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Magnitude homology equivalence of Euclidean sets

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Magnitude homology is an $\mathbb{R}^+$-graded homology theory of metric spaces that captures information on the complexity of geodesics. Here we address the question: when are two metric spaces magnitude homology equivalent, in the sense that there exist back-and-forth maps inducing mutually inverse maps in homology? We give a concrete geometric necessary and sufficient condition in the case of closed Euclidean sets. Along the way, we introduce the convex-geometric concepts of inner boundary and core, and prove a strengthening for closed convex sets of the classical theorem of Carathéodory.

1 Introduction

Magnitude homology is a homology theory of enriched categories [15]. For ordinary categories, it specialises to the standard homology of categories, which itself includes group homology and poset homology. But what has sparked the most interest in magnitude homology (as catalogued in [14]) is that it provides a homology theory of metric spaces, taking advantage of Lawvere's insight that metric spaces can be viewed as enriched categories [11].

The magnitude homology of metric spaces is a genuinely metric invariant. For example, whereas topological homology detects the existence of holes, magnitude homology detects their diameter (Theorem 5.7 of [10]). Whereas the homology of a topological space is trivial if it is contractible, the magnitude homology of a metric space is trivial if it is convex (Corollary 8.2 below, originally proved by Kaneta and Yoshinaga and, independently, by Jubin). A theorem of Asao (Theorem 5.3 of [1]) states that the second magnitude homology group of a metric space X is nontrivial if X contains a closed geodesic. Gomi proposes a slogan: 'The more geodesics are unique, the more magnitude homology is trivial' [7, p. 5].

The story of magnitude homology began with graphs. Hepworth and Willerton [8] defined the magnitude homology of a graph and established its basic properties, treating graphs as metric spaces in which the distance between two vertices is the number of edges in a shortest path between them. Later, Leinster and Shulman extended their definition to a large class of enriched categories, including metric spaces [15]. In this work, we suppress the enriched categorical context, working directly and explicitly with metric spaces.

Magnitude homology is not the first homology theory for metric spaces. Persistent homology, central to topological data analysis, is also such a theory. It is natural to compare the two theories, as has been done by Otter [17] and Cho [4]. Here we just note that magnitude homology and persistent homology capture quite different information about a space, and that work is underway to use magnitude homology in the analysis of networks (Giusti and Menara [6]).

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Like persistent homology, the magnitude homology of metric spaces X is a *graded* homology theory. There is one group $H_{n,\ell}(X)$ for each integer $n \geq 0$ and real number $\ell \geq 0$, where ℓ is to be regarded as a length scale.

As Hepworth and Willerton pointed out in the introduction to [8], magnitude homology is similar in spirit to Khovanov homology, a graded homology theory of links. The graded Euler characteristic of Khovanov homology is the Jones polynomial, and the graded Euler characteristic of magnitude homology is the invariant of metric spaces known as magnitude [12; 13; 14].

Magnitude is the canonical measure of the size of an enriched category. For ordinary categories, under finiteness hypotheses, the magnitude is the Euler characteristic of the nerve. In particular, when a finite set is seen as a discrete category, magnitude is cardinality. For metric spaces, magnitude is geometrically highly informative. For example, for suitable spaces X, the asymptotics of the magnitude of the rescaled space tX as $t \to \infty$ are known to determine the Minkowski dimension, volume and surface area of X (Corollary 7.4 of Meckes [16], Theorem 1 of Barceló and Carbery [2], and Theorem 2(d) of Gimperlein and Goffeng [5], respectively).

The categorification theorem mentioned, that the Euler characteristic of magnitude homology is magnitude, only holds for *finite* metric spaces (Theorem 7.12 of [15]). Although there is currently no categorification theorem for nonfinite spaces, the intention is that magnitude homology is the categorification of magnitude, and shares with it the property of capturing important geometric features.

This paper addresses the question: when do two metric spaces have the same magnitude homology?

To answer this, we first need to make 'same' precise. For any homology theory of any kind of object, there are at least three possible meanings. The first is simple: just ask that our objects X and Y satisfy $H_n(X) \cong H_n(Y)$ for all $n \geq 0$. This is generally seen as too loose a relation, and it is too loose for us too. For example, Roff provided an example of metric spaces whose first magnitude homology groups are isomorphic but whose first singular homology groups are not (Section 4.6 of [19]). And in Remarks 7.11(iii) below, we describe two simple but nonhomeomorphic metric spaces X and Y such that $H_{n,\ell}(X) \cong H_{n,\ell}(Y)$ for all n and ℓ.

The second option is quasi-isomorphism: generate an equivalence relation on spaces by declaring them equivalent if there exists a map between them inducing an isomorphism in homology. The third is more demanding still: declare X and Y to be equivalent if there exist maps $X \rightleftarrows Y$ whose induced maps in homology are mutually inverse.

Here we take the third option, defining metric spaces X and Y to be *magnitude homology equivalent* if there are maps $X \rightleftarrows Y$ whose induced maps $H_{n,*}(X) \rightleftarrows H_{n,*}(Y)$ are mutually inverse for all $n \geq 1$. Our maps of metric spaces are those that are short (that is, 1-Lipschitz, or contractions or distance-decreasing in the nonstrict sense). When metric spaces are viewed as enriched categories, these are the enriched functors.

Our main theorem (Theorem 9.1) states that two closed subsets of $\mathbb{R}^N$ are magnitude homology equivalent if and only if they satisfy an entirely concrete geometric condition: that their 'cores' are isometric.

The core is easily defined. Two distinct points x and y of a metric space X are *adjacent* if there is no other point p between them (that is, satisfying $d(x,p) + d(p,y) = d(x,y)$). The *inner boundary* of X

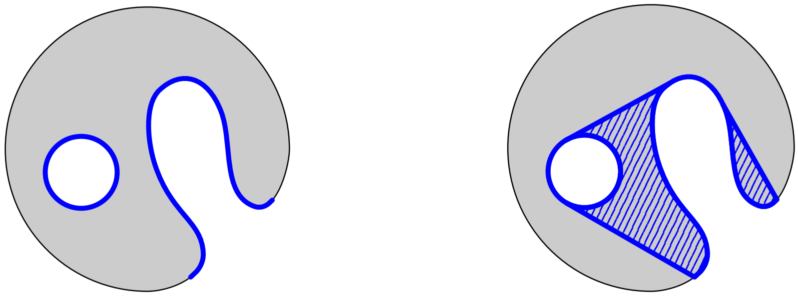


Figure 1: (a) A closed subset of $\mathbb{R}^2$, with its inner boundary the thick line; (b) the core of the same set, shaded (see Section 5).

is the set of all points adjacent to some other point (Figure 1(a)). For example, the inner boundary of a closed annulus is its inner bounding circle. Finally, for $X \subseteq \mathbb{R}^N$, the *core* of X is the intersection of X with the closed convex hull of its inner boundary (Figure 1(b)).

In fact, our main theorem says more. Let X and Y be subsets of Euclidean space. Magnitude homology equivalence of X and Y means that there exist maps $X \rightleftarrows Y$ whose induced maps $H_{n,*}(X) \rightleftarrows H_{n,*}(Y)$ are mutually inverse for *all* $n \geq 1$, but we show that this is also equivalent to them being mutually inverse for *some* $n \geq 1$.

This startling result is true not because magnitude homology is trivial, but because $\mathbb{R}^N$ and its subsets are in a certain sense rather simple metric spaces. For example, any path in $\mathbb{R}^N$ that is locally geodesic is globally geodesic. Magnitude homology reflects the complexity of geodesics in a space, so it is unsurprising that its behaviour on subsets of $\mathbb{R}^N$ is rather simple too.

For arbitrary metric spaces, magnitude homology can be much more complex, as has been thoroughly established in the case of graphs. For example, the magnitude homology groups of a subset of $\mathbb{R}^N$ are all free abelian, whereas Sazdanovic and Summers showed that every finitely generated abelian group arises as a subgroup of some magnitude homology group of some graph (Theorem 3.14 of [20]). The intrinsic complexity of the magnitude homology of graphs has been further analysed in recent work of Caputi and Collari [3].

The structure of this paper is as follows. In Section 2, we introduce *aligned* metric spaces, which are those in which the betweenness relation behaves as in subspaces of $\mathbb{R}^N$. This is the most general class of metric spaces that we consider. We show that alignedness is equivalent to the conjunction of two properties studied previously: being geodetic and having no 4-cuts.

The convex-geometric parts of this work require a strengthening of Carathéodory's classical theorem in the case of closed sets (Section 3). With that in hand, we study inner boundaries and cores (Sections 4 and 5). For example, the closed Carathéodory theorem is used to prove a fundamental result: any point in the convex hull of a closed set $X \subseteq \mathbb{R}^N$ is either in X itself or in the convex hull of its inner boundary (Proposition 4.5). This in turn is used to show that if X is not convex then every point of X has a unique closest point in its core — even though the core is not in general convex (Proposition 5.8).

We then review the magnitude homology of metric spaces, from the beginning (Section 6). The magnitude homology of aligned spaces (Section 7) is vastly simpler than the general case, thanks to the structure theorem of Kaneta and Yoshinaga (Theorem 7.2). For instance, this theorem implies that two maps of aligned spaces $f, g : X \to Y$ that agree on the inner boundary of X induce the same map in homology in positive degree. We improve slightly on their result, proving that the chain maps induced by f and g are chain homotopic (Theorem 7.4). But more importantly, Kaneta and Yoshinaga's theorem leads to a concrete geometric criterion for when two maps $X \rightleftarrows Y$ induce mutually inverse maps in magnitude homology (Theorem 7.10).

In Section 8, we introduce magnitude homology equivalence and prove that every closed, nonconvex subset of $\mathbb{R}^N$ is magnitude homology equivalent to its core (Theorem 8.6). Since taking the core is an idempotent process, this theorem provides a canonical representative for each magnitude homology equivalence class of closed Euclidean sets. And it is a crucial ingredient in our main theorem (Section 9), which gives several necessary and sufficient conditions for two closed Euclidean sets to be magnitude homology equivalent, one of which is that their cores are isometric. We also provide examples to show that for magnitude homology, the three notions of homological sameness discussed above are genuinely different.

2 Aligned spaces

Throughout this work, a *map* of metric spaces means one that is short in the following sense.

Definition 2.1 Let X and Y be metric spaces. A function $f : X \to Y$ is a *short map*, or just a *map*, if $d(f(x), f(x')) \leq d(x, x')$ for all $x, x' \in X$.

When metric spaces are viewed as enriched categories, these are the enriched functors.

An *isometry* is a map that is distance-preserving: $d(f(x), f(x')) = d(x, x')$. It need not be surjective.

For the rest of this section, let X be a metric space.

Definition 2.2 Let $x, y, z \in X$. We say that y is *between* x and z, and write $x \preceq y \preceq z$, if

$$d(x, z) = d(x, y) + d(y, z).$$

If also $x \neq y \neq z$, then y is *strictly between* x and z, written as $x \prec y \prec z$.

For $a, b \in X$, we define the *closed interval*

$$[a, b] = \{x \in X : a \preceq x \preceq b\}.$$

The intervals (a, b), $[a, b)$ and $(a, b]$ are defined similarly.

We will use two elementary facts without mention: first, that $[a, b]$ is topologically closed in X, and second, that if $x \preceq y \preceq y'$ and $x \preceq y' \preceq y$ then $y = y'$.

Definition 2.3 The space X is *aligned* if for all $n \geq 1$ and $x_0, \dots, x_n \in X$ satisfying $x_{i-1} \prec x_i \prec x_{i+1}$ whenever $0 < i < n$, we have

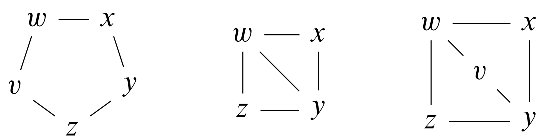
$$[x_0, x_n] = [x_0, x_1] \cup [x_1, x_2] \cup \dots \cup [x_{n-1}, x_n].$$

For example, Euclidean space $\mathbb{R}^N$ is aligned. Any subspace of an aligned space is also aligned.

We view graphs (taken to be connected and undirected) as metric spaces as follows: the points are the vertices, and the distance between two vertices is the number of edges in a shortest path connecting them.

Examples 2.4 (i) Any tree, seen as a metric space, is aligned.

(ii) None of the graphs



is aligned. In the first, $v \prec w \prec x$ and $w \prec x \prec y$, but

$$[v, y] = \{v, z, y\} \neq \{v, w, x, y\} = [v, w] \cup [w, x] \cup [x, y].$$

In both the second and the third, $x \prec y \prec z$ but $[x, y] \cup [y, z]$ is a proper subset of $[x, z]$.

We will use without mention the fact that in an aligned space,

$$[x_0, x_2] = [x_0, x_1] \cup [x_1, x_2]$$

holds not just when $x_0 \prec x_1 \prec x_2$, but also, slightly more generally, when $x_0 \preceq x_1 \preceq x_2$.

Lemma 2.5 *For points a, b, x, y of an aligned space, if $x, y \in [a, b]$ then $[x, y] \subseteq [a, b]$.*

The alignedness condition cannot be dropped. For example, the third graph of Examples 2.4(ii) has $w, y \in [x, z]$ but $v \in [w, y] \setminus [x, z]$.

Proof Let $x, y \in [a, b]$. Then $[a, b] = [a, y] \cup [y, b]$, so without loss of generality, $x \in [a, y]$. But then $[a, y] = [a, x] \cup [x, y]$, so $[x, y] \subseteq [a, y] \subseteq [a, b]$. □

Thus, intervals $[a, b]$ in an aligned space are convex in the following sense.

Definition 2.6 A subset A of X is *convex* in X if $[x, y] \subseteq A$ whenever $x, y \in A$. The *convex hull* $\mathrm{conv}(A)$ of A in X is the intersection of all convex subsets of X containing A. Its *closed convex hull* $\overline{\mathrm{conv}}(A)$ is the intersection of all closed convex subsets of X containing A.

Since an arbitrary intersection of convex sets is convex, $\mathrm{conv}(A)$ is the smallest convex set containing A, and similarly for $\overline{\mathrm{conv}}(A)$.

Remark 2.7 In $\mathbb{R}^N$, the closure of a convex set is closed, so the closed convex hull is the closure of the convex hull. Both these statements are false in an arbitrary aligned space. For example, take X to be the L-shaped subspace

$$\big[(0,0), (1,0)\big] \cup \big[(0,0), (0,1)\big]$$

of $\mathbb{R}^2$, and take $A = ((0,0), (1,0)] \cup \{(0,1)\}$. Then A is convex in X, but its closure is not. In fact, the closed convex hull of A in X is X itself.

Remark 2.8 In an arbitrary metric space X, convex hulls can be constructed as follows. For $A \subseteq X$, let $\mathrm{ic}(A)$ denote the *interval closure* of A, defined as

$$\mathrm{ic}(A) = \bigcup_{x,y \in A} [x, y].$$

Then

$$\mathrm{conv}(A) = \bigcup_{n \geq 0} \mathrm{ic}^n(A),$$

since the right-hand side is the smallest convex set containing A.

Intervals in aligned spaces embed into the real line:

Lemma 2.9 *For points a and b of an aligned space, the function $d(a, -) : [a, b] \to \mathbb{R}$ is an isometry. In particular, points of $[a, b]$ the same distance from a are equal.*

Proof Let $x, y \in [a, b]$. We must prove that $|d(a, y) - d(a, x)| = d(x, y)$.

By alignedness, $[a, b] = [a, x] \cup [x, b]$, so $y \in [a, x]$ or $y \in [x, b]$. Similarly, $x \in [a, y]$ or $x \in [y, b]$.

If $y \in [x, b]$ and $x \in [y, b]$ then $x = y$ and the result holds trivially. Otherwise, without loss of generality, $x \notin [y, b]$, so $x \in [a, y]$. Hence $d(a, y) = d(a, x) + d(x, y)$ and the result follows. □

The rest of this section relates alignedness to two conditions on metric spaces that appear in the magnitude homology literature (and were generalised from graph theory): being geodetic and having no 4-cuts. We will need this relationship only to show that a theorem of Kaneta and Yoshinaga applies to aligned spaces (see Theorem 7.2 below). The reader willing to take this on trust can omit the rest of this section.

Definition 2.10 The metric space X is *geodetic* if whenever $a, b \in X$ and $x, y \in [a, b]$, then either

(i) $a \preceq x \preceq y$ and $x \preceq y \preceq b$, or

(ii) $a \preceq y \preceq x$ and $y \preceq x \preceq b$.

Definition 2.11 A 4-*cut* in X is a 4-tuple (a, x, y, b) of points such that $a \prec x \prec y$ and $x \prec y \prec b$, and yet x, y are not both in $[a, b]$. The space X has *no* 4-*cuts* if no such tuple exists.

Examples 2.12 (i) Every subspace of $\mathbb{R}^N$ is geodetic with no 4-cuts.

(ii) The first graph of Examples 2.4(ii) is geodetic, but has a 4-cut (v, w, x, y).

(iii) The second graph of Examples 2.4(ii) has no 4-cuts but is not geodetic, since $w, y \in [x, z]$ but $w \notin [x, y]$ and $y \notin [x, w]$.

(iv) The third graph of Examples 2.4(ii) has a 4-cut (x, w, v, y) and is not geodetic (again, consider $w, y \in [x, z]$).

Proposition 2.13 *A metric space is aligned if and only if it is geodetic and has no 4-cuts.*

Proof First suppose that X is aligned. That X is geodetic follows from Lemma 2.9 and the fact that subspaces of $\mathbb{R}$ are geodetic. To prove that X has no 4-cuts, note that whenever $a \prec x \prec y$ and $x \prec y \prec b$, alignedness gives $[a, b] = [a, x] \cup [x, y] \cup [y, b]$, hence $x, y \in [a, b]$.

Conversely, suppose that X is geodetic and has no 4-cuts, and take points $x_0, \ldots, x_n$ satisfying $x_{i-1} \prec x_i \prec x_{i+1}$ whenever $0 < i < n$. We must show that $[x_0, x_n] = \bigcup_{i=1}^{n} [x_{i-1}, x_i]$. We use induction on n.

The case $n = 1$ is trivial. For $n = 2$, let $y \in [x_0, x_2]$. Applying the definition of geodetic to $y, x_1 \in [x_0, x_2]$ gives $y \in [x_0, x_1]$ or $y \in [x_1, x_2]$. Hence $[x_0, x_2] \subseteq [x_0, x_1] \cup [x_1, x_2]$. The opposite inclusion holds in any metric space (Lemma 4.13 of Leinster and Shulman [15]).

Now let $n \geq 3$. No 4-cuts gives $x_0 \prec x_2 \prec x_3$. Hence the list of points $x_0, x_2, x_3, \ldots, x_n$ satisfies the conditions of Definition 2.3, so by inductive hypothesis,

$$[x_0, x_n] = [x_0, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-1}, x_n].$$

Finally, $[x_0, x_2] = [x_0, x_1] \cup [x_1, x_2]$ by the $n = 2$ case. □

3 A Carathéodory theorem for closed sets

The classical Carathéodory theorem states that any point in the convex hull of a subset X of Euclidean space must, in fact, lie in the convex hull of some affinely independent subset of X. (See Theorem 1.1.4 of [21], for instance.) We will need a strengthening of this result for *closed* sets. It is very classical in flavour, but we have been unable to find it in the literature.

Theorem 3.1 *Let X be a closed subset of $\mathbb{R}^N$ and $a \in \operatorname{conv}(X)$. Then there exist $n \geq 0$ and affinely independent points $x_0, \ldots, x_n \in X$ such that $a \in \operatorname{conv}\{x_0, \ldots, x_n\}$ and*

$$\operatorname{conv}\{x_0, \ldots, x_n\} \cap X = \{x_0, \ldots, x_n\}. \tag{1}$$

The condition that X is closed cannot be dropped: consider $X = (-2, -1) \cup (1, 2) \subseteq \mathbb{R}$ and $a = 0$.

Proof First assume that X is compact. The result is trivial when $N = 0$, so let $N \geq 1$ and assume inductively that it holds for $N - 1$. If a can be expressed as a convex combination of elements of X that all lie in some proper affine subspace H of $\mathbb{R}^N$, we can apply the inductive hypothesis to $H \cap X$ and the result is proved. Assuming otherwise, Carathéodory's theorem implies that a is in the convex hull of some affinely independent subset of X, which must then have $N + 1$ elements.

Now write

$$\Delta^N = \Big\{(p_0, \ldots, p_N) \in \mathbb{R}^{N+1} : p_i \geq 0, \textstyle\sum p_i = 1\Big\},$$

and consider the maps

$$\begin{array}{ccccc} X^{N+1} & \xleftarrow{\pi} & X^{N+1} \times \Delta^N & \xrightarrow{\sigma} & \mathbb{R}^N, \\ (x_0, \ldots, x_N) & \longleftarrow & (x_0, \ldots, x_N, p_0, \ldots, p_N) & \longmapsto & \sum_{i=0}^{N} p_i x_i. \end{array}$$

Both maps are continuous and $X^{N+1} \times \Delta^N$ is compact, so the set

$$K = \big\{(x_0, \ldots, x_N) \in X^{N+1} : a \in \operatorname{conv}\{x_0, \ldots, x_N\}\big\} = \pi(\sigma^{-1}(a))$$

is compact, as well as nonempty. Moreover, the map

$$(\mathbb{R}^N)^{N+1} \to \mathbb{R}, \quad (z_0, \dots, z_N) \mapsto \mathrm{Vol}_N(\mathrm{conv}\{z_0, \dots, z_N\}),$$

is continuous, where Vol_N denotes N-dimensional volume. Hence $\mathrm{Vol}_N(\mathrm{conv}\{x_0, \dots, x_N\})$ attains a minimum at some point $(x_0, \dots, x_N)$ of K. Since $a \in \mathrm{conv}\{x_0, \dots, x_N\}$, the points $x_0, \dots, x_N$ do not lie in any proper affine subspace of $\mathbb{R}^N$, so $\mathrm{Vol}_N(\mathrm{conv}\{x_0, \dots, x_N\}) > 0$.

We now prove (1) for $x_0, \dots, x_N$. Certainly $\{x_0, \dots, x_N\} \subseteq \mathrm{conv}\{x_0, \dots, x_N\} \cap X$. Conversely, let $x \in \mathrm{conv}\{x_0, \dots, x_N\} \cap X$. We have

$$\mathrm{conv}\{x_0, \dots, x_N\} = \bigcup_{i=0}^{N} \mathrm{conv}\{x_0, \dots, x_{i-1}, x, x_{i+1}, \dots, x_N\},$$

so without loss of generality, $a \in \mathrm{conv}\{x, x_1, \dots, x_N\}$. Then $(x, x_1, \dots, x_N) \in K$, so

$$\mathrm{Vol}_N(\mathrm{conv}\{x, x_1, \dots, x_N\}) \geq \mathrm{Vol}_N(\mathrm{conv}\{x_0, x_1, \dots, x_N\}) \tag{2}$$

by minimality. But since $x \in \mathrm{conv}\{x_0, \dots, x_N\}$,

$$\mathrm{conv}\{x, x_1, \dots, x_N\} \subseteq \mathrm{conv}\{x_0, x_1, \dots, x_N\}, \tag{3}$$

so equality holds in (2). This and the fact that $\mathrm{Vol}_N(\mathrm{conv}\{x_0, \dots, x_N\}) > 0$ together imply that equality holds in (3). In particular, $x_0 \in \mathrm{conv}\{x, x_1, \dots, x_N\}$, and a short calculation using affine independence of $x_0, x_1, \dots, x_N$ then yields $x = x_0$. Hence $x \in \{x_0, \dots, x_N\}$, as required.

This proves the theorem when X is compact. Now consider the general case of a closed set X. Since $a \in \mathrm{conv}(X)$, we have $a \in \mathrm{conv}(F)$ for some finite $F \subseteq X$. Then $\mathrm{conv}(F)$ is compact, so $X' = \mathrm{conv}(F) \cap X$ is compact with $a \in \mathrm{conv}(X')$. So by the compact case, there are affinely independent points $x_0, \dots, x_n \in X'$ such that $a \in \mathrm{conv}\{x_0, \dots, x_n\}$ and $\mathrm{conv}\{x_0, \dots, x_n\} \cap X' = \{x_0, \dots, x_n\}$. Now $x_0, \dots, x_n \in X' \subseteq \mathrm{conv}(F)$, so $\mathrm{conv}\{x_0, \dots, x_n\} \subseteq \mathrm{conv}(F)$, giving

$$\mathrm{conv}\{x_0, \dots, x_n\} \cap X = \mathrm{conv}\{x_0, \dots, x_n\} \cap X' = \{x_0, \dots, x_n\}. \qquad \square$$

4 Inner boundaries

Our main theorem relates magnitude homology equivalence to two concrete geometric constructions: the inner boundary and the core. We introduce them in the next two sections.

Definition 4.1 Two points of a metric space are *adjacent* if they are distinct and there is no point strictly between them.

For example, when a (connected) graph is viewed as a metric space, two vertices are adjacent in this sense if and only if there is an edge between them.

Definition 4.2 The *inner boundary* ρX of a metric space X is the subset

$$\rho X = \{x \in X : x \text{ is adjacent to some point in } X\}.$$

Figure 1(a) and the following examples shed light on the choice of terminology.

Examples 4.3 (i) For a closed annulus in $\mathbb{R}^2$, the inner boundary is the inner bounding circle.

(ii) Similarly, when X is $\mathbb{R}^2$ with several disjoint open discs removed, the inner boundary is the union of their bounding circles.

(iii) The inner boundary of a finite metric space X with more than one point is X itself.

(iv) Let X be the L-shaped space of Remark 2.7. Then $\rho X = X \setminus \{(0,0)\}$. This shows that ρX need not be closed in X.

(v) A metric space X is said to be *Menger convex* if $\rho X = \varnothing$. Every convex or open subset of $\mathbb{R}^N$ is Menger convex. For closed sets in $\mathbb{R}^N$, Menger convexity is equivalent to convexity in the usual sense (Theorem 2.6.2 of [18]).

Remark 4.4 For $X \subseteq \mathbb{R}^N$, the inner boundary ρX is a subset of the topological boundary ∂X. But whereas ∂ is defined for subspaces of a topological space, ρ is defined for abstract metric spaces. Another difference between ρ and ∂ is that ρ is idempotent: $\rho\rho X = \rho X$ for all metric spaces X, as is easily shown.

Proposition 4.5 *Let X be a closed subset of $\mathbb{R}^N$. Then*

$$\operatorname{conv}(X) = X \cup \operatorname{conv}(\rho X) \quad \textit{and} \quad \overline{\operatorname{conv}}(X) = X \cup \overline{\operatorname{conv}}(\rho X).$$

Proof The second equation follows from the first by taking closures, and one inclusion of the first equation is immediate. It remains to prove that every point a of $\operatorname{conv}(X)$ is in $X \cup \operatorname{conv}(\rho X)$.

By Theorem 3.1 (closed Carathéodory), there exist $n \geq 0$ and affinely independent points $x_0, \dots, x_n \in X$ such that $a \in \operatorname{conv}\{x_0, \dots, x_n\}$ and

$$\operatorname{conv}\{x_0, \dots, x_n\} \cap X = \{x_0, \dots, x_n\}.$$

Whenever $i \neq j$, we have $[x_i, x_j] \cap X = \{x_i, x_j\}$, so x_i is adjacent to x_j. Hence if $n \geq 1$ then $x_0, \dots, x_n \in \rho X$ and $a \in \operatorname{conv}(\rho X)$, while if $n = 0$ then $a = x_0 \in X$. □

The inner boundary construction is not functorial in the obvious sense. For example, whenever Y is a closed convex subset of $\mathbb{R}^N$ and X is a closed but nonconvex subset of Y, we have $\rho X \neq \varnothing = \rho Y$ (by Examples 4.3(v)), so the inclusion $X \hookrightarrow Y$ cannot induce a map $\rho X \to \rho Y$. However, we now state conditions under which a map $f : X \to Y$ does restrict to a map $\rho X \to \rho Y$.

Lemma 4.6 *Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be maps of metric spaces.*

(i) *If $gf(x) = x$ for all $x \in \rho X$, then f sends adjacent points of X to adjacent points of Y and $f(\rho X) \subseteq \rho Y$;*

(ii) *If also $fg(y) = y$ for all $y \in \rho Y$, then f and g restrict to mutually inverse isometries $\rho X \rightleftarrows \rho Y$.*

Proof For (i), suppose that $gf(x) = x$ for all $x \in \rho X$, and let $x, x' \in X$ be adjacent. We must show that $f(x), f(x') \in Y$ are adjacent. Since x, x' are in ρX, they are fixed by gf. Since $x \neq x'$, it follows that

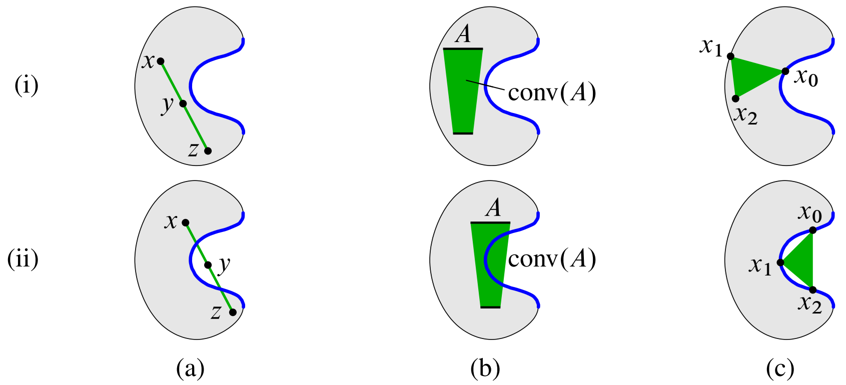


Figure 2: The two cases of (a) Lemma 4.7, (b) Lemma 4.9, and (c) Proposition 4.10. Here X is a closed subset of $E=\mathbb{R}^2$, with inner boundary the thick line. In (b), the subset A of X is the union of the two parallel line segments.

$f(x)\neq f(x')$. (Recall that distinctness is part of the definition of adjacency.) Now let $y\in[f(x),f(x')]$. By the triangle inequality, the fact that gf fixes x and x', and the shortness of f and g,

$$\begin{aligned} d(x,x') &\leq d(x,g(y))+d(g(y),x') = d(gf(x),g(y))+d(g(y),gf(x')) \\ &\leq d(f(x),y)+d(y,f(x')) = d(f(x),f(x')) \\ &\leq d(x,x'), \end{aligned}$$

so equality holds throughout. Hence

$$g(y)\in[x,x'],\quad d(x,g(y))=d(f(x),y),\quad d(g(y),x')=d(y,f(x')).$$

Since x and x' are adjacent, the first statement gives $g(y)\in\{x,x'\}$, then the others give $y\in\{f(x),f(x')\}$. Hence $f(x)$ and $f(x')$ are adjacent. It follows that $f(\rho X)\subseteq\rho Y$.

For (ii), it suffices to note that, by (i), $f(\rho X)\subseteq\rho Y$ and $g(\rho Y)\subseteq\rho X$. □

The rest of this section describes the interactions between the concepts of inner boundary and convex hull. It is needed for the proof of Proposition 5.9, but not for the main theorem.

The first two results address the question of whether the convex hull of points in a set crosses its inner boundary (Figure 2(a)–(b)).

Lemma 4.7 *Let E be an aligned metric space in which every closed interval $[a,b]$ is compact. Let X be a closed subset of E, let $x,z\in X$, and let $y\in[x,z]$. Then either*

(i) *$y\in X$, or*

(ii) *$[x,y)$ and $(y,z]$ both intersect ρX.*

Proof By hypothesis, $[x,y]$ is compact, and so is $[x,y]\cap X$ because X is closed; it is also nonempty. Hence it contains a point $y_0\in[x,y]\cap X$ that maximises $d(x,y_0)$. Similarly, let $y_1\in[y,z]\cap X$ maximise $d(y_1,z)$. Assuming that $y\notin X$, we have $y_0\prec y\prec y_1$ by Lemma 2.9.

We claim that y_0 is adjacent to y_1 in X. By alignedness and Lemma 2.9,

$$[y_0, y] \cup [y, y_1] = [y_0, y_1] \subseteq [x, y].$$

But $[y_0, y]$ and $[y, y_1]$ can only intersect X at y_0 and y_1, by maximality. Hence $[y_0, y_1] \cap X = \{y_0, y_1\}$. □

Remark 4.8 The closed intervals $[a, b]$ in an aligned space need not be compact; consider $[-1, 1]$ in $\mathbb{R} \setminus \{0\}$, for instance. But Lemma 2.9 implies that intervals $[a, b]$ in an aligned space E are totally bounded, so they are compact if E is complete.

Lemma 4.9 *Let E be an aligned metric space in which every closed interval $[a, b]$ is compact. Let X be a closed subset of E and $A \subseteq X$. Then either*

(i) $\mathrm{conv}(A) \subseteq X$, *or*

(ii) $\mathrm{conv}(A)$ *intersects* ρX.

Here, $\mathrm{conv}(A)$ denotes the convex hull in E (not X). Generally, we use conv to mean the convex hull in the largest space involved, which is typically $\mathbb{R}^N$.

Proof Suppose that $\mathrm{conv}(A) \cap \rho X = \varnothing$. By Remark 2.8, it is enough to prove that $\mathrm{ic}^n(A) \subseteq X$ for all $n \geq 0$. For $n = 0$, this is trivial. Let $n \geq 1$. For all $x, y \in \mathrm{ic}^{n-1}(A)$, we have $[x, y] \subseteq \mathrm{ic}^n(A) \subseteq \mathrm{conv}(A)$; but $\mathrm{conv}(A) \cap \rho X = \varnothing$, so $[x, y] \cap \rho X = \varnothing$. It follows from Lemma 4.7 that $[x, y] \subseteq X$ for all $x, y \in \mathrm{ic}^{n-1}(A)$. Thus, $\mathrm{ic}^n(A) \subseteq X$. □

The third and final result on convex hulls and inner boundaries is specific to Euclidean space, and describes a dichotomy (Figure 2(c)).

Proposition 4.10 *Let X be a closed subset of $\mathbb{R}^N$, and let $x_0, \ldots, x_n \in X$ be affinely independent points such that*

$$\mathrm{conv}\{x_0, \ldots, x_n\} \cap \rho X \subseteq \{x_0, \ldots, x_n\}. \tag{4}$$

Then either

(i) $\mathrm{conv}\{x_0, \ldots, x_n\} \subseteq X$, *or*

(ii) $\mathrm{conv}\{x_0, \ldots, x_n\} \cap X = \{x_0, \ldots, x_n\}$.

Conditions (i) and (ii) are mutually exclusive when $n > 0$. The affine independence hypothesis cannot be dropped: consider $X = [0, 1] \cup \{2\} \subseteq \mathbb{R}$ and $(x_0, x_1, x_2) = (0, 1, 2)$.

Proof Suppose that (ii) does not hold. Then we can choose a point $x \in \mathrm{conv}\{x_0, \ldots, x_n\} \cap X$ with $x \notin \{x_0, \ldots, x_n\}$. We must prove that $\mathrm{conv}\{x_0, \ldots, x_n\} \subseteq X$.

Define, for each $t \in [0, 1]$,

$$\Delta_t = \mathrm{conv}\{(1-t)x_0 + tx, \ldots, (1-t)x_n + tx\}.$$

First we claim that $\Delta_t \cap \rho X = \varnothing$ for all $t \in (0, 1]$.

Indeed, let $t \in (0,1]$. Since $x \in \operatorname{conv}\{x_0,\dots,x_n\}$, we have $\Delta_t \subseteq \operatorname{conv}\{x_0,\dots,x_n\}$ and therefore $\Delta_t \cap \rho X \subseteq \{x_0,\dots,x_n\}$ by (4). Supposing for a contradiction that $\Delta_t \cap \rho X \neq \varnothing$, it follows without loss of generality that $x_0 \in \Delta_t$, so that x_0 is a convex combination of $x_0,\dots,x_n,x$ with a nonzero coefficient of x. But x is in the convex hull of $x_0,\dots,x_n$, which are affinely independent, forcing $x = x_0$ and contradicting our assumption that $x \notin \{x_0,\dots,x_n\}$. This proves the claim that $\Delta_t \cap \rho X = \varnothing$.

In particular, $(x_i, x] \cap \rho X = \varnothing$ for each $i \in \{0,\dots,n\}$. Since $x_i, x \in X$, Lemma 4.7 then implies that $[x_i, x] \subseteq X$, giving $(1-t)x_i + tx \in X$ for each $t \in (0,1]$ and $i \in \{0,\dots,n\}$. Hence Δ_t is the convex hull of a subset of X, which allows us to apply Lemma 4.9 and deduce that $\Delta_t \subseteq X$, for each $t \in (0,1]$. And since X is closed, it follows that X contains $\Delta_0 = \operatorname{conv}\{x_0,\dots,x_n\}$. □

5 Cores

Here we introduce the key convex-geometric player in our main theorem. Our focus now is on subsets of Euclidean space $\mathbb{R}^N$. The notation conv and $\overline{\operatorname{conv}}$ will always refer to (closed) convex hulls in $\mathbb{R}^N$, and the notation $[a,b]$ means the straight line segment from a to b in $\mathbb{R}^N$.

Definition 5.1 The *core* of a subset $X \subseteq \mathbb{R}^N$ is $\operatorname{core}(X) = \overline{\operatorname{conv}}(\rho X) \cap X$.

An example of a core is shown in Figure 1(b). The following further examples build on Examples 4.3.

Examples 5.2 (i) The core of a closed annulus in $\mathbb{R}^2$ is the inner bounding circle.

(ii) Let X be $\mathbb{R}^2$ with an open disc and an open square removed, as in Figure 4(c) (see page 621). Then $\operatorname{core}(X)$ is as shown in Figure 4(a).

(iii) The core of a finite subset X of $\mathbb{R}^N$ with more than one point is X itself.

(iv) The core of the L-shaped space X of Remark 2.7 is X.

(v) A closed subset of $\mathbb{R}^N$ has empty core if and only if it is convex.

Inner boundaries are defined for abstract metric spaces, but cores are only defined for metric spaces embedded in $\mathbb{R}^N$. Nevertheless, the notion of core is intrinsic in the sense we now explain.

Write $\operatorname{aff}(X)$ for the affine hull of a set $X \subseteq \mathbb{R}^N$. The following lemma is classical.

Lemma 5.3 *Let $X \subseteq \mathbb{R}^N$, let $Y \subseteq \mathbb{R}^M$, and let $f : X \to Y$ be an invertible isometry. Then f extends uniquely to an invertible isometry $\operatorname{aff}(X) \to \operatorname{aff}(Y)$.*

Proof Existence is proved in Theorem 11.4 of Wells and Williams [23]. We sketch the uniqueness argument. Let $\bar{f}, \tilde{f} : \operatorname{aff}(X) \to \operatorname{aff}(Y)$ be isometries extending f. Then for each $a \in \operatorname{aff}(X)$, the set of points equidistant from $\bar{f}(a)$ and $\tilde{f}(a)$ is affine and contains Y (using the surjectivity of f), and therefore contains $\operatorname{aff}(Y)$. In particular, it contains $\bar{f}(a)$, so $\bar{f}(a) = \tilde{f}(a)$. □

We will consider the class of abstract metric spaces embeddable in $\mathbb{R}^N$.

Definition 5.4 A (*closed*) *Euclidean set* is a metric space isometric to a (closed) metric subspace of $\mathbb{R}^N$ for some $N \geq 0$.

Let X be a Euclidean set. Lemma 5.3 implies that $\mathrm{aff}(X)$, $\mathrm{conv}(\rho X)$, $\overline{\mathrm{conv}}(\rho X)$, $\mathrm{conv}(\rho X) \cap X$ and $\mathrm{core}(X)$ are all well defined as metric spaces, up to isometry. For example, if we embed X isometrically into $\mathbb{R}^N$ in one way and into $\mathbb{R}^M$ in another, then the core of the copy of X in $\mathbb{R}^N$ is isometric to the core of the copy of Y in $\mathbb{R}^M$. This is the sense in which the core is an intrinsic construction.

We also note that convexity is an intrinsic property of Euclidean sets X, being equivalent to the property that for all $x, y \in X$, there exists an isometry $[0, d(x,y)] \to X$ with $0 \mapsto x$ and $d(x,y) \mapsto y$.

Recall that the inner boundary construction is idempotent (Remark 4.4). We now show that the core construction is idempotent too, and relate the two idempotents.

Lemma 5.5 *For a Euclidean set* X*,*

(i) $\rho X \subseteq \mathrm{core}(X)$;

(ii) $\mathrm{core}(\rho X) = \rho X = \rho(\mathrm{core}(X))$;

(iii) $\mathrm{core}(\mathrm{core}(X)) = \mathrm{core}(X)$.

Proof This result will not be needed and the proof is elementary, so we just sketch it. Part (i) is immediate. The first identity in (ii) follows from ρ being idempotent, and the second is proved by a simple argument directly from the definitions, using the convexity of $\overline{\mathrm{conv}}(\rho X)$. Part (iii) follows from the definitions and the second identity in (ii). □

Lemma 4.6 provides conditions under which maps $X \rightleftarrows Y$ of metric spaces restrict to mutually inverse isometries between their inner boundaries. We now show that for Euclidean sets, such maps also induce an isometry between the cores.

Lemma 5.6 *Let* $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ *be maps between Euclidean sets. If* f *and* g *restrict to mutually inverse maps* $\rho X \rightleftarrows \rho Y$ *then they also restrict to mutually inverse maps* $\mathrm{core}(X) \rightleftarrows \mathrm{core}(Y)$.

Proof Suppose that f and g restrict to mutually inverse maps

$$\rho X \underset{g'}{\overset{f'}{\rightleftarrows}} \rho Y.$$

It follows from Lemma 5.3 that f' and g' extend uniquely to mutually inverse maps

$$\overline{\mathrm{conv}}(\rho X) \underset{G}{\overset{F}{\rightleftarrows}} \overline{\mathrm{conv}}(\rho Y).$$

It suffices to show that $f(x) = F(x)$ for all $x \in \mathrm{core}(X)$, and similarly for g. Evidently we need only prove the result for f.

Let $x \in \mathrm{core}(X)$. For each $y \in \rho Y$, the shortness of f implies that

$$d(f(x), y) \leq d(x, f'^{-1}(y)) = d\big(F(x), F(f'^{-1}(y))\big) = d(F(x), y).$$

Choosing an embedding of Y into $\mathbb{R}^M$, the set

$$V = \{b \in \mathbb{R}^M : d(f(x), b) \leq d(F(x), b)\}$$

therefore contains ρY. But V is either $\mathbb{R}^M$ or a closed half-space of it, so it is closed and convex; hence $\overline{\mathrm{conv}}(\rho Y) \subseteq V$. In particular, $F(x) \in V$, giving $f(x) = F(x)$. □

Remark 5.7 Note that f and g in the statement of the previous lemma need not be isometries, even though their respective restrictions are. For example, the retraction $X \to \mathrm{core}(X)$ constructed in the next proposition is not an isometry and yet, together with the inclusion $\mathrm{core}(X) \to X$, gives a pair of maps satisfying the hypotheses of Lemma 5.6.

For a nonempty convex closed subset C of $\mathbb{R}^N$, every point $x \in \mathbb{R}^N$ has a unique closest point $\pi(x) \in C$. This defines a short retraction $\pi : \mathbb{R}^N \to C$ of the inclusion $C \hookrightarrow \mathbb{R}^N$, called the *metric projection* onto C (Theorem 1.2.1 of [21]).

For a *non*convex closed subset X of $\mathbb{R}^N$, the inner boundary is nonempty (Examples 4.3(v)), so we have a metric projection map $\mathbb{R}^N \to \overline{\mathrm{conv}}(\rho X)$.

Proposition 5.8 *Let X be a* ***non****convex closed subset of $\mathbb{R}^N$. Then metric projection $\mathbb{R}^N \to \overline{\mathrm{conv}}(\rho X)$ restricts to a map $X \to \mathrm{core}(X)$.*

Thus, every point of X has a unique closest point in $\mathrm{core}(X)$.

Proof Write $\pi : \mathbb{R}^N \to \overline{\mathrm{conv}}(\rho X)$ for metric projection. We have to show that whenever $x \in X$, then also $\pi(x) \in X$. Certainly $[x, \pi(x)] \subseteq \overline{\mathrm{conv}}(X)$, so by Proposition 4.5,

$$[x, \pi(x)] = \big([x, \pi(x)] \cap X\big) \cup \big([x, \pi(x)] \cap \overline{\mathrm{conv}}(\rho X)\big).$$

Both sets in this union are closed in $\mathbb{R}^N$, and nonempty since x belongs to the first and $\pi(x)$ to the second. Since $[x, \pi(x)]$ is connected, $[x, \pi(x)] \cap X \cap \overline{\mathrm{conv}}(\rho X)$ contains some point y. Then $y \in \overline{\mathrm{conv}}(\rho X)$ with $d(x, y) \le d(x, \pi(x))$, which by definition of π implies that $y = \pi(x)$. Hence $\pi(x) \in X$. □

The final result of this section will not be needed for the main theorem, but is a basic property of cores.

Proposition 5.9 *Let X be a closed Euclidean set. Then* $\mathrm{core}(X) = \overline{\mathrm{conv}(\rho X) \cap X}$.

Proof The right-to-left inclusion is clear. For the converse, let $x \in \mathrm{core}(X)$ and $\varepsilon > 0$. We must prove that

$$B_\varepsilon(x) \cap \mathrm{conv}(\rho X) \cap X \neq \varnothing, \tag{5}$$

where $B_\varepsilon(x)$ denotes the open ball in $\mathbb{R}^N$. This is immediate if $B_\varepsilon(x) \cap \rho X \neq \varnothing$, so assume that $B_\varepsilon(x) \cap \rho X = \varnothing$.

We will use two properties of $B_\varepsilon(x) \cap X$. First, it is convex. For let $p, q \in B_\varepsilon(x) \cap X$. Then $[p, q] \subseteq B_\varepsilon(x)$, so $[p, q] \cap \rho X = \varnothing$. By Lemma 4.7 or Lemma 4.9, it follows that $[p, q] \subseteq X$, proving convexity.

The second property is that if $p \in B_\varepsilon(x) \cap X$ and $y \in \rho X$, then $[p, y] \cap B_\varepsilon(x) \subseteq X$. Let $q \in [p, y] \cap B_\varepsilon(x)$. Then $[p, q] \subseteq B_\varepsilon(x)$, so $[p, q) \cap \rho X = \varnothing$, and then Lemma 4.7 applied to $q \in [p, y]$ gives $q \in X$.

Since $x \in \overline{\mathrm{conv}}(\rho X)$, the set $\overline{B_{\varepsilon/2}(x)} \cap \mathrm{conv}(\rho X)$ is nonempty. We prove that it is a subset of X, which will imply property (5) and complete the proof.

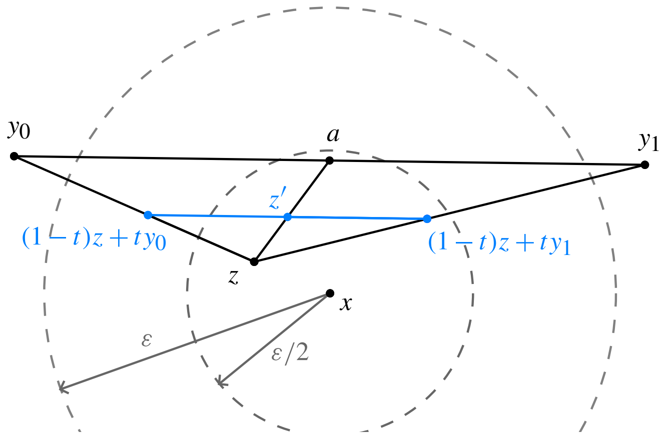


Figure 3: The proof of Proposition 5.9.

Thus, let $a \in \overline{B_{\varepsilon/2}(x)} \cap \mathrm{conv}(\rho X)$ (Figure 3). We must prove that $a \in X$. Choose a point $z \in \overline{B_{\varepsilon/2}(x)} \cap X$ minimising $d(a, z)$. It is enough to prove that $z = a$.

Since $a \in \mathrm{conv}(\rho X)$, we have $a = \sum_{i=0}^{n} p_i y_i$ for some $n \geq 0$, points $y_i \in \rho X$, and nonnegative p_i with $\sum p_i = 1$. Since $z \in B_\varepsilon(x)$, we can choose $t \in (0, 1]$ such that $(1-t)z + ty_i \in B_\varepsilon(x)$ for all $i \in \{0, \dots, n\}$.

The second property above implies that $[z, y_i] \cap B_\varepsilon(x) \subseteq X$, and in particular, $(1-t)z + ty_i \in X$, for each i. Hence by the first property,

$$\sum_{i=0}^{n} p_i((1-t)z + ty_i) \in X,$$

that is, the point $z' = (1-t)z + ta$ is in X. Moreover, $z, a \in \overline{B_{\varepsilon/2}(x)}$, so $z' \in \overline{B_{\varepsilon/2}(x)} \cap X$. But z was defined to be the point of $\overline{B_{\varepsilon/2}(x)} \cap X$ minimising the distance to a, and $d(a, z') = (1-t)d(a, z)$ with $t > 0$, so $z = a$. □

6 Magnitude homology of metric spaces

Here we review the definition of the magnitude homology of a metric space, first introduced in [15] as a special case of the magnitude homology of an enriched category.

Let $X = (X, d)$ be a metric space. Write $\mathbb{R}^+$ for the set of nonnegative real numbers.

Definition 6.1 Let $n \geq 0$. A *proper chain in X of degree n* is an $(n+1)$-tuple $\boldsymbol{x} = (x_0, \dots, x_n)$ of points in X such that $x_{i-1} \neq x_i$ whenever $1 \leq i \leq n$. Its *length* is $d(x_0, x_1) + \dots + d(x_{n-1}, x_n)$. Write $P_n(X)$ for the set of proper chains in X of degree n, and for $\ell \in \mathbb{R}^+$, write $P_{n,\ell}(X) = \{\boldsymbol{x} \in P_n(X) : \boldsymbol{x} \text{ has length } \ell\}$.

Let $\mathbb{Z}S$ denote the free abelian group on a set S.

Definition 6.2 For $n \geq 0$ and $\ell \in \mathbb{R}^+$, put $C_{n,\ell}(X) = \mathbb{Z} P_{n,\ell}(X)$. The *magnitude chain complex* of X is an $\mathbb{R}^+$-graded chain complex

$$C_{*,*}(X) = \bigoplus_{\ell \in \mathbb{R}^+} C_{*,\ell}(X).$$

The boundary map $\partial_n : C_{n,\ell}(X) \to C_{n-1,\ell}(X)$ is $\sum_{i=0}^{n} (-1)^i \partial_n^i$, where ∂_n^i is defined on generators by

$$\partial_n^i(x_0, \dots, x_n) = \begin{cases} (x_0, \dots, \widehat{x_i}, \dots, x_n) & \text{if } (x_0, \dots, \widehat{x_i}, \dots, x_n) \text{ has length } \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Here $(x_0, \dots, \widehat{x_i}, \dots, x_n)$ denotes the tuple resulting from removing the i-th entry of $(x_0, \dots, x_n)$. The *magnitude homology* $H_{*,*}(X)$ of X is the homology of $C_{*,*}(X)$.

Recall that maps of metric spaces are by definition short (Definition 2.1). Any such map $f : X \to Y$ induces a chain map $f_\# : C_{*,*}(X) \to C_{*,*}(Y)$, given on the generating set $P_{n,\ell}(X)$ by

$$f_\#(x_0, \dots, x_n) = \begin{cases} (f(x_0), \dots, f(x_n)) & \text{if } (f(x_0), \dots, f(x_n)) \in P_{n,\ell}(Y), \\ 0 & \text{otherwise.} \end{cases}$$

In turn, $f_\#$ induces a map $f_* : H_{*,*}(X) \to H_{*,*}(Y)$ in homology. Thus, $C_{*,*}$ is a functor from the category **Met** of metric spaces and their maps to the category of $\mathbb{R}^+$-graded chain complexes of abelian groups, and $H_{*,*}$ is a functor from **Met** to $(\mathbb{N} \times \mathbb{R}^+)$-graded abelian groups.

Remark 6.3 Some trivial cases are easily described. Clearly $H_{n,0}(X) = 0$ for all $n \geq 1$. Also, the boundary map $\partial_1 : C_{1,\ell}(X) \to C_{0,\ell}(X)$ is zero for any $\ell \in \mathbb{R}^+$, so

$$H_{0,\ell}(X) = \begin{cases} \mathbb{Z}X & \text{if } \ell = 0, \\ 0 & \text{otherwise.} \end{cases}$$

With only a little more effort, one shows that $H_{1,\ell}(X)$ is the free abelian group on the set of ordered pairs of adjacent points distance ℓ apart (Corollary 4.5 of [15]). In particular, a closed subset of $\mathbb{R}^N$ has trivial first magnitude homology if and only if it is convex.

The higher magnitude homology groups are more subtle. Even in the case of graphs, seen as metric spaces as in Section 2, the magnitude homology groups can have torsion (Corollary 5.12(3) of Kaneta and Yoshinaga [10]). Going further, Sazdanovic and Summers showed that every finitely generated abelian group arises as a subgroup of some magnitude homology group of some graph (Theorem 3.14 of [20]).

7 Magnitude homology of aligned spaces

When a space is aligned, its higher magnitude homology groups have a simple description similar to that of $H_{1,*}$ above. More exactly, Kaneta and Yoshinaga showed that all the magnitude homology groups of an aligned space are free, and they identified a basis, as follows.

Definition 7.1 Let X be a metric space, $n \geq 0$ and $\ell \in \mathbb{R}^+$. A *thin chain* of degree n and length ℓ is a proper chain $\boldsymbol{x} \in P_{n,\ell}(X)$ such that x_{i-1} is adjacent to x_i for all $i \in \{1, \dots, n\}$ and $x_i \notin [x_{i-1}, x_{i+1}]$ for all $i \in \{1, \dots, n-1\}$. Write

$$T_{n,\ell}(X) = \{\boldsymbol{x} \in P_{n,\ell}(X) : \boldsymbol{x} \text{ is thin}\}.$$

Any thin chain is a cycle, so there is a map of sets $T_{n,\ell}(X) \to H_{n,\ell}(X)$ assigning to each thin chain its homology class. This map extends uniquely to a homomorphism $\mathbb{Z}T_{n,\ell}(X) \to H_{n,\ell}(X)$.

Kaneta and Yoshinaga proved that when X is aligned, this canonical homomorphism is an isomorphism (Theorem 5.2 of [10]). Thus, the magnitude homology groups of an aligned space are freely generated by the thin chains.

In fact, they proved this under the more careful hypotheses that X is geodetic and has no 4-cuts at certain length scales. By Proposition 2.13, the cruder assumption of alignedness suffices. Although some of our results hold under more careful hypotheses too, we assume alignedness throughout in order to simplify the exposition.

The isomorphism $\mathbb{Z}T_{n,\ell} \cong H_{n,\ell}$ is natural in the following sense. A map $f : X \to Y$ induces a homomorphism $f_\star : \mathbb{Z}T_{n,\ell}(X) \to \mathbb{Z}T_{n,\ell}(Y)$, defined on generators by

$$f_\star(x_0, \dots, x_n) = \begin{cases} (f(x_0), \dots, f(x_n)) & \text{if } (f(x_0), \dots, f(x_n)) \in T_{n,\ell}(Y), \\ 0 & \text{otherwise.} \end{cases}$$

In this way, $\mathbb{Z}T_{n,\ell}$ becomes a functor from aligned spaces to abelian groups.

Theorem 7.2 (Kaneta and Yoshinaga) *Let $n \geq 0$ and $\ell \in \mathbb{R}^+$. For aligned spaces X, the canonical homomorphism $\mathbb{Z}T_{n,\ell}(X) \to H_{n,\ell}(X)$ is an isomorphism, natural in X.*

Proof The main statement follows from Theorem 5.2 of [10], using Proposition 2.13 above. The naturality is not stated explicitly in [10], but is readily checked. □

For $n \in \{0, 1\}$, Theorem 7.2 reproduces the descriptions of $H_{0,*}$ and $H_{1,*}$ at the end of Section 6, which do not require alignedness.

For any thin chain $(x_0, \dots, x_n)$ with $n \geq 1$, the points x_i are all in the inner boundary ρX. Hence the natural isomorphism of Theorem 7.2 gives:

Corollary 7.3 *Let $f, g : X \to Y$ be maps of aligned spaces. Let $n \geq 1$ and $\ell \in \mathbb{R}^+$. If $f(x) = g(x)$ for all $x \in \rho X$ then $f_* = g_* : H_{n,\ell}(X) \to H_{n,\ell}(Y)$.*

A thin chain in degree 0 is just a point, not necessarily in the inner boundary. By Remark 6.3, the maps $f_*, g_* : H_{0,*}(X) \to H_{0,*}(Y)$ are only equal when $f = g$.

Corollary 7.3 suggests an analogy: perhaps the condition that $f|_{\rho X} = g|_{\rho X}$ plays a similar role for magnitude homology of metric spaces as homotopy between maps plays for ordinary homology of topological spaces. We digress briefly (until Remark 7.8) to develop this idea, proving a stronger version of Corollary 7.3.

Theorem 7.4 *Let $f, g : X \to Y$ be maps of metric spaces, with X aligned. Let $\ell \in \mathbb{R}^+$. If $f(x) = g(x)$ for all $x \in \rho X$, then $f_\#, g_\# : C_{*,\ell}(X) \to C_{*,\ell}(Y)$ are chain homotopic in positive degree.*

To prove Theorem 7.4, we use Kaneta and Yoshinaga's notion of frame (Definition 3.3 of [10]).

Definition 7.5 For $0 \leq i \leq n$, a proper chain $\boldsymbol{x} = (x_0, \ldots, x_n) \in P_n(X)$ is *smooth* at i if $0 < i < n$ and $x_{i-1} \prec x_i \prec x_{i+1}$; otherwise, it is *singular* at i. Writing $0 = i_0 < i_1 < \cdots < i_k = n$ for the indices at which $\boldsymbol{x}$ is singular, the *frame* of $\boldsymbol{x}$ is $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$.

Recall that $\partial_n^i(\boldsymbol{x}) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$ when $\boldsymbol{x}$ is smooth at i.

Lemma 7.6 *Let X be an aligned space, let $\boldsymbol{x} = (x_0, \ldots, x_n) \in P_n(X)$, and let $i \in \{0, \ldots, n\}$. If $\boldsymbol{x}$ is smooth at i then $\partial_n^i(\boldsymbol{x})$ and $\boldsymbol{x}$ have the same frame.*

Proof This is proved by an elementary argument similar to the proof of Proposition 3.7 of Kaneta and Yoshinaga [10]. □

Proof of Theorem 7.4 We will define $\varphi : C_{n,\ell}(X) \to C_{n+1,\ell}(Y)$ for each $n \geq 0$ in such a way that

$$g_\# - f_\# = \partial\varphi + \varphi\partial : C_{n,\ell}(X) \to C_{n,\ell}(Y)$$

for each $n \geq 1$.

To this end, for each pair of distinct nonadjacent points $x, x' \in X$, choose a point xx' in the interval (x, x'). Let $n \geq 0$ and let $\boldsymbol{x} = (x_0, \ldots, x_n)$ be a proper chain, with frame $(x_0 = x_{i_0}, x_{i_1}, \ldots, x_{i_k} = x_n)$. If there is some $r \in \{1, \ldots, k\}$ such that $x_{i_{r-1}}$ is not adjacent to x_{i_r}, take the smallest such r. Then, by alignedness, there is a unique index $i_{r-1} < h \leq i_r$ such that $x_{i_{r-1}} x_{i_r} \in (x_{h-1}, x_h)$, and we put

$$\boldsymbol{x}' = (x_0, \ldots, x_{h-1}, x_{i_{r-1}} x_{i_r}, x_h, \ldots, x_n).$$

Otherwise, put $\boldsymbol{x}' = 0$. Finally, set $\varphi(\boldsymbol{x}) = (-1)^h (g_\# - f_\#)(\boldsymbol{x}')$, which defines φ on the generators of $C_{n,\ell}(X)$.

By Lemma 7.6, all the nonzero terms $\partial_n^j(\boldsymbol{x})$ in the sum $\partial(\boldsymbol{x}) = \sum (-1)^j \partial_n^j(\boldsymbol{x})$ have the same frame as $\boldsymbol{x}$. Using this, one finds that if the condition for $\boldsymbol{x}'$ to be nonzero holds, then the same is true for $\partial_n^j(\boldsymbol{x})$ whenever $\partial_n^j(\boldsymbol{x})$ is nonzero. In any case,

$$(\partial_n^j(\boldsymbol{x}))' = \begin{cases} \partial_{n+1}^j(\boldsymbol{x}') & \text{if } j < h, \\ \partial_{n+1}^{j+1}(\boldsymbol{x}') & \text{otherwise,} \end{cases}$$

where we make the convention that $0' = 0$. In turn, this implies that

$$\varphi(\partial_n^j(\boldsymbol{x})) = \begin{cases} -\partial_{n+1}^j \varphi(\boldsymbol{x}) & \text{if } j < h, \\ \partial_{n+1}^{j+1} \varphi(\boldsymbol{x}) & \text{otherwise.} \end{cases} \tag{6}$$

To prove that φ is a chain homotopy, it suffices to show that

$$(g_\# - f_\#)(\boldsymbol{x}) = (\partial\varphi + \varphi\partial)(\boldsymbol{x}). \tag{7}$$

If $\boldsymbol{x}' = 0$, then each component in the frame of $\boldsymbol{x}$ is adjacent to the next one, so $\boldsymbol{x}$ must equal its frame, and all the components of $\boldsymbol{x}$ are in ρX. In this case, it is clear that both sides of (7) are zero. The case where $\boldsymbol{x}'$ is nonzero is a routine calculation using (6) and the fact that $\partial_{n+1}^h(\boldsymbol{x}') = \boldsymbol{x}$. □

Remark 7.7 As an alternative proof, the existence of such a chain homotopy also follows from Theorem 7.2 and some general homological algebra. Let C be a chain complex of free abelian groups with free homology, and $Z_n = \ker \partial_n \subseteq C_n$, $B_n = \operatorname{im} \partial_{n+1} \subseteq C_n$ and $H_n = Z_n / B_n$ as usual. The H_n assemble into a complex H with trivial differentials. If $s_n : H_n \to Z_n$ is a section of the quotient map $q_n : Z_n \to H_n$ for each n, then the composites

$$H_n \xrightarrow{s_n} Z_n \hookrightarrow C_n \tag{8}$$

form a chain homotopy equivalence $H \simeq C$.

To see this, note that since a subgroup of a free abelian group is free, all of C_n, Z_n, B_n and H_n are free. In particular, B_{n-1} is free, so the short exact sequence

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

splits, giving $C_n \cong Z_n \oplus B_{n-1}$ for each n. It follows that C is the direct sum of complexes of the form

$$0 \longrightarrow B_n \hookrightarrow Z_n \longrightarrow 0.$$

Truncating this complex at Z_n gives a free resolution of H_n. Since H_n is free, it is also a free resolution of itself. The maps q_n and s_n give lifts of the identity on H_n to chain maps between these two resolutions, and are then inverse homotopy equivalences by the uniqueness up to chain homotopy of such lifts. Composing the maps s_n with the inclusions $Z_n \hookrightarrow C_n$ and then taking the direct sum over all n gives the desired chain homotopy equivalence $H \simeq C$.

For an aligned space X, Theorem 7.2 implies that $H_{*,\ell}(X)$ is free. We may take s_n to be the map that sends each thin chain to itself, giving a chain homotopy equivalence $\psi : H_{*,\ell}(X) \to C_{*,\ell}(X)$ as in (8), with homotopy inverse φ. Now if $f, g : X \to Y$ are equal on ρX then $f_\# \psi = g_\# \psi$ in positive degree, and so

$$f_\# \simeq f_\# \psi \varphi = g_\# \psi \varphi \simeq g_\#.$$

Remark 7.8 Here we return to the analogy between, on the one hand, the condition that two maps $X \rightrightarrows Y$ of metric spaces agree on the inner boundary of X, and, on the other, the condition that two maps $X \rightrightarrows Y$ of topological spaces are homotopic. The analogy is only loose, since equality on the inner boundary is not stable under 2-categorical composition, unlike topological homotopy. Indeed, there are examples of maps of metric spaces

$$X \xrightarrow{f} Y \underset{g'}{\overset{g}{\rightrightarrows}} Z$$

such that $g|_{\rho Y} = g'|_{\rho Y}$ but $(gf)|_{\rho X} \neq (g'f)|_{\rho X}$. (Take f to be the inclusion $\{0, 1\} \hookrightarrow [0, 1]$ and $g, g' : [0, 1] \to \mathbb{R}$ to be any two maps that differ at 0 or 1.) However, by Theorem 7.4, it is still the case that $(gf)_\#$ and $(g'f)_\#$ are chain homotopic in positive degree.

It is an open problem to find a compact description of the equivalence relation on maps of metric spaces that is generated by equality on the inner boundary and closed under 2-categorical composition.

Two maps that are equivalent in this sense are guaranteed by Theorem 7.4 to induce chain homotopic chain maps in positive degree.

Different connections between homotopy and magnitude homology have been investigated by Tajima and Yoshinaga [22].

This ends the digression on homotopy, and we return to the question of when two maps between aligned spaces induce the same map on homology in positive degree. From here on, we often suppress the length index ℓ, writing $H_n(X)$ for the $\mathbb{R}^+$-graded abelian group $\bigoplus_{\ell\in\mathbb{R}^+} H_{n,\ell}(X)$.

Proposition 7.9 *Let $e : X \to X$ be an endomorphism of an aligned metric space. The following are equivalent*:

(i) $e_* : H_n(X) \to H_n(X)$ *is the identity for all* $n \geq 1$.

(ii) $e_* : H_n(X) \to H_n(X)$ *is the identity for some* $n \geq 1$.

(iii) $e(x) = x$ *for all* $x \in \rho X$.

Proof That (i) implies (ii) is immediate. Now assume (ii), and choose such an n. To prove (iii), let $x \in \rho X$, and choose $x' \in X$ adjacent to x. Then the alternating $(n+1)$-tuple $(x, x', x, x', \dots)$ is a thin chain. By the naturality of the isomorphism in Theorem 7.2 and alignedness, $e_\star(\boldsymbol{x}) = \boldsymbol{x}$, giving $(x, x', \dots) = (e(x), e(x'), \dots)$ and so $e(x) = x$.

Lastly, assuming (iii), we prove (i). Let $n \geq 1$. By naturality and alignedness again, it is enough to prove that $e_\star(\boldsymbol{x}) = \boldsymbol{x}$ for each $\boldsymbol{x} = (x_0, \dots, x_n) \in T_{n,*}(X)$. But $x_0, \dots, x_n \in \rho X$ since $n \geq 1$, so $e(x_i) = x_i$ for each i. □

We now show that for two maps $X \rightleftarrows Y$ of metric spaces to be mutually inverse in positive degree magnitude homology is equivalent to a concrete geometric condition.

Theorem 7.10 *Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be maps of aligned metric spaces. The following are equivalent*:

(i) *The maps $H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y)$ are mutually inverse for all* $n \geq 1$.

(ii) *The maps $H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y)$ are mutually inverse for some* $n \geq 1$.

(iii) f *and* g *restrict to mutually inverse isometries* $\rho X \rightleftarrows \rho Y$.

Proof (i) $\Longrightarrow$ (ii) is trivial.

For (ii) $\Longrightarrow$ (iii), take $n \geq 1$ as in (ii). Then $(gf)_* : H_n(X) \to H_n(X)$ is the identity, so by Proposition 7.9, $gf(x) = x$ for all $x \in \rho X$. Similarly, $fg(y) = y$ for all $y \in \rho X$. Then (iii) follows from Lemma 4.6(ii).

(iii) $\Longrightarrow$ (i) follows from Proposition 7.9 applied to gf and fg. □

At this point, it follows easily that the positive-degree magnitude homology of a *non*convex closed Euclidean set is isomorphic to that of its core. Indeed, the retraction $X \to \mathrm{core}(X)$ of Proposition 5.8 and the inclusion $\mathrm{core}(X) \to X$ satisfy (iii) of the theorem by Lemma 5.5(i). We will return to this fact in the next section (see Theorem 8.6).

Remarks 7.11 The following counterexamples illustrate the essential role of the two opposing maps in Theorem 7.10.

(i) Spaces with isometric inner boundaries need not have isomorphic magnitude homology groups in positive degree. For example, consider $X = \{0, 1, 2, 3\}$ and $Y = \{0\} \cup [1, 2] \cup \{3\}$, metrised as subspaces of $\mathbb{R}$. Then $\rho X = \rho Y = X$. However,

$$T_{1,1}(X) = \{(0,1), (1,0), (1,2), (2,1), (2,3), (3,2)\},$$
$$T_{1,1}(Y) = \{(0,1), (1,0), (2,3), (3,2)\},$$

giving $H_{1,1}(X) = 6\mathbb{Z}$ but $H_{1,1}(Y) = 4\mathbb{Z}$ by Theorem 7.2.

(ii) There are examples of maps of aligned spaces $f : X \to Y$ such that $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for some but not all $n \geq 1$.

Indeed, let $X = \{0, 1, 2\} \subseteq \mathbb{R}$. Let $v_0, v_1, v_2 \in \mathbb{R}^2$ be the vertices of an equilateral triangle of edge length 1, and let Y be $\operatorname{conv}\{v_0, v_1, v_2\}$ with the open line segments (v_0, v_1) and (v_1, v_2) removed. Define $f : X \to Y$ by $f(i) = v_i$ for $i = 0, 1, 2$. We have

$$T_{1,1}(X) = \{(0,1), (1,0), (1,2), (2,1)\},$$
$$T_{1,1}(Y) = \{(v_0, v_1), (v_1, v_0), (v_1, v_2), (v_2, v_1)\},$$

with $T_{1,\ell}(X) = \varnothing = T_{1,\ell}(Y)$ for $\ell \neq 1$. Hence by Theorem 7.2, $f_* : H_{1,*}(X) \to H_{1,*}(Y)$ is an isomorphism. On the other hand,

$$T_{2,1}(X) = \{(0,1,0), (1,0,1), (1,2,1), (2,1,2)\},$$
$$T_{2,1}(Y) = \{(v_0, v_1, v_0), (v_0, v_1, v_2), (v_1, v_0, v_1), (v_1, v_2, v_1), (v_2, v_1, v_0), (v_2, v_1, v_2)\}.$$

Hence $H_{2,1}(X) = 4\mathbb{Z}$ and $H_{2,1}(Y) = 6\mathbb{Z}$, and so $f_* : H_{2,*}(X) \to H_{2,*}(Y)$ is not an isomorphism.

(iii) There are also examples of aligned spaces X and Y such that $H_{n,\ell}(X) \cong H_{n,\ell}(Y)$ for all $n \geq 0$ and $\ell \in \mathbb{R}^+$, but for which there is no map $X \to Y$ inducing isomorphisms in homology.

Let $X \subseteq \mathbb{R}^2$ be the unit circle centred at the origin, with the subspace metric. Let $Y \subseteq \mathbb{R}^2$ be the union of X and a circle of radius 1 centred at $(2, 0)$. In both spaces, the set of pairs of adjacent points distance ℓ apart has continuum cardinality when $\ell \leq 2$ and is empty otherwise. By considering chains of the form $(z, z', z, z', \dots)$, we deduce that $T_{n,\ell}(X)$ and $T_{n,\ell}(Y)$ have continuum cardinality for all $\ell \leq 2n$ and are empty otherwise. It follows from Theorem 7.2 that $H_{*,*}(X) \cong H_{*,*}(Y)$.

We now show that there is no map $f : X \to Y$ such that $f_* : H_{1,*}(X) \to H_{1,*}(Y)$ is an isomorphism of $\mathbb{R}^+$-graded abelian groups. Suppose for a contradiction that such a map f exists.

Theorem 7.2 implies that $H_{1,*}(X)$ is the free abelian group on the set of ordered pairs of adjacent points in X, and similarly for Y. It also implies that $f_\star : \mathbb{Z}T_{1,\ell}(X) \to \mathbb{Z}T_{1,\ell}(Y)$ is an isomorphism for each $\ell \in \mathbb{R}^+$. In particular, $f_\star$ is surjective, which since $\rho Y = Y$ implies that f is surjective. This is a contradiction: there is no short surjection $X \to Y$, since the diameter of Y is strictly larger than that of X.

8 Magnitude homology equivalence

Theorem 7.10 tells us when two maps of aligned metric spaces $X \rightleftarrows Y$ induce mutually inverse isomorphisms in magnitude homology. This raises a question: given only the spaces X and Y, when does such a pair of maps exist? This is the question answered by the main theorem, in the next section. Here we build up to it by considering inclusions and retractions. In keeping with Definition 2.1, a *retraction* is by definition short, and the term *retract* is used accordingly.

Proposition 8.1 *Let X be an aligned metric space and C a convex subset of X such that $\rho X \subseteq C$. Then the inclusion $C \hookrightarrow X$ induces an isomorphism $H_n(C) \to H_n(X)$ for all $n \geq 1$.*

Proof Write $\iota : C \hookrightarrow X$ for the inclusion. By Theorem 7.2, it is enough to show that for each $n \geq 1$ and $\ell \in \mathbb{R}^+$, the map

$$\iota_\star : \mathbb{Z}T_{n,\ell}(C) \to \mathbb{Z}T_{n,\ell}(X)$$

is an isomorphism.

It is injective if every thin chain in C is a thin chain in X, which is true by the convexity of C in X. It is surjective if every thin chain $(x_0, \dots, x_n)$ in X is a thin chain in C. Since $n \geq 1$, we have $x_i \in \rho X \subseteq C$ for each $i \in \{0, \dots, n\}$. Hence $(x_0, \dots, x_n)$ is a chain in C, and thin in C since it is thin in X. □

Taking $C = \varnothing$ in Proposition 8.1 gives the following corollary, which also follows directly from Theorem 7.2 (as Kaneta and Yoshinaga observed in the introduction to [10]), and was proved independently by Jubin as Theorem 7.2 of [9].

Corollary 8.2 *A Menger convex aligned metric space has trivial magnitude homology in positive degree.*

In particular, this corollary applies to any convex or open subset of $\mathbb{R}^N$.

Definition 8.3 Two metric spaces X and Y are *magnitude homology equivalent* if there exist maps $X \rightleftarrows Y$ inducing mutually inverse maps $H_n(X) \rightleftarrows H_n(Y)$ for all $n \geq 1$.

We do not require our maps to induce an isomorphism in degree 0, since by Remark 6.3, this would make X and Y isometric.

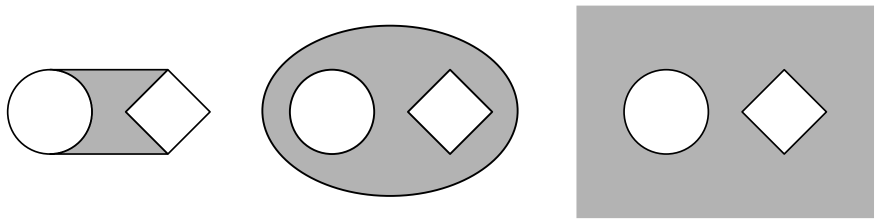
Example 8.4 All nonempty convex Euclidean sets are magnitude homology equivalent, by Corollary 8.2.

Proposition 8.5 *Let X be an aligned metric space and A a retract of X such that $\rho X \subseteq A$. Then A and X are magnitude homology equivalent.*

Proof Let $\iota : A \hookrightarrow X$ denote the inclusion, and choose a retraction $\pi : X \to A$. Then $\iota\pi(x) = x$ for all $x \in \rho X$, since $\rho X \subseteq A$. Hence by Corollary 7.3, the map

$$\iota_* \pi_* = (\iota\pi)_* : H_n(X) \to H_n(X)$$

is the identity for all $n \geq 1$. On the other hand, $\pi\iota = 1_A$, so $\pi_*\iota_*$ is the identity on $H_n(A)$. Hence π_* and ι_* are mutually inverse in positive degree. □



(a) (b) (c)

Figure 4: Three magnitude homology equivalent spaces, where the set S of Proposition 8.7 is the union of a disc and a filled square. In (a), $C = \overline{\mathrm{conv}}(S)$; in (b), C is a filled ellipse; in (c), $C = \mathbb{R}^2$.

Alternatively, Proposition 8.5 can be derived from Proposition 8.1 using the easily established fact that any retract of a metric space X is convex in X.

Theorem 8.6 *Every* ***non****convex closed Euclidean set is magnitude homology equivalent to its core.*

The core of a convex set is the empty space (Examples 5.2(v)), which is magnitude homology equivalent only to itself, so the nonconvexity condition cannot be dropped.

Proof Let $X \subseteq \mathbb{R}^N$ be a nonconvex closed set and $A = \mathrm{core}(X)$, which by Proposition 5.8 is a retract of X. Then apply Proposition 8.5. □

Since the core construction is idempotent (Lemma 5.5(iii)), Theorem 8.6 provides a canonical representative of each magnitude homology equivalence class of nonconvex closed Euclidean sets.

Proposition 8.5 generates many examples of magnitude homology equivalence:

Proposition 8.7 *Let $S \subseteq \mathbb{R}^N$. For nonempty convex closed sets $C \subseteq \mathbb{R}^N$ containing S, the magnitude homology equivalence class of $C \setminus S^\circ$ is independent of the choice of C.*

Proof We show that for every such C, the space $C \setminus S^\circ$ is magnitude homology equivalent to $X = \mathbb{R}^N \setminus S^\circ$. To do this, we apply Proposition 8.5 with $A = C \setminus S^\circ$. It remains to verify that $\rho X \subseteq A$ and that A is a retract of X.

First, $\rho X \subseteq \partial X = \partial S^\circ \subseteq C$, using Remark 4.4. Hence $\rho X \subseteq C \cap X = A$.

Now we show that A is a retract of X. Let π denote metric projection $\mathbb{R}^N \to C$. It is enough to prove that $\pi X \subseteq A$, so let $x \in X$. If $x \in C$ then $\pi(x) = x \in C \cap X = A$. If $x \notin C$ then $\pi(x) \in \partial C$ (a general property of metric projections), which since $S \subseteq C$ implies that $\pi(x) \in C \setminus S^\circ = A$. In either case, $\pi(x) \in A$. □

Figure 4 shows an example of Proposition 8.7.

Corollary 8.8 *For $\varnothing \neq S \subseteq \mathbb{R}^N$, the space $\overline{\mathrm{conv}}(S) \setminus S^\circ$ is magnitude homology equivalent to $\mathbb{R}^N \setminus S^\circ$.*

This follows immediately from Proposition 8.7, and implies in turn:

Corollary 8.9 *The boundary ∂C of a nonempty convex set $C \subseteq \mathbb{R}^N$ is magnitude homology equivalent to $\mathbb{R}^N \setminus C^\circ$.*

9 The main theorem

For our main theorem, recall that a closed Euclidean set is a metric space isometric to a closed subset of $\mathbb{R}^N$ for some $N \geq 0$, and that maps of metric spaces are taken to be short (Definition 2.1).

Theorem 9.1 *Let X and Y be nonempty closed Euclidean sets. The following are equivalent*:

(i) *X and Y are magnitude homology equivalent*; *that is*, *there exist maps* $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ *such that* $H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y)$ *are mutually inverse for all* $n \geq 1$.

(ii) *There exist maps* $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ *such that* $H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y)$ *are mutually inverse for some* $n \geq 1$.

(iii) *There exist maps* $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ *restricting to mutually inverse isometries* $\rho X \rightleftarrows \rho Y$.

(iv) *There exist maps* $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ *restricting to mutually inverse isometries* $\operatorname{core}(X) \rightleftarrows \operatorname{core}(Y)$.

(v) $\operatorname{core}(X)$ *and* $\operatorname{core}(Y)$ *are isometric.*

Most importantly, magnitude homology equivalence (i) is equivalent to the concrete geometric (v).

Proof Conditions (i)–(iii) are equivalent for all aligned spaces, by Theorem 7.10. (iii) $\Longrightarrow$ (iv) follows from Lemma 5.6, and (iv) $\Longrightarrow$ (v) is trivial.

The statement (v) $\Longrightarrow$ (i) follows from Theorem 8.6 in the case where X and Y are both nonconvex. By Examples 5.2(v), the only other possibility is that X and Y are both convex, in which case (i) holds by Example 8.4. □

Remarks 9.2 (i) The equivalent conditions of Theorem 9.1 are strictly stronger than the condition that X and Y have isometric inner boundaries, by Remarks 7.11(i).

(ii) The equivalent conditions of Theorem 9.1 are also strictly stronger than the condition that there is a map $f : X \to Y$ such that $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all $n \geq 1$. (And this in turn is stronger than the condition that $H_n(X) \cong H_n(Y)$ for all $n \geq 1$: Remarks 7.11(iii).)

Indeed, let $X = \{0\} \cup [1,2] \cup \{3\}$ and $Y = \{0,1,2\}$, and define $f : X \to Y$ by $f(0) = 0$, $f[1,2] = \{1\}$ and $f(3) = 2$. For $n \geq 1$, the thin chains in X and Y are given by

$$T_{n,n}(X) = \{(0,1,0,1,\ldots),(1,0,1,0,\ldots),(2,3,2,3,\ldots),(3,2,3,2,\ldots)\},$$
$$T_{n,n}(Y) = \{(0,1,0,1,\ldots),(1,0,1,0,\ldots),(1,2,1,2,\ldots),(2,1,2,1,\ldots)\},$$

and $T_{n,\ell}(X) = \varnothing = T_{n,\ell}(Y)$ when $\ell \neq n$. In all cases, the map $f_\star : \mathbb{Z}T_{n,\ell}(X) \to \mathbb{Z}T_{n,\ell}(Y)$ is a bijection, so $f_* : H_{n,\ell}(X) \to H_{n,\ell}(Y)$ is an isomorphism. However, $\operatorname{core}(X) = X$ and $\operatorname{core}(Y) = Y$, which are not isometric.

In other words, magnitude homology equivalence is a stronger property than quasi-isomorphism in positive degree. Finding a concrete geometric description of quasi-isomorphism for magnitude homology remains an open question.

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