb{N} : N_1(\ell) \geq 0 \text{ or } N_2(\ell) \geq 0\} \rangle,$$

which clearly implies the inclusion $\mathcal{S}_{\Gamma_1+\Gamma_2} \subset \mathcal{S}_{\Gamma_1} + \mathcal{S}_{\Gamma_2}$. □

3.1.1 In particular, when $\mathcal{S}_{\Gamma_1} = \mathcal{S}_{\Gamma_2}$, we get identity in (3.1.2) implying $\mathcal{S}_{2\Gamma_1} := \mathcal{S}_{\Gamma_1+\Gamma_1} = \mathcal{S}_{\Gamma_1}$. More generally, for an arbitrary SSR graph Γ and $k \in \mathbb{N}^*$, we have $\mathcal{S}_{k\Gamma} = \mathcal{S}_\Gamma$, where

$$k\Gamma := \underbrace{\Gamma + \Gamma + \cdots + \Gamma}_{k \text{ times}}.$$

Remark 3.1.3 This shows that the representation of a numerical semigroup $\mathcal{S}$ as the semigroup associated with an SSR graph is not unique. Moreover, one can construct an infinite family of representatives, which do not share immediate topological properties. In particular, the associated semigroup we have considered does not characterize completely the topological type of the singularity.

3.2 Example: representatives of $G(p, q)$

In this part we discuss the representability of the numerical semigroups $G(p, q)$.

Let $(C, 0) \subset (\mathbb{C}^2, 0)$ be an irreducible plane curve singularity defined by the germ of the analytic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. $(C, 0) \subset (\mathbb{C}^2, 0)$ admits a minimal good embedded resolution whose associated graph Γ_f is a connected, negative definite tree with an extra arrow representing the strict transform of C. The link of $(C, 0) \subset (\mathbb{C}^2, 0)$ is an algebraic knot $K \subset S^3$, whose isotopy type can be completely characterized by many invariants such as: embedded resolution graph, semigroup of $(C, 0)$, Puiseaux pairs, Newton pairs, linking pairs or the Alexander polynomial of the knot $K \subset S^3$. More details can be found in general references such as [5; 21].

Now assume that $(C, 0) \subset (\mathbb{C}^2, 0)$ has exactly one Puiseaux pair (p, q), which means that the normal form of the defining equation is exactly $x^p + y^q = 0$. In this case, the graph Γ_f is shown in Figure 1 where the decorations can be determined from (p, q). In fact, if we introduce the numbers $0 < \omega_p < p$ and $0 < \omega_q < q$ uniquely determined by the Diophantine equation $pq - \omega_p q - \omega_q p = 1$, then the negative continued fractions $p/\omega_p = [u_1, \ldots, u_k]$ and $q/\omega_q := [v_1, \ldots, v_l]$ give the corresponding decorations of Γ_f.

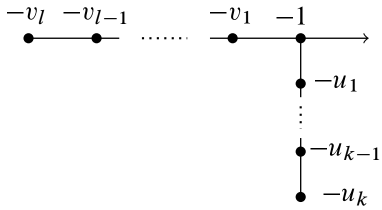

Figure 1: The graph Γ_f.

Another important invariant of an irreducible plane curve singularity is its numerical semigroup $\mathcal{S}_f$. In this case, this is $G(p,q)$. Moreover, by [8] $\mathcal{S}_f$ can be written as

(3.2.1)
$$\mathcal{S}_f = \{\ell \in \mathbb{N} : N(\ell) \geq 0\},$$

where $N(\ell) := \ell - \lceil \omega_p \ell / p \rceil - \lceil \omega_q \ell / q \rceil$ is the quasilinear function associated with the Seifert structure $Sf = (-1, (p, \omega_p), (q, \omega_q))$ of S^3, specified by the negative definite plumbing graph $\widetilde{\Gamma}_f := \Gamma_f \setminus \{\text{arrow}\}$. Thus we have concluded the following.

Corollary 3.2.2 *The numerical semigroup $G(p,q)$ is representable.*

Note that $G(p,q)$ can be represented with SSR graphs having more than two legs as well, by using $\widetilde{\Gamma}_f$ and Section 3.1.1. Indeed, for an arbitrary $k \geq 2$, $G(p,q)$ can be represented as the semigroup associated with the SSR graph $k\widetilde{\Gamma}_f$, defined by the Seifert invariants $Sf = (-k, k \times (p, \omega_p), k \times (q, \omega_q))$. (Here the notation means that (p, ω_p), as well as (q, ω_q) appears k times.)

Remark 3.2.3 Now, by Theorem 2.3.12 and Corollary 3.2.2, the representability of strongly flat semigroups is completely clarified.

3.3 Bounds for representable semigroups

Recall that $\mathcal{S}/k := \{\ell \in \mathbb{N} : k\ell \in \mathcal{S}\}$ is the quotient semigroup of the numerical semigroup $\mathcal{S}$ by k. Note that we can also extend this concept to any submonoid of $\mathbb{N}$.

Next we analyze $\mathcal{S}/k$ from the perspective of the quasilinear function. Let $\mathcal{S}$ be a representable semigroup and $N(\ell)$ its associated quasilinear function and $k \in \mathbb{N}^*$. We set $N^{(k)}(l) := N(kl)$. Then the semigroup associated with $N^{(k)}$, given by

$$\{\ell \in \mathbb{N} : N(k\ell) \geq 0\} = \{\ell \in \mathbb{N} : k\ell \in \mathcal{S}_N\},$$

is exactly $\mathcal{S}/k$. Moreover, one can prove that the quotient is also representable.

Lemma 3.3.1 *The semigroup $\mathcal{S}/k$ is representable for any representable semigroup $\mathcal{S}$ and $k \in \mathbb{N}^*$.*

Proof We represent $\mathcal{S}$ by a normal weighted homogeneous surface singularity $(X,0)$ with minimal good dual resolution graph Γ whose central vertex is denoted by E_0. Then the local algebra is expressed as $R_X = \bigoplus_{\ell \geq 0} H^0(E_0, \mathcal{O}_{E_0}(D^{(\ell)}))$, see (2.3.1) and the definition of $D^{(\ell)}$ therein. Then the Veronese

Figure 2: A representation of $G(3,5)$ and $G(3,4,5)$.

subring $R_X^{(k)} := \bigoplus_{\ell\geq 0} H^0(E_0, \mathcal{O}_{E_0}(D^{(k\ell)}))$ is also normal and therefore corresponds to a normal weighted homogeneous surface singularity. Moreover, the associated semigroup is exactly $\mathcal{S}/k$. □

Remark 3.3.2 For the previous lemma one can give a combinatorial proof which explicitly constructs from a representative Γ of $\mathcal{S}$ a representative $\Gamma^{(k)}$ of $\mathcal{S}/k$.

We fix $k \geq 1$ and assume that Γ has Seifert invariants $Sf = (-b_0, (\alpha_i, \omega_i)_{i=1}^n)$. Then the induced quasilinear function of the quotient $\mathcal{S}/k$ is expressed as

$$N^{(k)}(\ell) = N(k\ell) = b_0 k\ell - \sum_{i=1}^n \left\lceil \frac{k\ell\omega_i}{\alpha_i} \right\rceil.$$

For every $i = 1, \dots, n$ we consider $0 \leq r_i < \alpha_i$ satisfying $k\omega_i \equiv r_i \pmod{\alpha_i}$. Then one gets

$$N^{(k)}(\ell) = \left(k|e| + \sum_{i=1}^n \frac{r_i}{\alpha_i} \right)\ell - \sum_{i=1}^n \left\lceil \frac{r_i}{\alpha_i} \right\rceil.$$

First of all, notice that $k|e| + \sum_{i=1}^n r_i/\alpha_i \in \mathbb{Z}_{>0}$ and the new orbifold Euler number $e^{(k)} = ke$ is negative since $e < 0$. Hence, we can associate with $N^{(k)}$ a negative definite star-shaped graph $\Gamma^{(k)}$ with Seifert invariants $Sf^{(k)} = \big(ke - \sum_{i=1}^n r_i/\alpha_i, (\alpha_i, r_i)_{i=1}^n\big)$ which represents the quotient semigroup $\mathcal{S}/k$.

Note that if $r_i = 0$ then in $\Gamma^{(k)}$ there is no leg of type (α_i, r_i). In particular, if $\alpha_i \mid k$ for all i, then $\Gamma^{(k)}$ consists of a single vertex with self-intersection number ke.

Example 3.3.3 As an illustration of the previous construction we consider the semigroup $G(3,5)$. We claim that it can be represented by the graph on the left-hand side of Figure 2 (see Section 3.2). By Remark 3.3.2 the associated quasilinear function of the quotient semigroup $G(3,5)/2$ is written as $N^{(2)}(\ell) = 2\ell - 2\lceil \ell/5 \rceil - 2\lceil 2\ell/3 \rceil$ which provides the graph drawn on the right-hand side of Figure 2. Moreover, since the set of gaps of this quotient is $\{1,2\}$, we have $G(3,5)/2 = G(3,4,5)$.

In the sequel, we will give "bounds" for representable semigroups. It will be crucial in the next section for the characterization of flat semigroups.

Let Γ be a SSR graph with Seifert invariants

(3.3.4) $$Sf = \big(-b_0, s_1 \times (\alpha_1, \omega_1), \dots, s_n \times (\alpha_n, \omega_n)\big),$$

where $(\alpha_i, \omega_i) \neq (\alpha_j, \omega_j)$ for different indices $i \neq j$. Here s_i stands for the "multiplicity" of the (α_i, ω_i)-type leg. Then, with regard to its associated semigroup $\mathcal{S}_\Gamma$ we obtain the following result.

Theorem 3.3.5 *One has the inclusions*

$$G(\alpha, s_1\alpha_{-1}, \dots, s_n\alpha_{-n}) \subset \mathcal{S}_\Gamma \subset G(\alpha, s_1\alpha_1^*, \dots, s_n\alpha_n^*)/\mathfrak{o}, \tag{3.3.6}$$

where $\alpha_i^* := \alpha/\alpha_i$, $\alpha_{-i} = \mathrm{lcm}_{j\neq i}(\alpha_j)$ *and the bounds are considered as submonoids of* $\mathbb{N}$.

Proof We will first prove the upper bound. Using (2.2.3) and the definition of $N(\ell)$ one writes

$$\mathfrak{o}\ell = \alpha N(\ell) + \sum_{i=1}^{n} s_i\alpha_i^* \cdot \alpha_i\, f\left(\frac{\omega_i\ell}{\alpha_i}\right), \tag{3.3.7}$$

where $f(x) = \lceil x\rceil - x$ for any $x \in \mathbb{Q}$. Note that if $N(\ell) \geq 0$ then the right-hand side of (3.3.7) is a nonnegative linear form of the generators α and $\{s_i\alpha_i^*\}_{i=1}^n$. This implies that

$$\mathcal{S}_\Gamma \subset \bigl\{\ell \in \mathbb{N} : \mathfrak{o}\ell \in G(\alpha, s_1\alpha_1^*, \dots, s_n\alpha_n^*)\bigr\} = G(\alpha, s_1\alpha_1^*, \dots, s_n\alpha_n^*)/\mathfrak{o},$$

where $G(\alpha, s_1\alpha_1^*, \dots, s_n\alpha_n^*)$ is a submonoid of $\mathbb{N}$, not necessarily a numerical semigroup.

In order to see the lower bound we can proceed as follows. For any fixed $i \in \{1, \dots, n\}$ the definition of the orbifold Euler number gives the expression

$$s_i\frac{\omega_i}{\alpha_i} = b_0 + e - \sum_{j\neq i} s_j\frac{\omega_j}{\alpha_j},$$

which is used to deduce

$$\begin{aligned} N(s_i\alpha_{-i}) &= b_0 s_i\alpha_{-i} - \sum_{j\neq i} s_j\left\lceil s_i\alpha_{-i}\frac{\omega_j}{\alpha_j}\right\rceil - s_i\left\lceil \alpha_{-i}\left(s_i\frac{\omega_i}{\alpha_i}\right)\right\rceil \\ &= b_0 s_i\alpha_{-i} - s_i\sum_{j\neq i} s_j\alpha_{-i}\frac{\omega_j}{\alpha_j} - s_i\left(b_0\alpha_{-i} - \sum_{j\neq i}\alpha_{-i}s_j\frac{\omega_j}{\alpha_j} + \lceil e\alpha_{-i}\rceil\right) \\ &= -s_i\lceil e\alpha_{-i}\rceil \geq 0. \end{aligned}$$

Furthermore, one has $N(\ell_1+\ell_2) \geq N(\ell_1) + N(\ell_2)$ and $N(\alpha) = \mathfrak{o} > 0$ (Proposition 2.3.5(c)) which imply that $G(\alpha, s_1\alpha_{-1}, \dots, s_n\alpha_{-n}) \subset \mathcal{S}_\Gamma$. □

4 Representability of flat semigroups

We consider a special case in (3.3.6) and characterize the representable semigroups that realize the bounds. It turns out that these are exactly the *flat semigroups*. In this section we provide their characterization, prove that they are representable and discuss about their "canonical geometric representatives".

4.1

Consider the inclusions of (3.3.6) and assume that $\mathfrak{o} = 1$, $\alpha_{-i} = \alpha_i^*$ and $\gcd(s_i, \alpha_i) = 1$ for every i. In this case the bounds are, in fact, numerical semigroups, they coincide and (3.3.6) becomes an identity. On the other hand, condition $\alpha_{-i} = \alpha_i^*$ can be achieved exactly when the numbers α_i are pairwise relatively primes. Hence we get $\alpha_{-i} = \alpha_i^* = \prod_{j\neq i} \alpha_j$ for which we will use the unified notation $\widehat{\alpha}_i$.

In summary, one deduces the following consequence: if the graph Γ defined by the Seifert invariants $Sf = \big(-b_0, s_1 \times (\alpha_1, \omega_1), \dots, s_n \times (\alpha_n, \omega_n)\big)$ satisfies $\mathfrak{o} = 1$, the numbers $\{\alpha_i\}_{i=1}^n$ are pairwise relatively prime integers and $\gcd(s_i, \alpha_i) = 1$ for every i, then

$$\textbf{(4.1.1)} \qquad \mathcal{S}_\Gamma = G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n).$$

Moreover, in this case $\mathcal{S}_\Gamma$ is a flat semigroup. Indeed, using the notation from Section 2.1.2 one gets $q_0 = \gcd(\{s_j\}_j)$ and $q_i = \alpha_i$ for any $i \geq 1$. This implies that $\widetilde{q}_0 = \alpha_1 \dots \alpha_n = \alpha$, $\widetilde{q}_i = q_0 \cdot \widehat{\alpha}_i$ for $i \geq 1$, $\hat{s}_0 = 1$ and $\hat{s}_i = s_i/q_0$ for $i \geq 1$, hence, it clearly satisfies the flatness condition (see Definition 2.1.4) at $i = 0$.

Now, we start with a presentation $G(a_0, \dots, a_n)$ of a flat semigroup and assume that $\hat{s}_0 = 1$. Note that $\{a_0, \dots, a_n\}$ is not necessarily the minimal set of generators. In particular, when $n = 1$, for the next construction we have to use a presentation with at least three generators, e.g., $G(a_0a_1, a_0, a_1)$.

The chosen set of generators can be read as $\{\widetilde{q}_0, \hat{s}_1\widetilde{q}_1, \dots, \hat{s}_n\widetilde{q}_n\}$ where we define the numbers $q_i := \gcd(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$, $\hat{q}_i := \prod_{j\neq i} q_j$ and $\hat{s}_i := a_i/\hat{q}_i$ for all $i \in \{0, \dots, n\}$. Note that $\hat{s}_i$ is an integer since $\gcd(q_i, q_j) = 1$ for every $i \neq j$, and it follows that $\gcd(\hat{s}_i, q_i) = 1$. Setting $\alpha_i := q_i$ for $i \geq 1$ and $s_i := q_0 \cdot \hat{s}_i$ for every $i \geq 0$ one gets that $\{\alpha_i\}_i$ are pairwise relatively primes and $\gcd(s_i, \alpha_i) = 1$ if $i \geq 1$. Moreover, one identifies $\widetilde{q}_0 = \alpha$ and $\hat{s}_i\widetilde{q}_i = s_i\widehat{\alpha}_i$ for every $i \geq 1$, therefore, the semigroup is presented in the form of (4.1.1).

The previous argument provides the "arithmetical" characterization of flat semigroups which was also proved in [16].

Theorem 4.1.2 [16] *$\mathcal{S}$ is a flat semigroup if and only if there exist pairwise relatively prime integers $\alpha_i \geq 2$ ($i \in \{1, \dots, n\}$) such that $\mathcal{S}$ can be presented as $G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)$ where $\alpha = \alpha_1 \cdots \alpha_n$, $\widehat{\alpha}_i = \prod_{j\neq i} \alpha_j$ and $\gcd(\alpha_i, s_i) = 1$ for every i.*

Next, we show that, once the presentation (4.1.1) is fixed, there exists a canonical way to represent a flat semigroup as a semigroup associated with an SSR graph.

Theorem 4.1.3 *Every flat semigroup is representable.*

Proof Let $\mathcal{S} = G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)$ be a fixed presentation of the flat semigroup $\mathcal{S}$. We would like to find the appropriate $b_0 \geq 1$ and $\omega_1, \dots, \omega_n$ such that $0 < \omega_i < \alpha_i$ and the Seifert invariants

$$\textbf{(4.1.4)} \qquad Sf = (-b_0, s_1 \times (\alpha_1, \omega_1), \dots, s_n \times (\alpha_n, \omega_n))$$

define a negative definite star-shaped plumbing graph Γ with $\mathfrak{o} = \alpha|e| = 1$. Note that this later condition is equivalent with $b_0 - \sum_i s_i\omega_i/\alpha_i = 1/\alpha$.

If we ignore the s_i-multiplicities, then the Diophantine equation

$$\alpha\Big(\widetilde{b}_0 - \sum_i \widetilde{w}_i/\alpha_i\Big) = 1$$

has a unique solution $\widetilde{b}_0, \widetilde{\omega}_1, \dots \widetilde{\omega}_n$; see [7]. In addition, if $\widetilde{\omega}_i$ is divisible by s_i for every i, then one writes $\widetilde{b}_0 - \sum_i s_i \frac{\widetilde{\omega}_i / s_i}{\alpha_i} = 1/\alpha$, hence by setting $\omega_i := \widetilde{\omega}_i / s_i$ the construction is finished. However, in general, this divisibility does not hold and we have to perturb the initial solution $\widetilde{b}_0, \widetilde{\omega}_1, \dots, \widetilde{\omega}_n$ as follows.

Note that for arbitrary $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ we can write

$$\frac{1}{\alpha} = \widetilde{b}_0 + k_1 + k_2 + \cdots + k_n - \sum_{i=1}^{n} \frac{k_i \alpha_i + \widetilde{\omega}_i}{\alpha_i}.$$

Since the positive integers s_i and α_i are relatively prime, we choose k_i to be the unique nonnegative solution of the equation $k_i \alpha_i + \widetilde{\omega}_i \equiv 0 \pmod{s_i}$ such that $0 \leq k_i < s_i$. In this case $k_i \alpha_i + \widetilde{\omega}_i$ is divisible by s_i, hence we can define

$$\omega_i := (k_i \alpha_i + \widetilde{\omega}_i)/s_i \in \mathbb{N} \quad \text{for every } i \qquad \text{and} \qquad b_0 := \widetilde{b}_0 + k_1 + \cdots + k_n > 0.$$

This yields that

$$0 < \omega_i < \frac{1}{s_i}((s_i - 1)\alpha_i + \alpha_i) = \alpha_i \quad \text{for every } 1 \leq i \leq n \qquad \text{and} \qquad e = b_0 - \sum_{i=1}^{n} s_i \frac{\omega_i}{\alpha_i} = \frac{1}{\alpha},$$

hence we get $\mathfrak{o} = 1$, which finishes the proof. □

Remark 4.1.5 In the case where $s_i = 1$ for any i, the graph constructed in the proof of Theorem 4.1.3 is the canonical representative of a strongly flat semigroup. This motivates the following definition.

Definition 4.1.6 We say that the representative constructed in Theorem 4.1.3 is a *canonical representative* of the flat semigroup $\mathcal{S}$ associated with its presentation $G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)$.

Remark 4.1.7 (a) The next example illustrates how a canonical representative depends on the presentation $G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)$ of the flat semigroup. Once the presentation is fixed, it is unique from the previous proof.

Consider the numerical semigroup $\mathcal{S}$ generated by $a_0 = 6$, $a_1 = 15$ and $a_2 = 20$. One can check that $\mathcal{S}$ is flat in the first two generators, i.e.,

$$\hat{s}_0 = \hat{s}_1 = 1.$$

Hence, if we first present it as $G(2\cdot 3, 5\cdot 3, 10\cdot 2)$, then this provides a canonical representative with Seifert invariants $Sf = (-6, 5\times(2,1), 10\times(3,1))$. On the other hand, if we present it as $G(5\cdot 3, 2\cdot 3, 4\cdot 5)$, we get a graph with Seifert invariants $Sf = (-3, 4\times(3,1), 2\times(5,4))$. One can also choose a nonminimal presentation such as $G(2\cdot 3\cdot 5, 1\cdot 15, 2\cdot 10, 1\cdot 6)$. In this case, the associated canonical representative is defined by the Seifert invariants $Sf = (-2, (2,1), 2\times(3,1), (5,4))$.

(b) If we run the procedure of Theorem 4.1.3 for the strongly flat semigroup $G(p,q)$ presented as $G(pq, p, q)$, then the resultant canonical representative is defined by the Seifert invariants

$$Sf = (-b_0, (p, \omega_1), (q, \omega_2)),$$

where b_0, ω_1, ω_2 satisfy

$$pqb_0 - q\omega_1 - p\omega_2 = 1.$$

The last identity immediately implies that $b_0 = 1$ and the corresponding graph is $\widetilde{\Gamma}_f$, see Section 3.2. In other words, the canonical representative is S^3 with the corresponding Seifert structure.

4.2

In the sequel, we discuss some properties of the canonical representatives of flat semigroups.

Proposition 4.2.1 *A canonical representative of a flat semigroup is numerically Gorenstein.*

Proof Let Γ be the canonical representative associated with a fixed presentation of a flat semigroup and consider its Seifert invariants as in (4.1.4). By Lemma 2.2.6 one has to check that $\gamma \in \mathbb{Z}$ and $\gamma \equiv \omega_i' \pmod{\alpha_i}$ for every $1 \le i \le n$.

For the first, we recall that for a canonical representative we have $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ and $\mathfrak{o} = 1$, hence $|e| = 1/\alpha$. Then, from (2.2.4) we get

$$\gamma = \frac{1}{|e|}\cdot\Big(d-2-\sum_{\text{all }\alpha_j}\frac{1}{\alpha_j}\Big) = \alpha\cdot\Big(d-2-\sum_{i=1}^{n} s_i\frac{1}{\alpha_i}\Big) = (d-2)\alpha - \sum_{i=1}^{n} s_i\widehat{\alpha}_i \in \mathbb{Z}.$$

Note that $\gamma \equiv \omega_i' \pmod{\alpha_i}$ is equivalent with $\omega_i\gamma \equiv 1 \pmod{\alpha_i}$; see Section 2.2.2. The previous calculation gives the expression

$$\omega_i\gamma = \alpha_i\Big((d-2)\widehat{\alpha}_i - \sum_{\substack{k=1\\k\neq i}}^{n} s_k \prod_{\substack{j=1\\ j\notin\{i,k\}}}^{n}\alpha_j\Big)\omega_i - s_i\widehat{\alpha}_i\omega_i,$$

which tells us that $\omega_i\gamma \equiv -s_i\widehat{\alpha}_i\omega_i \pmod{\alpha_i}$. On the other hand, the identity $\mathfrak{o} = 1$ reads as

$$1 = \alpha|e| = \alpha\Big(b_0 - \sum_{k=1}^{n} s_k\frac{\omega_k}{\alpha_k}\Big) = \alpha_i\Big(b_0\widehat{\alpha}_i - \sum_{\substack{k=1\\k\neq i}}^{n} s_k\omega_k \prod_{\substack{j=1\\ j\notin\{i,k\}}}^{n}\alpha_j\Big) - s_i\widehat{\alpha}_i\omega_i,$$

which implies $-s_i\widehat{\alpha}_i\omega_i \equiv 1 \pmod{\alpha_i}$. Hence, $\omega_i\gamma \equiv 1 \pmod{\alpha_i}$ for any $i \in \{1,\dots,n\}$. □

We can apply Proposition 2.3.9 to deduce that a flat semigroup is symmetric. Furthermore, in this case the Frobenius number simplifies to $f_S = \alpha + \gamma$. This, expressed by the minimal set of generators reproves the formula (2.1.6) from [16, Theorem 2.5].

Theorem 4.2.2 *If S is a flat semigroup, minimally generated by $a_0, a_1, \dots, a_n$ $(n \ge 1)$, then*

(4.2.3)
$$f_S = \sum_{i=0}^{n}(q_i - 1)a_i - \prod_{i=0}^{n} q_i,$$

where $q_i = \alpha_i = \gcd(a_0, \dots, a_{i-1}, a_{i+1}\dots, a_n)$.

Proof Using the previous discussion and the notation $\alpha := \prod_{i=1}^n \alpha_i$, after a possible permutation of the generators, we can assume that $(a_0, a_1, \dots, a_n) = (\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)$. Then one has

$$\begin{aligned}\gamma + \alpha &= \frac{1}{|e|}\Big(d - 2 - \sum_{i=1}^n s_i \frac{1}{\alpha_i}\Big) + \alpha = \alpha\Big(d - 2 - \sum_{i=1}^n s_i \frac{1}{\alpha_i} + 1\Big) = d\alpha - \sum_{i=1}^n s_i\widehat{\alpha}_i - \alpha \\ &= \sum_{i=1}^n s_i\alpha_i\widehat{\alpha}_i - \sum_{i=1}^n s_i\widehat{\alpha}_i - \alpha = \sum_{i=1}^n (\alpha_i - 1)s_i\widehat{\alpha}_i + (\alpha_0 - 1)\alpha - \alpha_0\alpha \\ &= \sum_{i=0}^n (\alpha_i - 1)a_i - \prod_{i=0}^n \alpha_i.\end{aligned}$$

□

4.3 The geometric canonical representatives

In this section we construct explicit equations for weighted homogeneous surface singularities whose link (or minimal good dual resolution graph) is a canonical representative of a flat semigroup.

4.3.1 The universal abelian cover and the action of H

Lemma 4.3.1 *If Γ is the canonical representative of a flat semigroup $G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)$, then one has*

$$H \simeq \bigoplus_{i=1}^n \mathbb{Z}_{\alpha_i}^{s_i-1}.$$

Proof Recall that the canonical representative Γ associated with the given presentation is defined by the Seifert invariants $Sf = (-b_0, s_1 \times (\alpha_i, \omega_i), \dots, s_n \times (\alpha_n, \omega_n))$ where b_0 and ω_i are constructed by Theorem 4.1.3.

Let E_0 be the base element associated with the central node of Γ, and for simplicity, we will denote by $E_{j(i)}$ ($i \in \{1, \dots, n\}$, $j \in \{1, \dots, s_i\}$) the base elements associated with the end-vertices. The classes in $H = L'/L$ of the corresponding dual base elements will be denoted by $g_0 := [E_0^*]$ and $g_{j(i)} := [E_{j(i)}^*]$. Then the group H can be presented as

$$H = \Big\langle g_0, \{g_{j(i)}\}_{i,j} \;\Big|\; b_0 \cdot g_0 = \sum_{i=1}^n \sum_{j=1}^{s_i} \omega_i \cdot g_{j(i)};\ g_0 = \alpha_i \cdot g_{j(i)}\ (1 \le i \le n,\ 1 \le j \le s_i)\Big\rangle;$$

see Neumann [13].

Since $\mathfrak{o} = 1$, one gets $g_0 = 0$ and the relations simplify to $\sum_{i=1}^n \sum_{j=1}^{s_i} \omega_i \cdot g_{j(i)} = 0$ and $\alpha_i \cdot g_{j(i)} = 0$. From the former relation we deduce that $\widehat{\alpha}_i\omega_i \cdot \sum_{j=1}^{s_i} g_{j(i)} = 0$, while the latter gives $\alpha_i \cdot \sum_{j=1}^{s_i} g_{j(i)} = 0$. These imply that the order of $\sum_{j=1}^{s_i} g_{j(i)}$ divides both $\widehat{\alpha}_i\omega_i$ and α_i. Since $\{\alpha_i\}_i$ are pairwise relatively prime, this is possible if and only if $\sum_{j=1}^{s_i} g_{j(i)} = 0$ for any $i \in \{1, \dots, n\}$. Therefore, we get

$$\textbf{(4.3.2)} \qquad H \simeq \bigoplus_{i=1}^n \Big\langle g_{1(i)}, \dots, g_{s_i(i)} \;\Big|\; \alpha_i \cdot g_{j(i)} = 0;\ \sum_{j(i)=1}^{s_i} g_{j(i)} = 0\ (1 \le j \le s_i)\Big\rangle \simeq \bigoplus_{i=1}^n \mathbb{Z}_{\alpha_i}^{s_i-1}.$$

This completes the proof. □

Now, let $(X, 0)$ be a weighted homogeneous surface singularity whose minimal good dual resolution graph is Γ. Then there exists the universal abelian cover $(X^{ab}, 0) \to (X, 0)$ that induces an unramified

Galois covering $X^{ab}\setminus\{0\}\to X\setminus\{0\}$ with Galois group $H\simeq H_1(M,\mathbb{Z})$ (M is the link of $(X,0)$), i.e., $(X,0)=(X^{ab}/H,0)$. Furthermore, by a theorem of Neumann [13] (see also [11, 5.1.40]) this universal abelian cover $(X^{ab},0)$ is a Brieskorn complete intersection singularity, which can be given by

$$\textbf{(4.3.3)}\qquad \Big\{z=(z_{j(i)})\in\mathbb{C}^d \;\Big|\; f_k:=\sum_{i=1}^{n}\sum_{j(i)=1}^{s_i} c^k_{j(i)} z_{j(i)}^{\alpha_i}=0,\ k=1,\dots,d-2\Big\},$$

where the $(d-2)\times d$ matrix $(c^k_{j(i)})$ has full rank. Here $d:=s_1+\cdots+s_n$ and the variable $z_{j(i)}$ is assigned to the end-vertex $E_{j(i)}$ for any $i=1,\dots,n$, $j=1,\dots,s_i$. In the following, we define the H-action on X^{ab}.

Consider the Pontrjagin dual $\widehat{H}:=\operatorname{Hom}(H,S^1)$ of H and let $\theta:H\to\widehat{H}$, $[l']\mapsto e^{2\pi i(l',\cdot)}$ be the isomorphism between H and $\widehat{H}$. The H acts on $\mathbb{C}^d$ by the diagonal action

$$\operatorname{diag}(\chi_{j(i)})_{i,j}:H\to \operatorname{Diag}(d)\subset GL_d(\mathbb{C}),$$

where $\chi_{j(i)}:=e^{2\pi i(E^*_{j(i)},\cdot)}\in\widehat{H}$ is the character corresponding to $[E^*_{j(i)}]$. Since equations f_k are eigenvectors we obtain an induced action on X^{ab} too. By [13, Theorem 2.1] this action is free from the origin and the orbit space $(X^{ab}/H,0)\simeq(X,0)$ (for the right choice of $c^k_{j(i)}$).

Remark 4.3.4 Although the complex structure of $(X^{ab},0)$ depends on the choice of matrix $(c^k_{j(i)})$, its link M^{ab} (as a Seifert 3-manifold) is independent. Furthermore, by the result of [6, Theorem 7.2] (see also [11, 5.1.17]) one can show that in our case the resolution graph Γ^{ab} of $(X^{ab},0)$ inherits from Γ the following structure (in the sequel all invariants of M^{ab} will be marked by $\star^{ab}$):

- If $s_i=1$ then the leg in Γ with (α_i,ω_i) induces a leg in Γ^{ab} with $\alpha_i^{ab}=\alpha_i$ and this will appear with multiplicity $s_i^{ab}=\prod_{l=1}^n\alpha_l^{s_l-1}$.
- If $s_i>1$ then for any $j(i)\in\{1,\dots,s_i\}$ one gets $\alpha^{ab}_{j(i)}=1$ and $s^{ab}_{j(i)}=\alpha_i^{s_i-2}\prod_{l\neq i}\alpha_l^{s_l-1}$. Note that these legs completely disappear since $\alpha^{ab}_{j(i)}=1$, however, their multiplicity contributes to the genus of the central fiber in M^{ab}.
- More precisely, one gets $g^{ab}=1+\frac{1}{2}\prod_{l=1}^n\alpha_l^{s_l-1}\big(\sum_{i,s_i>1}s_i\big(1-\frac{1}{\alpha_i}\big)-2\big)$. Furthermore, b_0^{ab} and ω_i^{ab} can be determined using the following formulae: $\omega_i^{ab}\widehat{\alpha}_i\equiv-1\ (\operatorname{mod}\alpha_i)$ and $-e^{ab}=\prod_{l=1}^n\alpha_l^{s_l-2}$.

We emphasize that M^{ab} is a rational homology sphere if and only if $s_i=1$ for any i. Or, equivalently, Γ is the canonical representative of a strongly flat semigroup by Remark 4.1.5.

4.3.2 The equations of $(X,0)$ Now we study the induced action on the polynomial ring $R=\mathbb{C}[(z_{j(i)})_{i,j}]$ and the invariant subring R^H of R. By considering the generators of the invariant monomials and their relations in R^H, in the following we will study the possible equations for the analytic types of $(X,0)$.

Since the characters $\chi_{j(i)}$ generate $\widehat{H}$, they satisfy the relations from (4.3.2), namely one has

$$\textbf{(4.3.5)}\qquad \chi_{j(i)}^{\alpha_i}=1\quad\text{and}\quad\prod_{j(i)=1}^{s_i}\chi_{j(i)}=1\qquad\text{for any } i=1,\dots,n.$$

A monomial $z^a:=\prod_i\prod_{j(i)}z_{j(i)}^{a_{j(i)}}$ is in R^H if and only if $\prod_{j(i)}\chi_{j(i)}^{a_{j(i)}}=1$ for any i. From (4.3.5) one deduces that the monomials $z_{j(i)}^{\alpha_i}$ and $\prod_{j(i)=1}^{s_i}z_{j(i)}$ are in R^H for any i and $j(i)$. If there are any other generators, divided with the invariant monomials already listed, they must have the form of $\prod_{j(i)=1}^{s_i} z_{j(i)}^{a_{j(i)}}$ for some i with the exponents $(a_{j(i)}) = (a_1, \dots, a_{s_i-1}, 0)$ where $0 \le a_{j(i)} < \alpha_i$. We claim that in this case $a_{j(i)} = 0$ for $j(i) = 1, \dots, s_i - 1$ also. Indeed, one has the identity $\prod_{j(i)=1}^{s_i-1} \chi_{j(i)}^{a_{j(i)}} = 1$ which implies that $\sum_{j(i)=1}^{s_i-1} a_{j(i)} E_{j(i)}^* \in L$. In particular, $\sum_{j(i)=1}^{s_i-1} a_{j(i)} g_{j(i)} = 0 \in H$. Then by the isomorphism from Lemma 4.3.1 and the assumptions on $a_{j(i)}$ conclude that $a_{j(i)} = 0$.

In summary, the generators associated with i are as follows: if $s_i = 1$ then $z_i := z_{j(i)}$ is a generator; in the case $s_i > 0$ we get $w_{j(i)} := z_{j(i)}^{\alpha_i}$ for $j(i) = 1, \dots, s_i$ and $w_i := \prod_{j(i)=1}^{s_i} z_{j(i)}$. Then R^H can be presented as $\mathbb{C}[z_i, w_i, w_{j(i)}]/I$ where the ideal I is given by the relations

$$\textbf{(4.3.6)}\qquad \begin{cases} \sum_{i,s_i=1} c_i^k z_i^{\alpha_i} + \sum_{i,s_i\neq 1} \sum_{j(i)} c_{j(i)}^k w_{j(i)} = 0, & k = 1, \dots, d-2, \\ w_i^{\alpha_i} = \prod_{j(i)=1}^{s_i} w_{j(i)}, & \text{for every } i \text{ with } s_i > 1, \end{cases}$$

providing us the equations for the possible analytic types of $(X, 0)$. Note that the first type of relation comes from the equations (4.3.3) of X^{ab}, the second type of equations is given by the relations of the monoid algebra $\mathbb{C}[M_i]$ associated with the affine monoid

$$M_i = \big\langle t_1 := (\alpha_i, 0, \dots, 0), t_2 := (0, \alpha_i, 0, \dots, 0), \dots, (0, \dots, 0, \alpha_i), t_{s_i+1} := (1, 1, \dots, 1)\big\rangle \subset \mathbb{Z}_{\ge 0}^{s_i}$$

for every i with $s_i > 1$. In other words, for a fixed i the corresponding relations generate the toric ideal I_i where $\mathbb{C}[M_i] \simeq \mathbb{C}[w_{j(i)}, w_i]/I_i$. In fact, in our case we have $I_i = \big(w_i^{\alpha_i} - \prod_{j(i)=1}^{s_i} w_{j(i)}\big)$. Indeed, the generators of the relations in terms of the monoid generators have the form of

$$\sum_{l\in I} a_l t_l = \sum_{m\in J} b_m t_m + b_{s_i+1} t_{s_i+1}$$

for some $a_l, b_m, b_{s_i+1} \ge 0$ where $I, J \subset \{1, \dots, s_i\}$ and $I \cap J = \varnothing$. Looking at the coordinates, this implies that $b_m = 0$, $I = \{1, \dots, s_i\}$ and $a_l \alpha_i = b_{s_i+1}$, which gives us the only generator $\sum_{l=1}^{s_i} t_l = \alpha_i t_{s_i+1}$.

Note that many of the variables $w_{j(i)}$ can be eliminated and the number of equations in (4.3.6) can be reduced. In the following, we will distinguish three cases depending on the number $K := \#\{i : s_i = 1\}$ of legs with multiplicity 1.

<u>I: $K = 0$</u> In this case we have the linear system $\{\sum_{i,s_i\neq 1} \sum_{j(i)} c_{j(i)}^k w_{j(i)} = 0\}_{k=1,\dots,d-2}$. Associated with two fixed, not necessarily different indices i_1 and i_2 we choose $j_0(i_1) \in \{1, \dots, s_{i_1}\}$ and $j_0(i_2) \in \{1, \dots s_{i_2}\}$. For simplicity, we set the notation $x := w_{j_0(i_1)}$ and $y := w_{j_0(i_2)}$. Then, since the coefficients $\{c_{j(i)}^k\}$ are generic (i.e., the matrix of the system has rank $d-2$), the other variables can be expressed linearly in terms of x and y. Therefore, we get that $(X, 0) \subset (\mathbb{C}^{n+2}, 0)$ is defined by

$$\textbf{(4.3.7)}\qquad w_i^{\alpha_i} = \prod_{j(i)=1}^{s_i} (a_{j(i)} x + b_{j(i)} y), \quad i = 1, \dots n,$$

where $a_{j_0(i_1)} = b_{j_0(i_2)} = 1$, $a_{j_0(i_2)} = b_{j_0(i_1)} = 0$ and the other $a_{j(i)}, b_{j(i)} \in \mathbb{C}$ are generic coefficients.

<u>II: $K = 1$</u> We simply write z for the variable associated with the only i with $s_i = 1$, also set $\alpha := \alpha_i$. On the other hand, we fix $i_0 \in \{i : s_i > 1\}$ and one of its associated variables will be denoted by $x := w_{j_0(i_0)}$.

The other variables are linearly expressed with z^α and x, hence the equations of $(X,0)\subset\mathbb{C}^{n+1}$ are

(4.3.8) $$w_i^{\alpha_i}=\prod_{j(i)=1}^{s_i}(a_{j(i)}z^\alpha+b_{j(i)}x),\quad i\in\{i:s_i>1\},$$

where $a_{j_0(i_0)}=0$, $b_{j_0(i_0)}=1$ and the others are generic.

<u>III: $K>1$</u> In the last case, we choose i_1,i_2 with $s_{i_1}=s_{i_2}=1$ and denote their associated variables by $x:=z_{i_1}$ and $y:=z_{i_2}$. Then, one can express $z_i^{\alpha_i}$ for $i\in\{i:s_i=1\}$ and $i\neq i_1,i_2$, as well as the variables $w_{j(i)}$ for every $i\in\{i:s_i>1\}$ and $j(i)$, linearly in terms of $x^{\alpha_{i_1}}$ and $y^{\alpha_{i_2}}$. Therefore we get that $(X,0)\subset(\mathbb{C}^n,0)$ is defined by

(4.3.9) $$\begin{cases} z_i^{\alpha_i}=p_ix^{\alpha_{i_1}}+q_iy^{\alpha_{i_2}}, & i\in\{i:s_i=1\}\setminus\{i_1,i_2\};\\ w_i^{\alpha_i}=\prod_{j(i)=1}^{s_i}(a_{j(i)}x^{\alpha_{i_1}}+b_{j(i)}y^{\alpha_{i_2}}), & i\in\{i:s_i>1\},\end{cases}$$

where $p_i,q_i,a_{j(i)},b_{j(i)}$ are generic coefficients.

Remark 4.3.10 We emphasize that in all three cases the representatives $(X,0)$ are complete intersections. In particular, they are Gorenstein and Proposition 4.2.1 follows automatically.

Example 4.3.11 Consider the flat semigroup $G(6,15,20)$ discussed in Remark 4.1.7. We look at its (last) canonical representative, which is defined by the Seifert invariants $Sf=(-2,(2,1),2\times(3,1),(5,4))$. Then, by case III of the above construction, we get a family of suspension hypersurface singularities defined by $\big(X_{a_i,b_i}=\{f(x,y,z)=(a_1x^2+b_1y^5)(a_2x^2+b_2y^5)+z^3=0\},0\big)\subset(\mathbb{C}^3,0)$ where $a_i,b_i\in\mathbb{C}$ are generic coefficients. In particular, if we consider the hypersurface singularity defined by equation $x^4+y^{10}+z^3=0$ for example, we can check that its associated semigroup is $G(6,15,20)$. The other canonical representatives considered in Remark 4.1.7 provide (e.g., by case I) other families of complete intersections with the same associated semigroup.

5 The characterization of representable semigroups

5.1 Perturbation of the Seifert invariants and the characterization

In this section, we prove that representable semigroups are exactly the quotients of flat semigroups.

To prove this result, two technical steps are needed: first we show that every Seifert invariant (α,ω) can be "perturbed" without affecting the quasilinear function $N(\ell)$; the second step claims that the Seifert invariants $(-b_0,s_1\times(\alpha_1,\omega_1),\dots,s_n\times(\alpha_n,\omega_n))$ can be changed to $(-b_0,s_1\times(\alpha_1',\omega_1'),\dots,s_n\times(\alpha_n',\omega_n'))$ in such a way that the associated semigroup remain stable, while $\alpha_1',\dots\alpha_n'$ will be pairwise relatively prime and $\gcd(\alpha_i',s_i)=1$.

We start our discussion with the characterization of the latter case.

Theorem 5.1.1 *Let Γ be an SSR graph defined by the Seifert invariants*

$$Sf=\big(-b_0,s_1\times(\alpha_1,\omega_1),\dots,s_n\times(\alpha_n,\omega_n)\big).$$

If the numbers $\alpha_i \geq 2$ are pairwise relatively prime integers and $\gcd(\alpha_i, s_i) = 1$ for every i, then its associated semigroup is a quotient of a flat semigroup. In fact, $\mathcal{S}_\Gamma = G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)/\mathfrak{o}$.

Proof We observe that $\gcd(\alpha_i, \mathfrak{o}) \neq 1$ implies $\gcd(\alpha_i, s_i) \neq 1$, hence by the assumptions we must have $\gcd(\alpha_i, \mathfrak{o}) = 1$ for every $i \in \{1, \dots, n\}$. Indeed, for a fixed i the expression $\mathfrak{o} = \alpha b_0 - \sum_{j \neq i} s_j\omega_j\widehat{\alpha}_j - s_i\omega_i\widehat{\alpha}_i$ implies the identity $s_i\omega_i\widehat{\alpha}_i \equiv 0 \pmod{\gcd(\alpha_i, \mathfrak{o})}$, which simplifies to $s_i\omega_i \equiv 0 \pmod{\gcd(\alpha_i, \mathfrak{o})}$, since $\gcd(\alpha_i, \alpha_j) = 1$ for $j \neq i$. Then, by multiplying the last congruence with ω'_i and using (2.2.1) yields that $s_i \equiv 0 \pmod{\gcd(\alpha_i, \mathfrak{o})}$, which supports our claim.

Now, using similar ideas as in the proof of Theorem 4.1.3, we consider the nonnegative integers $k_1, \dots, k_n$ as the solutions of the equations

(5.1.2) $$k_1\alpha_1 + \omega_1 \equiv \cdots \equiv k_n\alpha_n + \omega_n \equiv 0 \pmod{\mathfrak{o}},$$

with $0 \leq k_i < \mathfrak{o}$ for every $1 \leq i \leq n$. These, on one hand, allow us to define the numbers

$$\widetilde{\omega}_i = \frac{1}{\mathfrak{o}}(k_i\alpha_i + \omega_i) \quad \text{for every } 1 \leq i \leq n,$$

which satisfy $\gcd(\widetilde{\omega}_i, \alpha_i) = 1$ and $0 < \widetilde{\omega}_i < \alpha_i$. On the other hand, they imply that $b_0 + \sum_{i=1}^n s_ik_i \equiv 0 \pmod{\mathfrak{o}}$ too, hence one can define the new central decoration as $\widetilde{b}_0 = \bigl(b_0 + \sum_{i=1}^n s_ik_i\bigr)/\mathfrak{o}$. Indeed, from (5.1.2), one gets $\sum_{i=1}^n \alpha s_ik_i + s_i\omega_i\widehat{\alpha}_i \equiv 0 \pmod{\mathfrak{o}}$. Furthermore, the definition of $\mathfrak{o}$ reads as $\mathfrak{o} = \alpha b_0 - \sum_i s_i\omega_i\widehat{\alpha}_i$ which, applied to the previous equation, gives $\alpha\bigl(b_0 + \sum_i s_ik_i\bigr) \equiv 0 \pmod{\mathfrak{o}}$. Since $\gcd(\alpha, \mathfrak{o}) = 1$ we deduce that $\bigl(b_0 + \sum_i s_ik_i\bigr) \equiv 0 \pmod{\mathfrak{o}}$.

We consider the SSR graph $\widetilde{\Gamma}$ defined by the Seifert invariants

$$Sf = \bigl(-\widetilde{b}_0, s_1 \times (\alpha_1, \widetilde{\omega}_1), \dots, s_n \times (\alpha_n, \widetilde{\omega}_n)\bigr).$$

In the following, any of the numerical data associated with $\widetilde{\Gamma}$ will be distinguished by the "tilde" notation, e.g., $\tilde{e}$ will stand for the orbifold Euler number of $\widetilde{\Gamma}$. The first observation is that $\tilde{e} = e/o < 0$, hence $\widetilde{\Gamma}$ is negative definite. Furthermore, $\widetilde{\alpha} = \alpha$ and one implies that $\tilde{\mathfrak{o}} = 1$. Then, by the assumptions of the theorem we conclude that $\widetilde{\Gamma}$ is the canonical representative of the flat semigroup $\mathcal{S}_{\widetilde{\Gamma}} = G(\alpha, s_1\widehat{\alpha}_1, \dots, s_n\widehat{\alpha}_n)$.

On the other hand, if we look at the quasilinear function $\widetilde{N}(\ell) = \widetilde{b}_0\ell - \sum_{i=1}^n s_i\lceil\widetilde{\omega}_i\ell/\alpha_i\rceil$ associated with $\mathcal{S}_{\widetilde{\Gamma}}$, one can see that $\widetilde{N}^{(\mathfrak{o})}(\ell) = \widetilde{N}(\mathfrak{o}\ell) = N(\ell)$. Hence $\mathcal{S}_\Gamma = \mathcal{S}_{\widetilde{\Gamma}}/\mathfrak{o}$. □

Lemma 5.1.3 *Let $r \in \mathbb{Q}_{>0}$ be arbitrary. For every $M > 0$ there exists $r_M \in \mathbb{Q}$ such that*

(5.1.4) $$\lceil r\ell\rceil = \lceil r'\ell\rceil \quad \textit{for every } \ell \in \mathbb{N},\ \ell \leq M \textit{ and } r' \in (r_M, r].$$

Proof Since $r \in \mathbb{Q}_{>0}$ one writes $r = \omega/\alpha$, where $\omega, \alpha \in \mathbb{N}$ and $\gcd(\omega, \alpha) = 1$. For a fixed $\ell \in \mathbb{N}$ we introduce the notation $x := \lceil\omega\ell/\alpha\rceil$ for simplicity. Notice that if $\ell = 0$ then (5.1.4) holds for any r', so in the sequel we assume that $\ell \neq 0$. By the properties of the function $t \mapsto \lceil t\rceil$ one knows

$$x - 1 + \frac{1}{\alpha} \leq \frac{\omega\ell}{\alpha} \leq x.$$

This implies that for any $r' \in \bigl(r - \frac{1}{\alpha\ell}, r\bigr]$ we have $x \geq r\ell \geq r'\ell > r\ell - \frac{1}{\alpha} \geq x - 1$, hence $\lceil r'\ell\rceil = x = \lceil r\ell\rceil$.

Thus, for a fixed ℓ we have constructed an interval for r' such that (5.1.4) is satisfied. When ℓ varies in $[0, M]$ we consider the intersection of these intervals

$$\bigcap_{0<\ell\leq M}\left(r-\frac{1}{\alpha\ell}, r\right] = \left(r-\frac{1}{\alpha M}, r\right] =: (r_M, r]. \qquad \square$$

Lemma 5.1.5 *Let $N\colon \mathbb{Z}\to\mathbb{Z}$, $N(\ell)=b_0\ell-\sum_{i=1}^n s_i\lceil\omega_i\ell/\alpha_i\rceil$ be a quasilinear function of a representable semigroup. Then for every $M\in\mathbb{N}$ there exists a modification $N'\colon\mathbb{Z}\to\mathbb{Z}$, $N'(\ell)=b_0\ell-\sum_{i=1}^n s_i\lceil\omega_i'\ell/\alpha_i'\rceil$ ($\omega_i',\alpha_i'\in\mathbb{Z}_{>0}$, $0<\omega_i'<\alpha_i'$ and $\gcd(\omega_i',\alpha_i')=1$), such that $\alpha_1',\dots,\alpha_n'$ are pairwise relatively prime, $\gcd(\alpha_i',s_i)=1$, and it satisfies*

$$N'(\ell)=N(\ell)\quad \textit{for every } \ell\in\mathbb{N} \textit{ and } \ell\leq M.$$

Proof Let Γ be the graph corresponding to $N(\ell)$. We will say that the i-th block of Γ consists of the s_i number of legs with Seifert invariant (α_i,ω_i). For a fixed $M\in\mathbb{N}^*$ we will construct the modification by induction on i.

First, we describe the inductive step. Assume that the $(i-1)$-th block is already modified. This means that the new Seifert invariants $(\alpha_1',\omega_1'),\dots,(\alpha_{i-1}',\omega_{i-1}')$ are constructed so that α_t' are pairwise relatively primes and $\gcd(\alpha_t',s_t)=1$ for any $t\in\{1,\dots,i-1\}$.

Then, for a large enough $k\in\mathbb{N}$, by Lemma 5.1.3 one finds a rational number $r'\in(r_{M,i}',\omega_i/\alpha_i)$ of the form $r'=x/(k\alpha_1'\dots\alpha_{i-1}'s_i+1)$ ($x\in\mathbb{N}$) satisfying $\lceil r'\ell\rceil=\lceil\omega_i\ell/\alpha_i\rceil$ for any $\ell\leq M$. This, written as $\omega_i'/\alpha_i':=r'$ (where $\gcd(\omega_i',\alpha_i')=1$) gives us the perturbation. Since α_i' is a divisor of $k\alpha_1'\dots\alpha_{i-1}'s_i+1$, it is relatively prime to all α_t' with $t\leq i-1$ and s_i. In this way, we get the i-th block of the modified graph with s_i legs, all having the Seifert invariant (α_i',ω_i').

For $i=1$ we can start by distinguishing two cases:

I. If $\gcd(\alpha_1,s_1)=1$ then we do not modify them and we set $(\alpha_1',\omega_1'):=(\alpha_1,\omega_1)$.

II. If $\gcd(\alpha_1,s_1)\neq1$ then we do the same as in the inductive step. Thus, we get a rational number of the form $r'=x/(ks_1+1)$ ($x\in\mathbb{N}$) that satisfies $\lceil r'\ell\rceil=\lceil\omega_1\ell/\alpha_1\rceil$ for any $\ell\leq M$, which provides $\omega_1'/\alpha_1':=r'$.

From the construction, it can be seen that $N(\ell)=N'(\ell)$ for any $\ell\leq M$. $\square$

Theorem 5.1.6 *A numerical semigroup is representable if and only if it is a quotient of a flat semigroup.*

Proof By Lemma 3.3.1 and Theorem 4.1.3 one concludes that a quotient of a flat semigroup is representable. For the converse, let S be a representable semigroup and consider one of its representative Γ defined by the Seifert invariants $(-b_0, s_1\times(\alpha_1,\omega_1),\dots,s_n\times(\alpha_n,\omega_n))$. Let M be the maximum of the largest generator (in the minimal set of generators) and the Frobenius number f_S. If we apply Lemma 5.1.5 for this fixed M, we get a new graph Γ' defined by the perturbed Seifert invariants $(-b_0, s_1\times(\alpha_1',\omega_1'),\dots,s_n\times(\alpha_n',\omega_n'))$ satisfying that $\alpha_1',\dots\alpha_n'$ are pairwise relatively prime integers and $\gcd(\alpha_i',s_{i'}')=1$. Moreover, the associated semigroup $\mathcal{S}_{\Gamma'}$ coincides with $\mathcal{S}$. Indeed, $N(\ell)=N(\ell')$ for all the integers $\ell\leq f_S$ implies that $\mathcal{S}_{\Gamma'}\subset\mathcal{S}$. On the other hand, $N(\ell)=N(\ell')$ for all the integers smaller than the largest generator deduces that $\mathcal{S}\subset\mathcal{S}_{\Gamma'}$. Finally, Theorem 5.1.1 applied to Γ' clarifies the statement. $\square$

Example 5.1.7 Consider the nonflat semigroup $G(4,6,7,9)$. We claim that it is representable and one of its representative is given by the SSR graph with Seifert invariants $Sf=(-2, 2\times(2,1), 2\times(4,1), (5,1))$ (calculations were performed using GAP). Following our previous discussions, a perturbation of the Seifert invariants can be chosen as follows.

Let us first consider the two legs with $(2,1)$. Note that $M=9$ and $(2,1)$ can be changed to, e.g., $(11,5)$ since $\frac{5}{11}\in\left(\frac{1}{2}-\frac{1}{18}=\frac{4}{9},\frac{1}{2}\right)$. Similarly, $(4,1)$ can be changed to $(13,3)$. These perturbations give us a new graph defined by the Seifert invariants $Sf=(-2, 2\times(11,5), 2\times(13,3), (5,1))$, which has $\mathfrak{o}=307$ and satisfies the assumptions of Theorem 5.1.1. Finally, we have

$$G(4,6,7,9)=\tfrac{1}{307}G(110,130,143).$$

5.2 Further speculations and remarks

We would like to propose a general question and emphasize some directions opened by the problem studied in this article.

In the theory of numerical semigroups one knows by [18] (see also [19]) that every numerical semigroup can be presented as one half of a symmetric numerical semigroup. More generally, for a fixed $k\geq 2$, every semigroup can be presented as one over k of a symmetric numerical semigroup; see [20]. Note that Example 3.3.3 is also an example of the construction given by Rosales and García-Sánchez in [18].

As a consequence, by applying Lemma 3.3.1, it follows that if we can prove that every symmetric semigroup is representable, then every numerical semigroup is representable. This approach naturally poses the following question.

Question 5.2.1 *Is there a symmetric numerical semigroup which is not representable*?

By the knowledge of the authors, there is no good understanding how the symmetric property of semigroups incarnates on the level of the representatives, or how does it fit in the "flat" classification theme of Raczunas and Chrząstowski-Wachtel.

For example, one can construct representable semigroups which are not symmetric, but they have a numerically Gorenstein representative:

Example 5.2.2 (nonsymmetric numerically Gorenstein case [7]) Let Γ be the graph defined by the Seifert invariants

$$Sf=(-2,(2,1),(2,1),(3,1),(3,1),(7,1),(7,1),(84,1)).$$

Then $Z_K=(86,43,43,29,29,13,13,2)$, hence Γ is numerically Gorenstein. In addition, one can compute that $\gamma=85$, $1/|e|=28$ and $\check{s}=28$ (see (2.3.8)), thus the Frobenius number equals $f_{\mathcal{S}_\Gamma}=85$. In addition $N(6)=N(85-6)=-1$ which implies $6, f_{\mathcal{S}_\Gamma}-6\notin\mathcal{S}_\Gamma$, hence it is not symmetric.

One the other hand, using the proof of Proposition 2.3.9 one can construct numerical semigroups which are symmetric, but not flat, as shown by the next example. Therefore, the set of symmetric semigroups is rather bigger than the set of the flat ones.

Example 5.2.3 (symmetric but not flat) Let Γ be the graph defined by the Seifert invariants

$$Sf = (-2, (35, 13), (35, 13), (21, 13), (21, 13)).$$

One can verify, e.g., with GAP, that $\mathcal{S}_\Gamma = G(8, 21, 35)$. Hence $\mathcal{S}_\Gamma$ is not a flat semigroup, but an almost flat. However, it is symmetric.

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