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Suspension splittings of 5-dimensional Poincaré duality complexes and their applications

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Let X be a connected, orientable, 5-dimensional Poincaré duality complex with torsion-free $H_1(X;\mathbb{Z})$. We show that ΣX is homotopy equivalent to a wedge of recognisable spaces and study to what extent its homotopy type is determined by algebraic data. These results are then used to compute the unstable cohomotopy groups $\pi^3(X)$ and $\pi^3(X;\mathbb{Z}/k)$ as well as give partial information about the cohomotopy set $\pi^2(X)$.

1 Introduction

Smooth, simply connected, closed, orientable 5-manifolds are well understood. In fact, the classification of these objects was started by Smale [30] and completed by Barden [3] in the 1960s (see also Zhubr [37] for another perspective). Later, Stöcker [34] would extend this to a classification of simply connected, 5-dimensional Poincaré duality complexes up to oriented homotopy type.

Comparatively little is known about nonsimply connected 5-manifolds. Since every finitely presentable group appears as the fundamental group of a smooth, closed, orientable 5-manifold [1], such manifolds are unclassifiable. Even the homotopy types of these manifolds are not well understood.

However, results for nonsimply connected 5-manifolds are starting to appear in the literature. These include the recent papers by Hambleton and Su [10], where certain 5-manifolds M with $\pi_1(M)=\mathbb{Z}/2$ are classified, and by Kreck and Su [22], where certain 5-manifolds with free fundamental groups are considered. Despite this, there seems to be very little in the literature relating to nonsimply connected Poincaré duality complexes in dimension 5.

In this paper we study the suspension splitting problem for connected, orientable, 5-dimensional Poincaré duality complexes X for which $H_1(X;\mathbb{Z})$ is torsion-free. The homology groups of such a space are given by

(1)

i	0	1	2	3	4	5
$H_i(X)$	$\mathbb{Z}$	$\mathbb{Z}^m$	$\mathbb{Z}^n\oplus G$	$\mathbb{Z}^n$	$\mathbb{Z}^m$	$\mathbb{Z}$

where $G=\bigoplus_{i=1}^{\ell}\mathbb{Z}/t_i\mathbb{Z}$ is a torsion group. We assume each $t_i=p_i^{r_i}$ for some prime p_i. Let

$$T=\{\{p_1^{r_1},\dots,p_\ell^{r_\ell}\}\}$$

be the collection containing all t_i's, where repeated t_i's are allowed.

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In order to state the main theorem we need to set some notation. Firstly, for $n \geq 2$ and $k \geq 1$ we let $P^n(k) = S^{n-1} \cup_k e^n$ be the Moore space obtained by attaching an n-cell to S^{n-1} by a degree-k map. Then it is known that (see [4] or Lemma 2.2)

$$\pi_4(P^3(2^r)) \cong \begin{cases} \mathbb{Z}/4\langle \xi_1 \rangle & \text{if } r = 1, \\ \mathbb{Z}/2\langle \xi_r \rangle \oplus \mathbb{Z}/2\langle \iota_r \circ \eta^2 \rangle & \text{if } r > 1, \end{cases}$$

where $\eta^2 \colon S^4 \to S^2$ is the composite $S^4 \xrightarrow{\eta} S^3 \xrightarrow{\eta} S^2$ of Hopf maps, $\iota_r \colon S^2 \to P^3(2^r)$ is the inclusion of the bottom cell, and ξ_r is a lift of $\eta \colon S^4 \to S^3$ through the pinch map $P^3(2^r) \to S^3$. Secondly, let $\mathbb{CP}^2(2^r)$ be the mapping cone of the composite $S^3 \xrightarrow{\eta} S^2 \xrightarrow{\iota_r} P^3(2^r)$, and define ϵ_r to be the composite

$$\epsilon_r \colon S^4 \xrightarrow{\xi_r} P^3(2^r) \hookrightarrow \mathbb{CP}^2(2^r).$$

In Lemma 2.3 we show that the suspension of ϵ_r generates a $\mathbb{Z}/2$ summand in $\pi_5(\Sigma\mathbb{CP}^2(2^r))$.

Our main theorem states that ΣX is homotopy equivalent to a wedge of recognisable spaces. Denote the d-fold wedge sum of a space A by $A^{\vee d}$. Given a sequence $\{f_i \colon S^4 \to A_i\}_{i=1}^d$ of maps, let $f_1 \perp \cdots \perp f_d$ denote the composite

$$f_1 \perp \cdots \perp f_d \colon S^4 \xrightarrow{\text{comult}} \bigvee_{i=1}^d S^4 \xrightarrow{\bigvee_i f_i} \bigvee_{i=1}^d A_i. \tag{2}$$

Theorem 1.1 *Let X be a 5-dimensional orientable Poincaré duality complex with $H_1(X;\mathbb{Z})$ torsion-free. Then there is a homotopy equivalence*

$$\Sigma X \simeq (S^2)^{\vee m} \vee (S^3)^{\vee n_3} \vee (S^4)^{\vee n_4} \vee (S^5)^{\vee n_5} \vee \bigvee_{t_i \in T'} P^4(t_i) \vee (\Sigma\mathbb{CP}^2)^{\vee b} \vee \bigvee_{j=1}^c \Sigma\mathbb{CP}^2(2^{r_j}) \vee \Sigma C_f$$

for some nonnegative integers n_3, n_4, n_5, b, c, where T' and $\{2^{r_j}\}$ are subcollections of T, and C_f is the mapping cone of one of the following maps:

(1) $x\eta \perp y\xi_r \perp z\epsilon_s \colon S^4 \to (S^3)^{\vee x} \vee P^3(2^r)^{\vee y} \vee \mathbb{CP}^2(2^s)^{\vee z}$,

(2) $x\eta^2 \perp y\xi_r \perp z\epsilon_s \colon S^4 \to (S^2)^{\vee x} \vee P^3(2^r)^{\vee y} \vee \mathbb{CP}^2(2^s)^{\vee z}$,

(3) $x(\iota \circ \eta^2) \perp y\xi_r \perp z\epsilon_s \colon S^4 \to P^3(2^q)^{\vee x} \vee P^3(2^r)^{\vee y} \vee \mathbb{CP}^2(2^s)^{\vee z}$,

(4) $y(\iota \circ \eta^2 + \xi_r) \perp z\epsilon_s \colon S^4 \to P^3(2^r)^{\vee y} \vee \mathbb{CP}^2(2^s)^{\vee z}$

for some $x, y, z \in \{0, 1\}$. Furthermore, if $y = z = 1$ then $s < r$.

These results complete a sequence of recent studies into suspension splittings of low-dimensional manifolds and Poincaré duality complexes. In particular, suspension splittings of 4-manifolds were considered by So and Theriault [31], and later by Li [23]. Suspension splittings of 6-manifolds were studied by Huang [15], and also by Cutler and So [8]. Suspension splittings of certain 7-manifolds are obtained in [16], and for highly connected manifolds of higher dimension in [14]. Other recent examples include results on the homotopy types of suspensions of toric manifolds [7; 12] and flag manifolds [19].

A triple suspension splitting of certain 5-manifolds was previously obtained by Huang [13, Propositions 3.7, 4.7]. Furthermore, shortly after our paper was announced, Li and Zhu [25] independently announced suspension splitting results for 5-manifolds. We compare our results with those of [25] below.

We explain in Section 5 to what extent the homotopy type of ΣX is determined by algebraic information contained in $H^*(X)$. An interesting case is when X is a closed, orientable 5-manifold and we have access to its characteristic classes. All manifolds in this paper will be smooth, with Stiefel–Whitney classes $w_i(X)$ and Pontryagin class $p_1(X)$.

The next theorem gives information about the attaching map f which appears in Theorem 1.1 in the case X is a 5-manifold. We say that f *contains* η if case (1) is realised and $x = 1$, and that f *contains* ξ_r or ϵ_s if case (1), (2), (3) or (4) is realised and $y = 1$ or $z = 1$, respectively.

Theorem 1.2 *Let X be a closed, orientable 5-manifold with torsion-free $H_1(X;\mathbb{Z})$ and let C_f be the mapping cone described in Theorem 1.1. Then the following statements hold.*

(1) *X is spin if and only if*

$$\Sigma X \simeq (S^2)^{\vee m} \vee (S^3)^{\vee n} \vee (S^4)^{\vee n} \vee (S^5)^{\vee m} \vee \bigvee_{t_i \in T} P^4(t_i) \vee S^6.$$

(2) *If X is nonspin and $w_3(X) \neq 0$, then f contains exactly one of ξ_1 or ϵ_1. Furthermore,*

- *if f contains ξ_1, then it contains no ϵ_s for any $s \geq 1$;*
- *if $w_2(X) \cdot w_3(X) \neq 0$, then f contains ξ_1 if and only if $p_1(X)$ is divisible by 2.*

(3) *If X is nonspin and $w_3(X) = 0$, then f must contain one of η, ξ_r, ϵ_s. Furthermore,*

- *if either of ξ_r or ϵ_s appears, then $r, s > 1$;*
- *the class $w_2(X)$ survives to exactly the E_r-page of the mod 2 Bockstein spectral sequence if and only if either ϵ_r appears, or ξ_r appears and no ϵ_s does for $s < r$;*
- *the class $w_2(X)$ survives to the E_∞-page of the mod 2 Bockstein spectral sequence if and only if $f \simeq \eta$.*

Actually, every Poincaré duality complex has algebraically defined Stiefel–Whitney classes, and Theorem 1.2 is a special case of more general results found in Propositions 5.2, 5.4, and 5.5.

In addition to the theoretical interest of suspension splittings, they have many useful applications. In the present paper we develop two. Firstly, the splitting of Theorem 1.1 induces decompositions of $h^*(X)$ for any generalised cohomology theory $h^*(\cdot)$. Our result in this regard is stated in Section 6.1 and is similar in spirit to the cohomological decomposition results for 4-manifolds given in [31], and for 6-manifolds given in [8].

Our second set of applications is concerned with the calculation of *unstable* cohomotopy groups of X. In Section 6.2 we describe how induced decompositions of the stable cohomotopy groups of X can be leveraged to obtain information in the unstable range. Cohomotopy groups of manifolds have been considered in many places in the literature. Along with Pontrjagin's classical work relating to

3-manifold cohomotopy groups, there are the more recent results of Kirby, Melvin, and Teichner [20], and of Taylor [35] computing $\pi^2(X)$ for a 4-manifold X. More recently, the group $\pi^4(X)$ has been considered by Konstantis [21] when X is a closed spin 5-manifold.

For a 5-manifold X, the most interesting cohomotopy group to study is $\pi^3(X)=[X,S^3]$, where the group structure is that induced by the Lie group structure of S^3. We also consider the cohomotopy sets with coefficients $\pi^q(X;\mathbb{Z}/k)=[X,P^{q+1}(k)]$. If $q\geq 4$, then $\pi^q(X;\mathbb{Z}/k)$ has a canonical group structure, but otherwise $\pi^q(X;\mathbb{Z}/k)$ is a priori only a pointed set. In Propositions 6.5 and 6.6 we obtain the following result.

Theorem 1.3 *Let X be a 5-dimensional CW complex. Then stabilisation induces a group isomorphism*

$$\pi^3(X)\cong\pi^3_S(X).$$

Moreover, for any $k\geq 1$, stabilisation induces a bijection

$$\pi^3(X;\mathbb{Z}/k)\cong\pi^3_S(X;\mathbb{Z}/k),$$

which equips $\pi^3(X;\mathbb{Z}/k)$ with a group structure.

Since stable cohomotopy (with $\mathbb{Z}$ or $\mathbb{Z}/k$ coefficients) is a generalised cohomology theory, both $\pi^*_S(X)$ and $\pi^*_S(X;\mathbb{Z}/k)$ are subject to the splitting result given in Theorem 6.1. Thus Theorem 1.3 leads to the following.

Corollary 1.4 *If X is a 5-dimensional orientable Poincaré duality complex with $H_1(X;\mathbb{Z})$ torsion-free, then there are group isomorphisms*

$$\pi^3(X)\cong\pi^3_S(S^3)^{\oplus n_4}\oplus\pi^3_S(S^4)^{\oplus n_5}\oplus\bigoplus_{t'_i\in T'}\pi^3_S(P^3(t'_i))\oplus\pi^3_S(\mathbb{CP}^2)^{\oplus b}\oplus\bigoplus_{j=1}^{c}\pi^3_S(\mathbb{CP}^2(2^{r_j}))\oplus\pi^3_S(C_f)$$

and

$$\begin{aligned}\pi^3(X;\mathbb{Z}/k)\cong\pi^3_S(S^3;\mathbb{Z}/k)^{\oplus n_4}\oplus\pi^3_S(S^4;\mathbb{Z}/k)^{\oplus n_5}\oplus\bigoplus_{t'_i\in T'}\pi^3_S(P^3(t'_i);\mathbb{Z}/k)\oplus\pi^3_S(\mathbb{CP}^2;\mathbb{Z}/k)^{\oplus b}\\ \oplus\bigoplus_{j=1}^{c}\pi^3_S(\mathbb{CP}^2(2^{r_j});\mathbb{Z}/k)\oplus\pi^3_S(C_f;\mathbb{Z}/k).\end{aligned}$$

Information on the summands appearing on the right-hand side of these isomorphisms is given in Sections 6.2.1 and 6.2.2.

Finally, we give partial information about the set $\pi^2(X)$. Our results owe a heavy debt to Taylor's work [35], but appear to be new. In light of Theorem 1.3, the next result implies that $\pi^2(X)$ is often determined by stable data.

Theorem 1.5 *Let X be a closed, orientable, 5-dimensional manifold.*

(1) *If X is simply connected, then there is a bijection*

$$\pi^2(X)\cong H^2(X)\times\pi^3(X).$$

(2) *If $H_1(X;\mathbb{Z})$ is torsion-free and $H_2(X;\mathbb{Z})$ is torsion, then $\eta\colon S^3 \to S^2$ induces a bijection*

$$\pi^2(X) \cong \pi^3(X).$$

Further results regarding $\pi^2(X)$ are found in Section 6.2.3. We remark that a class of nonsimply connected 5-manifolds satisfying the assumptions of Theorem 1.5(2) is studied in [22, Theorem 1.3].

A further comparison with Li and Zhu's work [25] is in order. Using similar methods, [25] includes a suspension splitting of smooth 5-manifolds similar to our Theorem 1.1, but under the more general assumption that $H_1(X)$ contains no 2- or 3-torsion. A point of departure is that Li and Zhu draw from the theory of elementary Chang complexes [4, §10], whereas our methods take advantage of the theory of Poincaré duality and characteristic classes, leading to some refinements in Section 5 not found in [25] (compare, for instance, the spin case of Theorem 1.2). Both papers offer applications to cohomotopy and generalised cohomology: our attention to cohomotopy is more extensive, including results on $\pi^2(X)$ and cohomotopy sets with coefficients not considered in [25], while [25] includes precise splittings induced at the level of K- and KO-theory which we do not derive here beyond a general statement for generalised cohomology in Section 6.1.

The paper is structured as follows. Section 2 contains a selection of lemmas as well as computations of certain homotopy groups which will be necessary in the sequel. In Section 3 we outline the proof of Theorem 1.1 and establish notation which will be used in the longer, more technical Section 4. This section is broken into two subsections which contain the details necessary to complete the proof of Theorem 1.1. In Section 5 we analyse the homotopy type of ΣX using Poincaré duality and prove Theorem 1.2.

Applications of Theorem 1.1 are given in Section 6. Generalised cohomology groups of X are evaluated in Section 6.1, and certain cohomotopy groups of X are calculated in Section 6.2. A proof of Theorem 1.5 is found in this final subsection, as well as the details required to prove Theorem 1.3 and Corollary 1.4.

2 Preliminaries

Throughout the paper we assume that all spaces are path-connected and equipped with basepoints which are preserved by all maps. All homology and cohomology groups are reduced and taken with integer coefficients unless otherwise specified.

2.1 Homotopy types of mapping cones

Given maps $f_i\colon S^n \to A_i$ for $1 \le i \le a$, write $C(f_1, \dots, f_a)$ for the mapping cone of the map

$$\sum_{i=1}^{a} (j_i \circ f_i)\colon S^n \to \bigvee_{i=1}^{a} A_i,$$

where $j_k\colon A_k \hookrightarrow \bigvee_{i=1}^{a} A_i$ is the inclusion. Observe that in the notation introduced in (2) we have $\sum_{i=1}^{a} (j_i \circ f_i) \simeq f_1 \perp \cdots \perp f_a$. Thus $C(f_1, \dots, f_a)$ is also the mapping cone of $f_1 \perp \cdots \perp f_a$.

Lemma 2.1 *Suppose each A_i is the suspension of a connected CW complex and each map f_i is a suspension. Then for any map $g\colon A_k \to A_l$, there is a homotopy equivalence*

$$C(f_1,\dots,f_a) \simeq C(f_1,\dots,f_{l-1}, f_l + (g\circ f_k), f_{l+1},\dots,f_a).$$

Proof Without loss of generality, we may assume that $k=1$ and $l=a=2$. Let $c\colon S^n \to S^n \vee S^n$ be the comultiplication. Since f_1 is a suspension, the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f_1} & A_1 \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ S^n\vee S^n & \xrightarrow{f_1\vee f_1} & A_1\vee A_1 \end{array}$$

homotopy commutes and fits in the middle of the following homotopy commutative diagram:

$$\begin{array}{ccccccccc} S^n & \xrightarrow{c} & S^n\vee S^n & \xrightarrow{f_1\vee f_2} & A_1\vee A_2 & & \xrightarrow{\Phi} & & A_1\vee A_2 \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c\vee 1} & & \downarrow{\scriptstyle c\vee 1} & & & & \| \\ S^n\vee S^n & \xrightarrow{1\vee c} & S^n\vee S^n\vee S^n & \xrightarrow{f_1\vee f_1\vee f_2} & A_1\vee A_1\vee A_2 & \xrightarrow{1\vee g\vee 1} & A_1\vee A_2\vee A_2 & \xrightarrow{1\vee\nabla} & A_1\vee A_2 \end{array}$$

where ∇ is the folding map. Here, the left-hand square homotopy commutes due to the coassociativity of the comultiplication c, and Φ is defined by the right-hand triangle.

Let F, $F'\colon S^n \to A_1\vee A_2$ be the maps defined by $F=(f_1\vee f_2)\circ c \simeq \jmath_1\circ f_1 + \jmath_2\circ f_2$ and $F' = \jmath_1\circ f_1 + \jmath_2\circ(f_2 + g\circ f_1)$. Since F' is obtained by following the diagram anticlockwise, we obtain a homotopy $\Phi\circ F\simeq F'$. This witnesses the homotopy commutativity of the following diagram in which the rows are cofibre sequences and $\widetilde{\Phi}$ is an induced map of cofibres:

$$\begin{array}{ccccc} S^n & \xrightarrow{F} & A_1\vee A_2 & \longrightarrow & C(f_1,f_2) \\ \| & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\widetilde{\Phi}} \\ S^n & \xrightarrow{F'} & A_1\vee A_2 & \longrightarrow & C(f_1, f_2+g\circ f_1) \end{array}$$

Since Φ is a homology equivalence, the five lemma implies that $\widetilde{\Phi}$ is as well. Since all spaces are simply connected, Whitehead's theorem implies that $\widetilde{\Phi}$ is a homotopy equivalence. □

2.2 Some homotopy groups

For $n\ge 3$ and $r,s\ge 1$ let $\imath_{rs}\colon P^n(2^r)\to P^n(2^s)$ be defined as follows. In case $r\ge s$, it is defined as the induced map of cofibres in the diagram

(3)
$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{2^r} & S^{n-1} & \xrightarrow{\imath_r} & P^n(2^r) \\ \downarrow{\scriptstyle 2^{r-s}} & & \| & & \downarrow{\scriptstyle \imath_{rs}} \\ S^{n-1} & \xrightarrow{2^s} & S^{n-1} & \xrightarrow{\imath_s} & P^n(2^s) \end{array}$$

where $\iota_r\colon S^{n-1}\to P^n(2^r)$ is the inclusion of the bottom cell. When $r<s$, the map ι_{rs} is similarly defined by the diagram

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{2^r} & S^{n-1} & \xrightarrow{\iota_r} & P^n(2^r) \\ \| & & \downarrow{\scriptstyle 2^{s-r}} & & \downarrow{\scriptstyle \iota_{rs}} \\ S^{n-1} & \xrightarrow{2^s} & S^{n-1} & \xrightarrow{\iota_s} & P^n(2^s) \end{array} \tag{4}$$

Lemma 2.2 *Let $\eta\colon S^{n+1}\to S^n$ be the Hopf map.*

(i) *If t is odd, then $\pi_4(P^4(t))=\pi_5(P^4(t))=0$.*

(ii) *If $t=2^r$, then $\pi_4(P^4(2^r))\cong\mathbb{Z}/2\langle\iota_r\circ\eta\rangle$ and*

$$\pi_4(P^3(2^r))\cong\begin{cases}\mathbb{Z}/4\langle\xi_1\rangle & \text{for } r=1,\\ \mathbb{Z}/2\langle\xi_r\rangle\oplus\mathbb{Z}/2\langle\iota_r\circ\eta^2\rangle & \text{for } r\geq 2,\end{cases}$$

where ξ_r is a lift of $\eta\colon S^4\to S^3$ through $P^3(2^r)$ and satisfies

$$2\xi_1\simeq\iota_1\circ\eta^2 \quad\text{and}\quad \xi_r=\iota_{1r}\circ\xi_1.$$

(iii) *The suspension $\Sigma\colon\pi_n(P^{n-1}(2^r))\to\pi_{n+1}(P^n(2^r))$ is an isomorphism for $n\geq 4$.*

Proof The homotopy groups in (i) and (ii) are given in [4, Chapter 11.1 and Theorem 11.5.9], where the Moore space $P^n(t)$ is denoted $M(\mathbb{Z}/t,n-1)$, with the special notation ΣP_t for $P^3(t)$.

For $n\geq 5$, (iii) follows from the Freudenthal suspension theorem. We need to show that it holds for $n=4$. By [4, Proposition 11.1.12], the double suspension

$$\pi_4(P^3(2^r))\xrightarrow{\Sigma}\pi_5(P^4(2^r))\xrightarrow{\Sigma}\pi_6(P^5(2^r))$$

is an isomorphism. Since the second Σ is an isomorphism, so is the first. □

For $r\geq 1$, let $\mathbb{CP}^2(2^r)$ be the mapping cone of $S^3\xrightarrow{\eta}S^2\xrightarrow{\iota_r}P^3(2^r)$ and let $\jmath_r\colon P^3(2^r)\to\mathbb{CP}^2(2^r)$ be the inclusion.

Lemma 2.3 *Let $q\colon\Sigma\mathbb{CP}^2\to S^5$ and $q_r\colon\Sigma\mathbb{CP}^2(2^r)\to S^5$ be the quotient maps. Then*

(i) $\pi_4(\Sigma\mathbb{CP}^2)=\pi_4(\Sigma\mathbb{CP}^2(2^r))=0$;

(ii) $\pi_5(\Sigma\mathbb{CP}^2)\cong\mathbb{Z}\langle\alpha\rangle$ *and* $\pi_5(\Sigma\mathbb{CP}^2(2^r))\cong\mathbb{Z}\langle\alpha_r\rangle\oplus\mathbb{Z}/2\langle\epsilon_r\rangle$, *where α and α_r satisfy*

$$q_*(\alpha)=(q_r)_*(\alpha_r)=2\in\pi_5(S^5),$$

and $\epsilon_r=\Sigma\jmath_r\circ\xi_r$;

(iii) *for $r,s\geq 1$ there exists a map $\jmath_{rs}\colon\Sigma\mathbb{CP}^2(2^r)\to\Sigma\mathbb{CP}^2(2^s)$ such that $\jmath_{rs}\circ\Sigma\jmath_r\simeq\Sigma\jmath_s\circ\iota_{rs}$ and*

$$\jmath_{rs}\circ\epsilon_r\simeq\begin{cases}* & \text{if } r>s,\\ \epsilon_s & \text{if } r\leq s.\end{cases}$$

Proof We first show (i) and (ii). Applying the Blakers–Massey theorem to the cofibration

$$S^4 \xrightarrow{\iota_r\circ\eta} P^4(2^r) \xrightarrow{\Sigma j_r} \Sigma\mathbb{C}\mathbb{P}^2(2^r)$$

yields an exact sequence

$$(5)\quad \pi_5(S^4) \xrightarrow{(\iota_r\circ\eta)_*} \pi_5(P^4(2^r)) \xrightarrow{(\Sigma j_r)_*} \pi_5(\Sigma\mathbb{C}\mathbb{P}^2(2^r)) \xrightarrow{\partial} \pi_4(S^4) \\ \xrightarrow{(\iota_r\circ\eta)_*} \pi_4(P^4(2^r)) \xrightarrow{(\Sigma j_r)_*} \pi_4(\Sigma\mathbb{C}\mathbb{P}^2(2^r)) \to 0.$$

By Lemma 2.2, $\pi_4(P^4(2^r)) \cong \mathbb{Z}/2\langle \iota_r\circ\eta\rangle$ so the second $(\iota_r\circ\eta)_*$ is surjective. It follows that $\pi_4(\Sigma\mathbb{C}\mathbb{P}^2(2^r))$ is trivial and $\pi_5(\Sigma\mathbb{C}\mathbb{P}^2(2^r))$ contains a $\mathbb{Z}$ summand with a generator α_r satisfying $\partial(\alpha_r) = 2 \in \pi_4(S^4)$. Since the boundary map ∂ is induced by the quotient map q_r, there is a homotopy $q_r\circ\alpha_r \simeq 2\colon S^5 \to S^5$.

We extract from (5) the short exact sequence

$$0 \longrightarrow \operatorname{coker}(\iota_r\circ\eta)_* \xrightarrow{(\Sigma j_r)_*} \pi_5(\Sigma\mathbb{C}\mathbb{P}^2(2^r)) \xrightarrow{\partial} \operatorname{Im}(\partial) \cong \mathbb{Z}\langle\alpha_r\rangle \longrightarrow 0,$$

where $(\iota_r\circ\eta)_*$ is the first map appearing in (5). We will show that $\operatorname{coker}(\iota_r\circ\eta)_* \cong \mathbb{Z}/2$, which then implies that $\pi_5(\Sigma\mathbb{C}\mathbb{P}^2(2^r)) \cong \mathbb{Z}\oplus\mathbb{Z}/2$.

In (5), the first $(\iota_r\circ\eta)_*$ takes the generator η of $\pi_5(S^4)\cong\mathbb{Z}/2$ to $\iota_r\circ\eta^2 \in \pi_5(P^4(2^r))$. By Lemma 2.2

$$\pi_5(P^4(2^r)) \cong \begin{cases} \mathbb{Z}/4\langle\xi_1\rangle & \text{if } r=1, \\ \mathbb{Z}/2\langle\xi_r\rangle \oplus \mathbb{Z}/2\langle\iota_r\circ\eta^2\rangle & \text{if } r\geq 2. \end{cases}$$

When $r\geq 2$, we immediately obtain $\operatorname{coker}(\iota_r\circ\eta)_* \cong \mathbb{Z}/2\langle\xi_r\rangle$. When $r=1$, as $\iota_1\circ\eta^2 \simeq 2\xi_1$, $\operatorname{coker}(\iota_1\circ\eta)_*$ is generated by the quotient image of ξ_1, which has order two. Let $\epsilon_r = \Sigma j_r\circ\xi_r$ for $r\geq 1$. Then $\epsilon_r \in \pi_5(\Sigma\mathbb{C}\mathbb{P}^2(2^r))$ has order two and we have

$$\pi_5(\Sigma\mathbb{C}\mathbb{P}^2(2^r)) \cong \mathbb{Z}\langle\alpha_r\rangle \oplus \mathbb{Z}/2\langle\epsilon_r\rangle.$$

Similarly, one applies the Blakers–Massey theorem to the cofibration

$$S^4 \xrightarrow{\eta} S^3 \xrightarrow{\Sigma j} \Sigma\mathbb{C}\mathbb{P}^2$$

to see that $\pi_4(\Sigma\mathbb{C}\mathbb{P}^2)$ is trivial and $\pi_5(\Sigma\mathbb{C}\mathbb{P}^2)\cong\mathbb{Z}\langle\alpha\rangle$, where the generator α satisfies $q\circ\alpha\simeq 2\colon S^5\to S^5$.

Next we show (iii). Given $r,s\in\mathbb{N}$, consider the diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{\eta} & S^3 & \xrightarrow{\iota_r} & P^4(2^r) \\ \downarrow{\scriptstyle 2^t} & & \downarrow{\scriptstyle 2^t} & & \downarrow{\scriptstyle \iota_{rs}} \\ S^4 & \xrightarrow{\eta} & S^3 & \xrightarrow{\iota_s} & P^4(2^s) \end{array}$$

where $t=0$ for $r\geq s$ and $t=s-r$ for $r<s$. The left square homotopy commutes since η here is a suspension, and the right square homotopy commutes due to (3) and (4). Extend the diagram so as

to obtain

$$
\text{(6)}\qquad
\begin{array}{ccccc}
S^4 & \xrightarrow{\imath_r\circ\eta} & P^4(2^r) & \xrightarrow{\Sigma\jmath_r} & \Sigma\mathbb{C}\mathbb{P}^2(2^r) \\
\downarrow{\scriptstyle 2^t} & & \downarrow{\scriptstyle \imath_{rs}} & & \downarrow{\scriptstyle \jmath_{rs}} \\
S^4 & \xrightarrow{\imath_s\circ\eta} & P^4(2^s) & \xrightarrow{\Sigma\jmath_s} & \Sigma\mathbb{C}\mathbb{P}^2(2^s)
\end{array}
$$

where the rows are cofibre sequences, and $\jmath_{rs}$ is an induced map. We show that $\jmath_{rs}$ has the asserted property.

There is a string of homotopies

$$\jmath_{rs}\circ\epsilon_r \simeq \jmath_{rs}\circ(\Sigma\jmath_r\circ\xi_r) \simeq \jmath_{rs}\circ\Sigma\jmath_r\circ(\imath_{1r}\circ\xi_1) \simeq (\Sigma\jmath_s\circ\imath_{rs})\circ\imath_{1r}\circ\xi_1,$$

where the first two homotopies are due to the definitions of ϵ_r and ξ_r, and the last is due to the right square of (6). Since

$$\imath_{rs}\circ\imath_{1r} \cong \begin{cases} 2^{r-s}\imath_{1s} & \text{if } r > s, \\ \imath_{1s} & \text{if } r \le s, \end{cases}$$

the composite $\jmath_{rs}\circ\epsilon_r$ is null homotopic if $r > s$, and is homotopic to ϵ_s if $r \le s$. □

Remark 2.4 The torsion generator $\epsilon_r \in \pi_5(\Sigma\mathbb{C}\mathbb{P}^2(2^r))$ is a suspension since $\Sigma\jmath_r$ and ξ_r are both suspensions by Lemma 2.2.

2.3 A test for the nullity of Whitehead products

For spaces A and B, let

$$[1,1]\colon \Sigma A\wedge B \to \Sigma A \vee \Sigma B$$

denote the universal Whitehead product, that is, the Whitehead product of the inclusions of ΣA and ΣB into $\Sigma A\vee\Sigma B$, whose cofibre is $\Sigma A\times\Sigma B$.

Lemma 2.5 *For $n \ge 2$ let $f\colon S^n \to \Sigma A\wedge B$ be a map, and let C be the mapping cone of the composite*

$$S^n \xrightarrow{f} \Sigma A\wedge B \xrightarrow{[1,1]} \Sigma A\vee\Sigma B.$$

Suppose $\Sigma A\wedge B$ is $(n-1)$-connected and $H_n(\Sigma A\wedge B;\mathbb{Z})$ is a cyclic group. If all cup products in $H^*(C;R)$ are trivial for any ring R, then $f\simeq *$.

Proof Since $\Sigma A\wedge B$ is $(n-1)$-connected, the Hurewicz theorem implies that f is null homotopic if and only if the induced map

$$f_*\colon H_n(S^n;\mathbb{Z}) \to H_n(\Sigma A\wedge B;\mathbb{Z})$$

is trivial. By assumption $H_n(\Sigma A\wedge B;\mathbb{Z}) \cong \mathbb{Z}/k$ for $k\in\{2,3,\dots,\infty\}$, where $\mathbb{Z}/\infty$ means $\mathbb{Z}$. The universal coefficient theorem then implies that f_* is trivial if and only if

$$f^*\colon H^n(\Sigma A\wedge B;\mathbb{Z}/k) \to H^n(S^n;\mathbb{Z}/k)$$

is trivial. We show that the latter holds under the assumption.

Consider the commutative diagram

$$\begin{array}{ccccc} S^n & \xrightarrow{f} & \Sigma A\wedge B & \xrightarrow{q} & C_f \\ \| & & \downarrow{\scriptstyle [1,1]} & & \downarrow \\ S^n & \xrightarrow{[1,1]\circ f} & \Sigma A\vee \Sigma B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \delta} \\ * & \longrightarrow & \Sigma A\times \Sigma B & = & \Sigma A\times \Sigma B \end{array}$$

where q is the quotient map, δ is an induced map, and all columns and rows are cofibre sequences. Apply $H^*(-;\mathbb{Z}/k)$ and suppress coefficients to obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^n(C_f) & \xrightarrow{q^*} & H^n(\Sigma A\wedge B) & \xrightarrow{f^*} & H^n(S^n) \\ & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon'} & & \\ & & H^{n+1}(\Sigma A\times \Sigma B) & = & H^{n+1}(\Sigma A\times \Sigma B) & & \\ & & \downarrow{\scriptstyle \delta^*} & & \downarrow & & \\ & & H^{n+1}(C) & \longrightarrow & H^{n+1}(\Sigma A\vee \Sigma B) & & \end{array}$$

where ϵ and ϵ' are connecting maps. Let $\alpha\in H^n(\Sigma A\wedge B;\mathbb{Z}/k)\cong\mathbb{Z}/k$ be a generator. To show the triviality of f we need to show that α is in $\operatorname{Im}(q^*)$.

Because the inclusion $\Sigma A\vee\Sigma B\to\Sigma A\times\Sigma B$ induces a surjection in cohomology, notice that ϵ' is injective and maps $H^n(\Sigma A\wedge B)\cong H^{n+1}(\Sigma A\wedge\Sigma B)$ onto the subgroup of $H^{n+1}(\Sigma A\times\Sigma B)$ generated by cup products of the form $\mathrm{pr}_1^*(x)\cup\mathrm{pr}_2^*(y)$, where $\mathrm{pr}_1\colon\Sigma A\times\Sigma B\to\Sigma A$ and $\mathrm{pr}_2\colon\Sigma A\times\Sigma B\to\Sigma B$ are projections. Since all cup products in $H^*(C)$ are assumed to be trivial, we therefore have $\delta^*(\epsilon'(\alpha))=0$. Now $\epsilon=\epsilon'\circ q^*$ is injective, so there is a unique class $\tilde\alpha\in H^n(C_f)$ such that $\epsilon(\tilde\alpha)=\epsilon'(\alpha)$. In particular $\alpha=q^*(\tilde\alpha)$. □

3 Strategy for proving the main theorem

Suppose the homology groups of X are given by (1). Let $\varphi\colon\bigvee_{i=1}^m S^1\to M$ be a map inducing an isomorphism on $H_1(\cdot)$, and let X' be the mapping cone of φ. Then the homology groups of X' are given by

i	0	1	2	3	4	5
$H_i(X')$	$\mathbb{Z}$	0	$\mathbb{Z}^n\oplus G$	$\mathbb{Z}^n$	$\mathbb{Z}^m$	$\mathbb{Z}$

Using the argument in [31, Lemma 5.1] we obtain a homotopy equivalence

$$\Sigma X\simeq\bigvee_{i=1}^m S^2\vee\Sigma X'. \tag{7}$$

Hence it suffices to study the homotopy type of $\Sigma X'$. Since $\Sigma X'$ is simply connected, it has a minimal cell structure which can be constructed as follows.

- Let $W_2 = \bigvee_{i=1}^{n} S^3 \vee \bigvee_{j=1}^{\ell} P^4(t_j)$.
- For $3 \le i \le 5$, let W_i be the mapping cone of a map

(8) $$\varphi_i\colon \bigvee_{j=1}^{n_i} S^i \longrightarrow W_{i-1},$$

where $n_i = \operatorname{rank}(H_i(X))$ is the i-th Betti number of X and φ_i induces a trivial homomorphism in homology.

- Then $W_5 \simeq \Sigma X'$ by [2, Theorem 7.3.2].

We will show that each of the W_i is homotopy equivalent to a wedge of recognisable spaces, and hence obtain Theorem 1.1. We begin with W_3.

Lemma 3.1 *There is a homotopy equivalence*

$$W_3 \simeq \bigvee_{i=1}^{n} (S^3 \vee S^4) \vee \bigvee_{j=1}^{\ell} P^4(t_j).$$

Proof Since W_3 is the mapping cone of $\varphi_3\colon \bigvee_{i=1}^{n} S^3 \to W_2$ given in (8), it suffices to show that φ_3 is null homotopic. Consider the commutative diagram

$$\begin{array}{ccc} \pi_3\big(\bigvee_{i=1}^{n} S^3\big) & \xrightarrow{[\varphi_3]_*} & \pi_3(W_2) \\ \big\downarrow h & & \big\downarrow h \\ H_3\big(\bigvee_{i=1}^{n} S^3\big) & \xrightarrow{(\varphi_3)_*} & H_3(W_2) \end{array}$$

where the vertical maps labelled h are Hurewicz morphisms. Since W_2 is 2-connected, the Hurewicz theorem implies that the two h's are isomorphisms. Since $(\varphi_3)_*$ is trivial by assumption, $\varphi_3 \simeq [\varphi_3]_*(\mathrm{id})$ is null homotopic. □

The calculations needed for W_4 and W_5 are significantly longer and are completed in the next section.

4 Homotopy types of W_4 and W_5

Let $\mathscr{W}$ be the collection of spaces that are of the form

$$(S^3)^{\vee n_3} \vee (S^4)^{\vee n_4} \vee (S^5)^{\vee n_5} \vee \bigvee_{i=1}^{a} P^4(2^{r_i}) \vee \bigvee_{t\in\mathcal{T}} P^4(t) \vee (\Sigma\mathbb{C}\mathbb{P}^2)^{\vee b} \vee \bigvee_{j=1}^{c} \Sigma\mathbb{C}\mathbb{P}^2(2^{s_j}),$$

where n_3, n_4, n_5, b are nonnegative integers, $\mathcal{T}$ is a collection of odd numbers, and $\{r_i\}_{i=1}^a$ and $\{s_j\}_{j=1}^c$ are sequences of positive integers such that $r_i \le r_{i+1}$ and $s_j \le s_{j+1}$. We will show that W_4 is homotopy equivalent to a wedge in $\mathscr{W}$ in Lemma 4.5, and compute the homotopy type of W_5 in Lemma 4.8.

We begin with a useful lemma for later calculations.

Lemma 4.1 *Fix a wedge $W \in \mathscr{W}$. Let $\varphi\colon S^d \to W$ be a map where $d = 4$ or 5, and let C_φ be its mapping cone. If φ induces the trivial morphism in homology and all cup products in $\widetilde{H}^*(C_\varphi; R)$ are trivial for any ring R, then there is a homotopy*

$$\varphi \simeq \sum_A \varphi(A),$$

where A runs over all wedge summands in W such that $A \ne S^d$, and each $\varphi(A)$ is

$$\varphi(A)\colon S^d \xrightarrow{\varphi} W \xrightarrow{\text{pinch}} A \hookrightarrow W.$$

Proof Let $\mathscr{A}$ be the collection of wedge summands in W. Since each wedge summand A is a suspension and is at least 2-connected, the Hilton–Milnor theorem implies that

$$\varphi \simeq \sum_{A \in \mathscr{A}} \varphi(A) + \sum_{\substack{A, A' \in \mathscr{A} \\ A \ne A'}} w(A, A'),$$

where $w(A, A')\colon S^d \to \Sigma^{-1} A \vee A'$ is the composite

$$w(A, A')\colon S^d \xrightarrow{f} \Sigma^{-1} A \wedge A' \xrightarrow{[1,1]} A \vee A' \hookrightarrow W$$

of a map $f\colon S^d \to \Sigma^{-1} A \wedge A'$, the Whitehead product $[1, 1]$, and the inclusion $A \vee A' \hookrightarrow W$.

For those $A = S^d$, the map $\varphi(A)$ is null homotopic since φ_* is trivial in homology by assumption. It remains to show that each $w(A, A')$ is null homotopic.

Suppose $d = 4$. Note that each $\Sigma^{-1} A \wedge A'$ is at least 4-connected. Hence $f\colon S^4 \to \Sigma^{-1} A \wedge A'$ is null homotopic and so is $w(A, A')$.

Suppose $d = 5$. Consider the composite

$$\varphi_{A,A'}\colon S^5 \xrightarrow{\varphi} W \xrightarrow{\text{pinch}} A \vee A'.$$

We have

$$\varphi_{A,A'} \simeq \pi \circ \varphi(A) + \pi \circ \varphi(A') + [1, 1] \circ f,$$

where $\pi\colon W \to A \vee A'$ is the pinch map. Let C_A, $C_{A'}$, $C_{AA'}$ and C_w be the mapping cones of $\pi \circ \varphi(A)$, $\pi \circ \varphi(A')$, $\varphi_{A,A'}$ and $[1, 1] \circ f$, respectively. Since cup products in $\widetilde{H}^*(C_\varphi; R)$ are trivial, so are the cup products in $\widetilde{H}^*(C_{AA'}; R)$. Note that $\pi \circ \varphi(A)$ and $\pi \circ \varphi(A')$ are suspensions, implying that cup products in $\widetilde{H}^*(C_A; R)$ and $\widetilde{H}^*(C_{A'}; R)$ are trivial. Then cup products in $\widetilde{H}^*(C_w; R)$ are also trivial by [8, Lemma 2.5] (the dimension assumption of the lemma is not necessary). Since $\Sigma^{-1} A \wedge A'$ satisfies the conditions of Lemma 2.5, f is null homotopic and so is $w(A, A')$. □

4.1 The homotopy type of W_4

Since W_4 is the mapping cone of $\varphi_4\colon \bigvee_{i=1}^m S_i^4 \to W_3$ given in (8), it can be obtained by adding a 5-cell to W_3 iteratively. Observe that

$$W_3 \simeq \bigvee_{j=1}^{n} (S^3 \vee S^4) \vee \bigvee_{k=1}^{\ell} P^4(t_k)$$

is a wedge in $\mathscr{W}$. To prove that W_4 is in $\mathscr{W}$, it suffices to show that the complex obtained by attaching a 5-cell is always homotopy equivalent to a wedge in $\mathscr{W}$.

Consider a wedge $W \in \mathscr{W}$ and label its 3-spheres, that is

$$W = \bigvee_{i=1}^{n_3} S_i^3 \vee \bigvee_{j=1}^{n_4} S^4 \vee \bigvee_{k=1}^{n_5} S^5 \vee \bigvee_{u=1}^{a} P^4(2^{r_u}) \vee \bigvee_{t\in\mathcal{T}} P^4(t) \vee \bigvee_{l=1}^{b} \Sigma\mathbb{C}P^2 \vee \bigvee_{v=1}^{c} \Sigma\mathbb{C}P^2(2^{s_v}). \tag{9}$$

Given any map $\varphi\colon S^4 \to W$, we claim that its mapping cone C_φ is also a wedge in $\mathscr{W}$ if φ induces the trivial morphism in homology.

Lemma 4.2 *Let W be a wedge in $\mathscr{W}$ as in (9) and let $\varphi\colon S^4 \to W$ be a map. If φ induces the trivial morphism in homology, then*

$$\varphi \simeq \sum_{i=1}^{n_3} \varphi(S_i^3) + \sum_{u=1}^{a} \varphi(P^4(2^{r_u})).$$

Proof Notice that all cup products in $\widetilde{H}^*(C_\varphi)$ are trivial as $H^i(C_\varphi) = 0$ for $i = 1, 2$. By Lemma 4.1 it suffices to show that the components

$$\varphi(S^5), \quad \varphi(P^4(t)), \quad \varphi(\Sigma\mathbb{C}P^2), \quad \varphi(\Sigma\mathbb{C}P^2(2^s))$$

are null homotopic. This holds due to the fact that $\pi_4(S^5) = 0$ and Lemmas 2.2 and 2.3. □

Recall from Lemma 2.2 that

$$\pi_4(S^3) \cong \mathbb{Z}/2\langle\eta\rangle \quad \text{and} \quad \pi_4(P^4(2^r)) \cong \mathbb{Z}/2\langle\iota_r \circ \eta\rangle,$$

where η is the suspended Hopf map and $\iota_r\colon S^3 \to P^4(2^r)$ is the inclusion of the bottom cell. For $1 \le i \le n_3$ and $1 \le u \le a$, let η_i and $\bar{\eta}_u$ be the composites

$$\eta_i\colon S^4 \xrightarrow{\eta} S_i^3 \hookrightarrow W \quad \text{and} \quad \bar{\eta}_u\colon S^4 \xrightarrow{\iota_{r_u}\circ\eta} P^4(2^{r_u}) \hookrightarrow W.$$

Then Lemma 4.2 implies that there are some $\mathbb{Z}/2$-coefficients x_i, y_u such that

$$\varphi \simeq \sum_{i=1}^{n_3} x_i\eta_i + \sum_{u=1}^{a} y_u\bar{\eta}_u. \tag{10}$$

Following the idea in [8; 31], we may choose a different map $\varphi'\colon S^4 \to W$ such that $C_\varphi \simeq C_{\varphi'}$.

Lemma 4.3 *Let $C(x_1,\dots,x_{n_3};y_1,\dots,y_a)$ be the mapping cone of $\sum_{i=1}^{n_3} x_i\eta_i+\sum_{u=1}^{a} y_u\overline{\eta}_u$. Then $C(x_1,\dots,x_{n_3};y_1,\dots,y_a)$ is homotopy equivalent to*

(i) $C(x_{\sigma(1)},\dots,x_{\sigma(n_3)};y_1,\dots,y_a)$ *for any permutation* $\sigma\colon\{1,\dots,n_3\}\to\{1,\dots,n_3\}$;

(ii) $C(x_1,\dots,x_{l-1},x_k+x_l,x_{l+1},\dots,x_{n_3};y_1,\dots,y_a)$ *for some distinct* $k,l\in\{1,\dots,n_3\}$;

(iii) $C(x_1,\dots,,x_{n_3};y_1,\dots,y_{l-1},x_k+y_l,y_{l+1},\dots,y_a)$ *for some* $k\in\{1,\dots,n_3\}$ *and* $l\in\{1,\dots,a\}$;

(iv) $C(x_1,\dots,,x_{n_3};y_1,\dots,y_{l-1},y_k+y_l,y_{l+1},\dots,y_a)$ *for some distinct* $k,l\in\{1,\dots,a\}$ *such that* $k\ge l$.

Proof First we prove (i). Let $\varphi'=\sum_{i=1}^{n_3} x_{\sigma(i)}\eta_i+\sum_{u=1}^{a} y_u\overline{\eta}_u$, and let $\sigma\colon W\to W$ be the map which permutes the wedge summands by mapping each S^3_i to $S^3_{\sigma(i)}$ and mapping the other wedge summands onto themselves by the identity map. Then there is a commutative diagram of cofibre sequences

$$\begin{array}{ccccc} S^4 & \xrightarrow{\varphi} & W & \longrightarrow & C(x_1,\dots,x_{n_3};y_1,\dots,y_a) \\ \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tilde{\sigma}} \\ S^4 & \xrightarrow{\varphi'} & W & \longrightarrow & C(x_{\sigma(1)},\dots,x_{\sigma(n_3)};y_1,\dots,y_a) \end{array}$$

where $\tilde{\sigma}$ is an induced map. Since σ is a homology isomorphism, so is $\tilde{\sigma}$ by the five lemma. Consequently, Whitehead's theorem implies that $\tilde{\sigma}$ is a homotopy equivalence.

Next we prove (ii) to (iv). Use Lemma 2.1 and take $g\colon A_k\to A_l$ to be

- the identity map $A_k\to A_l$ if $A_k=A_l$;
- the inclusion $\iota_{r_l}\colon S^3\to P^4(2^{r_l})$ if $A_k=S^3$ and $A_l=P^4(2^{r_l})$;
- the map $\iota_{r_kr_l}\colon P^4(2^{r_k})\to P^4(2^{r_l})$ in (3) if $A_k=P^4(2^{r_k})$, $A_l=P^4(2^{r_l})$ and $k\ge l$.

Then the homotopy equivalences follow immediately. □

Lemma 4.4 *Under the assumption of Lemma 4.2, C_φ is a wedge in $\mathscr{W}$.*

Proof By (10) we have $C_\varphi\simeq(x_1,\dots,x_{n_3};y_1,\dots,y_a)$ for some $\mathbb{Z}/2$ coefficients x_i and y_j. We divide the proof into 3 cases.

Case 1: If all x_i's and y_j's are zero, then φ is null homotopic and $C_\varphi\simeq W\vee S^5$ is in $\mathscr{W}$.

Case 2: Suppose $x_i=1$ for some $i\in\{1,\dots,n_3\}$. By Lemma 4.3(i) we assume it is x_1. If there are nonzero coefficients x_j and y_u in (10), then Lemma 4.3(ii) and (iii) implies that $C(x_1,\dots,x_{n_3};y_1,\dots,y_a)$ is homotopy equivalent to

$$C(x_1,\dots,x_{j-1},x_j+x_1,x_{j+1},\dots,x_{n_3};y_1,\dots,y_{u-1},y_u+x_1,y_{u+1},\dots,y_a).$$

Note that x_j+x_1 and y_u+x_1 are both zero. Repeat the argument to annihilate other nonzero coefficients. In the end we have $C_\varphi\simeq C(1,0,\dots,0;0,\dots,0)$. Observe that it is the mapping cone of η_1 and is homotopy equivalent to

$$(S^3)^{\vee(n_3-1)}\vee(S^4)^{\vee n_4}\vee(S^5)^{\vee n_5}\vee\bigvee_{u=1}^{a}P^4(2^{r_u})\vee\bigvee_{t\in\mathcal{T}}P^4(t)\vee(\Sigma\mathbb{C}\mathbb{P}^2)^{\vee(b+1)}\vee\bigvee_{v=1}^{c}\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v}).$$

Therefore C_φ is in $\mathscr{W}$.

Case 3: Suppose $x_i=0$ for all i and $y_u=1$ for some $u\in\{1,\dots,a\}$. Let μ be the largest index of those nonzero y_u. Then $C_\varphi=C(0,\dots,0;y_1,\dots,y_\mu,0,\dots,0)$ where $y_\mu=1$. Use Lemma 4.3(iv) to annihilate other nonzero y_u. In the end we have

$$C_\varphi\simeq C(0,\dots,0;0,\dots,0,y_\mu,0,\dots,0),$$

which is the mapping cone of $\tilde{\eta}_\mu$ and is homotopy equivalent to

$$(S^3)^{\vee n_3}\vee(S^4)^{\vee n_4}\vee(S^5)^{\vee n_5}\vee\bigvee_{\substack{1\le u\le a\\ u\neq\mu}}P^4(2^{r_u})\vee\bigvee_{t\in\mathcal{T}}P^4(t)\vee(\Sigma\mathbb{C}\mathbb{P}^2)^{\vee b}\vee\bigvee_{v=1}^{c}\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v})\vee\Sigma\mathbb{C}\mathbb{P}^2(2^{r_\mu}).$$

Therefore C_φ is in $\mathscr{W}$. □

Now we have all the ingredients to show that W_4 is in $\mathscr{W}$.

Lemma 4.5 *There is a homotopy equivalence*

$$W_4\simeq(S^3)^{\vee n_3}\vee(S^4)^{\vee n_4}\vee(S^5)^{\vee n_5}\vee\bigvee_{u=1}^{a}P^4(2^{r_u})\vee\bigvee_{t\in\mathcal{T}}P^4(t)\vee(\Sigma\mathbb{C}\mathbb{P}^2)^{\vee b}\vee\bigvee_{v=1}^{c}\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v})$$

for some integers $n_3,n_4,n_5,a,b,c,r_u,s_v$ and some collection $\mathcal{T}$ of odd numbers.

Proof Let $\{Y_i\}_{1\le i\le m}$ be a sequence of spaces where $Y_0=W_3$ and Y_i is the mapping cone of

$$f_i\colon S_i^4\hookrightarrow\bigvee_{j=1}^{m}S_j^4\xrightarrow{\varphi_4}W_3\hookrightarrow Y_{i-1}$$

for $1\le i\le m$. Then $Y_m\simeq W_4$. We prove that each Y_i is in $\mathscr{W}$ by induction on i.

By Lemma 3.1, $Y_0=W_3$ is homotopy equivalent to

$$W_3\simeq\bigvee_{j=1}^{n}(S^3\vee S^4)\vee\bigvee_{t\in T}P^4(t),$$

where $T=\{p^r\}$ is a collection of powers of primes. Let

$$T_{\mathrm{odd}}=\{p^r\in T\mid p\text{ is an odd prime}\}\quad\text{and}\quad T_{\mathrm{even}}=\{p^r\in T\mid p=2\}$$

be the subsets consisting of powers of odd and even primes, respectively. Further, let

$$T_{\mathrm{even}}=\{2^{r_1},\dots,2^{r_a}\}$$

such that $r_u\le r_{u+1}$. Therefore $Y_0\simeq\bigvee_{j=1}^{m}(S^3\vee S^4)\vee\bigvee_{u=1}^{a}P^4(2^{r_u})\vee\bigvee_{t\in T_{\mathrm{odd}}}P^4(t)$ and hence is in $\mathscr{W}$.

Assume that Y_i is in $\mathscr{W}$. Let Y_{i+1} be the mapping cone of $f_{i+1}\colon S^4 \to Y_i$. Since φ_4 induces the trivial morphism in homology, so does f_i. Lemmas 4.2 and 4.4 imply that Y_{i+1} is in $\mathscr{W}$.

By induction all Y_i' s are in $\mathscr{W}$ and, in particular, so is $W_4 = Y_m$. □

4.2 The homotopy type of W_5

Recall that W_5 is the mapping cone of $\varphi_5\colon S^5 \to W_4$ given in (8) such that φ_5 induces the trivial morphism in homology. Since $W_5 \simeq \Sigma X'$ is a suspension, all cup products in $\widetilde{H}^*(W_5)$ are trivial.

Lemma 4.6 *Label the* 3*-spheres and* 4*-spheres in* W_4 *by* S_i^3 *and* S_j^4*. Then*

$$\varphi_5 \simeq \sum_{i=1}^{n_3} \varphi(S_i^3) + \sum_{j=1}^{n_4} \varphi(S_j^4) + \sum_{u=1}^{a} \varphi(P^4(2^{r_u})) + \sum_{v=1}^{c} \varphi(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v})).$$

Furthermore, each $\varphi(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v}))$ *is in* $\mathbb{Z}/2\langle\epsilon_{s_v}\rangle$.

Proof Apply Lemma 4.1 to obtain $\varphi_5 \simeq \sum_{A\neq S^5} \varphi(A)$. By Lemma 2.2 each $\varphi(P^4(t))$ is null homotopic for t odd, so we have

$$\varphi_5 \simeq \sum_{i=1}^{n_3} \varphi(S_i^3) + \sum_{j=1}^{n_4} \varphi(S_j^4) + \sum_{u=1}^{a} \varphi(P^4(2^{r_u})) + \sum_{k=1}^{b} \varphi(\Sigma\mathbb{C}\mathbb{P}^2) + \sum_{v=1}^{c} \varphi(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v})).$$

By Lemma 2.3 each $\varphi(\Sigma\mathbb{C}\mathbb{P}^2) \simeq A\alpha$ and $\varphi(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v})) \simeq B\alpha_{s_v} + C\epsilon_{s_v}$ for some $A, B \in \mathbb{Z}$ and $C \in \mathbb{Z}/2$. It remains to show that $A = B = 0$.

Let $q_{s_v}\colon \Sigma\mathbb{C}\mathbb{P}^2(2^{s_v}) \to S^5$ be the quotient map. Observe that the composite

$$S^5 \xrightarrow{\varphi(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v}))} W_4 \xrightarrow{\text{pinch}} \Sigma\mathbb{C}\mathbb{P}^2(2^{s_v}) \xrightarrow{q_{s_v}} S^5$$

is homotopic to $q_{s_v} \circ \alpha_{s_v}$ which has degree 2 by Lemma 2.3. The induced morphism

$$\varphi(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v}))_*\colon H_5(S^5) \cong \mathbb{Z} \to H_5(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v})) \cong \mathbb{Z}$$

is given by $1 \mapsto \pm 2B$. Since by assumption φ_5 induces the trivial homology morphism, so does $\varphi(\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v}))$ and B has to be zero. A similar argument shows that $A = 0$. □

Lemma 4.6 says that φ_5 decomposes into a sum of maps that factor through the wedge summands $P^4(2^{r_u})$, S_i^3, S_j^4, $\Sigma\mathbb{C}\mathbb{P}^2(2^{s_v})$ of W_4. Define $\tilde{\eta}_i, \tilde{\xi}_j, \tilde{\epsilon}_k\colon S^5 \to W_4$ as follows:

- $\tilde{\eta}_i$ is $\begin{cases} S^5 \xrightarrow{\imath\circ\eta^2} P^4(2^{r_i}) \hookrightarrow W_4 & \text{for } 1 \le i \le a, \\ S^5 \xrightarrow{\eta^2} S^3_{i-a} \hookrightarrow W_4 & \text{for } a+1 \le i \le a+n_3, \\ S^5 \xrightarrow{\eta} S^4_{i-a-n_3} \hookrightarrow W_4 & \text{for } a+n_3+1 \le i \le a+n_3+n_4. \end{cases}$
- $\tilde{\xi}_j$ is $S^5 \xrightarrow{\xi_{r_j}} P^4(2^{r_j}) \hookrightarrow W_4$ for $1 \le j \le a$.
- $\tilde{\epsilon}_k$ is $S^5 \xrightarrow{\epsilon_{s_k}} \Sigma\mathbb{C}\mathbb{P}^2(2^{s_k}) \hookrightarrow W_4$ for $1 \le k \le c$.

To keep track of which wedge summands these maps factor through, we define $\{r(i)\}_{i=1}^{1+n_3+n_4}$ and $\{s(j)\}_{j=1}^{c}$ by

$$r(i)=\begin{cases} r_i & \text{for } 1\le i\le a,\\ \omega & \text{for } a+1\le i\le a+n_3,\\ \omega+1 & \text{for } a+n_3+1\le i\le a+n_3+n_4,\end{cases}\qquad\text{and}\qquad s(j)=s_j\quad\text{for } 1\le j\le c,$$

where ω is the first infinite ordinal. By Lemmas 2.2 and 2.3, we have

$$2\tilde{\eta}_i\simeq *,\quad 2\tilde{\epsilon}_k\simeq *,\quad 2\tilde{\xi}_j\simeq\begin{cases}\tilde{\eta}_j & \text{if } r(j)=1,\\ * & \text{if } r(j)>1.\end{cases}\tag{11}$$

It follows that the attaching map φ_5 can be written as a linear combination

$$\varphi_5\simeq\sum_{i=1}^{a+n_3+n_4}x_i\tilde{\eta}_i+\sum_{j=1}^{a}y_j\tilde{\xi}_j+\sum_{k=1}^{c}z_k\tilde{\epsilon}_k\tag{12}$$

for some coefficients $x_i,y_j,z_k\in\{0,1\}$. We now establish a lemma similar to Lemma 4.3 for φ_5.

Lemma 4.7 *Let $\boldsymbol{x}=(x_i)_{1\le i\le a+n_3+n_4}$, $\boldsymbol{y}=(y_j)_{1\le j\le a}$, and $\boldsymbol{z}=(z_k)_{1\le k\le c}$ be sequences of nonnegative integers, and let $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ be the mapping cone of*

$$\sum_{i=1}^{a+n_3+n_4}x_i\tilde{\eta}_i+\sum_{j=1}^{a}y_j\tilde{\xi}_j+\sum_{k=1}^{c}z_k\tilde{\epsilon}_k.$$

Then the homotopy type of $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ is restricted by the following conditions:

(i) *If $x_i=2$, then $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\simeq C(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_{a+n_3+n_4};\boldsymbol{y};\boldsymbol{z})$.*

(ii) *If $y_j=2$ with $r(j)=1$, then*

$$C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\simeq C(x_1,\dots,x_{j-1},x_j+1,x_{j+1},\dots,x_{a+n_3+n_4};y_1,\dots,y_{j-1},0,y_{j+1},\dots,y_a;\boldsymbol{z}).$$

(iii) *If $y_j=3$ with $r(j)=1$, then*

$$C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\simeq C(x_1,\dots,x_{j-1},x_j+1,x_{j+1},\dots,x_{a+n_3+n_4};y_1,\dots,y_{j-1},1,y_{j+1},\dots,y_a;\boldsymbol{z}).$$

(iv) *If $y_j=2$ with $r(j)>1$, then $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\simeq C(\boldsymbol{x};y_1,\dots,y_{j-1},0,y_{j+1},\dots,y_a;\boldsymbol{z})$.*

(v) *If $z_k=2$, then $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\simeq C(\boldsymbol{x};\boldsymbol{y};z_1,\dots,z_{k-1},0,z_{k+1},\dots,z_c)$.*

Furthermore, $C(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\simeq C(\boldsymbol{x}',\boldsymbol{y}',\boldsymbol{z}')$ if any of the following conditions is satisfied:

(vi) *$\boldsymbol{y}'=\boldsymbol{y}$, $\boldsymbol{z}'=\boldsymbol{z}$, and $\boldsymbol{x}'=(x_1,\dots,x_{l-1},x_k+x_l,x_{l+1},\dots,x_{a+n_3+n_4})$ for some distinct $k,l\in\{1,\dots,a+n_3+n_4\}$ and $r(k)\ge r(l)$.*

(vii) *$\boldsymbol{x}'=\boldsymbol{x}$, $\boldsymbol{z}'=\boldsymbol{z}$, and $\boldsymbol{y}'=(y_1,\dots,y_{l-1},y_k+y_l,y_{l+1},\dots,y_a)$ for some distinct $k,l\in\{1,\dots,a\}$ and $r(k)\le r(l)$.*

(viii) *$\boldsymbol{x}'=\boldsymbol{x}$, $\boldsymbol{y}'=\boldsymbol{y}$, and $\boldsymbol{z}'=(z_1,\dots,z_{l-1},z_k+z_l,z_{l+1},\dots,z_c)$ for some distinct $k,l\in\{1,\dots,c\}$ and $s(k)\le s(l)$.*

(ix) $\boldsymbol{x}' = \boldsymbol{x}$, $\boldsymbol{y}' = \boldsymbol{y}$, *and* $\boldsymbol{z}' = (z_1, \dots, z_{l-1}, y_k + z_l, z_{l+1}, \dots, z_c)$ *for some distinct* $k, l \in \{1, \dots, c\}$ *and* $r(k) \le s(l)$.

Proof Homotopy equivalences (i)–(v) follow immediately from (11). (In the case $y_j = 3$ with $r(j) = 1$, we have $3\tilde{\xi}_j \simeq \tilde{\eta}_j + \tilde{\xi}_j$, so (iii) follows.) Equivalences (vi)–(ix) follow from Lemma 2.1 by taking $g\colon A_k \to A_l$ to be

- the identity map $A_k \to A_l$ if $A_k = A_l$,
- the Hopf map $\eta\colon S^4 \to S^3$ if $A_k = S^4$ and $A_l = S^3$,
- the inclusion $\iota_{r_l}\colon S^3 \to P^4(2^{r_l})$ if $A_k = S^3$ and $A_l = P^4(2^{r_l})$,
- $\iota_{r_k r_l}\colon P^4(2^{r_k}) \to P^4(2^{r_l})$ if $A_k = P^4(2^{r_k})$ and $A_l = P^4(2^{r_l})$,
- $\iota_{r_l} \circ \eta$ if $A_k = S^4$ and $A_l = P^4(2^{r_l})$,
- $\jmath_{s_k s_l}$ if $A_k = \Sigma\mathbb{CP}^4(2^{s_k})$ and $A_l = \Sigma\mathbb{CP}^2(2^{s_l})$,
- $\jmath_{s_l} \circ \iota_{r_k s_l}$ if $A_k = P^4(2^{r_k})$ and $A_l = \Sigma\mathbb{CP}^2(2^{s_l})$,

where $\iota_{rs}\colon P^4(2^r) \to P^4(2^s)$ is the map given in (3) and (4), $\jmath_s\colon P^4(2^s) \to \Sigma\mathbb{CP}^2(2^s)$ is the inclusion, and $\jmath_{rs}\colon \Sigma\mathbb{CP}^2(2^r) \to \Sigma\mathbb{CP}^2(2^s)$ is the map given in Lemma 2.3. □

Lemma 4.8 *Given the coefficients* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ *of zeros and ones from* (12), *there exist*

$$u \in \{1, \dots, a + n_3 + n_4\}, \quad v \in \{1, \dots, a\}, \quad w \in \{1, \dots, c\}$$

such that W_5 *is homotopy equivalent to the mapping cone of*

$$x_u \tilde{\eta}_u + y_v \tilde{\xi}_v + z_w \tilde{\epsilon}_w\colon S^5 \to W_4.$$

Furthermore, if $y_v = 1$ *and* $z_w = 1$ *then* $s(w) < r(v)$.

Proof By definition, $W_5 = C(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. To prove the first part of the lemma, it suffices to show $C(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \simeq C(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ where

(a) $\boldsymbol{x}' = (0, \dots, 0, x_u, 0, \dots, 0)$,

(b) $\boldsymbol{y}' = (0, \dots, 0, y_v, 0, \dots, 0)$,

(c) $\boldsymbol{z}' = (0, \dots, 0, z_w, 0, \dots, 0)$.

For (a), let $I = \{i \mid x_i = 1\}$ be the index set of those nonzero $x_i = 1$ in $\boldsymbol{x}$. If I has at most one element then we are done. Otherwise take u to be the largest index in I and use Lemma 4.7(i) and (vi) to annihilate other nonzero x_i by x_u as in the proof of Lemma 4.7.

For (b), let $J = \{j \mid y_j = 1\}$ be the index set of those nonzero y_j in $\boldsymbol{y}$. If J has more than one element, then take v to be the smallest index in J and use Lemma 4.7(ii), (iv) and (vii) to annihilate other nonzero y_j by y_v. Part (c) can be proved similarly using Lemma 4.7(v) and (viii).

Next we prove the second part of the lemma. Suppose $y_v = 1$ and $z_w = 1$ with $s(w) \ge r(v)$. Use Lemma 4.7(v) and (ix) to annihilate z_w by y_v. □

We are now ready to prove the main theorem.

Proof of Theorem 1.1 By homotopy equivalence (7), we have $\Sigma X \simeq (S^2)^{\vee m} \vee W_5$. Hence it suffices to show that W_5 is homotopy equivalent to $\Sigma C_f \vee \bigvee_i P_i$ where f is one of the maps (1), (2), (3), or (4), and each P_i is either a sphere, a Moore space, $\Sigma\mathbb{CP}^2$, or $\Sigma\mathbb{CP}^2(2^r)$.

By Lemma 4.8, W_5 is homotopy equivalent to the mapping cone of

$$x_u\tilde{\eta}_u + y_v\tilde{\tilde{\xi}}_v + z_w\tilde{\epsilon}_w \tag{13}$$

for some $u \in \{1, \dots, a+n_3+n_4\}, v \in \{1, \dots, a\}$ and $w \in \{1, \dots, c\}$, and $x_u, y_v, z_w \in \{0, 1\}$ such that $s(w) < r(v)$ if $y_v = z_w = 1$. Suppose $u \le a$. By definition $\tilde{\eta}_u, \tilde{\tilde{\xi}}_v$ and $\tilde{\epsilon}_w$ factor through the summands $P^4(2^{r_u})$, $P^4(2^{r_v})$ and $\Sigma\mathbb{CP}^2(2^{s_w})$ in W_4. If $u \ne v$ then the map (13) factors through

$$(x_u\imath_{r_u} \circ \eta^2) \perp (y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w}) \colon S^5 \to P^4(2^{r_u})^{\vee x_u} \vee P^4(2^{r_v})^{\vee y_v} \vee \Sigma\mathbb{CP}^2(2^{s_w})^{\vee z_w}.$$

Note that this map is the suspension of $(x_u\eta_{r_u}) \perp (y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w})$, so $W_5 \simeq \Sigma C_f \vee \bigvee_i P_i$ with f of the form (3). If $u = v$ then the map (13) factors through

$$(y_v\imath_{r_v} \circ \eta^2 + y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w}) \colon S^5 \to P^4(2^{r_v})^{\vee y_v} \vee \Sigma\mathbb{CP}^2(2^{s_w})^{\vee z_w}.$$

It is the suspension of $(y_v\eta_{r_v} + y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w})$, so $W_5 \simeq \Sigma C_f \vee \bigvee_i P_i$ with f of the form (4).

Suppose $u > a$. If $r(u) = \omega$ then $\tilde{\eta}_u$ factors through $S^3_u \subset W_4$ and the map (13) factors through

$$(x_u\eta^2) \perp (y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w}) \colon S^5 \to (S^3_u)^{\vee x_u} \vee P^4(2^{r_v})^{\vee y_v} \vee \Sigma\mathbb{CP}^2(2^{s_w})^{\vee z_w}.$$

It is the suspension of $(x_u\eta^2) \perp (y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w})$, so $W_5 \simeq \Sigma C_f \vee \bigvee_i P_i$ with f of the form (2). If $r(u) = \omega + 1$ then $\tilde{\eta}_u$ factors through $S^4_u \subset W_4$ and the map (13) factors through

$$(x_u\eta) \perp (y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w}) \colon S^5 \to (S^4_u)^{\vee x_u} \vee P^4(2^{r_v})^{\vee y_v} \vee \Sigma\mathbb{CP}^2(2^{s_w})^{\vee z_w}.$$

It is the suspension of $(x_u\eta) \perp (y_v\xi_{r_v}) \perp (z_w\epsilon_{s_w})$, so $W_5 \simeq \Sigma C_f \vee \bigvee_i P_i$ with f of the form (1). □

5 Analysis of the splitting

In this section we discuss algebraic constraints on the form of the splitting given in Theorem 1.1 which arise from Poincaré duality. The case in which X is a closed 5-manifold is especially tractable, since information about the decomposition of ΣX can be read off from the easily available algebraic and geometric information contained in its characteristic classes. However, most of the analysis we provide depends only on Poincaré duality, so has wider application. We assume throughout that X is an orientable Poincaré duality 5-complex, but shall place assumptions on its homology only when necessary.

For such an X, the cup product pairing $H^k(X;\mathbb{Z}/2) \otimes H^{5-k}(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$ is non-degenerate and gives rise to Wu classes $\upsilon_i(X) \in H^i(X;\mathbb{Z}/2)$ which are characterised by the equation

$$\langle \mathrm{Sq}(u), [X] \rangle = \langle \upsilon(X) \cup u, [X] \rangle,$$

where $u \in H^*(X;\mathbb{Z}/2)$, Sq is the total Steenrod square, and $\upsilon(X) = 1 + \upsilon_1(X) + \cdots$ is the total Wu class. Stiefel–Whitney classes $w_i(X) \in H^i(X;\mathbb{Z}/2)$ are then defined by writing

$$\mathrm{Sq}(\upsilon(X)) = 1 + w_1(X) + w_2(X) + \cdots. \tag{14}$$

Orientability implies that $\upsilon_1(X) = w_1(X) = 0$, and hence that $w_2(X) = \upsilon_2(X)$. Because Sq^i vanishes in $H^*(X;\mathbb{Z}/2)$ for $i \geq 3$, each of $\upsilon_3(X), \upsilon_4(X)$ and $\upsilon_5(X)$ vanishes. Consequently, (14) becomes

$$\upsilon_2(X) + \mathrm{Sq}^1\, \upsilon_2(X) + \mathrm{Sq}^2\, \upsilon_2(X) = w_2(X) + w_3(X) + w_4(X) + w_5(X),$$

meaning that

$$w_3(X) = \mathrm{Sq}^1\, w_2(X), \quad w_4(X) = w_2(X)^2, \quad w_5(X) = 0.$$

Thus the entire information of the Stiefel–Whitney classes of X is determined by $w_2(X)$ and the action of the Steenrod algebra on $H^*(X;\mathbb{Z}/2)$.

When X is a closed manifold, the Stiefel–Whitney classes defined above agree with the Stiefel–Whitney classes of its tangent bundle. Its Euler class vanishes, so the only additional piece of information in this case comes in the form of its first Pontryagin class $p_1(X) \in H^4(X;\mathbb{Z})$. This satisfies $\rho_2(p_1(X)) = w_2(X)^2$, where ρ_2 is reduction mod 2 [6, Theorem 1.5].

Proposition 5.1 *Let X be a Poincaré duality 5-complex.*

(1) *The operation*

$$\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$$

is given by cupping with $w_2 = w_2(X)$.

(2) *The operation*

$$\mathrm{Sq}^2\,\mathrm{Sq}^1\colon H^2(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$$

is given by cupping with $w_3 = w_3(X)$.

(3) *If $H_1(X;\mathbb{Z})$ contains no 2-torsion, then the operation*

$$\mathrm{Sq}^2\colon H^2(X;\mathbb{Z}/2) \to H^4(X;\mathbb{Z}/2)$$

is given by cupping with $w_2 = w_2(X)$.

Proof (1) This is implied by the agreement of the second Stiefel–Whitney and Wu classes of X. See (14).

(2) For $x \in H^2(X;\mathbb{Z}/2)$ use (1) to get $\mathrm{Sq}^1(w_2 \cdot x) = \mathrm{Sq}^1\,\mathrm{Sq}^2\, x = \mathrm{Sq}^3\, x = 0$. On the other hand, this gives

$$0 = \mathrm{Sq}^1(w_2 \cdot x) = w_3 \cdot x + w_2 \cdot \mathrm{Sq}^1\, x = w_3 \cdot x + \mathrm{Sq}^2\,\mathrm{Sq}^1\, x.$$

We conclude that $\mathrm{Sq}^2\,\mathrm{Sq}^1\, x = w_3 \cdot x$.

(3) The cup product pairing $H^1(X;\mathbb{Z}/2)\otimes H^4(X;\mathbb{Z}/2)\to\mathbb{Z}/2$ is nondegenerate. Thus for $x\in H^2(X;\mathbb{Z}/2)$ we have that $\mathrm{Sq}^2 x$ is equal to $w_2\cdot x$ if and only if $a\cdot(\mathrm{Sq}^2 x+w_2\cdot x)=0$ for all $a\in H^1(X;\mathbb{Z}/2)$. For such an a we have $\mathrm{Sq}^2 a=0$ for dimension reasons, and $\mathrm{Sq}^1 a=0$ holds because of the assumption. Thus $\mathrm{Sq}^2(a\cdot x)=a\cdot\mathrm{Sq}^2(x)$, and hence

$$a\cdot(\mathrm{Sq}^2 x+w_2\cdot x)=\mathrm{Sq}^2(a\cdot x)+w_2\cdot(a\cdot x)=(\mathrm{Sq}^2+w_2)\cdot a\cdot x=0,$$

since $(\mathrm{Sq}^2+w_2)\cdot H^3(X;\mathbb{Z}/2)=0$ by (1). □

As we explain below, the operations

(15) $$\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2)\to H^5(X;\mathbb{Z}/2),\quad \mathrm{Sq}^2\,\mathrm{Sq}^1\colon H^2(X;\mathbb{Z}/2)\to H^5(X;\mathbb{Z}/2)$$

give direct information about the components of the map f which appears in Theorem 1.1. On the other hand, Proposition 5.1 shows that the vanishing of these operations is equivalent to the vanishing of $w_2(X), w_3(X)$, respectively. In particular, the information contained in (15) is equivalent to that contained in these Stiefel–Whitney classes, which already hold significant geometric and algebraic importance.

Proposition 5.2 *Let X be an Poincaré duality 5-complex with torsion-free $H_1(X;\mathbb{Z})$. In the notation of Theorem 1.1, the map f contains a component η, ξ_r or ϵ_r for some r if and only if $w_2(X)\neq 0$.*

Proof The point is that each of these maps is detected by $\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2)\to H^5(X;\mathbb{Z}/2)$. This is obvious for η, and it follows for ξ_r and ϵ_r from the properties listed in Lemma 2.2 and Lemma 2.3. As discussed above, the Sq^2 is trivial if and only if $w_2(X)$ vanishes. □

Corollary 5.3 *Let X be a Poincaré duality 5-complex with torsion-free $H_1(X;\mathbb{Z})$.*

(1) *If ΣX splits off a copy of $\Sigma\mathbb{CP}^2$ or $\Sigma\mathbb{CP}^2(2^r)$, then f has an η, ξ_r or ϵ_r component.*

(2) *If $w_2(X)=0$, then no $\Sigma\mathbb{CP}^2$ or $\Sigma\mathbb{CP}^2(2^r)$ splits off ΣX.*

Proof (1) A copy of $\Sigma\mathbb{CP}^2$ or $\Sigma\mathbb{CP}^2(2^r)$ splitting off ΣX would indicate that

$$\mathrm{Sq}^2: H^2(X;\mathbb{Z}/2)\to H^4(X;\mathbb{Z}/2)$$

is nontrivial. Because of Proposition 5.1, this implies that $w_2(X)\neq 0$.

(2) By Proposition 5.1, $\mathrm{Sq}^2 : H^2(X;\mathbb{Z}/2)\to H^4(X;\mathbb{Z}/2)$, and therefore $\mathrm{Sq}^2 : H^3(\Sigma X;\mathbb{Z}/2)\to H^5(\Sigma X;\mathbb{Z}/2)$, are trivial. □

Since $w_3(X)=\mathrm{Sq}^1 w_2(X)$, the vanishing of $w_2(X)$ implies that of $w_3(X)$. However, when $w_2(X)\neq 0$, the vanishing of $w_3(X)$ has certain implications for the topology of X.

Proposition 5.4 *Let X be a Poincaré duality 5-complex with $w_2(X)\neq 0$. Then f can contain exactly one of ξ_1, ϵ_1, and this occurs if and only if $w_3(X)\neq 0$. If this holds, then there is $u\in H^2(X;\mathbb{Z}/2)$ with $w_2(X)\cdot\mathrm{Sq}^1 u\neq 0$, and f contains ϵ_1 if and only if $w_2(X)\cdot u\neq 0$. If f contains ξ_1, then it contains no ϵ_s for any $s\geq 1$.*

Proof Evidently, the presence of ξ_1 or ϵ_1 appearing in f is equivalent to $\mathrm{Sq}^2\,\mathrm{Sq}^1$ acting nontrivially in $H^*(X;\mathbb{Z}/2)$. By Proposition 5.1, this is equivalent to $w_3(X)\neq 0$. In this case there is $u\in H^2(X;\mathbb{Z}/2)$ satisfying

$$0\neq w_3(X)\cdot u=\mathrm{Sq}^2\,\mathrm{Sq}^1\,u=w_2(X)\cdot\mathrm{Sq}^1\,u.$$

Now, both ξ_1 and ϵ_1 cannot appear together, since Theorem 1.1 states that if f contains both ξ_r and ϵ_s, then $s<r$. In particular, if ξ_1 appears in f then no ϵ_s appears.

To distinguish between the two cases, observe that u must correspond to an element in $H^*(C_f;\mathbb{Z}/2)$ which is represented by either the bottom cell of $P^3(2)$, or by the bottom cell of $\mathbb{CP}^2(2)$. In the first case, $\mathrm{Sq}^2\,u=u^2=0$, while $\mathrm{Sq}^2\,u\neq 0$ holds in the second. Appealing again to Proposition 5.1 we complete the proof of the statement. □

There is more that can be said in the case $w_2(X)\neq 0$ and $w_3(X)=0$. Let $\{(E_r^*(X),d_r)\}_{r\in\mathbb{N}}$ be the mod 2 cohomology Bockstein spectral sequence for X. Let also $\beta_r\colon H^*(X;\mathbb{Z}/2^r)\to H^{*+1}(X;\mathbb{Z}/2)$ be the Bockstein connecting map associated with the short exact sequence

$$0\to\mathbb{Z}/2\to\mathbb{Z}/2^{r+1}\to\mathbb{Z}/2^r\to 0.$$

If $i\colon\mathbb{Z}/2\to\mathbb{Z}/2^r$ is the inclusion, then $i_\#\beta_r=d_r$ under the identification of d_r given in [27, Proposition 10.4]. In particular, $\beta_1=\mathrm{Sq}^1$, so $w_3(X)=0$ is the statement that $w_2(X)$ survives past the first page of the Bockstein spectral sequence.

Now assume that w_2 survives to the r-th page of the Bockstein spectral sequence. Then there is a class $w_2^{(r)}\in H^2(X;\mathbb{Z}/2^r)$ with $\rho_2 w_2^{(r)}=w_2$. Because $H^5(X;\mathbb{Z})$ is free abelian, $\beta_r(H^4(X;\mathbb{Z}/2^r))=0$, so for any $x\in H^2(X;\mathbb{Z}/2^r)$,

$$0=\beta_r(w_2^{(r)}\cdot x)=\beta_r(w_2^{(r)})\cdot\rho_2(x)+w_2\cdot\beta_r x=\beta_r(w_2^{(r)})\cdot\rho_2(x)+\mathrm{Sq}^2\,\beta_r(x).$$

That is,

$$\mathrm{Sq}^2\,\beta_r(x)=\beta_r(w_2^{(r)})\cdot\rho_2(x).$$

The right-hand side of this equation is independent of the choice of $w_2^{(r)}$. In any case, if $\beta_r(w_2^{(r)})=0$, then $\mathrm{Sq}^2\,\beta_r$ acts trivially on $H^*(X;\mathbb{Z}/2^r)$. On the other hand, if $d_r w_2\neq 0$, then necessarily $\beta_r(w_2^{(r)})\neq 0$, and the operation $\mathrm{Sq}^2\,\beta_r$ is nontrivial. From this discussion we have the following.

Proposition 5.5 *Let X be a Poincaré duality 5-complex with $H_1(X;\mathbb{Z})$ torsion-free and $w_2(X)\neq 0$. Write $r(X)\in\mathbb{N}\cup\{\infty\}$ for the greatest integer for which $w_2(X)$ survives to the $E_{r(X)}$-page of the mod 2 Bockstein spectral sequence. A necessary and sufficient condition that $r(X)>1$ is that $w_3(X)=0$.*

(1) *$r(X)<\infty$ if and only if exactly one of $\xi_{r(X)},\epsilon_{r(X)}$ appears in f, but no ξ_s or ϵ_s for $s<r(X)$ does. If $\epsilon_{r(X)}$ appears, then there is $u\in H^2(X;\mathbb{Z}/2^{r(X)})$ with $\mathrm{Sq}^2\,\beta_{r(X)}(u)\neq 0\neq\mathrm{Sq}^2\,\rho_2(u)$. If $\xi_{r(X)}$ appears, then no ϵ_s appears for any $s>r(X)$.*

(2) *$r(X)=\infty$ if and only if $f\simeq\eta$. If this holds, then there is $u\in H^3(X;\mathbb{Z})$ such that $\mathrm{Sq}^2\,\rho_2(u)\neq 0$.*

Proof Only the finer points need be explained. In (1) these are dealt with by appealing to the part of Theorem 1.1 that states that if f contains both ξ_r and ϵ_s, then $s < r$.

For (2), it has been shown above that f contains no ξ_r or ϵ_s for any $r, s \geq 1$. However, Proposition 5.2 then states that f must have an η component. There are four possible options for f listed in Theorem 1.1, and given the constraints just mentioned, $f \simeq \eta$ is the only one which may be realised. □

Next, components $\eta^2, \iota_r \circ \eta^2$ appearing in f can be detected using secondary cohomology operations. In particular, this is possible using the secondary operation based on the relation $\mathrm{Sq}^2\,\mathrm{Sq}^2 + \mathrm{Sq}^1(\mathrm{Sq}^2\,\mathrm{Sq}^1) = 0$ [11, p. 149]. Assuming that $H_1(X;\mathbb{Z})$ contains no 2-torsion, Proposition 5.1 may be used to construct this operation as

$$\Theta\colon \{x \in H^2(X;\mathbb{Z}/2) \mid x \cdot w_2 = 0 = x \cdot w_3\} \to H^5(X;\mathbb{Z}/2)/(w_2 \cdot H^3(X;\mathbb{Z}/2)).$$

Unfortunately, use of this operation is only feasible when $w_2 = 0$. In this case it is an operation

$$\Theta\colon H^2(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2). \tag{16}$$

Alone it cannot distinguish between $\eta^2, \iota_r \circ \eta^2$, but this may be accomplished by means of a higher Bockstein operator β_r.

Proposition 5.6 *Let X be a Poincaré duality 5-complex which has torsion-free $H_1(X)$ and $w_2(X) = 0$. If f has a component of either η^2 or $\iota_r \circ \eta^2$, then there is $u \in H^2(X;\mathbb{Z}/2)$ satisfying $\Theta(u) \neq 0$. If $\beta_r u = 0$ for all r, then u detects η^2. If $\beta_r u \neq 0$ for some $r \geq 2$, then u detects $\iota_r \circ \eta^2$.* □

Combining the operations above makes it possible also to detect a $\iota_r \circ \eta^2 + \xi_r$ ($r \geq 2$) component of f.

Proof of Theorem 1.2 (1) We refer to Theorem 1.1. Proposition 5.2 explains that f can only contain components η^2 or $\iota_r \circ \eta^2$, both of which are detected by the secondary operation Θ of (16). Consequently, the complex ΣC_f has at most three cells, and splits up further if and only if Θ evaluates trivially on its cohomology. With this reduction, it is easy to follow the method of [26, p. 32] to see that X being a manifold forces Θ to be trivial.

(2) The first part of the statement is covered by Proposition 5.4. For the second, observe that

$$w_2(X) \cdot w_3(X) = w_2(X) \cdot \mathrm{Sq}^1\, w_2(X) \neq 0.$$

Thus, still following Proposition 5.4, ξ_1 is detected if $w_2(X)^2 = 0$. However, $w_2(X)^2 = \rho_2(p_1(X))$ holds for any 5-dimensional manifold. Clearly $\rho_2(p_1(X)) = 0$ if and only if $p_1(X)$ is divisible by 2.

(3) See Proposition 5.5. □

Example 5.6.1 Let S_g be a closed orientable surface of genus g, and Y a closed, orientable 3-manifold with $H_1(Y;\mathbb{Z})$ torsion-free. Form $X = S_g \times Y$. Then X is spin with torsion-free homology. Theorem 1.2 says that

$$\Sigma X \simeq \Sigma S_g \vee \Sigma Y \vee \Sigma(S_g \wedge Y)$$

splits as a wedge of spheres. Clearly this requires that ΣY is a wedge of spheres. □

Example 5.6.2 Let Y be a nonspin 4-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}^{m-1}$ and let $X = S^1 \times Y$. Then

$$\Sigma X \simeq S^2 \vee \Sigma Y \vee \Sigma^2 Y.$$

The homology of X is torsion free, so part (3) of Theorem 1.2 is in effect with $w_2(X)$ surviving to the E_∞-page of the Bockstein spectral sequence. Thus $f \simeq \eta$. This should be compared with the decomposition of ΣY given in [31, Theorem 1.1], which implies the same result. □

For simply connected X, the maps f have been determined completely by Stöcker [34]. In the case that X is a smooth 5-manifold, much of this was already implicit in Barden's classification results [3]. However, Stöcker's results also extend to simply connected Poincaré duality complexes. We end this section by reviewing these known results.

In [3] Barden defines simply connected, oriented 5-manifolds X_0, X_{-1}, X_∞, M_∞, and X_k, M_k for $k \geq 2$, and proves the following.

Theorem 5.7 (Barden [3, Theorem 2.3]) *Every simply connected, closed, orientable, smooth 5-manifold is diffeomorphic to a manifold of the form*

$$X_{j,k_1,\dots,k_n} = X_j \# M_{k_1} \# \cdots \# M_{k_n},$$

where $-1 \leq j \leq \infty$ and $1 < k_1 \leq k_2 \leq \cdots \leq k_n$ are such that either k_i divides k_{i+1} or $k_{i+1} = \infty$. □

We have $X_0 = S^5$ and $M_\infty = S^2 \times S^3$. The manifolds M_k are of the form

$$M_k \simeq (P^2(k) \vee P^2(k)) \cup_\omega e^5,$$

where ω is a Whitehead product [34]. All of these manifolds split apart after a single suspension. The other manifolds are more interesting.

- $X_{-1} = \mathrm{SU}_3/\mathrm{SO}_3$ is the Wu manifold. Jie Wu [36, Example 6.15] has observed that

$$X_{-1} \simeq P^3(2) \cup_{\xi_1} e^5.$$

- $X_\infty = S^2 \tilde{\times} S^3$ is the total space of the nontrivial S^3-bundle over S^2, classified by a generator of $\pi_1(\mathrm{SO}_4) \cong \mathbb{Z}/2$. Following [17] we check that

$$X_\infty \simeq (S^2 \vee S^3) \cup_{\phi_\infty} e^5,$$

where $\phi_\infty = \jmath_3 \circ \eta + [\iota_2, \iota_3]$.

- X_k for $k \geq 1$ is a nonspin manifold which, according to [34], is of the form

$$X_k \simeq (P^3(2^k) \vee P^3(2^k)) \cup_{\phi_k} e^5,$$

where $\phi_k = \omega + \jmath_1 \circ \xi_k$ with ω a Whitehead product.

In addition to these manifolds there are certain interesting nonsmoothable, simply connected Poincaré duality 5-complexes. One such is the Gitler–Stasheff example [9]

$$M'_\infty \simeq (S^2 \vee S^3) \cup_\varphi e^5,$$

where $\varphi = [\iota_2, \iota_3] + j_2 \circ \eta^2$. The others we consider are those constructed by Stöcker [34]:

- $M'_k \simeq (P^2(k) \vee P^2(k)) \cup_{\varphi'_k} e^5$ for $k \geq 2$, where $\varphi'_k = \omega + \iota_k \circ \eta^2$ with ω a Whitehead product;
- $X'_k = (P^3(2^k) \vee P^3(2^k)) \cup_{\phi'_k} e^5$ for $k \geq 1$, where $\phi'_k = \omega + j_1(\xi_k + \iota_{2^k} \circ \eta^2)$ with ω a Whitehead product.

Theorem 5.8 (Stöcker [34, Theorem 10.1]) *Every simply connected, 5-dimensional, orientable Poincaré duality complex is homotopy equivalent to a connected sum of the spaces defined above.* □

Corollary 5.9 *If X is a closed, orientable, simply connected 5-manifold, then f can contain only η or ξ_r for some $r \geq 1$. If X is an orientable, simply connected Poincaré duality 5-complex, then f can contain only $\eta, \eta^2, \xi_r, \iota_r \circ \eta^2$, or $(\xi_r + \iota_r \circ \eta^2)$ for some $r \geq 1$.* □

6 Applications

6.1 Generalised cohomology

As a simple application of our main theorem we explain how to evaluate a given cohomology theory h^* on any Poincaré duality complex satisfying the assumptions of Theorem 1.1. For example h^* could be singular cohomology, complex or real K-theory, or cobordism. We will give applications in the sequel when h^* is stable cohomotopy.

Theorem 6.1 *Let X be a connected, orientable, 5-dimensional Poincaré duality complex with homology as in (1), and let h^* be a reduced generalised cohomology theory. Then*

$$h^*(X) \cong \bigoplus^{m} h^*(S^1) \oplus \bigoplus^{n_3} h^*(S^2) \oplus \bigoplus^{n_4} h^*(S^3) \oplus \bigoplus^{n_5} h^*(S^4) \oplus \bigoplus_{t'_i \in T'} h^*(P^3(t'_i)) \oplus \bigoplus^{b} h^*(\mathbb{CP}^2) \oplus \bigoplus_{j=1}^{c} h^*(\mathbb{CP}^2(2^{r_j})) \oplus h^*(C_f),$$

where the integers $n_3, n_4, n_5, a, b, c, t'_i, r_j$ are explained in Theorem 1.1.

Proof The lemma follows immediately from the string of isomorphisms

$$h^*(X) \cong h^{*+1}(\Sigma X) \cong h^{*+1}\Big(\bigvee_i \Sigma P_i\Big) \cong \bigoplus_i h^{*+1}(\Sigma P_i) \cong \bigoplus_i h^*(P_i),$$

where the first and the third isomorphisms are due to the suspension isomorphisms of h^* and the second isomorphism is due to the suspension splitting $\Sigma X \simeq \bigvee_i \Sigma P_i$ in Theorem 1.1. Since we work only with finite wedge sums, we do not need to assume that h^* satisfies the wedge axiom to obtain the third isomorphism. □

There are similar decompositions of $h_*(M)$ for any generalised homology theory h_* which we will leave to the reader to spell out. Interesting applications of this are when h_* is K-homology or stable homotopy.

6.2 Cohomotopy groups and sets

In this section we consider the problem of computing the cohomotopy sets $\pi^n(X) = [X, S^n]$ and $\pi^n(X;\mathbb{Z}/k) = [X, P^{n+1}(k)]$ for X a Poincaré duality 5-complex satisfying the assumptions of our main theorem. We start by briefly recalling the definitions of these sets and the construction of group operations on them when X is any CW complex.

In general, $\pi^n(X)$ is only a pointed set. However, if X is a CW complex of dimension $\leq 2n-2$, then $\pi^n(X)$ carries a natural abelian group structure, which can defined as follows. Suppose maps $f, g\colon X \to S^n$ are given. Because the inclusion $S^n \vee S^n \hookrightarrow S^n \times S^n$ is $(2n-1)$-connected, there is a unique compression of $(f, g)\colon X \to S^n \times S^n$ into a map $\theta_{f,g}\colon X \to S^n \vee S^n$. We define

$$f + g = \nabla \circ \theta_{f,g}\colon X \to S^n, \tag{17}$$

where $\nabla\colon S^n \vee S^n \to S^n$ is the folding map, and put $-f = (-1) \circ f$. Borsuk [5] shows that these definitions equip $\pi^n(X)$ with an abelian group structure (see Spanier [32] for full proofs).

There is also a second description of this group structure which is due to Taylor [35], and can be useful to know. If $\dim X \leq 2n-2$, then by a standard argument we have $\pi^n(X) \cong [X, \mathrm{PK}_{2n-2}(S^n)]$, where $\mathrm{PK}_i(S^n)$ denotes the i-th Postnikov section of S^n. Owing essentially to the Freudenthal suspension theorem, $\mathrm{PK}_{2n-2}(S^n)$ is an infinite loop space. Hence $\pi^n(X)$ is an abelian group in each of these cases.

This viewpoint has some advantages. For example, it makes clear that while working in this dimension range, the function

$$\Sigma\colon \pi^n(X) \to \pi^{n+1}(\Sigma X) \tag{18}$$

induced by the suspension map $\sigma\colon S^n \to \Omega S^{n+1}$ is compatible with group operations on both sides. This gives a well-defined suspension *homomorphism*.

In fact, this leads to yet another description of the group $\pi^n(X)$. Namely, the suspension (18) is bijective when $\dim X \leq 2n-2$, and we may turn $\pi^n(X)$ into a group by requiring it to be a homomorphism. We've already explained why this gives the same operation as Taylor's. Showing that it agrees with Borsuk's original definition (17) is the easiest way to establish the equivalence of all three definitions.

Now, Borsuk's original definition (17) has its merits. In cases $n = 1, 3, 7$ there is one further group structure on $\pi^n(X)$, which comes from the H-space multiplication on S^n. Because the multiplication represents an extension of the folding map ∇ over $S^n \times S^n$, we find further agreement between this operation and the three set out above.

It is also possible to define cohomotopy groups with coefficients. These were introduced originally by Peterson [29] via the definition

$$\pi^n(X;\mathbb{Z}/k) = [X, P^{n+1}(k)].$$

As above, $\pi^n(X;\mathbb{Z}/k)$ is a group when X is a CW complex of dimension $\leq 2n-2$. Moreover, the maps $\pi^n(X) \to \pi^n(X;\mathbb{Z}/k)$ and $\pi^n(X;\mathbb{Z}/k) \to \pi^{n+1}(X)$ which are induced by the inclusion $S^n \hookrightarrow P^{n+1}(k)$, and the pinch map $P^{n+1}(k) \to S^{n+1}$, respectively, are group homomorphisms in this range. More recently, these groups have been studied by Li, Pan, and Wu [24], who called them *modular cohomotopy groups*.

The cohomotopy and modular cohomotopy groups are related in the usual long exact sequence which starts

$$\pi^m(X) \xrightarrow{\times k} \pi^m(X) \to \pi^m(X;\mathbb{Z}/k) \to \pi^{m+1}(X) \xrightarrow{\times k} \pi^{m+1}(X) \to \cdots$$

and extends infinitely to the right, where $m \geq \max\{(\dim X+1)/2, 3\}$ (these sets may fail to be groups in the bottom degree). For example, when $\dim X = 5$, the following sequence is exact

$$\pi^3(X) \xrightarrow{\times k} \pi^3(X) \to \pi^3(X;\mathbb{Z}/k) \to \pi^4(X) \to \cdots \to \pi^5(X) \xrightarrow{\times k} \pi^5(X) \to \pi^5(X;\mathbb{Z}/k) \to 0.$$

The next statement is a simple application of the Freudenthal theorem which will be used in the sequel.

Proposition 6.2 *If X is a CW complex of dimension $\leq 2n-2$, then stabilisation induces isomorphisms*

$$\pi^n(X) \cong \pi^n_S(X), \quad \pi^n(X;\mathbb{Z}/k) \cong \pi^n_S(X;\mathbb{Z}/k),$$

where $\pi^n_S(X)$ ($\pi^{n+1}_S(X;\mathbb{Z}/k)$) is the n-th stable cohomotopy group (with coefficients) of X. □

Now, let us turn to computation. For this we specialise to X being a 5-dimensional Poincaré duality complex satisfying the assumptions of Theorem 1.1. Of course,

$$\pi^n(X) = 0 = \pi^n(X;\mathbb{Z}/k), \quad n \geq 6,$$

for dimension reasons. We also compute

$$\pi^5(X) \cong H^5(X;\mathbb{Z}) \cong \mathbb{Z}, \quad \pi^5(X;\mathbb{Z}/k) \cong H^5(X;\mathbb{Z}/k) \cong \mathbb{Z}/k$$

using the Hopf theorem. In this range, $\pi^4(X)$ and $\pi^4(X;\mathbb{Z}/k)$ are also groups, and their structure is described below using results known in the literature.

On the other hand, $\pi^3(X)$ and $\pi^3(X;\mathbb{Z}/k)$ fall outside of Borsuk's range, and must be dealt with separately. While the group structure on S^3 induces a group structure on $\pi^3(X)$, a priori $\pi^3(X;\mathbb{Z}/k)$ is only a pointed set. As it turns out, $\pi^3(X;\mathbb{Z}/k)$ does carry a canonical group structure. This, along with the group $\pi^3(X)$, is described in Sections 6.2.1 and 6.2.2. We give information about $\pi^2(X)$ in Section 6.2.3.

As for $\pi^4(X)$, we have a well-known result of Steenrod [33], which gives a short exact sequence

(19) $$0 \to H^5(X;\mathbb{Z}/2)/\mathrm{Sq}^2(H^3(X;\mathbb{Z})) \to \pi^4(X) \to H^4(X;\mathbb{Z}) \to 0.$$

Under our standing assumption that $H_1(X;\mathbb{Z}) \cong H^4(X;\mathbb{Z})$ is torsion-free, the sequence splits (as observed by Taylor [35, Example 6.3], this splitting holds under slightly more general conditions). We can also replace $H^3(X;\mathbb{Z})$ with the mod 2 reduced group. Thus:

Proposition 6.3 *There is a group isomorphism*

$$\pi^4(X) \cong H^4(X;\mathbb{Z}) \oplus H^5(X;\mathbb{Z}/2)/\mathrm{Sq}^2(H^3(X;\mathbb{Z}/2)).$$

In case X is a closed 5-manifold, this may be written

$$\pi^4(X) \cong \begin{cases} H^4(X;\mathbb{Z}) & \text{if } X \text{ is nonspin,} \\ H^4(X;\mathbb{Z}) \oplus \mathbb{Z}/2 & \text{if } X \text{ is spin.} \end{cases}$$

This situation has been studied in detail by Konstantis [21], who has explicitly described a splitting map for Steenrod's exact sequence (19). Konstantis gets some mileage out of the fact that $\pi^4(X) \cong [X, \mathbb{HP}^\infty]$, and elements of the latter set are in one-to-one correspondence with isomorphism classes of quaternionic line bundles over X. Extensions to higher dimensions have recently been performed in [18], where the group $\pi^n(X^{n+1})$ is studied for X^{n+1} a (possibly nonorientable) $(n+1)$-manifold.

To introduce coefficients, consider the exact sequence

$$\cdots \to \pi^4(X) \xrightarrow{\times k} \pi^4(X) \to \pi^4(X;\mathbb{Z}/k) \to \pi^5(X) \xrightarrow{\times k} \pi^5(X) \to \cdots.$$

Since multiplication by k on the torsion-free group $\pi^5(X) \cong \mathbb{Z}$ is injective, the map $\pi^4(X) \to \pi^4(X;\mathbb{Z}/k)$ is onto. Hence $\pi^4(X;\mathbb{Z}/k) \cong \pi^4(X)/(k \cdot \pi^4(X))$. Using Proposition 6.3 we thus state the following.

Proposition 6.4 *For an odd integer $k \geq 1$,*

$$\pi^4(X;\mathbb{Z}/k) \cong H^4(X;\mathbb{Z}/k).$$

For an even integer $k \geq 2$,

$$\pi^4(X;\mathbb{Z}/k) \cong H^4(X;\mathbb{Z}/k) \oplus H^5(X;\mathbb{Z}/2)/\mathrm{Sq}^2(H^3(X;\mathbb{Z}/2)).$$

The reader may wish to compare these with the results in [24], where for a prime p and integer $r \geq 1$ the short exact sequence

$$0 \to H^{n+2p-3}(Y;\mathbb{Z}/p)/\mathcal{P}^1(H^{n-1}(Y;\mathbb{Z}/p^r)) \to \pi^n(Y;\mathbb{Z}/p^r) \to H^n(Y;\mathbb{Z}/p^r) \to 0$$

is constructed for any complex Y with $\dim Y \leq n+2p-3$ and any $n \geq 2p-1$.

6.2.1 The group $\pi^3(X)$ The group structure on $\pi^3(X)$ comes from the Lie multiplication on S^3, as it is outside of the dimension range of Borsuk's constructions. For the same reason, Proposition 6.2 is not immediately applicable.

Proposition 6.5 *There is a group isomorphism*

$$\pi^3(X) \cong \pi^3_S(X) \cong \pi^3_S(S^3)^{\oplus n_4} \oplus \pi^3_S(S^4)^{\oplus n_5} \oplus \bigoplus_{t_i' \in T'} \pi^3_S(P^3(t_i')) \oplus \pi^3_S(\mathbb{CP}^2)^{\oplus b} \oplus \bigoplus_{j=1}^{c} \pi^3_S(\mathbb{CP}^2(2^{r_j})) \oplus \pi^3_S(C_f),$$

where

- $\pi^3_S(S^3) \cong \mathbb{Z}$ *and* $\pi^3_S(S^4) \cong \mathbb{Z}/2$,
- $\pi^3_S(P^3(t)) \cong \mathbb{Z}/t$,
- $\pi^3_S(\mathbb{CP}^2) = 0$,
- $\pi^3_S(\mathbb{CP}^2(2^r)) \cong \mathbb{Z}/2^{r+1}$.

Proof Since S^3 is an H-space, the suspension map $\sigma\colon S^3 \to \Omega S^4$ has a left homotopy inverse. Consequently, the induced map

$$\Sigma\colon \pi^3(X) \to \pi^4(\Sigma X) \tag{20}$$

is injective. On the other hand, its surjectivity is already covered by the Freudenthal theorem. Thus (20) is bijective.

Now, although σ is not an H-map, the obstruction to it being one is represented by a mapping $S^3 \wedge S^3 \cong S^6 \to \Omega S^4$. Since X is 5-dimensional, the obstruction vanishes when applied to any mapping $X \to S^3$. It follows that (20) is an isomorphism of groups.

Finally, ΣX is 6-dimensional, so Proposition 6.2 may now be applied to $\pi^4(\Sigma X)$ to complete the proof.

As for the bullet points, the groups $\pi^3_S(S^n) = \pi^S_n(S^3)$ are well known, and $\pi^3_S(P^3(t)) \cong \mathbb{Z}/t$ and $\pi^3_S(\mathbb{CP}^2) = 0$ are easily obtained.

For $\pi^3_S(\mathbb{CP}^2(2^r))$, apply π^3_S to the cofibre sequence $S^3 \to P^3(2^r) \to \mathbb{CP}^2(2^r)$. The first map factors through the inclusion $S^2 \hookrightarrow P^3(2^r)$, so induces the trivial map on both π^3_S and π^4_S. Thus we obtain a short exact sequence

$$0 \leftarrow \mathbb{Z}/2^r \leftarrow \pi^3_S(\mathbb{CP}^2(2^r)) \leftarrow \mathbb{Z}/2 \leftarrow 0. \tag{21}$$

On the other hand, there is a cofibre sequence $\mathbb{CP}^2 \to \mathbb{CP}^2(2^r) \to S^3$ whose exact sequence gives an epimorphism $\pi^3_S(S^3) \cong \mathbb{Z} \to \pi^3_S(\mathbb{CP}^2(2^r))$. It follows from this that $\pi^3_S(\mathbb{CP}^2(2^r))$ is generated by a single element, and hence that (21) cannot split. But $\mathrm{Ext}(\mathbb{Z}/2^r, \mathbb{Z}/2) \cong \mathbb{Z}/2$, so the only other possibility is $\pi^3_S(\mathbb{CP}^2(2^r)) \cong \mathbb{Z}/2^{r+1}$. □

We do not give detailed information about the group $\pi^3_S(C_f)$, since it depends so much on the exact form of the attaching map f.

6.2.2 The set $\pi^3(X;\mathbb{Z}/k)$ Recall that the suspension map

$$\sigma\colon P^{n+1}(k) \to \Omega P^{n+2}(k)$$

is $(2n-1)$-connected. Thus $\Sigma\colon \pi^n(X;\mathbb{Z}/k) \to \pi^{n+1}(\Sigma X;\mathbb{Z}/k)$ is bijective for $\dim X \le 2n-2$ and is surjective for $\dim X = 2n-1$. This was partially recorded in Proposition 6.2 as the statement that if X is a 5-dimensional CW complex, then $\Sigma\colon \pi^n(X;\mathbb{Z}/k) \to \pi^{n+1}(\Sigma X;\mathbb{Z}/k)$ is an isomorphism for $n \ge 4$. The following extends this range by one dimension.

Proposition 6.6 *Let X be a 5-dimensional CW complex. Then for any integer $k \ge 1$ the suspension $\Sigma\colon \pi^3(X;\mathbb{Z}/k) \to \pi^4(\Sigma X;\mathbb{Z}/k)$ is a bijection.*

Proof Write $P^q = P^q(k)$ and identify $\Omega\Sigma P^4$ with the James construction $J(P^4)$. Then the suspension σ is identified with the inclusion $P^4 = J_1(P^4) \hookrightarrow J(P^4)$. Letting $J_2(P^4)$ denote the second stage of the canonical filtration on $J(P^4)$, we have $[X, J(P^4)] \cong [X, J_2(P^4)]$ for dimension reasons. Furthermore, it is well known that this space sits in a cofibration

$$\Sigma P^3 \wedge P^3 \xrightarrow{[\mathbb{1},\mathbb{1}]} P^4 \to J_2(P^4),$$

where $[\mathbb{1},\mathbb{1}]$ is the Whitehead product $\nabla\circ[1,1]$. Thus the 6-skeleton of $J_2(P^4)$ is of the form $P^4\cup_\alpha e^6$, where α is the restriction of $[\mathbb{1},\mathbb{1}]$ to the bottom cell of $\Sigma P^3\wedge P^3$. Let $\iota\colon S^3\to P^4$ be the inclusion and consider the diagram

$$\begin{array}{ccc} \Sigma S^2\wedge S^2 & \xrightarrow{\Sigma\iota\wedge\iota} & \Sigma P^3\wedge P^3 \\ \downarrow{\scriptstyle [\mathbb{1},\mathbb{1}]} & & \downarrow{\scriptstyle [\mathbb{1},\mathbb{1}]} \\ S^3 & \xrightarrow{\iota} & P^4 \end{array}$$

The diagram homotopy commutes by naturality of the Whitehead product and shows that α factors through $[\mathbb{1},\mathbb{1}]\colon \Sigma S^3\wedge S^3\to S^3$. However, since S^3 is an H-space, this latter Whitehead product is trivial. The 6-skeleton of $J_2(P^4)$ is then given by $P^4\vee S^6$, and it follows that the induced map

$$\sigma_*\colon [X,P^4]\to[X,J(P^4)]\cong[X,P^4\vee S^6]\cong[X,P^4]$$

is bijective. □

Proposition 6.2 gives $\pi^4(\Sigma X;\mathbb{Z}/k)\cong\pi^4_S(\Sigma X;\mathbb{Z}/k)$. Combining this with Proposition 6.6, we record the following.

Corollary 6.7 *If X is a 5-dimensional CW complex, then for any $k\geq 1$, the set*

$$\pi^3(X;\mathbb{Z}/k)\cong\pi^3_S(X;\mathbb{Z}/k)$$

carries a canonical group structure.

Finally, Theorem 6.1 yields a decomposition result for this group.

Proposition 6.8 *Let X be as in Theorem 1.1. Then for any $k\geq 1$ there is a group isomorphism*

$$\pi^3(X;\mathbb{Z}/k)\cong\pi^3_S(S^3;\mathbb{Z}/k)^{\oplus n_4}\oplus\pi^3_S(S^4;\mathbb{Z}/k)^{\oplus n_5}\oplus\bigoplus_{t_i'\in T'}\pi^3_S(P^3(t_i');\mathbb{Z}/k)$$
$$\oplus\,\pi^3_S(\mathbb{CP}^2;\mathbb{Z}/k)^{\oplus b}\oplus\bigoplus_{j=1}^{c}\pi^3_S(\mathbb{CP}^2(2^{r_j});\mathbb{Z}/k)\oplus\pi^3_S(C_f;\mathbb{Z}/k),$$

where

- $\pi^3_S(S^3;\mathbb{Z}/k)=\mathbb{Z}/k$,
- $\pi^3_S(S^4;\mathbb{Z}/k)=\begin{cases}0 & \text{if } k \text{ is odd},\\ \mathbb{Z}/2 & \text{if } k \text{ is even},\end{cases}$
- $\pi^3_S(P^3(t);\mathbb{Z}/k)\cong\mathbb{Z}/\gcd(t,k)$,
- $\pi^3_S(\mathbb{CP}^2;\mathbb{Z}/k)=0$,
- $\pi^3_S(\mathbb{CP}^2(2^r);\mathbb{Z}/k)=\mathbb{Z}/\gcd(2^{r+1},k)$.

Proof We compute the component groups. The first two are given in Lemma 2.2, and for the third, we have $\pi^3_S(P^3(t);\mathbb{Z}/k)=\pi^S_3(P^4(t);\mathbb{Z}/k)$. A universal coefficient theorem shows that $\pi^S_3(P^4(t);\mathbb{Z}/k)\cong\pi^S_3(P^4(t))\otimes\mathbb{Z}/k$, which gives the claimed result.

As for the fourth group, there is an exact sequence

$$0 \leftarrow \pi_S^3(\mathbb{CP}^2; \mathbb{Z}/k) \leftarrow \pi_S^3(S^4; \mathbb{Z}/k) \xleftarrow{\eta^*} \pi_S^3(S^3; \mathbb{Z}/k) \leftarrow \cdots.$$

The group $\pi_S^3(S^4; \mathbb{Z}/k)$ vanishes if k is odd. If k is even, then it is $\mathbb{Z}/2$, and the generators given in Lemma 2.2 show that the η^* appearing in the sequence is onto.

For the last group use the exact sequence

$$\cdots \to \pi^3(\mathbb{CP}^2(2^r)) \xrightarrow{\times k} \pi^3(\mathbb{CP}^2(2^r)) \to \pi_S^3(\mathbb{CP}^2(2^r); \mathbb{Z}/k) \to \pi^4(\mathbb{CP}^2(2^r)) \xrightarrow{\times k} \pi^4(\mathbb{CP}^2(2^r)) \to \cdots.$$

Since $\pi^4(\mathbb{CP}^2(2^r)) \cong \mathbb{Z}$, the right-most arrow is injective. On the left-hand side, it was shown in Proposition 6.5 that $\pi^3(\mathbb{CP}^2(2^r)) \cong \mathbb{Z}/2^{r+1}$. This means that

$$\cdots \to \mathbb{Z}/2^{r+1} \xrightarrow{\times k} \mathbb{Z}/2^{r+1} \to \pi_S^3(\mathbb{CP}^2(2^r); \mathbb{Z}/k) \to 0$$

is exact, and this gives the result immediately. □

6.2.3 The set $\pi^2(X)$ There is no natural group structure on $\pi^2(X)$. However, this set does carry a $\pi^3(X)$-action which contains useful information. In general, the computation of $\pi^2(X)$ is a difficult problem, and a full discussion is well outside the scope of this modest section. We will content ourselves with sketching a few details, most of which are already contained in Taylor's paper [35].

We begin by considering the diagram

$$\begin{array}{ccccccccc}
S^1 & \longrightarrow & S^3 & \xrightarrow{\eta} & S^2 & \xrightarrow{i} & \mathbb{CP}^\infty & \xrightarrow{\rho} & \mathbb{HP}^\infty \\
\downarrow & & \downarrow & & \downarrow & & \| & & \downarrow \\
\Omega\mathbb{CP}^\infty & \longrightarrow & \Omega E_6 & \longrightarrow & F & \longrightarrow & \mathbb{CP}^\infty & \xrightarrow{\tilde{\rho}} & E_6
\end{array} \tag{22}$$

The top row is a well-known fibre sequence in which i classifies a generator of $H^2(S^2)$ and ρ is induced by the extension of scalars $\mathbb{C} \subseteq \mathbb{H}$. The bottom row is obtained thus. We denote by E_6 the sixth Postnikov section of $\mathbb{HP}^\infty$, with nonvanishing homotopy groups only in degrees $4, 5, 6$. The arrow $\mathbb{HP}^\infty \to E_6$ is taken to be the natural map, and $\tilde{\rho}$ is defined to be the composite of ρ with this map. The space F denotes the homotopy fibre of $\tilde{\rho}$. Thus the bottom row of (22) is a homotopy fibration sequence. We let $S^2 \to F$ be an induced map of homotopy fibres, and complete the left-most vertical arrows by looping.

The goal is to apply the functor $[X, -]$ to the diagram and use the exactness of the rows to study $[X, S^2] = \pi^2(X)$. We leverage the following facts. Firstly, the inclusion of the bottom cell $S^4 \hookrightarrow \mathbb{HP}^\infty$ is 7-connected, as is the map $\mathbb{HP}^\infty \to E_6$. Consequently

$$\pi^4(X) \cong [X, \mathbb{HP}^\infty] \cong [X, E_6] \quad \text{and} \quad \pi^3(X) \cong [X, \Omega E_6].$$

Furthermore, $S^2 \to F$ is 6-connected, so $\pi^2(X) \cong [X, F]$.

The last fact we will use is that E_6 is an infinite loop space. Indeed, the 7-connected map $S^4 \to E_6$ induces isomorphisms on the first three nontrivial homotopy groups of S^4. All these groups lie in the stable range, so for any $N \geq 0$ the space E_6 is homotopy equivalent to the N-fold loop space on a

Postnikov section approximating the $(N+6)$-type of S^{4+N}. This is particularly important because it allows us to use the following theorem of Taylor.

Theorem 6.9 (Taylor [35, Theorem 5.2]) *Let B and C be homotopy associative H-spaces and $p\colon E \to B$ the homotopy fibre of a map $w\colon B \to C$. Suppose that X is a space. Then, for each $\alpha \in [X, B]$ such that $w_*(\alpha) = * \in [X, C]$, there is a group homomorphism*

$$\psi_\alpha\colon [X, \Omega B] \to [X, \Omega C],$$

and a bijection from $\operatorname{coker}(\psi_\alpha)$ *onto* $(p_*)^{-1}(\alpha) \subseteq [X, E]$. □

Let $h\colon \pi^2(S^2) \to H^2(X;\mathbb{Z})$ be the map defined by $h(\alpha) = \alpha^*(s_2)$, where $s_2 \in H^2(S^2;\mathbb{Z})$ is a fixed generator. Also let $(-)^2\colon H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z})$ denote the squaring map $u \mapsto u^2$. Finally, with ρ_2 denoting reduction mod 2, introduce the secondary operation

$$\Theta\colon \{u \in H^2(X;\mathbb{Z}) \mid \rho_2(u^2) = 0\} \to H^5(X;\mathbb{Z}/2)/(\mathrm{Sq}^2\, H^3(X;\mathbb{Z}/2)), \tag{23}$$

which is based on the Adem relation $\mathrm{Sq}^2(\mathrm{Sq}^2\,\rho_2) = 0$. The indeterminacy is unwieldy, but we will only need to use it when $\mathrm{Sq}^2\, H^3(X;\mathbb{Z}/2) = 0$.

Proposition 6.10 *Let X be a 5-dimensional CW complex.*

(1) *If $\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$ is onto, then there is an exact sequence of sets*

$$* \to [X, S^3] \to [X, S^2] \xrightarrow{h} H^2(X;\mathbb{Z}) \xrightarrow{(-)^2} H^4(X;\mathbb{Z}).$$

(2) *If $\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$ is trivial, then there is an exact sequence of sets*

$$* \to [X, S^3] \to [X, S^2] \xrightarrow{h} H^2(X;\mathbb{Z}) \xrightarrow{(-)^2 \oplus \theta} H^4(X;\mathbb{Z}) \oplus H^5(X;\mathbb{Z}/2),$$

where

$$\theta(u) = \begin{cases} 0 & \text{if } u^2 \neq 0, \\ \Theta(u) & \text{if } u^2 = 0. \end{cases}$$

Proof This is obtained in both cases by applying $[X, -]$ to the top row of (22) to get

$$[X, S^1] \longrightarrow [X, S^3] \xrightarrow{\eta_*} [X, S^2] \xrightarrow{i_*} [X, \mathbb{CP}^\infty] \xrightarrow{\rho_*} [X, \mathbb{HP}^\infty].$$

On the left, the fact that the fibre inclusion $S^1 \to S^3$ is null-homotopic implies that the image of $[X, S^1]$ in $[X, S^3]$ contains only the constant map. Moving to the right, it is clear that h is the map induced by the inclusion $i\colon S^2 \hookrightarrow \mathbb{CP}^\infty$ under the identification $[X, \mathbb{CP}^\infty] \cong H^2(X)$. On the far right, we have $[X, \mathbb{HP}^\infty] \cong \pi^4(X)$, and this group is described by the short exact sequence (19).

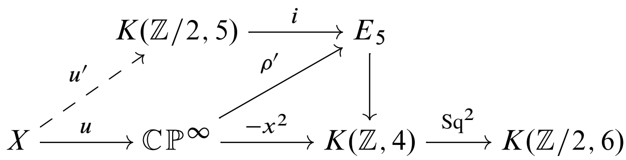
In case (1) we have $\pi^4(X) \cong H^4(X)$. The map $\rho\colon \mathbb{CP}^\infty \to \mathbb{HP}^\infty$ classifies the bundle $\gamma_{\mathbb{C}} \oplus \overline{\gamma}_{\mathbb{C}}$, where $\gamma_{\mathbb{C}}$ is the canonical line bundle over $\mathbb{CP}^\infty$ and $\overline{\gamma}_{\mathbb{C}}$ its conjugate. The composite $\mathbb{CP}^\infty \xrightarrow{\rho} \mathbb{HP}^\infty \to K(\mathbb{Z}, 4)$ returns the second Chern class of $\gamma_{\mathbb{C}} \oplus \overline{\gamma}_{\mathbb{C}}$, which is $-c_1(\gamma_{\mathbb{C}})^2$. Since $c_1(\gamma_{\mathbb{C}})$ generates $H^2(\mathbb{CP}^\infty)$, we identify the squaring map. Removal of the minus sign does not affect exactness.

In case (2), the sequence (19) may not split. However, we do not need to understand the full structure of $\pi^4(X)$ to verify what is being claimed. Thus we replace it with $H^4(X) \oplus H^5(X;\mathbb{Z}/2)$. The first factor will play the same role as above, and for the exactness claim it will be sufficient to understand the action of $\theta\colon H^2(X) \to H^5(X;\mathbb{Z}/2)$ on those $u \in H^2(X)$ satisfying $u^2 = 0$. We will show that if $u \in H^2(X)$, then $\rho_*(u)$ is null-homotopic if and only if $u^2 = 0 = \theta(u)$.

To this end, we replace $\mathbb{HP}^\infty$ with its fifth Postnikov section E_5. We have

$$[X, \mathbb{HP}^\infty] \cong [X, E_6] \cong [X, E_5],$$

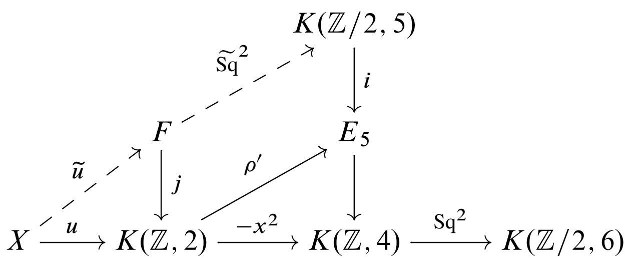
and the situation is

(24) 

The map ρ' is obtained by projecting $\tilde\rho$ from diagram (22) to E_5, and, as explained above, $-x^2 = -c_1(\gamma_{\mathbb{C}})^2$ is what is obtained upon projecting ρ' to $K(\mathbb{Z},4)$. Since $\rho' \circ u$ is the projection of $\rho_*(u) = \rho \circ u \in [X, \mathbb{HP}^\infty]$ to $[X, E_5]$, it is this composite which we need to understand. When $u^2 = 0$, the map $\rho' \circ u$ lifts to $u'\colon X \to K(\mathbb{Z}/2, 5)$, and $\mathrm{Sq}^2(H^3(X;\mathbb{Z}/2)) = 0$ means that $\rho' \circ u \simeq *$ if and only if $u' = 0$. Clearly we may identify

$$\theta(u) = u' \in H^5(X;\mathbb{Z}/2).$$

To proceed it is convenient to identify $\mathbb{CP}^\infty \simeq K(\mathbb{Z},2)$ and regard the map $-x^2\colon K(\mathbb{Z},2) \to K(\mathbb{Z}/4)$ as an unstable cohomology operation. Let F be its homotopy fibre and consider the diagram



The relation $\mathrm{Sq}^2(-x^2) = -2x^3 = 0$ gives rise to a colifting $\widetilde{\mathrm{Sq}}^2\colon F \to K(\mathbb{Z}/2, 5)$, which is constructed so that $i \circ \widetilde{\mathrm{Sq}}^2 \simeq -\rho' \circ j$ [11, p. 56]. Similarly, since $u^2 = 0$, we have $\tilde u\colon X \to F$ lifting u. Because $\widetilde{\mathrm{Sq}}^2 \circ \tilde u$ has order 2, we have $i \circ \widetilde{\mathrm{Sq}}^2 \circ \tilde u \simeq \rho' \circ u$. Hence

$$\theta(u) = \widetilde{\mathrm{Sq}}^2 \circ \tilde u.$$

On the other hand, the null composition $\mathrm{Sq}^2 \circ (-x^2) \simeq *$ gives rise to a secondary operation

$$\Theta'\colon \{u \in H^2(X) \mid u^2 = 0\} \to H^5(X;\mathbb{Z}/2),$$

whose indeterminacy vanishes. This acts on its domain as $\Theta'(u) = \widetilde{\mathrm{Sq}}^2 \circ \tilde{u}$, so

$$\Theta'(u) = \theta(u).$$

However, for $u \in H^2(X)$ we have $\mathrm{Sq}^2(\rho_2(u)) = \rho_2(u^2)$, meaning that $\Theta(u)$ is defined whenever $\Theta'(u)$ is. Clearly $\Theta(u) = \Theta'(u)$ in this case. □

Remark 6.11 For $u \in H^2(X;\mathbb{Z})$ satisfying $u^2 = 0$, consider the sequence

$$X \xrightarrow{u} \mathbb{CP}^\infty \xrightarrow{x^2} K(\mathbb{Z},4) \xrightarrow{\mathrm{Sq}^2} K(\mathbb{Z}/2,6).$$

Each pair of consecutive arrows composes to a null map, so the functional cohomology operation

$$\mathrm{Sq}^2_u\colon H^4(\mathbb{CP}^\infty;\mathbb{Z}) \to H^5(X;\mathbb{Z}/2)/\mathrm{Sq}^2(H^3(X;\mathbb{Z}/2))$$

is defined [28, Chapter 16.1]. Inspecting (24), we see that $\theta(u) = \mathrm{Sq}^2_u(x^2) \in H^5(X;\mathbb{Z}/2)$.

Combining Proposition 6.10 and Taylor's Theorem 6.9, standard methods now yield the following.

Proposition 6.12 *Let X be a 5-dimensional CW complex.*

(1) *The map $\eta_*\colon \pi^3(X) \to \pi^2(X)$ is injective with image $h^{-1}(0) = \{\alpha \in \pi^2(X) \mid \alpha^*(s_2) = 0\}$ where $s_2 \in H^2(S^2;\mathbb{Z})$ is a generator.*

(2) *If $\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$ is onto, then there is a pairwise-disjoint decomposition*

$$\pi^2(X) = \bigcup \{h^{-1}(u) \mid u \in H^2(X),\ u^2 = 0\}.$$

If $u \in H^2(X;\mathbb{Z})$ satisfies $u^2 = 0$, then $h^{-1}(u) \subseteq \pi^2(X)$ is nonempty, and

$$h^{-1}(u) \cong \pi^3(X)/\psi_u(H^1(X)),$$

where ψ_u is the group homomorphism given in Theorem 6.9.

(3) *If $\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$ is trivial, then there is a pairwise-disjoint decomposition*

$$\pi^2(X) = \bigcup \{h^{-1}(u) \mid u \in H^2(X),\ u^2 = 0 = \Theta(u)\}.$$

If $u \in H^2(X;\mathbb{Z})$ and $u^2 = 0 = \Theta(u)$, then $h^{-1}(u) \subseteq \pi^2(X)$ is nonempty and

$$h^{-1}(u) \cong \pi^3(X)/\psi_u(H^1(X)),$$

where ψ_u is the group homomorphism given in Theorem 6.9.

Remark 6.13 For X as in Proposition 6.12 and $u \in H^2(X;\mathbb{Z})$ lifting to $\tilde{u}\colon X \to S^2$, there is an explicit description of the homomorphism ψ_u given in Taylor [35, Section 5]. Since $\Omega\rho \simeq *$, this gives for any $\alpha \in H^1(S;\mathbb{Z})$ that the map $\psi_u(\alpha)$ is the composite

$$X \xrightarrow{\bar{\Delta}} X \wedge X \xrightarrow{\alpha\wedge\tilde{u}} S^1 \wedge S^2 \xrightarrow{\phi} S^3$$

for some map ϕ. The map ϕ must extend over $S^1 \wedge \mathbb{CP}^2$ and hence must have even degree. In fact, according to [35, Corollary 5.5, Lemma 6.5], ϕ has degree ± 2.

We have two applications for Proposition 6.12, which we give below. In each case it shows that the set $\pi^2(X)$ is determined by stable data.

Corollary 6.14 *If X is an orientable, 5-dimensional Poincaré duality complex, $H_1(X;\mathbb{Z})$ is torsion-free, and $H_2(X;\mathbb{Z})$ is torsion, then $\eta\colon S^3 \to S^2$ induces a bijection $\pi^2(X) \cong \pi^3(X)$.*

Proof Because $H_2(X)$ is torsion, duality gives $H^2(X) \cong H_3(X) = 0$ (compare (1)). Thus the statement follows from part (1) of Proposition 6.12. □

We remark that a class of smooth, nonsimply connected 5-manifolds satisfying the assumption of Corollary 6.14 is studied in [22, Theorem 1.3].

Corollary 6.15 *Let X be a simply connected, orientable, 5-dimensional Poincaré duality complex.*

(1) *If $\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$ is onto, then*
$$\pi^2(X) = \pi^3(X) \times H^2(X;\mathbb{Z}).$$

(2) *If $\mathrm{Sq}^2\colon H^3(X;\mathbb{Z}/2) \to H^5(X;\mathbb{Z}/2)$ is trivial, then*
$$\pi^2(X) \cong \pi^3(X) \times \{x \in H^2(X;\mathbb{Z}) \mid \Theta(x) = 0\},$$
where $\Theta\colon H^2(X;\mathbb{Z}) \to H^5(X;\mathbb{Z}/2)$ is the secondary operation (23).

Proof The hypothesis of simple connectivity gives $H^1(X) = 0$, meaning that each of the ψ_u homomorphisms is trivial. Since $H^4(X) \cong H_1(X) = 0$, the condition $u^2 = 0$ on $u \in H^2(X)$ is vacuous. Thus in light of Proposition 6.12 the result is clear. □

Proof of Theorem 1.5 The first statement is implied by Corollary 6.15. When X is nonspin, part (1) of this corollary applies and the result is immediate. Thus we reduce to the case in which X is spin. Then the secondary operation $\Theta\colon H^2(X;\mathbb{Z}) \to H^5(X;\mathbb{Z}/2)$ is defined without indeterminacy. It is known that Θ detects η^2, and, as has been discussed in Corollary 5.9, Barden's classification result (Theorem 5.7) implies that for this reason Θ vanishes on the cohomology of any simply connected 5-manifold.

The second statement of the theorem is a special case of Corollary 6.14. □

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