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Profinite rigidity properties of central extensions of 2-orbifold groups

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We extend Wilkes' results on the profinite rigidity of Seifert fibre spaces to the setting of central extensions of 2-orbifold groups with higher-rank centre. We prove that both rigid and nonrigid phenomena arise in this setting and that the nonrigid phenomena are transient in the sense that if $\widehat{G}_1 \cong \widehat{G}_2$, then $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.

1 Introduction

One of the central pursuits of group theory has always been the search for ways of distinguishing between nonisomorphic groups. An invariant that is important from both a theoretical and computational viewpoint is the set $\mathcal{C}(G)$ of (isomorphism types of) all finite quotients of a given group G. Studying this invariant it quickly becomes obvious that it is reasonable to assume the *residual-finiteness* of G — that every nontrivial element of G is nontrivial in some finite quotient.

One can ask whether or not considering just the *set* $\mathcal{C}(G)$ of finite quotients is a good enough invariant, or whether the additional structure of homomorphisms between them should be "remembered" somehow. This leads to the definition of the *profinite completion* $\widehat{G}$ of a group G. Most remarkably, for finitely generated groups G and H their completions $\widehat{G}$ and $\widehat{H}$ are isomorphic if and only if $\mathcal{C}(G) = \mathcal{C}(H)$ — see [9].

As with any invariant, a natural question is: which objects does it distinguish (uniquely)?

Meta question 1 *Which finitely generated residually finite groups G are such that, for any finitely generated residually finite group H,*

$$\widehat{G} \cong \widehat{H} \iff G \cong H?$$

A group G with this property is called (*absolutely*) *profinitely rigid*. It is not too difficult to give examples which are *not* profinitely rigid. Baumslag gave examples of two nonisomorphic semidirect products $(\mathbb{Z}/25) \rtimes \mathbb{Z}$, which share profinite completions — see [2]. Also, related to our article, Hempel [15] gave examples of mapping tori for self-homeomorphisms of surfaces which produced nonhomeomorphic 3-manifolds with isomorphic profinite completions.

On the other hand, $\mathbb{Z}$ and other finitely generated abelian groups are profinitely rigid; going further one can find more examples which are virtually solvable, but it was not until [4] that we got examples of profinitely rigid *full-sized* groups — examples which do not satisfy a group law (since they contain F_2). This is indicative of the fact that Meta question 1 is very difficult in general which prompts modifying it to the following Meta question 2.

MSC2020: primary 20E26; secondary 20E18.

Meta question 2 *Given a class $\mathscr{D}$ of finitely generated residually finite groups, which groups $G \in \mathscr{D}$ are such that, for any $H \in \mathscr{D}$,*

$$\widehat{G} \cong \widehat{H} \iff G \cong H.$$

Groups G with this property are called *profinitely rigid within the class $\mathscr{D}$*. Meta question 2 is often much more approachable because one can use structure theorems for groups in $\mathscr{D}$. For example [3] showed that Fuchsian groups are distinguished from each other (and other lattices in connected Lie groups) by their profinite completions. Wilton [35; 36] showed that every finitely generated free group and surface group is distinguished by profinite completion from all other limit groups. This was recently extended by Morales [23] to the class of finitely generated residually free groups.

Significant advances have been made in the study of profinite rigidity properties of 3-manifold groups — for a survey see Reid's article [28]. Among others, Funar [13] — using classical work of Stebe — showed that torus bundles with Sol geometry are an ample source of nonisomorphic pairs of groups with isomorphic profinite completions; Wilton and Zalesskii [37] proved that profinite completions distinguish the eight geometries of closed orientable 3-manifolds with infinite fundamental groups.

Finally, particularly relevant to this article, Wilkes [33] showed that the fundamental groups of Seifert fibre spaces are distinguished from each other apart from the examples given by Hempel. The present work aims at extending a part of this result by considering central group extensions with kernel $\mathbb{Z}^n$ for $n > 1$ and the quotient being a 2-orbifold group.

A related question was also studied by Ma and Wang [20] — the authors were interested in extending Wilkes' results to the setting of 4-dimensional Seifert fibre spaces with the fibre a 2-torus and the base a closed orientable 2-orbifold. Their results contradict our Theorem B, which is due to a flaw in their Theorem 2; the authors are aware of this fact.

Original results

The first result of this paper restricts the study of profinite rigidity within the class of groups with free abelian centre where the quotient is an infinite closed orientable 2-orbifold group: the only case we need to consider is when the kernel and the quotient are fixed.

Theorem A *Let n_1, n_2 be natural numbers and Δ_1, Δ_2 be infinite fundamental groups of closed orientable 2-orbifolds. Let G_1 and G_2 be central extensions $\mathbb{Z}^{n_1}$-by-Δ_1 and $\mathbb{Z}^{n_2}$-by-Δ_2, respectively. If $\widehat{G}_1 \cong \widehat{G}_2$ then $n_1 = n_2$ and $\Delta_1 \cong \Delta_2$.*

Proposition 2.9 of [5] is a similar result obtained independently.

The main result is Theorem B which — apart from a single family of exceptions — determines for a fixed $n > 1$ and a fixed infinite closed orientable 2-orbifold group Δ whether the central extensions with kernel $\mathbb{Z}^n$ and quotient Δ are distinguished from each other by their profinite completions. In particular, it implies that for $n > m$ they all are.

The result depends on the definition of *Smith coefficients d_j* associated to the m-tuple of orders of cone points. The exceptional family which Theorem B doesn't classify is characterised by $d_{m-(n-1)} = 12$.

Definition 1.1 (Smith coefficients) Let $P = (p_1, \dots, p_m)$ be an m-tuple of natural numbers. We define the *Smith coefficients* $d_1, \dots, d_m$ associated to P as

$$d_j := \frac{\gcd(p_{i_1} p_{i_2} \cdots p_{i_j} \text{ for } 1 \le i_1 < i_2 < \cdots < i_j \le m)}{\gcd(p_{i_1} p_{i_2} \cdots p_{i_{j-1}} \text{ for } 1 \le i_1 < i_2 < \cdots < i_{j-1} \le m)}, \tag{1}$$

where for $j = 1$ the denominator is defined to be 1.

Theorem B *Let Δ be an infinite fundamental group of a closed orientable 2-orbifold with $m \ge 0$ cone points of orders $p_1, \dots, p_m$, and let $d_1, \dots, d_m$ be the Smith coefficients associated to $(p_1, \dots, p_m)$. For $n > 1$ the following hold:*

(1) *If $n > m$, or $d_{m-(n-1)} \in \{1, 2, 3, 4, 6\}$, then the nonisomorphic central extensions of $\mathbb{Z}^n$ by Δ are distinguished from each other by their profinite completions.*

(2) *If $n \le m$ and $d_{m-(n-1)} \notin \{1, 2, 3, 4, 6, 12\}$, then there exist nonisomorphic central extensions G_1, G_2 of $\mathbb{Z}^n$ by Δ with $\widehat{G}_1 \cong \widehat{G}_2$.*

The third result shows that in fact all examples of lack of rigidity come from the phenomenon described by Baumslag [2] — that $\widehat{G}_1 \cong \widehat{G}_2$ implies in this context that $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.

Theorem C *Let Δ be an infinite fundamental group of a closed orientable 2-orbifold. Let $n > 1$ and G_1, G_2 be central extensions of $\mathbb{Z}^n$ by Δ such that $\widehat{G}_1 \cong \widehat{G}_2$. Then $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.*

Note that in Theorems B and C we made the assumption that $n > 1$. The theorems could be also stated for $n = 1$, but this case has already been covered by [15; 33].

Structure of this article

Section 1 states the main questions of this paper and gives the related results as context for them; it states the original results and discusses the structure of this article.

Section 2 contains the necessary technical background and minor lemmas: Section 2.1 introduces profinite completions and the implications of goodness to residual-finiteness; Section 2.2 discusses 2-orbifold groups, their automorphisms and centres; Section 2.3 introduces the needed results of group cohomology and computes the cohomology groups of the 2-orbifold groups in question.

In particular, in Section 2.2.2 we show that the extensions in question are residually finite and that the topology induced on the kernel is the full profinite topology. In Section 2.2.3 we quote important results on automorphisms of 2-orbifold groups and their profinite completions, specifying how they act on the maximal torsion elements. In Section 2.2.4 we show that the 2-orbifold groups we consider and their profinite completions are centreless. Section 2.3.3 gives an explicit form of the standard cohomological Hopf's formula, which we later need for calculations. Finally, Section 2.3.5 computes the cohomology groups of the 2-orbifold groups we consider and also their profinite completions, as well as the actions of their automorphism groups on the cohomology groups themselves.

Section 3 translates the questions of isomorphism of central extensions of $\mathbb{Z}^n$ by 2-orbifold groups and isomorphism of their profinite completions to questions about orbits of certain actions by matrices on sets of matrices.

Section 4 explores the translated question and gives a thorough classification of when distinct orbits of the first action collapse to one orbit under the second action.

Finally, Section 5 states the results back in the setting of distinguishing nonisomorphic central extensions of $\mathbb{Z}^n$ by 2-orbifold groups and proves them.

2 Background

2.1 Profinite completions

Since this article is not intended as a thorough introduction to the subject of profinite groups and profinite completions only the very basic definitions are given here. For a proper introduction to the topic the reader is referred to the lecture notes of Reid [27; 28], the recent book of Wilkes [34], and to the book of Ribes and Zalesskii [29] for a comprehensive reference.

2.1.1 Profinite completions The *profinite completion* of a group organises "the set of all finite quotients of a group" into an algebraic object which also "remembers" how the different quotients relate to each other.

In the following and throughout this article, $<$ and $\triangleleft$ denote the relations of being a subgroup and being a normal subgroup, respectively, without requiring that the containment is proper. Subscripts $<_{\text{fi}}$ and $\triangleleft_{\text{fi}}$ indicate that the subgroup is of finite index.

For each group G we can form an inverse system $\mathcal{N} = \{G/N \mid N \triangleleft_{\text{fi}} G\}$ of its canonical finite quotients. It is indeed an inverse system since for any finite index normal subgroups $N < M$ of G we have a canonical map $\varphi_{NM} : G/N \to G/M$ and given any $N, M \triangleleft_{\text{fi}} G$, the intersection $N \cap M$ is also a finite index normal subgroup of G.

Definition 2.1 Given a group G together with the inverse system of its canonical finite quotients G/N, we define its *profinite completion* to be the inverse limit

$$\widehat{G} := \varprojlim_{N \triangleleft_{\text{fi}} G} G/N.$$

The limit $\widehat{G}$ may be realised as a subgroup of the direct product $\prod_{N \triangleleft_{\text{fi}} G} G/N$ which may be treated as a compact space by giving the finite groups G/N the discrete topology. This makes $\widehat{G}$ into a compact Hausdorff topological group.

From the canonical maps $\psi_N : G \to G/N$ we get a canonical map $\mathsf{h} : G \to \widehat{G}$ which is injective if and only if the group G is residually finite, i.e., the intersection of its finite index subgroups is the trivial subgroup. The image $\mathsf{h}(G)$ is dense in $\widehat{G}$.

The following striking result has become standard — see [9] for the original paper or Corollary 3.2.8 of [29] for a more general version. We write $\mathcal{C}(G)$ for the set of isomorphism classes of finite quotients of G.

Theorem 2.2 *Let G and H be finitely generated. Then $\mathcal{C}(G) = \mathcal{C}(H)$ if and only if $\widehat{G} \cong \widehat{H}$.*

2.1.2 Profinite completions and extensions of groups We will need to know what happens to an extension of groups under profinite completion. The following is taken from Proposition 3.2.5 of [29]. Here and throughout $\overline{H}$ denotes the closure of $\mathsf{h}(H)$ in $\widehat{G}$ for $H < G$.

Proposition 2.3 *Given a short exact sequence* $1 \to N \to G \to Q \to 1$ *of any groups, the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathsf{h}_G|_N} & & \big\downarrow{\scriptstyle \mathsf{h}_G} & & \big\downarrow{\scriptstyle \mathsf{h}_Q} & & \\ 1 & \longrightarrow & \overline{N} & \longrightarrow & \widehat{G} & \xrightarrow{\hat{\pi}} & \widehat{Q} & \longrightarrow & 1 \end{array}$$

commutes and its bottom row is exact.

In the case where N is residually finite and *good* in the sense of Serre (see [31] and Definition 2.30 here) we can say more about what happens to N-by-Q extensions after profinite completion. The result is proved as Corollary 6.2 in [14], and relies on Exercise 2.b of Section I.2.6 in [31].

Proposition 2.4 *Let N be a residually finite, finitely generated group and Q be a good residually finite group. Then any N-by-Q extension G is residually finite and induces the full profinite topology on N.*

Thus the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathsf{h}_N} & & \big\downarrow{\scriptstyle \mathsf{h}_G} & & \big\downarrow{\scriptstyle \mathsf{h}_Q} & & \\ 1 & \longrightarrow & \widehat{N} & \longrightarrow & \widehat{G} & \xrightarrow{\hat{\pi}} & \widehat{Q} & \longrightarrow & 1 \end{array}$$

commutes and has exact rows.

2.2 2-orbifold groups

2.2.1 2-orbifold groups When a group G acts on an n-manifold M properly discontinuously and freely, the quotient space $G\backslash M$ is naturally a manifold and we get a covering $M \twoheadrightarrow G\backslash M$. *Orbifolds* are objects that describe the quotient accurately when the action has nontrivial finite isotropy groups.

For us it is natural to set aside the technicalities associated to orbifolds and consider only their fundamental groups. Moreover, we restrict ourselves to only the geometric orbifolds.

Definition 2.5 Let M be one of $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{S}^n$. A group $G < \mathrm{Isom}(M)$ is called an *n-orbifold group* if its action on M is properly discontinuous. We say that an n-orbifold group G is *orientable* if $G < \mathrm{Isom}^+(M)$. We call a 2-orbifold group *closed* if the quotient topological space $G\backslash M$ is homeomorphic to a closed surface S.

Every discrete lattice in $\mathrm{PSL}_2(\mathbb{R})$ (i.e., a *Fuchsian group*) is a 2-orbifold group, but only the *cocompact* lattices are closed 2-orbifold groups.

The closed orientable 2-orbifold groups have presentations which are convenient to work with, which were known already by Klein [17], and derived from Poincaré's work on fundamental convex polygons for Fuchsian groups — see [26].

Proposition 2.6 *A closed orientable 2-orbifold group G has a presentation*

$$\Big\langle x_1,\dots,y_g,a_1,\dots,a_m \mid \prod_{i=1}^{g}[x_i,y_i]\cdot a_1\cdots a_m=1,\ a_i^{p_i}=1 \text{ for } i=1,\dots,m\Big\rangle, \tag{2}$$

where g is the genus of the quotient surface $G\backslash M$, while m is the number of orbits of singular points and p_i is the size of the stabilisers of the i-th orbit.

We introduce the following definition for brevity. The reader should be warned that the term *nice* 2-orbifold group is by no means standard.

Definition 2.7 A 2-orbifold group is called *nice* if it is closed and orientable, infinite and not isomorphic to $\mathbb{Z}^2$.

In fact, almost all of the closed orientable 2-orbifold groups are nice.

Proposition 2.8 *A closed orientable 2-orbifold group $G<\mathrm{Isom}^+(M)$ with an underlying surface of genus g and $m\geq 0$ cone points of orders $p_1,\dots,p_m$ is nice if and only if one of the following conditions holds:*

- $g>1$.
- $g=1$, *and* $m\geq 1$.
- $g=0$, *and* $m\geq 4$.
- $g=0$, *and* $m=3$ *and* $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\leq 1$.

Proof The group G is infinite if and only if the orbifold Euler characteristic is nonpositive, which is $\chi=2-2g-\sum_{i=1}^{m}\big(1-\frac{1}{p_i}\big)$. The only case that needs to be excluded is $G\cong\mathbb{Z}^2$ which is $g=1$ and $m=0$. □

We will later use the following fact, which tells us that 2-orbifold groups are virtually surface groups. It was proved in [7; 10] for Fuchsian groups, (or [12] in the spherical case corrected by [8]); it is also a consequence of Theorem 2.5 of [30].

Proposition 2.9 *Let Δ be a 2-orbifold group. Then, there is a finite index subgroup $\Sigma<\Delta$ isomorphic to a surface group.*

We will need to know the torsion elements of nice 2-orbifold groups.

Proposition 2.10 *Let Δ be a nice 2-orbifold group with presentation*

$$\Big\langle x_1,\dots,y_g,a_1,\dots,a_m \mid \prod[x_i,y_i]\cdot a_1\cdots a_m=1,\ a_i^{p_i}=1 \text{ for } i=1,\dots,m\Big\rangle.$$

Then any torsion element of Δ is conjugate to a power of one of the a_i.

Proof As an oriented 2-orbifold group, Δ is a subgroup of $\mathrm{Isom}^+(M)$ and since it's infinite $M=\mathbb{R}^2$ or $\mathbb{H}^2$. Any torsion element of $\mathrm{Isom}^+(M)$ must fix a point in M. Since the action away from the lifts of cone points is free, the fixed point of the torsion element must be one of the lifts of cone points and thus the torsion element is conjugate to a power of one of the a_i. □

2.2.2 Residual finiteness of central extensions of 2-orbifold groups In order to study profinite rigidity of extensions $\mathbb{Z}^n$-by-Δ we need to know that they are residually finite, and that they induce the full topology on the kernel. This is a consequence of Proposition 2.4 and the following facts: 2-orbifold groups are good in the sense of Serre — see Proposition 3.6 of [14]; they are residually finite as a consequence of Maltsev's theorem [22] — see Section III.7 of [19].

Proposition 2.11 *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence such that $N = \mathbb{Z}^n$ and Q is a closed orientable 2-orbifold group. Then G is residually finite and $\overline{N} = \widehat{N}$ in $\widehat{G}$.*

2.2.3 Automorphisms and torsion elements of Δ and $\widehat{\Delta}$ It will be important for us to study the automorphisms of nice 2-orbifold groups and their profinite completions. Theorem 2.12 covers the case of Fuchsian groups while the non-Fuchsian nice 2-orbifolds are covered by Proposition 2.13.

Theorem 2.12 (Theorem 5.1 of [3]) *Let G be a finitely generated Fuchsian group. Then every finite subgroup of $\widehat{G}$ is conjugate to a subgroup of G and if two maximal finite subgroups of G are conjugate in $\widehat{G}$, then they are already conjugate in G.*

Proposition 2.13 (Proposition 4.3 of [33]) *Let G be a nice 2-orbifold group isomorphic to a subgroup of* $\mathrm{Isom}^+(\mathbb{R}^2)$. *Then every torsion element of $\widehat{G}$ is conjugate to a torsion element of G, and if two torsion elements of G are conjugate in $\widehat{G}$, then they are already conjugate in G.*

Theorem 2.12 and Proposition 2.13 together with Proposition 2.10 show that the generators a_i of a nice 2-orbifold group G (as in presentation (2)) are sent under any automorphism $\phi : \Delta \to \Delta$ (respectively, under any automorphism $\phi : \widehat{\Delta} \to \widehat{\Delta}$) to

$$^{g_i}(a_{\sigma(i)}^{k_i}) := g_i a_{\sigma(i)}^{k_i} g_i^{-1},$$

where $g_i \in \Delta$ (respectively, $g_i \in \widehat{\Delta}$) and σ is a permutation in $\mathrm{Sym}(m)$.

It will prove useful to show that any permutation of a_i which respects the orders p_i is actually possible.

Proposition 2.14 *Let $\sigma \in \mathrm{Sym}(m)$ be a permutation of $\{1, 2, \dots, m\}$ such that $p_i = p_{\sigma(i)}$ for all $i = 1, \dots, m$. Then there exists an automorphism $\phi \in \mathrm{Aut}(\Delta)$ such that $\phi(a_i) = {}^{g_i}(a_{\sigma(i)})$ for some $g_i \in \Delta$.*

Proof It is enough to show this for σ being a single transposition, say $\sigma = (ij)$. Then

$$\begin{aligned} a_1 \cdots a_i \cdots a_j \cdots a_m &= a_1 \cdots a_{i-1} \cdot {}^{a_i}(a_{i+1} \cdots a_j) \cdot a_i \cdot a_{j+1} \cdots a_m \\ &= a_1 \cdots a_{i-1} \cdot {}^{a_i}a_j \cdot {}^{a_i a_j^{-1}}(a_{i+1} \cdots a_{j-1}) \cdot a_i \cdot a_{j+1} \cdots a_m, \end{aligned}$$

so we can define a homomorphism of Δ by setting the following images of generators a_i (and keeping generators x_i, y_i fixed):

$$\begin{array}{lll} a_1 \mapsto a_1, \quad \dots, & a_{i-1} \mapsto a_{i-1}, & a_i \mapsto {}^{a_i}a_j, \\ a_{i+1} \mapsto {}^{a_i a_j^{-1}}(a_{i+1}), \quad \dots, & a_{j-1} \mapsto {}^{a_i a_j^{-1}}(a_{j-1}), & a_j \mapsto a_i, \\ a_{j+1} \mapsto a_{j+1}, \quad \dots, & a_m \mapsto a_m. & \end{array}$$

All of the relators are sent to relators and each of the generators is in the image, so the map is surjective. Since Δ is finitely generated residually finite, it is Hopfian and so ϕ is an automorphism. □

2.2.4 Trivial centres of Δ and $\widehat{\Delta}$ The fact that nice 2-orbifolds have trivial centres is well known. It is more difficult though to extend this to the profinite completion, which is the content of Proposition 2.16.

Proposition 2.15 *Let Δ be a nice 2-orbifold group. Then $Z(\Delta) = 1$.*

Proof Consider Δ as a subgroup of $\mathrm{Isom}^+(M)$ where $M = \mathbb{R}^2$ or $\mathbb{H}^2$. Then, take $z \in Z(\Delta)$, assume that $z \neq 1$. If z has a fixed point, then it acts as a nonidentity rotation and has only one fixed point, and the whole group Δ has to fix it too. This would mean that Δ is finite cyclic, which is a contradiction.

If z has no fixed points, it must have a translation axis, which would be fixed by Δ (not necessarily pointwise). This would mean that Δ is virtually cyclic, which is also excluded. □

Proposition 2.16 *Let Δ be a nice 2-orbifold group. Then $Z(\widehat{\Delta}) = 1$.*

Proof First, if $z \in Z(\widehat{\Delta})$ is a torsion element, then by Theorem 2.12 and Proposition 2.13 it is conjugate to a torsion element in Δ, which we showed in Proposition 2.15 to be centerless. Thus z is of infinite order.

By Proposition 2.9 any 2-orbifold group Δ contains a surface group Σ as a finite-index subgroup. Since $\widehat{\Sigma}$ is a finite index subgroup of $\widehat{\Delta}$, then for any $g \in \widehat{\Delta}$ there is some $k \in \mathbb{N}$ such that $g^k \in \widehat{\Sigma}$. In particular, if $z \in Z(\widehat{\Delta})$ and $z^k \in \widehat{\Sigma}$, we have $z^k \in Z(\widehat{\Sigma})$.

Now, if Σ is a fundamental group of a surface of genus > 1, then $\widehat{\Sigma}$ is centerless as shown in Proposition 18 of [1]. This would imply that $z^k = 1$, which is a contradiction.

If Σ is of genus 0, then Δ is a finite group, which can't be the case as Δ is nice.

If Σ is of genus 1, then Δ is a 2-dimensional crystallographic group — a discrete, cocompact subgroup of $\mathrm{Isom}^+(\mathbb{R}^2)$. There are exactly five orientation-preserving crystallographic groups in dimension 2 and they are split extensions $\mathbb{Z}^2 \rtimes (\mathbb{Z}/d)$ for $d = 1, 2, 3, 4, 6$ where the action is by rotation of square or hexagonal grid; see Section 3.2 of [32]. The semidirect product structure carries over to the profinite completion where a direct computation shows that the completions are centerless unless $d = 1$. □

The only infinite closed orientable 2-orbifold group which is not nice, i.e., $\mathbb{Z}^2$, obviously does not satisfy the previous propositions. However, it turns out that for a central $\mathbb{Z}^n$-by-$\mathbb{Z}^2$ extension G unless $G \cong \mathbb{Z}^{n+2}$ the centre is the kernel $\mathbb{Z}^n$ of the extension. The same holds for the profinite completions of these extensions.

Proposition 2.17 *Any central extension of $\mathbb{Z}^n$ by $\mathbb{Z}^2$ is isomorphic to $H_k \times \mathbb{Z}^{n-1}$ for some integer $k \geq 0$, where H_k has presentation*

$$\langle a, b, c \mid [a, c] = [b, c] = 1,\ [a, b] = c^k \rangle.$$

Furthermore, unless $k = 0$, we have

$$Z(H_k) = \langle c \rangle, \quad Z(\widehat{H}_k) = \overline{\langle c \rangle}.$$

Proof The fact that any central extension admits such a presentation is a consequence of Hopf's formula and the fact that $H^2(\mathbb{Z}^2;\mathbb{Z}^n)\cong\mathbb{Z}^n$. The number k is such that the cohomology class in $\mathbb{Z}^n$ is a k-th multiple of a primitive vector in $\mathbb{Z}^n$.

The group H_k can also be described as the set $\mathbb{Z}^3$ with multiplication given by

$$(x_1,y_1,z_1)*(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2+kx_1y_2).$$

Similarly, $\widehat{H}_k$ is isomorphic to the set $\widehat{\mathbb{Z}}^3$ with the same multiplication formula. In both cases

$$[(1,0,0),(x,y,z)]=(0,0,ky),$$
$$[(0,1,0),(x,y,z)]=(0,0,-kx).$$

This implies that unless $k=0$, the centre is generated by $(0,0,1)$, as both $\mathbb{Z}$ and $\widehat{\mathbb{Z}}$ are torsion-free. □

2.2.5 Distinguishing 2-orbifold groups by profinite completions Here we cite a strong result which we will later make use of. It was stated as Corollary 4.2 of [33], where it was proved using Theorem 1.1 of [3].

Theorem 2.18 *Let O_1, O_2 be closed 2-orbifolds. If there is an isomorphism $\widehat{\pi_1(O_1)}\cong\widehat{\pi_1(O_2)}$, then $\pi_1(O_1)\cong\pi_1(O_2)$.*

Our definition of a nice 2-orbifold requires being closed, so Theorem 2.18 applies to it.

2.3 Group cohomology

This section lists the main results of group cohomology which we use later. An introduction to group cohomology can be found in the book of Brown [6] or the lecture notes of Löh [18] and the book of Wilkes [34].

2.3.1 Group extensions With N-by-Q group extensions, it is important to properly distinguish between their isomorphism classes of groups G fitting in short exact sequences $1\to N\to G\to Q\to 1$ and the equivalence classes of such sequences.

Since we are dealing with *central extensions* we give the definitions in this context and denote the first group in a short exact sequence by 0 rather than 1.

Definition 2.19 Let $0\to M\xrightarrow{\iota_i} G_i\xrightarrow{\pi_i} Q\to 1$ be two group extensions, with $i=1,2$. We say that they are *equivalent* if there is an isomorphism $\widetilde{\phi}:G_1\to G_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\widetilde{\phi}} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q & \longrightarrow & 1 \end{array}$$

commutes.

If there exists an isomorphism $\widetilde{\phi} : G_1 \to G_2$ and automorphisms $\Phi : M \to M$ and $\phi : Q \to Q$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \widetilde{\phi}} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & M & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q & \longrightarrow & 1 \end{array}$$

commutes, we say that the extensions are *similar*.

Given an abelian group M and a group Q we denote by $\mathcal{E}(Q; M)$ the set of equivalence classes of *central* M-by-Q extensions and by $\overline{\mathcal{E}}(Q; M)$ the set of similarity classes of such extensions.

We finish this section by noting that in our setting the similarity classes are in bijection with the isomorphism classes because of the next standard result. It comes from the fact that for a central M-by-Q extension G where $Z(Q) = 1$ the copy of M in G is not just central, but it is equal to the centre $Z(G)$.

Proposition 2.20 *Let M be an abelian group and Q be a group with $Z(Q) = 1$. Then, there is a bijection between the set $\overline{\mathcal{E}}(Q; M)$ of similarity classes of central extensions of M by Q and the set of isomorphism classes of groups G such that $Z(G) \cong M$ and $G/Z(G) \cong Q$.*

2.3.2 $H^2(Q; M)$ classifies extensions The most important result for us is how the second cohomology group classifies the equivalence classes of group extensions with given kernel and quotient. The reader is referred to Section IV.3 of [6] for details.

Proposition 2.21 *For an abelian group M and a group Q, there is a natural bijection between the set $\mathcal{E}(Q; M)$ of* ***equivalence*** *classes of central extensions of M by Q and the second cohomology group $H^2(Q; M)$ of Q with coefficients in M, treated as a trivial Q-module.*

The naturality of this correspondence is explained in detail in Proposition 2.22, which we adapt from Theorem 1.5.13 of [18]. It also follows from Exercise 1 in Section IV.3 of [6].

Proposition 2.22 *Let M_1 be a Q_1-module, M_2 a Q_2-module, and $\phi : Q_1 \to Q_2$ and $\Phi : M_1 \to M_2$ be homomorphisms such that*

$$\Phi(q \cdot m) = \phi(q) \cdot \Phi(m).$$

Given cohomology classes $[\zeta_i] \in H^2(Q_i; M_i)$ and extension classes

$$[0 \to M_i \xrightarrow{\iota_i} G_i \xrightarrow{\pi_i} Q_i \to 1],$$

which they represent, the following are equivalent:

(1) $\Phi_*([\zeta_1]) = \phi^*([\zeta_2])$ *as elements of $H^2(Q_1; M_2)$.*

(2) *There is a homomorphism $\widetilde{\phi} : G_1 \to G_2$ such that the diagram* (3) *commutes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \widetilde{\phi}} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & M_2 & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q_2 & \longrightarrow & 1 \end{array} \tag{3}$$

This naturality has an important consequence for us.

Corollary 2.23 *Let Q be a group and let M be a Q-module with trivial action. Then the groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(Q)$ *act (on the left) on $H^2(Q;M)$ by*

$$\Phi\cdot[\zeta]=\Phi_*([\zeta]),\quad \phi\cdot[\zeta]=(\phi^{-1})^*([\zeta])$$

and these actions commute. Furthermore, the orbits of the action of $\mathrm{Aut}(M)\times\mathrm{Aut}(Q)$ *on $H^2(Q;M)$ are in bijection with the set $\bar{\mathcal{E}}(Q;M)$ of similarity classes of central extensions of M by Q.*

2.3.3 Cohomological Hopf's formula In Section 2.3.2 we discussed the naturality of how $H^2(Q;M)$ classifies the equivalence classes of group extensions, but we will later need an explicit way of computing the action of $\mathrm{Aut}(Q)$ on $H^2(Q;M)$ done with a specific free resolution — one coming from the presentation complex. In order to do that we introduce a version of cohomological Hopf's formula applied to the partial resolution coming from the presentation complex of Q.

The main tools for this are the inflation-restriction exact sequence and the natural correspondence between $H^1(Q;M)$ and the group of homomorphisms $Q\to M$.

The following statement of the inflation-restriction exact sequence was taken from Proposition 1.6.6 of [24] and it holds in the context of modules M with possibly nontrivial action of G. We use superscript M^G to denote the G-invariants submodule.

Theorem 2.24 (inflation-restriction exact sequence) *Given a short exact sequence of groups* $1\to N\to G\to Q\to 1$ *and a G-module M, there is an exact sequence*

$$0\to H^1(Q;M^N)\to H^1(G;M)\to H^1(N;M)^Q\to H^2(Q;M^N)\to H^2(G;M),$$

where the maps $H^k(Q;M^N)\to H^k(G;M)$ are the ***inflation*** *maps, the maps $H^k(G;M)\to H^k(N;M)^Q$ are the* ***restriction*** *maps and the map $\mathrm{Tr}:H^1(N;M)^Q\to H^2(Q;M^N)$ is the* ***transgression*** *map given by the following procedure.*

Given a 1-cocycle $g:N\to M$ whose cohomology class is Q-invariant, there is a 1-cochain $f:G\to M$ such that

(1) $f|_N=g$,

(2) $(d^*f):G\times G\to M$ *is constant on the cosets of $N\times N$,*

(3) $\mathrm{im}(d^*f)$ *lies in M^N.*

Since (d^*f) is constant on the cosets of $N\times N$, we can consider it as a function $Q\times Q\to M^N$ and set

$$\mathrm{Tr}([g]):=[(d^*f):Q\times Q\to M^N]\in H^2(Q;M^N).$$

Lemma 2.25 is a well-known elementary result.

Lemma 2.25 *Let M be an abelian group treated as a trivial G-module. Then there is a natural bijection*

$$H^1(Q;M)\to\{\text{homomorphisms } f:Q\to M\}.$$

Finally, we can state the explicit version of cohomological Hopf's formula. While the existence of the isomorphism in it is well known, the explicit isomorphism that we need is hard to find in the literature.

Proposition 2.26 (cohomological Hopf's formula) *Let Q be a group with presentation*

$$\langle a_i : i \in I \mid r_j : j \in J \rangle$$

and M be an abelian group treated as a trivial Q-module. Let F be the free group on the set $\{a_i : i \in I\}$ and R be the smallest normal subgroup of F containing the set $\{r_j : j \in J\}$.

Let $C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z}$ be any partial free resolution of the presentation resolution with $C_2 = \bigoplus_{j \in J} \mathbb{Z}Q \cdot \boldsymbol{r}_j$ and $C_1 = \bigoplus_{i \in I} \mathbb{Z}Q \cdot \boldsymbol{a}_i$.

Then the map

$$\frac{\{F\text{-invariant homs. } R \to M\}}{\{\text{restrictions of homs. } F \to M\}} \xrightarrow{\Psi} \frac{\ker(\operatorname{Hom}_{\mathbb{Z}Q}(C_2, M) \to \operatorname{Hom}_{\mathbb{Z}Q}(C_3, M))}{\operatorname{im}(\operatorname{Hom}_{\mathbb{Z}Q}(C_1, M) \to \operatorname{Hom}_{\mathbb{Z}Q}(C_2, M))},$$

where $\Psi([g : R \to M]) = [\boldsymbol{r}_j \mapsto g(r_j) : j \in J]$, is a natural bijection.

Proof We use the inflation-restriction exact sequence from Theorem 2.24 for the short exact sequence $1 \to R \to F \xrightarrow{\pi} Q \to 1$ and M treated as an F-module.

Since F is of cohomological dimension 1, we get $H^2(F; M) = 0$ and so the transgression map $\operatorname{Tr} \colon H^1(R; M)^Q \to H^2(Q; M)$ is surjective with $\ker \operatorname{Tr}$ being equal to the image of $H^1(F; M)$ under the restriction map. Now, $H^1(R; M)^Q$ is the set of F-invariant homomorphisms $R \to M$, while $H^1(F; M)$ is the set of all homomorphisms $F \to M$. This means that sending $[g : R \to M]$ to $\operatorname{Tr}([g]) \in H^2(Q; M)$ gives a natural isomorphism

$$\frac{\{F\text{-invariant homs. } R \to M\}}{\{\text{restrictions of homs. } F \to M\}} \to H^2(Q; M).$$

We now need to compose this natural isomorphism with the natural isomorphism

$$\Theta : H^2(Q; M) \to \frac{\ker(\operatorname{Hom}_{\mathbb{Z}Q}(C_2, M) \to \operatorname{Hom}_{\mathbb{Z}Q}(C_3, M))}{\operatorname{im}(\operatorname{Hom}_{\mathbb{Z}Q}(C_1, M) \to \operatorname{Hom}_{\mathbb{Z}Q}(C_2, M))}$$

coming from computing the cohomology with different resolutions of $\mathbb{Z}$ by $\mathbb{Z}Q$-modules, namely the presentation resolution (top row) and the bar resolution (bottom row), related by the chain maps shown in the diagram

$$\begin{array}{ccccccc}
\bigoplus_{j \in J} \mathbb{Z}Q \cdot \boldsymbol{r}_i & \longrightarrow & \bigoplus_{i \in I} \mathbb{Z}Q \cdot \boldsymbol{a}_i & \longrightarrow & \mathbb{Z}Q & \longrightarrow & \mathbb{Z} \\
\downarrow{\scriptstyle \Theta_2} & & \downarrow{\scriptstyle \Theta_1} & & \| & & \| \\
\bigoplus_{q_1, q_2 \in Q} \mathbb{Z}Q \cdot [q_1 | q_2] & \longrightarrow & \bigoplus_{q \in Q} \mathbb{Z}Q \cdot [q] & \longrightarrow & \mathbb{Z}Q \cdot [\cdot] & \longrightarrow & \mathbb{Z}
\end{array}$$

Defining Θ_1 is straightforward: we set $\boldsymbol{a}_i \mapsto [a_i]$. Defining Θ_2 poses a bigger challenge. First, notice that for a relation $r_j = a_{i_1} a_{i_2} \cdots a_{i_l}$ we have

$$d(\boldsymbol{r}_j) = \boldsymbol{a}_{i_1} + a_{i_1} \cdot \boldsymbol{a}_{i_2} + \cdots + a_{i_1} a_{i_2} \cdots a_{i_{l-1}} \cdot \boldsymbol{a}_{i_l}.$$

Let's define Θ_2 by "triangulating the relation disc" as

$$\Theta_2(\boldsymbol{r}_j) = [1|a_{i_1}] + [a_{i_1}|a_{i_2}] + [a_{i_1}a_{i_2}|a_{i_3}] + \cdots + [a_{i_1}a_{i_2}\cdots a_{i_{l-1}}|a_{i_l}].$$

Then we get

$$\begin{aligned}(d \circ \Theta_2)(\boldsymbol{r}_j) &= (1 \cdot [a_{i_1}] - [a_{i_1}] + [1]) + (a_{i_1} \cdot [a_{i_2}] - [a_{i_1}a_{i_2}] + [a_{i_1}]) \\ &\qquad + (a_{i_1}a_{i_2} \cdot [a_{i_3}] - [a_{i_1}a_{i_2}a_{i_3}] + [a_{i_1}a_{i_2}]) + \cdots \\ &\qquad + (a_{i_1}a_{i_2}\cdots a_{i_{l-1}} \cdot [a_{i_l}] - [1] + [a_{i_1}a_{i_2}\cdots a_{i_{l-1}}]) \\ &= 1 \cdot [a_{i_1}] + a_{i_1} \cdot [a_{i_2}] + a_{i_1}a_{i_2} \cdot [a_{i_3}] + \cdots + a_{i_1}a_{i_2}\cdots a_{i_{l-1}} \cdot [a_{i_l}] \\ &= (\Theta_1 \circ d)(\boldsymbol{r}_j),\end{aligned}$$

which means that Θ_* is indeed a chain map and since it extends an isomorphism $\mathbb{Z}Q \to \mathbb{Z}Q \cdot [\,\cdot\,]$, it extends to a chain equivalence.

Now, we need to compute the map Θ. Given $h \in C^2(Q; M)$ we get

$$\Theta(h)(\boldsymbol{r}_j) = h(1, a_{i_1}) + h(a_{i_1}, a_{i_2}) + \cdots + h(a_{i_1}a_{i_2}\cdots a_{i_{l-1}}, a_{i_l}).$$

Finally, let's compute the composition $\Theta \circ \mathrm{Tr}$. Given an F-invariant homomorphism $g : R \to M$ and choosing a 1-cochain $f : F \to M$ such that $\mathrm{Tr}([g]) = [(d^* f)]$ we get

$$\begin{aligned}\Theta(\mathrm{Tr}(g))(\boldsymbol{r}_j) &= \mathrm{Tr}(g)(1, a_{i_1}) + \cdots + \mathrm{Tr}(g)(a_{i_1}a_{i_2}\cdots a_{i_{l-1}}, a_{i_l}) \\ &= (1 \cdot f(a_{i_1}) - f(1 \cdot a_{i_1}) + f(1)) + (a_{i_1} \cdot f(a_{i_2}) - f(a_{i_1} \cdot a_{i_2}) + f(a_{i_1})) + \cdots \\ &\qquad + (a_{i_1}\cdots a_{i_{l-1}} \cdot f(a_{i_l}) - f(a_{i_1}\cdots a_{i_l}) + f(a_{i_1}\cdots a_{i_{l-1}})) \\ &= f(1) + 1 \cdot f(a_{i_1}) + a_{i_1} \cdot f(a_{i_1}a_{i_2}) + \cdots + a_{i_1}\cdots a_{i_{l-1}} \cdot f(a_{i_l}) - f(r_j) \\ &= (d^* f)(\boldsymbol{r}_j) - g(r_j).\end{aligned}$$

Thus we get that $\Psi([g]) = -\Theta \circ \mathrm{Tr}$. Furthermore, since Θ and Tr are natural isomorphisms, so is Ψ. □

2.3.4 Cohomology of profinite groups For profinite groups the usual group cohomology does not capture the topological data. However, if we require the appropriate maps to be continuous, we do get valuable information.

It turns out that for reasons coming from homological algebra, it is not obvious which topological modules one can allow to make the "profinite homology and cohomology" well-behaved theories. To avoid complications, we follow the definitions of Section 6.2 of [29] specified to finite coefficient modules, but first we need to define the completed group algebra $\widehat{\mathbb{Z}}[\![\Gamma]\!]$ for a profinite group Γ.

Definition 2.27 Given a profinite group Γ, we define the *completed group algebra* $\widehat{\mathbb{Z}[\Gamma]} = \widehat{\mathbb{Z}}[\![\Gamma]\!]$ to be the limit

$$\widehat{\mathbb{Z}}[\![\Gamma]\!] := \varprojlim_{\substack{N \in \mathbb{N} \\ U \lhd_o \Gamma}} (\mathbb{Z}/N)[\Gamma/U].$$

Now, we need to define two types of topological $\widehat{\mathbb{Z}}[\![\Gamma]\!]$-modules.

Definition 2.28 For a profinite group Γ a topological $\widehat{\mathbb{Z}}[\![\Gamma]\!]$-module M is called *discrete* if it has discrete topology and *profinite* if it has profinite topology.

This allows us to give a definition for the cohomology of profinite groups. Being of homological algebra nature it is somewhat abstract, but it ensures all functorial properties that we require from cohomology.

Definition 2.29 For a profinite group Γ and a *discrete* $\widehat{\mathbb{Z}}[\![\Gamma]\!]$-module M we set

$$H^k_{\mathrm{prof}}(\Gamma; M) := \mathrm{Ext}^k_{\widehat{\mathbb{Z}}[\![\Gamma]\!]}(\widehat{\mathbb{Z}}; M).$$

In practice, we can compute $H^k_{\mathrm{prof}}(\Gamma; M)$ by taking a projective resolution of $\widehat{\mathbb{Z}}$ by profinite $\widehat{\mathbb{Z}}[\![\Gamma]\!]$-modules of length at least $k+1$, applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}[\![\Gamma]\!]}(-, M)$ and computing the homology of the resulting chain complex.

In principle one can take the analogue of the bar resolution for the projective resolution required in Definition 2.29, but this would not be convenient for calculations in practice. However, for *good* groups it turns out to be feasible to convert a "small" free resolution for G into a "small" free resolution for $\widehat{G}$. We describe it in Lemma 2.31, which we take from Proposition 3.14 of [33], but first we define goodness.

Definition 2.30 A group G is called *good* if the canonical map

$$\mathsf{h}^* : H^i_{\mathrm{prof}}(\widehat{G}; M) \to H^i(G; M)$$

is an isomorphism for any finite G-module M and any $i \in \mathbb{N}$.

Lemma 2.31 *Let G be a good group with a partial resolution $(C_i)_{0 \le i \le n}$ of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}G$-modules*

$$C_i = \bigoplus_{j=1}^{l_i} \mathbb{Z}G \cdot \boldsymbol{x}_{ij}.$$

Then $(\widehat{C}_i)_{0 \le i \le n}$ is a partial resolution of $\widehat{\mathbb{Z}}$ by free $\widehat{\mathbb{Z}}[\![\widehat{G}]\!]$-modules, where

$$\widehat{C}_i = \bigoplus_{j=1}^{l_i} \widehat{\mathbb{Z}}[\![\widehat{G}]\!] \cdot \boldsymbol{x}_{ij}.$$

Finally, we need to note that among the mentioned "functorial properties we require from cohomology" are the following two.

Firstly, the inflation-restriction exact sequence from Theorem 2.24 exists for (continuous) exact sequences of profinite groups.

Secondly, given a profinite group Γ and a *finite* group M, the group $H^2_{\mathrm{prof}}(\Gamma; M)$ is in natural bijection with the equivalence classes of *profinite* extensions of M by Γ — these are the short exact sequences

$$0 \to M \xrightarrow{\iota} E \xrightarrow{\pi} \Gamma \to 1,$$

where E is a profinite group and the maps ι and π are continuous homomorphisms. Analogously, we define *equivalence* and *similarity* of profinite extensions by requiring that the isomorphisms in Definition 2.19 be continuous.

2.3.5 Cohomology of nice 2-orbifold groups In this section we compute the cohomology groups for nice 2-orbifold groups, taking coefficient modules with trivial action.

We base our computation on a partial free resolution of length 3 given in Proposition 6.6 of [33]. It should be noted that this result was known much earlier: Majumdar [21] gives this resolution and computes all integral homology and cohomology groups, Patterson [25] computes the second cohomology groups of Fuchsian groups and i.a. uses it to study certain extensions of them, and Ellis and Williams [11] study the cohomology groups and extensions of generalised triangle groups. The cohomology rings of Fuchsian groups were also computed in [16] using a spectral sequence argument.

While we could quote these results together with the universal coefficient theorem to compute the cohomology groups $H^2(\Delta;\mathbb{Z}^n)$, doing it in a concrete manner makes it easier to later compute the actions $\mathrm{Aut}(\mathbb{Z}^n)\times\mathrm{Aut}(\Delta)\curvearrowright H^2(\Delta;\mathbb{Z}^n)$ and its profinite counterpart $\mathrm{Aut}(\widehat{\mathbb{Z}}^n)\times\mathrm{Aut}(\widehat{\Delta})\curvearrowright H^2_{\mathrm{prof}}(\widehat{\Delta};(\mathbb{Z}/N)^n)$, which we need for classifying group extensions up to isomorphism.

In the following results and throughout the rest of the paper we will use a bold font to denote basis elements of free modules.

Proposition 2.32 *The cellular chain complex for the Cayley complex $C_{\mathcal{P}}=\widetilde{K_{\mathcal{P}}}$ of the presentation* (2) *for a nice 2-orbifold group,*

$$\underbrace{\mathbb{Z}\Delta\{\boldsymbol{r}_0,\dots,\boldsymbol{r}_m\}}_{C_2}\xrightarrow{d}\underbrace{\mathbb{Z}\Delta\{\boldsymbol{x}_1,\dots,\boldsymbol{y}_g,\boldsymbol{a}_1,\dots,\boldsymbol{a}_m\}}_{C_1}\xrightarrow{d}\underbrace{\mathbb{Z}\Delta}_{C_0}\xrightarrow{d}\mathbb{Z},$$

can be resolved one dimension further by setting $C_3=\mathbb{Z}\Delta\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_m\}$ where the map $d:C_3\to C_2$ is defined as

$$d(\boldsymbol{z}_i)=(a_i-1)\cdot\boldsymbol{r}_i.$$

Because the extensions we study are central, it will prove useful to look at the chain complex in Proposition 2.32 after applying the functor $\mathbb{Z}\otimes_{\mathbb{Z}\Delta}(-)$. Indeed any homomorphism $\phi:C\to M$ from a $\mathbb{Z}\Delta$-module C to a *trivial* $\mathbb{Z}\Delta$-module M must satisfy $\phi(g\cdot c)=\phi(c)$ for any $g\in\Delta$ and $c\in C$. Thus the functor $\mathrm{Hom}_{\mathbb{Z}\Delta}(-,M):\mathbb{Z}\Delta\text{-}\mathsf{Modules}\to\mathsf{AbelianGroups}^{\mathrm{op}}$ factors as shown in the diagram

(4)
$$\begin{array}{ccc}\mathbb{Z}\Delta\text{-}\mathsf{Modules} & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}\Delta}(-,M)} & \mathsf{AbelianGroups}^{\mathrm{op}}\\ & {\scriptstyle\mathbb{Z}\otimes_{\mathbb{Z}\Delta}(-)}\searrow \quad \nearrow{\scriptstyle\mathrm{Hom}_{\mathbb{Z}}(-,M)} & \\ & \mathsf{AbelianGroups} & \end{array}$$

Proposition 2.33 *The functor $\mathbb{Z}\otimes_{\mathbb{Z}\Delta}(-):\mathbb{Z}\Delta\text{-}\mathsf{Modules}\to\mathsf{AbelianGroups}$ applied to the complex $C_3\to C_2\to C_1\to C_0$ from Proposition 2.32 gives a complex $D_3\to D_2\to D_1\to D_0$ which together* *with maps* $1\otimes(-) : C_i \to \mathbb{Z}\otimes_{\mathbb{Z}\Delta} C_i$ *form the diagram*

$$\begin{array}{ccccccc} \underbrace{\mathbb{Z}\Delta\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_m\}}_{C_3} & \xrightarrow{d} & \underbrace{\mathbb{Z}\Delta\{\boldsymbol{r}_0,\dots,\boldsymbol{r}_m\}}_{C_2} & \xrightarrow{d} & \underbrace{\mathbb{Z}\Delta\{\boldsymbol{x}_1,\dots,\boldsymbol{a}_m\}}_{C_1} & \xrightarrow{d} & \underbrace{\mathbb{Z}\Delta}_{C_0} \\ \downarrow{\scriptstyle 1\otimes(-)} & & \downarrow{\scriptstyle 1\otimes(-)} & & \downarrow{\scriptstyle 1\otimes(-)} & & \downarrow{\scriptstyle 1\otimes(-)} \\ \underbrace{\mathbb{Z}\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_m\}}_{D_3} & \xrightarrow{0} & \underbrace{\mathbb{Z}\{\boldsymbol{r}_0,\dots,\boldsymbol{r}_m\}}_{D_2} & \xrightarrow{d} & \underbrace{\mathbb{Z}\{\boldsymbol{x}_1,\dots,\boldsymbol{a}_m\}}_{D_1} & \xrightarrow{0} & \underbrace{\mathbb{Z}}_{D_0} \end{array} \tag{5}$$

which commutes. The maps $D_3 \to D_2$ *and* $D_1 \to D_0$ *are zero, and the map* $d : D_2 \to D_1$ *is given by*

$$\boldsymbol{r}_0 \mapsto \sum_{i=1}^{m} \boldsymbol{a}_i \quad \text{and} \quad \boldsymbol{r}_i \mapsto p_i \cdot \boldsymbol{a}_i.$$

Proof The boundary maps $D_3 \to D_2$ and $D_1 \to D_0$ in the bottom row became zero maps, because $d(\boldsymbol{z}_i) = (a_i - 1)\cdot \boldsymbol{r}_i$ and $1\otimes(a_i-1)\cdot\boldsymbol{r}_i = 0\cdot\boldsymbol{r}_i$, and similarly for the map in degree 1. □

It is worth noting that the map $d : \mathbb{Z}\{\boldsymbol{r}_0,\dots,\boldsymbol{r}_m\} \to \mathbb{Z}\{\boldsymbol{x}_1,\dots,\boldsymbol{a}_m\}$ is described by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & p_1 & 0 & \cdots & 0 \\ 1 & 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & p_m \end{pmatrix}. \tag{6}$$

Proposition 2.34 *Given a nice* 2*-orbifold group* Δ *with presentation* (2), *the second cohomology group* $H^2(\Delta; M)$ *of* Δ *with trivial coefficient module* M *is*

$$H^2(\Delta; M) \cong \bigoplus_{i=0}^{m} M\cdot \boldsymbol{r}_i^* / (\boldsymbol{r}_0^* + p_i\cdot\boldsymbol{r}_i^* \text{ for } i = 1,\dots,m). \tag{7}$$

Proof Applying functor $\operatorname{Hom}_{\mathbb{Z}\Delta}(-, M)$ to the partial resolution

$$C_3 \to C_2 \to C_1 \to C_0$$

from Proposition 2.32 factors through the functor $\mathbb{Z}\otimes_{\mathbb{Z}\Delta}(-)$, as shown in Proposition 2.33, so we only need to apply the functor $\operatorname{Hom}_{\mathbb{Z}}(-, M)$ to the chain complex $D_3 \to D_2 \to D_1 \to D_0$ of Proposition 2.33. This gives us the chain complex

$$\underbrace{M\{\boldsymbol{z}_1^*,\dots,\boldsymbol{z}_m^*\}}_{\operatorname{Hom}(D_3,M)} \xleftarrow[0]{} \underbrace{M\{\boldsymbol{r}_0^*,\dots,\boldsymbol{r}_m^*\}}_{\operatorname{Hom}(D_2,M)} \xleftarrow[d^*]{} \underbrace{M\{\boldsymbol{x}_1^*,\dots,\boldsymbol{a}_m^*\}}_{\operatorname{Hom}(D_1,M)} \xleftarrow[0]{} \underbrace{M}_{\operatorname{Hom}(D_0,M)}.$$

Now, the kernel of the map $d^* : \mathrm{Hom}(D_2, M) \to \mathrm{Hom}(D_3, M)$ is the whole domain $\mathrm{Hom}(D_2, M)$ and so $H^2(\Delta; M)$ is the cokernel of d^*. Since all of $\boldsymbol{x}_i^*$ and $\boldsymbol{y}_i^*$ are mapped to zero, this cokernel is precisely

$$\bigoplus_{i=0}^{m} M \cdot \boldsymbol{r}_i^* / (\boldsymbol{r}_0^* + p_i \cdot \boldsymbol{r}_i^* \text{ for } i = 1, \dots, m). \qquad \square$$

Now we can compute the action $\mathrm{Aut}(M) \times \mathrm{Aut}(\Delta) \curvearrowright H^2(\Delta; M)$, as introduced in Corollary 2.23.

Proposition 2.35 *Let $\Phi \in \mathrm{Aut}(M)$ and $\phi \in \mathrm{Aut}(\Delta)$ be automorphisms, and $[\zeta] = \left[\sum_{i=0}^{m} m_i \cdot \boldsymbol{r}_i^*\right]$ be an element of $H^2(\Delta; M)$. Then*

$$\Phi \cdot [\zeta] = \left[\sum_{i=0}^{m} \Phi(m_i) \cdot \boldsymbol{r}_i^*\right] \quad \text{and} \quad \phi^{-1} \cdot [\zeta] = \phi^*([\zeta]) = \kappa \cdot \left[m_0 \cdot \boldsymbol{r}_0^* + \sum_{i=1}^{m} m_i \cdot \boldsymbol{r}_{\sigma(i)}\right],$$

where $\phi(a_i) = {}^{g_i}(a_{\sigma(i)}^{k_i})$ and $\kappa \in \mathbb{Z}$ is such that $\kappa \equiv k_i \bmod p_i$.

Furthermore if $M \cong \mathbb{Z}^r \oplus T$ with $r \geq 1$ and T being torsion, then $\kappa \in \mathbb{Z}^\times = \{\pm 1\}$.

The proof of this theorem follows the proof of Proposition 6.6 of [33], with the exception of changing $\mathbb{Z}$ to M.

Proof The action of Φ is the usual action of the automorphism group of the coefficient module.

For computing ϕ^*, denote by F the free group generated by the letters $x_1, \dots, y_g, a_1, \dots, a_m$ and by R the kernel of $v : F \to \Delta$, so that the sequence

$$1 \to R \to F \xrightarrow{v} \Delta \to 1$$

is exact. To compute the effect that ϕ induces on $H^2(\Delta; M)$, we want to use the naturality of cohomological Hopf's formula as in Proposition 2.26 and compute it on the set

$$\frac{\{F-\text{invariant homomorphisms } f : R \to M\}}{\{\text{restrictions of homomorphisms } g : F \to M\}}.$$

We need to choose a map $\widetilde{\phi} : F \to F$ such that the diagram

$$\begin{array}{ccc} F & \xrightarrow{\widetilde{\phi}} & F \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\ \Delta & \xrightarrow{\phi} & \Delta \end{array}$$

commutes. For easy calculation we choose $\widetilde{\phi}(a_i) = {}^{g_i}(a_{\sigma(i)}^{k_i})$ on the generators a_i and extend it to the other generators in any way which makes the diagram commute. This is a valid choice as $v({}^{g_i}(a_{\sigma(i)}^{k_i})) = {}^{g_i}(a_{\sigma(i)}^{k_i})$ as elements of Δ.

Now, for $i = 1, \dots, m$ we can easily compute the effect on the r_i:

$$r_i = a_i^{p_i} \overset{\widetilde{\phi}}{\longmapsto} ({}^{g_i}a_{\sigma(i)}^{k_i})^{p_i} = {}^{g_i}(a_{\sigma(i)}^{p_i})^{k_i} = {}^{g_i}(r_{\sigma(i)}^{k_i}).$$

This, perhaps surprisingly, gives us enough information to compute the action on $H^2(\Delta; M)$. On the level of the chain D_* (see Proposition 2.33) we get

$$\begin{aligned}\phi_*(\boldsymbol{a}_i) &= k_i \cdot \boldsymbol{a}_{\sigma(i)},\\ \phi_*(\boldsymbol{r}_i) &= p_i \cdot \boldsymbol{r}_{\sigma(i)} \quad \text{for } i = 1,2,3,\\ \phi_*(\boldsymbol{r}_0) &= \kappa \cdot \boldsymbol{r}_0 + \sum_{i=1}^{m} \mu_i \cdot \boldsymbol{r}_{\sigma(i)} \quad \text{for some } \kappa, \mu_i \in \mathbb{Z}.\end{aligned}$$

Since ϕ_* and the boundary maps d commute, we get

$$\begin{aligned}(\phi_* \circ d)(\boldsymbol{r}_0) &= \phi_*(\boldsymbol{a}_1 + \ldots + \boldsymbol{a}_m) = \sum_{i=1}^{m} k_i \cdot \boldsymbol{a}_{\sigma(i)},\\ (d \circ \phi_*)(\boldsymbol{r}_0) &= d\Big(\kappa \cdot \boldsymbol{r}_0 + \sum_{i=1}^{m} \mu_i \cdot \boldsymbol{r}_{\sigma(i)}\Big) = \sum_{i=1}^{m} (\kappa + p_i \mu_i) \cdot \boldsymbol{a}_{\sigma(i)},\end{aligned}$$

so we get $k_i = \kappa + p_i \mu_i$ for $i = 1, \ldots, m$.

To see what happens at the cohomology level, let's compute what happens after applying $\mathrm{Hom}(-, M)$. We will write $m \cdot \boldsymbol{r}_i^*$ for the unique map $C_2 \to M$ which sends $\boldsymbol{r}_i$ to m and all other $\boldsymbol{r}_j$ to 0:

$$\begin{aligned}\phi^*(m \cdot \boldsymbol{r}_0^*) &= \kappa \cdot m \cdot \boldsymbol{r}_0^*,\\ \phi^*(m \cdot \boldsymbol{r}_i^*) &= \mu_i \cdot m \cdot \boldsymbol{r}_0^* + k_i \cdot m \cdot \boldsymbol{r}_{\sigma(i)}^*,\\ d^*(m \cdot \boldsymbol{a}_i^*) &= m \cdot \boldsymbol{r}_0^* + p_i \cdot m \cdot \boldsymbol{r}_i^*.\end{aligned}$$

Finally, let's compute the action of ϕ^* on cohomology:

$$\begin{aligned}\phi^*([m \cdot \boldsymbol{r}_0^*]) &= \kappa \cdot [m \cdot \boldsymbol{r}_0^*],\\ \phi^*([m \cdot \boldsymbol{r}_i^*]) &= \mu_i \cdot [m \cdot \boldsymbol{r}_0^*] + k_i \cdot [m \cdot \boldsymbol{r}_{\sigma(i)}^*]\\ &= \mu_i \cdot [m \cdot \boldsymbol{r}_0^*] + (\kappa + p_i \mu_i) \cdot [m \cdot \boldsymbol{r}_{\sigma(i)}^*]\\ &= \mu_i \cdot [m \cdot \boldsymbol{r}_0^*] + \kappa \cdot [m \cdot \boldsymbol{r}_{\sigma(i)}^*] + \mu_i \cdot [p_i \cdot m \cdot \boldsymbol{r}_{\sigma(i)}^*]\\ &= \kappa \cdot [m \cdot \boldsymbol{r}_{\sigma(i)}^*] + \mu_i \cdot [m \cdot \boldsymbol{r}_0^* + p_i \cdot m \cdot \boldsymbol{r}_{\sigma(i)}^*]\\ &= \kappa \cdot [m \cdot \boldsymbol{r}_{\sigma(i)}^*] + \mu_i \cdot [m \cdot \boldsymbol{r}_0^* + p_{\sigma(i)} \cdot m \cdot \boldsymbol{r}_{\sigma(i)}^*]\\ &= \kappa \cdot [m \cdot \boldsymbol{r}_{\sigma(i)}^*] + \mu_i \cdot [d^*(m \cdot \boldsymbol{a}_{\sigma(i)})]\\ &= \kappa \cdot [m \cdot \boldsymbol{r}_{\sigma(i)}^*].\end{aligned}$$

Thus, the action is by multiplication by $\kappa \in \mathbb{Z}$ together with permutation of the generators $\boldsymbol{r}_i^*$ for the cocycles $Z^2(B; M)$.

Taking the $(m!)$-th power of ϕ to get rid of the permutation σ, we get

$$(\phi^{m!})^*[\zeta] = \kappa^{m!} \cdot [\zeta]$$

and if $M \cong \mathbb{Z}^r \oplus T$, then $H^2(\Delta; M)$ contains a copy of $\mathbb{Z}^r$ too and so $\kappa = \pm 1$ since $(\phi^{m!})^*$ is an automorphism. □

Now, we can do the very similar calculations for the profinite cohomology defined in Section 2.3.4.

Proposition 2.36 *Given a nice* 2*-orbifold group* Δ *with presentation* (2) *and a finite trivial* $\widehat{\Delta}$*-module* M*, the second profinite cohomology group is*

$$H^2_{\mathrm{prof}}(\widehat{\Delta}; M) = \bigoplus_{i=0}^{m} M \cdot \boldsymbol{r}_i^* / (\boldsymbol{r}_0^* + p_i \cdot \boldsymbol{r}_i^* \text{ for } i = 1, \ldots, m). \tag{8}$$

Proof The partial free resolution from Proposition 2.32 gives rise to a partial free resolution (9) of $\widehat{\mathbb{Z}}$ by $\widehat{\mathbb{Z}}[\![\widehat{\Delta}]\!]$-modules by Lemma 2.31. We can use it to compute the $H^2_{\mathrm{prof}}(\widehat{\Delta}; M)$:

$$\widehat{C}_3 \xrightarrow{\hat{d}_3} \widehat{C}_2 \xrightarrow{\hat{d}_2} \widehat{C}_1 \xrightarrow{\hat{d}_1} \widehat{C}_0 \xrightarrow{\hat{d}_0} \widehat{\mathbb{Z}}. \tag{9}$$

Applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}[\![\widehat{\Delta}]\!]}(-, M)$ to the resolution (9) factors through the functor $\widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{\Delta}]\!]} (-)$, similarly to Proposition 2.33.

Thus, we only need to apply the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}}(-, M)$ to the chain complex

$$\widehat{D}_3 \to \widehat{D}_2 \to \widehat{D}_1 \to \widehat{D}_0$$

of $\widehat{D}_i = \widehat{\mathbb{Z}} \otimes_{\widehat{\mathbb{Z}}[\![\widehat{\Delta}]\!]} \widehat{C}_i$, where incidently $\widehat{D}_i$ is the profinite completion of the abelian group D_i from Proposition 2.33 treated as a trivial $\widehat{\mathbb{Z}}[\![\widehat{\Delta}]\!]$-module.

In particular, we get the commutative diagram

$$\begin{array}{ccccccc}
\underbrace{R\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\}}_{\widehat{C}_3} & \xrightarrow{\hat{d}} & \underbrace{R\{\boldsymbol{r}_0, \ldots, \boldsymbol{r}_m\}}_{\widehat{C}_2} & \xrightarrow{\hat{d}} & \underbrace{R\{\boldsymbol{x}_1, \ldots, \boldsymbol{a}_m\}}_{\widehat{C}_1} & \xrightarrow{\hat{d}} & \underbrace{R}_{\widehat{C}_0} \\
\downarrow {\scriptstyle 1\otimes(-)} & & \downarrow {\scriptstyle 1\otimes(-)} & & \downarrow {\scriptstyle 1\otimes(-)} & & \downarrow {\scriptstyle 1\otimes(-)} \\
\underbrace{\widehat{\mathbb{Z}}\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\}}_{\widehat{D}_3} & \xrightarrow{0} & \underbrace{\widehat{\mathbb{Z}}\{\boldsymbol{r}_0, \ldots, \boldsymbol{r}_m\}}_{\widehat{D}_2} & \xrightarrow{\hat{d}} & \underbrace{\widehat{\mathbb{Z}}\{\boldsymbol{x}_1, \ldots, \boldsymbol{a}_m\}}_{\widehat{D}_1} & \xrightarrow{0} & \underbrace{\widehat{\mathbb{Z}}}_{\widehat{D}_0}
\end{array} \tag{10}$$

where we write R for $\widehat{\mathbb{Z}}[\![\widehat{\Delta}]\!]$.

Applying the functor $\mathrm{Hom}_{\widehat{\mathbb{Z}}}(-, M)$ to the bottom row of diagram (10) gives us the chain complex

$$\underbrace{M\{\boldsymbol{z}_1^*, \ldots, \boldsymbol{z}_m^*\}}_{\mathrm{Hom}(\widehat{D}_3, M)} \xleftarrow[0]{} \underbrace{M\{\boldsymbol{r}_0^*, \ldots, \boldsymbol{r}_m^*\}}_{\mathrm{Hom}(\widehat{D}_2, M)} \xleftarrow[\hat{d}^*]{} \underbrace{M\{\boldsymbol{x}_1^*, \ldots, \boldsymbol{a}_m^*\}}_{\mathrm{Hom}(\widehat{D}_1, M)} \xleftarrow[0]{} \underbrace{M}_{\mathrm{Hom}(\widehat{D}_0, M)} . \tag{11}$$

Now, the kernel of the map $\mathrm{Hom}(\widehat{D}_2, M) \to \mathrm{Hom}(\widehat{D}_3, M)$ is the whole $\mathrm{Hom}(\widehat{D}_2, M)$ and so $H^2_{\mathrm{prof}}(\widehat{\Delta}; M)$ is the cokernel of $\hat{d}^*$. Since all of $\boldsymbol{x}_i^*$ and $\boldsymbol{y}_i^*$ are mapped to zero by $\hat{d}^*$ this cokernel is precisely

$$\bigoplus_{i=0}^{m} M \cdot \boldsymbol{r}_i^* / (\boldsymbol{r}_0^* + p_i \cdot \boldsymbol{r}_i^* \text{ for } i = 1, \ldots, m). \qquad \square$$

Proposition 2.37 *Let $\Phi \in \operatorname{Aut}(M)$ and $\phi \in \operatorname{Aut}(\widehat{\Delta})$ be automorphisms, and $[\zeta] = \left[\sum_{i=0}^m m_i \cdot \boldsymbol{r}_i^*\right]$ be an element of $H^2_{\mathrm{prof}}(\widehat{\Delta}; M)$. Then*

$$\Phi \cdot [\zeta] = \left[\sum_{i=0}^m \Phi(m_i) \cdot \boldsymbol{r}_i^*\right] \quad \text{and} \quad \phi^{-1} \cdot [\zeta] = \phi^*([\zeta]) = \kappa \cdot \left[m_0 \cdot \boldsymbol{r}_0^* + \sum_{i=1}^m m_i \cdot \boldsymbol{r}_{\sigma(i)}^*\right],$$

where $\phi(a_i) = {}^{g_i}(a_{\sigma(i)}^{k_i})$ and $\kappa \in \widehat{\mathbb{Z}}^\times$ is such that $\kappa \equiv k_i \bmod p_i$.

The proof of Proposition 2.37 is the very same calculation as the proof of Proposition 2.35 with the only difference being that $\widehat{\mathbb{Z}}^\times$ is now a much larger group than $\mathbb{Z}^\times = \{\pm 1\}$.

3 Matrix correspondence

In this section we introduce the "matrix correspondence" — a conglomerate of a few minor results and our main tool for studying the central extensions of $\mathbb{Z}^n$ by nice 2-orbifold groups. Essentially, it translates the problem of profinite rigidity among these extensions to a question regarding multiplying matrices and permuting columns.

Proposition 3.1 (matrix correspondence) *Let Δ be a nice 2-orbifold group with m cone points of order $p_1, \dots, p_m$.*

(1) *There is a $\mathrm{GL}_n(\mathbb{Z}) \times \operatorname{Aut}(\Delta)$-equivariant canonical injective homomorphism*

$$\Psi : H^2(\Delta; \mathbb{Z}^n) \to \varprojlim_{N \in \mathbb{N}} H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n)$$

given by sending a cohomology class $[\zeta]$ representing an equivalence class of central $\mathbb{Z}^n$-by-Δ extensions

$$[0 \to \mathbb{Z}^n \to G \to \Delta \to 1]$$

to the consistent system $([\zeta_N])_N$ of cohomology classes representing the equivalence classes of corresponding extensions

$$[0 \to (\mathbb{Z}/N)^n \to \widehat{G}/N\widehat{\mathbb{Z}}^n \to \widehat{\Delta} \to 1].$$

(2) *Cohomology classes $[\zeta_1]$ and $[\zeta_2]$ in $H^2(\Delta; \mathbb{Z}^n)$ represent extension classes $[0 \to \mathbb{Z}^n \to G_i \to \Delta \to 1]$ with $G_1 \cong G_2$ if and only if they lie in the same orbit of $\mathrm{GL}_n(\mathbb{Z}) \times \operatorname{Aut}(\Delta) \curvearrowright H^2(\Delta; \mathbb{Z}^n)$.*

Furthermore, $\widehat{G}_1 \cong \widehat{G}_2$ if and only if the elements $\Psi([\zeta_1])$ and $\Psi([\zeta_2])$ lie in the same orbit of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \operatorname{Aut}(\widehat{\Delta}) \curvearrowright \varprojlim H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n)$.

(3) *We can identify $H^2(\Delta; \mathbb{Z}^n)$ with a quotient of $\mathbb{Z}^{n \times (m+1)}$ as*

$$A_\Delta^{(n)} := \mathbb{Z}^{n \times (m+1)} / (\boldsymbol{r}_0^* + p_i \boldsymbol{r}_i^* \text{ for } i = 1, \dots, m),$$

where $\boldsymbol{r}_0^*, \boldsymbol{r}_1^*, \dots, \boldsymbol{r}_m^*$ denote generators of $\mathbb{Z}^{n \times (m+1)}$ treated as a free $\mathbb{Z}^n$- module, where $x \cdot \boldsymbol{r}_i^*$ will denote the matrix having vector x as the i-th column and zeros everywhere else.

Similarly, we can identify $\varprojlim H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n)$ with $\widehat{A}_\Delta^{(n)}$.

Furthermore, the orbits of the actions

$$\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(\Delta) \curvearrowright A_\Delta^{(n)} \quad \textit{and} \quad \mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(\widehat{\Delta}) \curvearrowright \widehat{A}_\Delta^{(n)}$$

are the same as, respectively, those of $\mathrm{GL}_n(\mathbb{Z}) \times \Sigma \curvearrowright A_\Delta^{(n)}$*, and those of* $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \Sigma \curvearrowright \widehat{A}_\Delta^{(n)}$*, where*

$$\Sigma = \{\sigma \in \mathrm{Sym}(m) \mid p_{\sigma(i)} = p_i \text{ for } i = 1, \dots, m\}$$

acts by permuting the columns by $\sigma(\boldsymbol{r}_i^*) = \boldsymbol{r}_{\sigma(i)}^*$.

Proof of part (1) The map $m_N : H^2(\Delta; \mathbb{Z}^n) \to H^2(\Delta; (\mathbb{Z}/N)^n)$ coming from reduction of the coefficients modulo N is canonical and thus equivariant under the action of $\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(\Delta)$. Now, by goodness of Δ, the canonical map

$$\iota_N : H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n) \to H^2(\Delta; (\mathbb{Z}/N)^n)$$

is an isomorphism, so we can form the composition

$$\iota_N^{-1} \circ m_N : H^2(\Delta; \mathbb{Z}^n) \to H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n).$$

These maps are canonical and thus consistent with the transition maps

$$m_{N_1 N_2} : \widehat{H}(\widehat{\Delta}; (\mathbb{Z}/N_1)^n) \to H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N_2)^n),$$

thus giving a canonical map to $\varprojlim H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n)$.

The maps m_N and ι_N correspond to the following maps of extensions and thus the composition $\iota_N^{-1} \circ m_N$ maps $[\zeta]$ to $[\zeta_N]$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & G & \longrightarrow & \Delta & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \bmod N} & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & (\mathbb{Z}/N)^n & \longrightarrow & G/N\mathbb{Z}^n & \longrightarrow & \Delta & \longrightarrow & 1 \\
 & & \big\| & & \big\downarrow{\scriptstyle \mathsf{h}_{G/N\mathbb{Z}^n}} & & \big\downarrow{\scriptstyle \mathsf{h}_\Delta} & & \\
0 & \longrightarrow & (\mathbb{Z}/N)^n & \longrightarrow & \widehat{G}/N\widehat{\mathbb{Z}}^n & \longrightarrow & \widehat{\Delta} & \longrightarrow & 1
\end{array}$$

□

Proof of part (2) By Proposition 2.20, the isomorphism classes of groups G with centre $Z(G) \cong \mathbb{Z}^n$ and quotient $G/Z(G) \cong \Delta$ are in bijection with the set $\overline{\mathcal{E}}(\Delta; \mathbb{Z}^n)$ of similarity classes of central extensions of $\mathbb{Z}^n$ by Δ. Also, this set is in bijection with the set of orbits of the action $\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(\Delta) \curvearrowright H^2(\Delta; \mathbb{Z}^n)$. Thus $G_1 \cong G_2$ if and only if the cocycles $[\zeta_1]$ and $[\zeta_2]$ lie in the same orbit.

Furthermore, since $\widehat{G}_i = \varprojlim \widehat{G}_i/N\widehat{\mathbb{Z}}^n$, a consistent set of isomorphisms $\widetilde{\phi}_N : \widehat{G}_1/N\widehat{\mathbb{Z}}^n \to \widehat{G}_2/N\widehat{\mathbb{Z}}^n$ gives an isomorphism $\widetilde{\phi} : \widehat{G}_1 \to \widehat{G}_2$. If $(\widehat{\Phi}, \phi) \in \mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(\Delta)$ is such that $(\widehat{\Phi}, \phi) \cdot [\zeta_{1,N}] = [\zeta_{2,N}]$ then this gives a consistent set of isomorphisms $(\widetilde{\phi}_N)$, so $\widehat{G}_1 \cong \widehat{G}_2$.

Conversely, if $\widetilde{\phi} : \widehat{G}_1 \to \widehat{G}_2$ is an isomorphism then it must give an automorphism $\widehat{\Phi}$ of the kernel $\widehat{\mathbb{Z}}^n$ and an automorphism ϕ of the quotient $\widehat{\Delta}$. Then clearly $(\widehat{\Phi}, \phi) \cdot [\zeta_{1,N}] = [\zeta_{2,N}]$ for all N and so indeed

$\Psi([\zeta_1])$ and $\Psi([\zeta_2])$ lie in the same orbit of the action

$$\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(\widehat{\Delta}) \curvearrowright \varprojlim H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n). \qquad \square$$

Proof of part (3) Propositions 2.34 and 2.36 computed the second cohomology group $H^2(\Delta; M)$ for trivial Δ-modules M and $H^2_{\mathrm{prof}}(\widehat{\Delta}; M)$ for finite trivial $\widehat{\Delta}$-modules M. Thus we indeed have $H^2(\Delta; \mathbb{Z}^n) \cong A^{(n)}_\Delta$ via a natural isomorphism. Similar computation gives a canonical isomorphism $\varprojlim H^2_{\mathrm{prof}}(\widehat{\Delta}; (\mathbb{Z}/N)^n) \cong \widehat{A}^{(n)}_\Delta$ and the map Ψ can be identified with $\mathsf{h} : A^{(n)}_\Delta \to \widehat{A}^{(n)}_\Delta$.

Finally, we need to use the explicit formulas from Propositions 2.35 and 2.37 for the actions of $\mathrm{Aut}(\Delta)$ and $\mathrm{Aut}(\widehat{\Delta})$ to show that the orbits do not change when we restrict the actions to smaller subgroups — respectively, $\mathrm{GL}_n(\mathbb{Z}) \times \Sigma$ and $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \Sigma$. An element $\phi^{-1} \in \mathrm{Aut}(\Delta)$ acts by

$$\phi^{-1} \cdot [\zeta] = \phi^*([\zeta]) = \kappa \cdot \left[m_0 \cdot \boldsymbol{r}_0^* + \sum_{i=1}^m m_i \cdot \boldsymbol{r}^*_{\sigma(i)} \right],$$

where $\phi(a_i) = {}^{g_i}(a_{\sigma(i)}^{k_i})$ and $\kappa \in \mathbb{Z}^\times$ is such that $\kappa \equiv k_i \mod p_i$. Thus, for all $[\zeta] \in H^2(\Delta; \mathbb{Z}^n)$ we have $(\mathrm{Id}, \phi^{-1}) \cdot [\zeta] = (\kappa \cdot \mathrm{Id}, \sigma) \cdot [\zeta]$, and thus the orbits of $\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(\Delta)$ and of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \Sigma$ are indeed the same. Similarly, an automorphism $\phi^{-1} \in \mathrm{Aut}(\widehat{\Delta})$ acts by the very same formula with the exception that κ is now allowed to be in a much bigger group $\widehat{\mathbb{Z}}^\times$. Finally, $\kappa \cdot \mathrm{Id}$ is an element of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ and thus the orbits of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \mathrm{Aut}(\widehat{\Delta})$ are the same as those of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \Sigma$. $\square$

4 Using the matrix correspondence

This section aims at understanding the group $A^{(n)}_\Delta$ introduced in Proposition 3.1 together with the actions of $\mathrm{GL}_n(\mathbb{Z})$ and Σ on it. The question we investigate is: how are the orbits of $\mathrm{GL}_n(\mathbb{Z}) \times \Sigma \curvearrowright A^{(n)}_\Delta$ related to the orbits of $\mathrm{GL}_n(\widehat{\mathbb{Z}}) \times \Sigma \curvearrowright \widehat{A}^{(n)}_\Delta$?

Throughout this section Δ will be any nice 2-orbifold group with cone points of orders $p_1, \dots, p_m$. Let $A^{(n)}_\Delta$ be as defined in Proposition 3.1, i.e., a quotient of the additive group $\mathbb{Z}^{n\times(m+1)}$ of $n\times(m+1)$ matrices with integer coefficients by relations $\boldsymbol{r}_0^* + p_i \boldsymbol{r}_i^*$ for $i = 1, \dots, m$ on columns.

4.1 General structure of $A^{(n)}_\Delta$

The next proposition enables us to understand better the structure of $A^{(n)}_\Delta$ and the action of $\mathrm{GL}_n(\mathbb{Z}) \times \Sigma$ on it.

Proposition 4.1 *The projection $\mathbb{Z}^{n\times(m+1)} \to \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n$ descends to a short exact sequence*

$$0 \to \mathbb{Z}^n \to A^{(n)}_\Delta \xrightarrow{q} \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n \to 0,$$

which is equivariant under the action of $\mathrm{GL}_n(\mathbb{Z}) \times \Sigma$. Also, the group Σ acts trivially on the kernel $\mathbb{Z}^n = \ker\big(q : A^{(n)}_\Delta \to \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n\big)$.

Proof To make following the proof easier, all mentioned groups and maps between them are shown in the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^n\{\boldsymbol{r}_0^* + p_i\boldsymbol{r}_i^*\} & = & \mathbb{Z}^n\{\boldsymbol{r}_0^* + p_i\boldsymbol{r}_i^*\} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathbb{Z}^n \oplus \big(\bigoplus_{i=1}^m p_i\mathbb{Z}^n\big) & \longrightarrow & \mathbb{Z}^{n\times(m+1)} & \longrightarrow & \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & A_\Delta^{(n)} & \xrightarrow{q} & \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{12}
$$

All the relations $\boldsymbol{r}_0^* + p_i\boldsymbol{r}_i^*$ are sent to 0 by the projection map, so we get a map $A_\Delta^{(n)} \twoheadrightarrow \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n$. The kernel of this map is the quotient of $\mathbb{Z}^n \oplus \big(\bigoplus_{i=1}^m p_i\mathbb{Z}^n\big)$ by the kernel

$$\mathbb{Z}^n\{\boldsymbol{r}_0^* + p_i\boldsymbol{r}_i^* \mid i = 1, \dots, m\} = \ker(\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_\Delta^{(n)}),$$

which is isomorphic to $\mathbb{Z}^n$. □

The next result computes the group $A_\Delta^{(n)}$ (which is not an original result — see [16; 21]) and relates this computation to the quotient map $q : A_\Delta^{(n)} \to \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n$.

Proposition 4.2 *There is an isomorphism*

$$\bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n \xrightarrow{\alpha} \bigoplus_{j=1}^m (\mathbb{Z}/d_j)^n,$$

where $(d_1, d_2, \dots, d_m)$ are the Smith coefficients of $(p_1, \dots, p_m)$ (see Definition 1.1).

There is also an isomorphism

$$A_\Delta^{(n)} \xrightarrow{\beta} \mathbb{Z}^n \oplus \Bigg(\bigoplus_{j=1}^{m-1} (\mathbb{Z}/d_j)^n\Bigg).$$

Furthermore, α and β are equivariant with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ and this identification gives the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & A_\Delta^{(n)} & \xrightarrow{q} & \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\cdot d_m} & \mathbb{Z}^n \oplus \big(\bigoplus_{j=1}^{m-1} (\mathbb{Z}/d_j)^n\big) & \longrightarrow & \bigoplus_{j=1}^m (\mathbb{Z}/d_j)^n & \longrightarrow & 0
\end{array}
$$

Proof The fact that α is an isomorphism and that d_j are as in (1) comes from computation of the Smith normal form using matrix minors. Now, $A_\Delta^{(n)}$ is isomorphic to $\mathbb{Z}^n \otimes_{\mathbb{Z}} \operatorname{coker}(\mathbb{Z}^m \to \mathbb{Z}^{m+1})$ where the map is given by the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_m \end{pmatrix}.$$

We can first use the row and column operations which give us α to get the diagonal matrix with d_j for $j = 1, 2, \ldots, m$ and do the column operations showed in

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{m-1} & 0 \\ 0 & 0 & \cdots & 0 & d_m \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{m-1} & 0 \\ -d_m & -d_m & \cdots & -d_m & d_m \end{pmatrix}. \tag{13}$$

Now, since $d_j \mid d_m$ for all $j = 1, 2, \ldots, m$ we can fully eliminate the bottom row with row operations affecting only the bottom row, which means that the first m basis elements for $\mathbb{Z}^{m+1}$ are not changed, whence the diagram in the statement actually commutes.

Finally, because we only manipulated the $\mathbb{Z}^n$-bases, the maps are indeed $\mathrm{GL}_n(\mathbb{Z})$-equivariant. □

Proposition 4.1 allowed us to understand better the torsion part of $A_\Delta^{(n)}$, but it will be necessary to understand the torsion-free part too. Proposition 4.3 helps with this. It comes from the notion of *Euler class* defined for Seifert fibre spaces.

Proposition 4.3 *Let* $\widetilde{E} : \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^n$ *be defined by*

$$\widetilde{E} : \sum_{i=0}^{m} x_i \cdot \boldsymbol{r}_i^* \mapsto -d_m \cdot x_0 + \sum_{i=1}^{m} \frac{d_m}{p_i} \cdot x_i .$$

Then $\widetilde{E}$ *factors through* $\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_\Delta^{(n)}$ *giving* $E : A_\Delta^{(n)} \to \mathbb{Z}^n$*. Furthermore, the map is surjective and equivariant with respect to the action of* $\mathrm{GL}_n(\mathbb{Z}) \times \Sigma$ (*with* Σ *acting trivially*).

Proof First of all, notice that

$$d_m = p_1 \cdots p_m / \gcd(p_1 \cdots p_{m-1}, p_1 \cdots p_{m-2} p_m, \ldots, p_2 p_3 \cdots p_m) = \operatorname{lcm}(p_1, \ldots, p_m),$$

so each d_m / p_i is an integer and the map $\widetilde{E} : \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^n$ is actually well defined. All of the relations are sent to zero by the map

$$E(x \cdot \boldsymbol{r}_0^* + p_i \cdot x \cdot \boldsymbol{r}_i^*) = -d_m \cdot x + \frac{d_m}{p_i} \cdot p_i \cdot x = 0,$$

so $\widetilde{E}$ factors through $\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_\Delta^{(n)}$ giving $E : A_\Delta^{(n)} \to \mathbb{Z}^n$. The map $\widetilde{E}$ is clearly equivariant with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ and also with respect to the action of Σ since elements of Σ permute the columns which have the same p_i. Since the quotient $\mathbb{Z}^{n\times(m+1)} \twoheadrightarrow A_\Delta^{(n)}$ is also equivariant, the induced map E is equivariant too. Now, for surjectivity, notice that $\gcd(d_m/p_1,\dots,d_m/p_m) = 1$ so there exist integers k_i such that $\sum k_i\cdot(d_m/p_i) = 1$. Hence, for any $v \in \mathbb{Z}^n$,

$$E\left(\sum_{i=1}^{m} k_i\cdot v\cdot \boldsymbol{r}_i^*\right) = v,$$

so the map is indeed surjective. □

4.2 Orbits collapsing

Using the quotient map shown in Proposition 4.1 we can show that some orbits of $\mathrm{GL}_n(\mathbb{Z})\times\Sigma \curvearrowright A_\Delta^{(n)}$ become identified in $\mathrm{GL}_n(\widehat{\mathbb{Z}})\times\Sigma \curvearrowright \widehat{A}_\Delta^{(n)}$. This is the content of Proposition 4.5, but before we can prove it, we need Lemma 4.4 to know when the effect of acting by a profinite matrix $\widehat{\Phi}\in\mathrm{GL}_n(\widehat{\mathbb{Z}})$ on some $[A]\in A_\Delta^{(n)} < \widehat{A}_\Delta^{(n)}$ gives a matrix $\widehat{\Phi}\cdot[A]\in\widehat{A}_\Delta^{(n)}$ which still lies in the copy of $A_\Delta^{(n)}$.

Lemma 4.4 *Let $[A]\in A_\Delta^{(n)}$ and let $\widehat{\Phi}$ be a profinite matrix in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$. Then $\widehat{\Phi}\cdot[A]$ is in $A_\Delta^{(n)}$ if and only if $\widehat{\Phi}\cdot E([A])\in\mathbb{Z}^n$ for the map $E : A_\Delta^{(n)}\to\mathbb{Z}^n$ defined in Proposition 4.3.*

Proof Consider the composition

$$(E\circ\beta^{-1}) : \mathbb{Z}^n \oplus \left(\bigoplus_{j=1}^{m-1}(\mathbb{Z}/d_j)^n\right) \twoheadrightarrow \mathbb{Z}^n$$

as shown in the diagram

$$\begin{array}{ccccc} \mathbb{Z}^n\oplus\left(\bigoplus_{j=1}^{m-1}(\mathbb{Z}/d_j)^n\right) & \xrightarrow[\cong]{\beta^{-1}} & A_\Delta^{(n)} & \xrightarrow{E} & \mathbb{Z}^n \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathbb{Z}}^n\oplus\left(\bigoplus_{j=1}^{m-1}(\mathbb{Z}/d_j)^n\right) & \xrightarrow[\cong]{\hat{\beta}^{-1}} & \widehat{A}_\Delta^{(n)} & \xrightarrow{\widehat{E}} & \widehat{\mathbb{Z}}^n \end{array} \tag{14}$$

Its surjectivity implies that the kernel must be precisely the torsion subgroup $\bigoplus_{j=1}^{m-1}(\mathbb{Z}/d_j)^n$. Thus we can regard $E\circ\beta^{-1}$ as a projection onto the $\mathbb{Z}^n$ direct factor with possibly some automorphism of $\mathbb{Z}^n$.

Now, the bottom row is equivariant with respect to the action of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$, so

$$\widehat{\Phi}\cdot E([A])\in\mathbb{Z}^n \implies \widehat{E}(\widehat{\Phi}\cdot[A])\in\mathbb{Z}^n,$$

which means that the $\widehat{\mathbb{Z}}^n$-coordinate of $\widehat{\Phi}\cdot\beta^{-1}([A])$ is actually in $\mathbb{Z}^n$, but the set of all such elements is precisely the set $\beta(A_\Delta^{(n)})\subset\hat{\beta}(\widehat{A}_\Delta^{(n)})$. □

Proposition 4.5 *Assume that $n>1$ and that there exists a natural number d different from 1, 2, 3, 4 and 6, such that at least n of the numbers p_i are divisible by d. Then there exist $[A],[B]\in A_\Delta^{(n)}$ and a matrix $\widehat{\Phi}\in\mathrm{GL}_n(\widehat{\mathbb{Z}})$ such that $\widehat{\Phi}\cdot[A]=[B]$, but there is no $\Phi\in\mathrm{GL}_n(\mathbb{Z})$ and $\sigma\in\Sigma$ such that $\Phi\cdot[A]=\sigma\cdot[B]$.*

Proof In Proposition 4.1 we defined the map $q : A_\Delta^{(n)} \twoheadrightarrow \bigoplus_{i=1}^m (\mathbb{Z}/p_i)^n$. Setting l to be the number of p_i divisible by d, we can quotient further to get

$$\mathbb{Z}^{n\times(m+1)} \xrightarrow{p} A_\Delta^{(n)} \xrightarrow{q} \bigoplus_{i=1}^{m} (\mathbb{Z}/p_i)^n \xrightarrow{q'} \bigoplus_{i=1}^{l} (\mathbb{Z}/d)^n.$$

The map q' is clearly equivariant with respect to the action of $\mathrm{GL}_n(\mathbb{Z})$ and since Σ can permute columns without changing their respective p_i, we get a quotient action of Σ on $\bigoplus_{i=1}^l (\mathbb{Z}/d)^n$ by column permutation.

We can assume that the p_i which are divisible by d are $p_1, \dots, p_l$. Now, let a be an integer coprime to d and such that $a \not\equiv \pm 1 \bmod p_n$ — here is where we use that $d \neq 1, 2, 3, 4, 6$ and that $d \mid p_n$ — and consider the $n\times(m+1)$ matrix

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{Z}^{n\times(m+1)}.$$

Now, take any matrix $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ such that

- $\widehat{\Phi} \cdot E([A]) = E([A])$ — in particular, $\widehat{\Phi} \cdot [A] \in A_\Delta^{(n)}$;
- $\det \widehat{\Phi} \not\equiv \pm 1 \bmod d$.

We set $[B] := \widehat{\Phi} \cdot [A]$ and show that there is no $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ and $\sigma \in \Sigma$ such that

$$\Phi \cdot (q' \circ q)([A]) = \sigma \cdot (q' \circ q)([\widehat{\Phi} \cdot [A]]) = \sigma \cdot (q' \circ q)([B]).$$

The matrix $(q' \circ q)([A])$ is the $n\times n$ identity matrix with $l - n$ zero columns appended on the right. Thus $\Phi \cdot (q' \circ q)([A])$ is the matrix $(\Phi \bmod d)$ appended by $l - n$ zero columns. Note that $(\Phi \bmod d)$ can't have any zero columns. In a similar spirit $(q' \circ q)(\widehat{\Phi} \cdot [A])$ is the matrix $(\widehat{\Phi} \bmod d)$ — whose columns are all nonzero — followed by $l - n$ zero columns. This implies that σ sends the last $l - n$ columns to themselves and possibly permutes the first n ones.

Thus we get that $\Phi \equiv \sigma \cdot \widehat{\Phi} \bmod d$, which in turn implies that $\det \Phi \equiv \det \sigma \cdot \det \widehat{\Phi}$, but $\det \Phi \equiv \pm 1$ and $\det \sigma \equiv \pm 1$, while $\widehat{\Phi}$ was chosen specifically so that $\det \widehat{\Phi} \not\equiv \pm 1$. □

4.3 Orbits separated

Analysing the action of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ closely we will show that sometimes orbits do not change.

Lemma 4.6 *Given any matrix $A \in \mathbb{Z}^{n\times(m+1)}$, we can find a matrix $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ such that $A' = \Phi A$ is an upper-triangular matrix, i.e., that $A'_{ij} = 0$ for $i > j$.*

Proof We can multiply by permutation matrices from $\mathrm{GL}_n(\mathbb{Z})$ to permute rows of A and multiply by elementary matrices to perform elementary row operations (adding an integer multiple of a row to another one). Following Euclid's algorithm, we can in this way make the first column have at most one nonzero entry — equal to the gcd of its entries — in the top place. We can do the same for the next column, not doing any operations that would change the first row. Proceeding inductively, we get an upper-triangular matrix. □

In the following $\mathrm{SL}_n^{\pm 1}(R)$ denotes the group of $n \times n$ matrices with coefficients in R and determinant equal to ± 1.

Lemma 4.7 *In the commuting diagram*

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}}) \\ & \overset{v_1}{\searrow} & \downarrow v_2 \\ & & \mathrm{GL}_n(\mathbb{Z}/d) \end{array}$$

with v_1 and v_2 being reductions modulo d, the image of v_1 is the same as the image of v_2. In particular, given a matrix $\widehat{\Phi}$ with profinite entries whose determinant is ± 1, we can find an integer matrix Φ with determinant ± 1 such that $\widehat{\Phi} \equiv \Phi \bmod d$.

Furthermore, if the first column of $\widehat{\Phi}$ is $(\pm 1, 0, \dots, 0)^T$, then we can require that the first column of Φ is also $(\pm 1, 0, \dots, 0)^T$ (with the same choice of sign).

Proof The image of $v_2 : \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/d)$ is inside $\mathrm{SL}_n^{\pm 1}(\mathbb{Z}/d)$, and the image of $v_1 : \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/d)$ is $\mathrm{SL}_n^{\pm 1}(\mathbb{Z}/d)$, so two images must be the same.

For the second part, notice that if the first column of $\widehat{\Phi}$ is $(\pm 1, 0, \dots, 0)^T$ then

$$\widehat{\Phi} = \begin{pmatrix} \pm 1 & (\hat{\boldsymbol{v}}')^T \\ \mathbb{0} & \widehat{\Phi}' \end{pmatrix},$$

where $\det \widehat{\Phi}' = \pm 1$. Now take a matrix $\Phi' \in \mathrm{GL}_{n-1}(\mathbb{Z})$ such that $\Phi' \equiv \widehat{\Phi}'$ modulo d, and $\boldsymbol{v}' \in \mathbb{Z}^{n-1}$ such that $\boldsymbol{v}' \equiv \hat{\boldsymbol{v}}'$. Then the matrix

$$\Phi = \begin{pmatrix} \pm 1 & (\boldsymbol{v}')^T \\ \mathbb{0} & \Phi' \end{pmatrix}$$

is in $\mathrm{GL}_n(\mathbb{Z})$ and $\Phi \equiv \widehat{\Phi}$ modulo d. □

Having proved Lemmas 4.6 and 4.7, we can prove our main result about orbits remaining distinct.

The reason that the statement counts j such that $d_j \notin \{1, 2, 3, 4, 6\}$ is that 1, 2, 3, 4, 6 are the only numbers t such that $(\mathbb{Z}/t)^\times$ has at most two elements.

Proposition 4.8 *Let Δ be a nice 2-orbifold with $m \geq 0$ cone points of order $p_1, \dots, p_m$. Let $A_\Delta^{(n)} \cong \mathbb{Z}^n \oplus \left(\bigoplus_{j=1}^{m-1} (\mathbb{Z}/d_j)^n\right)$ as in Proposition 4.2 and let*

$$k = 1 + |\{1 \leq j \leq m-1 \mid d_j \notin \{1, 2, 3, 4, 6\}\}|.$$

Let $[A], [B] \in A_\Delta^{(n)}$ be related by $\widehat{\Phi} \cdot [A] = [B]$ for some $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$.

Then, if $n > k$, we can find a matrix $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ with integer coefficients such that $\Phi \cdot [A] = \widehat{\Phi} \cdot [A] = [B]$.

Proof First, transform A and B to their upper-triangular form with respect to the decomposition $A_\Delta^{(n)} \cong \mathbb{Z}^n \oplus \bigl(\bigoplus_{j=1}^{m-1} (\mathbb{Z}/d_{m-j})^n\bigr)$ — here we wrote $A_\Delta^{(n)}$ as a quotient of $\mathbb{Z}^n \cdot s_0^* \oplus \mathbb{Z}^n \cdot s_{m-1}^* \oplus \cdots \oplus \mathbb{Z}^n \cdot s_1^*$ and the matrices A' and B' were chosen to be upper-triangular with respect to this basis $(s_0^*, s_{m-1}^*, \dots, s_1^*)$. Let $\Phi_A, \Phi_B \in \mathrm{GL}_n(\mathbb{Z})$ be the matrices used to perform this upper-triangularisation, i.e., such that $\Phi_A \cdot A$ and $\Phi_B \cdot B$ are upper-triangular. With that we get

$$\begin{aligned} \widehat{\Phi} \cdot [A] = [B] &\iff (\Phi_B \widehat{\Phi} \Phi_A^{-1}) \cdot [\Phi_A \cdot A] = [\Phi_B \cdot B] \\ &\iff \widehat{\Phi}' \cdot [A'] = [B'], \end{aligned}$$

where $\widehat{\Phi}' = (\Phi_B^{-1} \widehat{\Phi} \Phi_A^{-1})$, $A' = \Phi_A \cdot A$ and $B' = \Phi_B \cdot B$.

Now, let $A' = x_0 \cdot s_0^* + x_{m-1} \cdot s_{m-1}^* + \cdots + x_1 \cdot s_1^*$ and similarly $B' = y_0 \cdot s_0^* + y_{m-1} \cdot s_{m-1}^* + \cdots + y_1 \cdot s_1^*$. Then,

$$\widehat{\Phi}' \cdot [A'] = [B'] \iff \widehat{\Phi}' x_0 = y_0,\ \widehat{\Phi}' x_j \equiv y_j \bmod d_j \text{ for } j = 1, \dots, m-1.$$

Now comes the crucial bit of the proof.

Consider $\widehat{\Phi}'' \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ obtained from $\widehat{\Phi}'$ by multiplying the last column by $\kappa \in \widehat{\mathbb{Z}}^\times$. We claim that if $\kappa \equiv 1 \bmod d_{m-(n-1)}$, then still $\widehat{\Phi}'' \cdot [A'] = [B']$.

There are two cases to check here.

1. For $j = 0, m-1, m-2, \dots, m-(n-2)$, the bottom coordinate of x_j is zero (as A' is upper-triangular). Then, changing the last column of $\widehat{\Phi}'$ does not affect the result of multiplication; in fact $\widehat{\Phi}'' \cdot x_j = y_j$ with genuine equality, not just congruence.
2. For $j = m-(n-1), m-n, \dots, 1$, we have that d_j divides $d_{m-(n-1)}$ and so $\widehat{\Phi}'' \equiv \widehat{\Phi}' \bmod d_j$ as $\kappa \equiv 1 \bmod d_{m-(n-1)}$. In particular, $\widehat{\Phi}'' \cdot x_j \equiv \widehat{\Phi}' \cdot x_j = y_j \bmod d_j$.

Thus we know that indeed $\widehat{\Phi}'' \cdot [A'] = [B']$.

Now, notice that we have some freedom in the choice of κ. We only assumed that $\kappa \in \widehat{\mathbb{Z}}^\times$ and that $\kappa \equiv 1 \bmod d_{m-(n-1)}$. Since $d_{m-(n-1)}$ is in the set $\{1, 2, 3, 4, 6\}$, the integers coprime to $d_{m-(n-1)}$ are $\pm 1 \bmod d_{m-(n-1)}$, so in particular $\det \widehat{\Phi}' \equiv \pm 1 \bmod d_{m-(n-1)}$. Since $\det \widehat{\Phi}'' = \kappa \cdot \det \widehat{\Phi}'$, we can choose $\kappa = \pm (\det \widehat{\Phi}')^{-1}$ so that $\det \widehat{\Phi}'' = \pm 1$, i.e., $\widehat{\Phi}'' \in \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}})$.

At this point we know that $\widehat{\Phi}'' \cdot [A'] = [B']$ for some $\widehat{\Phi}'' \in \mathrm{SL}_n^{\pm 1}(\widehat{\mathbb{Z}})$. By Lemma 4.7, we can choose $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ such that $\Phi \equiv \widehat{\Phi}'' \bmod d_{m-1}$ and additionally if the first column of $\widehat{\Phi}''$ is $\pm(1, 0, \dots, 0)^T$, then we can require the first column of Φ to be the same.

Now, since $d_j \mid d_{m-1}$ for all $j = 1, \dots, m-1$, we have

$$\Phi \cdot x_j \equiv \widehat{\Phi}'' \cdot x_j = y_j \mod d_j$$

for all $j \neq 0$. For $j = 0$ notice that if $x_0 \neq 0$, then the first column of $\widehat{\Phi}''$ must be $\pm(1, 0, \dots, 0)^T$ and it is the same as the first column of Φ, so $\Phi \cdot x_0 = y_0$. If $x_0 = y_0 = 0$, then we get $\Phi \cdot x_0 = y_0$ trivially.

Finally, $\Phi \cdot [A'] = [B']$ gives $(\Phi_B^{-1} \Phi \Phi_A) \cdot [A] = [B]$. □

We can use the same idea of modifying the last column of $\widehat{\Phi}$ which was used in Proposition 4.8 to show that in fact the elements of $A_\Delta^{(n)}$ lying in the same orbit of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ actually become members of the same orbit of $\mathrm{GL}_{n+1}(\mathbb{Z}) \curvearrowright A_\Delta^{(n+1)}$ after appending with a row of zeros on the bottom.

Proposition 4.9 *The map $f : \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^{(n+1)\times(m+1)}$ which appends a row of zeros to a matrix from $\mathbb{Z}^{n\times(m+1)}$ thereby making it an $(n+1)\times(m+1)$ matrix induces a map $\bar{f} : A_\Delta^{(n)} \to A_\Delta^{(n+1)}$. Furthermore, if $[A],[B] \in A_\Delta^{(n)}$ are such that $\widehat{\Phi}\cdot[A] = [B]$ for some $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$, then there exists $\Phi \in \mathrm{GL}_{n+1}(\mathbb{Z})$ such that $\Phi\cdot\bar{f}([A]) = \bar{f}([B])$.*

Proof Take Φ_A, Φ_B such that $[A'] = \Phi_A\cdot[A]$ and $[B'] = \Phi_B\cdot[B]$ are upper-triangular as matrices in $\mathbb{Z}^n \oplus \big(\bigoplus_{j=1}^{m-1}(\mathbb{Z}/d_j)^n\big)$. Then let $\widehat{\Phi}' = \Phi_B\widehat{\Phi}\Phi_A^{-1}$ so that $\widehat{\Phi}'\cdot[A'] = [B']$. Take $\widehat{\Phi}''$ to be the block diagonal matrix

$$\begin{pmatrix} \widehat{\Phi} & \mathbb{0} \\ \mathbb{0}^T & (\det\widehat{\Phi})^{-1} \end{pmatrix}.$$

Since the bottom rows of $\bar{f}([A'])$ and $\bar{f}([B'])$ are both $\mathbb{0}^T$, we get $\widehat{\Phi}''\cdot\bar{f}([A']) = \bar{f}([B'])$. Also, $\det\widehat{\Phi}'' = 1$, so we can choose a matrix $\Phi \in \mathrm{GL}_{n+1}(\mathbb{Z})$ such that $\Phi \equiv \widehat{\Phi}'' \bmod d_{m-1}$ and such that if the first column of $\widehat{\Phi}''$ is $\pm(1,0,\ldots,0)^T$, then so is the first column of Φ. As before, this implies that $\Phi\cdot\bar{f}([A']) = \bar{f}([B'])$ and so

$$\begin{pmatrix} \Phi_B^{-1} & \mathbb{0} \\ \mathbb{0}^T & 1 \end{pmatrix}\cdot\Phi\cdot\begin{pmatrix} \Phi_A & \mathbb{0} \\ \mathbb{0}^T & 1 \end{pmatrix}\cdot\bar{f}([A]) = \bar{f}([B]). \qquad \square$$

4.4 Summary of results

Now we are ready to summarise our state of knowledge.

Proposition 4.10 *Let $n > 1$. Let Δ be a nice 2-orbifold with cone points of orders $p_1,\ldots,p_m$. Let $d_1,\ldots,d_m$ be as in Proposition 4.2 — in particular, $d_j \mid d_{j+1}$. Then, exactly one of the following statements holds:*

1. *$n > m$ or $d_{m-(n-1)} \in \{1,2,3,4,6\}$, and $\widehat{\Phi}\cdot[A] = [B]$ for some $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ implies that $\Phi\cdot[A] = [B]$ for some $\Phi \in \mathrm{GL}_n(\mathbb{Z})$.*
2. *$n \le m$ and $d_{m-(n-1)} \notin \{1,2,3,4,6,12\}$, and there exist $[A],[B] \in A_\Delta^{(n)}$ such that $\widehat{\Phi}\cdot[A] = [B]$ for some $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$, but there are no $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ and $\sigma \in \Sigma$ such that $\Phi\cdot[A] = \sigma\cdot[B]$.*
3. *$n \le m$ and $d_{m-(n-1)} = 12$.*

Proof Following Proposition 4.8 let

$$k = 1 + |\{j \le m-1 \mid d_j \notin \{1,2,3,4,6\}\}|.$$

If $n > m$, then clearly $k < n$. If $d_{m-(n-1)} \in \{1,2,3,4,6\}$, then

$$d_1,\ldots,d_{m-(n-1)} \in \{1,2,3,4,6\}$$

and so

$$\{j \le m-1 \mid d_j \notin \{1,2,3,4,6\}\} \subset \{m-(n-2),\dots,m-1\},$$

so $k \le |\{m-(n-2),\dots,m-1\}|+1 = n-1 < n$. Thus, the hypotheses of Proposition 4.8 are satisfied so there indeed does exist $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ such that $\Phi\cdot[A]=[B]$.

Alternatively, if $n \le m$ and $d_{m-(n-1)} \notin \{1,2,3,4,6,12\}$, then there is some prime power $p^\alpha \notin \{2,3,4\}$ such that $p^\alpha \mid d_{m-(n-1)}$. Since

$$d_{m-(n-1)} \mid d_{m-(n-2)} \mid \cdots \mid d_{m-1} \mid d_m,$$

this implies that p^α divides at least n of $p_1,\dots,p_m$. By Proposition 4.5, this implies that there are some $[A],[B] \in A_\Delta^{(n)}$ and $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ such that $[B]=\widehat{\Phi}\cdot[A]$, but for no $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ and $\sigma \in \Sigma$ do we have $\Phi\cdot[A]=\sigma\cdot[B]$.

The only number which is not in $\{1,2,3,4,6\}$ and is not divisible by a prime power different to 2, 3 and 4 is 12, so if conditions (1) and (2) do not hold, we must have $d_{m-(n-1)}=12$. □

Remark 4.11 The attentive reader certainly noticed the "classification" is not conclusive if $d_{m-(n-1)}=12$. The reason for this is that in this case there may be no $\Phi \in \mathrm{GL}_n(\mathbb{Z})$ such that $\Phi\cdot[A]=\widehat{\Phi}\cdot[A]=[B]$, but there is a *pair* $(\Phi,\sigma) \in \mathrm{GL}_n(\mathbb{Z}) \times \mathrm{Aut}(\Delta)$ such that $\Phi\cdot[A]=\sigma\cdot[B]$. The author spent considerable time trying to understand this case, but was not able to arrive at a full classification.

5 Distinguishing central extensions $\mathbb{Z}^n$-by-Δ by their finite quotients

Finally, having done the "calculations", we can go on to proving our main result — deciding which central extensions of finitely generated free abelian groups by infinite 2-orbifold groups are distinguished from each other by their finite quotients.

Section 5.1 reduces the question to distinguishing between central extensions of the same $\mathbb{Z}^n$ by the same 2-orbifold group Δ. Section 5.2 solves this reduced question.

5.1 Kernel and quotient must agree

Theorem A *Let n_1, n_2 be natural numbers and Δ_1, Δ_2 be infinite fundamental groups of closed orientable 2-orbifolds. Let G_1 and G_2 be central extensions $\mathbb{Z}^{n_1}$-by-Δ_1 and $\mathbb{Z}^{n_2}$-by-Δ_2, respectively. If $\widehat{G}_1 \cong \widehat{G}_2$ then $n_1 = n_2$ and $\Delta_1 \cong \Delta_2$.*

Proof In Proposition 2.11 we showed that such central extensions are residually finite and that we get commutative diagrams of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{n_i} & \longrightarrow & G_i & \longrightarrow & \Delta_i & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{\mathbb{Z}}^{n_i} & \longrightarrow & \widehat{G}_i & \longrightarrow & \widehat{\Delta}_i & \longrightarrow & 1 \end{array}$$

where all of the vertical maps are the natural inclusions.

Firstly, let's exclude the case that $\Delta \cong \mathbb{Z}^2$ and that G_1 is isomorphic to $\mathbb{Z}^{n_1+2}$. Then also $G_2 \cong \mathbb{Z}^{n_1+2}$ and so Δ is abelian, and thus it is $\mathbb{Z}^2$. The kernel of a surjection from $\mathbb{Z}^{n_1+2}$ to $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}^{n_1}$, so the theorem statement holds.

In Section 2.2.4 we studied the centres of 2-orbifold groups and their profinite completions, showing in Proposition 2.16 that for any nice 2-orbifold group Δ we have $Z(\widehat{\Delta}) = 1$. Since the extensions $0 \to \widehat{\mathbb{Z}}^{n_i} \to \widehat{G}_i \to \widehat{\Delta}_i \to 1$ are also central, we get that $Z(\widehat{G}_i) = \widehat{\mathbb{Z}}^{n_i}$. In Proposition 2.17 we showed that in the case of $\Delta \cong \mathbb{Z}^2$ also $Z(\widehat{G}_i) = \widehat{\mathbb{Z}}^{n_i}$ unless $G_i \cong \mathbb{Z}^{n_i+2}$, which was covered in the previous paragraph. This gives

$$\widehat{\mathbb{Z}}^{n_1} \cong Z(\widehat{G}_1) \cong Z(\widehat{G}_2) \cong \widehat{\mathbb{Z}}^{n_2},$$
$$\widehat{\Delta}_1 \cong G_1/Z(\widehat{G}_1) \cong G_2/Z(\widehat{G}_2) \cong \widehat{\Delta}_2.$$

Now, finitely generated abelian groups are distinguished by their profinite completions, so we get $n_1 = n_2$.

The isomorphism $\Delta_1 \cong \Delta_2$ follows directly from Theorem 2.18. □

5.2 Distinguishing central extensions $\mathbb{Z}^n$-by-Δ fixing n and Δ

In this section, we state our main results for distinguishing central extensions of a fixed $\mathbb{Z}^n$ by a fixed infinite closed 2-orbifold group Δ.

Theorem B *Let Δ be an infinite fundamental group of a closed orientable 2-orbifold with $m \geq 0$ cone points of orders $p_1, \dots, p_m$, and let $d_1, \dots, d_m$ be the Smith coefficients associated to $(p_1, \dots, p_m)$. For $n > 1$ the following hold:*

1. *If $n > m$, or $d_{m-(n-1)} \in \{1, 2, 3, 4, 6\}$, then the nonisomorphic central extensions of $\mathbb{Z}^n$ by Δ are distinguished from each other by their profinite completions.*
2. *If $n \leq m$ and $d_{m-(n-1)} \notin \{1, 2, 3, 4, 6, 12\}$, then there exist nonisomorphic central extensions G_1, G_2 of $\mathbb{Z}^n$ by Δ with $\widehat{G}_1 \cong \widehat{G}_2$.*

Proof First, let's solve the case of $\Delta \cong \mathbb{Z}^2$. By Proposition 2.17, the central extensions of $\mathbb{Z}^n$ by $\mathbb{Z}^2$ are isomorphic to $H_k \times \mathbb{Z}^{n-1}$, where H_k has presentation $\langle a, b, c \mid [a,c] = [b,c] = 1,\ [a,b] = c^k \rangle$. The abelianisation of H_k is $\mathbb{Z}^{n+1} \oplus (\mathbb{Z}/k)$, so it is different for different values of k. Abelianisation is detected by profinite completions, so central extensions of $\mathbb{Z}^n$ by $\mathbb{Z}^2$ are distinguished by their profinite completions.

Now, let $\Delta \ncong \mathbb{Z}^2$. For $j = 1, 2$ let $0 \to \mathbb{Z}^n \xrightarrow{\iota_j} G_j \xrightarrow{\pi_j} \Delta \to 1$ be central extensions. From Proposition 3.1 we know that $G_1 \cong G_2$ if and only if the representatives $[A_j] \in A_\Delta^{(n)}$ lie in the same orbit of $\mathrm{GL}_n(\mathbb{Z}) \times \Sigma \curvearrowright A_\Delta^{(n)}$; furthermore, $\widehat{G}_1 \cong \widehat{G}_2$ if and only if $\widehat{\Phi} \cdot [A_1] = \sigma \cdot [A_2]$ for some $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ and $\sigma \in \Sigma$.

Proposition 4.10 tells us when these conditions coincide given the hypotheses (1) and (2). □

Finally, we prove Theorem C from the introduction.

Theorem C *Let Δ be an infinite fundamental group of a closed orientable 2-orbifold. Let $n > 1$ and G_1, G_2 be central extensions of $\mathbb{Z}^n$ by Δ such that $\widehat{G}_1 \cong \widehat{G}_2$. Then $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$.*

Proof Firstly, in the case of $\Delta \cong \mathbb{Z}^2$ we have $\widehat{G}_1 \cong \widehat{G}_2 \iff G_1 \cong G_2$, so the statement holds.

The short exact sequences $0 \to \mathbb{Z}^n \xrightarrow{\iota_j} G_j \xrightarrow{\pi_j} \Delta \to 1$ stabilise to the sequences

$$0 \to \mathbb{Z}^n \times \mathbb{Z} \xrightarrow{\iota_j \times \mathrm{Id}_{\mathbb{Z}}} G_j \times \mathbb{Z} \xrightarrow{\pi_j \times 1} \Delta \to 1$$

represented by $\bar{f}([A_j])$ where $[A_j] \in A_\Delta^{(n)}$ represents the original extension, and where $\bar{f} : A_\Delta^{(n)} \to A_\Delta^{(n+1)}$ is the map defined in Proposition 4.9. It is induced from $f : \mathbb{Z}^{n\times(m+1)} \to \mathbb{Z}^{(n+1)\times(m+1)}$ by appending an $n\times(m+1)$ matrix with a zero row at the bottom.

Now, $\widehat{G}_1 \cong \widehat{G}_2$, so $\widehat{\Phi} \cdot [A_1] = \sigma \cdot [A_2] = [\sigma \cdot A_2]$ for some $\widehat{\Phi} \in \mathrm{GL}_n(\widehat{\mathbb{Z}})$ and $\sigma \in \Sigma$. In Proposition 4.9 it was shown that $\widehat{\Phi} \cdot [A] = [B]$ implies that $\Phi \cdot \bar{f}([A]) = \bar{f}([B])$ for some $\Phi \in \mathrm{GL}_{n+1}(\mathbb{Z})$. In the present case, it means that $\Phi \cdot \bar{f}([A_1]) = \bar{f}([\sigma \cdot A_2]) = \sigma \cdot \bar{f}([A_2])$ and hence that $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$. □

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References

[1] **M P Anderson**, *Exactness properties of profinite completion functors*, Topology 13 (1974) 229–239 MR

[2] **G Baumslag**, *Residually finite groups with the same finite images*, Compositio Math. 29 (1974) 249–252 MR

[3] **M R Bridson**, **M D E Conder**, **A W Reid**, *Determining Fuchsian groups by their finite quotients*, Isr. J. Math. 214 (2016) 1–41

[4] **M R Bridson**, **D B McReynolds**, **A W Reid**, **R Spitler**, *Absolute profinite rigidity and hyperbolic geometry*, Ann. of Math. (2) 192:3 (2020) 679–719 MR

[5] **M R Bridson**, **A W Reid**, **R Spitler**, *Absolute profinite rigidity, direct products, and finite presentability*, preprint (2023) arXiv 2312.06058v1

[6] **K S Brown**, *Cohomology of groups*, Graduate Texts in Mathematics 87, Springer (1982) MR

[7] **R G Burns**, **D Solitar**, *The indices of torsion-free subgroups of Fuchsian groups*, Proc. Amer. Math. Soc. 89:3 (1983) 414–418 MR

[8] **T C Chau**, *A note concerning Fox's paper on Fenchel's conjecture*, Proc. Amer. Math. Soc. 88:4 (1983) 584–586 MR

[9] **J D Dixon**, **E W Formanek**, **J C Poland**, **L Ribes**, *Profinite completions and isomorphic finite quotients*, J. Pure Appl. Algebra 23:3 (1982) 227–231 MR

[10] **A L Edmonds**, **J H Ewing**, **R S Kulkarni**, *Torsion free subgroups of Fuchsian groups and tessellations of surfaces*, Invent. Math. 69:3 (1982) 331–346 MR

[11] **G Ellis**, **G Williams**, *On the cohomology of generalized triangle groups*, Comment. Math. Helv. 80:3 (2005) 571–591 MR

[12] **R H Fox**, *On Fenchel's conjecture about F-groups*, Mat. Tidsskr. B 1952 (1952) 61–65 MR

[13] **L Funar**, *Torus bundles not distinguished by TQFT invariants*, Geom. Topol. 17:4 (2013) 2289–2344 MR

[14] **F Grunewald**, **A Jaikin-Zapirain**, **P A Zalesskii**, *Cohomological goodness and the profinite completion of Bianchi groups*, Duke Math. J. 144:1 (2008) 53–72 MR

[15] **J Hempel**, *Some 3-manifold groups with the same finite quotients*, preprint (2014) arXiv 1409.3509v2

[16] **S Hughes**, *Cohomology of Fuchsian groups and non-Euclidean crystallographic groups*, Manuscripta Math. 170:3-4 (2023) 659–676 MR

[17] **F Klein**, *Neue Beiträge zur Riemann'schen Functionentheorie*, Math. Ann. 21:2 (1883) 141–218 MR

[18] **C Löh**, *Group cohomology*, lecture notes (2019) Available at https://www.uni-r.de/Fakultaeten/nat_Fak_I/loeh/teaching/grouphom_ss19/lecture_notes.pdf

[19] **R C Lyndon**, **P E Schupp**, *Combinatorial group theory*, Ergebnisse der Math. 89, Springer (1977) MR

[20] **J Ma**, **Z Wang**, *On profinite rigidity of 4-dimensional Seifert manifolds*, preprint (2023) arXiv 2304.01808v1

[21] **S Majumdar**, *A free resolution for a class of groups*, J. London Math. Soc. (2) 2 (1970) 615–619 MR

[22] **A Maltsev**, *On isomorphic matrix representations of infinite groups*, Rec. Math. [Mat. Sb.] N.S. 8/50 (1940) 405–422 MR In Russian.

[23] **I Morales**, *On the profinite rigidity of free and surface groups*, Math. Ann. 390:1 (2024) 1507–1540 MR

[24] **J Neukirch**, **A Schmidt**, **K Wingberg**, *Cohomology of number fields*, Grundl. Math. Wissen. 323, Springer (2013) MR

[25] **S J Patterson**, *On the cohomology of Fuchsian groups*, Glasgow Math. J. 16:2 (1975) 123–140 MR

[26] **H Poincaré**, *Papers on Fuchsian functions*, Springer (1985) MR

[27] **A W Reid**, *Profinite properties of discrete groups*, from "Groups St Andrews 2013" (C M Campbell, M R Quick, E F Robertson, C M Roney-Dougal, editors), London Math. Soc. Lecture Note Ser. 422, Cambridge Univ. Press (2015) 73–104 MR

[28] **A W Reid**, *Profinite rigidity*, from "Proceedings of the International Congress of Mathematicians, II" (B Sirakov, P N de Souza, M Viana, editors), World Sci., Hackensack, NJ (2018) 1193–1216 MR

[29] **L Ribes**, **P Zalesskii**, *Profinite groups*, Ergebnisse der Math. 40, Springer (2000) MR

[30] **P Scott**, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15:5 (1983) 401–487 MR

[31] **J-P Serre**, *Galois cohomology*, Springer (2002) MR

[32] **A Szczepański**, *Geometry of crystallographic groups*, Algebra and Discrete Mathematics 4, World Sci., Hackensack, NJ (2012) MR

[33] **G Wilkes**, *Profinite rigidity for Seifert fibre spaces*, Geom. Dedicata 188 (2017) 141–163 MR

[34] **G Wilkes**, *Profinite groups and residual finiteness*, EMS Press, Berlin (2024) MR

[35] **H Wilton**, *Essential surfaces in graph pairs*, J. Amer. Math. Soc. 31:4 (2018) 893–919 MR

[36] **H Wilton**, *On the profinite rigidity of surface groups and surface words*, C. R. Math. Acad. Sci. Paris 359 (2021) 119–122 MR

[37] **H Wilton**, **P Zalesskii**, *Distinguishing geometries using finite quotients*, Geom. Topol. 21:1 (2017) 345–384 MR

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