Algebraic & Geometric Topology

Volume 26 (2026)

RO(G)-graded Bredon cohomology of Euclidean configuration spaces

Daniel Dugger and Christy Hazel

msp

msp

Algebraic & Geometric Topology **26**:2 (2026) 437–484
DOI: 10.2140/agt.2026.26.437
Published: February 11, 2026

RO(G)-graded Bredon cohomology of Euclidean configuration spaces

DANIEL DUGGER AND CHRISTY HAZEL

Let G be a finite group and let V be a G-representation. We investigate the RO(G)-graded Bredon cohomology with constant integral coefficients of the space of ordered configurations in V. In the case that V contains a trivial subrepresentation, we show that the cohomology is free as a module over the cohomology of a point, and we give a generators-and-relations description of the ring structure. In the case that V does not contain a trivial representation, we give a computation of the module structure that works as long as a certain vanishing condition holds in the Bredon cohomology of a point. We verify this vanishing condition holds in the case that $\dim(V) \geq 3$ and G is either a cyclic group C_n or the symmetric group on three letters.

1 Introduction

Let G be a finite group and let $\underline{\mathbb{Z}}$ denote the constant coefficient Mackey functor whose value is $\mathbb{Z}$. The RO(G)-graded Bredon cohomology of a G-space X is the graded group $H^\star(X;\underline{\mathbb{Z}}) = \bigoplus_{W \in \mathrm{RO}(G)} H^W(X;\underline{\mathbb{Z}})$, which becomes a multigraded ring after fixing an ordered basis of irreducible representations (see Section 2.1 for discussion of this point). If V is a G-representation let $\mathrm{OC}_q(V)$ denote the *ordered configuration space* of tuples $(x_1, \dots, x_q) \in V^q$ such that $x_i \neq x_j$ for all $i \neq j$. The G-action on $\mathrm{OC}_q(V)$ is inherited from the action on V. Our goal in this paper is to compute the groups $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$, together with the ring structure when possible. While we are not able to completely solve this problem in all cases, we have the following results:

(1) We compute the additive structure, as a module over $H^\star(\mathrm{pt};\underline{\mathbb{Z}})$, whenever $V \supseteq 1$ (i.e., V contains a copy of the 1-dimensional trivial representation). We also give a generators-and-relations description of the cohomology ring. These methods work for any finite group G.

MSC2020: 55N91, 55P91.

(2) For the case where $V \not\supseteq 1$, we give a calculation of the additive structure that depends on certain hypotheses specific to the pair (G, V). We do not know how widely these hypotheses are satisfied, though we verify them in the cases where $\dim V \geq 3$ and G is either C_n ($n \in \mathbb{Z}_+$) or the symmetric group Σ_3. When $G = C_2$ we are also able to do the calculations for $\dim V = 2$. For the ring structure, we give an approach that works for $G = C_2$ but may be difficult to carry out in general.

To explain the results in more detail, let us first recall the nonequivariant situation. The spaces $\mathrm{OC}_q(\mathbb{R}^n)$ have been much-studied, classically by Arnold [2] when $n = 2$ and more generally by Cohen [5, Chapter III, Sections 6–7] building off of work from [9]. (Some of these classical papers use the notation $F(\mathbb{R}^n, q)$ for the ordered configuration space.) There are also many modern references, just a few of which are [10; 14; 23]. For $n \geq 2$ the integral cohomology ring of $\mathrm{OC}_q(\mathbb{R}^n)$ is the quotient of the polynomial ring $\mathbb{Z}[\omega_{ij} \mid 1 \leq i \neq j \leq q]$ by the three sets of relations

$$\begin{aligned} \omega_{ij} &= (-1)^n \omega_{ji}, \\ \omega_{ij}^2 &= 0, \\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} &= 0. \end{aligned}$$

The third of these will be called the *Arnold relation*, as it seems to have first appeared in [2]. Often the first relation is omitted and the generators are just taken to be ω_{ij} for $i < j$, but having both ω_{ij} and ω_{ji} is convenient for the third relation and will also be convenient when we generalize to the equivariant situation. (Readers curious about the $n = 1$ case can look at Section 6.4.)

Since the ω_{ij} classes are easy to describe, we do this here. One has maps $\widetilde{\omega}_{ij} : \mathrm{OC}_q(\mathbb{R}^n) \to S^{n-1}$ sending a configuration $\underline{x} = (x_1, \dots, x_q)$ to $(x_i - x_j)/|x_i - x_j|$, and ω_{ij} is just the pullback of the fundamental class from the cohomology of the sphere: $\omega_{ij} = \widetilde{\omega}_{ij}^*(\iota_{n-1})$. The first two of the three relations then follow immediately. There are different ways to derive the Arnold relation (the original in [2] using differential forms), but we will give a very geometric/topological approach in Section 3 below.

The above presentation implies that $H^*(\mathrm{OC}_q(\mathbb{R}^n))$ is nonzero only in degrees that are multiples of $n-1$, and in those degrees it is free abelian. It is not exactly transparent from the relations, but the rank of $H^{i(n-1)}(\mathrm{OC}_q(\mathbb{R}^n))$ is the Stirling number of the first kind $c(q, q-i)$. Here $c(a, b)$ is the number of permutations of a that can be expressed as a product of b disjoint cycles (see Section 3.8 for other descriptions of the Stirling numbers). The total rank of $H^*(\mathrm{OC}_q(\mathbb{R}^n))$ is therefore $q!$. In Section 6.5 we will see that both of these facts have interesting explanations in terms of equivariant cohomology, an observation that is due to Proudfoot [20].

Now let us move to Bredon cohomology for G-equivariant spaces. We write $\mathbb{M}$ for $H^\star(\mathrm{pt}; \underline{\mathbb{Z}})$, which is the ground ring for our $\mathrm{RO}(G)$-graded theory. Every orthogonal representation W has a so-called Euler class $a_W \in \mathbb{M}^W$ (see Section 2.6). These classes have the properties that $a_W \cdot x = x \cdot a_W$ for all $x \in \mathbb{M}$, $a_{W_1 \oplus W_2} = a_{W_1} \cdot a_{W_2}$, and $a_1 = 0$. Note that the last two imply that $a_W = 0$ whenever $W \supseteq 1$.

Before stating our results let us recall that all representations are assumed to be orthogonal, and if $W_1 \subseteq W_2$ then $W_2 - W_1$ denotes the orthogonal complement. If $W \supseteq 1$ then when we write $W - 1$ we assume that a specific 1-dimensional trivial subrepresentation has been chosen in W.

Our first main result is as follows:

Theorem 5.2 If $V \supseteq 1$ and $\dim V \geq 2$ then $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$ is the quotient of the free $\mathbb{M}$-algebra generated by classes ω_{ij} of degree $V-1$, $1 \leq i \neq j \leq q$, subject to the relations

$$\begin{aligned} \omega_{ij} &= (-1)^{|V|}\omega_{ji} + a_{V-1}, \\ \omega_{ij}^2 &= a_{V-1}\omega_{ij}, \\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} &= a_{V-1}(\omega_{ij} + \omega_{jk} + \omega_{ki}) - a_{V-1}^2. \end{aligned}$$

Observe that if $V \supseteq 2$ then the Euler classes a_{V-1} all vanish and the above relations reduce to the nonequivariant form.

Despite the similarity of the above result with the nonequivariant version, our proof is notably different in style. In the nonequivariant setting the map $\mathrm{OC}_{q+1}(\mathbb{R}^n) \to \mathrm{OC}_q(\mathbb{R}^n)$ that forgets the last point in the configuration is a fibration whose fiber is (up to homotopy) a wedge of spheres, and so one can use the Serre spectral sequence to inductively do the computations. In the equivariant setting we are hampered by the fact that the Serre spectral sequence is a much less usable tool, with even basic computations seeming to require an extensive knowledge of cohomology with local coefficients. Instead of going this route we study the "motive" $H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+$ and build this up inductively via a collection of cofiber sequences. This only works because the boundary maps in these cofiber sequences turn out to always vanish, by somewhat of a miracle.

For the case where $V \not\supseteq 1$ we can use essentially the same techniques, but here we get less lucky and the "miracle" does not come for free. We are able to prove that the boundary maps vanish in some familiar cases, but so far not in general. To state our result here, let $S(V)$ be the unit sphere inside the representation V. These spheres are key to the calculations because the $\widetilde{\omega}_{ij}$ maps take the form $\mathrm{OC}_q(V) \to S(V)$. The issue we run into is that not much is known about $H^\star(S(V);\underline{\mathbb{Z}})$ in the case $V \not\supseteq 1$, as this object is intricately related to $H^\star(\mathrm{pt};\underline{\mathbb{Z}})$, which also can be mysterious.

Theorem 4.17 Let G be a finite group and suppose that V is a G-representation such that $\dim V \geq 2$ and $V^G = 0$. Additionally, assume for all $\ell \in \mathbb{Z}$ that in the sequence

$$H^{\ell V-\ell}(\mathrm{pt}) \xrightarrow{\cdot a_V} H^{(\ell+1)V-\ell}(\mathrm{pt}) \xrightarrow{\cdot a_V} H^{(\ell+2)V-\ell}(\mathrm{pt})$$

the first map is surjective and the second map is injective. (These conditions are equivalent to the statement $H^{\ell V-(\ell-1)}(S(V)) = 0$ for all $\ell \in \mathbb{Z}$.) Then there is a splitting of $H\underline{\mathbb{Z}}$-modules

$$H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+ \simeq \bigvee_{j=0}^{q-1} \Big(H\underline{\mathbb{Z}} \wedge \Sigma^{j(V-1)}(S(V)_+)\Big)^{a(q,j)},$$

where $a(q,j)$ is an alternating sum of Stirling numbers given in Definition 4.16 in Section 4.

This theorem gives an additive splitting for $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$ in terms of shifted copies of $H^\star(S(V);\underline{\mathbb{Z}})$, but the latter remains a black box. The hypotheses of Theorem 4.17 are an interesting property of the ground ring $H^\star(\mathrm{pt};\underline{\mathbb{Z}})$ that seems to be worthy of further investigation. As test cases we verify that these hypotheses hold when $\dim V \geq 3$ and G is either C_n $(n \geq 1)$ or Σ_3 — see Appendix B for details. The case $\dim V = 2$ is an anomaly and the hypotheses are almost never satisfied here when $\ell = -2$; again, details are in Appendix B. Despite this, in the special case $G = C_2$ we verify that Theorem 4.17 still holds when $\dim V = 2$. See Proposition 4.20.

We are likewise unable to give a simple presentation of the multiplicative structure on $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$ in the case $V \not\supseteq 1$, at least for general G. In Section 5.3 we discuss some of the issues involved and give a solution when $G = C_2$.

Our primary motivation in this paper was the study of Bredon cohomology, with configuration spaces being a convenient test area for computations. But the results also offer an interesting perspective on some nonequivariant phenomena. The relations defining the singular cohomology rings of the spaces $\mathrm{OC}_q(\mathbb{R}^n)$ only depend on the parity of n, but because the degrees of the generators depend on n it is not obvious how to give a direct comparison between the rings. For example, it is of course not true that there is a map of spaces $\mathrm{OC}_q(\mathbb{R}^n) \to \mathrm{OC}_q(\mathbb{R}^{n+2})$ that induces an isomorphism on singular cohomology, or a map $\mathrm{OC}_q(\mathbb{R}^n) \to \mathrm{OC}_q(\mathbb{R}^{n+1})$ that induces an isomorphism with mod 2 coefficients.

However, the use of Bredon cohomology *does* lead to some direct connections here. For example, take $G = C_2$ and $V = \mathbb{R}^n \oplus \mathbb{R}_-$, where $\mathbb{R}^n$ has the trivial action and $\mathbb{R}_-$ has the sign action. Write H^*_{sing} for ordinary singular cohomology of nonequivariant spaces. The inclusion $i : \mathrm{OC}_q(\mathbb{R}^n) \hookrightarrow \mathrm{OC}_q(V)$ *does* induce an interesting map on Bredon cohomology, and the diagram

$$\begin{array}{ccc} & & H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}}) \\ & \swarrow^{i^*} & \downarrow \psi \\ H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n)) \otimes \mathbb{M} \xrightarrow{\cong} H^\star(\mathrm{OC}_q(\mathbb{R}^n);\underline{\mathbb{Z}}) & & H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^{n+1});\mathbb{Z}) \end{array}$$

where ψ is the "forgetful map" from Bredon to singular cohomology allows *some* information to be passed between $H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n))$ and $H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^{n+1}))$. One can play a similar game when V has two or more copies of $\mathbb{R}_-$ added on instead of one. This phenomenon had already been noticed in the context of Borel equivariant cohomology in the work of Proudfoot and his collaborators [6; 18; 19; 20], and we include discussion of the Bredon case not because it leads to any groundbreaking insights but just as a demonstration of the inner-workings of our computations. See Section 6 for discussion of this topic.

Remark 1.1 For other recent work on Bredon homology and configuration spaces, see [4]. That paper focuses on Bredon homology of *unordered* configuration spaces, and so the results are in a somewhat different direction than the ones here.

1.2 Notational conventions

Throughout this paper G denotes a finite group and V, W denote finite-dimensional, orthogonal G-representations. The dimension is written $|V|$ or $\dim V$, depending on context. If W is an orthogonal subrepresentation of V, then we write $V - W$ for the orthogonal complement. In the case that V contains a one-dimensional trivial representation, we will write $V-1$ to indicate the orthogonal complement of some choice of trivial subrepresentation. For a representation V we write $S(V)$ for the unit sphere in V, $D(V)$ for the closed unit disk in V, and S^V for the one-point compactification of V. We will sometimes use the fact that $D(V)/S(V) \cong S^V$, for example via the isomorphism $x \mapsto \tan\bigl(\frac{\pi}{2}|x|\bigr)\cdot x$ which is canonical in V.

To the representation V we assign numerical invariants $d(V)$ and $e(V)$ that appear throughout the paper. The definitions are in Sections 2.6 and 2.15.

We use $\star$ to denote RO(G)-gradings and $*$ to denote integer gradings.

2 Background

In this section we review background information about Bredon cohomology, and also establish some fundamental results about both the Bredon cohomology of a point and the Bredon cohomology of spheres.

2.1 RO(G)-graded Bredon cohomology

For foundational material on RO(G)-graded Bredon cohomology we refer the reader to [17]. We follow the now-common practice of using $*$ to denote integer gradings and $\star$ to denote RO(G)-gradings, so that the notation $H^\star(X;\underline{\mathbb{Z}})$ indicates $\star \in \mathrm{RO}(G)$.

We will use that $H^W(-;\underline{\mathbb{Z}})$ is represented by the space $\mathrm{AG}(S^W)$, the free abelian group on S^W (suitably topologized) with the basepoint ∞ as the zero element. This fact is due to dos Santos [22], building on earlier work of Lima-Filho [16]. The assignment $W \mapsto \mathrm{AG}(S^W)$ defines an equivariant ring spectrum denoted $H\underline{\mathbb{Z}}$.

Defining RO(G)-graded cohomology rings requires a certain amount of care, and it is common practice to sweep some of the subtleties under the rug. But since signs will be important for us, we need to give a brief synopsis. Our overall approach follows that of [7] but with a few modifications.

Fix once and for all an ordered collection of irreducible representations $I_1,\dots,I_r$ giving a basis for RO(G). For each j fix a dual of S^{I_j} in the equivariant stable homotopy category and denote it S^{-I_j}. For $n_1,\dots,n_r \ge 0$ define

$$S^{\pm n_1 I_1 \pm \dots \pm n_r I_r} = (S^{\pm I_1})^{\wedge(n_1)} \wedge (S^{\pm I_2})^{\wedge(n_2)} \wedge \dots \wedge (S^{\pm I_r})^{\wedge(n_r)}.$$

For $\alpha \in \mathrm{RO}(G)$ we write S^α for $S^{m_1 I_1 + \dots + m_r I_r}$ where the m_i are the unique integers for which $\alpha = \sum m_i I_i$. Define

$$H^\alpha(X;\underline{\mathbb{Z}}) = [S^{-\alpha} \wedge X_+, H\underline{\mathbb{Z}}] \quad \text{and} \quad H^\star(X;\underline{\mathbb{Z}}) = \bigoplus_{\alpha \in \mathrm{RO}(G)} H^\alpha(X;\underline{\mathbb{Z}}).$$

For a representation V write $\underline{V}$ for the corresponding element of RO(G). Note that S^V and $S^{\underline{V}}$ are homeomorphic, but not canonically. Likewise, the group $H^V(X;\underline{\mathbb{Z}}) = [S^{-V} \wedge X_+, H\underline{\mathbb{Z}}]$ is isomorphic to $H^{\underline{V}}(X;\underline{\mathbb{Z}})$, but again not canonically. If $\underline{V}=\sum n_j I_j$, define a *rigidification* of V to be an isomorphism $V \xrightarrow{\cong} \bigoplus_j I_j^{\oplus n_j}$. Such a rigidification determines a homeomorphism $S^V \xrightarrow{\cong} S^{\underline{V}}$. By a *rigid representation* we mean a representation equipped with a chosen rigidification.

To define the multiplication on $H^\star(X;\underline{\mathbb{Z}})$ we need to choose identifications $\phi_{\alpha,\beta}\colon S^{-\alpha}\wedge S^{-\beta} \xrightarrow{\cong} S^{-(\alpha+\beta)}$ for all $\alpha,\beta \in \mathrm{RO}(G)$. Then for $x\in H^\alpha(X)$ and $y\in H^\beta(X)$ we define xy to be the composite

$$\begin{array}{l} S^{-(\alpha+\beta)}\wedge X_+ \\ \big\downarrow \phi_{\alpha,\beta}^{-1}\wedge\Delta \\ S^{-\alpha}\wedge S^{-\beta}\wedge X_+\wedge X_+ \xrightarrow{1\wedge t\wedge 1} S^{-\alpha}\wedge X_+\wedge S^{-\beta}\wedge X_+ \xrightarrow{x\wedge y} H\underline{\mathbb{Z}}\wedge H\underline{\mathbb{Z}} \xrightarrow{\mu} H\underline{\mathbb{Z}} \end{array}$$

We choose the isomorphisms ϕ essentially as described in [7] but with one variation. Said briefly, they are obtained by the following rules:

- Commute any $S^{\pm I_j}$ past $S^{\pm I_k}$ for $j\neq k$.
- Allow any S^{I_j} to annihilate an S^{-I_j} that is next to it, via the duality maps.
- Every time we commute $S^{\pm I_j}$ past $S^{\pm I_k}$ for $j\neq k$ we multiply by the sign $(-1)^{|I_j|\cdot|I_k|}$.

The third rule was *not* used in [7], but putting it in has the effect of making the connection to nonequivariant topology cleaner (more on this in a moment). The inclusion of these signs was a topic of [8]. For the resulting product to be associative one needs the ϕ maps to satisfy a certain coherence condition (see [7]), but that is true with these choices. For $x\in H^\alpha(X)$ and $y\in H^\beta(X)$ one also finds the skew-commutativity rule

(2.1.1) $$xy = yx\cdot(-1)^{|\alpha|\cdot|\beta|},$$

where if $\alpha=\sum n_j I_j$ then $|\alpha|=\sum n_j\cdot \dim I_j$.

Remark 2.2 If one uses H^V rather than $H^{\underline{V}}$ then the need for the ϕ-maps disappears, because one has the canonical isomorphism $S^V\wedge S^W\cong S^{V\oplus W}$. To extend this to virtual representations one then needs for the notation H^{V-W} to depend on both V and W and not just on the difference in $\mathrm{RO}(G)$; such a group would be better written as $H^{(V,W)}$. One then obtains a theory that is not exactly "$\mathrm{RO}(G)$-graded" but where the indexing is on pairs of representations. In this setting products like xy and yx live in different groups — e.g., $H^{V_1\oplus V_2}$ rather than $H^{V_2\oplus V_1}$ — and so are not just related by a sign. Rather, they are related by a certain twist isomorphism induced by the twist isomorphism on vector spaces. This kind of "fancy" grading is indeed what people often mean when they say "$\mathrm{RO}(G)$-graded". On the one hand, it is an appealing Gordian-knot style approach to the situation. On the other hand, in practical computations one often wants to work with a concrete $\mathbb{Z}^r$-graded ring with formulas such as (2.1.1), rather than a behemoth indexed by all pairs of representations.

Here is an example of how these issues play out in practice. In Proposition 2.18 below we will describe the cohomology ring of S^W for any representation W. To give the answer in the usual language of algebra, e.g., via generators and relations, one is pushed into the context of honest $\mathrm{RO}(G)$-gradings. The downside of that context is that there is no canonical generator in $\widetilde{H}^{\underline{V}}(S^V)$. If we instead use the "fancy" RO(G)-grading then we have available the canonical element in $\widetilde{H}^V(S^V)$, but the complexity of the grading makes it awkward to give a complete algebraic description of the answer. As a compromise we could assume that V is a rigid representation, which gives a canonical generator in $\widetilde{H}^{\underline{V}}(S^V)$, but the extra assumption of rigidity can itself feel awkward and unsatisfying.

We refer the reader to [15, Remark 4.11; 17, Chapter XIII] for further discussion of these issues. The approach we will take is something like the following: merge the perspectives of both the honest and fancy RO(G)-gradings whenever convenient, but keep in the backs of our minds that when using the former one might need to add assumptions about rigidity and when using the latter one might need to add in the effects of certain isomorphisms.

As a final remark on this topic we recall the forgetful map ψ from equivariant homotopy to non-equivariant homotopy. At the basic level of spaces this sends a G-space X to the underlying space without the G-action, inducing the evident map $\psi : [X,Y]_G \to [X,Y]$ on homotopy classes. Note that $\psi(S^V) \cong S^{|V|}$, but not canonically. Even $\psi(S^{\underline{V}}) \cong S^{|V|}$ is not canonical. However, if we fix once and for all an orientation on each I_j then we at least get homeomorphisms $S^{I_j} \cong S^{|I_j|}$ that are canonical up to homotopy, and therefore resulting homotopy equivalences $\psi(S^{\underline{V}}) \simeq S^{|V|}$ (again canonical up to homotopy). One then obtains induced maps $\psi : H^\alpha(X;\underline{\mathbb{Z}}) \to H^{|\alpha|}_{\mathrm{sing}}(X;\mathbb{Z})$. Unlike in [7], these ψ's assemble to give a ring homomorphism — this is because of the signs that were included in the ϕ-maps above, and is the justification for incorporating them.

2.3 Computational tools

We will need the following two properties of $H^\star(-;\underline{\mathbb{Z}})$.

Proposition 2.4 (quotient lemma) *Let X be a G-CW complex. Then $\widetilde{H}^n(X;\underline{\mathbb{Z}}) \cong \widetilde{H}^n_{\mathrm{sing}}(X/G;\mathbb{Z})$.*

The above is a standard property, following from $H^n(X;\underline{\mathbb{Z}}) = [X, \mathrm{AG}(S^n)]_G = [X/G, \mathrm{AG}(S^n)]$ since the G-action on $\mathrm{AG}(S^n)$ is trivial.

Proposition 2.5 *If X is a space with trivial G-action and the graded group $H^*_{\mathrm{sing}}(X)$ is free abelian, then $H^\star(X;\underline{\mathbb{Z}}) \cong H^*_{\mathrm{sing}}(X) \otimes \mathbb{M}$.*

Proof This follows immediately from the usual cellular spectral sequence, which can be regarded as a spectral sequence of $\mathbb{M}$-modules. All differentials vanish except d_1. □

2.6 Euler classes and the cohomology of a point

A representation W has an associated "Euler class" $a_W \in \mathbb{M}^W$, and when W is G-oriented there is an "orientation class" $u_W \in \mathbb{M}^{W-|W|}$. These classes are described in [11, Section 3] but most likely date earlier and are standard constructions. We review them here and establish some basic properties that seem not to be in the literature.

Let $a_W : S^0 \to S^W$ be the map that sends the basepoint to ∞ and the nonbasepoint to 0. This gives an element in $\pi_{-W}(S^0)$. If we compose with the canonical inclusion $S^W \hookrightarrow \mathrm{AG}(S^W)$ then we also get a cohomology class in $\widetilde{H}^W(S^0;\underline{\mathbb{Z}}) \cong H^W(\mathrm{pt};\underline{\mathbb{Z}})$. In a slight abuse of notation we will denote both the homotopy class and the cohomology class by a_W. In either setting, a_W is commonly called the *Euler class* of W.

If $1 \subseteq W$ then the map $S^0 \to S^W$ is equivariantly null and so $a_W = 0$. The following result generalizes this by precisely determining the order of a_W in all other cases:

Proposition 2.7 *The group $H^W(\mathrm{pt};\underline{\mathbb{Z}})$ is generated by a_W and has order equal to*

$$\gcd\{\#(G/H) \mid H \le G, W^H \neq 0\}.$$

In particular, if $1 \subseteq W$ then the gcd is 1 and so $H^W(\mathrm{pt};\underline{\mathbb{Z}}) = 0$.

Proof of Proposition 2.7 We first show a_W generates the group. We use the isomorphisms

$$H^W(\mathrm{pt};\underline{\mathbb{Z}}) \cong [S^0, \mathrm{AG}(S^W)]_G \cong \pi_0(\mathrm{AG}(S^W)^G).$$

Elements of the object on the right are represented by G-equivariant finite formal sums $\sum n_i[x_i]$ with $x_i \in S^W$ and $n_i \in \mathbb{Z}$, with the understanding that the term $[\infty]$ represents the zero element and so can be dropped from any formal sum. The straight-line homotopy that contracts $S^W - \{\infty\}$ to 0 then shows that every formal sum can be deformed to be a multiple of $[0]$. The element a_W is the formal sum $1[0]$ and so our cohomology group is cyclic and a_W is a generator.

We can also use this isomorphism to see why a_W is annihilated by the gcd. Suppose that $H \le G$ and $W^H \neq 0$. Let $x \in W^H - \{0\}$, and set $\alpha = \sum_{gH \in G/H}[gx]$. This is a G-equivariant formal sum that doesn't involve the term $[0]$. The homotopy that contracts all points in $S^W - \{0\}$ to ∞ shows that $\alpha \sim 0$, whereas the homotopy that contracts all points in $S^W - \{\infty\}$ to 0 shows that $\alpha \sim \#(G/H)[0]$. So we find that $0 = \#(G/H) \cdot a_W$. Since this holds for all possible H, a_W is annihilated by the gcd from the statement of the proposition.

To show the order is exactly equal to the gcd, we use the isomorphisms

$$H^W(\mathrm{pt};\underline{\mathbb{Z}}) \cong H_{-W}(\mathrm{pt};\underline{\mathbb{Z}}) \cong \widetilde{H}_0(S^W;\underline{\mathbb{Z}}).$$

To compute this group, we build a G-CW complex Y with a weak equivalence $Y \to X = S^W$ and then use the cellular chain complex of Y. Start by putting in two fixed 0-cells — the points 0 and ∞ — which gives a 0-skeleton Y_0 and an inclusion $Y_0 \to X$ such that $\pi_0(Y_0^H) \to \pi_0(X^H)$ surjective for every H. Next, for every $H \le G$ such that $W^H \neq 0$ pick an element $v \in W^H - 0$ and add the equivariant 1-cell $G/H \times e_v$ to Y_0, where e_v is an ordinary 1-cell that connects the two 0-cells. Then add a second equivariant 1-cell $G/H \times e_{-v}$ with the same boundary. This creates a 1-skeleton Y_1 together with a map $Y_1 \to X$ that sends each $e_{\pm v}$ to the ray from 0 to ∞ that passes through $\pm v$. The induced maps are such that $\pi_0(Y_1^H) \to \pi_0(X^H)$ is an isomorphism and $\pi_1(Y_1^H) \to \pi_1(X^H)$ is a surjection, for every H (for the latter, remember that each X^H is a sphere and so has π_1 equal to either 0 or $\mathbb{Z}$ depending on the dimension). Continue by adding 2-cells and higher; the exact details are irrelevant to the H_0 calculation.

The reduced Bredon chain complex for Y is then

$$\cdots \to \bigoplus_{\substack{H \leq G \\ W^H \neq 0}} \mathbb{Z}^2 \to \mathbb{Z} \to 0,$$

with the generators of the summands corresponding to $H \leq G$ each being sent to $\#(G/H)$ in the target. So $\widetilde{H}_0(S^W)$ is as claimed. □

For convenience of future use, set $\mathcal{D}(W) = \{\#(G/H) \mid H \leq G, W^H \neq 0\}$ and $d(W) = \gcd \mathcal{D}(W)$.

Remark 2.8 The following example shows that the gcd in Proposition 2.7 cannot be replaced by a min. When $G = \Sigma_3$, let σ be the one-dimensional sign representation and let λ be the irreducible representation on $\mathbb{R}^2 = \mathbb{C}$ that permutes the cube roots of unity. Take $W = \sigma \oplus \lambda$. Then $\mathcal{D}(W) = \{2, 3, 6\}$ and so $d(W) = 1$.

Here is another useful description of $d(W)$:

Proposition 2.9 *$d(W) = \gcd\{\#(G/H) \mid H \leq G$ and G/H embeds into $W - \{0\}\}$.*

Proof Let $\mathcal{E}(W)$ be the set from the statement of the proposition. Clearly $\mathcal{E}(W) \subseteq \mathcal{D}(W)$. But if $H \leq G$ and $W^H \neq 0$, pick $x \in W^H - \{0\}$ and set $J = \mathrm{Stab}(x)$. Then $H \subseteq J$ and G/J embeds into $W - \{0\}$. Since $\#(G/J) \cdot \#(J/H) = \#(G/H)$, this proves that every element of $\mathcal{D}(W)$ is a multiple of an element of $\mathcal{E}(W)$. Since $\mathcal{E}(W) \subseteq \mathcal{D}(W)$, the two sets will have the same gcd. □

Example 2.10 Let the generator of the cyclic group C_n act on $W = \mathbb{R}^2$ by rotation through $\frac{2\pi}{n}$ radians. Then $d(W) = n$, and so this example shows that the order of $H^W(\mathrm{pt}; \underline{\mathbb{Z}})$ can be arbitrarily large.

More generally, if G is arbitrary and the action on $W - \{0\}$ is free, then $d(W) = \#G$.

In contrast to the above example, we have the following result for odd-dimensional representations:

Proposition 2.11 *If $\dim W$ is odd then $d(W)$ is either 1 or 2. Equivalently, for any representation W the integer $1 - (-1)^{|W|}$ annihilates a_W.*

It is possible to give a proof of this result that uses only Smith theory and elementary algebra — we include this in Appendix A below. There is also a surprisingly simple approach that instead uses only Bredon cohomology — see Remark 2.20. The Bredon proof is contingent on getting certain signs correct, and so we have also included the Smith theory approach as a reality check.

Corollary 2.12 *For any G-space X, any $x \in H^\star(X; \underline{\mathbb{Z}})$, and any W, we have the strict commutativity $a_W x = x a_W$.*

Proof It suffices to prove this when x is homogeneous, say of degree $\alpha \in \mathrm{RO}(G)$. Then skew-commutativity says that $a_W x = (-1)^{|\alpha| \cdot |W|} x a_W$. If $|W|$ is even the sign is $+1$, but if $|W|$ is odd then by Proposition 2.11 $a_W = -a_W$ and so the sign can be interpreted as $+1$ even if it isn't. □

As one final observation on this topic we record the following:

Proposition 2.13 *For any representations W_1 and W_2, the multiplication pairing*

$$H^{W_1}(\mathrm{pt};\underline{\mathbb{Z}})\otimes H^{W_2}(\mathrm{pt};\underline{\mathbb{Z}})\to H^{W_1\oplus W_2}(\mathrm{pt};\underline{\mathbb{Z}})$$

is an isomorphism.

Proof Recall the isomorphism $H^W(\mathrm{pt};\underline{\mathbb{Z}})\cong\pi_0\big(\mathrm{AG}(S^W)^G\big)$, with elements of the last group represented by formal sums. In terms of these formal sums, the pairing takes a sum $\sum m_i[x_i]$ on S^{W_1} and a sum $\sum n_j[y_j]$ on S^{W_2} and sends this to the formal sum $\sum_{i,j} m_in_j[(x_i,y_j)]$, with the ordered pair suitably interpreted if either coordinate is ∞. From this description one sees immediately that $a_{W_1}\otimes a_{W_2}$ is sent to $a_{W_1\oplus W_2}$, and so the pairing is a surjection. It then suffices to check that the orders of the domain and codomain are equal. But since $(W_1\oplus W_2)^H=W_1^H\oplus W_2^H$ it follows at once that $d(W_1\oplus W_2)=\gcd\{d(W_1),d(W_2)\}$, and this is precisely the order of $\mathbb{Z}/d(W_1)\otimes\mathbb{Z}/d(W_2)$ (note that we are using Proposition 2.7 in several places here). □

By the above work we understand the groups $H^\alpha(\mathrm{pt};\underline{\mathbb{Z}})$ when α is a positive element of $\mathrm{RO}(G)$ — i.e., α is an actual representation. We will occasionally also need to know about the case where α is a *negative* element of $\mathrm{RO}(G)$, but that case is much easier:

Proposition 2.14 *If W is a nonzero G-representation then $H^{-W}(\mathrm{pt};\underline{\mathbb{Z}})=0$. Moreover, if $W\neq 1$ then $H^{1-W}(\mathrm{pt};\underline{\mathbb{Z}})=0$.*

Proof For the first part we use that $H^{-W}(\mathrm{pt};\underline{\mathbb{Z}})\cong\widetilde{H}^{-W}(S^0;\underline{\mathbb{Z}})\cong\widetilde{H}^0(S^W;\underline{\mathbb{Z}})\cong\widetilde{H}^0_{\mathrm{sing}}(S^W/G)=0$. The third isomorphism is by the quotient lemma (Proposition 2.4), and the last equality holds because S^W is connected and therefore S^W/G is as well.

Similarly, use that S^W is the unreduced suspension $\Sigma_uS(W)$ to get that

$$H^{1-W}(\mathrm{pt})=\widetilde{H}^1(S^W)\cong\widetilde{H}^1(\Sigma_uS(W))=\widetilde{H}^1_{\mathrm{sing}}(\Sigma_u(S(W)/G))=\widetilde{H}^0_{\mathrm{sing}}(S(W)/G).$$

As long as $W\neq 1$ the space $S(W)/G$ will be connected and so the above group vanishes. □

2.15 The orientation classes

We say that a representation V is *orientable* if each element of G acts on V with positive determinant. This is equivalent to saying that the induced G-action on $H_d^{\mathrm{sing}}(S^V)$ is trivial where $d=\dim V$, and also to the analogous statement for cohomology. An *orientation* of V is just an orientation of the underlying vector space, which we regard as a choice of generator for $H_d^{\mathrm{sing}}(S^V)$.

When V is orientable the forgetful map $\psi\colon H_d(S^V)\to H_d^{\mathrm{sing}}(S^V)$ is an isomorphism. This is explained in [11, Example 3.10]. A choice of orientation for V therefore determines an element $\mathfrak{o}_V\in H_d(S^V)$. We can obtain a corresponding element in cohomology via the isomorphisms in the diagram

$$\begin{array}{ccccccccc}
H_d(S^V) & \xrightarrow{\cong} & [S^d,S^V\wedge H\underline{\mathbb{Z}}] & \xrightarrow{\cong} & [S^0,S^{V-d}\wedge H\underline{\mathbb{Z}}] & \xrightarrow{\cong} & H^{V-d}(\mathrm{pt}) & & \\
\downarrow{\scriptstyle\psi} & & \downarrow & & \downarrow & & \downarrow & \searrow^{\psi} & \\
H_d^{\mathrm{sing}}(S^V) & = & [G_+\wedge S^d,S^V\wedge H\underline{\mathbb{Z}}] & \xrightarrow{\cong} & [G_+,S^{V-d}\wedge H\underline{\mathbb{Z}}] & \xrightarrow{\cong} & H^{V-d}(G) & = & H^0_{\mathrm{sing}}(\mathrm{pt})
\end{array}$$

Here the vertical maps are induced by $G_+ \to S^0$. The element in $H^{V-d}(\mathrm{pt})$ obtained by pushing $\mathfrak{o}_V$ across the top row is denoted u_V. Note that $\psi(u_V) = 1$.

Remark 2.16 Warning: The ψ at the right of the above diagram is not well defined unless V is oriented, since it requires an identification $S^V \cong S^d$ in the classical stable homotopy category. So the chosen orientation of V is coming up both in the choice of generator and in the definition of ψ, which is why the choices "cancel out" to give $\psi(u_V) = 1$. The orientation is also used in the identification $H^{V-d}(G) = H^0_{\mathrm{sing}}(\mathrm{pt})$. The diagram itself only depends on a choice of (classical) orientation of V, but the assumption that V is orientable as a representation is needed to know that the maps from the top groups to the bottom ones are isomorphisms (otherwise the top groups are all zero).

Let G_V be the isotropy group of V, and define $e(V) = \#(G/G_V)$. This measures the size of a generic orbit in V; alternatively, it is the degree of the branched cover $S^V \to S^V/G$. Note that $e(V \oplus W)$ is always a common multiple of $e(V)$ and $e(W)$. As an example to see how the e-invariant compares to other invariants we have considered, let $G = \Sigma_3$, let σ be the sign representation, and let λ be the irreducible representation on $\mathbb{R}^2$ as the symmetries of an equilateral triangle. We have

$$\begin{aligned} \mathcal{D}(\sigma) &= \{2,6\}, & d(\sigma) &= 2, & e(\sigma) &= 2; \\ \mathcal{D}(\lambda) &= \{3,6\}, & d(\lambda) &= 3, & e(\lambda) &= 6; \\ \mathcal{D}(\sigma \oplus \lambda) &= \{2,3,6\}, & d(\sigma \oplus \lambda) &= 1, & e(\sigma \oplus \lambda) &= 6. \end{aligned}$$

It is always true that $e(V) \in \mathcal{D}(V)$, and so $d(V) \mid e(V)$.

If V is oriented then $H^{d-V}(\mathrm{pt}) \cong \mathbb{Z}$ and is generated by an element denoted $\frac{e(V)}{u_V}$, which has the evident property that

$$u_V \cdot \frac{e(V)}{u_V} = e(V) \cdot 1 \in H^0(\mathrm{pt}).$$

This follows from using the isomorphisms $H^{d-V}(\mathrm{pt}) \cong \widetilde{H}^d(S^V) \cong \widetilde{H}^d_{\mathrm{sing}}(S^V/G)$ and then proving that the map on degree-d singular cohomology induced by the projection $S^V \to S^V/G$ is $\mathbb{Z} \to \mathbb{Z}$ sending a generator to $e(V)$ times a generator. Alternatively, the class $\frac{e(V)}{u_V}$ arises as the transfer of a certain class $e_V \in H^\star(G/G_V)$; see [12, Definition 9.9.7 and Lemma 9.9.10]. We will not need the class e_V in what follows.

2.17 The cohomology of representation spheres

We next turn to the cohomology of spheres S^W. The inclusion $S^W \hookrightarrow \mathrm{AG}(S^W)$ gives a canonical element $\iota_W \in \widetilde{H}^W(S^W; \underline{\mathbb{Z}})$, and the RO($G$)-graded suspension theorem implies that additively $H^\star(S^W; \underline{\mathbb{Z}})$ is the free $\mathbb{M}$-module generated by 1 and ι_W. It only remains to specify the product structure:

Proposition 2.18 *$H^\star(S^W; \underline{\mathbb{Z}})$ is the free $\mathbb{M}$-algebra generated by ι_W subject to the relation $\iota_W^2 = a_W \iota_W$.*

Again, if $1 \subseteq W$ then $a_W = 0$ and we obtain the familiar relation $\iota_W^2 = 0$ from nonequivariant topology. Also note that "R-algebra" typically means a ring homomorphism $R \to S$ where R acts centrally, and in our context "centrally" means in the skew-commutative sense of (2.1.1). But also recall that $a_W\iota_W = \iota_W a_W$ by Corollary 2.12.

Proof of Proposition 2.18 Write $\iota = \iota_W$ and $K_W = \mathrm{AG}(S^W)$. The product ι^2 is represented by the composite

$$S^W \xrightarrow{\Delta} S^W \wedge S^W \xrightarrow{\iota\wedge\iota} K_W \wedge K_W \xrightarrow{\mu} K_{2W}.$$

The key observation is that Δ is homotopic to the map

$$S^W = S^W \wedge S^0 \xrightarrow{\mathrm{id}\wedge a_W} S^W \wedge S^W$$

and also to the map $a_W \wedge \mathrm{id}$. Since the composite $\iota \circ a_W$ is the cohomology class $a_W \in \mathbb{M}$, the claim about Δ immediately yields $\iota^2 = a_W\iota = \iota a_W$.

To see the homotopy, use $D(W)/S(W)$ as our model for S^W. The homotopy is

$$H : D(W)/S(W) \times I \to D(W)/S(W) \wedge D(W)/S(W)$$

given by $H(x,t) = (x, tx)$ (or, in the second case, by $H(x,t) = (tx, x)$). □

We will also need to know how the antipodal map $A : S^W \to S^W$ acts on cohomology. This is another instance where the Euler class plays a role. Using the model of S^W as the one-point compactification, we define $A(\underline{x}) = -\underline{x}/|\underline{x}|^2$, which is equivariant and satisfies $0 \mapsto \infty$, $\infty \mapsto 0$. Note that A is not a pointed map, and so does not preserve the reduced cohomology. Instead the following is true:

Proposition 2.19 *For $A : S^W \to S^W$ the antipodal map we have $A^*(\iota_W) = (-1)^{|W|+1}\iota_W + a_W$.*

Proof Let $j_0, j_\infty : \mathrm{pt} \to S^W$ be the maps sending the basepoint to 0 or ∞, respectively. Then $\iota_W \circ j_\infty$ is the inclusion of the basepoint, so $j_\infty^*(\iota_W) = 0$. On the other hand, $j_0^*(\iota_W) = a_W$. The proof will use these formulas together with $A \circ j_0 = j_\infty$ and $A \circ j_\infty = j_0$.

We know that $H^W(S^W) = \mathbb{M}^W \oplus \widetilde{H}^W(S^W)$, with the first summand generated by a_W and the second by ι_W. Since $A^*(\iota_W) \in H^W(S^W)$ we can therefore write $A^*(\iota_W) = ka_W + m\iota_W$ for some $k, m \in \mathbb{Z}$ (really k is only defined modulo $d(W)$). Applying the forgetful map ψ into nonequivariant singular cohomology, we have $\psi(\iota_W) = \iota$, which is a generator of $H^{|W|}_{\mathrm{sing}}(S^{|W|})$, and $\psi(a_W) = 0$. This shows that $m = (-1)^{|W|+1}$, the usual nonequivariant degree of the antipodal map on $S^{|W|}$. Next consider

$$0 = j_\infty^*(\iota_W) = j_0^* A^*(\iota_W) = kj_0^*(a_W) + mj_0^*(\iota_W) = (k+m)a_W.$$

So $ka_W = -ma_W$ and we obtain

$$A^*(\iota_W) = -ma_W + m\iota_W = (-1)^{|W|+1}\iota_W + (-1)^{|W|}a_W \tag{2.19.1}$$

(we have switched the order of the terms in the last equality). This is almost what we want — we just need to show $(-1)^{|W|}a_W = a_W$. To do so, we can also consider

$$a_W = j_0^*(\iota_W) = j_\infty^*(A^*\iota_W) = (-1)^{|W|+1} j_\infty^*(\iota_W) + (-1)^{|W|} j_\infty^*(a_W) = (-1)^{|W|}a_W. \tag{2.19.2}$$

So we may rewrite (2.19.1) as $A^*(\iota_W) = (-1)^{|W|+1}\iota_W + a_W$. □

Remark 2.20 Notice that (2.19.2) shows that $1-(-1)^{|W|}$ annihilates a_W. This proves Proposition 2.11.

One can also consider the cohomology of the spheres $S(W)$. If $W \supseteq 1$ then $S(W) \cong S^{W-1}$, but otherwise $S(W)$ has no fixed point and so cannot be the 1-point compactification of any representation. The inclusion of $S(W)$ into $D(W)$ gives a cofiber sequence of pointed spaces $S(W)_+ \to D(W)_+ \to S^W$, but since $D(W)$ is contractible we can also write it in the form

$$S(W)_+ \to S^0 \xrightarrow{a_W} S^W. \tag{2.20.1}$$

The resulting long exact sequence in reduced cohomology takes the form

$$\cdots \leftarrow \mathbb{M}^{\alpha+1} \xleftarrow{a_W} \mathbb{M}^{\alpha+1-W} \leftarrow H^\alpha(S(W)) \leftarrow \mathbb{M}^\alpha \xleftarrow{a_W} \mathbb{M}^{\alpha-W} \leftarrow \cdots.$$

So the groups $H^\star(S(W))$ are extensions of $\ker a_W$ and $\operatorname{coker} a_W$. Determining these groups seems to be difficult in general, but the following is one useful fact:

Proposition 2.21 *If $V \supseteq W$ then $H^V(S(W)) = 0$.*

Proof The piece of the relevant long exact sequence is

$$\cdots \leftarrow \mathbb{M}^{V+1} \xleftarrow{a_W} \mathbb{M}^{V+1-W} \leftarrow H^V(S(W)) \leftarrow \mathbb{M}^V \xleftarrow{a_W} \mathbb{M}^{V-W} \leftarrow \cdots.$$

But $\mathbb{M}^V$ is generated by a_V, and $a_V = a_W \cdot a_{V-W}$. So the map $\mathbb{M}^{V-W} \to \mathbb{M}^V$ is surjective. On the other side, since $V+1-W = (V-W)+1$ is a representation containing 1 we know from Proposition 2.7 that $\mathbb{M}^{V-W+1} = 0$. So $H^V(S(W)) = 0$ by the long exact sequence. □

We will be particularly interested in the group $H^{W-1}(S(W))$, and for this we have

$$\begin{array}{ccccccccc}
\mathbb{M}^W & \xleftarrow{a_W} & \mathbb{M}^0 & \longleftarrow & H^{W-1}(S(W)) & \longleftarrow & \mathbb{M}^{W-1} & \longleftarrow & \mathbb{M}^{-1} \\
\cong\uparrow & & \cong\uparrow & & & & & & \Vert \\
\mathbb{Z}/d(W) & \twoheadleftarrow & \mathbb{Z} & & & & & & 0
\end{array}$$

So we get a short exact sequence $0 \leftarrow \mathbb{Z} \leftarrow H^{W-1}(S(W)) \leftarrow \mathbb{M}^{W-1} \leftarrow 0$, which is split but not naturally.

If $W \supseteq 1$ then $\mathbb{M}^W = 0$, and we define a *fundamental class* for $S(W)$ to be an element of $H^{W-1}(S(W))$ that maps to a generator in $\mathbb{M}^0$. If $W \supseteq 2$ then $\mathbb{M}^{W-1} = 0$ and there are only two such classes, but in general there will be $2 \cdot d(W-1)$ of them. Note that applying ψ to a fundamental class gives a fundamental class in singular cohomology.

In the case $W \not\supseteq 1$ then the kernel of $\mathbb{M}^0 \to \mathbb{M}^W$ is still a copy of $\mathbb{Z}$, but it is generated by $d(W) \cdot 1$. We define a *semifundamental class* to be an element of $H^{W-1}(S(W))$ that maps to a generator of this kernel. Applying ψ to a semifundamental class gives $\pm d(W) \cdot 1$ in singular cohomology. (Semifundamental classes will be needed in Section 4.19 below).

Remark 2.22 When $W \supseteq 1$ a basepoint can be chosen in $S(W)$, thus the inclusion $\mathbb{M}^{W-1} \hookrightarrow H^{W-1}(S(W))$ is split and one obtains an isomorphism $\widetilde{H}^{W-1}(S(W)) \cong \mathbb{Z}$. However, be warned that the choice of basepoint is not canonical and so neither is this isomorphism; this can be a source of some confusion. If $W \supseteq 2$ then this issue goes away, since different choices of basepoint are homotopic.

3 Cohomology relations on configuration spaces

In this section we return to the configuration spaces $\mathrm{OC}_q(V)$, where V is an orthogonal G-representation. Note that $\mathrm{OC}_q(V)^H \cong \mathrm{OC}_q(V^H)$ for all subgroups $H \subseteq G$. A configuration of points $(x_1, \dots, x_q)$ will often be denoted $\underline{x}$ for short.

For each $i, j \le q$ recall the map $\widetilde{\omega}_{ij} : \mathrm{OC}_q(V) \to S(V)$ given by

$$\widetilde{\omega}_{ij}(\underline{x}) = \frac{x_i - x_j}{\|x_i - x_j\|}.$$

Observe that $\widetilde{\omega}_{ij}$ and $\widetilde{\omega}_{ji}$ differ by postcomposition with the antipodal map on $S(V)$.

For the rest of this section we assume $V \supseteq 1$. We fix a specific copy of 1 inside V, and $V - 1$ denotes the orthogonal complement. Fix a unit vector $e_1 \in 1 \subseteq V$ and regard this as the basepoint of $S(V)$. Stereographic projection gives an isomorphism $f : S^{V-1} \to S(V)$ that sends $\infty \mapsto e_1$ and $0 \mapsto -e_1$ and that equals the identity on $S(V - 1)$. In the future, whenever we identify $S(V)$ and S^{V-1} we use this isomorphism.

The map f induces $f^* : \widetilde{H}^{V-1}(S(V)) \to \widetilde{H}^{V-1}(S^{V-1})$ and we let ι_{V-1} be the preimage of the canonical generator (we will sometimes drop the subscript). Observe that in the diagram

$$\begin{array}{ccc} S(V) & \xrightarrow{i} & \mathrm{AG}(S(V)) \\ f \uparrow \cong & & \cong \uparrow \mathrm{AG}(f) \\ S^{V-1} & \xrightarrow{i} & \mathrm{AG}(S^{V-1}) \end{array}$$

the composite $S(V) \to \mathrm{AG}(S^{V-1})$ is ι_{V-1}.

Define $\omega_{ij} = \widetilde{\omega}_{ij}^*(\iota_{V-1})$. We prove the following properties of the classes ω_{ij}:

Proposition 3.1 $\quad \omega_{ij}^2 = a_{V-1}\omega_{ij} = \omega_{ij} a_{V-1}$ *and* $\omega_{ji} = (-1)^{|V|}\omega_{ij} + a_{V-1}$.

Proof The first relation follows at once from Proposition 2.18, and then the second from Corollary 2.12. For the third we use $\widetilde{\omega}_{ji} = A \circ \widetilde{\omega}_{ij}$ where A is the antipodal map on $S(V)$. Then

$$\omega_{ji} = \widetilde{\omega}_{ji}^*(\iota) = \widetilde{\omega}_{ij}^*(A^*\iota) = \widetilde{\omega}_{ij}^*((-1)^{|V|}\iota + a_{V-1}) = (-1)^{|V|}\omega_{ij} + a_{V-1},$$

where in the third equality we used Proposition 2.19. □

Corollary 3.2 $\quad \omega_{ij} \cdot \omega_{ji} = 0.$

Proof We just compute

$$\begin{aligned} \omega_{ij}\omega_{ji} = \omega_{ij}\big((-1)^{|V|}\omega_{ij} + a_{V-1}\big) &= (-1)^{|V|}\omega_{ij}^2 + \omega_{ij} a_{V-1} \\ &= (-1)^{|V|}\omega_{ij} a_{V-1} + \omega_{ij} a_{V-1} \\ &= (1 - (-1)^{|V|-1}) \cdot \omega_{ij} a_{V-1}. \end{aligned}$$

Now use that $1 - (-1)^{|V|-1}$ annihilates a_{V-1}, by Proposition 2.11. □

There is an alternative proof of the above identity that is more geometric — see Remark 3.7 for details.

We next work towards establishing the Arnold relation. The following seems to be the most "geometric" form of the relation:

Proposition 3.3 *For any triple of indices i, j, k the relation*

$$(\omega_{ij}-\omega_{ik})\cdot(\omega_{ji}-\omega_{jk})=0$$

holds in $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$.

Before giving the proof let us observe that the relation can be rewritten as follows:

Corollary 3.4 *For any triple i, j, k the relation*

$$\omega_{ij}\omega_{jk}+\omega_{jk}\omega_{ki}+\omega_{ki}\omega_{ij}=a_{V-1}(\omega_{ij}+\omega_{jk}+\omega_{ki})-a_{V-1}^2$$

holds in $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$.

Proof Multiply out the relation from Proposition 3.3 using distributivity and use that $\omega_{ij}\omega_{ji}=0$ from Corollary 3.2. This yields

$$\begin{aligned}0&=-\omega_{ij}\omega_{jk}-\omega_{ik}\omega_{ji}+\omega_{ik}\omega_{jk}\\&=-\omega_{ij}\omega_{jk}-((-1)^{|V|}\omega_{ki}+a_{V-1})((-1)^{|V|}\omega_{ij}+a_{V-1})+(-1)^{|V|-1}\omega_{jk}\omega_{ik}\\&=-\omega_{ij}\omega_{jk}-\omega_{ki}\omega_{ij}-a_{V-1}((-1)^{|V|}\omega_{ij}+(-1)^{|V|}\omega_{ki}+a_{V-1})+(-1)^{|V|-1}\omega_{jk}\omega_{ik}\\&=-\omega_{ij}\omega_{jk}-\omega_{ki}\omega_{ij}+a_{V-1}(\omega_{ij}+\omega_{ki}-a_{V-1})+(-1)^{|V|-1}\omega_{jk}\omega_{ik}.\end{aligned}$$

We have used Proposition 3.1 in the second equality, and in the fourth we have twice used that $a_{V-1}=(-1)^{|V|-1}a_{V-1}$ by Proposition 2.11. We are also using in several places that a_{V-1} is central (Corollary 2.12). Next we use Proposition 3.1 again to replace ω_{ik} with $(-1)^{|V|}\omega_{ki}+a_{V-1}$, and this gives us

$$0=-\omega_{ij}\omega_{jk}-\omega_{ki}\omega_{ij}-\omega_{jk}\omega_{ki}+a_{V-1}(\omega_{ij}+\omega_{ki}+\omega_{jk}-a_{V-1}).\qquad\square$$

Remark 3.5 The natural Σ_3-action on $\mathrm{OC}_3(V)$ induces an action on the cohomology ring $H^\star(\mathrm{OC}_3(V);\underline{\mathbb{Z}})$. In the nonequivariant setting, the Arnold relation is readily checked to be the unique quadratic relation that is invariant under the Σ_3-action (up to multiplication by scalars, of course). The Serre spectral sequence for the fibration $\mathrm{OC}_3(V)\to\mathrm{OC}_2(V)$ (forget the last point in the configuration) shows there must be a nontrivial quadratic relation, and then equivariance says there is only one possibility. This is a way to derive the Arnold relation without doing any geometry at all.

However, in our equivariant setting the Arnold relation from Corollary 3.4 is *not* the unique quadratic relation that is equivariant: removing the constant term a_{V-1}^2 yields another relation with the same equivariance properties.

Proof of Proposition 3.3 For $a,b\in V$ with $a\neq b$ write $u(a,b)$ for the unit vector pointing from b towards a, i.e., $u(a,b)=\frac{a-b}{\|a-b\|}$. For an ordered configuration $\underline{x}=(x_1,\dots,x_q)$ write $x_{ij}=u(x_i,x_j)$. With this notation the cohomology class ω_{ij} may be regarded as the map $\mathrm{OC}_q(V)\to\mathrm{AG}(S(V))$ sending $\underline{x}$ to $[x_{ij}]$.

For any indices i, j, k there is a map $\mathrm{OC}_q(V) \to \mathrm{OC}_3(V)$ that sends $\underline{x}$ to the ordered triple (x_i, x_j, x_k). The class ω_{12} on $\mathrm{OC}_3(V)$ pulls back to ω_{ij} and so forth, so it suffices to prove the desired relation on $\mathrm{OC}_3(V)$ with $i = 1$, $j = 2$, and $k = 3$.

Given an affine hyperplane H in V, define a "side" of H to be the union of H with one of the two components of $V - H$. So H divides V into two "sides", which are closed subsets whose intersection is H. We will be concerned with two points being on the same side of H or on opposite sides of H — but note that these notions are not negations of each other since one of our two points could be in H.

Given a configuration $\underline{x}$, let H_{rs} be the affine hyperplane in V that passes through x_r and is perpendicular to $x_r - x_s$. Define $U_+ \subseteq \mathrm{OC}_3(V)$ to be the subset of those configurations $\underline{x}$ where x_2 and x_3 lie on the same side of H_{12}. Likewise, define $U_- \subseteq \mathrm{OC}_3(V)$ to be the subset of those configurations $\underline{x}$ where x_2 and x_3 lie on opposite sides of H_{12}. Note that U_+ and U_- are preserved by the G-action, and clearly $U_+ \cup U_- = \mathrm{OC}_3(V)$.

We will prove that

- $\omega_{12} - \omega_{13}$ is zero when restricted to U_+, and therefore lifts to a class $\alpha \in H^\star(\mathrm{OC}_3(V), U_+)$,
- $\omega_{21} - \omega_{23}$ is zero when restricted to U_-, and therefore lifts to a class $\beta \in H^\star(\mathrm{OC}_3(V), U_-)$.

From these it follows that $(\omega_{12} - \omega_{13})(\omega_{21} - \omega_{23})$ lifts to the product $\alpha\beta \in H^\star(\mathrm{OC}_3(V), U_+ \cup U_-)$, and is therefore zero.

The class $\omega_{12} - \omega_{13}$ is the map $\mathrm{OC}_q(V) \to \mathrm{AG}(S(V))$ that sends a configuration $\underline{x}$ to $[x_{12}] - [x_{13}]$. But if we restrict to U_+ then the line segment connecting x_2 to x_3 does not pass through x_1, and so we have the homotopy $H : U_+ \times I \to \mathrm{AG}(S(V))$ given by

$$H(\underline{x}, t) = [x_{12}] - [u(x_1, tx_2 + (1-t)x_3)].$$

Then $H_0 = \omega_{12} - \omega_{13}$ and H_1 is the zero class.

The proof for $\omega_{21} - \omega_{23}$ restricted to U_- is very similar, but this time we move x_3 to x_1 via the straight line and use that it does not pass through x_2. □

Remark 3.6 Say that "ABC forms a ray" if C is on $\overrightarrow{AB}$ and is outside the interval $\overline{AB}$. The above argument can also be written with U_+ the set of configurations $\underline{x}$ such that $x_2x_1x_3$ is not a ray, and U_- the set of configurations x such that $x_1x_2x_3$ is not a ray.

Remark 3.7 The above style of argument also gives an amusing alternative proof of Corollary 3.2. Fix a trivial subrepresentation $1 \subseteq V$ and let H be the orthogonal complement. Let U_+ and U_- be the subspaces of configurations $\underline{x}$ where x_{ij} is on one of the two sides of H. We leave the remaining details to the reader.

3.8 Review of Stirling numbers

The unsigned Stirling number of the first kind $c(a, b)$ can be described in any of the following ways:

(1) The number of permutations of a that can be expressed as a product of b disjoint cycles (a fixed point counts as a cycle here).

(2) The number of ways of placing a distinguishable points on b indistinguishable circles, with no circle left empty.

(3) The absolute value of the coefficient of x^b in $x(x-1)(x-2)\cdots(x-a+1)$.

(4) The sum of all $(a-b)$-fold products of distinct elements from the set $\{1,2,\dots,a-1\}$.

(5) The number of ways of selecting $a-b$ elements in the upper triangular part of an $a\times a$ matrix so that no two are in the same column.

(6) The number of permutations of a that can be written as a product of $a-b$ transpositions and no fewer.

Note that $c(a,a)=1$ and $c(a,a-1)=\binom{a}{2}$, and that $\sum_b c(a,b)=a!$. Moreover, the Stirling numbers satisfy the recursion relation $c(a+1,b)=ac(a,b)+c(a,b-1)$ which can be regarded as a variant of Pascal's identity for binomial coefficients.

A standard result says that the Stirling numbers are connected to the cohomology of configuration spaces:

Proposition 3.9 *The cohomology groups $H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n))$ are nonzero only in degrees that are multiples of $n-1$, and $H^{i(n-1)}_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n))\cong\mathbb{Z}^{c(q,q-i)}$. Consequently, the total rank of $H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n))$ is $q!$.*

Proof A classical result is that $H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n))$ is the quotient of the exterior algebra $\Lambda_{\mathbb{Z}}(\omega_{ij})_{1\le i<j\le q}$ modulo the Arnold relations. The Arnold relations allow us to rewrite any product $\omega_{ij}\omega_{kl}$ in which the maximal index that appears is repeated — for example, $\omega_{24}\omega_{34}$ — as a sum of monomials that do not have that property. Consequently, an additive basis is given by monomials of the form

$$\omega_{1j_{11}}\cdots\omega_{1j_{1r_1}}\cdot\omega_{2j_{21}}\cdots\omega_{2j_{r_2}}\cdots \tag{3.9.1}$$

with the following properties:

- ω_{ij} appears only when $i<j$, and appears at most once.
- The set of indices $\{j_{a*}\}$ does not intersect the set of indices $\{j_{b*}\}$, for all choices of a and b.

If we imagine a $q\times q$ matrix in which every row reads $[1,2,3,\dots,q]$, then the choice of j_{a*} indices can be obtained by circling entries of the a-th row in the upper triangular part of the matrix. The second condition above is the condition that no two circled entries are in the same column, so the number of such monomials coincides with description (5) of the Stirling numbers. □

The equivariant analog of the above result is as follows:

Proposition 3.10 *Let R be the quotient of the free skew-commutative $\mathbb{M}$-algebra generated by classes ω_{ij} for $1\le i\ne j\le q$ subject to the relations*

- $\omega_{ij}=(-1)^{|V|}\omega_{ji}+a_{V-1}$,
- $\omega_{ij}^2=a_{V-1}\omega_{ij}$,
- $\omega_{ij}\omega_{jk}+\omega_{jk}\omega_{ki}+\omega_{ki}\omega_{ij}=a_{V-1}(\omega_{ij}+\omega_{jk}+\omega_{ki})-a_{V-1}^2$.

Then as a left $\mathbb{M}$-module R is free with a basis that has $c(q,q-i)$ generators in degree $i(V-1)$ for all i.

Proof The proof is essentially the same as the nonequivariant case. The basis consists of the same monomials in the ω_{ij} described in the proof of Proposition 3.9. □

4 The homology of configuration spaces

In this section we study the (co)homology of ordered configurations of k points in a G-representation V. Our computation will proceed roughly as follows. Observe

$$\mathrm{OC}_q(V) = V^q - \bigcup_{1\le i<j\le q} D_{i,j},$$

where $D_{i,j}$ is the subspace of V^q where the i-th and j-th components are equal. We can inductively remove these subspaces, noting at each stage the resulting space (after smashing with $H\underline{\mathbb{Z}}$) is weakly equivalent to a wedge of certain suspensions of the unit sphere $S(V)$. The inductive step will rely on the vanishing of particular maps from the unit sphere, as seen in Theorem 4.4 below.

4.1 V-arrangements and a motivic splitting

In this section we fix a finite-dimensional G-representation V. We prove a splitting theorem for certain complements in V^q of unions of subrepresentations. The configuration space $\mathrm{OC}_q(V)$ will be an example of such a space, but we have need for the extra generality. We start by introducing some necessary conditions on the subrepresentations that are removed.

Definition 4.2 *Fix $q \ge 1$ and let $H_1, \dots, H_m \subseteq V^q$ be a collection of subrepresentations. We say this collection is a **V-arrangement** if for every $r \ge 1$ and every set of indices $1 \le i_1, \dots, i_r \le m$, there is an isomorphism of representations $H_{i_1} \cap \cdots \cap H_{i_r} \cong V^s$ for some $s \ge q - r$.*

In a V-arrangement each H_i will be either all of V^q or else isomorphic to V^{q-1}. The collection of all $D_{i,j} = \{\underline{x} \in V^q \mid x_i = x_j\}$ is an example of a V-arrangement. The empty collection of subspaces ($m = 0$) is also an example of a V-arrangement. Any subcollection of a V-arrangement is another V-arrangement. We also have need of the following intersection property:

Proposition 4.3 *If $H_1, \dots, H_m \subseteq V^q$ is a V-arrangement, then $H_1 \cap H_m, \dots, H_{m-1} \cap H_m \subseteq H_m$ is also a V-arrangement.*

Proof There are two cases, depending on whether $H_m = V^q$ or $H_m \cong V^{q-1}$. The first is trivial. In the second case, an r-fold intersection of the $H_i \cap H_m$ is an $(r+1)$-fold intersection of the original H_i, and so is isomorphic to V^s for some $s \ge q - (r+1) = (q-1) - r$. So the necessary condition is satisfied. □

Now we come to the core result. It involves an awkward vanishing condition which will be investigated in more detail in Section 4.5.

Theorem 4.4 *Let $q \ge 1$, let $m \ge 1$, and let $H_1, \dots, H_m \subseteq V^q$ be a V-arrangement. Set X equal to $V^q - (H_1 \cup \cdots \cup H_m)$. Suppose that for all $\ell \in \mathbb{Z}$ we have*

$$\Big[S(V)_+, H\underline{\mathbb{Z}} \wedge \Sigma^{\ell(V-1)+V}\big(S(V)_+\big)\Big] = 0$$

(this condition depends only on V). Then $H\underline{\mathbb{Z}} \wedge X_+$ is equivalent (as an $H\underline{\mathbb{Z}}$-module) to a wedge of summands of the form $H\underline{\mathbb{Z}} \wedge \Sigma^{j(V-1)}\big(S(V)_+\big)$ for $j = 0, 1, \dots, q-1$.

Proof We proceed by induction on m. When $m=1$ there are two cases: $H_1 = V^q$ and $H_1 \cong V^{q-1}$. In the former $X=\varnothing$ and $H\underline{\mathbb{Z}} \wedge X_+ = *$, which is the empty wedge of the given summands. In the latter case we have $X = V^q - H_1 \cong V^{q-1} \times (V-0) \simeq S(V)$, and so $H\underline{\mathbb{Z}} \wedge X_+ \simeq H\underline{\mathbb{Z}} \wedge S(V)_+$. This completes the base case.

For the inductive step we fix $m \geq 2$ and assume the desired splitting holds for all V-arrangements of size $m-1$ or smaller. Let $H_1, \dots, H_m \subseteq V^q$ be a V-arrangement. If $H_m = V^q$ then $X=\varnothing$ and the result is trivial, so we may assume $H_m \neq V^q$. Set $Y = V^q - (H_1 \cup \cdots \cup H_{m-1})$ and observe that $H\underline{\mathbb{Z}} \wedge Y_+$ decomposes as desired by the induction hypothesis. Let

$$Z = H_m - ((H_1 \cap H_m) \cup \cdots \cup (H_{m-1} \cap H_m)).$$

We can also apply the induction hypothesis to obtain a similar splitting for $H\underline{\mathbb{Z}} \wedge Z_+$, by Proposition 4.3. Note that the splitting for $H\underline{\mathbb{Z}} \wedge Z_+$ will only have summands $H\underline{\mathbb{Z}} \wedge \Sigma^{j(V-1)}\big(S(V)_+\big)$ for $j \leq q-2$, since $H_m \cong V^{q-1}$.

Note that Z is a closed submanifold of Y. Let $N_Y Z$ denote the normal bundle of Z in Y. This bundle is trivial: it is the pullback of the normal bundle of H_m in V^q under the inclusion $Z \hookrightarrow H_m$, and $N_{V^q}(H_m)$ is trivial because H_m is a linear subspace of V^q. The fibers are isomorphic to the orthogonal complement of H_m in V^q, which (up to isomorphism) is V.

The submanifold $Z \hookrightarrow Y$ has a nonequivariant tubular neighborhood by [13, Corollary III.2.3]. The argument for this can be modified to produce an equivariant tubular neighborhood using standard equivariant techniques (e.g., intersecting all G-translates of a neighborhood produces an equivariant neighborhood, and a nonequivariant shrinking of a bundle down into an open neighborhood of the zero section can be made equivariant by a suitable "averaging" over G). So one obtains an equivariant tubular neighborhood of Z in Y. Using the Pontryagin–Thom collapse map outside this tubular neighborhood, we have a homotopy cofiber sequence

$$(Y-Z)_+ \hookrightarrow Y_+ \to \mathrm{Th}(N_Y Z),$$

where the rightmost term is the Thom space. Since $N_Y Z$ is trivial we have

$$\mathrm{Th}(N_Y Z) \cong \Sigma^V Z_+.$$

Smashing our cofiber sequence with $H\underline{\mathbb{Z}}$ yields the corresponding homotopy cofiber sequence of $H\underline{\mathbb{Z}}$-modules

$$H\underline{\mathbb{Z}} \wedge (Y-Z)_+ \to H\underline{\mathbb{Z}} \wedge Y_+ \to H\underline{\mathbb{Z}} \wedge \Sigma^V(Z_+).$$

Our plan is to show the right map is null, which will give us the equivalence

$$H\underline{\mathbb{Z}} \wedge (Y-Z)_+ \simeq (H\underline{\mathbb{Z}} \wedge Y_+) \vee \Sigma^{-1}(H\underline{\mathbb{Z}} \wedge \Sigma^V(Z_+)).$$

This will complete the proof because $Y-Z=X$ and we have splittings for the two terms on the right by the induction hypothesis.

The map we need to analyze is

$$\begin{array}{ccc} H\underline{\mathbb{Z}} \wedge Y_+ & \longrightarrow & H\underline{\mathbb{Z}} \wedge \mathrm{Th}(N_Y Z) \\ \wr\| & & \wr\| \\ \bigvee_j H\underline{\mathbb{Z}} \wedge \Sigma^{j(V-1)}(S(V)_+) & & H\underline{\mathbb{Z}} \wedge \Sigma^V(Z_+) \\ & & \wr\| \\ & & \Sigma^V\big(\bigvee_i H\underline{\mathbb{Z}} \wedge \Sigma^{i(V-1)}(S(V)_+)\big) \end{array} \tag{4.4.1}$$

where $0 \le j \le q-1$ and $0 \le i \le q-2$. Note that in the wedges there might be multiple summands corresponding to each value of j or i. This map is null if and only if each component — corresponding to choosing individual summands in the domain and codomain — is null. But these components lie in the groups

$$\begin{aligned} &[H\underline{\mathbb{Z}} \wedge S^{j(V-1)} \wedge S(V)_+, \Sigma^V H\underline{\mathbb{Z}} \wedge S^{i(V-1)} \wedge S(V)_+]_{H\underline{\mathbb{Z}}} \\ &\qquad\cong [S^{j(V-1)} \wedge S(V)_+, \Sigma^V H\underline{\mathbb{Z}} \wedge S^{i(V-1)} \wedge S(V)_+] \\ &\qquad\cong [S(V)_+, H\underline{\mathbb{Z}} \wedge S^{(i-j)(V-1)+V} \wedge S(V)_+] \end{aligned}$$

and these groups vanish by assumption. □

4.5 The vanishing criterion

We now investigate the vanishing hypothesis that appeared in Theorem 4.4. We will see that this follows from a more concrete condition that only involves the Bredon cohomology of a point, and that this is satisfied in many cases of interest.

Definition 4.6 *Let V be an orthogonal G-representation. We say V satisfies the* ***vanishing requirement*** *for G if for all $\ell \in \mathbb{Z}$*

- $H^{\ell V-\ell}(\mathrm{pt}) \xrightarrow{\cdot a_V} H^{(\ell+1)V-\ell}(\mathrm{pt})$ *is surjective, and*
- $H^{(\ell+1)V-\ell}(\mathrm{pt}) \xrightarrow{\cdot a_V} H^{(\ell+2)V-\ell}(\mathrm{pt})$ *is injective.*

The terminology "vanishing requirement" comes from the following:

Proposition 4.7 *A representation V satisfies the vanishing requirement if and only if $H^{\ell(V-1)+V}(S(V))$ vanishes for all $\ell \in \mathbb{Z}$.*

Proof This follows readily from the cofiber sequence $S(V)_+ \to S^0 \xrightarrow{a_V} S^V$. □

The vanishing requirement implies the hypothesis of Theorem 4.4:

Proposition 4.8 *Suppose V satisfies the vanishing requirement. Then*

$$\big[S(V)_+, H\underline{\mathbb{Z}} \wedge \Sigma^{\ell(V-1)+V}\big(S(V)_+\big)\big] = 0$$

for all $\ell \in \mathbb{Z}$.

Proof We use the usual cofiber sequence $S(V)_+ \to S^0 \xrightarrow{a_V} S^V$ of (2.20.1) and then smash with $H\underline{\mathbb{Z}} \wedge S^{\ell(V-1)+V}$ to get the cofiber sequence

$$H\underline{\mathbb{Z}} \wedge \Sigma^{\ell(V-1)+V}\big(S(V)_+\big) \to H\underline{\mathbb{Z}} \wedge S^{\ell(V-1)+V} \xrightarrow{a_V} H\underline{\mathbb{Z}} \wedge S^{\ell(V-1)+2V}.$$

Applying $[S(V)_+, -]$ then yields a long exact sequence

$$H^{\ell(V-1)+2V-1}(S(V)) \to \big[S(V)_+, H\underline{\mathbb{Z}} \wedge \Sigma^{\ell(V-1)+V}\big(S(V)_+\big)\big] \to H^{\ell(V-1)+V}(S(V)).$$

Since V satisfies the vanishing condition, the left and right groups are zero, and thus the middle group must be zero. □

Which representations satisfy the vanishing requirement? Here is the first class of examples:

Proposition 4.9 *If $V \supseteq 1$ and $V \neq 2$ then V satisfies the vanishing requirement.*

Proof Since $V \supseteq 1$ we have $a_V = 0$, so the vanishing requirement is equivalent to the statement that $H^{(\ell+1)V-\ell}(\mathrm{pt}) = 0$ for all $\ell \in \mathbb{Z}$. If we set $W = V - 1$ then this says $H^{1+(\ell+1)W}(\mathrm{pt}) = 0$ for all $\ell \in \mathbb{Z}$. When $\ell \geq -1$ this is by Proposition 2.7. When $\ell \leq -2$ this is by Proposition 2.14, using that $W \neq 1$. □

Verifying the vanishing requirement for representations not containing 1 seems to be more difficult, as it requires computing significant portions of the ring $\mathbb{M}$. Such computations are sparse in the literature and have only been done for a few families of groups. In the examples that we do know, the vanishing requirement is always satisfied as long as $\dim(V) \geq 3$. We summarize this here:

Proposition 4.10 *Let G be either a cyclic group C_n ($n \geq 2$) or the symmetric group Σ_3. Then any orthogonal G-representation V with $\dim(V) \geq 3$ satisfies the vanishing requirement.*

Proof In each setting the proof amounts to an analysis of the detailed computations of $\mathbb{M}$ available in the literature. This is somewhat lengthy, though the main aspect that is "hard" is organizing the known facts about $\mathbb{M}$. We go through the arguments in detail in Appendix B. □

4.11 The case when $V \supseteq 1$

We have already proven that such V satisfy the vanishing requirement, so we obtain the following:

Proposition 4.12 *If $V \supsetneq 1$ then $H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+$ is weakly equivalent, as an $H\underline{\mathbb{Z}}$-module, to a wedge sum of copies of $H\underline{\mathbb{Z}} \wedge S^{j(V-1)}$ where $0 \leq j \leq q$.*

Proof Recall that the space $\mathrm{OC}_q(V)$ can be constructed inductively by removing the $\binom{q}{2}$ subspaces $D_{i,j} = \{\underline{x} \in V^q \mid x_i = x_j\}$. The subspaces $D_{i,j}$ form a V-arrangement. Thus for $V \neq 2$ the result follows from Theorem 4.4 and Proposition 4.9 after noting $S(V) \simeq S^{V-1}$ and $H\underline{\mathbb{Z}} \wedge S(V)_+ \simeq H\underline{\mathbb{Z}} \vee (H\underline{\mathbb{Z}} \wedge S^{V-1})$.

The case $V = 2$ must be handled separately, but follows trivially from the fact that the forgetful map $\Psi : H\underline{\mathbb{Z}}\text{-Mod} \to H\mathbb{Z}\text{-Mod}$, when restricted to the thick subcategory generated by trivial suspensions of $H\underline{\mathbb{Z}}$, is a triangulated equivalence. □

Theorem 4.13 *Let V be an orthogonal G-representation such that $V \supsetneq 1$. Then*

$$H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+ \simeq \bigvee_{j=0}^{q} (H\underline{\mathbb{Z}} \wedge S^{j(V-1)})^{c(q,q-j)}$$

as $H\underline{\mathbb{Z}}$-modules, where $c(q,q-j)$ are the unsigned Stirling numbers of the first kind (see Section 3.8).

Proof By Proposition 4.12 there exist nonnegative integers b_j such that

$$H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+ \simeq \bigvee_{j=0}^{q} (H\underline{\mathbb{Z}} \wedge S^{j(V-1)})^{b_j}.$$

We just need to show $b_j = c(q,q-j)$. Let $n = \dim(V)$. The forgetful functor $\Psi : H\underline{\mathbb{Z}}\text{-Mod} \to H\mathbb{Z}\text{-Mod}$ preserves wedge sums and sends $H\underline{\mathbb{Z}} \wedge X$ to $H\mathbb{Z} \wedge X$ (where in the latter case we have forgotten the G-action on X), so applying this to $H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+$ gives an underlying equivalence

$$H\mathbb{Z} \wedge \mathrm{OC}_q(\mathbb{R}^n)_+ \simeq \bigvee_{j=0}^{q} (H\mathbb{Z} \wedge S^{j(n-1)})^{b_j}.$$

But we know the homology of $\mathrm{OC}_q(\mathbb{R}^n)$ from Proposition 3.9, and thus it must be that $b_j = c(q,q-j)$. □

Corollary 4.14 *If $V \supsetneq 1$ then $H^\star(\mathrm{OC}_q(V); \underline{\mathbb{Z}})$ is a free $\mathbb{M}$-module generated by elements in degrees $j(V-1)$ for $0 \le j \le q$, with $c(q,q-j)$ elements in degree $j(V-1)$.*

Proof This is immediate from Theorem 4.13, using that $H^\star(X;\underline{\mathbb{Z}}) \cong [H\underline{\mathbb{Z}} \wedge X_+, \Sigma^\star H\underline{\mathbb{Z}}]_{H\underline{\mathbb{Z}}}$. □

4.15 The case when $V \not\supseteq 1$

For this case we only have results when V satisfies the vanishing requirement of Definition 4.6. When that is satisfied, Theorem 4.4 gives us an additive splitting for the cohomology of $\mathrm{OC}_q(V)$ in terms of the cohomology of suspensions of $S(V)_+$. To calculate the multiplicity of the different pieces we can again appeal to the forgetful functor to nonequivariant topology. The extra basepoint in $S(V)_+$ leads to a slightly different form of results compared to the $V \supseteq 1$ case. In particular, we get an alternating sum of Stirling numbers in the answer.

Definition 4.16 *Define $a(q,j)$ to be the following alternating sum of Stirling numbers of the first kind:*

$$\begin{aligned} a(q,j) &= c(q,q-j) - c(q,q-(j-1)) + c(q,q-(j-2)) - \cdots + (-1)^j c(q,q) \\ &= \sum_{i=0}^{j} (-1)^i c(q,q-(j-i)). \end{aligned}$$

Theorem 4.17 *Let G be a finite group and suppose V is a G-representation that satisfies the vanishing requirement. Then*

$$H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+ \simeq \bigvee_{j=0}^{q-1} \big(H\underline{\mathbb{Z}} \wedge \Sigma^{j(V-1)}(S(V)_+)\big)^{a(q,j)},$$

where $a(q,j)$ is the alternating sum defined in Definition 4.16.

Proof The ordered configuration space $\mathrm{OC}_q(V)$ is the complement in V^q of the V-arrangement $D_{i,j} = \{\underline{x} \in V^q \mid x_i = x_j\}$. Thus, as long as V satisfies the vanishing requirement, Theorem 4.4 gives that

$$H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+ \simeq \bigvee_{j=0}^{q-1} \bigl(H\underline{\mathbb{Z}} \wedge \Sigma^{j(V-1)}\bigl(S(V)_+\bigr)\bigr)^{b_j}$$

for some b_j. We just need to show $b_j = a(q, j)$.

Let $n = \dim(V)$ and apply the forgetful functor $\Psi : H\underline{\mathbb{Z}}\text{-Mod} \to H\mathbb{Z}\text{-Mod}$ to get an underlying equivalence

$$\begin{aligned} H\mathbb{Z} \wedge \mathrm{OC}_q(\mathbb{R}^n)_+ &\simeq \bigvee_{j=0}^{q-1} \bigl(H\mathbb{Z} \wedge \Sigma^{j(n-1)}\bigl(S_+^{n-1}\bigr)\bigr)^{b_j} \\ &\simeq \bigvee_{j=0}^{q-1} (H\mathbb{Z} \wedge (S^{j(n-1)} \vee S^{(j+1)(n-1)}))^{b_j} \\ &\simeq H\mathbb{Z}^{b_0} \vee \bigvee_{j=1}^{q-1} (H\mathbb{Z} \wedge S^{j(n-1)})^{b_j + b_{j-1}}. \end{aligned}$$

Using the classical computation given in Proposition 3.9 we see $b_0 = c(q, q)$ and $b_{j-1} + b_j = c(q, q-j)$ for $j > 0$. We can then inductively solve to get

$$b_j = c(q, q-j) - c(q, q-(j-1)) + c(q, q-(j-2)) + \cdots,$$

which is exactly the sequence $a(q, j)$ defined above. □

Corollary 4.18 *In the setting of Theorem 4.17 there is a decomposition of $\mathbb{M}$-modules*

$$H^\star(\mathrm{OC}_q(V); \underline{\mathbb{Z}}) \cong \bigoplus_{j=0}^{q-1} \Sigma^{j(V-1)} H^\star(S(V); \underline{\mathbb{Z}})^{\oplus a(q,j)}.$$

Proof This is immediate. □

4.19 The case of $G = C_2$

We demonstrate our results with a complete discussion of the spaces $\mathrm{OC}_q(V)$ for all nontrivial C_2-representations V. Let σ denote the sign representation on $\mathbb{R}^1$, and note all finite-dimensional orthogonal representations satisfy $V \cong s1 \oplus t\sigma$ for some $s, t \geq 0$.

For the case $s \geq 1$ Corollary 4.14 gives that $H^\star(\mathrm{OC}_q(V); \underline{\mathbb{Z}})$ is a free $\mathbb{M}$-module and provides a precise description of the degrees and multiplicities of the generators. So we instead focus on $s = 0$, where $V = t\sigma$.

If $t \geq 3$ then V satisfies the vanishing criterion by Proposition 4.10, and so Corollary 4.18 applies. For this to be useful we need to know $H^\star(S(V); \underline{\mathbb{Z}})$, but this is well known. If we set

$$\mathbb{E}_t = \mathbb{M}[u_{2\sigma}^{-1}]/(a_\sigma^t) = \mathbb{Z}[u_{2\sigma}^{\pm 1}, a_\sigma]/(2a_\sigma, a_\sigma^t)$$

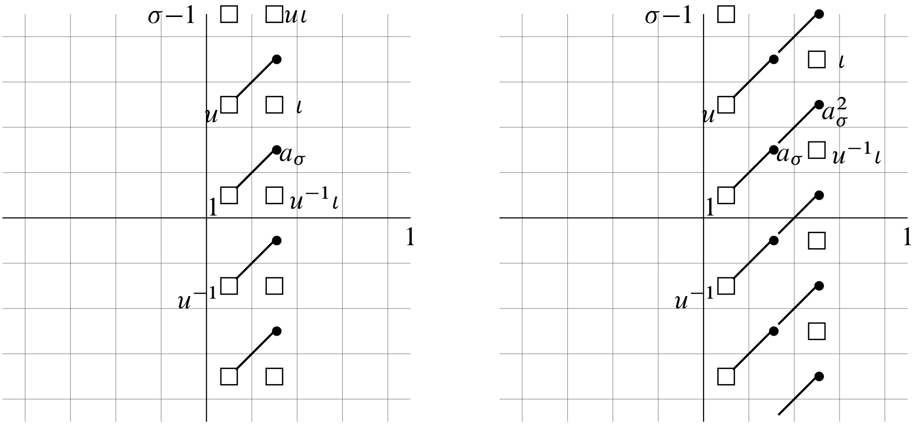

Figure 1: The rings $H^\star(S(2\sigma);\underline{\mathbb{Z}})$ and $H^\star(S(3\sigma);\underline{\mathbb{Z}})$.

then we have

$$H^\star(S(t\sigma);\underline{\mathbb{Z}}) = \mathbb{E}_t[\iota]/(a_\sigma\iota,\iota^2),$$

where ι is a semifundamental class (see Section 2.17) of degree $t\sigma-1$. Figure 1 shows pictures for $t=2$ and $t=3$. We have used motivic indexing for the picture, so that the index $s+t\sigma$ is drawn in spot $(s+t,t)$ on the grid; this explains the labeling of the axes. Squares denote copies of $\mathbb{Z}$ and dots denote copies of $\mathbb{Z}/2$. Diagonal lines represent multiplication by a_σ, e.g., the line connecting u to $a_\sigma u$. Note that the forgetful map satisfies $\psi(u)=1$, $\psi(a_\sigma)=0$, and $\psi(\iota)=2$.

The representation $V=2\sigma$ does not satisfy the vanishing hypothesis, but we will see below that Corollary 4.18 still holds in this case. So it is convenient to use this as an example. The splitting of Corollary 4.18 says that additively we have

$$H^\star(\mathrm{OC}_3(2\sigma);\underline{\mathbb{Z}}) \cong H^\star(S(2\sigma);\underline{\mathbb{Z}}) \oplus \Sigma^{2\sigma-1}H^\star(S(2\sigma);\underline{\mathbb{Z}}) \oplus \Sigma^{2\sigma-1}H^\star(S(2\sigma);\underline{\mathbb{Z}}).$$

The picture for this is in Figure 2. The number 2's shown remind us that there are *two* copies in those spots (e.g., two red dots in the (2,1) box), and the coloring on the second two summands is just to help distinguish them from the first. The ring structure for this example will be discussed in Section 5.3 below.

Proposition 4.20 *When $G=C_2$ and $V=2\sigma$ the conclusions of Theorem 4.17 and Corollary 4.18 still hold.*

Proof We only give a sketch. Set $V=2\sigma$ and consider $\mathrm{OC}_q(V)$. Following the method of Theorem 4.4 we will prove the vanishing of all of the maps that come up. These maps have components living in the groups

$$X_j = [S(V)_+, H\underline{\mathbb{Z}}\wedge S^{j(V-1)+V}\wedge S(V)_+]$$

for various values of $j\in\mathbb{Z}$, and we will consider the forgetful map

$$\psi : X_j \to [S^1_+, H\mathbb{Z}\wedge S^{j+2}\wedge S^1_+]_e = X_j^e$$

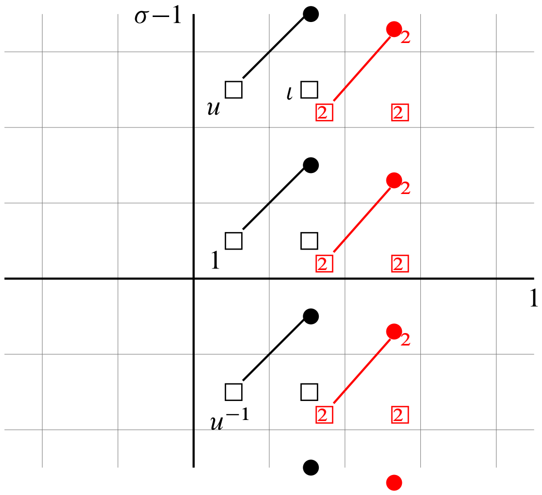

Figure 2: The additive structure of $H^\star(\mathrm{OC}_3(2\sigma);\underline{\mathbb{Z}})$.

where the target X_j^e is maps in the ordinary stable homotopy category. Using the cofiber sequence $S(V)_+ \to S^0 \to S^V$ in the codomain shows that X_j sits in a long exact sequence

$$\cdots \xrightarrow{\cdot a_{2\sigma}} H^{2V-1+j(V-1)}(S(V)) \to X_j \to H^{V+j(V-1)}(S(V)) \xrightarrow{\cdot a_{2\sigma}} \cdots.$$

The cohomology groups of $S(V)$ are as shown in Figure 1 and one sees that the two groups sandwiching the X_j are zero except in the cases $j=-1,-2,-3$. In these cases one gets that X_j is $\mathbb{Z}$, $\mathbb{Z}^2$, and $\mathbb{Z}$ (respectively), and in each case the forgetful map ψ is injective: this is by a diagram chase using that the same is true for the relevant groups in $H^\star(S(V))$ sandwiching it, and because the maps labeled a_σ and $a_{2\sigma}$ become zero under ψ. So we can prove that our elements of X_j are zero by proving that they are sent to zero under ψ.

Now we do something slightly clever. Let $W = 1 \oplus \sigma$. Considering the same method for building up $\mathrm{OC}_q(W)$, we already know that all of the equivariant maps vanish by Propositions 4.8 and 4.9. So applying ψ to them yields the zero maps, but these are exactly the same as the maps we needed to prove were zero. □

5 The product structure on cohomology

In this section we study the multiplicative structure on the cohomology ring $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$, following our work on the additive structure in the previous section. In the case $V \supseteq 1$ we can give a presentation of the ring as an $\mathbb{M}$-algebra: this is Theorem 5.2 from the introduction, which we prove here. The proof uses the generators and relations we have produced in Section 3, the additive results from Section 4, and the known calculations in the nonequivariant setting from [5].

The case $V \not\supseteq 1$ is intrinsically more difficult: we discuss some of the considerations in Section 5.3 below.

5.1 The case $V \supseteq 1$

We now come to one of the main results mentioned in the introduction:

Theorem 5.2 *If $V \supseteq 1$ then $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$ is the quotient of the free $\mathbb{M}$-algebra generated by classes ω_{ij} of degree $V-1$, $1 \le i \ne j \le q$, subject to the relations*

$$\omega_{ij} = (-1)^{|V|}\omega_{ji} + a_{V-1},$$
$$\omega_{ij}^2 = a_{V-1}\omega_{ij},$$
$$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = a_{V-1}(\omega_{ij} + \omega_{jk} + \omega_{ki}) - a_{V-1}^2.$$

Proof We start with the equivalence of $H\underline{\mathbb{Z}}$-modules

$$H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+ \simeq \bigvee_i (H\underline{\mathbb{Z}} \wedge S^{i(V-1)})^{c(q,q-i)}$$

from Theorem 4.13. From this we get that

$$\begin{aligned} H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}}) &= [\mathrm{OC}_q(V)_+, \Sigma^\star H\underline{\mathbb{Z}}] \\ &= [H\underline{\mathbb{Z}} \wedge \mathrm{OC}_q(V)_+, \Sigma^\star H\underline{\mathbb{Z}}]_{H\underline{\mathbb{Z}}} \\ &\cong \bigoplus_i \widetilde{H}^\star(S^{i(V-1)})^{\oplus c(q,q-i)} \\ &= \bigoplus_i \Sigma^{i(V-1)}\mathbb{M}^{\oplus c(q,q-i)}. \end{aligned}$$

Let R be the free skew-commutative $\mathbb{M}$-algebra generated by the classes ω_{st} subject to the relations listed in the theorem statement. By Proposition 3.10, R is a free $\mathbb{M}$-module with the same number of basis elements in the same degrees as $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$, and so these are isomorphic free $\mathbb{M}$-modules. Furthermore, the work in Section 3 gives us a map of $\mathbb{M}$-algebras $f : R \to H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$. We just need to show f is an isomorphism. The difficulty here is that we have two free bases in the picture: one for R consisting of products of ω_{st} classes, and one for $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$ coming from Theorem 4.13, which is a basis we know almost nothing about. The nontrivial part of the proof involves watching these two bases interact.

It is enough to prove that f maps a free basis in the domain to a free basis in the target. All basis elements are in degrees $j(V-1)$ for $j = 0, 1, \dots, q-1$, so we can focus on these degrees. Let $T = \bigoplus_{j\in\mathbb{Z}} \mathbb{M}^{j(V-1)}$, regarded as a $\mathbb{Z}$-graded subring of $\mathbb{M}$. If $j < 0$ then $\mathbb{M}^{j(V-1)} = 0$ by Proposition 2.14, and $T^0 = \mathbb{M}^0 = \mathbb{Z}$. If $j > 0$ then $T^j = \mathbb{M}^{j(V-1)} \cong \mathbb{Z}/d(V-1)$ by Propositions 2.7 and 2.13. In fact those results show that $T = \mathbb{Z}[a_{V-1}]/(d(V-1)\cdot a_{V-1})$. Observe that the $\mathbb{Z}$-graded subrings

$$R^{*(V-1)} \subseteq R \quad \text{and} \quad H^{*(V-1)}(\mathrm{OC}_q(V);\underline{\mathbb{Z}}) \subseteq H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$$

are naturally graded T-modules, and f is a map of T-modules.

Using the $\mathbb{M}$-module decomposition of $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$ given above, we have that

$$\begin{aligned} H^{j(V-1)}(\mathrm{OC}_q(V);\underline{\mathbb{Z}}) &\cong \Bigl(\bigoplus_{i=0}^{q-1} \Sigma^{i(V-1)}\mathbb{M}^{\oplus c(q,q-i)}\Bigr)^{j(V-1)} \\ &= \bigoplus_{i=0}^{q-1} (\mathbb{M}^{(j-i)(V-1)})^{\oplus c(q,q-i)} \\ &= \bigoplus_{i=0}^{q-1} (T^{j-i})^{\oplus c(q,q-i)}. \end{aligned}$$

Using our description of T we can therefore rewrite the above isomorphism as

(5.2.1) $$H^{j(V-1)}(\mathrm{OC}_q(V);\underline{\mathbb{Z}}) \cong \mathbb{Z}^{c(q,q-j)} \oplus (\mathbb{Z}/d(V-1))^{\oplus(c(q,q-(j+1))+\cdots+c(q,1))}.$$

Write f' for f restricted to the degrees $*(V-1)$, regarded as a map of graded T-modules. For convenience denote the domain and codomain of f' by M and N. Note that M and N are both free and finitely generated, with generators in the same degrees, and that T is nonnegatively graded. The proof from here goes as follows:

(1) We prove that f' is an isomorphism if we quotient out by the torsion.

(2) Quotienting by the torsion is the same as quotienting by the ideal $I \subseteq T$ of positive elements, so (1) says that $f' : M/IM \to N/IN$ is an isomorphism.

(3) Standard commutative algebra then implies that f' is an isomorphism (e.g., prove this by inducting up the degrees).

The only part that needs further justification is (1). For this, consider the triangle of ring maps

$$\begin{array}{ccc} R & \xrightarrow{f} & H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}}) \\ & \searrow & \downarrow \psi \\ & & H^*_{\mathrm{sing}}(\mathrm{OC}_q(V);\mathbb{Z}) \end{array}$$

The diagonal map takes the generators ω_{st} to the corresponding singular cohomology classes. We know this map is surjective because we know the ω_{st} classes generate the singular cohomology. We similarly know ψ is surjective. Note that all of the torsion summands in (5.2.1) map to zero under ψ, since the codomain is torsion-free. In a specific degree $j(V-1)$, our triangle looks like

(5.2.2) $$\begin{array}{ccc} \mathbb{Z}^{c(q,q-j)} \oplus \text{torsion} & \xrightarrow{f} & \mathbb{Z}^{c(q,q-j)} \oplus \text{torsion} \\ & \searrow\!\!\searrow & \downarrow\!\downarrow \psi \\ & & \mathbb{Z}^{c(q,q-j)} \end{array}$$

After quotienting by the torsion subgroups the diagonal and vertical maps become isomorphisms, so f becomes an isomorphism as well. □

5.3 The case $V \not\supseteq 1$

The first difficulty we encounter here is that $H^\star(\mathrm{OC}_q(V);\underline{\mathbb{Z}})$ is not a free $\mathbb{M}$-module. It is a direct sum of shifted copies of $H^\star(S(V);\underline{\mathbb{Z}})$, but we don't actually have a nice description of the cohomology of $S(V)$. As one example, if $G=C_2$ and $V=\mathbb{R}^n$ with the antipodal action then $H^\star(S(V))$ is not a finitely generated $\mathbb{M}$-module. So describing these rings with generators and relations is in some sense the wrong thing to do.

It is tempting to replace the ground ring $\mathbb{M}$ with $H^\star(S(V);\underline{\mathbb{Z}})$ and make all descriptions relative to that. This can be done (at least in some cases), but it is somewhat unnatural. First of all we must choose a "reference map" $\mathrm{OC}_q(V)\to S(V)$. We have all of the ω_{ij} for $1\le i\ne j\le q$, but choosing one from this list feels arbitrary. But okay, let us reluctantly choose the ω_{12} map as our fixed reference.

As an extended example let us now focus on $\mathrm{OC}_3(2\sigma)$, whose additive structure is given in Figure 2 from Section 4. Note that we have not yet chosen specific generators for the red terms in $H^{2\sigma-1}$ (all cohomology groups in this discussion are of the space $\mathrm{OC}_3(2\sigma)$). We will use the forgetful map $\psi : H^{2\sigma-1}\to H^1_{\mathrm{sing}}$ to help with this.

Observe the Euler class a_σ fits into a cofiber sequence $S^0 \xrightarrow{a_\sigma} S^\sigma \to C_{2+}\wedge S^1$. Smashing with $\mathrm{OC}_3(2\sigma)$ then yields the standard forgetful long exact sequence

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & H^0_{\mathrm{sing}} & \longrightarrow & H^{\sigma-1} & \xrightarrow{a_\sigma} & H^{2\sigma-1} & \xrightarrow{\psi} & H^1_{\mathrm{sing}} & \longrightarrow & H^\sigma & \longrightarrow & H^{2\sigma} & \longrightarrow \cdots \\
& & & \| & & \| & & \| & & \| & & \| & \\
& & & 0 & & \mathbb{Z}^3 & & \mathbb{Z}^3 & & \mathbb{Z}/2 & & \mathbb{Z}^2 &
\end{array}$$

So ψ is an inclusion where the image has index 2. A little work shows that

$$\mathrm{Im}(\psi)=\langle \omega_{12}-\omega_{13},\ \omega_{12}-\omega_{23},\ 2\omega_{12}\rangle.$$

Note that these generators are not canonical; it would be better to describe $\mathrm{Im}(\psi)$ as being spanned by all the differences of ω-classes together with all doubles of ω-classes. However, we have chosen generators that are convenient with respect to our original "reference frame" of ω_{12}. Since $\psi(\iota)=2\omega_{12}$, we can fix the two red generators $A,B\in H^{2\sigma-1}$ by requiring that they map to $\omega_{12}-\omega_{13}$ and $\omega_{12}-\omega_{23}$.

To understand products we need to look at $\psi : H^{4\sigma-2}\to H^2$, and again access this via the forgetful long exact sequence

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^1_{\mathrm{sing}} & \longrightarrow & H^{3\sigma-2} & \xrightarrow{a_\sigma} & H^{4\sigma-2} & \xrightarrow{\psi} & H^2_{\mathrm{sing}} & \longrightarrow & H^{3\sigma-1} & \longrightarrow \cdots \\
& & & \| & & \| & & \| & & \| & \\
& & & \mathbb{Z}/2 & & \mathbb{Z}^2 & & \mathbb{Z}^2 & & (\mathbb{Z}/2)^2 &
\end{array}$$

Since ψ is injective we can use it to detect products. Note that $\psi(A)$ and $\psi(B)$ are degree 1 and therefore square to zero, so A^2 and B^2 must also vanish. We also compute

$$\begin{aligned}
\psi(AB)=\psi(A)\psi(B)=(\omega_{12}-\omega_{13})(\omega_{12}-\omega_{23}) &= -\omega_{13}\omega_{12}-\omega_{12}\omega_{23}+\omega_{13}\omega_{23}\\
&= -\omega_{31}\omega_{12}-\omega_{12}\omega_{23}-\omega_{23}\omega_{31}=0.
\end{aligned}$$

So AB must be zero, and likewise for BA by skew-commutativity.

Likewise, $\psi(\iota A) = \psi(\iota)\psi(A) = 2\omega_{12}(\omega_{12} - \omega_{13}) = -2\omega_{12}\omega_{13}$ and $\psi(\iota B) = -2\omega_{12}\omega_{23}$. Since H^2_{sing} is generated by $\omega_{12}\omega_{13}$ and $\omega_{12}\omega_{23}$, and the quotient by $\mathrm{Im}(\psi)$ is $(\mathbb{Z}/2)^2$, this calculates that $\mathrm{Im}(\psi)$ is generated by $2\omega_{12}\omega_{13}$ and $2\omega_{12}\omega_{23}$. So ιA and ιB may be taken to be the generators in this degree.

Putting everything together, we have established

$$H^\star(\mathrm{OC}_3(2\sigma); \underline{\mathbb{Z}}) \cong \mathbb{E}_2[\iota, A, B]/(A^2, B^2, AB, a_\sigma \iota, \iota^2),$$

where as usual the polynomial ring is interpreted to be skew-commutative and not strictly commutative. The forgetful map has $\psi(\iota) = 2\omega_{12}$, $\psi(A) = \omega_{12} - \omega_{13}$, and $\psi(B) = \omega_{12} - \omega_{23}$.

The above example generalizes to the following result:

Proposition 5.4 *Let $G = C_2$ and let $\mathbb{E}_r = \mathbb{M}[u^{-1}]/(a_\sigma^r)$. For $n \geq 2$ let $T \subseteq H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n))$ be the subring generated by the elements $\omega_{ij} - \omega_{kl}$ and $2\omega_{ij}$ (for any i, j, k, l). Then there is an isomorphism*

$$H^\star(\mathrm{OC}_q(n\sigma); \underline{\mathbb{Z}}) \cong \mathbb{E}_n \otimes T,$$

where the generators of T listed above are regarded as lying in degree $n\sigma - 1$ and the ring structure on the tensor product is the skew-commutative one.

6 Comparing configuration spaces

The ring presented in Theorem 5.2 can be regarded as a deformation of the classical singular cohomology of the configuration space $\mathrm{OC}_q(\mathbb{R}^n)$, with the symbol a_{V-1} as the deformation parameter. The paper [24] introduced and studied a closely related ring, namely the ring of functions $\mathrm{Func}(\Sigma_q, \mathbb{Z})$ (actually this is just one example of the rings studied in [24], which worked in the more general context of hyperplane arrangements). Varchenko and Gelfand proved that $\mathrm{Func}(\Sigma_q, \mathbb{Z})$ can be presented as the quotient of $\mathbb{Z}[e_{ij} \mid 1 \leq i \neq j \leq q]$ by the relations

- $e_{ij} = 1 - e_{ji}$,
- $e_{ij}^2 = e_{ij}$,
- $e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} = e_{ij} + e_{jk} + e_{ki} - 1$.

The similarity between the above presentation and the one from Theorem 5.2 is transparent, but why are these two rings related?

Essentially the same question has been answered by Proudfoot and his collaborators [20; 18; 6] (working in the cases of $\mathbb{Z}/2$-, $\mathbb{Q}$-, and $\mathbb{Z}$-coefficients, respectively). They explain the relation between three things: the Varchenko–Gelfand ring, the cohomology ring of $\mathrm{OC}_q(\mathbb{R}^n)$, and the Borel-equivariant cohomology of spaces $\mathrm{OC}_q(V)$ for certain choices of group G (either C_2 or S^1) and representation V.

In this section we will review some of this story and retell it in the context of Bredon cohomology, using the results of our computations. On the one hand, the Bredon story does not contain any fundamentally new ideas or improve on the Borel story in any significant way (in fact in some ways our version is worse, as we will see). On the other hand, the Bredon version has a few interesting aspects that demonstrate some features of the theory.

The heart of the matter involves the inclusions $\mathrm{OC}_q(V) \hookrightarrow \mathrm{OC}_q(V \oplus W)$ for various representations V and W. In (reduced) singular cohomology these maps are always zero, but in equivariant cohomology they can be nonzero. The space $\mathrm{OC}_q(\mathbb{R}^1)$ (where $\mathbb{R}^1$ has the trivial action) is homotopy equivalent to Σ_q, and so $H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^1))$ is just the ring $\mathrm{Func}(\Sigma_q, \mathbb{Z})$ concentrated in degree 0. We will see (following Proudfoot) that studying the inclusions $\mathrm{OC}_q(\mathbb{R}^1) \hookrightarrow \mathrm{OC}_q(\mathbb{R}^1 \oplus W)$ for certain W leads to the desired connection between Theorem 5.2 and the Varchenko–Gelfand ring.

6.1 The general comparison theorem for representations

Fix representations V and W where $V \supseteq 1$. Consider the inclusion $u : \mathrm{OC}_q(V) \hookrightarrow \mathrm{OC}_q(V \oplus W)$. The Bredon cohomology of the codomain has generators $\omega_{ij}(V \oplus W)$ in degree $V + W - 1$, whereas for the domain the generators $\omega_{ij}(V)$ lie in degree $V - 1$. Here is the fundamental computation:

Proposition 6.2 *We have $u^*(\omega_{ij}(V \oplus W)) = \omega_{ij}(V) \cdot a_W$. Consequently, if $W \supseteq 1$ then u^* is the zero map on reduced cohomology.*

Proof This is a consequence of the commutative diagram

$$\begin{array}{ccc} \mathrm{OC}_q(V \oplus W) & \xrightarrow{\widetilde{\omega}_{ij}(V \oplus W)} & S(V \oplus W) \\ \uparrow & & \uparrow \\ \mathrm{OC}_q(V) & \xrightarrow{\widetilde{\omega}_{ij}(V)} & S(V) \end{array}$$

Since $V \supseteq 1$ we have $S(V) \cong S^{V-1}$ and $S(V \oplus W) \cong S^{V-1+W}$. So the result follows formally from Lemma 6.3 below. □

Lemma 6.3 *Let K and L be representations where $K \supseteq 1$. Then one has*

$$j^*(\iota_{K \oplus L}) = \iota_K \cdot a_L,$$

where $j : S^K \hookrightarrow S^{K \oplus L}$ is the evident inclusion.

Proof Use the commutative diagram

$$\begin{array}{ccccc} S^K & \xrightarrow{j} & S^{K \oplus L} & \xrightarrow{\iota_{K \oplus L}} & \mathrm{AG}(S^{K \oplus L}) \\ \| & & & & \uparrow \mu \\ S^K \wedge S^0 & \xrightarrow{\mathrm{id} \wedge a_L} & S^K \wedge S^L & \xrightarrow{\iota_K \wedge \iota_L} & \mathrm{AG}(S^K) \wedge \mathrm{AG}(S^L). \end{array}$$ □

When $W \not\supseteq 1$ the map u^* from Proposition 6.2 has the potential to be nonzero and to contain useful information. We will see some examples below.

6.4 The Varchenko–Gelfand ring

Let $H_{ij} \subseteq \mathbb{R}^q$ be the hyperplane defined by $x_i = x_j$. Then $\mathrm{OC}_q(\mathbb{R}) = \mathbb{R}^q - \bigcup_{i,j} H_{ij}$, and this space is a union of contractible components which are in bijective correspondence with elements of Σ_q: the component of a configuration $\underline{x}$ is indexed by the unique permutation σ that puts the elements of $\underline{x}$ in ascending order, in the sense that $x_{\sigma^{-1}(1)} < x_{\sigma^{-1}(2)} < \cdots$. (The use of σ^{-1} in the subscripts provides that $x_i < x_j$ implies $\sigma(i) < \sigma(j)$.) Varchenko–Gelfand [24] consider the ring of locally constant integer-valued functions on $\mathrm{OC}_q(\mathbb{R})$, which by the above bijection is the same as the ring $\mathrm{Func}(\Sigma_q, \mathbb{Z})$. It is also $H^0(\mathrm{OC}_q(\mathbb{R}); \mathbb{Z})$.

For each $i \neq j$ [24] defines the *Heaviside function* e_{ij} to have the value 1 on all elements of $\mathbb{R}^q$ on the side of H_{ij} where $x_i > x_j$, and to take the value 0 on the other side of H_{ij}. The relations

$$e_{ij}^2 = e_{ij}, \quad e_{ij} = 1 - e_{ji}$$

are self-evident. If we instead think of these as functions $\Sigma_q \to \mathbb{Z}$, then for $\sigma \in \Sigma_q$ the above says that $e_{ij}(\sigma)$ is 1 if $\sigma(i) > \sigma(j)$ and 0 otherwise.

Observe that e_{ij} basically coincides with the cohomology class $\omega_{ij} \in H^0(\mathrm{OC}_q(\mathbb{R}))$ that we have previously defined, if one takes $-1 \in S(\mathbb{R})$ as the basepoint so that it represents 0 in $\mathrm{AG}(S(\mathbb{R}))$.

The analog of the Arnold relation is proven as follows. Given a permutation σ, if $\sigma(i) < \sigma(j)$ and $\sigma(j) < \sigma(k)$ then it must be the case that $\sigma(i) < \sigma(k)$. This shows that the product $e_{ji}e_{kj}e_{ik}$ must be the zero function. Rewrite this as

$$0 = (1-e_{ij})(1-e_{jk})(1-e_{ki}) = 1 - e_{ij} - e_{jk} - e_{ki} + e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} - e_{ij}e_{jk}e_{ki}$$

and then use that the final cubic term must be zero by exactly the same reasoning as before (we are grateful to Nick Proudfoot for explaining this to us).

In [24] it is proven that $\mathrm{Func}(\Sigma_q, \mathbb{Z})$ is the quotient of the polynomial ring $\mathbb{Z}[e_{ij}]$ by the three types of relations listed above. Here we have produced the map $\mathbb{Z}[e_{ij}]/\sim \to \mathrm{Func}(\Sigma_q, \mathbb{Z})$. One way to justify that this is an isomorphism is via the following steps:

- Check by the same argument as in the proof of Proposition 3.10 that the domain is a free abelian group whose rank is $q!$, as is true for the codomain.
- Prove surjectivity as follows: For $\sigma \in \Sigma_q$ let $\delta_\sigma : \Sigma_q \to \mathbb{Z}$ send σ to 1 and all other permutations to zero. Then the δ_σ functions are a $\mathbb{Z}$-basis for $\mathcal{F}(\Sigma_q, \mathbb{Z})$. For a given σ let P_σ be the product of $\binom{q}{2}$ factors, where for every $i < j$ we include the factor e_{ij} if $\sigma(i) > \sigma(j)$, and the factor $1 - e_{ij}$ if $\sigma(i) < \sigma(j)$. As an example, for the permutation $\sigma = \left(\begin{smallmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{smallmatrix}\right)$ we would have

$$P_\sigma = (1 - e_{12})e_{13}e_{23}.$$

Observe that $P_\sigma(\sigma) = 1$. If α is another permutation then $P_\sigma(\alpha) = 0$ unless the relative order of all pairs under α exactly matches the relative order under σ — but this can only happen if $\alpha = \sigma$. So $P_\sigma = \delta_\sigma$, and this proves surjectivity.

Varchenko and Gelfand considered the increasing filtration F_* of the ring $R = \operatorname{Func}(\Sigma_q, \mathbb{Z})$ where F_r is spanned by monomials in the e_{ij} of degree less than or equal to r. Thus we have

$$\mathbb{Z}\langle 1\rangle = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_q = R$$

and $F_r \cdot F_s \subseteq F_{r+s}$. Therefore $\operatorname{gr} R$ is a graded ring, and they observed that

$$\operatorname{gr} R \cong H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n)) \quad \text{with } F_i/F_{i-1} \cong H^{i(n-1)}_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}^n))$$

(really they did this for $n = 2$, though the observation is of course valid for higher n as well).

For a filtered ring as above, the *Rees ring* $\operatorname{Rees}(R)$ is the subring $\bigoplus_i t^i F_i \subseteq R[t]$. One has

$$\operatorname{Rees}(R)/(t) \cong \operatorname{gr} R \quad \text{and} \quad \operatorname{Rees}(R)/(t-1) \cong R.$$

We depict this in diagram form as

$$\begin{array}{ccc} & \operatorname{Rees}(R) & \\ {\scriptstyle t=1} \swarrow & & \searrow {\scriptstyle t=0} \\ R & & \operatorname{gr}(R) \end{array}$$

The relations among the e_{ij} classes become the following homogeneous relations that hold in $\operatorname{Rees}(R)$:

$$\begin{aligned} e_{ij} = 1 - e_{ji} &\rightsquigarrow [te_{ij}] = t - [te_{ji}], \\ e_{ij}^2 = e_{ij} &\rightsquigarrow [te_{ij}]^2 = t \cdot [te_{ij}], \\ e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} = e_{ij} + e_{jk} + e_{ki} - 1 &\rightsquigarrow \\ [te_{ij}][te_{jk}] + [te_{jk}][te_{ki}] + [te_{ki}][te_{ij}] &= t[te_{ij}] + t[te_{jk}] + t[te_{ki}] - t^2. \end{aligned}$$

Note that we are writing $[te_{ij}]$ for the evident class in $\operatorname{Rees}(R)_1$ to make it clear that this is not a multiple of t in the Rees ring.

The reader will note that these relations in the Rees ring are almost the same as those in Theorem 5.2 — with t replaced by a_{V-1} — though there are some sign differences that we will discuss in the next section.

6.5 The denouement

Let W be a representation such that G acts freely on $W - \{0\}$. Examples to keep in mind are when $G = C_2$ and $W = \mathbb{R}$ with the sign action, and when $G = C_n$ and $W = \mathbb{R}^2$ with the generator of G acting as counterclockwise rotation by $\frac{2\pi}{n}$ radians. Note that if W exists then $d(W) = \#G$, where $d(W)$ is the greatest common divisor defined in Proposition 2.7; so by Proposition 2.11 if $\#G > 2$ then W must be even-dimensional.

We will consider the inclusion

$$u : \mathrm{OC}_q(\mathbb{R}) \hookrightarrow \mathrm{OC}_q(\mathbb{R} \oplus W),$$

where the G-action on $\mathbb{R}$ is the trivial one. So u is the inclusion of the fixed set. Consider the diagram

$$\begin{array}{ccc} & H^\star(\mathrm{OC}_q(\mathbb{R}\oplus W)) & \\ \overset{u^*}{\swarrow} & & \overset{\psi}{\searrow} \\ H^\star(\mathrm{OC}_q(\mathbb{R})) & & H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}\oplus W)) \\ \uparrow\cong & & \\ H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}))\otimes\mathbb{M} = \mathrm{Func}(\Sigma_q,\mathbb{Z})\otimes\mathbb{M} & & \end{array}$$

The vertical isomorphism is from Proposition 2.5. The ring map ψ sends the generators $\omega_{ij}(\mathbb{R}\oplus W)$ to the classical generators ω_{ij}, and so ψ is surjective. The element $e_{ij}\otimes 1\in \mathrm{Func}(\Sigma_q,\mathbb{Z})\otimes\mathbb{M}$ maps to $\omega_{ij}(\mathbb{R})$ in $H^\star(\mathrm{OC}_q(\mathbb{R}))$ by inspection, and we know $u^*(\omega_{ij}(\mathbb{R}\oplus W))=a_W\cdot\omega_{ij}(\mathbb{R})$ by Proposition 6.2.

All of the interesting phenomena are in the degrees nW for $n\in\mathbb{Z}$, so we restrict our attention to those. In fact the groups are zero when $n<0$ (this uses Proposition 2.14), so we focus on $n\geq 0$. The following table shows the additive generators of $H^\star(\mathrm{OC}_q(\mathbb{R}\oplus W))$ in these degrees:

0	1	2	3	$\cdots$
1	ω_{ij} a_W	$\omega_{ij}\omega_{mn}$ $a_W\omega_{ij}$ a_W^2	$\omega_{ij}\omega_{mn}\omega_{st}$ $a_W\omega_{ij}\omega_{mn}$ $a_W^2\omega_{ij}$ a_W^3	$\cdots$

Here the ω_{ij} classes and their products are all torsion-free, whereas all classes with a_W are annihilated by $d(W)$. If one formally substitutes $\omega_{ij}=a_We_{ij}$ then this starts to look like the Rees ring for $\mathrm{Func}(\Sigma_q,\mathbb{Z})$ associated to the Varchenko–Gelfand filtration, but there are two differences. One small but important difference is that the class a_W is torsion. The other difference is that when W is odd-dimensional the relations in $H^{*W}(\mathrm{OC}_q(\mathbb{R}\oplus W))$ relating ω_{ij} to ω_{ji} have signs that differ from those in the Rees ring. (Remember that G acts freely on $W-\{0\}$ and so — from the discussion at the beginning of this subsection — the case where W is odd-dimensional only occurs when $G=C_2$ and then $d(W)=2$. While having $d(W)=2$ still does not give us the Rees ring, it forces all signs on the a_W classes to be irrelevant.)

We can redraw our diagram as

$$\begin{array}{ccc} & H^{*W}(\mathrm{OC}_q(\mathbb{R}\oplus W)) & \\ \overset{a_W=1}{\swarrow\!\!\swarrow} & & \overset{a_W=0}{\searrow\!\!\searrow} \\ \mathrm{Func}(\Sigma_q,\mathbb{Z}/d(W)) & & H^*_{\mathrm{sing}}(\mathrm{OC}_q(\mathbb{R}\oplus W)) \end{array}$$

The object on the bottom left is purely combinatorial, whereas the one on the bottom right is topological; we will use the diagram to pass information between them. Whether or not the object on top is isomorphic to the Rees ring is immaterial to the rest of our discussion, but it plays a comparable role. We can regard this object as a graded module over $R = \mathbb{Z}[a_W]/(d(W)a_W)$, and as such it is a free module. Let b_i denote the number of generators of rank i, and pretend for the moment that we don't know these numbers.

The two arrows are the result of applying $(-) \otimes_R R/(a_W - 1)$ and $(-) \otimes_R R/(a_W)$. In the latter case, $R/(a_W)$ is a graded R-module and so the target is also graded (as we know). So the b_i are the same as the ranks of the singular cohomology groups. In the former case, $R/(a_W - 1)$ is not graded and we instead only get a filtration defined by letting F_i be the image of the i-th graded piece. By inspection this is precisely the Varchenko–Gelfand filtration, and it follows formally that F_i/F_{i+1} is a free $\mathbb{Z}/d(W)$-module of rank b_i.

As a consequence of the above we deduce that

- The total rank of $H^*_{\text{sing}}(\mathrm{OC}_q(\mathbb{R} \oplus W))$ is the same as the rank of $\mathrm{Func}(\Sigma_q, \mathbb{Z}/d(W))$ as a $\mathbb{Z}/d(W)$-module, which is manifestly equal to $q!$.
- The rank of $H^{i\cdot|W|}_{\text{sing}}(\mathrm{OC}_q(\mathbb{R} \oplus W))$ is equal to the rank of F_i/F_{i+1} as a $\mathbb{Z}/d(W)$-module, which is a purely combinatorial object. From here it is "only" a matter of combinatorics to identify this with the appropriate Stirling number.

This is not exactly an independent calculation of $H^*_{\text{sing}}(\mathrm{OC}_q(\mathbb{R} \oplus W))$ in that it presupposes knowing the freeness of the upper object in our diagram, but it does give a satisfying explanation of how the machinery of equivariant cohomology provides combinatorial interpretations of the cohomology of $\mathrm{OC}_q(\mathbb{R} \oplus W)$.

Following Proudfoot and his collaborators, it is tempting to push these methods to incorporate the Σ_q-actions that exist everywhere. Certainly Σ_q acts on the three objects in our diagram, and the action is preserved by the maps. It would be nice to conclude, for example, that the total cohomology ring of $\mathrm{OC}_q(\mathbb{R} \oplus W)$ is isomorphic — as a Σ_q-module — to $\mathrm{Func}(\Sigma_q, \mathbb{Z})$. Unfortunately, the fact that a_W is torsion plays against us here.

There are a couple of approaches to removing the torsion condition on a_W. One is to look at an inverse limit system where the torsion gets larger and larger: for example, $G = C_n$ and $W = \mathbb{R}^2$ with the rotation-by-$\frac{2\pi}{n}$ representation, and let $n \to \infty$. A related — and more satisfying — approach is to take $G = S^1$ with $W = \mathbb{R}^2$ the analogous "rotation" representation. This is the approach taken in [6; 18] for Borel cohomology; it can also be done in the Bredon setting, but it takes us outside the context of finite groups that has been the subject of the present paper.

Appendix A A proof for odd-dimensional representations

Here we give an elementary proof (avoiding Bredon cohomology) of Proposition 2.11. That is, we show that if V is an odd-dimensional representation of G over $\mathbb{R}$ then the gcd of $\mathcal{D}(V) = \{\#(G/H) \mid V^H \neq 0\}$ is either 1 or 2. This implies the Euler class satisfies either $a_V = 0$ or $2a_V = 0$. We use Smith theory as one step of the analysis, so the proof is not entirely algebraic.

Proof of Proposition 2.11 First observe that $d(V) = \gcd \mathcal{D}(V)$ is a divisor of $\#G$ by the definition. Next we prove $d(V)$ must be a power of 2.

Let p be an odd prime dividing $\#G$ and let $H \leq G$ be a Sylow p-subgroup. Smith theory implies that $\chi(S^V) \equiv \chi(S^{V^H}) \bmod p$ (one reference is [17, IV.1.5]). But $\chi(S^V) = 0$ since $\dim V$ is odd, so p divides $\chi(S^{V^H})$. In particular, $V^H \neq 0$ and therefore $\#(G/H) \in \mathcal{D}(V)$. Note p does not divide $\#(G/H)$, and so p does not divide $\gcd \mathcal{D}(V)$. We have proven this for all odd primes dividing $\#G$. Thus, $d(V)$ is a power of 2.

We now break into cases based on the parity of $\#G$:

Case 1: *$\#G$ is odd.* Since $d(V)$ is a power of 2 and $d(V) \mid \#G$, we see $d(V) = 1$.

Case 2*a*: *$\#G$ is a power of* 2. We will prove the stronger result that either $1 \in \mathcal{D}(V)$ or $2 \in \mathcal{D}(V)$.

Since $\mathcal{D}(V_1 \oplus V_2) = \mathcal{D}(V_1) \cup \mathcal{D}(V_2)$, it suffices to prove the result when V is irreducible. In that case $\mathrm{Hom}_G(V, V)$ is a real division algebra and so is equal to one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$; V is accordingly classified as "real", "complex", or "quaternionic". In the latter two cases V must be even-dimensional over $\mathbb{R}$, so in fact our V is "real". The complexification $V_{\mathbb{C}}$ of an irreducible "real" representation is also irreducible; this follows from $\mathrm{End}_G(V_{\mathbb{C}}) \cong \mathrm{End}_G(V)_{\mathbb{C}} \cong \mathbb{C}$.

Since $V_{\mathbb{C}}$ is irreducible we have $\dim_{\mathbb{C}} V_{\mathbb{C}}$ divides $\#G$. But this says $\dim V$ divides $\#G$, so $\dim V$ is a power of 2. Since $\dim V$ is also odd, $\dim V = 1$. The representation of G on V is therefore specified by a map $G \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^*$, or really a map $G \to \mathbb{Z}/2$ since the image must lie in the torsion elements. Either this map is trivial or else the kernel is index 2, which says that either $1 \in \mathcal{D}(V)$ or $2 \in \mathcal{D}(V)$. So $\gcd \mathcal{D}(V)$ is 1 or 2 here.

Case 2*b*: *$\#G$ is even.* Let P be a Sylow 2-subgroup of G. By Case 2a, applied to V as a P-representation, either $V^P \neq 0$ or there is an index 2 subgroup $Q \subseteq P$ such that $V^Q \neq 0$. In the first case we have that $\#(G/P) \in \mathcal{D}(V)$, therefore $\mathcal{D}(V)$ contains an odd number, and since we already showed the gcd is a power of 2 it must be exactly 1. In the second case we have that $\#(G/Q) \in \mathcal{D}(V)$, i.e., $\mathcal{D}(V)$ contains twice an odd number. Since the gcd is a power of 2, it can only be 1 or 2. □

Appendix B Verifying the vanishing requirement

In this section we discuss the matter of checking the vanishing requirement. Recall that the requirement for (G, V) is that in the sequence

$$H^{\ell V-\ell}(\mathrm{pt}) \xrightarrow{a_V} H^{(\ell+1)V-\ell}(\mathrm{pt}) \xrightarrow{a_V} H^{(\ell+2)V-\ell}(\mathrm{pt}) \tag{B.0.1}$$

the first map is surjective and the second map is injective for all $\ell \in \mathbb{Z}$. By Proposition 4.9 the requirement always holds if $V^G \neq 0$ and $V \neq 2$, so here we always focus on $V^G = 0$.

When $\ell = 0$ the conditions are satisfied because we know all three groups by Proposition 2.7 and can check it directly. When $\ell = -1$ the middle group is zero, and so the conditions are trivially satisfied. When $\ell = -2$ the sequence is

$$H^{2-2V}(\mathrm{pt}) \xrightarrow{a_V} H^{2-V}(\mathrm{pt}) \xrightarrow{a_V} H^2(\mathrm{pt}) = 0.$$

Since the last group is zero the conditions will be satisfied if and only if the middle group is zero. But if V is orientable and $\dim V = 2$ we know $H^{2-V}(\mathrm{pt}) \neq 0$. For this reason we will typically only be able to verify the conditions when $\dim V \neq 2$.

Remark B.1 If $\alpha \in RO(G)$ we can consider the Mackey functor $\underline{H}^{\alpha}(\mathrm{pt})$, which is a $\underline{\mathbb{Z}}$-module. So the composite

$$H^{\alpha}(\mathrm{pt}) \xrightarrow{\mathrm{Res}} H^{\alpha}(G) \xrightarrow{\mathrm{Tr}} H^{\alpha}(\mathrm{pt})$$

is multiplication by $\#G$. The middle group is isomorphic to $H^{|\alpha|}_{\mathrm{sing}}(\mathrm{pt})$ and is therefore zero when $|\alpha| \neq 0$. It follows that when $|\alpha| \neq 0$ the group $H^{\alpha}(\mathrm{pt})$ is torsion and annihilated by $\#G$. The index $(\ell+1)V - \ell$ of the middle group from (B.0.1) can have rank 0 only when $\ell = -2$ and $\dim V = 2$. Avoiding this case therefore ensures that the middle group is annihilated by $\#G$.

In the ring $\mathbb{M} = H^{\star}(\mathrm{pt}; \underline{\mathbb{Z}})$ the portion graded by virtual representations $W + n$ and $-W + k$ for all (nonvirtual) representations W and all $n, k \in \mathbb{Z}$ is somewhat more accessible then the entirety of $\mathbb{M}$. We call this the *regular portion* of $\mathbb{M}$. Note that the regular portion is not a subring. The direct sum of terms $\mathbb{M}^{W+n}$ for W a representation and $n \in \mathbb{Z}$ will be called the "positive" part of $\mathbb{M}$, and the direct sum of terms $\mathbb{M}^{-W+n}$ for $n \in \mathbb{Z}$ will be called the "negative" part. Note that these terms only refer to the regular region. Recall that $\mathbb{M}^{W+n} = 0$ for $n > 0$ or $n < -\dim W$, and likewise $\mathbb{M}^{n-W} = 0$ for $n \leq 0$ or $n > \dim W$ (see Propositions 2.7 and 2.14 for parts of this; the other parts follow from suspension isomorphisms and cellular homology). So the nonvanishing groups are concentrated in positive and negative "cones".

For a particular representation V, let $\mathbb{M}(V)$ be the subring of $\mathbb{M}$ consisting of all gradings $aV + b$ for $a, b \in \mathbb{Z}$. This is contained in the regular portion of $\mathbb{M}$. Observe that the vanishing requirements concern only these subrings, and in particular do not involve the "irregular" portion of $\mathbb{M}$.

Our goal in this appendix is to prove Proposition 4.10, which says that the vanishing requirement is always satisfied if $\dim V \geq 3$ and G is either a cyclic group or Σ_3. This depends on the calculations of $H^{\star}(\mathrm{pt})$ for cyclic groups that were done in [3], and the analogous calculations for Σ_3 that were done in [25].

B.2 Background on cyclic groups

Let C_n be the cyclic group with n elements. Let $\lambda(k)$ denote $\mathbb{R}^2 = \mathbb{C}$ with the generator of C_n acting as multiplication by $e^{\frac{2\pi i k}{n}}$. In addition to $\lambda(k)$ only depending on k modulo n, one also has $\lambda(k) \cong \lambda(n-k)$ (using complex conjugation for the isomorphism). It is known that $RO(C_n)$ has a basis consisting of $1, \lambda(1), \ldots, \lambda(\lfloor \frac{n-1}{2} \rfloor)$ and — when n is even — the sign representation σ. We take this as our ordered basis for the $RO(C_n)$-grading.

It turns out we don't need to consider all of the regular portion of $RO(C_n)$ to verify the vanishing requirement. Using results from [1; 3], we need only consider representations comprised of the irreducible representations $\lambda(d)$ for $d \mid n$ (and σ if n is even). The following proposition makes this reduction precise:

Proposition B.3 *Suppose $V = m_0\sigma + \sum_k m_k\lambda(k)$ where each $m_i \geq 0$ (take $m_0 = 0$ if n isn't even). Let V' be the representation given by replacing each $\lambda(k)$ with $\lambda((k,n))$, where as usual (k,n) is the* *gcd. That is,* $V' = m_0\sigma + \sum_k m_k\lambda((k,n))$*. Then the vanishing condition holds for* V *if and only if the vanishing condition holds for* V'*.*

Proof Define $\mathrm{RO}_0(C_n)$ to be the subring of $\mathrm{RO}(C_n)$ consisting of $\alpha \in \mathrm{RO}(C_n)$ such that $|\alpha^H| = 0$ for all subgroups H of C_n. It is straightforward to verify $|V^H| = |(V')^H|$ for all H and so $V' - V \in \mathrm{RO}_0(C_n)$. It follows from [1, Theorem B] that there exists an invertible element $\mu \in H^{V'-V}(\mathrm{pt})$. Multiplication by μ gives an isomorphism from $H^V(\mathrm{pt})$ to $H^{V'}(\mathrm{pt})$. Both groups are isomorphic to $\mathbb{Z}/d(V)$ and generated by a_V and $a_{V'}$, respectively (note $d(V) = d(V')$ because $|V^H| = |(V')^H|$ for all H). Thus $\mu a_V = b a_{V'}$ and $\mu^{-1} a_{V'} = c a_V$ for some $b, c \in \mathbb{Z}/d(V)$ satisfying $bc = 1$.

Now consider the commutative diagram

$$\begin{array}{ccccc}
H^{\ell V-\ell}(\mathrm{pt}) & \xrightarrow{a_V} & H^{(\ell+1)V-\ell}(\mathrm{pt}) & \xrightarrow{a_V} & H^{(\ell+2)V-\ell}(\mathrm{pt}) \\
\cong\big\downarrow \mu^{\ell} & & \cong\big\downarrow \mu^{\ell+1} & & \cong\big\downarrow \mu^{\ell+2} \\
H^{\ell V'-\ell}(\mathrm{pt}) & \xrightarrow{b a_{V'}} & H^{(\ell+1)V'-\ell}(\mathrm{pt}) & \xrightarrow{b a_{V'}} & H^{(\ell+2)V'-\ell}(\mathrm{pt})
\end{array}$$

Note the maps in the top row are injective and then surjective, respectively, if and only if the corresponding maps in the bottom row are. Lastly note that since $b \in \mathbb{Z}/d(V)$ is a unit, multiplication by $b a_{V'}$ is injective (surjective) if and only if multiplication by $a_{V'}$ is injective (surjective). □

Moving forward we only consider the regular portion of $\mathbb{M}$ where the representations W are direct sums of copies of 1 and $\lambda(d)$ where d is a proper divisor of n (and σ if n is even).

It is easiest to explain the Basu–Dey calculation [3] by first looking at the case $G = C_{p^e}$ where p is prime. Here we have the irreducible representations 1 and $\lambda_i = \lambda(p^i)$ for $0 \le i \le e-1$, and if p is even we also have the sign representation σ. We take $1, \lambda_0, \dots, \lambda_{e-1}, (\sigma)$ as our chosen ordered basis for the multigrading (again, note that this is smaller than the full $\mathrm{RO}(G)$-grading but sufficient for our purposes).

Write a_i for the Euler class and u_i for the orientation class of λ_i. The paper [3] proves that when p is odd the positive cone of $H^\star(\mathrm{pt})$ is

$$\mathbb{Z}[a_0, \dots, a_{e-1}, u_0, \dots, u_{e-1}]/R,$$

where R is the ideal generated by the relations

$$p^{e-i} a_i = 0 \quad \text{(Euler class torsion relations)},$$
$$p^{j-i} a_i u_j = a_j u_i \ \text{ for } i < j \quad \text{(gold relations)}.$$

From this it is easy to see that any particular $H^\alpha(\mathrm{pt})$ in the positive cone is cyclic, generated by the unique monomial in this degree having the form $a_{i_1}^{f_1} \cdots a_{i_r}^{f_r} u_{j_1}^{h_1} \cdots u_{j_s}^{h_s}$ where all $f_k > 0$, $h_k > 0$, and $i_1 < i_2 < \dots < i_r \le j_1 < \dots < j_s$ (briefly, no index on an a-class is larger than an index on a u-class). The additive order of this generator is p^{e-i_r}, or ∞ when no a-classes appear in the monomial. Note in particular that within the positive cone $H^\alpha(\mathrm{pt})$ is torsion if $|\alpha| > 0$ and torsion-free if $|\alpha| = 0$.

For $p=2$ one instead has that $H^\star(\mathrm{pt})$ is $\mathbb{Z}[a_0,\dots,a_{e-1},a_\sigma,u_0,\dots,u_{e-1}]/R$ where R is the ideal generated by the relations

$$2^{e-i}a_i=0,\ 2a_\sigma=0 \quad \text{(Euler class torsion relations)},$$
$$a_\sigma^2=a_{e-1} \quad \text{(follows from } 2\sigma\cong\lambda_{e-1}),$$
$$2^{j-i}a_iu_j=a_ju_i \ \text{ for } i<j \quad \text{(gold relations)}.$$

Note that the generator a_{e-1} can be eliminated from this presentation, but it is convenient not to do that. Again, one checks that in each particular degree $H^\alpha(\mathrm{pt})$ is cyclic generated by the unique monomial in this degree having the form

$$a_\sigma^\epsilon a_{i_1}^{f_1}\cdots a_{i_r}^{f_r}u_{j_1}^{h_1}\cdots u_{j_s}^{h_s},$$

where all $f_k>0$, $h_k>0$, $\epsilon\in\{0,1\}$, and $i_1<i_2<\cdots<i_r\le j_1<\cdots<j_s$. The additive order of this class is 2 if $\epsilon\neq 0$, ∞ if no a-classes appear, and 2^{e-i_r} otherwise.

Now let us describe how the above picture changes for the group $G=C_n$. We restrict the $\mathrm{RO}(C_n)$-grading to representations that are direct sums of the irreducibles σ (when n is even) and $\lambda(d)$ for $d\mid n$. Basu–Dey [3] prove that the positive cone (or really, the region of the positive cone indexed by this portion of $\mathrm{RO}(G)$) is generated by the classes $a_{\lambda(d)}$ and $u_{\lambda(d)}$ for all $d\mid n$, $d\neq n$, together with a_σ when n is even, subject to the relations

$$\frac{n}{d}\cdot a_{\lambda(d)}=0 \quad \text{(Euler class torsion relations)},$$
$$\frac{d}{(d,s)}\cdot a_{\lambda(s)}u_{\lambda(d)}=\frac{s}{(d,s)}\cdot a_{\lambda(d)}u_{\lambda(s)} \quad \text{(gold relations)},$$
$$2a_\sigma=0 \ \text{ and } \ a_\sigma^2=a_{\lambda(\frac{n}{2})} \quad \text{(when } n \text{ is even)}.$$

Here (d,s) is the gcd of d and s.

As shorthand, let P_n denote the multigraded ring described by the above generators and relations. It is somewhat complicated to work with, and in particular does not have the property that the group in any fixed degree is generated by a monomial in the a- and u-classes. But if we localize at a fixed prime number it becomes much simpler. Fix a prime $p\in\mathbb{Z}$ and set $P_{n,p}=P_n\otimes\mathbb{Z}_{(p)}$. Then every homogeneous component of $P_{n,p}$ is a cyclic group, generated by a monomial in the a- and u-classes. The key observation is that in the gold relations at most one of the two coefficients $\frac{d}{(d,s)}$ and $\frac{s}{(d,s)}$ can have p as a factor. For more detail, let $\nu_p(x)$ denote the p-adic valuation of an integer x. Observe the following:

- $a_{\lambda(d)}=0$ in $P_{n,p}$ whenever $\nu_p(d)=\nu_p(n)$, using the Euler class torsion relations.
- Whenever $\nu_p(s)=\nu_p(d)$ then $a_{\lambda(s)}$ and $u_{\lambda(d)}$ commute in $P_{n,p}$ up to a unit multiple from $\mathbb{Z}_{(p)}$, by the gold relation.
- If $\nu_p(s)>\nu_p(d)$ then the gold relation implies that $a_{\lambda(s)}u_{\lambda(d)}$ is a $\mathbb{Z}_{(p)}$-multiple of $a_{\lambda(d)}u_{\lambda(s)}$ in $P_{n,p}$.

In the last two points we can actually arrange for the multiple to be in $\mathbb{Z}$ rather than just $\mathbb{Z}_{(p)}$, using that the a-classes are torsion.

Using the above observations, every monomial in the a's and u's is an integral multiple of one of the form

$$a_\sigma^\epsilon a_{\lambda(i_1)}^{f_1} \cdots a_{\lambda(i_r)}^{f_r} u_{\lambda(j_1)}^{h_1} \cdots u_{\lambda(j_s)}^{h_s},$$

where $\nu_p(i_k) < \nu_p(n)$ for all k, all $f_k > 0$, all $h_k > 0$, $\epsilon \in \{0, 1\}$,

$$\nu_p(i_1) \le \nu_p(i_2) \le \cdots \le \nu_p(i_r) \le \nu_p(j_1) \le \cdots \le \nu_p(j_s),$$

and where a_σ only occurs in the case $p = 2$ and n even. The additive order of this monomial is 2 if $p = 2$ and $\epsilon = 1$, otherwise it is $p^{\nu_p(n)-\nu_p(i_r)}$.

For $\alpha \in \mathrm{RO}(G)$ let us call $|\alpha^G|$ the *fixed-set-dimension* of α. The a-classes all live in degrees where the fixed-set-dimension is zero, and the u-classes live in degrees where the fixed-set-dimension is -2. So the nonzero classes in the positive cone only occur in degrees that have even fixed-set-dimension.

We will also need information about the negative cone of $H^\star(\mathrm{pt})$, but this is not explicitly calculated in [3]. However, the structure we need can be inferred from the positive cone using equivariant Anderson duality and a little legwork. We are grateful to the referee for outlining this part of the argument.

Recall $G = C_n$. An easy calculation gives that $\underline{H}^{s-\lambda(1)}(\mathrm{pt}) \cong \widetilde{\underline{H}}^s(S^{\lambda(1)})$ is zero for $s \ne 2$ and is $\underline{\mathbb{Z}}^*$ for $s = 2$. By the uniqueness of equivariant Eilenberg–Mac Lane spectra this implies that $\Sigma^{2-\lambda(1)} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}^*$. Equivariant Anderson duality says $I(H\underline{\mathbb{Z}}) \simeq H\underline{\mathbb{Z}}^*$, and putting the two together gives

$$I(H\underline{\mathbb{Z}}) \simeq \Sigma^{2-\lambda(1)} H\underline{\mathbb{Z}}.$$

For every $\alpha \in \mathrm{RO}(G)$ this yields the existence of canonical short exact sequences

$$0 \to \mathrm{Ext}(H_{\alpha+\lambda(1)-3}(X), \mathbb{Z}) \to H^\alpha(X) \to \mathrm{Hom}(H_{\alpha+\lambda(1)-2}(X), \mathbb{Z}) \to 0,$$

necessarily split (assuming X finite type) because the term on the right is always free. When $X = \mathrm{pt}$ we can use $H_\beta(\mathrm{pt}) = H^{-\beta}(\mathrm{pt})$ to rewrite this sequence as

$$0 \to \mathrm{Ext}(H^{3-\lambda(1)-\alpha}, \mathbb{Z}) \to H^\alpha \to \mathrm{Hom}(H^{2-\lambda(1)-\alpha}, \mathbb{Z}) \to 0 \tag{B.3.1}$$

(where we are now abbreviating $H^\beta(\mathrm{pt}) = H^\beta$). If $|\alpha| \ne 0$ then $H^{2-\lambda(1)-\alpha}$ is torsion and therefore the Hom-term is zero.

For an introduction to Anderson duality in the equivariant setting (focusing on the group C_2), see [21].

The sequence (B.3.1) almost lets us relate the negative cone to the positive cone, but things are a little off. Note that if $\alpha = s - W$ for W a representation and $s \in \mathbb{Z}$, then the cohomology groups appearing in the two ends of (B.3.1) are not in the regular portion of $H^\star(\mathrm{pt})$ at all if W doesn't contain $\lambda(1)$, and certainly not in the positive cone. In a moment we will explain a technique for shifting them into the positive cone.

The above short exact sequences are compatible with multiplication by the classes a_V (which come from the equivariant stable homotopy groups of spheres), in the sense that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Ext}(H_{\alpha+\lambda(1)-3}(X),\mathbb{Z}) & \longrightarrow & H^{\alpha}(X) & \longrightarrow & \mathrm{Hom}(H_{\alpha+\lambda(1)-2}(X),\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{Ext}((\cdot a_V),\mathbb{Z})} & & \downarrow{\scriptstyle \cdot a_V} & & \downarrow{\scriptstyle \mathrm{Hom}((\cdot a_V),\mathbb{Z})} \\ \mathrm{Ext}(H_{\alpha+V+\lambda(1)-3}(X),\mathbb{Z}) & \longrightarrow & H^{\alpha+V}(X) & \longrightarrow & \mathrm{Hom}(H_{\alpha+V+\lambda(1)-2}(X),\mathbb{Z}) \end{array}$$

Note that multiplication by a_V on homology reduces the degrees by V. We will use this diagram when $X = \mathrm{pt}$, reindexing the homology groups as cohomology groups like we did above.

It will be useful to adopt the following terminology: $H^\star(\mathrm{pt})$ is *V-nice* in degree α if the sequence $H^{\alpha-V} \xrightarrow{a_V} H^{\alpha} \xrightarrow{a_V} H^{\alpha+V}$ consists of cyclic torsion groups, the first map is surjective, and the second map is injective.

Lemma B.4 *Suppose $0 \notin \{|\alpha|, |\alpha+V|, |\alpha-V|\}$. Then if $H^\star(\mathrm{pt})$ is V-nice in degree $3-\alpha-\lambda(1)$, it is also V-nice in degree α.*

Proof We know that $H^{2-\alpha-\lambda(1)}$ is torsion, since the dimension of the index is $-|\alpha|$ and this is nonzero (see Remark B.1). Likewise, $H^{2-\alpha-V-\lambda(1)}$ and $H^{2-\alpha+V-\lambda(1)}$ are torsion. So in the Anderson duality short exact sequences for $H^{\alpha-V}$, H^{α} and $H^{\alpha+V}$, the Hom-terms all vanish. We therefore have

$$\begin{array}{ccccc} H^{\alpha-V} & \longrightarrow & H^{\alpha} & \longrightarrow & H^{\alpha+V} \\ \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\ \mathrm{Ext}(H^{3+V-\alpha-\lambda(1)}) & \longrightarrow & \mathrm{Ext}(H^{3-\alpha-\lambda(1)}) & \longrightarrow & \mathrm{Ext}(H^{3-\alpha-V-\lambda(1)}) \end{array}$$

where $\mathrm{Ext}(A)$ is short for $\mathrm{Ext}(A,\mathbb{Z})$. Now use that a surjection (resp. injection) of cyclic torsion groups becomes an injection (resp. surjection) of cyclic torsion groups upon applying $\mathrm{Ext}(-,\mathbb{Z})$. □

Next we explain the technique we mentioned above for shifting by $\lambda(1)$:

Lemma B.5 *If $|\beta| > 2$ then $H^{\beta-\lambda(1)} \xrightarrow{\cdot a_{\lambda(1)}} H^{\beta}$ is an isomorphism. Consequently, if $|\alpha - V| > 2$ then $H^\star(\mathrm{pt})$ is V-nice in degree α if and only if it is V-nice in degree $\alpha - \lambda(1)$.*

Proof For convenience just write $\lambda = \lambda(1)$. The second statement is immediate from the first, using the commutative diagram

$$\begin{array}{ccccc} H^{\alpha-V-\lambda} & \xrightarrow{\cdot a_V} & H^{\alpha-\lambda} & \xrightarrow{\cdot a_V} & H^{\alpha+V-\lambda} \\ {\scriptstyle \cdot a_\lambda}\downarrow{\scriptstyle\cong} & & {\scriptstyle \cdot a_\lambda}\downarrow{\scriptstyle\cong} & & {\scriptstyle \cdot a_\lambda}\downarrow{\scriptstyle\cong} \\ H^{\alpha-V} & \xrightarrow{\cdot a_V} & H^{\alpha} & \xrightarrow{\cdot a_V} & H^{\alpha+V} \end{array}$$

Let $j_0, j_\infty : \mathrm{pt} \to S^\lambda$ be the inclusion of the points 0 and ∞. Recall that ∞ is our basepoint, so $\widetilde{H}^\beta(S^\lambda) = \ker(j_\infty)^*$. The map $(j_0)^* : \widetilde{H}^\beta(S^\lambda) \to H^\beta(\mathrm{pt})$ is isomorphic to $H^{\beta-\lambda} \xrightarrow{\cdot a_\lambda} H^\beta$.

Regard S^λ as the unreduced suspension $\Sigma_u S(\lambda)$, and let $G \hookrightarrow S(\lambda)$ be the canonical embedding of the n-th roots of unity (recall $G = C_n$). We have the cofiber sequence $\Sigma_u G \to S^\lambda \to S^2 \wedge G_+$, and we also have the cover of $\Sigma_u G$ by its upper and lower cones C_+G and C_-G. The following diagram has the long exact sequence for the cofiber sequence horizontally, and the Mayer–Vietoris sequence for the cover vertically:

$$\begin{array}{ccccccc} & & & H^\beta(G) & & \\ & & & \uparrow & & \\ & & & H^\beta(\mathrm{pt})\oplus H^\beta(\mathrm{pt}) & & \\ & & & \uparrow & & \\ H^{\beta-2}(G) & \longrightarrow & H^\beta(S^\lambda) & \longrightarrow & H^\beta(\Sigma_u G) & \longrightarrow & H^{\beta-1}(G) \\ & & & \uparrow & & \\ & & & H^{\beta-1}(G) & & \end{array}$$

Now $H^\gamma(G)\neq 0$ only when $|\gamma|=0$, so if $|\beta|>2$ then all of the $H^\star(G)$ groups in the diagram vanish. So the composite $H^\beta(S^\lambda)\to H^\beta(\mathrm{pt})\oplus H^\beta(\mathrm{pt})$ is an isomorphism. One readily checks that this map is $(j_\infty)^*\oplus(j_0)^*$, and then it follows by algebra that $(j_0)^*:\widetilde{H}^\beta(S^\lambda)\to H^\beta(\mathrm{pt})$ is an isomorphism. □

We can now prove our result about the vanishing requirement for cyclic groups:

Proposition B.6 *Let $n\geq 2$ and let V be a finite-dimensional C_n-representation with* $\dim V\geq 3$. *Then the vanishing requirement holds for (C_n,V).*

Proof We will first focus on the case where n is odd, and then at the end describe the modifications needed for n even. Proposition 4.9 showed that the vanishing requirement is satisfied when $1\subseteq V$, so assume $1\not\subseteq V$. For every $\ell\in\mathbb{Z}$ we need to check that in the sequence

$$H^{\ell V-\ell}(\mathrm{pt})\xrightarrow{\cdot a_V}H^{(\ell+1)V-\ell}(\mathrm{pt})\xrightarrow{\cdot a_V}H^{(\ell+2)V-\ell}(\mathrm{pt})$$

the first map is surjective and the second is injective. We will analyze the cases $\ell\geq 0$ and $\ell<0$ separately, starting with the first. For $\ell\geq 0$ we will actually prove the stronger statement that for any representation W such that $1\not\subseteq W$, and any $s\geq 0$, the sequence (B.6.1) satisfies (1) and (2) below:

$$H^{W-s-V}(\mathrm{pt})\xrightarrow{\cdot a_V}H^{W-s}(\mathrm{pt})\xrightarrow{\cdot a_V}H^{W+V-s}(\mathrm{pt}). \tag{B.6.1}$$

(1) If s is odd and $V\subseteq W$ then all the groups are zero, and so both maps are isomorphisms.

(2) When s is even suppose additionally that for every irreducible representation λ that appears in V, having multiplicity m, one has $\left(m+\frac{s}{2}\right)\lambda\subseteq W$. Then the first map is surjective and the second map is an isomorphism.

This stronger statement will be needed to handle the negative cone, in the second part of the proof. Observe that for $\ell\geq 0$ the choice $W=(\ell+1)V$ and $s=\ell$ clearly satisfies either (1) or (2), and so the above statements imply the vanishing requirement.

If s is odd then the fixed set dimensions of $W-s$, $W+V-s$, and $W-V-s$ are all odd, and since $V\subseteq W$ the corresponding cohomology groups all lie in the positive cone; so all of the groups are zero. This verifies (1).

Next assume s is even. For (2), surjectivity and injectivity of maps can be verified by checking these locally at each prime. So fix a prime p and look at everything with $\mathbb{Z}_{(p)}$ coefficients (we will suppress the coefficients in the notation). Write $V=\bigoplus_{j=1}^{r} m_j\lambda_{i_j}$ with all $m_j>0$ and $i_1<i_2<\cdots<i_r$. The group $H^{W-s}(\mathrm{pt})$ is generated by a monomial M in the a-classes and u-classes, where no p-adic valuation on an a-index is larger than a p-adic valuation on a u-index. The total number of u-classes in M must be exactly $\frac{s}{2}$, in order to get a fixed-set dimension of $-s$. Since $W\supseteq \bigl(m_j+\frac{s}{2}\bigr)\lambda_{i_j}$, the total number of a_{i_j} and u_{i_j} classes in the monomial must be $m_j+\frac{s}{2}$. Since there can be at most $\frac{s}{2}$ of the u_{i_j}-classes, there must be at least m_j of the a_{i_j}-classes. As this holds for every j, we have proven that M is a multiple of a_V. This proves that the first map in our sequence is surjective.

The monomial $M\cdot a_V$ lies in $H^{W+V-s}(\mathrm{pt})$, and since M was a multiple of a_V we know that M and $M\cdot a_V$ have exactly the same a- and u-classes appearing — it's just that in the latter some of the multiplicities are larger. In particular, $M\cdot a_V$ inherits from M the property that no p-adic valuation on an a-index is larger than the p-adic valuation of a u-index. Hence, $M\cdot a_V$ is a generator for $H^{W+V-s}(\mathrm{pt})$. Likewise, the fact that M and $M\cdot a_V$ contain the same a-classes implies that they have the same additive order. We have therefore shown that the second map in our sequence is a surjection between groups of the same order, and hence an isomorphism. We have now verified (2) locally at each prime p, and therefore globally as well.

At this point we have proven the vanishing requirement for $\ell\geq 0$.

The vanishing requirement for $\ell=-1$ is trivial, since the middle group in our sequence is then $H^1(\mathrm{pt})=0$. So suppose now that $\ell=-r$ where $r\geq 2$.

The case $r=2$ needs to be handled separately. Here we are looking at $H^{2-2V}\xrightarrow{a_V}H^{2-V}\xrightarrow{a_V}H^2$, and since $H^2=0$ the vanishing requirement is equivalent to the statement $H^{2-V}=0$. But Anderson duality gives $H^{2-V}\cong \mathrm{Ext}(H^{V-\lambda(1)+1},\mathbb{Z})$. However, by Lemma B.5 the map $H^{V-\lambda(1)+1}\xrightarrow{a_\lambda}H^{V+1}$ is an isomorphism, and the latter group is zero by Proposition 2.7. This completes the $r=2$ case.

Now assume $r\geq 3$. By (B.6.1) with $W=(r-1)V$ and $s=(r-3)$, we know that $H^\star(\mathrm{pt})$ is V-nice in degree $(r-1)V-(r-3)$ (all of the groups in the sequence are indexed by α with $|\alpha|>0$ and are therefore torsion cyclic). Then by Lemma B.5 it follows that $H^\star(\mathrm{pt})$ is also V-nice in degree $(r-1)V-(r-3)-\lambda(1)$ (one needs to check that $(r-2)|V|-(r-3)>2$, but this is fine using $|V|\geq 3$ and $r\geq 3$). Finally, Lemma B.4 now implies that $H^\star(\mathrm{pt})$ is V-nice in degree $r-(r-1)V$. Note to apply this lemma with $\alpha=r-(r-1)V$ one needs to check that $0\notin\{r-(r-1)|V|, r-(r-2)|V|, r-r|V|\}$, and there is a slight issue for the middle term when $r=3$ and $|V|=3$; but in fact this case cannot occur, since we are assuming $|V|\geq 3$ and $V\not\supseteq 1$, which forces $|V|\geq 4$. This completes the proof of the vanishing requirement for the negative cone.

The n even case is essentially the same, with the only difference coming from σ. One proves the analogs of (1)–(2) from (B.6.1) but where the hypothesis "$\bigl(m+\frac{s}{2}\bigr)\lambda\subseteq W$" for (2) is replaced by:

- Whenever λ is an irreducible two-dimensional representation contained in V with multiplicity m, then $(m+\frac{s}{2})\lambda \subseteq W$.
- If σ is contained in V with multiplicity m then $(m+s)\sigma$ is contained in W.

The same proofs as before work almost verbatim.

The positive-cone case of the vanishing requirement follows immediately from this. The negative-cone case is then almost verbatim as for n odd, but with one extra consideration. When $r=3$ and $|V|=3$ the hypotheses of Lemma B.4 are not satisfied and one must look a little more closely. Here $V=\sigma+\lambda(d)$ for some d. Write $\lambda=\lambda(1)$. The steps are as follows.

In the diagram

$$\begin{array}{ccccc} H^{V} & \xrightarrow{\cdot a_V} & H^{2V} & \xrightarrow{\cdot a_V} & H^{3V} \\ \uparrow{\scriptstyle \cdot a_\lambda} & & \uparrow{\scriptstyle \cdot a_\lambda} & & \uparrow{\scriptstyle \cdot a_\lambda} \\ H^{V-\lambda} & \xrightarrow{\cdot a_V} & H^{2V-\lambda} & \xrightarrow{\cdot a_V} & H^{3V-\lambda} \end{array}$$

the top horizontal maps are isomorphisms by direct computation (this follows from Proposition 2.7). The vertical maps are isomorphisms by Lemma B.5. Now write down the following Anderson duality diagram, where the columns are all short exact:

$$\begin{array}{ccccc} \mathrm{Hom}(H^{3V-\lambda-1},\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H^{2V-\lambda-1},\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H^{V-\lambda-1},\mathbb{Z}) \\ \uparrow & & \uparrow & & \uparrow \\ H^{3-3V} & \longrightarrow & H^{3-2V} & \longrightarrow & H^{3-V} \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Ext}(H^{3V-\lambda}) & \longrightarrow & \mathrm{Ext}(H^{2V-\lambda}) & \longrightarrow & \mathrm{Ext}(H^{V-\lambda}) \end{array}$$

The bottom horizontal maps are isomorphisms by what has already been proven. The groups $H^{3V-\lambda-1}$ and $H^{2V-\lambda-1}$ are torsion, and so the corresponding Hom-terms vanish. It follows that H^{3-2V} is torsion and therefore maps into the torsion subgroup of H^{3-V}, which is the image of $\mathrm{Ext}(H^{V-\lambda})$. It now follows that in the center horizontal row the first map is surjective and the second injective, which is what we needed to check. □

B.7 The case $G=\Sigma_3$

We end by verifying the vanishing requirement for the noncyclic group Σ_3. For this group the calculations are taken from [25]. For our ordered basis of $\mathrm{RO}(\Sigma_3)$ we take 1, λ, σ, where σ is the sign representation and λ is the action on $\mathbb{R}^2$ via the symmetries of an equilateral triangle. The Euler classes a_λ and a_σ have orders 3 and 2, respectively. Note that the product $a_\lambda a_\sigma$ must then be zero, which simplifies some things as we will have far fewer classes to manage.

The representations 2σ and $\lambda+\sigma$ are orientable, so we have orientation classes $u_{2\sigma}$ and $u_{\lambda+\sigma}$. It turns out that the class $u_{\lambda+\sigma}/u_{2\sigma}$ also exists; we denote this as $u_{\lambda-\sigma}$. Note that we do not need to consider the class $u_{2\lambda}$, even though 2λ is orientable, because it arises as the product $u_{2\lambda} = u_{\lambda-\sigma}^2 u_{2\sigma}$. Also note $u_{\lambda+\sigma} = u_{\lambda-\sigma} u_{2\sigma}$, so we only need to record how the classes $u_{\lambda-\sigma}$ and $u_{2\sigma}$ interact with elements.

We also have the following elements:

- Elements $2/u_{2\sigma}^k$ and $3/u_{\lambda-\sigma}^k$ for $k \geq 1$, which generate $\mathbb{Z}$ summands;
- Elements $\Sigma^1(1/(u_{2\sigma}^k a_\sigma^l))$ for $k, l \geq 1$, which generate $\mathbb{Z}/2$ summands. The degree of this class is $1 - k(2\sigma - 2) - l\sigma$ (note in particular the "1+" part — the Σ^1 appears in the notation to remind us that the degree of this element is one more than the expected degree of $1/(u_{2\sigma}^k a_{2\sigma}^l)$);
- Elements $\Sigma^1(1/(u_{\lambda-\sigma}^k a_\lambda^l))$ for $k, l \geq 1$, which generate $\mathbb{Z}/3$ summands.

Multiplication by $u_{2\sigma}$, $u_{\lambda-\sigma}$, a_σ, and a_λ respects the fraction notation in the expected ways. For example

$$u_{2\sigma} \cdot \frac{2}{u_{2\sigma}^k} = \frac{2}{u_\lambda^{k-1}}, \quad a_\sigma \cdot \frac{2}{u_{2\sigma}^k} = 0,$$

$$u_{2\sigma} \cdot \Sigma^1\left(\frac{1}{u_{2\sigma}^k a_\sigma^l}\right) = \begin{cases} \Sigma^1\left(\dfrac{1}{u_{2\sigma}^{k-1} a_\sigma^l}\right) & \text{if } k \geq 2, \\ 0 & \text{if } k = 1, \end{cases}$$

$$a_\sigma \cdot \Sigma^1\left(\frac{1}{u_{2\sigma}^k a_\sigma^l}\right) = \begin{cases} \Sigma^1\left(\dfrac{1}{u_{2\sigma}^k a_\sigma^{l-1}}\right) & \text{if } l \geq 2, \\ 0 & \text{if } l = 1. \end{cases}$$

Note the slogan "if the resulting fraction is not one of the allowed ones, then the product is zero". The products $a_\lambda^l \cdot 2/u_{2\sigma}^k$, $u_\lambda^l \cdot 2/u_{2\sigma}^k$, $u_\lambda^l \cdot \Sigma^1(1/(u_{2\sigma}^k a_\sigma^l))$ are nonzero and generate $\mathbb{Z}/3$, $\mathbb{Z}$, and $\mathbb{Z}/2$ summands, respectively. We have similar formulas for how the classes $3/u_{\lambda-\sigma}^k$ and $\Sigma^1(1/(u_{\lambda-\sigma}^k a_\lambda^l))$ behave when multiplied by Euler and orientation classes.

For an element $r(1) + s(\lambda) + t(\sigma) \in RO(\Sigma_3)$ we call r the *fixed-point index*, s the *λ-index*, and t the *σ-index*. Observe the elements a_λ and a_σ have fixed-point index zero, $u_{\lambda-\sigma}$ has fixed-point index one, and $u_{2\sigma}$ has fixed-point index two.

As a $\mathbb{Z}[u_{2\sigma}, u_{\lambda-\sigma}, a_\sigma, a_\lambda]$-module, $\mathbb{M}$ splits into two pieces $\mathbb{M}_0$ and $\mathbb{M}_1$, which we will analyze separately to verify the vanishing requirement. We will only describe these pieces empirically. The chart for $\mathbb{M}_0$ is shown in Figure 3. To read this chart just note that:

- Triangles represent $\mathbb{Z}/3$'s and dots represent $\mathbb{Z}/2$'s. The red classes generate copies of $\mathbb{Z}$.
- Multiplication by a_λ moves up and to the right, whereas multiplication by a_σ moves down and to the right; both are drawn with blue lines. Note this is a projection of a multidimensional figure onto a plane, and so blue rays that look like they intersect usually don't.
- The fixed-point index is constant on connected components of the chart, and is indicated in gray square brackets. Moving down in the diagram increases this index by 1, and moving left increases it by 2.
- A "knight's move" of right-one/down-two preserves the fixed-point index, and there are instances where two seemingly different components run into each other. For example, $a_\sigma^4 u_{\lambda-\sigma}^2$ and $a_\lambda^2 u_{2\sigma}$ are both in

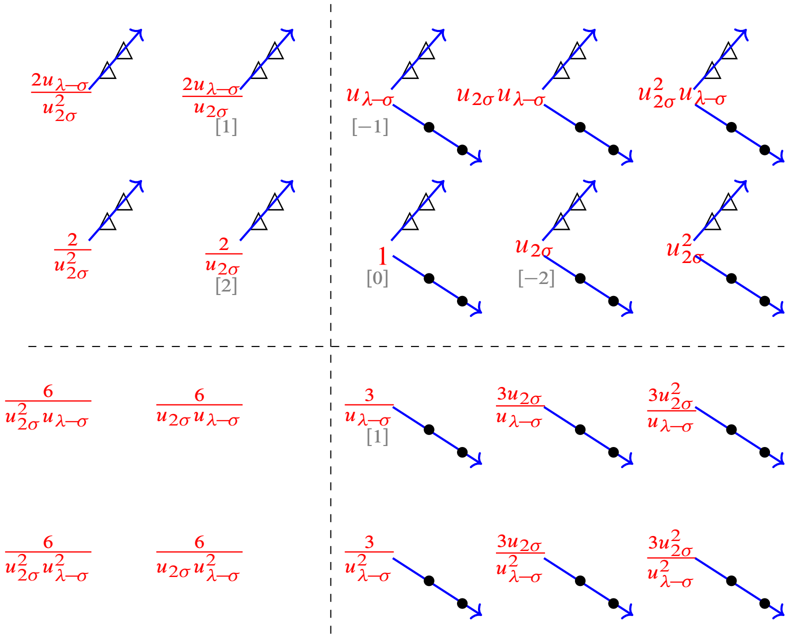

Figure 3: The module $\mathbb{M}_0$.

degree $-2+2\lambda+2\sigma$. The first class generates a $\mathbb{Z}/2$ and the second a $\mathbb{Z}/3$, so these combine to give a $\mathbb{Z}/6$ in that degree. This phenomenon only happens in the upper right quadrant of the chart.

- The dashed lines separate the chart into four quadrants. Observe that everything in the lower-right has negative λ-index and positive σ-index, whereas everything in the upper left has positive λ-index and negative σ-index. Thus, nothing in these two regions lies in the regular portion of $\mathbb{M}$. The other two quadrants have a mixed collection of indices, from this particular perspective.

The chart for $\mathbb{M}_1$ is shown in Figure 4. The main points to keep in mind are:

- As in the previous chart, triangles represent $\mathbb{Z}/3$'s and dots represent $\mathbb{Z}/2$'s. The a_σ multiplications are down-and-to-the-right, and the a_λ multiplications are up-and-to-the-right.
- The red classes are just placeholders, and don't actually exist in $\mathbb{M}$ (or said differently, they are all zero).
- Every nonzero element in this chart is infinitely divisible by either a_λ or a_σ.
- The fixed-point index (shown in gray brackets) behaves similarly to Figure 3, and there is again the issue — this time in the lower left quadrant of the diagram — of the a_σ- and a_λ-rays sometimes running into each other to create a $\mathbb{Z}/6$.
- Observe that every class in $\mathbb{M}_1$ has a negative λ-index or a negative σ-index.

The nonzero elements in the first quadrant of the $\mathbb{M}_0$ chart are sums and multiples of elements of the form

(i) $u_{\lambda-\sigma}^P u_{2\sigma}^Q a_\sigma^R$, lying in degree $P\lambda+(2Q+R-P)\sigma-(P+2Q)$, and

(ii) $u_{\lambda-\sigma}^P u_{2\sigma}^Q a_\lambda^R$, lying in degree $(P+R)\lambda+(2Q-P)\sigma-(P+2Q)$.

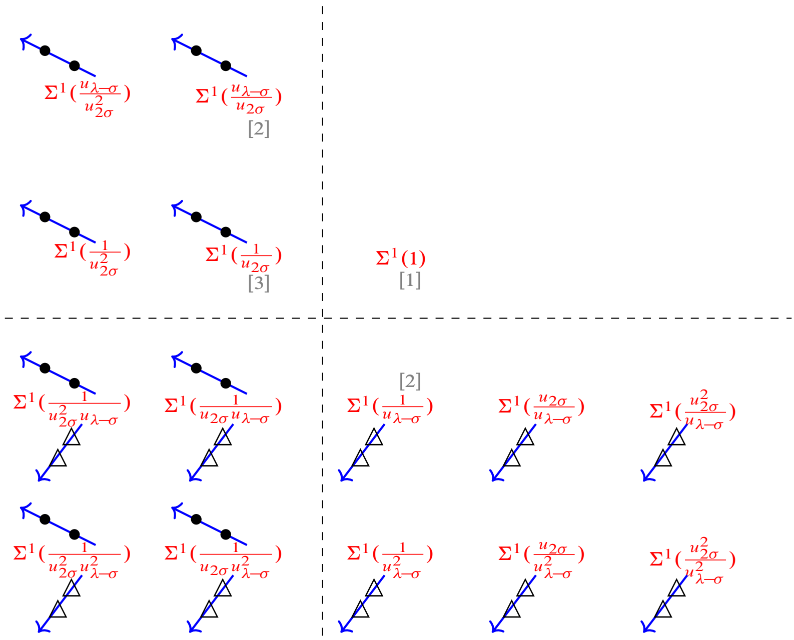

Figure 4: The module $\mathbb{M}_1$.

Since $P, Q, R \geq 0$ we observe that $-$(fixed point index) $\geq \lambda$-index in case (i), and $-$(fixed point index) $\geq$ σ-index in case (ii).

Let $V = i\lambda + j\sigma$ with $i, j \geq 0$ and assume $2i + j \neq 2$ to avoid the dim $V = 2$ anomaly. The vanishing requirement is that for each $\ell \in \mathbb{Z}$ the two maps

$$? \xrightarrow{a_\lambda^i a_\sigma^j} H^{(\ell+1)(i\lambda+j\sigma)-\ell}(\mathrm{pt}) \xrightarrow{a_\lambda^i a_\sigma^j} ??$$

are surjective and injective, respectively. When $\ell \geq 0$ the middle group is in the first quadrant of the $\mathbb{M}_0$ chart, and comparing to the previous paragraph we will satisfy the index inequalities only when $i = 0$ (putting us in case (i) with $P = 0$ and $\ell = 2Q$), or when $j = 0$ (putting us in case (ii) with $2Q - P = 0$ and $\ell = P + 2Q = 4Q$). In the first of these cases the group is $\mathbb{Z}/2$ generated by $u_{2\sigma}^{\ell/2} a_\sigma^{(\ell+1)j-\ell}$; injectivity and surjectivity are clear from the chart once one notes that $(\ell+1)j - \ell \geq j \geq 1$. In the second of the cases, $\ell = 4Q$ and the group is $\mathbb{Z}/3$ generated by $u_{\lambda-\sigma}^{2Q} u_{2\sigma}^{Q} a_\lambda^{(4Q+1)i-2Q}$. Here $(4Q+1)i - 2Q \geq i \geq 1$ and so it is clear from the chart that the outgoing map is injective and the incoming map is surjective.

When $\ell = -1$ the middle group is $H^1(\mathrm{pt}) = 0$, so the conditions are satisfied.

Now we deal with the case $\ell \leq -2$. A little work shows that $\mathbb{M}_0$ is zero in the degree $(\ell+1)(i\lambda+j\sigma)-\ell$: the idea is that since the λ-index and σ-index are both nonpositive with at least one negative, a nonzero class could only appear in the lower left quadrant of the chart, or maybe on the bottom line of the upper left quadrant. These possibilities are quickly ruled out except for the single case $V = 2\sigma$, $\ell = -2$: this is the only case where the middle group has a nonzero piece in $\mathbb{M}_0$. But this case is contrary to our hypothesis that $2i + j \neq 2$.

So we only need consider $\mathbb{M}_1$ now. Degrees where the λ-index and σ-index are both nonpositive are all in the lower-left quadrant. The analysis is very similar to the $\ell > 0$ case, in that one quickly finds the middle group to be zero except when either i or j is zero, and in those remaining cases one readily sees that the required conditions are always satisfied.

The conclusion is that the vanishing requirement holds for every representation with $\dim V \neq 2$.

Acknowledgements

We are grateful to Nick Proudfoot for conversations regarding [6; 18; 19; 20]. Hazel is also thankful to Mike Hill for some helpful conversations about Bredon cohomology. We are grateful to the referee for a careful reading and for suggestions in Appendix B.

References

[1] **V Angeltveit**, *The Picard group of the category of C_n-equivariant stable homotopy theory* (2021) arXiv 2109.04625

[2] **V I Arnold**, *The cohomology ring of the group of dyed braids*, Mat. Zametki 5 (1969) 227–231 MR In Russian; translated in Math. Notes 5 (1969) 138–140

[3] **S Basu**, **P Dey**, *Equivariant cohomology for cyclic groups*, Int. Math. Res. Not. 2025:12 (2025) art. id. rnaf150 MR

[4] **E Belmont**, **J D Quigley**, **C Vogeli**, *Bredon homological stability for configuration spaces of G-manifolds* (2023) arXiv 2311.02459

[5] **F R Cohen**, **T J Lada**, **J P May**, *The homology of iterated loop spaces*, Lecture Notes in Mathematics 533, Springer (1976) MR

[6] **G Dorpalen-Barry**, **N Proudfoot**, **J Wang**, *Equivariant cohomology and conditional oriented matroids*, Int. Math. Res. Not. 2024:11 (2024) 9292–9322 MR

[7] **D Dugger**, *Coherence for invertible objects and multigraded homotopy rings*, Algebr. Geom. Topol. 14:2 (2014) 1055–1106 MR

[8] **D Dugger**, **B I Dundas**, **D C Isaksen**, **P A Østvær**, *The multiplicative structures on motivic homotopy groups*, Algebr. Geom. Topol. 24:3 (2024) 1781–1786 MR

[9] **E Fadell**, **L Neuwirth**, *Configuration spaces*, Math. Scand. 10 (1962) 111–118 MR

[10] **E Getzler**, *Resolving mixed Hodge modules on configuration spaces*, Duke Math. J. 96:1 (1999) 175–203 MR

[11] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) 184:1 (2016) 1–262 MR

[12] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *Equivariant stable homotopy theory and the Kervaire invariant problem*, New Mathematical Monographs 40, Cambridge Univ. Press (2021) MR

[13] **A A Kosinski**, *Differential manifolds*, Pure and Applied Mathematics 138, Academic Press, Boston, MA (1993) MR

[14] **G I Lehrer**, **L Solomon**, *On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes*, J. Algebra 104:2 (1986) 410–424 MR

[15] **L G Lewis, Jr**, *The* RO(G)*-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/p$ actions*, from “Algebraic topology and transformation groups” (Göttingen, 1987) (T tom Dieck, editor), Lecture Notes in Math. 1361, Springer (1988) 53–122 MR

[16] **P Lima-Filho**, *On the equivariant homotopy of free abelian groups on G-spaces and G-spectra*, Math. Z. 224:4 (1997) 567–601 MR

[17] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics 91, Amer. Math. Soc., Providence, RI (1996) MR

[18] **D Moseley**, *Equivariant cohomology and the Varchenko–Gelfand filtration*, J. Algebra 472 (2017) 95–114 MR

[19] **D Moseley**, **N Proudfoot**, **B Young**, *The Orlik–Terao algebra and the cohomology of configuration space*, Exp. Math. 26:3 (2017) 373–380 MR

[20] **N Proudfoot**, *The equivariant Orlik–Solomon algebra*, J. Algebra 305:2 (2006) 1186–1196 MR

[21] **N Ricka**, *Equivariant Anderson duality and Mackey functor duality*, Glasg. Math. J. 58:3 (2016) 649–676 MR

[22] **P F dos Santos**, *A note on the equivariant Dold–Thom theorem*, J. Pure Appl. Algebra 183:1-3 (2003) 299–312 MR

[23] **D P Sinha**, *The* (*non-equivariant*) *homology of the little disks operad*, from "OPERADS 2009" (J-L Loday, B Vallette, editors), Sémin. Congr. 26, Soc. Math. France, Paris (2013) 253–279 MR

[24] **A N Varchenko**, **I M Gel'fand**, *Heaviside functions of a configuration of hyperplanes*, Funct. Anal. Its Appl. 21 (1987) 255–270

[25] **G Yan**, *The Green functor structure of* $RO(D_{2p})$*-graded cohomology of a point*, preprint (2023) arXiv 2310.16118

DANIEL DUGGER ddugger@uoregon.edu
Department of Mathematics, University of Oregon, Eugene, OR, United States

CHRISTY HAZEL hazelchristy@grinnell.edu
Department of Mathematics, Grinnell College, Grinnell, IA, United States

Received: February 7, 2024 Revised: November 25, 2024

Algebraic & Geometric Topology

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2026 is US $795/year for the electronic version, and $1170/year (+$80, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

Algebraic & Geometric Topology

Volume 26 Issue 2 (pages 411–824) 2026