Algebraic & Geometric Topology

Volume 26 (2026)

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Algebraic & Geometric Topology **26**:1 (2026) 397–410
DOI: 10.2140/agt.2026.26.397
Published: January 16, 2026

Profinite completions of products

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A source of difficulty in profinite homotopy theory is that the profinite completion functor does not preserve finite products. We provide a new, checkable criterion on prospaces X and Y that guarantees that the profinite completion of $X \times Y$ agrees with the product of the profinite completions of X and Y. Using this criterion, we show that profinite completion preserves products of étale homotopy types of qcqs schemes. This fills a gap in Chough's proof of the Künneth formula for the étale homotopy type of a product of proper schemes over a separably closed field.

0 Introduction

Write $\mathbf{Spc}_\pi$ for the ∞-category of π-finite spaces. Given a set Σ of primes, write $\mathbf{Spc}_\Sigma \subset \mathbf{Spc}_\pi$ for the full subcategory spanned by those π-finite spaces whose homotopy groups have orders divisible only by primes in Σ. Write

$$(-)^\wedge_\Sigma : \mathrm{Pro}(\mathbf{Spc}) \to \mathrm{Pro}(\mathbf{Spc}_\Sigma)$$

for the Σ-completion functor, i.e., the left adjoint to the inclusion $\mathrm{Pro}(\mathbf{Spc}_\Sigma) \subset \mathrm{Pro}(\mathbf{Spc})$. One source of difficulty in profinite homotopy theory is that the Σ-completion functor does not preserve finite limits, or even finite products (see [5, Remark 3.10; 15, Remark E.5.2.6]).

As far as we are aware, given connected spaces X and Y, the only general condition to check that the profinite completion of $X \times Y$ is the product of the profinite completions of X and Y is to check that the homotopy groups of X and Y are *good* in the sense of Serre [5, Proposition 3.9]. However, it is generally quite difficult to check if a group is good, and there are hard conjectures about whether or not naturally occurring groups are good. For example, Deligne and Morava's conjecture that the mapping class group $\Gamma_{g,n}$ of a genus-g curve with n marked points is good [19, problem on p. 94] is still open.

The purpose of this note is to provide the following new, checkable criterion on prospaces X and Y that guarantees that the natural map $(X \times Y)^\wedge_\Sigma \to X^\wedge_\Sigma \times Y^\wedge_\Sigma$ is an equivalence.

MSC2020: 55P99.

0.1 Theorem (Theorem 2.13) *Let X and Y be prospaces and let Σ be a set of primes. If X and Y are both in the smallest full subcategory of* $\mathrm{Pro}(\mathbf{Spc})$ *containing* $\mathrm{Pro}(\mathbf{Spc}_\pi)$ *that is closed under geometric realizations of simplicial objects, retracts, and cofiltered limits, then the natural map*

$$(X \times Y)^{\wedge}_{\Sigma} \to X^{\wedge}_{\Sigma} \times Y^{\wedge}_{\Sigma}$$

is an equivalence.

In fact, Theorem 2.13 is a bit more general. However, the more general statement requires introducing a bit of terminology, so we refer the reader to later in the text for the statement.

Now we state the two key applications. The first is that if one is already in the setting of profinite homotopy theory, then Σ-completion preserves products:

0.2 Corollary (Corollary 2.9) *Let Σ be a set of primes. Then the Σ-completion functor restricted to profinite spaces*

$$(-)^{\wedge}_{\Sigma} : \mathrm{Pro}(\mathbf{Spc}_\pi) \to \mathrm{Pro}(\mathbf{Spc}_\Sigma)$$

preserves products.

The second is that in the setting of étale homotopy theory, Σ-completion preserves finite products. Given a scheme X, write $\Pi^{\text{ét}}_{\infty}(X) \in \mathrm{Pro}(\mathbf{Spc})$ for the étale homotopy type of X. Our work with Barwick and Glasman [1, Theorems 10.2.3 and 12.5.1] provides a description of the (protruncated) étale homotopy type of a qcqs scheme as the geometric realization of an explicit simplicial profinite space. The existence of this presentation implies:

0.3 Corollary (Example 2.17) *Let Σ be a set of primes and let X and Y be qcqs schemes. Then the natural map of profinite spaces*

$$(\Pi^{\text{ét}}_{\infty}(X) \times \Pi^{\text{ét}}_{\infty}(Y))^{\wedge}_{\Sigma} \to \Pi^{\text{ét}}_{\infty}(X)^{\wedge}_{\Sigma} \times \Pi^{\text{ét}}_{\infty}(Y)^{\wedge}_{\Sigma}$$

is an equivalence.

0.4 Remark Corollary 0.3 fills a gap in Chough's proof of the Künneth formula for the étale homotopy type of a product of proper schemes over a separably closed field [7, Theorem 5.3]. Chough's proof cites the false claim that profinite completion preserves finite limits. However, what Chough actually uses is Corollary 0.3 (with Σ the set of all primes). In particular, the conclusion of [7, Theorem 5.3] remains valid.

In our work with Holzschuh and Wolf [9, §4], Corollary 0.3 is a key ingredient used to prove other Künneth formulas in étale homotopy theory.

0.5 Proof Strategy The first observation is that since protruncation preserves limits [8, Proposition 3.9], it suffices to prove a variant of Theorem 0.1 where X and Y are already protruncated. Since equivalences of protruncated spaces are detected on all truncations (which is not true for general prospaces), this simplifies the situation. Second, since Σ-completion preserves cofiltered limits, Theorem 0.1 would follow if we knew that geometric realizations preserved finite products in the ∞-categories of protruncated and Σ-profinite spaces. See Lemma 2.7 and Corollary 2.8 for the key categorical result explaining this.

With these reductions, the main technical step in our argument is to show the stronger claim that geometric realizations are *universal* in the ∞-categories of protruncated and Σ-profinite spaces. In fact, we show that the colimit of any diagram which can be evaluated as a finite colimit in every n-category for $n < \infty$ (see Definition 1.9) is universal in these ∞-categories. See Proposition 1.17 and Corollary 1.18.

0.6 Linear Overview Section 1 proves that geometric realizations are universal in the ∞-categories of protruncated and profinite spaces. In particular, geometric realizations preserve finite products in these ∞-categories. Section 2 proves Corollaries 0.2 and 0.3. It is immediate from [1, Theorem 10.2.3] that the protruncated étale homotopy type can be written as the geometric realization of a simplicial profinite space. However, for ease of reference we have provided a detailed explanation of this fact in the Appendix.

0.7 Conventions Throughout, we use the notational conventions of [8, §§1 and 3]. In an effort to keep this note short, we do not recapitulate them here.

1 Universality of colimits

In this section, we prove that geometric realizations are universal in the ∞-categories of protruncated and Σ-profinite spaces. We accomplish this by proving a more general fact: colimits over diagrams that can be computed as finite colimits when valued in an n-category (see Definition 1.9) are universal in the ∞-categories of protruncated and Σ-profinite spaces (Proposition 1.17 and Corollary 1.18).

The first observation is that finite colimits are universal in protruncated spaces.

1.1 Lemma *Let $\mathcal{C}$ be an ∞-category with pullbacks and finite colimits. If finite colimits are universal in $\mathcal{C}$, then finite colimits are universal in* $\mathrm{Pro}(\mathcal{C})$.

Proof By (the dual of) [12, Proposition 5.3.5.15], pullbacks, pushouts, and finite coproducts are computed "levelwise" in $\mathrm{Pro}(\mathcal{C})$. Thus the assumption that finite colimits are universal in $\mathcal{C}$ implies that pushouts and finite coproducts are universal in $\mathrm{Pro}(\mathcal{C})$. Since a functor that preserves pushouts and finite coproducts preserves finite colimits, we deduce that all finite colimits are universal in $\mathrm{Pro}(\mathcal{C})$. □

1.2 Example Finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc})$. For each integer $n \geq 0$, finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$.

1.3 Recollection A localization $L : \mathcal{C} \to \mathcal{D}$ is *locally cartesian* if for any cospan $X \to Z \leftarrow Y$ such that $X, Z \in \mathcal{D}$, the natural map $L(X \times_Z Y) \to X \times_Z L(Y)$ is an equivalence.

1.4 Example [8, Proposition 3.18] For any set Σ of primes, the localization

$$(-)^{\wedge}_{\Sigma} : \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \to \mathrm{Pro}(\mathbf{Spc}_{\Sigma})$$

is locally cartesian. However, $(-)^{\wedge}_{\Sigma}$ does not generally preserve finite products.

The following is immediate from the definitions:

1.5 Lemma *Let $\mathcal{I}$ be an ∞-category, $\mathcal{C}$ an ∞-category with pullbacks and $\mathcal{I}$-shaped colimits, and let $L : \mathcal{C} \to \mathcal{D}$ be a locally cartesian localization. If $\mathcal{I}$-shaped colimits are universal in $\mathcal{C}$, then $\mathcal{I}$-shaped colimits are universal in $\mathcal{D}$.*

1.6 Example Since the protruncation functor $\tau_{<\infty} : \mathrm{Pro}(\mathbf{Spc}) \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})$ preserves limits [8, Proposition 3.9], Example 1.2 and Lemma 1.5 show that finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$.

Now we formulate the key property of the category $\mathbf{\Delta}^{\mathrm{op}}$ that we need.

1.7 Definition Let $n \geq 0$ be an integer. A functor between ∞-categories $c : \mathcal{I} \to \mathcal{J}$ is *n-colimit-cofinal* if for every n-category $\mathcal{C}$ and functor $f : \mathcal{J} \to \mathcal{C}$, the following conditions are satisfied:

(1) The colimit $\mathrm{colim}_{\mathcal{J}} f$ exists if and only if the colimit $\mathrm{colim}_{\mathcal{I}} fc$ exists.

(2) If the colimit $\mathrm{colim}_{\mathcal{J}} f$ exists, then the natural map $\mathrm{colim}_{\mathcal{I}} fc \to \mathrm{colim}_{\mathcal{J}} f$ is an equivalence.

1.8 Example For an integer $n \geq 0$, write $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ for the full subcategory spanned by those nonempty linearly ordered finite sets of cardinality $\leq n+1$. By [10, Proposition A.1], the inclusion $\mathbf{\Delta}_{\leq n}^{\mathrm{op}} \subset \mathbf{\Delta}^{\mathrm{op}}$ is n-colimit-cofinal.

1.9 Definition Let $\mathcal{I}$ be an ∞-category. We say that $\mathcal{I}$ is *almost finite* if for each integer $n \geq 0$, there exists a *finite* ∞-category $\mathcal{I}_n$ and an n-colimit-cofinal functor $c_n : \mathcal{I}_n \to \mathcal{I}$.

Here are a number of important examples of almost finite ∞-categories.

1.10 Example If $\mathcal{I}$ is an ∞-category that admits a colimit-cofinal functor from a finite ∞-category, then $\mathcal{I}$ is almost finite.

1.11 Example For each $n \geq 0$, the category $\mathbf{\Delta}_{\leq n}^{\mathrm{op}}$ is a finite ∞-category [6, Example 6.5.3]. Hence the category $\mathbf{\Delta}^{\mathrm{op}}$ is almost finite: the inclusion $\mathbf{\Delta}_{\leq n}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is an n-colimit-cofinal functor from a finite ∞-category.

1.12 Definition Let K be a simplicial set. The *∞-category presented by* K is the image of K under the natural functor $\mathrm{s}\mathbf{Set} \to \mathbf{Cat}_{\infty}$ obtained by inverting the weak equivalences in the Joyal model structure. The *space presented by* K is the image of K under the natural functor $\mathrm{s}\mathbf{Set} \to \mathbf{Spc}$ obtained by inverting the weak equivalences in the Kan–Quillen model structure.

1.13 Example Let K be a simplicial set with finitely many simplices in each dimension and let $\mathcal{I}$ be the ∞-category presented by K. Then $\mathcal{I}$ is almost finite: we take $\mathcal{I}_n$ to be the ∞-category presented by the $(n+1)$-skeleton of $\mathrm{sk}_{n+1} K$ and $c_n : \mathcal{I}_n \to \mathcal{I}$ the functor induced by the inclusion $\mathrm{sk}_{n+1} K \subset K$.

1.14 Recollection A space X is *almost π-finite* if $\pi_0(X)$ is finite and all homotopy groups of X are finite. An almost π-finite space admits a presentation by a Kan complex with finitely many simplices in each dimension [15, Lemma E.1.6.5]. Hence:

1.15 Example As a special case of Example 1.13, every almost π-finite space is an almost finite ∞-category.

1.16 Definition Let $\mathcal{C}$ be an ∞-category with pullbacks. We say that *almost finite colimits are universal in* $\mathcal{C}$ if for each almost finite ∞-category $\mathcal{I}$, the ∞-category $\mathcal{C}$ admits $\mathcal{I}$-shaped colimits and $\mathcal{I}$-shaped colimits are universal in $\mathcal{C}$.

The argument that Lurie gives in the proof of [15, Theorem E.6.3.1] essentially shows that almost finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc}_\Sigma)$. However, the statement about universality of colimits is less general and the result is only stated when Σ is the set of all primes. We also need to know that almost finite colimits are also universal in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$. The strategy is the same as Lurie's proof: we use that equivalences are checked on truncations to reduce to the case of finite colimits.

1.17 Proposition *Almost finite colimits are universal in* $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$.

Moreover, Proposition 1.17 strengthens [15, Theorem E.6.3.1]:

1.18 Corollary *Let* Σ *be a set of primes. Then almost finite colimits are universal in* $\mathrm{Pro}(\mathbf{Spc}_\Sigma)$.

Proof of Corollary 1.18 Since the localization $(-)^\wedge_\Sigma : \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \to \mathrm{Pro}(\mathbf{Spc}_\Sigma)$ is locally cartesian [8, Proposition 3.18], this follows from Lemma 1.5 and Proposition 1.17. □

Proof of Proposition 1.17 Let $\mathcal{I}$ be an almost finite ∞-category, let $f : X \to Z$ be a morphism in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$, and let

$$g : \mathcal{I} \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})_{/Z}$$

be a diagram of protruncated spaces over Z. To prove the claim, it suffices to show that for each integer $n \geq 0$, the induced map

$$\tau_{\leq n}(\operatorname*{colim}_{i \in \mathcal{I}} X \times_Z g(i)) \to \tau_{\leq n}(X \times_Z \operatorname*{colim}_{i \in \mathcal{I}} g(i))$$

is an equivalence in $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$. Since $\mathcal{I}$ is almost finite, there exists a finite ∞-category $\mathcal{I}_{n+2}$ and $(n+2)$-colimit-cofinal functor $c_{n+2} : \mathcal{I}_{n+2} \to \mathcal{I}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} \tau_{\leq n}(X \times_Z gc_{n+2}(j)) & \xrightarrow{\sim} & \tau_{\leq n}(\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} X \times_Z gc_{n+2}(j)) & \longrightarrow & \tau_{\leq n}(X \times_Z \operatorname*{colim}_{j \in \mathcal{I}_{n+2}} gc_{n+2}(j)) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname*{colim}_{i \in \mathcal{I}} \tau_{\leq n}(X \times_Z g(i)) & \xrightarrow{\sim} & \tau_{\leq n}(\operatorname*{colim}_{i \in \mathcal{I}} X \times_Z g(i)) & \longrightarrow & \tau_{\leq n}(X \times_Z \operatorname*{colim}_{i \in \mathcal{I}} g(i))
\end{array}$$

(Here, the colimits in the leftmost column are computed in $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$.) Since $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$ is an $(n+1)$-category and $c_{n+2} : \mathcal{I}_{n+2} \to \mathcal{I}$ is $(n+2)$-colimit-cofinal, the leftmost vertical map is an equivalence. Thus the central vertical map is also an equivalence. Since $\mathcal{I}_{n+2}$ is finite and finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ (Example 1.6), the top right-hand horizontal map is an equivalence.

To complete the proof, it suffices to show that the rightmost vertical map is an equivalence. For this, consider the commutative square

$$\begin{array}{ccc} \operatorname*{colim}_{j \in \mathcal{I}_{n+2}} \tau_{\leq n+1} g c_{n+2}(j) & \xrightarrow{\sim} & \tau_{\leq n+1}(\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} g c_{n+2}(j)) \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{i \in \mathcal{I}} \tau_{\leq n+1} g(i) & \xrightarrow{\sim} & \tau_{\leq n+1}(\operatorname*{colim}_{i \in \mathcal{I}} g(i)) \end{array}$$

where the colimits in the left-hand column are computed in $\mathrm{Pro}(\mathbf{Spc}_{\leq n+1})$. Since $\mathrm{Pro}(\mathbf{Spc}_{\leq n+1})$ is an $(n+2)$-category and c_{n+2} is $(n+2)$-colimit-cofinal, the left-hand vertical map is an equivalence. Hence the right-hand vertical map is also an equivalence. As a consequence, the map

$$\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} g c_{n+2}(j) \to \operatorname*{colim}_{i \in \mathcal{I}} g(i)$$

is n-connected. Thus the base change

$$X \times_Z \operatorname*{colim}_{j \in \mathcal{I}_{n+2}} g c_{n+2}(j) \to X \times_Z \operatorname*{colim}_{i \in \mathcal{I}} g(i)$$

is also n-connected. Hence the map

$$\tau_{\leq n}(X \times_Z \operatorname*{colim}_{j \in \mathcal{I}_{n+2}} g c_{n+2}(j)) \to \tau_{\leq n}(X \times_Z \operatorname*{colim}_{i \in \mathcal{I}} g(i))$$

is an equivalence, as desired. □

The universality of geometric realizations implies that geometric realizations preserve finite products:

1.19 Lemma *Let $\mathcal{I}$ be a sifted ∞-category and let $\mathcal{C}$ be an ∞-category with finite limits and $\mathcal{I}$-shaped colimits. If $\mathcal{I}$-shaped colimits are universal in $\mathcal{C}$, then the functor*

$$\operatorname{colim}_{\mathcal{I}} : \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$$

preserves finite products.

Proof Let $X_\bullet, Y_\bullet : \mathcal{I} \to \mathcal{C}$ be functors. We have natural equivalences

$$\begin{aligned} \operatorname*{colim}_{i \in \mathcal{I}} X_i \times Y_i &\xrightarrow{\sim} \operatorname*{colim}_{(i,j) \in \mathcal{I} \times \mathcal{I}} X_i \times Y_j && (\mathcal{I} \text{ is sifted}) \\ &\simeq \operatorname*{colim}_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{I}} (X_i \times Y_j) \\ &\xrightarrow{\sim} \operatorname*{colim}_{i \in \mathcal{I}} (X_i \times \operatorname*{colim}_{j \in \mathcal{I}} Y_j) && (\mathcal{I}\text{-shaped colimits are universal}) \\ &\xrightarrow{\sim} (\operatorname*{colim}_{i \in \mathcal{I}} X_i) \times (\operatorname*{colim}_{j \in \mathcal{I}} Y_j) && (\mathcal{I}\text{-shaped colimits are universal}). \end{aligned}$$

□

1.20 Corollary *Let Σ be a set of primes. Then geometric realizations preserve finite products in the ∞-categories $\mathrm{Pro}(\mathbf{Spc}_\Sigma)$ and $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$.*

Proof Combine Proposition 1.17, Corollary 1.18, and Lemma 1.19. □

2 Completions of products

The goal of this section is to prove Corollaries 0.2 and 0.3. We do this by noting that in the setting of both of these results, the prospaces of interest can be written as geometric realizations of simplicial profinite spaces. We begin by axiomatizing the situation:

2.1 Definition Let $\mathcal{C}$ be an ∞-category, let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory, and let $X \in \mathcal{C}$. A $\mathcal{D}$*-resolution* of X is a simplicial object $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{D}$ together with an equivalence

$$X \simeq \operatorname{colim}\big(\, \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{D} \hookrightarrow \mathcal{C} \,\big).$$

We say that an object X *admits a* $\mathcal{D}$*-resolution* if there exists $\mathcal{D}$-resolution of X.

2.2 Notation Write $\mathbf{Set}_{\mathrm{fin}}$ for the category of finite sets and all maps.

Finite and almost π-finite spaces admit $\mathbf{Set}_{\mathrm{fin}}$-resolutions:

2.3 Lemma *Let X be a space which is finite*[1] *or almost π-finite. Then*:

(1) *As an object of* $\mathbf{Spc}$, *the space X admits a* $\mathbf{Set}_{\mathrm{fin}}$*-resolution.*

(2) *When regarded as a constant object of* $\mathrm{Pro}(\mathbf{Spc})$, *the space X admits a* $\mathbf{Set}_{\mathrm{fin}}$*-resolution.*

(3) *The protruncated space* $\tau_{<\infty}(X)$ *admits a* $\mathbf{Set}_{\mathrm{fin}}$*-resolution.*

(4) *The profinite space* $X^{\wedge}_{\pi}$ *admits a* $\mathbf{Set}_{\mathrm{fin}}$*-resolution.*

Proof For (1), if X is finite, then X can be written as the geometric realization of a simplicial set with finitely many nondegenerate simplices [21, Proposition 2.5].[2] If X is almost π-finite, then X can be written as the geometric realization of a Kan complex with finitely many simplices in each dimension [15, Lemma E.1.6.5].

Now note that (2)–(4) follow from (1) and that each functor in the diagram

$$\mathbf{Spc} \hookrightarrow \mathrm{Pro}(\mathbf{Spc}) \xrightarrow{\tau_{<\infty}} \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \xrightarrow{(-)^{\wedge}_{\pi}} \mathrm{Pro}(\mathbf{Spc}_{\pi})$$

preserves colimits. □

Algebraic geometry also gives rise to many examples of protruncated spaces admitting profinite resolutions.

2.4 Notation (shapes) Given an ∞-topos $\boldsymbol{X}$, we write $\Pi_\infty(\boldsymbol{X}) \in \mathrm{Pro}(\mathbf{Spc})$ for the *shape* of $\boldsymbol{X}$. We write $\Pi_{<\infty}(\boldsymbol{X})$ for the protruncation of $\Pi_\infty(\boldsymbol{X})$.

[1] I.e., in the smallest subcategory of $\mathbf{Spc}$ containing the point and the empty set and closed under pushouts. Equivalently, a space X is finite if and only if X is represented by a finite CW complex.

[2] In fact, every finite space is equivalent to the classifying space of a finite poset; see [12, Proposition 4.1.1.3(3) and Variant 4.2.3.16; 16, Tag 02MU; 18, Theorem 1].

2.5 Recall that an ∞-topos $\boldsymbol{X}$ is *spectral* in the sense of [1, Definition 9.2.1] if $\boldsymbol{X}$ is bounded coherent and the ∞-category $\mathrm{Pt}(\boldsymbol{X})$ of points of $\boldsymbol{X}$ has the property that every endomorphism of an object of $\mathrm{Pt}(\boldsymbol{X})$ is an equivalence. The most important example of a spectral ∞-topos is the étale ∞-topos of a qcqs scheme [1, Example 9.2.4]. Example A.9 explains why the protruncated shape $\Pi_{<\infty}(\boldsymbol{X})$ of a spectral ∞-topos admits a natural $\mathrm{Pro}(\mathbf{Spc}_\pi)$-resolution.

The next few results are the key categorical input we need. To state them, we axiomatize the abstract properties of the subcategory $\mathrm{Pro}(\mathbf{Spc}_\pi) \subset \mathrm{Pro}(\mathbf{Spc}_{<\infty})$.

2.6 Notation Let $\mathcal{C}$ be an ∞-category with geometric realizations and finite products, and let $L : \mathcal{C} \to \mathcal{D}$ be a localization. Assume that geometric realizations preserve finite products in both $\mathcal{C}$ and $\mathcal{D}$. Write $\mathcal{C}_{|\mathcal{D}|} \subset \mathcal{C}$ for the smallest full subcategory containing $\mathcal{D} \subset \mathcal{C}$ and closed under geometric realizations and retracts.

2.7 Lemma *In the setting of Notation 2.6, let $X_\bullet, Y_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be simplicial objects with colimits X and Y. Assume that for each $n \geq 0$, the natural map*

$$L(X_n \times Y_n) \to L(X_n) \times L(Y_n)$$

is an equivalence. Then the natural map $L(X \times Y) \to L(X) \times L(Y)$ is an equivalence.

Proof We compute

$$\begin{aligned}
L(X \times Y) &\simeq L(\operatorname*{colim}_{m \in \mathbf{\Delta}^{\mathrm{op}}} X_m \times \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} Y_n) \\
&\simeq L(\operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} X_n \times Y_n) && \text{(geometric realizations preserve finite products in } \mathcal{C}) \\
&\simeq \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} L(X_n \times Y_n) \\
&\xrightarrow{\sim} \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} L(X_n) \times L(Y_n) && \text{(assumption)} \\
&\simeq \operatorname*{colim}_{m \in \mathbf{\Delta}^{\mathrm{op}}} L(X_m) \times \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} L(Y_n) && \text{(geometric realizations preserve finite products in } \mathcal{D}) \\
&\simeq L(X) \times L(Y).
\end{aligned}$$

□

2.8 Corollary *In the setting of Notation 2.6*:

(1) *The full subcategory of $\mathcal{C} \times \mathcal{C}$ spanned by those objects (X, Y) such that the natural map*

$$L(X \times Y) \to L(X) \times L(Y)$$

is an equivalence is closed under geometric realizations and retracts.

(2) *If $X, Y \in \mathcal{C}_{|\mathcal{D}|}$, then the natural map $L(X \times Y) \to L(X) \times L(Y)$ is an equivalence.*

(3) *If $X, Y \in \mathcal{C}$ admit $\mathcal{D}$-resolutions, then the natural map $L(X \times Y) \to L(X) \times L(Y)$ is an equivalence.*

Proof Item (1) follows from Lemma 2.7 and the fact that equivalences are closed under retracts. Now (2) is an immediate consequence of (1) and the definition of $\mathcal{C}_{|\mathcal{D}|}$ as the closure of $\mathcal{D} \subset \mathcal{C}$ under geometric realizations and retracts. Finally, (3) is a special case of (2). □

We now record some consequences of Corollary 2.8.

2.9 Corollary *Let Σ be a set of prime numbers. Then the Σ-completion functor*

$$(-)^{\wedge}_{\Sigma} : \mathrm{Pro}(\mathbf{Spc}_{\pi}) \to \mathrm{Pro}(\mathbf{Spc}_{\Sigma})$$

preserves products.

Proof Since Σ-completion preserves cofiltered limits and the terminal object, it suffices to show that Σ-completion preserves binary products of profinite spaces. Again because Σ-completion preserves cofiltered limits, we are reduced to showing that if X and Y are π-finite spaces, then the natural map

$$(X \times Y)^{\wedge}_{\Sigma} \to X^{\wedge}_{\Sigma} \times Y^{\wedge}_{\Sigma}$$

is an equivalence. Since π-finite spaces admit $\mathbf{Set}_{\mathrm{fin}}$-resolutions (Lemma 2.3) and geometric realizations preserve finite products in $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ and $\mathrm{Pro}(\mathbf{Spc}_{\Sigma})$ (Corollary 1.20), the claim follows from Corollary 2.8. □

2.10 Warning The functor $(-)^{\wedge}_{\Sigma} : \mathrm{Pro}(\mathbf{Spc}_{\pi}) \to \mathrm{Pro}(\mathbf{Spc}_{\Sigma})$ does not generally preserve pullbacks, or even loop objects.

We now introduce a slight enlargement of the subcategory of protruncated spaces admitting a $\mathrm{Pro}(\mathbf{Spc}_{\pi})$-resolution on which profinite completion preserves finite products.

2.11 Notation Let

$$\mathrm{Pro}(\mathbf{Spc}_{<\infty})' \subset \mathrm{Pro}(\mathbf{Spc}_{<\infty})$$

denote the smallest full subcategory containing $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ and closed under geometric realizations, retracts, and cofiltered limits.

2.12 Observation (procompact spaces) Write $\mathbf{Spc}^{\omega} \subset \mathbf{Spc}$ for the full subcategory spanned by the compact objects.[3] Then by Lemma 2.3, the image of

$$\tau_{<\infty} : \mathrm{Pro}(\mathbf{Spc}^{\omega}) \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})$$

is contained in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})'$.

The following is the main result of this note.

2.13 Theorem *Let Σ be a set of primes and let $X, Y \in \mathrm{Pro}(\mathbf{Spc}_{<\infty})'$. Then the natural map*

$$(X \times Y)^{\wedge}_{\Sigma} \to X^{\wedge}_{\Sigma} \times Y^{\wedge}_{\Sigma}$$

is an equivalence.

[3] A space X is compact if and only if X is a retract of a finite space. In more classical terminology, a space X is compact if and only if X is represented by a *finitely dominated* CW complex.

Proof By Corollary 2.9, the Σ-completion functor

$$(-)^{\wedge}_{\Sigma} : \mathrm{Pro}(\mathbf{Spc}_{\pi}) \to \mathrm{Pro}(\mathbf{Spc}_{\Sigma})$$

preserves products. Hence it suffices to prove the claim in the special case where Σ is the set of all primes. For this, note that cofiltered limits preserve finite products in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ and $\mathrm{Pro}(\mathbf{Spc}_{\pi})$; moreover, the profinite completion functor preserves cofiltered limits. Thus it suffices to treat the case where X and Y are in the smallest full subcategory of $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ containing $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ and closed under geometric realizations and retracts. In this case, since geometric realizations preserve finite products in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ and $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ (Corollary 1.20), Corollary 2.8 completes the proof. $\square$

2.14 Example If X and Y are protruncated spaces that admit $\mathrm{Pro}(\mathbf{Spc}_{\pi})$-resolutions, then the natural map $(X \times Y)^{\wedge}_{\Sigma} \to X^{\wedge}_{\Sigma} \times Y^{\wedge}_{\Sigma}$ is an equivalence.

2.15 Example If $X, Y \in \mathbf{Spc}$ are compact, then the natural map $(X \times Y)^{\wedge}_{\Sigma} \to X^{\wedge}_{\Sigma} \times Y^{\wedge}_{\Sigma}$ is an equivalence.

We conclude by recording two applications of Theorem 2.13.

2.16 Corollary *Let Σ be a set of primes and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be spectral ∞-topoi. Then the natural map*

$$(\Pi_{\infty}(\boldsymbol{X}) \times \Pi_{\infty}(\boldsymbol{Y}))^{\wedge}_{\Sigma} \to \Pi_{\infty}(\boldsymbol{X})^{\wedge}_{\Sigma} \times \Pi_{\infty}(\boldsymbol{Y})^{\wedge}_{\Sigma}$$

is an equivalence.

Proof Example A.9 shows that $\Pi_{<\infty}(\boldsymbol{X})$ and $\Pi_{<\infty}(\boldsymbol{Y})$ admit $\mathrm{Pro}(\mathbf{Spc}_{\pi})$-resolutions. Since protruncation preserves products, the claim now follows from Theorem 2.13. $\square$

2.17 Example Let Σ be a set of primes and let X and Y be qcqs schemes. Then the natural map

$$(\Pi^{\text{ét}}_{\infty}(X) \times \Pi^{\text{ét}}_{\infty}(Y))^{\wedge}_{\Sigma} \to \Pi^{\text{ét}}_{\infty}(X)^{\wedge}_{\Sigma} \times \Pi^{\text{ét}}_{\infty}(Y)^{\wedge}_{\Sigma}$$

is an equivalence.

The second application is to ∞-operads. Recently, it has become clear that it is important to consider profinite completions of ∞-operads. For example, Boavida de Brito, Horel, and Robertson explained a beautiful relationship between the Grothendieck–Teichmüller group and the profinite completion of the genus-0 surface operad [5]. However, since profinite completion does not preserve products, profinite completions of ∞-operads generally cannot be computed "levelwise". In general, more care is required to work with profinite completions of ∞-operads; this is the topic of recent work of Blom and Moerdijk [3; 4].

An immediate consequence of Theorem 2.13 is that profinite completions *can* be computed levelwise more generally than was previously known:

2.18 Corollary *Let Σ be a set of primes and let $\{\mathcal{O}(n)\}_{n\geq 0}$ be an ∞-operad in spaces. Assume that for each $n \geq 0$, we have*

$$\tau_{<\infty}\mathcal{O}(n) \in \mathrm{Pro}(\mathbf{Spc}_{<\infty})'.$$

Then the levelwise Σ-completion $\{\mathcal{O}(n)^{\wedge}_{\Sigma}\}_{n\geq 0}$ defines an ∞-operad in $\mathrm{Pro}(\mathbf{Spc}_{\Sigma})$.

Appendix Classifying prospaces via geometric realizations

The purpose of this appendix is to explain why the protruncated shape of a spectral ∞-topos (e.g., the étale ∞-topos of a qcqs scheme) admits a presentation as a geometric realization of a simplicial profinite space (Example A.9). Using the description of the protruncated shape of a spectral ∞-topos as a protruncated classifying prospace given by Barwick–Glasman–Haine [1, Theorem 10.2.3], this is an exercise in the definitions. For the ease of the reader, we spell out the details here.

We make use of the description of ∞-categories as simplicial spaces.

A.1 Recollection (∞-categories as simplicial spaces) The nerve functor

$$\mathrm{N} : \mathbf{Cat}_\infty \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Spc}), \quad \mathcal{C} \mapsto \left[I \mapsto \mathrm{Fun}(I, \mathcal{C})^{\simeq} \right],$$

is fully faithful [11; 13, §1; 14, Proposition A.7.10; 15, §A.8.2; 20]. One can explicitly identify its image; objects in the image of this embedding are often called *complete Segal spaces* or *categories internal to spaces*. Under this embedding, the subcategory $\mathbf{Spc} \subset \mathbf{Cat}_\infty$ corresponds to the constant functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Spc}$. Moreover, the localization $\mathrm{B} : \mathbf{Cat}_\infty \to \mathbf{Spc}$ is given by geometric realization.

A.2 Notation Let $\mathcal{C}$ be an ∞-category with finite limits. We write

$$\mathrm{Cat}(\mathcal{C}) \subset \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$$

for the full subcategory spanned by the *categories internal to* $\mathcal{C}$. See [1, Definition 13.1.1; 17, Proposition 3.2.7] for the definition.

A.3 Notation Write $\mathbf{Cat}_{<\infty} \subset \mathbf{Cat}_\infty$ for the full subcategory spanned by those ∞-categories $\mathcal{C}$ for which there exists an integer $n \geq 0$ such that $\mathcal{C}$ is an n-category. Write $\mathbf{Cat}_{\infty,\pi} \subset \mathbf{Cat}_{<\infty}$ for the full subcategory spanned by those ∞-categories with the property that there are finitely many objects up to equivalence and all mapping spaces are π-finite.

A.4 Observation The nerve $\mathrm{N} : \mathbf{Cat}_\infty \xrightarrow{\sim} \mathrm{Cat}(\mathbf{Spc})$ restricts to equivalences

$$\mathbf{Cat}_{<\infty} \xrightarrow{\sim} \mathrm{Cat}(\mathbf{Spc}_{<\infty}) \quad \text{and} \quad \mathbf{Cat}_{\infty,\pi} \xrightarrow{\sim} \mathrm{Cat}(\mathbf{Spc}_{\pi}).$$

In order to describe protruncated classifying spaces via geometric realizations, it is useful to describe pro-∞-categories as category objects in prospaces.

A.5 Observation By [1, Proposition 13.1.12; 12, Proposition 5.3.5.11], the composite

$$\mathbf{Cat}_{<\infty} \xrightarrow[\mathrm{N}]{\sim} \mathrm{Cat}(\mathbf{Spc}_{<\infty}) \hookrightarrow \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{<\infty}))$$

extends along cofiltered limits to a fully faithful right adjoint

$$\mathrm{N} : \mathrm{Pro}(\mathbf{Cat}_{<\infty}) \hookrightarrow \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{<\infty})).$$

This functor restricts to a fully faithful right adjoint

$$\mathrm{N} : \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi}) \hookrightarrow \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{\pi})).$$

A.6 Remark We do not know if the embedding $\mathrm{N} : \mathbf{Cat}_\infty \hookrightarrow \mathbf{Cat}(\mathrm{Pro}(\mathbf{Spc}))$ extends along cofiltered limits to a fully faithful functor $\mathrm{Pro}(\mathbf{Cat}_\infty) \to \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}))$.

A.7 Observation It is immediate from the definitions that the following diagram of fully faithful right adjoints commutes:

$$\begin{array}{ccccccc}
\mathrm{Pro}(\mathbf{Spc}_\pi) & \underset{\perp}{\overset{\mathrm{B}^{\wedge}_{\pi}}{\rightleftarrows}} & \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi}) & \underset{\mathrm{N}}{\overset{\perp}{\rightleftarrows}} & \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_\pi)) & \overset{\perp}{\rightleftarrows} & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_\pi)) \\
{\scriptstyle (-)^{\wedge}_{\pi}} \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow & & {\scriptstyle (-)^{\wedge}_{\pi} \circ -} \uparrow \dashv \downarrow \\
\mathrm{Pro}(\mathbf{Spc}_{<\infty}) & \underset{\perp}{\overset{\mathrm{B}_{<\infty}}{\rightleftarrows}} & \mathrm{Pro}(\mathbf{Cat}_{<\infty}) & \underset{\mathrm{N}}{\overset{\perp}{\rightleftarrows}} & \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{<\infty})) & \overset{\perp}{\rightleftarrows} & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_{<\infty}))
\end{array}$$

The long composite left adjoints

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_{<\infty})) \to \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \quad \text{and} \quad \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_{\pi})) \to \mathrm{Pro}(\mathbf{Spc}_{\pi})$$

are simply the colimit functors. Since the diagram of left adjoints also commutes we deduce:

A.8 Corollary *Let $\mathcal{C} \in \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi})$ be a profinite ∞-category. Then there are natural equivalences*

$$\mathrm{B}_{<\infty}(\mathcal{C}) \simeq \mathrm{colim}\big(\, \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{N}(\mathcal{C})} \mathrm{Pro}(\mathbf{Spc}_\pi) \hookrightarrow \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \,\big),$$

$$\mathrm{B}^{\wedge}_{\pi}(\mathcal{C}) \simeq \mathrm{colim}\big(\, \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{N}(\mathcal{C})} \mathrm{Pro}(\mathbf{Spc}_\pi) \,\big).$$

A.9 Example Let X be a spectral ∞-topos. Through [1, Theorem 9.3.1], Barwick–Glasman–Haine refined the ∞-category $\mathrm{Pt}(X)$ of points of X to a profinite ∞-category

$$\widehat{\Pi}_{(\infty,1)}(X) \in \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi})$$

called the *stratified shape* of X. In [1, Theorem 10.2.3] they show that there is a natural equivalence

$$\Pi_{<\infty}(X) \xrightarrow{\sim} \mathrm{B}_{<\infty}(\widehat{\Pi}_{(\infty,1)}(X)).$$

That is, the protruncated shape of X can be recovered as the protruncated classifying space of the stratified shape $\widehat{\Pi}_{(\infty,1)}(X)$. Hence Corollary A.8 shows that the protruncated shape $\Pi_{<\infty}(X)$ admits a natural $\mathrm{Pro}(\mathbf{Spc}_\pi)$-resolution in the sense of Definition 2.1.

A.10 Remark Using the proétale topology [2], one can show that the protruncated étale homotopy type of a qcqs scheme admits a $\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})$-resolution.

To see this, first note that given a qcqs scheme X, the pullback functor $\nu^* : X_{\text{ét}} \to X_{\text{proét}}$ from the étale ∞-topos of X to the proétale ∞-topos of X is fully faithful when restricted to truncated objects. (This observation generalizes [2, Lemma 5.1.2 and Corollary 5.1.6].) As a result, the protruncated shapes of $X_{\text{ét}}$ and $X_{\text{proét}}$ are equivalent. Hence the protruncated étale homotopy type is a hypercomplete proétale cosheaf.

Recall that the w-contractible affine schemes form a basis for the proétale topology; in particular, every qcqs scheme admits a proétale hypercover by w-contractible affine schemes. Moreover, for each w-contractible affine scheme U, the prospace $\Pi^{\text{ét}}_{<\infty}(U)$ is naturally identified with the profinite set $\pi_0(U)$ of connected components of U. Hence the protruncated étale homotopy type

$$\Pi^{\text{ét}}_{<\infty} : \mathbf{Sch}^{\text{qcqs}} \to \text{Pro}(\mathbf{Spc}_{<\infty})$$

is the unique hypercomplete proétale cosheaf whose restriction to w-contractible affines is given by $U \mapsto \pi_0(U)$. Given a qcqs scheme X, choose a proétale hypercover $U_\bullet$ of X be w-contractible affines. Then the simplicial object $\pi_0(U_\bullet)$ is a $\text{Pro}(\mathbf{Set}_{\text{fin}})$-resolution of $\Pi^{\text{ét}}_{<\infty}(X)$.

Acknowledgements

We thank Luciana Basualdo Bonatto, Chang-Yeon Chough, Tim Holzschuh, Marcy Robertson, and Sebastian Wolf for many enlightening discussions around the contents of this note. We gratefully acknowledge support from the NSF Mathematical Sciences Postdoctoral Research Fellowship under Grant # DMS-2102957 and a grant from the Simons Foundation (816048, LC).

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Received: July 3, 2024 Revised: January 27, 2025

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 26 Issue 1 (pages 1–410) 2026