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BRANDON DOHERTY

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Symmetry in the cubical Joyal model structure

BRANDON DOHERTY

We study properties of the cubical Joyal model structures on cubical sets by means of a combinatorial construction which allows for convenient comparisons between categories of cubical sets with and without symmetries. In particular, we prove that the cubical Joyal model structures on categories of cubical sets with connections are cartesian monoidal. Our techniques also allow us to prove that the geometric product of cubical sets (with or without connections) is symmetric up to natural weak equivalence in the cubical Joyal model structure, and to obtain induced model structures for $(\infty, 1)$-categories on cubical sets with symmetries.

1 Introduction

Model structures on categories of *cubical sets* can be used to model higher categories. Much like the more familiar simplicial sets, cubical sets are presheaves on a small indexing category (a "cube category") which can be thought of as collections of cubes in all dimensions. Unlike in the case of simplicial sets, there are many possible choices of this indexing category, giving rise to categories of cubical sets whose cubes are related by various different kinds of structure maps.

In particular, models of ∞-groupoids (the *Grothendieck model structures* — see Cisinski [6; 7]) and $(\infty, 1)$-categories (the *cubical Joyal model structures* — see Doherty, Kapulkin, Lindsey, and Sattler [9]), Quillen equivalent to the standard simplicial models, have been established using *minimal cubical sets* and *cubical sets with connections*. These kinds of cubical sets have a relatively simple set of structure maps: *face maps*, *projections* which play the role of degeneracies, and in the latter case, an additional kind of degeneracy called *connections*. Other cube categories include *symmetries*, automorphisms corresponding to the natural symmetries of cubical shapes.

Precomposition with an inclusion of cube categories $i : \square_A \hookrightarrow \square_B$ defines a forgetful functor between the corresponding categories of cubical sets (which "forgets the additional structure maps of $\square_B$"), having both a left and a right adjoint. The aim of this paper is to study the cubical Joyal model structures by means of such comparison functors, specifically between the categories of cubical sets described above and those with symmetries. In doing so, we will also prove or reprove analogous results involving the Grothendieck model structures.

Our main results involve the *geometric product* of cubical sets, a monoidal product $\otimes$ having the useful property that a geometric product of cubes is again a cube; this is the most convenient monoidal product for parameterizing homotopy in the cubical Joyal model structures. One drawback of the geometric product, at least for the kinds of cubical sets on which cubical Joyal model structures have been established, is that

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it is not symmetric — in general, $X \otimes Y$ is not isomorphic to $Y \otimes X$. Our first main result establishes that the geometric product is "symmetric up to natural weak equivalence", for both minimal cubical sets and those with connections.

Theorem (see Theorem 4.3) *For cubical sets X and Y, the geometric products $X \otimes Y$ and $Y \otimes X$ are naturally weakly equivalent.*

Our results also allow us to obtain alternative models of $(\infty, 1)$-categories via model structures on categories of cubical sets with symmetries.

Theorem (see Theorem 4.7) *Cubical sets with symmetries and connections admit model structures for $(\infty, 1)$-categories left- and right-induced by the forgetful functor to minimal cubical sets, and Quillen equivalent to the cubical Joyal model structures.*

We also study the cartesian product of cubical sets and its relationship with the geometric product. In the absence of connections, the cartesian product is not homotopically well behaved (see Jardine [19, Remark 3.5] for instance). On the other hand, cubical sets with connections form a strict test category, implying compatibility of the cartesian product with the Grothendieck model structure. Our next main result generalizes this observation to the setting of $(\infty, 1)$-categories.

Theorem (see Corollary 5.10 and Theorem 5.12) *The cubical Joyal model structure on cubical sets with connections is monoidal with respect to the cartesian product. Moreover, the geometric and cartesian products are naturally weakly equivalent in the cubical Joyal model structure.*

The key combinatorial construction underlying all of our results is that of the *standard decomposition cubes*, which allow for certain inclusions of cubical sets to be expressed as transfinite composites of open-box fillings. Though similar in spirit to the open-box filling techniques used by Doherty [8] to analyze the relationship between minimal cubical sets and those with connections, this construction does not rely on intricate analysis of standard factorizations into generators of maps in cube categories, as the techniques of that paper did. Instead, it can be described by simple formulas (see Definition 3.12 and Lemma 3.13), and admits a more intuitive conceptual understanding (see Examples 3.16).

Related work

Model structures for ∞-groupoids on categories of cubical sets with symmetries have been the subject of considerable study, much of it motivated by applications to type theory; see, for instance, Isaacson [18], Awodey [1], Cavallo and Sattler [5], and Awodey, Cavallo, Coquand, Riehl and Sattler [2]. Induced model structures on one particular category of cubical sets with symmetries, for both ∞-groupoids and $(\infty, 1)$-categories, have been constructed by Hackney and Rovelli [14]; the model structures in that reference are induced by a functor from cubical to simplicial sets, whereas ours are induced by forgetful functors from symmetric cubical sets to those without symmetries.

The topic of the compatibility of cartesian products of cubical sets with model structures for ∞-groupoids has previously been studied in terms of the theory of strict test categories; see in particular the work of Maltsiniotis [21], in which it is shown that the cube category with faces, projections and connections is a strict test category, and that of Buchholtz and Morehouse [3], in which this result is extended to cube categories with additional structure maps.

Organization of the paper

In Section 2, we define the categories of cubical sets under consideration and discuss some of the basic constructions used in their study, including the geometric product. Most of this is background material, but we will also define some concepts of specific relevance to the present work and prove some basic combinatorial lemmas which will be of use in later sections. In Section 3, we introduce the adjoint triples $i_! \dashv i^* \dashv i_*$ which relate different categories of cubical sets, as well as the standard decomposition cubes which allow the units of the adjunctions $i_! \dashv i^*$ to be analyzed by open-box filling. In particular, Proposition 3.25 is our key technical result, of which all of this paper's main theorems are consequences. In Section 4 we apply Proposition 3.25 to show that, given certain conditions on the categories of cubical sets under consideration, the unit of the comparison adjunction $i_! \dashv i^*$ is a natural trivial cofibration in the cubical Joyal model structure. Using this, we show that the geometric product is symmetric up to natural weak equivalence, and construct induced model structures for $(\infty, 1)$-categories and ∞-groupoids on cubical sets with symmetries. In Section 5, we further apply Proposition 3.25 to show that the cubical Joyal model structure on cubical sets with connections is cartesian monoidal, and is naturally weakly equivalent to the geometric product.

2 Background

2.1 Categories of cubical sets

We begin by introducing the categories which will be our primary objects of study.

Definition 2.1 The *full cube category* $\Box_S$ is the full subcategory of Set on the objects $\{0,1\}^n, n \geq 0$. This category is generated under composition by the following classes of maps (where $\wedge$ and $\vee$, respectively, denote the minimum and maximum functions $\{0,1\}^2 \to \{0,1\}$):

- *faces* $\partial^n_{i,\varepsilon} : \{0,1\}^{n-1} \to \{0,1\}^n$ for $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ given by

$$\partial^n_{i,\varepsilon}(a_1, a_2, \ldots, a_{n-1}) = (a_1, a_2, \ldots, a_{i-1}, \varepsilon, a_i, \ldots, a_{n-1});$$

- *diagonals* $\delta^n_i : \{0,1\}^{n-1} \to \{0,1\}^n$ for $i = 1, \ldots, n-1$ given by

$$\delta^n_i(a_1, a_2, \ldots, a_{n-1}) = (a_1, a_2, \ldots, a_{i-1}, a_i, a_i, a_{i+1}, \ldots, a_{n-1});$$

- *projections* $\sigma^n_i : \{0,1\}^{n+1} \to \{0,1\}^n$ for $i = 1, 2, \ldots, n+1$ given by

$$\sigma^n_i(a_1, a_2, \ldots, a_{n+1}) = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1});$$

- *positive connections* $\gamma^n_{i,1} : \{0,1\}^{n+1} \to \{0,1\}^n$ for $i = 1, 2, \ldots, n$ given by

$$\gamma^n_{i,1}(a_1, a_2, \ldots, a_{n+1}) = (a_1, a_2, \ldots, a_{i-1}, a_i \wedge a_{i+1}, a_{i+2}, \ldots, a_{n+1});$$

- *negative connections* $\gamma^n_{i,0} : \{0,1\}^{n+1} \to \{0,1\}^n$ for $i = 1, 2, \ldots, n$ given by

$$\gamma^n_{i,0}(a_1, a_2, \ldots, a_{n+1}) = (a_1, a_2, \ldots, a_{i-1}, a_i \vee a_{i+1}, a_{i+2}, \ldots, a_{n+1});$$

- *transpositions* $\lambda^n_i : \{0,1\}^n \to \{0,1\}^n$ for $i = 1, 2, \ldots, n-1$ given by

$$\lambda^n_i(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n);$$

- *reversals* $\rho^n_i : \{0,1\}^n \to \{0,1\}^n$ for $i = 1, 2, \ldots, n$ given by

$$\rho^n_i(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_{i-1}, 1 - a_i, a_{i+1}, \ldots, a_n).$$

For clarity, the superscript n will typically be omitted when it is irrelevant or clear from context.

Analogously to the familiar simplicial identities, one could describe the composition of these maps in terms of *cubical identities*, as is done, for instance, in [13] for the faces, projections and connections. As this would be both unwieldy and unnecessary for the present work, we will not produce a full list of cubical identities here; instead, we simply record a few identities which will be of particular use in our calculations, all of which can easily be derived from the definitions above.

Lemma 2.2 *The maps above satisfy the identities*

- $\sigma_i \partial_{i,\varepsilon} = \mathrm{id}$;
- $\gamma_{i,\varepsilon}\partial_{i,\varepsilon} = \gamma_{i,\varepsilon}\partial_{i+1,\varepsilon} = \mathrm{id}$;
- $\gamma_{i,\varepsilon}\partial_{i,1-\varepsilon} = \gamma_{i,\varepsilon}\partial_{i+1,1-\varepsilon} = \partial_{i,1-\varepsilon}\sigma_i$. □

A *cube category* is a subcategory of $\square_S$ generated by some subset of the generating classes above which includes the faces and projections. We denote cube categories by $\square_A$, where A is a (possibly empty) subset of the set of symbols $\{\wedge, \vee, \Sigma, \rho, \delta\}$. For such a subset, $\square_A$ denotes the cube category generated by faces and projections, as well as

- positive connections if $\wedge \in A$;
- negative connections if $\vee \in A$;
- transpositions if $\Sigma \in A$;
- reversals if $\rho \in A$;
- diagonals if $\delta \in A$.

By composing transpositions, we obtain all automorphisms of the posets $\{0 \leq 1\}^n$, which act by permutation of coordinates. We will refer to these maps as *symmetries*, and speak of a cube category "containing symmetries" rather than "containing transpositions".

In this context we let $\mathcal{S}$ denote the entire set $\{\wedge, \vee, \Sigma, \rho, \delta\}$, so that an arbitrary cube category can be written as $\Box_A$ for $A \subseteq \mathcal{S}$. For ease of notation, when writing subsets of $\mathcal{S}$ in subscripts we will not use braces or commas, so that, for instance, the cube category with both kinds of connections is denoted by $\Box_{\wedge\vee}$. When viewing $\{0,1\}^n$ as an object of $\Box_A$, we will denote it by $\Box_A^n$.

Note that the positive connections and reversals together generate the negative connections, and vice versa; thus we will implicitly assume, when dealing with an arbitrary subset $A \subseteq \mathcal{S}$, that if A contains both ρ and $\wedge$ (resp. $\vee$) then A contains $\vee$ (resp. $\wedge$) as well.

The category of presheaves on $\Box_A$, i.e., contravariant functors from $\Box_A$ to Set, will be denoted by cSet_A. Objects in such a category are called *cubical sets*, and may be viewed as collections of cubes in each dimension; for $X \in \mathsf{cSet}$, the image under X of $\Box_A^n$ is denoted by X_n, and thought of as the set of n-cubes of X. By a standard abuse of notation, we write the representable presheaf $\Box_A(-, \Box_A^n)$ as $\Box_A^n$. We refer to this object as the *n-cube* in cSet_A; this may be viewed as consisting of a single cube of dimension n and all of its faces.

Examples 2.3 We list specific examples of notable cube categories.

- $\Box_\varnothing$ is the *minimal cube category*, having only faces and projections. The corresponding category of cubical sets $\mathsf{cSet}_\varnothing$ was originally studied by Kan [20], and more recently by Cisinski [6].
- $\Box_{\wedge\vee\Sigma\delta}$ is the subcategory of maps which respect the partial ordering on $\{0,1\}^n$ induced by the ordering on $\{0,1\}$ given by $0 \le 1$; in other words, it is the full subcategory of Poset on the powers of the interval $[1] = \{0 \le 1\}$. As this cube category is of particular interest, we will denote it $\Box_{\mathcal{P}}$ for ease of reference (so $\mathcal{P} = \{\wedge, \vee, \Sigma, \delta\}$).

Before continuing, we should comment on certain ambiguities of terminology. In many sources, in particular [9], the term "degeneracy" is used to refer specifically to the projections. Our terminology instead follows [4], as it is convenient to reserve "degeneracy" for a catch-all term for the strictly degree-decreasing maps in the Eilenberg–Zilber structures to be described in Section 2.3. Likewise, in many sources, the term "diagonal" refers to the maps $\{0,1\}^n \to \{0,1\}^{2n}$ which repeat an entire string of coordinates, i.e., those which send $(a_1, \ldots, a_n)$ to $(a_1, \ldots, a_n, a_1, \ldots, a_n)$; the well-known observation that diagonals and projections together generate the symmetries refers to these diagonals, not to those specified in our list above. Our usage instead follows [3], as we wish to regard the diagonals and symmetries as distinct generating classes.

We will denote the n-tuple $(0, \ldots, 0)$ by $\vec{0}_n$; likewise $(1, \ldots, 1)$ will be denoted by $\vec{1}_n$. When there is no risk of confusion we will omit the subscripts and simply write $\vec{0}$, $\vec{1}$.

We will occasionally represent cubical sets visually. For a 1-cube f, we draw

$$x \xrightarrow{f} y$$

to indicate $x = f\partial_{1,0}$ and $y = f\partial_{1,1}$. For a 2-cube s, we draw

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ z \arrow[r, "k"] & w \end{tikzcd}$$

to indicate $s\partial_{1,0} = f$, $s\partial_{1,1} = g$, $s\partial_{2,0} = h$, and $s\partial_{2,1} = k$. As for the convention when drawing 3-dimensional cubes, we use the ordering of axes

$$\begin{tikzcd} \cdot \arrow[rr, "1"] \arrow[dr, "3"] \arrow[ddd, "2"'] & & \cdot \\ & \cdot & \\ & & \\ \cdot & & \end{tikzcd}$$

For readability, we do not label 2- and 3-cubes.

Lastly, a degenerate 1-cube $x\sigma_1$ on x is represented by

$$\begin{tikzcd} x \arrow[r, equal] & x, \end{tikzcd}$$

while a 2- or 3-cube whose boundary agrees with that of a degenerate cube is assumed to be degenerate unless indicated otherwise. For instance, a 2-cube depicted as

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, "f"'] & x \arrow[d, "f"] \\ y \arrow[r, equal] & y \end{tikzcd}$$

represents $f\sigma_1$.

2.2 The geometric product

The cartesian product on Set restricts to a monoidal product $\otimes$ on each category $\square_A$. The faithfulness of the cartesian product implies the following useful result.

Lemma 2.4 *For any $A \subseteq \mathcal{S}$, the functor $\otimes : \square_A \times \square_A \to \square_A$ is faithful.* $\square$

We may extend this to a monoidal product on cSet_A, called the *geometric product*, by Day convolution. In other words, we define the geometric product $\otimes : \mathsf{cSet}_A \times \mathsf{cSet}_A \to \mathsf{cSet}_A$ by left Kan extension, as depicted below:

$$\begin{tikzcd} \square_A \times \square_A \arrow[r, "\otimes"] \arrow[d] & \square_A \arrow[r] & \mathsf{cSet}_A \\ \mathsf{cSet}_A \times \mathsf{cSet}_A \arrow[urr, "\otimes"'] & & \end{tikzcd}$$

The unit of this monoidal structure is the 0-cube $\square^0_A$.

From this construction we obtain a formula for the geometric product $X \otimes Y$ as a colimit over the cubes of X and Y:

$$X \otimes Y = \operatorname*{colim}_{\substack{x : \square^m_A \to X \\ y : \square^n_A \to Y}} \square^{m+n}_A.$$

Given a pair of cubes $x : \square_A^m \to X$, $y : \square_A^n \to Y$, the component of the colimit cone corresponding to the pair (x, y) is a cube $\square_A^{m+n} \to X \otimes Y$, which we denote by $x \otimes y$. As the notation suggests, this is also the image of (x, y) under the bifunctor $\otimes : \mathsf{cSet}_A \times \mathsf{cSet}_A \to \mathsf{cSet}_A$. The colimit formula above allows us to obtain the following characterization of $X \otimes Y$, generalizing [9, Proposition 1.24].

Proposition 2.5 *For $X, Y \in \mathsf{cSet}_A$, every cube of $X \otimes Y$ is of the form $(x \otimes y)\phi$ for some $x : \square_A^m \to X$, some $y : \square_A^n \to Y$, and some $\phi : \square_A^k \to \square_A^{m+n}$. Moreover, these cubes are subject to this identification: for x, y as above and $\psi : \square_A^{m'} \to \square^m$, $\psi' : \square_A^{n'} \to \square_A^n$, we have $(x \otimes y)(\psi \otimes \psi') = (x\psi) \otimes (y\psi')$.* □

Note that for most cube categories $\square_A$, the geometric product of cubical sets does not coincide with the cartesian product, and is not even a symmetric monoidal product. However, it is symmetric if $\Sigma \in A$, and coincides with the cartesian product if $\Sigma, \delta \in A$. (These results are well known, but we will reprove the latter as Proposition 5.6.)

In cases where the two do not coincide, the geometric product is better behaved than the cartesian product — for instance, in every cube category $\square_A$, for $m, n \geq 0$ we have $\square_A^m \otimes \square_A^n = \square_A^{m+n}$, while the analogous isomorphism generally does not hold for the cartesian product. Moreover, in the absence of connections, a cartesian product of cubes may not even have the homotopy type of a contractible space (see [19, Remark 3.5]).

Given a cubical set X, the left tensor $- \otimes X$ and the right tensor $X \otimes -$ define two functors $\mathsf{cSet}_A \to \mathsf{cSet}_A$; as discussed above, these are not isomorphic unless $\square_A$ contains symmetries. Both of these functors admit right adjoints, and we write $\underline{\hom}_L(X, -)$ for the right adjoint of the left tensor and $\underline{\hom}_R(X, -)$ for the right adjoint of the right tensor. (Of course, in the symmetric case these are naturally isomorphic.) Explicitly, these functors are given by $\underline{\hom}_L(X, Y)_n = \mathsf{cSet}(\square^n \otimes X, Y)$, $\underline{\hom}_R(X, Y)_n = \mathsf{cSet}(X \otimes \square^n, Y)$. Thus the monoidal structure on cSet_A given by the geometric product is closed.

2.3 Combinatorics of cubical sets

We next discuss the convenient combinatorial properties of certain categories of cubical sets, which are of use in studying their homotopy theory.

Proposition 2.6 [4, Corollary 7.9] *If $\delta \notin A$, then $\square_A$ is an Eilenberg–Zilber category, with the following structure:*

- *for all n, $\deg([1]^n) = n$;*
- *$(\square_A)_+$ is generated under composition by the faces, symmetries, and reversals;*
- *$(\square_A)_-$ is generated under composition by the projections, connections, symmetries, and reversals.* □

In keeping with standard usage in the theory of Eilenberg–Zilber categories, we will refer to the nonisomorphisms of $(\square_A)_-$ as *degeneracies*. Given a cubical set X and an m-cube $x \in X_m$, we say x is *degenerate* if it is in the image of the structure map $X_n \to X_m$ induced by some degeneracy $\square_A^m \to \square_A^n$, and *nondegenerate* otherwise.

Lemma 2.7 [9, Proposition 1.18] *Let $X \in \mathsf{cSet}_A$ for $A \subseteq \{\wedge, \vee\}$. Then each cube $x : \Box^n_A \to X$ can be written as $y\phi$ for a unique nondegenerate cube $y : \Box^m_A \to X$ and a unique map $\phi : \Box^n_A \to \Box^m_A$ in $(\Box_A)_-$.* □

In cubical sets with symmetries or reversals, but without diagonals, as well as those with symmetries and diagonals but without connections, an analogue of Lemma 2.7 holds, up to isomorphism; see [4, Theorems 7.9 and 8.12]. In our present work, however, we will have no need of this stronger result.

The following consequence of Lemma 2.7 will be of use in recognizing nondegenerate cubes.

Corollary 2.8 *Let $X \in \mathsf{cSet}_A$ with $A \subseteq \{\wedge, \vee\}$ and $x : \Box^m_A \to X$, $y : \Box^{m-1}_A \to X$, with x degenerate and y nondegenerate. Then it is not the case that $x\partial_{i,\varepsilon} = y$ for exactly one face map $\partial_{i,\varepsilon}$.*

Proof We note that by Lemma 2.7, we have $x = z\phi$ for a unique degeneracy $\phi : \Box^m_A \to \Box^n_A$ and a unique nondegenerate n-cube $z : \Box^n_A \to X$. In the case $n < m - 1$, consider a face $x\partial_{i,\varepsilon} = z\phi\partial_{i,\varepsilon}$; our assumption on m implies that $\phi\partial_{i,\varepsilon} : \Box^{m-1}_A \to \Box^n_A$ strictly decreases degree. By Proposition 2.6, we can factor $\phi\partial_{i,\varepsilon}$ as $\kappa\phi'$ for some κ in $(\Box_A)_+$ and ϕ' in $(\Box_A)_-$. Since $\kappa\phi'$ strictly decreases degree, ϕ' must be a degeneracy. Thus we see that in this case, all faces of x are degenerate.

In the case $m = n + 1$, then either $x = z\sigma_i$ for some $1 \le i \le n$, or $x = z\gamma_{i,\varepsilon}$ for some $1 \le i \le n - 1$ and $\varepsilon \in \{0, 1\}$. In this case, Lemma 2.2 shows that two distinct faces of x are equal to z; a straightforward calculation shows that all other faces of x are degenerate. □

In many cube categories of interest, there are unique factorizations of maps into standard forms, i.e., composites of generators satisfying detailed combinatorial characterizations which facilitate computation; see, for instance, [4, Fact 7.2; 13]. In the case of cubical sets with diagonals, such factorizations are not available, owing to that category's less convenient combinatorial properties (see [4, Theorem 8.12]). Nevertheless, we may obtain factorizations in all cube categories which will be of use in studying the corresponding categories of cubical sets; we now define the classes of maps which will appear in these factorizations.

Definition 2.9 For $A \subseteq \mathcal{S}$, $1 \le i \le n$, and $\varepsilon \in \{0, 1\}$, a map $\phi : \Box^m_A \to \Box^n_A$ in $\Box_A$ *fixes coordinate i at ε* if $\phi(\vec{a})_i = \varepsilon$ for all $\vec{a} \in \Box^m_A$. A morphism $\phi : \Box^m_A \to \Box^n_A$ in $\Box_A$ is *active* if it does not fix any coordinate.

Note that the definition of an active map in $\Box_A$ does not depend on A. Among the generating classes of maps listed in Definition 2.1, all except for the faces are active.

Lemma 2.10 *For $A \subseteq \mathcal{P}$, a map $\phi : \Box^m_A \to \Box^n_A$ in $\Box_A$ fixes coordinate i at 0 if and only if $\phi(\vec{1})_i = 0$, and fixes coordinate i at 1 if and only if $\phi(\vec{0})_i = 1$.*

Proof The "only if" direction is immediate from Definition 2.9. For the "if" direction, suppose $\phi(\vec{0})_i = 1$; then because ϕ is a poset map and $\vec{0}$ is minimal, for every $\vec{a} \in \Box^m_A$ we have $\phi(\vec{0}) \le \phi(\vec{a})$, implying in particular that $\phi(\vec{a})_i = 1$. The proof for the case where $\phi(\vec{1})_i = 0$ is similar. □

Corollary 2.11 *For $A \subseteq \mathcal{P}$, a map $\phi : \square^m_A \to \square^n_A$ in $\square_A$ is active if and only if it preserves the initial and terminal elements, i.e., $\phi(\vec{0}_m) = \vec{0}_n$ and $\phi(\vec{1}_m) = \vec{1}_n$.* □

Lemma 2.12 *For $A \subseteq \mathcal{S}$, every map $\phi : \square^m_A \to \square^n_A$ in $\square_A$ factors as $\partial\psi$, where ψ is active and ∂ is a (possibly empty) composite of face maps, with both ψ and ∂ unique. Moreover, $\partial = \partial_{i_1,\varepsilon_1} \cdots \partial_{i_p,\varepsilon_p}\psi$, where ψ is a unique active map and $i_1 > \cdots > i_p$ is the set of coordinates fixed by ϕ, with each i_k fixed at ε_k.*

Proof We will first prove the existence of the stated factorization. We define $\phi' : \square^m_A \to \square^n_A$ as the composite $\partial_{i_1,\varepsilon_1} \cdots \partial_{i_p,\varepsilon_p}\sigma_{i_p} \cdots \sigma_{i_1}\phi$; we will show that this is equal to ϕ. We may note that the map $\partial_{i_1,\varepsilon_p} \cdots \partial_{i_p,\varepsilon_p}\sigma_{i_p} \cdots \sigma_{i_1} : \square^n_A \to \square^n_A$ sends a tuple $\vec{a} \in \square^n_A$ to the tuple obtained by replacing $\vec{a}_{i_k}$ with ε_k for each $1 \leq k \leq p$. Thus, for any $\vec{a} \in \square^m_A$ we have $\phi'_{i_k} = \varepsilon_k$ for all $1 \leq k \leq p$; by assumption this means $\phi'(\vec{a})_{i_k} = \phi(\vec{a})_{i_k}$. For j not equal to any i_k we likewise have $\phi'(\vec{a})_j = \phi(\vec{a})_j$ as these coordinates are unaffected by composition with $\partial_{i_1,\varepsilon_1} \cdots \partial_{i_p,\varepsilon_p}\sigma_{i_p} \cdots \sigma_{i_1}$. Thus $\phi = \phi'$.

Now let $\psi = \sigma_{i_p} \cdots \sigma_{i_1}\phi$, so that $\phi = \partial_{i_1,\varepsilon_1} \cdots \partial_{i_p,\varepsilon_p}\psi$. This map does not fix any coordinate, since $\sigma_{i_p} \cdots \sigma_{i_1}$ projects away all the coordinates fixed by ϕ. Thus ψ is active.

To see that this factorization is unique, suppose that $\partial\psi = \partial'\psi'$, where ψ and ψ' are active and ∂ and ∂' are composites of face maps. By [13, Lemma 4.1], we may factor ∂ as $\partial_{i_1,\varepsilon_1} \cdots \partial_{i_p,\varepsilon_p}$ for some $i_1 > \cdots > i_p$ and $\varepsilon_1, \ldots, \varepsilon_p \in \{0,1\}$, and may similarly factor ∂' as $\partial_{i'_1,\varepsilon'_1} \cdots \partial_{i'_p,\varepsilon'_p}$. Then the set of coordinates fixed by $\partial\psi$ is precisely $\{i_1, \ldots, i_p\}$, with each i_k fixed at ε_k, and similarly the set of coordinates fixed by $\partial'\psi'$ is precisely $\{i'_1, \ldots, i'_{p'}\}$, with each i'_k fixed at ε'_k. Since $\partial\psi = \partial'\psi'$ by assumption, we thus see that the sets $\{i_1, \ldots, i_p\}$ and $\{i'_1, \ldots, i'_{p'}\}$ are equal; from this it follows that $p = p'$, and that $i_k = i'_k$ and $\varepsilon_k = \varepsilon'_k$ for all k. Thus $\partial = \partial'$; since all face maps are monomorphisms, it follows that $\psi = \psi'$. □

Corollary 2.13 *For $n \geq 2$, $1 \leq i < j \leq n$, and $\varepsilon, \varepsilon' \in \{0,1\}$, a map $\phi : \square^m \to \square^n$ factors through both $\partial_{i,\varepsilon}$ and $\partial_{j,\varepsilon'}$ if and only if it factors through the composite map $\partial_{j,\varepsilon'}\partial_{i,\varepsilon} = \partial_{i,\varepsilon}\partial_{j-1,\varepsilon'}$. In other words, the diagram*

$$\begin{tikzcd}
\square^{n-2}_A \arrow[r, "\partial_{j-1,\varepsilon'}"] \arrow[d, "\partial_{i,\varepsilon}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \square^{n-1}_A \arrow[d, "\partial_{i,\varepsilon}"] \\
\square^{n-1}_A \arrow[r, "\partial_{j,\varepsilon'}"] & \square^n_A
\end{tikzcd}$$

is a pullback. □

For a general $A \subseteq \mathcal{S}$, we write $\partial\square^n_A$ for the union of all proper faces of $\square^n_A$, i.e., the subobject of $\square^n_A$ consisting of all maps into $\square^n_A$ which factor through a face map on the left. We refer to $\partial\square^n_A$ as the *boundary* of the n-cube. Note that for cube categories not containing diagonals, this coincides with the standard definition of the boundary of a representable presheaf on an Eilenberg–Zilber category when cSet_A is equipped with the Eilenberg–Zilber structure of Proposition 2.6. The subobject of $\square^n_A$ given by the union of all faces except $\partial_{i,\varepsilon}$ will be denoted by $\sqcap^n_{A,i,\varepsilon}$ and referred to as the (i,ε)-*open box* of dimension n.

Our factorization results allow us to obtain an explicit description of boundaries and open boxes as coequalizers of coproducts of cubes of lower dimension, analogous to the characterization of boundaries of simplices given in [12, Section I.2].

Proposition 2.14 *In any cube category $\square_A$, for $n \geq 2$, the diagram*

$$\coprod_{\substack{1 \leq i < j \leq n \\ \varepsilon, \varepsilon' \in \{0,1\}}} \square_A^{n-2} \underset{\iota_{i,\varepsilon}\partial_{j-1,\varepsilon'}}{\overset{\iota_{j,\varepsilon'}\partial_{i,\varepsilon}}{\rightrightarrows}} \coprod_{\substack{1 \leq i \leq n \\ \varepsilon \in \{0,1\}}} \square_A^{n-1} \xrightarrow{\partial_{i,\varepsilon}} \partial\square_A^n$$

is a coequalizer. Moreover, the inclusion $\partial\square_A^n \hookrightarrow \square_A^n$ is the map from the coequalizer induced by the map

$$\coprod_{\substack{1 \leq i \leq n \\ \varepsilon \in \{0,1\}}} \square_A^{n-1} \to \square_A^n$$

acting as $\partial_{i,\varepsilon}$ on component (i, ε).

A similar description holds for any open box $\sqcap_{A,i,\varepsilon}^n$ where the coproducts are taken over all faces (or pairs of faces) other than $\partial_{i,\varepsilon}$.

Proof By a standard result, for any set X and family of subsets $(X_i)_{i \in I}$, there is a coequalizer diagram

$$\coprod_{\substack{i,j \in I \\ i \neq j \\ X_i \cap X_j \neq \varnothing}} X_i \cap X_j \rightrightarrows \coprod_{i \in I} X_i \longrightarrow \bigcup_{i \in I} X_i,$$

where one of the two morphisms between coproducts acts on the (i, j) component as the inclusion of $X_i \cap X_j$ into X_i composed with the coproduct inclusion at i, while the other acts as the inclusion into X_j composed with the coproduct inclusion at j. Moreover, in such cases the inclusion $\bigcup_{i \in I} X_i \hookrightarrow X$ is the coequalizer map induced by the map $\coprod_{i \in I} X_i \to X$ which acts on the i component as the inclusion of X_i.

An analogous result thus holds in any presheaf category for unions of subcomplexes of a given presheaf. In particular, for $n \geq 0$ we may apply this result to $\partial\square_A^n$ or $\sqcap_{A,i,\varepsilon}^n$, viewed as a union of faces of $\square_A^n$. In light of Corollary 2.13 (and the fact that for any i the $(i, 0)$- and $(i, 1)$-faces of $\square_A^n$ have empty intersection) we thus obtain the coequalizer diagram given in the statement. □

2.4 Model structures on cubical sets

Next we review certain model structures on categories of the form cSet_A with $A \subseteq \{\wedge, \vee\}$ which model the theories of ∞-groupoids and $(\infty, 1)$-categories. We begin with a model structure for ∞-groupoids, due to Cisinski.

Definition 2.15 A map of cubical sets is a *Kan fibration* if it has the right lifting property with respect to all open box fillings. A cubical set X is a *cubical Kan complex* if the map $X \to \square_A^0$ is a Kan fibration.

Theorem 2.16 (Cisinski) *For $A \subseteq \{\wedge, \vee\}$, the category cSet_A carries a cofibrantly generated model structure, referred to as the Grothendieck model structure, in which the cofibrations are the monomorphisms* *and the fibrations are the Kan fibrations. In particular, the fibrant objects of this model structure are the cubical Kan complexes.*

In all of these cases, the Grothendieck model structure on cSet_A *is the test model structure which arises from the fact that* $\Box_A$ *is a test category.*

Proof For the case $A = \varnothing$, see [6, Theorem 8.4.38]. For other $A \subseteq \{\wedge, \vee\}$, see [7, Theorem 1.17]. (The proof in this reference is for $A = \{\vee\}$, but the proof for other A is identical.) $\Box$

The definition of the fibrations in this model structure implies that the open box fillings form a set of generating trivial cofibrations; maps in the saturation of this set will also be referred to as *anodyne maps*.

We next consider models of $(\infty, 1)$-categories. In [9], it was shown that each of the categories cSet_A for $A \subseteq \{\wedge, \vee\}$ carries an analogue of the Joyal model structure on the category sSet of simplicial sets. To describe these model structures, we must first introduce certain concepts which play a key role in their definition; all of these definitions will be identical regardless of the specific cube category under consideration.

Definition 2.17 We recall standard terminology of the theory of cubical models of $(\infty, 1)$-categories.

- The *critical edge* of $\Box^n_A$ with respect to a face $\partial_{i,\varepsilon}$, denoted by $\partial^c_{i,\varepsilon}$, is the edge $\Box^1_A \to \Box^n_A$ identified under the isomorphisms $\Box^1_A \cong \Box^0_A \otimes \Box^1_A \otimes \Box^0_A$ and $\Box^n_A \cong \Box^{i-1}_A \otimes \Box^1_A \otimes \Box^{n-i}_A$ with $\overrightarrow{(1-\varepsilon)}_{i-1} \otimes \mathrm{id} \otimes \overrightarrow{(1-\varepsilon)}_{n-i}$. In other words, this is the edge between the two vertices of the n-cube having all coordinates other than the i-th equal to $1-\varepsilon$.

- For $n \geq 2$, $1 \leq i \leq n$, $\varepsilon \in \{0, 1\}$, the (i, ε)-*inner open box*, denoted by $\widehat{\sqcap}^n_{A,i,\varepsilon}$, is the quotient of $\sqcap^n_{A,i,\varepsilon}$ in which the critical edge $\partial^c_{i,\varepsilon}$ is made degenerate. The (i, ε)-*inner cube*, denoted by $\widehat{\Box}^n_{A,i,\varepsilon}$, is defined similarly. The (i, ε)-*inner open box inclusion* is the inclusion $\widehat{\sqcap}^n_{A,i,\varepsilon} \hookrightarrow \widehat{\Box}^n_{A,i,\varepsilon}$.

- The *invertible interval* K is the cubical set depicted below:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, equal] & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d, equal] \\
1 \arrow[r, equal] & 1 \arrow[r] & 0
\end{tikzcd}$$

- The class of *inner anodyne maps* is the saturation of the set of inner open box inclusions.

- An *inner fibration* is a map having the right lifting property with respect to the inner open box inclusions.

- An *isofibration* is a map having the right lifting property with respect to the endpoint inclusions $\Box^0_A \hookrightarrow K$.

- A *cubical quasicategory* is a cubical set X such that the map $X \to \Box^0$ is an inner fibration.

Theorem 2.18 [9] *For* $A \subseteq \{\wedge, \vee\}$, *the category* cSet_A *carries a model structure, referred to as the cubical Joyal model structure, in which*

- *the cofibrations are the monomorphisms*;
- *the fibrant objects are the cubical quasicategories*;
- *a map between fibrant objects is a fibration if and only if it is both an inner fibration and an isofibration.* $\Box$

From this, we can see that the inner open box inclusions and the endpoint inclusions $\square^0_A \hookrightarrow K$ form a set of *pseudogenerating trivial cofibrations*, i.e., that fibrations with fibrant codomain are characterized by the right lifting property with respect to these maps.

In both the Grothendieck and cubical Joyal model structures, cylinder objects may be constructed using the geometric product. In the Grothendieck model structure, a cylinder on X is given by $X \otimes \square^1_A$, while in the cubical Joyal model structure, a cylinder on X is given by $X \otimes K$. This gives rise to a natural concept of homotopy in each model structure; by standard constructions, weak equivalences can then be defined as maps inducing isomorphisms on homotopy classes of maps into fibrant objects.

More generally, we have the following result demonstrating the compatibility of these model structures on cubical sets with the geometric product.

Theorem 2.19 *For $A \subseteq \{\wedge, \vee\}$, the Grothendieck and cubical Joyal model structures on cSet_A are monoidal with respect to the geometric product.*

Proof For the Grothendieck model structure on $\mathsf{cSet}_\vee$, this is part of [7, Theorem 1.7]; identical proofs apply for other A. For the cubical Joyal model structure, this is [9, Corollary 4.11]. □

The following result will allow us to prove many statements about the Grothendieck model structure as immediate consequences of analogous statements about the cubical Joyal model structure.

Proposition 2.20 [9, Proposition 4.23(i)] *For $A \subseteq \{\wedge, \vee\}$, the Grothendieck model structure on cSet_A is a localization of the cubical Joyal model structure. In particular, every weak equivalence in the cubical Joyal model structure is a weak equivalence in the Grothendieck model structure.* □

3 Comparisons between categories of cubical sets

Our task in this section is to develop the technical tools which we will use to relate the categories of cubical sets on which the Grothendieck and cubical Joyal model structures have been established to those having symmetries or diagonals. In Section 3.1, we introduce the adjoint triples which will be used in these comparisons, and prove some of their basic properties. In Section 3.2 we describe our main combinatorial construction, the *standard decomposition cubes*, and analyze some of their basic properties. In Section 3.3, we show how the standard decomposition cubes can be used to prove that certain morphisms of cubical sets are trivial cofibrations, and we discuss some basic consequences of this result, including that the geometric product of cubical sets is symmetric up to a zigzag of natural weak equivalences in the cubical Joyal model structure.

3.1 The left Kan extension functor

For $A \subseteq B \subseteq \mathcal{S}$ we have an inclusion of cube categories $i : \square_A \hookrightarrow \square_B$. This induces a precomposition functor on the corresponding presheaf categories, denoted by $i^* : \mathsf{cSet}_B \to \mathsf{cSet}_A$. One may view this as a forgetful functor: for $X \in \mathsf{cSet}_B$, the cubical set $i^* X \in \mathsf{cSet}_A$ has the same set of cubes in each dimension as X, but lacks the structure maps corresponding to morphisms of $\square_B$ which are not present in $\square_A$. For instance, if $A = \varnothing$ and $B = \{\wedge\}$, then the connections of a cubical set $X \in \mathsf{cSet}_\wedge$ still appear as cubes of i^*X, but can no longer be identified as connections; they are, in general, nondegenerate.

The operations of left and right Kan extension define left and right adjoint functors to i^*, respectively; thus we obtain an adjoint triple $i_! \dashv i^* \dashv i_*$. In [8], these adjoint triples are considered in the setting of model structures for (∞, n)-categories on categories cSet_A^+ of cubical sets with markings. The techniques used in that paper, and results thereby obtained, generalize easily to the unmarked setting.

Specifically, the key idea of the proof of [8, Proposition 2.2] is to show that for any marked cubical set X, the unit $\eta_X : X \to i^*i_!X$ is anodyne, by verifying that it has the left lifting property with respect to *comical fibrations*, the naive fibrations of the model structures studied in that paper, and this is done by explicitly constructing lifts via filling of open boxes having specified faces marked. In the setting of unmarked cubical sets modeling $(\infty, 1)$-categories or ∞-groupoids, one can likewise verify that η_X has the right lifting property with respect to inner fibrations, by explicitly constructing lifts via filling inner open boxes. At any step of the proof of [8, Proposition 2.9] and the results on which it depends in which it is verified that an open box is *comical*, i.e., that its faces satisfy the marking conditions given in [10, Definition 2.1], the given proof specializes to a proof that its critical edge is degenerate (note that the critical edge is the unique edge which is required to be marked by these conditions). One may then use the analogue of [8, Proposition 2.29] to prove analogues of [8, Theorem 2.25 and Corollary 2.29] via arguments of homotopical algebra similar to those used in [8].

Proposition 3.1 (see [8, Proposition 2.9]) *For $A \subseteq B \subseteq \{\wedge, \vee\}$, the unit of $i_! : \mathsf{cSet}_A \rightleftarrows \mathsf{cSet}_B : i^*$ is a trivial cofibration in the cubical Joyal model structure on cSet_A.* □

Theorem 3.2 (see [8, Theorem 2.25 and Corollary 2.29]) *For $A \subseteq B \subseteq \{\wedge, \vee\}$, both of the adjunctions $i_! : \mathsf{cSet}_A \rightleftarrows \mathsf{cSet}_B : i^*$ and $i^* : \mathsf{cSet}_B \rightleftarrows \mathsf{cSet}_A : i_*$ are Quillen equivalences, where cSet_A and cSet_B are equipped with either their respective Grothendieck or cubical Joyal model structures. Moreover, in both cases, both of the left adjoints $i_!, i^*$ create weak equivalences.* □

Note that although the Quillen equivalence mentioned in Theorem 3.2 for the cubical Joyal model structures is stated as [8, Theorem A.24], the proof given in that paper is incorrect, as it relies on the assumption that the forgetful functor $|-| : \mathsf{cSet}_A^+ \to \mathsf{cSet}_A$ (which forgets markings on cubes) is left Quillen; this is not the case for the cubical Joyal model structure and the model structure on marked cubical sets considered in that work, as is correctly observed earlier in the same appendix of that paper.

The adjunctions $i_! \dashv i^*$ will be of the greatest relevance to our work; thus our immediate task is to characterize the left adjoints $i_!$. A standard result concerning left Kan extension functors between presheaf categories gives us the following.

Proposition 3.3 *For any $A \subseteq B \subseteq \mathcal{S}$, the diagram*

$$\begin{tikzcd} \square_A \arrow[r, "i"] \arrow[d] & \square_B \arrow[d] \\ \mathsf{cSet}_A \arrow[r, "i_!"] & \mathsf{cSet}_B \end{tikzcd}$$

commutes up to natural isomorphism, where the vertical functors are Yoneda embeddings. In particular, for all $n \geq 0$ we have $i_! \Box^n_A \cong \Box^n_B$. $\square$

Roughly speaking, for $X \in \mathsf{cSet}_A$, the presheaf $i_! X$ is obtained by freely adding the structure maps of $\Box_B$ to X in a manner compatible with the structure maps of X. More precisely, by the density theorem and the fact that $i_!$ preserves colimits as a left adjoint, we have

$$i_! X = \operatorname*{colim}_{\Box^n_A \to X} \Box^n_B.$$

From this, together with the standard description of a colimit as a coequalizer of coproducts, we obtain the following.

Proposition 3.4 *Let $A \subseteq B \subseteq \mathcal{S}$. For $X \in \mathsf{cSet}_A$, an n-cube of $i_! X$ is determined by a cube $x \in X_m$ for some m and a map $\psi : \Box^n_B \to \Box^m_B$; we write this as $x\psi$, or simply x in the case $\psi = \mathrm{id}$. These are subject to the identification $(x\phi)\psi = x(\phi\psi)$ for x as above, $\phi : \Box^n_A \to \Box^m_A$ and $\psi : \Box^{n'}_B \to \Box^n_B$. Structure maps of $\Box_B$ act by precomposition.*

Given a map $f : X \to Y$ in cSet_A, the map $i_! f : i_! X \to i_! Y$ sends each cube $x\phi$ to $f(x)\phi$. $\square$

Analyzing the adjunction $i_! \dashv i^*$ in view of this characterization of $i_!$, we obtain the following description of its unit.

Corollary 3.5 *For $X \in \mathsf{cSet}_A$ as above, the unit map $\eta : X \to i^* i_! X$ sends each cube x of X to x, viewed as a cube of $i^* i_! X$.* $\square$

We likewise obtain convenient characterizations of the actions of $i_!$ on some of the key objects and morphisms used in the study of the cubical Joyal model structures.

Lemma 3.6 *For any $A \subseteq B \subseteq \mathcal{S}$, the image under $i_!$ of a boundary inclusion, (inner) open box filling, or endpoint inclusion into K in cSet_A is the corresponding map in cSet_B.*

Proof In all cases, this follows from the fact that the relevant map is induced by a particular colimit defined according to a formula which does not depend on A.

For inclusions of boundaries and open boxes, we note that by Proposition 3.3 and the fact that $i_!$ preserves colimits as a left adjoint, $i_!$ sends the colimit diagrams and maps of Proposition 2.14 to their analogues in cSet_B.

For inner open box fillings, we note that the inner cube $\widehat{\Box}^n_{A,i,\varepsilon}$ is defined by the pushout diagram

$$\begin{tikzcd} \Box^1_A \arrow[r, "\partial^c_{i,\varepsilon}"] \arrow[d] & \Box^n_A \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ \Box^0_A \arrow[r] & \widehat{\Box}^n_{A,i,\varepsilon} \end{tikzcd}$$

where the top horizontal map picks out the critical edge with respect to $\partial_{i,\varepsilon}$. The inner open box $\widehat{\sqcap}^n_{A,i,\varepsilon}$ is defined by a similar pushout diagram, and the inclusion $\widehat{\sqcap}^n_{A,i,\varepsilon} \hookrightarrow \widehat{\Box}^n_{A,i,\varepsilon}$ is the map between pushouts induced by the inclusion $\sqcap^n_{A,i,\varepsilon} \hookrightarrow \square^n_A$. The fact that $i_!$ preserves this inner open box inclusion thus follows from the previously established fact that it preserves open box inclusions, together with a further application of Proposition 3.3 and the fact that it preserves colimits.

For endpoint inclusions into K, we may make a similar argument based on an explicit construction of K as a colimit of representable cubical sets. $\square$

Corollary 3.7 *For $A \subseteq B \subseteq \mathcal{S}$ with $A \subseteq \{\wedge, \vee\}$, the functor $i_! : \mathsf{cSet}_A \to \mathsf{cSet}_B$ preserves monomorphisms.*

Proof In these cases, the class of monomorphisms in cSet_A is the saturation of the set of boundary inclusions (see [9, Lemma 1.32]), so this is immediate from Lemma 3.6. $\square$

In the case where $A \subseteq \{\wedge, \vee\}$, the convenient combinatorial properties of cSet_A allow us to characterize the cubes of $i_! X$ in terms of the nondegenerate cubes of X.

Proposition 3.8 *For $A \subseteq B \subseteq \mathcal{S}$ with $A \subseteq \{\wedge, \vee\}$, and $X \in \mathsf{cSet}_A$, each cube of $i_! X$ is equal to $x\phi$ for a unique nondegenerate cube x of X and a unique active map ϕ in $\square_B$.*

Proof Since $\square_A$ is an Eilenberg–Zilber category by Proposition 2.6, we may proceed by induction on skeleta. We first note that $i_! \square^n_A = \square^n_B$ by Proposition 3.3. The statement for the case $X = \square^n_A$ thus follows from Lemma 2.12 and the fact that the nondegenerate cubes of $\square^n_A$ are precisely the composites of face maps (including the identity, viewed as an empty composite). Now suppose that the statement holds for some $X \in \mathsf{cSet}_A$, and that Y is obtained from X by adjoining a single nondegenerate n-cube y; in other words, suppose we have a pushout diagram

$$\begin{tikzcd}
\partial\square^n_A \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\
\square^n_A \arrow[r, "y"'] \arrow[ur, phantom, "\ulcorner" very near end] & Y
\end{tikzcd}$$

in cSet_A. Then $i_!$ preserves this pushout as a left adjoint. Applying Lemma 3.6 and restricting to level m for a given $m \geq 0$, we have a pushout diagram

$$\begin{tikzcd}
(\partial\square^n_B)_m \arrow[r] \arrow[d, tail] & (i_! X)_m \arrow[d, tail] \\
(\square^n_B)_m \arrow[r, "i_! y"'] \arrow[ur, phantom, "\ulcorner" very near end] & (i_! Y)_m
\end{tikzcd}$$

in Set.

Thus $(i_! Y)_m \cong (i_! X)_m \sqcup ((\square^n_B)_m \setminus (\partial\square^n_B)_m)$. By the induction hypothesis, every cube in the left component of this disjoint union is equal to $x\phi$ for a unique $k \geq 0$, a unique nondegenerate $x : \square^k_A \to X$, and a unique active $\phi : \square^m_B \to \square^k_B$. To characterize cubes in the second coproduct component, we may note that the complement $((\square^n_B)_m \setminus (\partial\square^n_B)_m)$ consists precisely of the active maps $\phi : \square^m_B \to \square^n_B$, and each such map, viewed as an m-cube of $\square^n_B$, is sent by $i_! y$ to $y\phi$. Thus the cubes $y\phi$ are distinct as elements of $(i_! Y)_m$, both from each other and from the cubes of $(i_! X)_m$. Thus the statement holds for Y.

Finally, we must show that the stated property is preserved by transfinite composition. Let κ be a limit ordinal, and consider a transfinite composite of inclusions $X_\lambda \hookrightarrow X_{\lambda'}$ for $\lambda < \lambda' < \kappa$, such that the statement holds for each X_λ. Denote the colimit object by X; then X is the union of all the X_λ, with the inclusion $X_\lambda \hookrightarrow X$ given by the colimit map. Then $i_!$ preserves this transfinite composite as a left adjoint, so that $i_!X$ is similarly the union of the $i_!X_\lambda$. Thus every cube y of $i_!X$ is contained in $i_!X_\lambda$ for some λ; it follows that $y = x\phi$ for some nondegenerate cube x of $X_\lambda \subseteq X$ and some active map ϕ. To see that x and ϕ are unique, suppose that $x\phi = x'\phi'$, where both x and x' are nondegenerate and both ϕ and ϕ' are active. Then there is some λ such that both x and x' are contained in X_λ. Since $i_!X_\lambda \hookrightarrow i_!X$ is a monomorphism by Corollary 3.7, we have $x\phi = x'\phi'$ in X_λ as well. Thus $x = x'$ and $\phi = \phi'$ by the uniqueness property in X_λ. □

Corollary 3.9 *For A, B, X as above, the nondegenerate cubes of $i^*i_!X$ are precisely those of the form $x\phi$ where x is a nondegenerate cube of X and ϕ is an active map not factoring on the right through any degeneracy in $\square_A$.* □

Corollary 3.10 *For A, B, X as above, the unit map $\eta : X \to i^*i_!X$ is a monomorphism. In particular, if $x, x' : \square^n_A \to X$ are distinct, then the corresponding cubes of $i^*i_!X$ are distinct as well.*

Proof This is immediate from Corollary 3.5 and Proposition 3.8. □

Finally, we consider the compatibility of $i_!$ with the geometric product of cubical sets.

Proposition 3.11 *For any $A \subseteq B \subseteq \mathcal{S}$, the adjunction $i_! : \mathsf{cSet}_A \rightleftarrows \mathsf{cSet}_B : i^*$ is strong monoidal with respect to the geometric product.*

Proof Following the proof structure of [18, Proposition 4.4], we observe that by Proposition 3.3, $i_!$ is the left Kan extension of the composite of i with the Yoneda embedding on $\square_B$, which sends each object $\square^n_A$ to $\square^n_B$. By [17, Theorem 5.1], the statement thus follows from the fact that this composite is strong monoidal. □

3.2 Standard decomposition cubes

Our next task is to define and study a combinatorial construction which is of use in relating different kinds of cubical sets, and which will play a key role in proving the main results of this section. For concreteness, throughout the remainder of this subsection we fix $A \subseteq B \subseteq \mathcal{S}$ with $\wedge \in A$. Of course, for the case $\vee \in A$, we could make parallel arguments involving negative connections.

Definition 3.12 For a map $\phi : \square^m_B \to \square^n_B$, we define the *standard decomposition cube* $N_k(\phi)$ to be the composite

$$\gamma_{n,1}(\phi \otimes \square^1_B \otimes \square^k_B) : \square^{m+1+k}_B \to \square^{n+k}_B.$$

Note that we have written $\square^1_B \otimes \square^k_B$ above rather than $\square^{1+k}_B$ because coordinate $m+1$ will play a distinct role in our computations.

We will also have use for the following more explicit characterization of maps $N_k(\phi)$.

Lemma 3.13 *Given an $(m+1+k)$-tuple $(\vec{a}, b, \vec{c}) = (a_1, \dots, a_m, b, c_1, \dots, c_k) \in \square^{m+1+k}$, $N_k(\phi)$ sends $(\vec{a}, b, \vec{c})$ to the $(n+k)$-tuple*

$$(\phi(\vec{a})_1, \dots, \phi(\vec{a})_{n-1}, \phi(\vec{a})_n \wedge b, c_1, \dots, c_k)$$

in $\square^{n+k}$. $\square$

Corollary 3.14 *For all $j, k \geq 0$ and $\phi : \square^m_B \to \square^n_B$, we have $N_{j+k}(\phi) = N_j(\phi) \otimes \square^k_B$. In particular, $N_k(\phi) = N_0(\phi) \otimes \square^k_B$ for all k.* $\square$

Lemma 3.15 *If ϕ is active, then so is $N_k(\phi)$ for any k.*

Proof From Lemma 3.13, it is immediate that a map of the form $N_k(\phi)$ does not fix coordinates 1 through $n-1$ if ϕ does not, and does not fix coordinates $n+1$ through $n+1+k$ regardless of ϕ. It thus remains to consider coordinate n. We may note that $N_k(\phi)(\vec{a}, 1, \vec{c})_n = \phi(\vec{a})_n \wedge 1 = \phi(\vec{a})_n$. Therefore, if ϕ does not fix coordinate n, then neither does $N_k(\phi)$. $\square$

The purpose of this definition is to facilitate proofs involving open box filling; in the most important cases, we will fill open boxes in subcomplexes of $i^*\square^n_B$ whose interiors will be standard decomposition cubes, with the missing face corresponding to $\phi \otimes \square^k_B$. Moreover, we will see that this open box is inner in the case where $B \subseteq \mathcal{P}$.

The rough intuition behind the definition of the standard decomposition cubes is as follows. We may view $\phi \otimes \square^k_B$ as a network of paths in the $(n+k)$-cube from its initial vertex $(\phi \otimes \square^k_B)(\vec{0})$ to its terminal vertex $(\phi \otimes \square^k_B)(\vec{1})$. The $(m+1, 0)$-face of $N_k(\phi)$ is $\partial_{m,0}\sigma_m(\phi \otimes \square^k_B)$, which corresponds to the network of paths which proceeds as prescribed by ϕ on the first $m-1$ coordinates, while leaving the m-th coordinate fixed at 0. The edges connecting the $(m+1, 0)$-face to the $(m+1, 1)$-face then proceed from each vertex of $\partial_{m,0}\sigma_m(\phi \otimes \square^k_B)$ to the corresponding vertex of $\phi \otimes \square^k_B$. Thus the $(m+1, 1)$-open box on $N_k(\phi)$ is obtained by "separating out" the m-th component of $\phi \otimes \square^k_B$ into an extra dimension.

Examples 3.16 In order to further develop the intuition behind the definition above, we illustrate various cubes of the form $N_0(\phi)$.

- For $\delta_1 : \square^1_B \to \square^2_B$, mapping a to (a, a), the cube $N_0(\delta_1) : \square^2_B \to \square^2_B$, corresponding to the map $(a, b) \mapsto (a, a \wedge b)$, is

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d, equal] & (1,0) \arrow[d] \\ (0,0) \arrow[r] & (1,1) \end{tikzcd}$$

We may view this face as witnessing a factorization of the diagonal edge $(0, 0) \to (1, 1)$ as a composite of the edges $(0, 0) \to (1, 0)$ and $(1, 0) \to (1, 1)$. Viewing the diagonal edge as a path of length 1 from $(0, 0)$ to $(1, 1)$ in the poset $\{0 \leq 1\}^2$, we may think of this factorization as being obtained by "separating out" the two components of this path: first advancing from 0 to 1 in the first component while keeping the second fixed, then advancing from 0 to 1 in the second component while keeping the first fixed. Note that δ_1 appears as the $(2, 1)$-face of this cube, and that the critical edge with respect to this face is degenerate.

- For $\lambda_{21} : \square^2_B \to \square^2_B$, mapping (a_1, a_2) to (a_2, a_1), the cube $N_0(\lambda_{21})$, corresponding to the map $(a_1, a_2, b) \mapsto (a_2, a_1 \wedge b)$, is

$$\begin{tikzcd}
(0,0) \arrow[rr, equal] \arrow[dd] \arrow[dr, equal] & & (0,0) \arrow[dd] \arrow[dr] & \\
& (0,0) \arrow[rr] \arrow[dd] & & (0,1) \arrow[dd] \\
(1,0) \arrow[rr, equal] \arrow[dr, equal] & & (1,0) \arrow[dr] & \\
& (1,0) \arrow[rr] & & (1,1)
\end{tikzcd}$$

Similarly to the previous example, we may view the cube λ_{21} as a family of paths from $(0,0)$ to $(1,1)$. Then the $(3,0)$-face of this cube is obtained by proceeding from 0 to 1 in the first component as prescribed by this family of paths, while keeping the second component fixed at 0. The edges connecting the $(3,0)$-face to the $(3,1)$-face then advance the second component from 0 to 1 whenever a 1 appears in the second component of the corresponding vertex of ϕ. Note that λ_{21} appears as the $(3,1)$-face of this cube, and that the critical edge with respect to this face is degenerate.

- For $\rho_1 : \square^1_B \to \square^1_B$, the cube $N_0(\rho_1)$, corresponding to the map $(a,b) \mapsto (1-a) \wedge b$, is

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d] & 0 \arrow[d, equal] \\
1 \arrow[r] & 0
\end{tikzcd}$$

Here the $(2,0)$-face holds the unique component fixed at 0, while the edges adjacent to this face proceed from 0 to 1 whenever the corresponding vertex of ρ_1 is 1. In this case, because the initial vertex of ρ_1 is 1, this does not result in the critical edge with respect to the $(2,1)$-face being degenerate, unlike the critical edges with respect to the $(m+1,1)$-faces in the previous examples.

We next study various combinatorial properties of standard decomposition cubes which will be of use in our open box filling constructions. We begin with certain identities satisfied by standard decomposition cubes associated to composite maps.

Lemma 3.17 *For any $\phi : \square^m_B \to \square^n_B$ and any $k \geq 0$, the following identities are satisfied*:

(1) $N_k(\phi\sigma_i) = N_k(\phi)\sigma_i$ *for all* $1 \leq i \leq m+1$;

(2) $N_k(\phi\gamma_{i,\varepsilon}) = N_k(\phi)\gamma_{i,\varepsilon}$ *for all* $1 \leq i \leq m$ *and* $\varepsilon \in \{0,1\}$;

(3) $N_k(\partial_{i,\varepsilon}\phi) = \partial_{i,\varepsilon}N_k(\phi)$ *for* $1 \leq i \leq n$ *and* $\varepsilon \in \{0,1\}$;

(4) $N_k(\partial_{n+1,0}\phi) = \partial_{n+1,0}\sigma_n(\phi \otimes \square^1_B \otimes \square^k_B)$;

(5) $N_k(\partial_{n+1,1}\phi) = \phi \otimes \square^1_B \otimes \square^k_B$.

Proof For item (1), we compute

$$\begin{aligned}N_k(\phi\sigma_i) &= \gamma_{n,1}(\phi\sigma_i \otimes \Box^1_B \otimes \Box^k_B)\\ &= \gamma_{n,1}(\phi \otimes \Box^1_B \otimes \Box^k_B)(\sigma_i \otimes \Box^1_B \otimes \Box^k_B)\\ &= N_k(\phi)\sigma_i.\end{aligned}$$

The calculation for item (2) is similar. For items (3)–(5) we compute

$$\begin{aligned}N_k(\partial_{i,\varepsilon}\phi) &= \gamma_{n+1,1}(\partial_{i,\varepsilon}\phi \otimes \Box^1_B \otimes \Box^k_B)\\ &= \gamma_{n+1,1}(\partial_{i,\varepsilon} \otimes \Box^1_B \otimes \Box^k_B)(\phi \otimes \Box^1_B \otimes \Box^k_B)\\ &= \gamma_{n+1,1}\partial_{i,\varepsilon}(\phi \otimes \Box^1_B \otimes \Box^k_B).\end{aligned}$$

For $1 \le i \le n$, this is

$$\partial_{i,\varepsilon}\gamma_{n,1}(\phi \otimes \Box^1_B \otimes \Box^k_B) = \partial_{i,\varepsilon}N_k(\phi).$$

This proves item (3). In the case $i = n+1$, we may apply Lemma 2.2 to the composite $\gamma_{n+1,1}\partial_{n+1,\varepsilon}$, thus obtaining items (4) and (5). □

We next analyze the faces of standard decomposition cubes in detail.

Lemma 3.18 *For any map $\phi : \Box^m_B \to \Box^n_B$, we may characterize the faces of $N_k(\phi)$ as follows.*

- *For $i \le m$ and $\varepsilon \in \{0,1\}$, we have $N_k(\phi)\partial_{i,\varepsilon} = N_k(\phi\partial_{i,\varepsilon})$. Moreover, in this case one of the following holds:*
  1. *$N_k(\phi)\partial_{i,\varepsilon}$ factors through a face map on the left;*
  2. *$N_k(\phi)\partial_{i,\varepsilon} = \psi \otimes \Box^1_B \otimes \Box^k_B$ for some active map $\psi : \Box^{m-1}_B \to \Box^{n-1}_B$;*
  3. *$N_k(\phi)\partial_{i,\varepsilon} = N_k(\psi)$ for some active map $\psi : \Box^{m-1}_B \to \Box^n_B$.*
- *$N_k(\phi)\partial_{m+1,0} = \partial_{n,0}\sigma_n\phi \otimes \Box^k_B$.*
- *$N_k(\phi)\partial_{m+1,1} = \phi \otimes \Box^k_B$.*
- *For $k \ge 1$ and $i = m+1+p$, $p \ge 1$, and $\varepsilon \in \{0,1\}$, we have $N_k(\phi)\partial_{i,\varepsilon} = \partial_{n+p,\varepsilon}N_{k-1}(\phi)$.*

Proof We first consider the case $i \le m$. In this case, using the functoriality of the monoidal product, we may compute

$$\begin{aligned}N(\phi)\partial_{i,\varepsilon} &= \gamma_{n,1}(\phi \otimes \Box^1_B \otimes \Box^k_B)\partial_{i,\varepsilon}\\ &= \gamma_{n,1}(\phi \otimes \Box^1_B)(\partial_{i,\varepsilon} \otimes \Box^1_B \otimes \Box^k_B)\\ &= \gamma_{n,1}(\phi\partial_{i,\varepsilon} \otimes \Box^1_B \otimes \Box^k_B)\\ &= N_k(\phi\partial_{i,\varepsilon}).\end{aligned}$$

To analyze this case further, we consider the set of coordinates fixed by $\phi\partial_{i,\varepsilon} : \Box^{m-1}_B \to \Box^n_B$. If this map fixes some coordinate $j < n$ at ε', then by Lemma 2.12 we have $\phi\partial_{i,\varepsilon} = \partial_{j,\varepsilon'}\psi$ for some $\psi : \Box^{m-1}_B \to \Box^{n-1}_B$.

Thus we may compute

$$\begin{aligned} N_k(\phi\partial_{i,\varepsilon}) &= N_k(\partial_{j,\varepsilon'}\psi) \\ &= \gamma_{n,1}(\partial_{j,\varepsilon'}\psi \otimes \square^1_B \otimes \square^k_B) \\ &= \gamma_{n,1}\partial_{j,\varepsilon'}(\psi \otimes \square^1_B \otimes \square^k_B) \\ &= \partial_{j,\varepsilon'}\gamma_{n-1,1}(\psi \otimes \square^1_B \otimes \square^k_B). \end{aligned}$$

We can make a similar calculation in the case where $\phi\partial_{i,\varepsilon}$ fixes coordinate n at 0. So in these cases, (1) holds.

Next suppose that the only coordinate fixed by $\phi\partial_{i,\varepsilon}$ is n, and this is fixed at 1. In this case, again applying Lemma 2.12, we have $\phi\partial_{i,\varepsilon} = \partial_{n,1}\psi$ for an active map $\psi : \square^{m-1}_B \to \square^{n-1}_B$. Thus we may compute

$$\begin{aligned} N_k(\phi\partial_{i,\varepsilon}) &= N_k(\partial_{n,1}\psi) \\ &= \gamma_{n,1}(\partial_{n,1}\psi \otimes \square^1_B \otimes \square^k_B) \\ &= \gamma_{n,1}\partial_{n,1}(\psi \otimes \square^1_B \otimes \square^k_B) \\ &= \psi \otimes \square^1_B \otimes \square^k_B. \end{aligned}$$

Thus (2) holds in this case.

Finally, we consider the case where $\phi\partial_{i,\varepsilon}$ does not fix any coordinate. By definition, this means that $\phi\partial_{i,\varepsilon}$ is active, so (3) holds in this case.

Now we consider the case $i = m+1$. For this case, we first note that $\partial_{m+1,\varepsilon} = \square^m_B \otimes \partial_{1,\varepsilon} \otimes \square^k_B$. Now observe that by the functoriality of the monoidal product, we have the commuting diagram

$$\begin{tikzcd} \square^m_B \otimes \square^0_B \otimes \square^k_B \arrow[r, "\square^m_B\otimes\partial_{1,\varepsilon}\otimes\square^k_B"] \arrow[d, "\phi\otimes\square^0_B\otimes\square^k_B"'] & \square^m_B \otimes \square^1_B \otimes \square^k_B \arrow[d, "\phi\otimes\square^1_B\otimes\square^k_B"] \\ \square^n_B \otimes \square^0_B \otimes \square^k_B \arrow[r, "\square^n_B\otimes\partial_{1,\varepsilon}\otimes\square^k_B"] & \square^n_B \otimes \square^1_B \otimes \square^k_B \end{tikzcd}$$

Thus we see that $(\phi \otimes \square^1_B \otimes \square^k_B)\partial_{m+1,\varepsilon} = \partial_{n+1,\varepsilon}(\phi \otimes \square^k_B)$. Postcomposing with $\gamma_{n,1}$ on the left side of this equation yields $N_k(\phi)\partial_{m+1,\varepsilon}$, while by Lemma 2.2, postcomposing with $\gamma_{n,1}$ on the right side yields either $\phi \otimes \square^k_B$ or $\partial_{n,0}\sigma_n(\phi \otimes \square^k_B)$ depending on the value of ε.

Finally, we consider the case $i > m+1$, i.e., $i = m+1+p$ for some $p > 0$; note that this case can only occur for $k \geq 1$. In this case, $\partial_{m+1+p,\varepsilon} = \square^m_B \otimes \square^1_B \otimes \partial_{p,\varepsilon}$. Thus we obtain the commuting diagram

$$\begin{tikzcd} \square^m_B \otimes \square^1_B \otimes \square^{k-1}_B \arrow[r, "\square^m_B\otimes\square^1_B\otimes\partial_{p,\varepsilon}"] \arrow[d, "\phi\otimes\square^1_B\otimes\square^{k-1}_B"'] & \square^m_B \otimes \square^1_B \otimes \square^k_B \arrow[d, "\phi\otimes\square^1_B\otimes\square^k_B"] \\ \square^n_B \otimes \square^1_B \otimes \square^{k-1}_B \arrow[r, "\square^n_B\otimes\square^1_B\otimes\partial_{p,\varepsilon}"] & \square^n_B \otimes \square^1_B \otimes \square^k_B \end{tikzcd}$$

similarly to the previous case. Thus we see that $(\phi \otimes \square^1_B \otimes \square^k_B)\partial_{m+1+p,\varepsilon} = \partial_{n+1+p,\varepsilon}(\phi \otimes \square^1_B \otimes \square^{k-1}_B)$. Postcomposing with $\gamma_{n,1}$, we obtain $\partial_{n+p,\varepsilon}\gamma_{n,1}(\phi \otimes \square^1_B \otimes \square^{k-1}_B) = \partial_{n+p,\varepsilon}N_{k-1}(\phi)$. □

The following lemma will be used to show that when ϕ is a poset map, the open box which will be filled to obtain $\phi \otimes \Box_B^k$ and $N_k(\phi)$ is inner.

Lemma 3.19 *If $B \subseteq \mathcal{P}$, then for any active map $\phi : \Box_B^m \to \Box_B^n$ and any $k \geq 0$, the critical edge of $N_k(\phi)$ with respect to $\partial_{m+1,1}$ is degenerate.*

Proof The critical edge with respect to $\partial_{m+1,1}$, viewed as a map $\Box_B^1 \to \Box_B^m \otimes \Box_B^1 \otimes \Box_B^k$ in $\Box_A$, sends $a \in \Box_B^1$ to the $(m+1+k)$-tuple $(\vec{0}_m, a, \vec{0}_k)$. Furthermore, our assumption that ϕ is active implies that $\phi(\vec{0}_m)_i = 0$ for all $1 \leq i \leq n$ by Corollary 2.11. Applying Lemma 3.13, we can therefore see that the map obtained by precomposing $N_k(\phi)$ with this critical edge sends $a \in \Box_B^1$ to

$$\begin{aligned}(\phi(\vec{0}_m)_1, \ldots, \phi(\vec{0}_m)_{n-1}, \phi(\vec{0}_m)_n \wedge a, \vec{0}_k) &= (0, \ldots, 0, 0 \wedge a, 0, \ldots, 0) \\ &= (0, \ldots, 0, 0, 0, \ldots, 0) \\ &= \vec{0}_{n+k}.\end{aligned}$$

So this composite is constant at $\vec{0}_{n+k}$; thus it factors through the projection $\Box_B^1 \to \Box_B^0$. $\Box$

We next consider certain special cases in which $N_k(\phi)$ is degenerate; both of these results follow easily from Lemma 3.13.

Lemma 3.20 *For any map $\phi : \Box_B^m \to \Box_B^n$, we have $N_k(\phi \otimes \Box_B^1) = (\phi \otimes \Box_B^1 \otimes \Box_B^k)\gamma_{m+1,1}$.* $\Box$

Lemma 3.21 *For all maps $\phi : \Box_B^m \to \Box_B^n$ and $k \geq 0$, we have $N_k(N_0(\phi)) = (N_0(\phi) \otimes \Box_B^k)\gamma_{m+1,1}$.* $\Box$

3.3 Decomposition-closed subcomplexes

Our next goal is to prove a technical result involving the standard decomposition cubes, which we will specialize in different ways to obtain our main results on the unit of the adjunction $i_! \dashv i^*$ and the cartesian product of cubical sets.

Recall that by Lemma 2.12, an arbitrary morphism $\psi : \Box_B^m \to \Box_B^n$ can be factored uniquely as an active map $\phi : \Box_B^m \to \Box_B^p$ followed by a (possibly empty) composite of face maps $\kappa : \Box_B^p \to \Box_B^n$. Such factorizations, and their interactions with the N construction, will play a key role in the proofs of this section. As such, we define the following properties of morphisms in $\Box_B$.

Definition 3.22 Given a morphism $\kappa\phi$ in $\Box_B$, where $\phi : \Box_B^m \to \Box_B^p$ is active and $\kappa : \Box_B^p \to \Box_B^n$ is a composite of face maps:

- the *base dimension* of $\kappa\phi$ is the value p;
- the *tail length* of $\kappa\phi$ is the maximal value of k such that $\phi = \phi' \otimes \Box_B^k$ for some $\phi' : \Box_B^{m-k} \to \Box_B^{p-k}$.

The following is an immediate consequence of Lemma 3.13.

Lemma 3.23 *For any active map ϕ and $k \geq 0$, the tail length of $N_k(\phi)$ is k.* $\Box$

We next define this section's central objects of study.

Definition 3.24 A *decomposition-closed subcomplex* of $i^*\square^n_B$ is a subcomplex $X \subseteq i^*\square^n_B$ containing the image of the unit map $\square^n_A \to i^*\square^n_B$, such that if a map $\kappa(\phi \otimes \square^k_B) : \square^m_B \otimes \square^k_B \to \square^p_B \to \square^n_B$, where $\phi : \square^m_B \to \square^{p-k}_B$ is active and $\kappa : \square^p_B \to \square^n_B$ is a (possibly empty) composite of face maps, is contained in X when viewed as an $(m+k)$-cube of $i^*\square^n_B$, then so is the $(m+1+k)$-cube $\kappa N_k(\phi)$.

For a given n, the most natural examples of decomposition-closed subcomplexes are $\square^n_A$ and $i^*\square^n_B$ itself. In Proposition 5.9 we will see that if $\Sigma, \delta \in B$, then for $m, n \geq 0$, the cartesian product $\square^m_A \times \square^n_A$ can also be viewed as a decomposition-closed subcomplex of $i^*\square^{m+n}_B$.

We can now state the main result of this section.

Proposition 3.25 *Given a pair of decomposition-closed subcomplexes $X \subseteq Y \subseteq i^*\square^n_B$, the inclusion $X \hookrightarrow Y$ is anodyne. Moreover, if $B \subseteq \mathcal{P}$, then the inclusion is inner anodyne.*

Our strategy for proving this result will be to construct the cubes of Y from those of X via repeated filling of open boxes, using the property of decomposition-closure and the combinatorial results of Section 3.2. To organize this construction, we factor the inclusion $X \hookrightarrow Y$ through a series of intermediate subcomplexes. Thus we now fix $n \geq 0$ and a pair of decomposition-closed subcomplexes $X \subseteq Y \subseteq i^*\square^n_B$. Our factorization of the inclusion $X \hookrightarrow Y$ will proceed in three steps: we will first filter the cubes of Y based on their base dimension, then their dimension, and finally their tail length. As we will be viewing maps in $\square_B$ as cubes of $i^*\square^n_B$, in light of Corollary 3.9 we will refer to such maps as "degenerate" if they factor on the right through a degeneracy in $\square_A$, even though $\square_B$ itself may not admit an Eilenberg–Zilber structure.

As the first step of our decomposition, for $0 \leq i \leq n$ we define W^i to be the subcomplex of Y consisting of all cubes of X, together with all cubes of Y having base dimension less than or equal to i. Thus $W^0 = X$, since the only cubes having base dimension 0 are the degeneracies of 0-cubes, all of which are contained in $\square^n_A$. Similarly, we have $W^n = Y$, since n is the largest base dimension which a map with codomain $\square^n_B$ can have. We may note that each W^i is still a decomposition-closed subcomplex of $i^*\square^n_B$, since for any map of the form $\kappa(\phi \otimes \square^k_B)$ with ϕ active and κ a composite of face maps, the base dimension of $\kappa N_k(\phi)$ is the same as that of $\kappa(\phi \otimes \square^k_B)$.

We thus obtain a sequence of inclusions

$$X = W^0 \hookrightarrow W^1 \hookrightarrow \cdots \hookrightarrow W^{n-1} \hookrightarrow W^n = Y.$$

To prove Proposition 3.25, therefore, it will suffice to show that each inclusion in the diagram above is (inner) anodyne. To this end, we fix $0 \leq i \leq n-1$, and focus on the inclusion $W^i \hookrightarrow W^{i+1}$.

As the next step of our factorization, for $j \geq 0$ we let $W^{i,j}$ denote the subcomplex of W^{i+1} consisting of

- all cubes of W^i;
- all cubes of Y of the form $\kappa\psi$ where $\kappa : \square^{i+1}_B \to \square^n_B$ is a composite of face maps and ψ is an active map $\square^{j'}_B \to \square^{i+1}_B$ for some $j' \leq j$;
- all cubes of Y of the form $\kappa N_k(\phi)$ for κ, ψ as above where $\psi = \phi \otimes \square^k_B$ for some active map $\phi : \square^{j'-k}_B \to \square^{i+1-k}_B$;
- degeneracies of the above.

To see that this indeed defines a subcomplex of W^{i+1}, note that it is closed under degeneracies by definition, and under faces by Lemma 3.18.

Similarly to the previous step, we see that $W^{i,0} = W^i$ because all 0-cubes of $i^*\Box^n_B$ are contained in $\Box^n_A$. Thus we obtain an infinite sequence of inclusions

$$W^i = W^{i,0} \hookrightarrow W^{i,1} \hookrightarrow \cdots \hookrightarrow W^{i,j} \hookrightarrow W^{i,j+1} \hookrightarrow \cdots.$$

Since every cube of W^{i+1} is contained in some subcomplex $W^{i,j}$, the union of the $W^{i,j}$, i.e., the colimit of the diagram above, is W^{i+1}. Therefore, because (inner) anodyne maps are closed under transfinite composition, to prove Proposition 3.25 it will suffice to prove that each inclusion $W^{i,j} \hookrightarrow W^{i,j+1}$ is inner anodyne. Therefore, we now fix some $j \geq 0$.

As the final step of our decomposition, for $0 \leq k \leq j+1$, we define $W^{i,j}_k$ to be the subcomplex of $W^{i,j+1}$ consisting of

- all cubes of $W^{i,j}$;
- all cubes of Y of the form $\kappa(\phi \otimes \Box^{k'}_B)$, where $\kappa : \Box^{i+1}_B \to \Box^n_B$ is a composite of face maps and $\phi : \Box^{j+1-k'}_B \to \Box^{i+1-k'}_B$ is active, for $k' \geq k$;
- all cubes of Y of the form $\kappa N_{k'}(\phi)$ for κ, ϕ as above;
- degeneracies of the above.

Once again, to see that this is a subcomplex of $W^{i,j+1}$ we note that it is closed under degeneracies by construction, and can be straightforwardly shown to be closed under faces using Lemma 3.18.

Similarly to the previous two factorizations, we may note that $W^{i,j}_{j+1} = W^{i,j}$, since the only active map ϕ with domain $\Box^0_B$ is the identity, and for this map and any composite of faces κ, the maps $\kappa(\phi \otimes \Box^{j+1}_B)$ and $\kappa N_k(\phi)$ are in $\Box_A$. Likewise, $W^{i,j}_0 = W^{i,j+1}$, and we have a sequence of inclusions

$$W^{i,j} = W^{i,j}_{j+1} \hookrightarrow W^{i,j}_j \hookrightarrow \cdots \hookrightarrow W^{i,j}_1 \hookrightarrow W^{i,j}_0 = W^{i,j+1}.$$

We have thus reduced our task to proving that each of the inclusions in the diagram above is (inner) anodyne; it is this task which we will take up by means of open-box filling. To this end, in addition to the values of n, i, j already fixed, let us also fix k satisfying $0 \leq k \leq j$.

Our task will be to analyze the inclusion $W^{i,j}_{k+1} \hookrightarrow W^{i,j}_k$. We may view this inclusion as adjoining to $W^{i,j}_{k+1}$ all cubes of Y of the form $\kappa(\phi \otimes \Box^k_B)$ where $\kappa : \Box^{i+1}_B \to \Box^n_B$ is a composite of face maps and $\phi : \Box^{j+1-k}_B \to \Box^{i+1-k}_B$ is active, as well as the corresponding cubes obtained by replacing $\phi \otimes \Box^k_B$ with $N_k(\phi)$, and degeneracies of these. Note, however, that some cubes of this form are already contained in $W^{i,j}_{k+1}$, such as those for which $\phi = \phi' \otimes \Box^1$ for some map ϕ' (as in that case, $\phi \otimes \Box^k_B = \phi' \otimes \Box^{k+1}_B$). Thus we begin by determining which new cubes must be adjoined to $W^{i,j}_{k+1}$ in order to construct $W^{i,j}_k$.

Lemma 3.26 *A cube $\kappa(\phi \otimes \Box^k_B)$, where $\kappa : \Box^{i+1}_B \to \Box^n_B$ is a composite of face maps and $\phi : \Box^{j+1-k}_B \to \Box^{i+1-k}_B$ is active, is present in $W^{i,j}_{k+1}$ if and only if one of the following cases holds*:

- *$\kappa(\phi \otimes \square_B^k)$ is a cube of X;*
- *$\phi \otimes \square_B^k$ is degenerate;*
- *$\phi \otimes \square_B^k = \phi' \otimes \square_B^{k+1}$ for some active map $\phi' : \square_B^{j-k} \to \square_B^{i-k}$;*
- *$\phi \otimes \square_B^k = N_p(\phi')$ for some map $\phi' : \square_B^{j-p} \to \square_B^{i+1-p}$.*

Moreover, in each of these cases $\kappa N_k(\phi)$ is present in $W_{k+1}^{i,j}$ as well.

Proof That $\kappa(\phi \otimes \square_B^k)$ is present in $W_{k+1}^{i,j}$ if and only if one of the listed cases holds is immediate from the definition of $W_{k+1}^{i,j}$. (Note that $\kappa(\phi \otimes \square_B^k)$ is degenerate if and only if $\phi \otimes \square_B^k$ is.) To show that $N_k(\phi)$ is present in $W^{m,k+1}$ in each of these cases, we consider each case in turn.

- Suppose $\kappa(\phi \otimes \square_B^k)$ is a cube of X. Then because X is decomposition-closed by assumption, $\kappa N_k(\phi)$ is contained in X, and thus in $W_{k+1}^{i,j}$.
- Suppose $\phi \otimes \square_B^k$ is degenerate; this implies that ϕ is a degeneracy of some $\phi' : \square_B^{j-k} \to \square_B^{i+1-k}$. That ϕ' is active is a straightforward consequence of Lemma 2.12. By Lemma 3.17, $N_k(\phi)$ is a degeneracy of $N_k(\phi')$. Since $\kappa N_k(\phi')$ is a cube of $W^{i,j} \subseteq W_{k+1}^{i,j}$, so is $N_k(\phi)$.
- Suppose $\phi \otimes \square_B^k = \phi' \otimes \square_B^{k+1}$ for some active map $\phi' : \square_B^{j-k} \to \square_B^{i-k}$. Then by Lemma 2.4, we have $\phi = \phi' \otimes \square_B^1$. (In other words, this is precisely the case in which ϕ has positive tail length.) Thus Lemma 3.20 implies that $\kappa N_k(\phi)$ is a degeneracy of $\kappa(\phi \otimes \square_B^k)$, hence is also contained in $W_{k+1}^{i,j}$.
- Finally, suppose that none of the previous cases hold; then the only remaining case in which $\kappa(\phi \otimes \square_B^k)$ could be contained in $W_{k+1}^{i,j}$ is if $\phi \otimes \square_B^k = N_p(\phi')$ for some map $\phi' : \square_B^{j-p} \to \square_B^{i+1-p}$. Because the previous case in particular does not hold, ϕ has tail length 0, so that $\phi \otimes \square_B^k$ has tail length k. By Lemma 3.23, it must therefore be the case that $p = k$. By Lemma 2.4 and Corollary 3.14, therefore, we have $\phi = N_0(\phi')$. Therefore, by Lemma 3.21, $\kappa N_k(\phi)$ is a degeneracy of $\kappa(\phi \otimes \square_B^k)$, and hence is also contained in $W_{k+1}^{i,j}$. $\square$

Thus the cubes of $W_k^{i,j}$ which we must construct by open box filling are those falling under none of the listed cases above. We next show that the necessary open boxes do indeed exist in $W_{k+1}^{i,j}$.

Lemma 3.27 *For an active map $\phi : \square_B^{j+1-k} \to \square_B^{i+1-k}$ and composite of faces $\kappa : \square_B^{i+1} \to \square_B^n$ such that $\kappa(\phi \otimes \square_B^k)$ is not present in $W_{k+1}^{i,j}$, all faces of $\kappa N_k(\phi)$ aside from its $(j+2-k, 1)$-face are present in $W_{k+1}^{i,j}$.*

Proof We analyze the faces of $\kappa N_k(\phi)$ using Lemma 3.18. We first consider a face $\kappa N_k(\phi)\partial_{p,\varepsilon}$ with $1 \leq p \leq j+1-k$; here Lemma 3.18 provides three possible cases.

In case (1), we have $N_k(\phi)\partial_{p,\varepsilon} = \partial_{q,\varepsilon'}\psi$ for some face map $\partial_{q,\varepsilon'} : \square_B^i \to \square_B^{i+1}$ and some map $\psi : \square_B^{j+1} \to \square_B^i$. It follows that $\kappa N_k(\phi)\partial_{p,\varepsilon} = \kappa\partial_{q,\varepsilon'}\psi$ has base dimension at most i, and is therefore contained in $W^i \subseteq W_{k+1}^{i,j}$.

In case (2), we have $N_k(\phi)\partial_{p,\varepsilon} = \psi \otimes \square_B^{k+1}$ for an active map $\psi : \square_B^{j-k} \to \square_B^{i-k}$, so $\kappa N_k(\phi)\partial_{p,\varepsilon} = \kappa(\psi \otimes \square_B^{k+1})$ is contained in $W_{k+1}^{i,j}$ by definition.

In case (3), $N_k(\phi)\partial_{i,\varepsilon} = N_k(\psi)$ for an active map $\psi : \Box_B^{j-k} \to \Box_B^{i+1-k}$, so $\kappa N_k(\phi)\partial_{i,\varepsilon} = \kappa N_k(\psi)$ is contained in $W^{i,j} \subseteq W_{k+1}^{i,j}$.

Finally, for faces $N_k(\phi)\partial_{p,\varepsilon}$ with $(p,\varepsilon) = (j+2-k, 0)$ or $p > j+2-k$, Lemma 3.18 shows that $N_k(\phi)\partial_{p,\varepsilon}$ factors through a face map on the left, so $\kappa N_k(\phi)\partial_{p,\varepsilon}$ is contained in $W^i \subseteq W_{k+1}^{i,j}$. □

We have now established sufficient results about decomposition-closed subcomplexes to prove the main result of this subsection.

Proof of Proposition 3.25 We first note that by the closure of (inner) anodyne maps under composition and transfinite composition, it suffices to show that each inclusion $W_{k+1}^{i,j} \hookrightarrow W_k^{i,j}$ is (inner) anodyne.

By the definition of $W_k^{i,j}$ and Lemma 3.26, constructing $W_k^{i,j}$ from $W_{k+1}^{i,j}$ amounts to adjoining to $W_{k+1}^{i,j}$ the cubes $\kappa(\phi \otimes \Box_B^k)$ and $\kappa N_k(\phi)$ for all active maps $\phi : \Box_B^{i+1-k} \to \Box_B^{j+1-k}$ not covered by any of the cases listed in the statement of that result. Moreover, we may note that in these cases, $\kappa(\phi \otimes \Box_B^k)$ and $\kappa N_k(\phi)$ are nondegenerate as cubes of $i^*\Box_B^n$; to see this, we may note that $\kappa(\phi \otimes \Box_B^k)$ is nondegenerate by assumption, and that by Lemmas 3.18 and 3.27 it appears as exactly one face of $\kappa N_k(\phi)$. Thus we can apply Corollary 2.8 to show that $\kappa N_k(\phi)$ is nondegenerate as well.

By Lemma 3.27, for each such κ, ϕ, the faces of $\kappa N_k(\phi)$ other than its $(j+2-k, 1)$-face $\kappa\phi \otimes \Box_B^k$ form a $(j+2-k, 1)$-open box in $W_{k+1}^{i,j}$. Thus we may obtain $W_k^{i,j}$ from $W_{k+1}^{i,j}$ by filling all of these open boxes; in other words, we have a pushout diagram

$$\begin{tikzcd} \coprod \sqcap_{A,j+2-k,1}^{j+2} \arrow[r] \arrow[d] & W_{k+1}^{i,j} \arrow[d] \\ \coprod \Box_A^{j+2} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & W_k^{i,j} \end{tikzcd}$$

where the coproduct is taken over all composites of face maps $\kappa : \Box_B^{i+1} \to \Box_B^n$ and active maps $\phi : \Box_B^{j+1-k} \to \Box_B^{i+1-k}$ such that $\kappa(\phi \otimes \Box_B^k)$ is not contained in $W_{k+1}^{i,j}$. For such a pair (κ, ϕ), the (κ,ϕ)-component of the bottom horizontal map picks out the cube $\kappa N_k(\phi)$, while that of the top horizontal map picks out the $(j+2-k, 1)$-open box on $\kappa N_k(\phi)$. Thus $W_{k+1}^{i,j} \hookrightarrow W_k^{i,j}$ is anodyne.

Moreover, by Lemma 3.19, if $B \subseteq \mathcal{P}$ then these open boxes are inner. Thus we could replace the diagram above with one in which the left vertical map is a coproduct of inclusions $\widehat{\sqcap}_{A,j+2-k,1}^{j+2} \hookrightarrow \widehat{\Box}_{A,j+2-k,1}^{j+2}$, thereby showing that $W_{k+1}^{i,j} \hookrightarrow W_k^{i,j}$ is inner anodyne. □

4 The unit of the left Kan extension adjunction

We now consider a generalization of Proposition 3.1 which follows straightforwardly from Proposition 3.25.

Proposition 4.1 *For $A \subseteq B \subseteq \mathcal{S}$, with $A \subseteq \{\wedge, \vee\}$ and B containing at least one element of $\{\wedge, \vee\}$, the unit $\eta : \mathrm{id}_{\mathsf{cSet}_A} \Rightarrow i^* i_!$ of the adjunction $i_! \dashv i^*$ induced by the inclusion $i : \Box_A \hookrightarrow \Box_B$ is a natural trivial cofibration in the Grothendieck model structure on cSet_A. Moreover, if $B \subseteq \mathcal{P}$, then η is a natural trivial cofibration in the cubical Joyal model structure on cSet_A.*

Proof Corollary 3.10 shows that the unit of $i_! \dashv i^*$ is a cofibration, thus it remains only to prove that it is a weak equivalence. The case $B \subseteq \{\wedge, \vee\}$ is given by Proposition 3.1. For our remaining cases, we first note that $i^* i_!$ preserves pushouts as a left adjoint, and preserves monomorphisms by Corollary 3.7 and the fact that i^* is a right adjoint. Therefore, by a standard induction on skeleta argument involving the gluing lemma (see [8, Lemma 1.6]), it will suffice to consider the components of the unit at representable cubical sets.

First suppose $\wedge \in A$; note that since $A \subseteq B$, in this case the condition that B contains at least one of $\wedge$, $\vee$ is satisfied automatically. In this case, the statement that each map $\square^n_A \hookrightarrow i^*\square^n_B$ is a trivial cofibration follows from Proposition 3.25, taking $X = \square^n_A$, $Y = i^*\square^n_B$. For the case $\vee \in A$, we can prove the claim by a similar argument, where the cubes $N_k(\phi)$ are constructed using negative connections.

Finally, we consider the case in which $A = \varnothing$ and B is not assumed to be a subset of $\{\wedge, \vee\}$. For concreteness, assume $\wedge \in B$; the case $\vee \in B$ is analogous.

In this case, we factor the inclusion $\square_\varnothing \hookrightarrow \square_B$ as a composite of inclusions $j : \square_\varnothing \hookrightarrow \square_\wedge$ and $l : \square_\wedge \hookrightarrow \square_B$. Then the adjunction $i_! \dashv i^*$ is the composite of the adjunctions $j_! \dashv j^*$ and $l_! \dashv l^*$. Denote the unit of $j_! \dashv j^*$ by η^j and the unit of $l_! \dashv l^*$ by η^l. Then the component of the unit of $i_! \dashv i^*$ at an object $X \in \mathsf{cSet}_\varnothing$ is the composite

$$X \xhookrightarrow{\eta^j_X} j^* j_! X \xhookrightarrow{j^*\eta^l_{j_!X}} j^* l^* l_! j_! X \cong i^* i_! X.$$

We have shown that η^j is a natural weak equivalence in $\mathsf{cSet}_\varnothing$, and that η^l is a natural weak equivalence in $\mathsf{cSet}_\wedge$. Moreover, j^* preserves weak equivalences by Theorem 3.2. Thus the composite depicted above is a weak equivalence. $\square$

Remark 4.2 It is natural to wonder whether Proposition 4.1 can be generalized — for instance, whether the unit of the adjunction $i_! \dashv i^*$ induced by a proper inclusion $\square_A \hookrightarrow \square_B$ can be a trivial cofibration in the cubical Joyal model structure on cSet_A if $\square_B$ contains reversals, or in either the cubical Joyal or the Grothendieck model structure if $A = \varnothing$ and $\square_B$ does not contain connections. In fact, neither of these results holds; in all cases it can be verified that the component of the unit at a representable cube of dimension 1 or 2 is not a weak equivalence.

The remainder of this section will be devoted to exploring the consequences of Proposition 4.1. We first apply it to show that although the monoidal structure given by the geometric product on cSet_A for $A \subseteq \{\wedge, \vee\}$ is not symmetric, it is "symmetric up to natural weak equivalence" in the cubical Joyal model structure.

Theorem 4.3 *For $A \subseteq \{\wedge, \vee\}$ and $X, Y \in \mathsf{cSet}_A$, we have a zigzag of weak equivalences in the cubical Joyal model structure relating $X \otimes Y$ and $Y \otimes X$, natural in X and Y.*

Proof Again, let $B \subseteq \mathcal{P}$ such that $A \subseteq B$ and B contains Σ and at least one of $\wedge$, $\vee$, and consider $i : \square_A \hookrightarrow \square_B$. By Proposition 4.1 we have a natural trivial cofibration $X \otimes Y \hookrightarrow i^* i_!(X \otimes Y)$. The codomain of this map is isomorphic to $i^*(i_! X \otimes i_! Y)$ by Proposition 3.11; this, in turn, is isomorphic to

$i^*(i_! Y \otimes i_! X)$ since the geometric product in cSet_B is symmetric. A further application of Proposition 3.11 shows that this is naturally isomorphic to $i^* i_!(Y \otimes X)$, and a further application of Proposition 4.1 provides a natural trivial cofibration $Y \otimes X \hookrightarrow i^* i_!(Y \otimes X)$. □

We now consider the use of Proposition 4.1 to induce model structures on categories cSet_A where A contains at least one kind of connection, but is not a subset of $\{\wedge, \vee\}$. In particular, we will thus obtain models for $(\infty, 1)$-categories using some kinds of cubical sets with symmetries.

We begin by reviewing the definition of an induced model structure.

Definition 4.4 Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : U$ be an adjunction between model categories. The model structure on C is *left induced* by F if F creates cofibrations and weak equivalences. Likewise, the model structure on D is *right induced* by U if U creates fibrations and weak equivalences.

Note that for a given adjunction $\mathsf{C} \rightleftarrows \mathsf{D}$ and a given model structure on D, the left-induced model structure on C is unique, if one exists, since the definition determines its cofibrations and weak equivalences. Likewise, for a given model structure on C, the right-induced model structure on D is unique, if one exists.

Our constructions will follow from the application of established results on the existence of induced model structures, enabled by Proposition 4.1. For ease of reference, we package the model-categorical reasoning involved into the following general result.

Proposition 4.5 *Let* C *be a combinatorial model category*, D *a locally presentable category*, *and* $F : \mathsf{D} \to \mathsf{C}$ *a functor admitting a left adjoint* L *and a right adjoint* R. *Suppose that the composite functor* FL *preserves cofibrations*, *and the unit* $\mathrm{id}_{\mathsf{C}} \Rightarrow FL$ *is a natural weak equivalence. Then* D *admits both a left-induced model structure* D_l *and a right-induced model structure* D_r. *Moreover*,

- *the adjunctions* $L \dashv F$ *and* $F \dashv R$ *are Quillen with respect to both* D_r *and* D_l, *with* $L \dashv F$ *being a Quillen equivalence in both cases*;
- *the adjunction* $\mathrm{id}_{\mathsf{D}} : \mathsf{D}_r \rightleftarrows \mathsf{D}_l : \mathrm{id}_{\mathsf{D}}$ *is a Quillen equivalence.*

Proof The composite functor FL preserves cofibrations by assumption, and preserves weak equivalences by the assumption that $\mathrm{id}_{\mathsf{C}} \Rightarrow FL$ is a natural weak equivalence together with two-out-of-three. Thus FL is a left Quillen functor; the existence of the right-induced model structure, and the fact that $L \dashv F$ and $F \dashv R$ are Quillen adjunctions with respect to this model structure, follow by [11, Theorem 2.3]. To see that $L \dashv F$ is a Quillen equivalence between C and D_r, we note that the definition of a right-induced model structure and our assumption on the unit of $L \dashv F$ imply that hypothesis (c) of [16, Corollary 1.3.16] is satisfied.

Next we show that the left-induced model structure D_l exists and that $L \dashv F$ and $F \dashv R$ are Quillen adjunctions with respect to this model structure as well. For this, we note that our assumption on the unit of $L \dashv F$, together with [14, Remark 1.11], implies that $L \dashv F \dashv R$ is a homotopy idempotent string in the sense of [14, Definition 1.10]; this, together with our previous observation that FL is left Quillen, allows us to apply [14, Theorem 1.13].

That $\mathrm{id}_{\mathsf{D}} : \mathsf{D}_r \rightleftarrows \mathsf{D}_l : \mathrm{id}_{\mathsf{D}}$ is a Quillen equivalence follows from [14, Proposition 1.16]. We then see that $L : \mathsf{C} \rightleftarrows \mathsf{D}_l : F$ is a Quillen equivalence as the composite of $L : \mathsf{C} \rightleftarrows \mathsf{D}_r : F$ with $\mathrm{id}_{\mathsf{D}} : \mathsf{D}_r \rightleftarrows \mathsf{D}_l : \mathrm{id}_{\mathsf{D}}$. □

In the case of right-induced model structures, generating sets of cofibrations, trivial cofibrations, and anodyne maps can easily be obtained from those of the original model category via the following well-known result.

Proposition 4.6 (see [15, Theorem 11.3.2]) *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction, with $\mathcal{C}$ a model category, and let I be a class of generating cofibrations (resp. generating trivial cofibrations, pseudogenerating trivial cofibrations) for $\mathcal{C}$. If the right-induced model structure on $\mathcal{D}$ exists, then FI is a class of generating cofibrations (resp. generating trivial cofibrations, pseudogenerating trivial cofibrations) for this model structure.*

Proof We consider the case of generating cofibrations; the other cases are similar. A morphism f in $\mathcal{D}$ is a trivial fibration in the right-induced model structure if and only if Uf has the right lifting property with respect to all maps of I. This, in turn, holds if and only if f has the right lifting property with respect to all maps of FI. □

We now construct the desired model structures on cubical sets.

Theorem 4.7 *Let $\Box_A$ be a cube category containing at least one kind of connection. Then cSet_A admits the following model structures:*

- *the* ***right-induced Grothendieck model structure****, in which weak equivalences and fibrations are created by $i^* : \mathsf{cSet}_A \to \mathsf{cSet}_\varnothing$ from the Grothendieck model structure on $\mathsf{cSet}_\varnothing$;*
- *the* ***left-induced Grothendieck model structure****, in which weak equivalences and cofibrations are created by $i^* : \mathsf{cSet}_A \to \mathsf{cSet}_\varnothing$ from the Grothendieck model structure on $\mathsf{cSet}_\varnothing$.*

Moreover, if $\Box_A$ does not contain reversals, then cSet_A admits the following model structures:

- *the* ***right-induced cubical Joyal model structure****, in which weak equivalences and fibrations are created by $i^* : \mathsf{cSet}_A \to \mathsf{cSet}_\varnothing$ from the cubical Joyal model structure on $\mathsf{cSet}_\varnothing$;*
- *the* ***left-induced cubical Joyal model structure****, in which weak equivalences and cofibrations are created by $i^* : \mathsf{cSet}_A \to \mathsf{cSet}_\varnothing$ from the cubical Joyal model structure on $\mathsf{cSet}_\varnothing$.*

Furthermore:

- *for each induced model structure, the adjunction $i_! : \mathsf{cSet}_\varnothing \rightleftarrows \mathsf{cSet}_A : i^*$ is a Quillen equivalence;*
- *for each induced model structure, the adjunction $i^* : \mathsf{cSet}_A \rightleftarrows \mathsf{cSet}_\varnothing : i_*$ is a Quillen adjunction;*
- *in each case, the identity adjunction on cSet_A is a Quillen equivalence between the left- and right-induced model structures, with the right-induced model structure as the domain of the left adjoint.*

Proof We verify the hypotheses of Proposition 4.5 with respect to $i^* : \mathsf{cSet}_A \to \mathsf{cSet}_\varnothing$. The composite functor $i^* i_!$ preserves cofibrations by Corollary 3.7 and the fact that i^* preserves monomorphisms as a right adjoint, while the unit of $i_! \dashv i^*$ is a natural weak equivalence by Proposition 4.1. □

We devote the remainder of this section to analysis of the model structures of Theorem 4.7.

Definition 4.8 (see [6, Definitions 8.1.23 and 8.1.30]) A monomorphism $X \to Y$ in a category of cubical sets cSet_A is *normal* if for every $n \geq 0$, the automorphism group of $\square_A^n$ acts freely on the nondegenerate elements of $Y_n \setminus X_n$.

A cubical set $X \in \mathsf{cSet}_A$ is *normal* if $\varnothing \hookrightarrow X$ is a normal monomorphism, i.e., if the automorphism group of each $\square_A^n$ acts freely on the nondegenerate cubes of X.

In particular, we may note that if $\square_A$ does not contain symmetries or reversals, then all automorphism groups of objects of $\square_A$ are trivial, so that all monomorphisms of cSet_A are normal.

Proposition 4.9 *The left-induced model structures of Theorem 4.7 have monomorphisms as their cofibrations. In the right-induced model structures of Theorem 4.7, the cofibrations are generated by the set of boundary inclusions. In particular, if A does not contain diagonals, then the cofibrations of the right-induced model structures are the normal monomorphisms.*

Proof For the left-induced model structures, we note that by the definition of i^* it is immediate that a map f in cSet_A is a monomorphism if and only if $i^* f$ is a monomorphism.

The characterization of the generating cofibrations in the right-induced model structure is immediate from Lemma 3.6 and Proposition 4.6. In the case of cubical sets without diagonals, it follows that the cofibrations are the normal monomorphisms by [4, Theorem 7.9; 6, Proposition 8.1.35]. □

Corollary 4.10 *For A as in the statement of Theorem 4.7, if $\delta \notin A$, then the cofibrant objects of both right-induced model structures on cSet_A are the normal cubical sets.* □

Proposition 4.11 *In the right-induced Grothendieck model structures, the open box inclusions form a generating set of trivial cofibrations. In the right-induced cubical Joyal model structures, the inner open box inclusions and endpoint inclusions into K form a set of pseudogenerating trivial cofibrations.*

Proof As in the proof of Proposition 4.9, this is immediate from Lemma 3.6 and Proposition 4.6. □

Remark 4.12 For any $A \subseteq B \subseteq \mathcal{S}$ with $A \subseteq \{\wedge, \vee\}$, we could instead construct the induced model structures on cSet_B of Theorem 4.7 using $i^* : \mathsf{cSet}_B \to \mathsf{cSet}_A$. To see that the model structure thus produced is independent of A, we may first note that by Theorem 3.2, the class of weak equivalences in cSet_B created by $i^* : \mathsf{cSet}_B \to \mathsf{cSet}_A$ coincides with that created by $i^* : \mathsf{cSet}_B \to \mathsf{cSet}_\varnothing$. We could then apply the argument of Proposition 4.9 verbatim to obtain identical characterizations of both model structures' cofibrations.

All of the cube categories under consideration here, except for those which contain diagonals but not symmetries, are shown to be test categories in [3, Corollary 3]. With this in mind, we may obtain an alternative characterization of the left-induced Grothendieck model structures of Theorem 4.7 in these cases.

Proposition 4.13 *Let $\square_A$ be a cube category containing at least one kind of connection which is also a test category. Then the left-induced Grothendieck model structure of Theorem 4.7 coincides with the test model structure on cSet_A.*

Proof Both the test model structure and the left-induced Grothendieck model structure have monomorphisms as their cofibrations (the latter by Proposition 4.9), so it suffices to show that the weak equivalences of the test model structure are created by $i^* : \square_A \to \square_\varnothing$. For this, by [6, Theorem 4.2.23] (implication $(b'') \Rightarrow (d)$) it suffices to show that $i^*\square^n_A$ is contractible in the Grothendieck model structure on $\mathsf{cSet}_\varnothing$ for all n. This, in turn, follows from Proposition 4.1 and the contractibility of $\square^n_\varnothing$ in the Grothendieck model structure. $\square$

Remark 4.14 In the case $A \subseteq \{\wedge, \vee\}$, we have Quillen equivalences $T : \mathsf{cSet}_A \rightleftarrows \mathsf{sSet} : U$ between the cubical Joyal (resp. Grothendieck) model structure on cSet_A and the Joyal (resp. Quillen) model structure on sSet [9, Theorems 6.1 and 6.26]. The left adjoint T, the *triangulation functor*, which sends each cube $\square^n_A$ to the simplicial set $(\Delta^1)^n$, creates the weak equivalences of cSet_A in both cases. Thus, for any inclusion of cube categories $i : \square_A \hookrightarrow \square_B$ with $A \subseteq \{\wedge, \vee\}$, the weak equivalences of the induced model structures on cSet_B are created by the composite functor $Ti^* : \mathsf{cSet}_B \to \mathsf{sSet}$.

The triangulation functor can be similarly defined for any cube category $\square_B$ not containing reversals. For the case $B = \{\vee, \Sigma, \delta\}$ (and similarly for the isomorphic case $B = \{\wedge, \Sigma, \delta\}$), [5, Theorem 7.8 and Corollary 7.23] shows that $T : \mathsf{cSet}_B \to \mathsf{sSet}$, where sSet is equipped with the Quillen model structure, creates the weak equivalences of the test model structure. Thus T creates the same class of weak equivalences as i^* by Proposition 4.13. In the case $B = \mathcal{P}$, it can similarly be shown that T creates the weak equivalences of the test model structure using the characterization of T given in [22].

In general, it is an open question whether $T : \mathsf{cSet}_B \to \mathsf{sSet}$ (where sSet is equipped with the Quillen or Joyal model structure as appropriate) creates the weak equivalences of the induced model structures on cSet_B. In the case $B = \mathcal{P}$, proving this for the left-induced cubical Joyal model structure would be equivalent to proving that it coincides with that constructed in [14, Proposition 2.3] under the name $\mathsf{cSet}_{(m,1)}$, which also has monomorphisms as its cofibrations and has weak equivalences created by triangulation.

5 Cartesian monoidality

We now consider the Cartesian product of cubical sets, with the aim of showing that the cubical Joyal model structures on categories cSet_A with A a nonempty subset of $\{\wedge, \vee\}$ are cartesian monoidal. We will do this using a natural map between the geometric and cartesian products, which we will show to be a trivial cofibration in the cubical Joyal model structure. Our techniques will also allow us to obtain a new proof of cartesian monoidality for the Grothendieck model structures on these categories. (That the Grothendieck model structures are cartesian monoidal was previously known, as a consequence of the fact that in these cases $\square_A$ is a strict test category — see [21, Proposition 4.3] as well as [7, Theorem 1.7].)

We begin with a general result concerning pushout products in model categories.

Lemma 5.1 *Let $\mathcal{C}$ be a model category equipped with a bifunctor $\odot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Suppose the following are true*:

- *if i and j are cofibrations in $\mathcal{C}$, then so is the pushout product $i \mathbin{\widehat{\odot}} j$;*
- *the bifunctor $\odot$ preserves the cofibrations and trivial cofibrations of $\mathcal{C}$ in each variable.*

Then a pushout product of cofibrations $i \mathbin{\widehat{\odot}} j$ is a trivial cofibration if either i or j is trivial.

Proof Let $i : A \to B$ and $j : X \to Y$ be a pair of cofibrations in $\mathcal{C}$, and suppose that j is trivial; the case where i is trivial is similar. The pushout product $i \mathbin{\widehat{\odot}} j$ is a cofibration by assumption, so we need only show that it is a weak equivalence.

We first note that $A \odot j : A \odot X \to A \odot Y$ is a trivial cofibration by assumption. Thus its pushout $B \odot X \to A \odot Y \cup_{A \odot X} B \odot X$ is a trivial cofibration as well. Likewise, $B \odot j : B \odot X \to B \odot Y$ is a trivial cofibration by assumption.

Now consider the commuting diagram

$$\begin{tikzcd} B \odot X \arrow[r] \arrow[dr, "B \odot j"'] & A \odot X \cup_{A \odot X} B \odot X \arrow[d, "i \mathbin{\widehat{\odot}} j"] \\ & B \odot Y \end{tikzcd}$$

It follows that $i \mathbin{\widehat{\odot}} j$ is a weak equivalence by two-out-of-three. □

5.1 Comparison of the geometric and cartesian products

We now proceed to our analysis of the geometric and cartesian products of cubical sets. Throughout this subsection, fix $A \subseteq B \subseteq \mathcal{P}$, with $\Sigma, \delta \in B$. This implies, in particular, that for each $n \geq 0$ there is a map $D_n : \square^n_B \to \square^{2n}_B$ sending $(a_1, \ldots, a_n)$ to $(a_1, \ldots, a_n, a_1, \ldots, a_n)$. (For $n = 0$ this is the identity.)

To aid in understanding the maps we will use in our proofs, we define the following composites of degeneracy morphisms for $m, n \geq 0$:

- The map $\sigma^F_m : \square^{m+n}_A \to \square^n_A$ deletes the first m components, sending $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ to $(b_1, \ldots, b_n)$.
- The map $\sigma^L_n : \square^{m+n}_A \to \square^m_N$ deletes the last n components, sending $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ to $(a_1, \ldots, a_m)$.

(In particular, both σ^F_0 and σ^L_0 are identities.) By straightforward calculations, we can obtain the following lemmas involving the maps defined above.

Lemma 5.2 *For any $\phi : \square^m_B \to \square^n_B$, we have $D_n \phi = (\phi \otimes \phi) D_m$.* □

Lemma 5.3 *For any integers $m, n \geq 0$, the map $\sigma^L_n \otimes \sigma^F_m : \square^{m+n}_A \otimes \square^{m+n}_A \to \square^m_A \otimes \square^n_A$ is a retraction of D_{m+n}.* □

Note that we may take either of the dimension variables in the lemma above to be 0, obtaining the result for any n, the maps $\sigma^L_n, \sigma^F_n : \square^{2n}_A \to \square^n_A$ are retractions of D_n.

We begin by defining the comparison map which will be our object of study. For any $A \subseteq \mathcal{S}$, given $X, Y \in \mathsf{cSet}_A$, we define the projection map π_X to be the composite $X \otimes Y \to X \otimes \square^0_A \cong X$. We similarly define a projection map $\pi_Y : X \otimes Y \to Y$.

Then for a pair of cubes $x : \square^m_A \to X$, $y : \square^n_A \to Y$, the map π_X sends the $(m+n)$-cube $x \otimes y$ of $X \otimes Y$ to $x\sigma^L_n$, while π_Y sends $x \otimes y$ to $y\sigma^F_m$. In the case of a pair of 1-cubes x and y, with initial and terminal vertices x_0, x_1, y_0, y_1, we illustrate $x \otimes y$ and its images under the projection maps to X and Y as

$$\begin{tikzcd} x_0 \otimes y_0 \arrow[r, "x \otimes y_0"] \arrow[d, "x_0 \otimes y"'] \arrow[dr, phantom, "x \otimes y"] & x_1 \otimes y_0 \arrow[d, "x_1 \otimes y"] \\ x_0 \otimes y_1 \arrow[r, "x \otimes y_1"'] & x_1 \otimes y_1 \end{tikzcd} \qquad \begin{tikzcd} x_0 \arrow[r, "x"] \arrow[d, equal] \arrow[dr, phantom, "x\sigma_2"] & x_1 \arrow[d, equal] \\ x_0 \arrow[r, "x"'] & x_1 \end{tikzcd} \qquad \begin{tikzcd} y_0 \arrow[r, equal] \arrow[d, "y"'] \arrow[dr, phantom, "y\sigma_1"] & y_0 \arrow[d, "y"] \\ y_1 \arrow[r, equal] & y_1 \end{tikzcd}$$

These images are the degeneracies $x\sigma^L_1 = x\sigma_2$ and $y\sigma^F_1 = y\sigma_1$, respectively.

The projections π_X, π_Y induce a natural map $X \otimes Y \to X \times Y$, sending a cube $x \otimes y$ as above to the pair $(x\sigma^L_n, y\sigma^F_m)$. It is this natural map which will be our key tool in comparing the geometric and cartesian products.

For $B \subseteq \mathcal{S}$ with $A \subseteq B$ and $\Sigma, \delta \in B$, we also define a natural map $X \times Y \to i^* i_!(X \otimes Y)$, as follows: given a pair of cubes $x : \square^n_A \to X$, $y : \square^n_A \to Y$, we send the pair (x, y) to $(x \otimes y)D_n$. We now verify that this defines a valid map of cubical sets.

Proposition 5.4 *For every $X, Y \in \mathsf{cSet}_A$, the assignment given above defines a map of cubical sets $X \times Y \to i^* i_!(X \otimes Y)$.*

Proof We must show that the given assignment is compatible with the structure maps of $X \times Y$. To that end, consider $x : \square^n_A \to X$, $y : \square^n_A \to Y$, and a map $\phi : \square^m_A \to \square^n_A$. Then, applying Proposition 2.5 and Lemma 5.2, we can see that $(x, y)\phi = (x\phi, y\phi)$ is mapped to

$$(x\phi \otimes y\phi)D_m = (x \otimes y)(\phi \otimes \phi)D_m = (x \otimes y)D_n\phi. \qquad \square$$

Lemma 5.5 *The composite $X \otimes Y \to X \times Y \to i^* i_!(X \otimes Y)$ of the two maps defined above is equal to the unit of the adjunction $i_! \dashv i^*$.*

Proof Given $x : \square^m_A \to X$ and $y : \square^n_A \to Y$, by Proposition 2.5 and Lemma 5.3 the image of $x \otimes y$ under the composite map is

$$(x\sigma^L_n \otimes y\sigma^F_m)D_{m+n} = (x \otimes y)(\sigma^L_n \otimes \sigma^F_m)D_{m+n} = x \otimes y.$$

Thus the composite and the unit agree on all cubes of the form $x \otimes y$; by Proposition 2.5, this is sufficient to show that they agree on all cubes of $X \otimes Y$. $\square$

This result provides us with a proof of the well-known fact that in categories of cubical sets with symmetries and diagonals, the geometric and cartesian products coincide.

Proposition 5.6 *If $A \subseteq \mathcal{S}$ with $\Sigma, \delta \in A$, then the natural map $X \otimes Y \to X \times Y$ is an isomorphism.*

Proof In this case, we may apply Lemma 5.5 with $A = B$, so that $i_! \dashv i^*$ is the identity adjunction on cSet_A. Thus we see that the composite $X \otimes Y \to X \times Y \to X \otimes Y$ is the identity on $X \otimes Y$.

It thus remains to consider the composite $X \times Y \to X \otimes Y \to X \times Y$. Given $x : \square^n_A \to X$, $y : \square^n_A \to Y$, the map $X \times Y \to X \otimes Y$ sends the pair (x, y) to $(x \otimes y)D_n$. The map $X \otimes Y \to X \times Y$ then sends this cube to $(x\sigma^L_n, y\sigma^F_n)D_n$. By definition, this is equal to $(x\sigma^L_n D_n, y\sigma^F_n D_n)$; by Lemma 5.3, this is equal to (x, y). Thus this composite is the identity on $X \times Y$. $\square$

5.2 Cartesian monoidality of cubical Joyal model structures

Our next goal is to use the comparison map between the geometric and cartesian products to prove our desired cartesian monoidality results for the cubical Joyal model structures. Throughout this subsection, fix A, B as in the previous subsection, with the additional assumptions that A is a nonempty subset of $\{\wedge, \vee\}$ and $\rho \notin B$. For concreteness, we will assume $\wedge \in A$; as in Section 4, the case where $\vee \in A$ follows by parallel arguments involving negative connections.

Lemma 5.7 *For all $X, Y \in \mathsf{cSet}_A$, the morphisms $X \otimes Y \to X \times Y$ and $X \times Y \to i^*i_!(X \otimes Y)$ are monomorphisms.*

Proof For $X \otimes Y \to X \times Y$, this is immediate from Corollary 3.10 and Lemma 5.5.

To see that $X \times Y \to i^*i_!(X \otimes Y)$ is a monomorphism, suppose that for some cubes $x, x' : \square^n_A \to X$, $y, y' : \square^n_A \to Y$, the pairs (x, y) and (x', y') are mapped to the same cube of $i^*i_!(X \otimes Y)$ — in other words, that $(x \otimes y)D_n = (x' \otimes y')D_n$. Applying the map $i^*i_!\pi_X$, we see that $x\sigma^L_n D_n = x'\sigma^L_n D_n$ in $i^*i_!X$. By Lemma 5.3, it follows that $x = x'$ as cubes of $i^*i_!X$; thus $x = x'$ in X by Corollary 3.10. A similar proof shows that $y = y'$. $\square$

In particular, considering the case where X and Y are representable, we have for any $m, n \geq 0$ a composable pair of monomorphisms

$$\square^{m+n}_A \to \square^m_A \times \square^n_A \to i^*\square^{m+n}_B.$$

We aim to show that this composable pair of monomorphisms witnesses $\square^m_A \times \square^n_A$ as a decomposition-closed subcomplex of $i^*i_!\square^{m+n}_B$, in order to apply the results of Section 3.3. To do this, we will first characterize the image in $i^*\square^{m+n}_B$ of $\square^m_A \times \square^n_A$.

Proposition 5.8 *The image of the map $\square^m_A \times \square^n_A \to i^*\square^{m+n}_B$ consists of all morphisms ϕ in $\square_B$ with codomain $\square^{m+n}_B$ such that $\sigma^F_m\phi$ and $\sigma^L_n\phi$ are in $\square_A$.*

Proof We first show that any k-cube ϕ of $i^*\square^{m+n}_B$ which satisfies the given criterion when considered as a morphism in $\square_B$ is contained in the image of $\square^m_A \times \square^n_A$. In this case, the pair $(\sigma^L_n\phi, \sigma^F_m\phi)$ defines a k-cube of $\square^m_A \times \square^n_A$, which is mapped to $(\sigma^L_n\phi \otimes \sigma^F_m\phi)D_k$. Using Proposition 2.5 and Lemma 5.2, we may calculate

$$(\sigma^L_n\phi \otimes \sigma^F_m\phi)D_k = (\sigma^L_n \otimes \sigma^F_m)(\phi \otimes \phi)D_k = (\sigma^L_n \otimes \sigma^F_m)D_{m+n}\phi = \phi.$$

Next we will show that all cubes in the image of $\square_A^m \times \square_A^n$ satisfy the given criterion. Consider a pair of k-cubes $\phi : \square_A^k \to \square_A^m, \phi' : \square_A^k \to \square_A^n$. The k-cube (ϕ, ϕ') of $\square_A^m \times \square_A^n$ is mapped to $(\phi \otimes \phi')D_k$. A straightforward calculation involving Lemmas 5.2 and 5.3 shows that the diagram

$$\begin{tikzcd} \square_B^k \arrow[r, "D_k"] \arrow[dr, equal] & \square_B^k \otimes \square_B^k \arrow[r, "\phi\otimes\phi'"] \arrow[d, "\sigma_k^F"] & \square_B^m \otimes \square_B^n \arrow[d, "\sigma_m^F"] \\ & \square_B^k \arrow[r, "\phi'"] & \square_B^n \end{tikzcd}$$

commutes. Thus $\sigma_m^F(\phi \otimes \phi')D_k = \phi'$. A similar proof shows $\sigma_n^L(\phi \otimes \phi')D_k = \phi$. $\square$

Proposition 5.9 *For any $m, n \geq 0$, the maps $\square_A^{m+n} \to \square_A^m \times \square_A^n \to i^*\square_B^{m+n}$ witness $\square_A^m \times \square_A^n$ as a decomposition-closed subcomplex of $i^*\square_B^{m+n}$.*

Proof The two maps are monomorphisms by Lemma 5.7, and compose to the unit of $i_! \dashv i^*$ by Lemma 5.5. Thus it remains to show that if a cube $\kappa(\phi \otimes \square_B^k)$ of $i^*\square_B^{m+n}$, where ϕ is active and κ is a composite of face maps, is contained in the image of $\square_A^m \otimes \square_A^n$, then so is $\kappa N_k(\phi)$. By Proposition 5.8, this is equivalent to showing that if $\sigma_m^F \kappa(\phi \otimes \square_B^k)$ and $\sigma_n^L \kappa(\phi \otimes \square_B^k)$ are contained in $\square_A$, then so are $\sigma_m^F \kappa N_k(\phi)$ and $\sigma_n^L \kappa N_k(\phi)$.

We first note that for any face composite $\kappa : \square^p \to \square^{m+n}$ there exist commuting diagrams

$$\begin{tikzcd} \square_B^p \arrow[r, "\sigma_{p-p'}^L"] \arrow[d, "\kappa"'] & \square_B^{p'} \arrow[d, "\kappa'"] \\ \square_B^{m+n} \arrow[r, "\sigma_n^L"] & \square_B^m \end{tikzcd} \qquad \begin{tikzcd} \square_B^p \arrow[r, "\sigma_{p-p''}^F"] \arrow[d, "\kappa"'] & \square_B^{p''} \arrow[d, "\kappa''"] \\ \square_B^{m+n} \arrow[r, "\sigma_m^F"] & \square_B^n \end{tikzcd}$$

for some face composites $\kappa' : \square_B^{p'} \to \square_B^m$, $\kappa'' : \square_B^{p''} \to \square_B^n$. Thus the statement we aim to prove can be rephrased as: if $\kappa'\sigma_{p-p'}^L(\phi \otimes \square_B^k)$ and $\kappa''\sigma_{p-p''}^F(\phi \otimes \square_B^k)$ are in $\square_A$, then so are $\kappa'\sigma_{p-p'}^L N_k(\phi)$ and $\kappa''\sigma_{p-p''}^F N_k(\phi)$.

Moreover, for any face composite $\kappa : \square_B^r \to \square_B^s$ and any $\psi : \square_B^q \to \square_B^r$, the composite $\kappa\psi$ is in $\square_A$ if and only if ψ is in $\square_A$ (because every face map is in $\square_A$, and has a retraction in $\square_A$.) Thus it will suffice to prove the following: for any morphism $\psi : \square_B^q \to \square_B^r$ and any $k \geq 0$ and $0 \leq i \leq r + k$, if $\sigma_i^F(\psi \otimes \square_B^k)$ is in $\square_A$, then so is $\sigma_i^F N_k(\psi)$, and likewise, if $\sigma_i^L(\psi \otimes \square_B^k)$ is in $\square_A$ then so is $\sigma_i^L N_k(\psi)$. Throughout our proof, a tuple $(a_1, \ldots, a_q, b, c_1, \ldots, c_k)$ in $\square_B^{q+1+k}$ will be abbreviated by $(\vec{a}, b, \vec{c})$.

First we prove the statement involving postcomposition with σ_i^F. If $i < r$, then $\sigma_i^F = \sigma_i^F \otimes \square_B^k$, so that $\sigma_i^F(\psi \otimes \square_B^k) = \sigma_i^F \psi \otimes \square_B^k$. This map is in $\square_A$ by assumption, therefore $\sigma_i^F \psi \otimes \square_B^1 \otimes \square_B^k$ is in $\square_A$ as well. Hence, the composite $\gamma_{r-i,1}(\sigma_i^F \psi \otimes \square_B^1 \otimes \square_B^k)$ is also in $\square_A$. Using Lemma 3.13, we may compute that this composite sends a tuple $(\vec{a}, b, \vec{c})$ to

$$(\psi(\vec{a})_{i+1}, \ldots, \psi(\vec{a})_{r-1}, \psi(\vec{a})_r \wedge b, c_1, \ldots, c_k).$$

We may likewise compute that this is the image of $(\vec{a}, b, \vec{c})$ under $\sigma_i^F N_k(\psi)$.

If $i \geq r$, then $\sigma_i^F N_k(\psi)$ sends $(\vec{a}, b, \vec{c})$ to

$$(c_{i-r+1}, \ldots, c_k).$$

Thus this map is the degeneracy $\sigma^F_{q+1+i-r}$.

We now consider the case of postcomposition with σ_i^L. In this case, if $1 \leq i \leq k$, then we may note that $\sigma_i^L = \Box_B^r \otimes \sigma_i^L$. We thus have $\sigma_i^L(\psi \otimes \Box_B^k) = \psi \otimes \sigma_i^L = (\psi \otimes \Box_B^{k-i})\sigma_i^L$.

So $(\psi \otimes \Box_B^{k-i})\sigma_i^L$ is in $\Box_A$; precomposing with a section of σ_i^L, we observe that $\psi \otimes \Box_B^{k-i}$ is in $\Box_A$, hence so is $\psi \otimes \Box_B^{1+k}$. It follows that the composite $\sigma_i^L \gamma_{r,1}(\psi \otimes \Box_B^{1+k}) = \sigma_i^L N_k(\phi)$ is in $\Box_A$ as well.

If $k < i \leq r+k$, then $\sigma_i^L N_k(\psi)$ sends $(\vec{a}, b, \vec{c})$ to

$$(\psi(\vec{a})_1, \ldots, \psi(\vec{a})_{r-(i-k)}).$$

Thus this map is equal to $\sigma_i^L(\psi \otimes \Box_B^k)$, and is therefore in $\Box_A$ by assumption. □

Corollary 5.10 *For all $X, Y \in \mathsf{cSet}_A$, the natural maps $X \otimes Y \to X \times Y$ and $X \times Y \to i^* i_!(X \otimes Y)$ are trivial cofibrations.*

Proof Both maps are monomorphisms by Lemma 5.7, so we need only prove that they are weak equivalences. Because the bifunctors $\otimes, \times$, and $i^* i_! \otimes$ all preserve cofibrations and colimits in each variable, by Proposition 2.6 and a standard induction on skeleta argument it suffices to consider the case where both X and Y are representable. This case is immediate from Propositions 3.25 and 5.9. □

Corollary 5.11 *The bifunctor $\times : \mathsf{cSet}_A \to \mathsf{cSet}_A$ preserves weak equivalences in each variable.*

Proof This is immediate from Theorem 2.19 and Corollary 5.10. □

Theorem 5.12 *The cubical Joyal model structure on cSet_A is cartesian monoidal.*

Proof Since all objects are cofibrant, we need only prove that the cartesian product on cSet_A satisfies the pushout product axiom. That a pushout product of cofibrations is a cofibration follows from the corresponding result for monomorphisms of sets. Thus the pushout product axiom follows from Lemma 5.1 and Corollary 5.11. □

We likewise obtain a new proof of the following previously known result.

Theorem 5.13 *The Grothendieck model structure on cSet_A is cartesian monoidal.*

Proof The natural map $X \otimes Y \to X \times Y$ is a trivial cofibration in the Grothendieck model structure by Proposition 2.20 and Corollary 5.10, so we may proceed exactly as in the proof of Corollary 5.11 and Theorem 5.12. □

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BRANDON DOHERTY bcd23b@fsu.edu

Department of Mathematics, Florida State University, Tallahassee, FL, United States

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