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Explicit formulas for the Hattori–Stong theorem and applications

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We employ combinatorial techniques to present an explicit formula for the coefficients of Chern classes involved in the Hattori–Stong integrability conditions. We also give an evenness condition for the signature of stably almost-complex manifolds in terms of Chern numbers. As an application, it can be shown that the signature of a $2n$-dimensional stably almost-complex manifold whose possibly nonzero Chern numbers being c_n and $c_i c_{n-i}$ is even, which particularly rules out the existence of such structure on rational projective planes. Some other related results and remarks are also discussed.

1 Introduction

Let E and F be two complex vector bundles over a topological space X, $\wedge^k(\cdot)$ the k-th exterior power operation and

$$\wedge_t(E) := 1 + \sum_{k\geq 1} \wedge^k(E)t^k \quad (1 := \text{trivial line bundle}).$$

The operation $\wedge_t(\cdot)$ in K-theory satisfies

$$\wedge_t(E-F) = \wedge_t(E)[\wedge_t(F)]^{-1} \in K(X)[\![t]\!]. \tag{1-1}$$

The K-theory operations $\gamma^k(\cdot)$ are defined by

$$1 + \sum_{k\geq 1} \gamma^k(E-F)t^k := \wedge_{\frac{t}{1-t}}(E-F) \in K(X)[\![t]\!]. \tag{1-2}$$

The importance of these operations $\gamma^k(\cdot)$ can be appreciated from at least two aspects. Firstly, these $\gamma^k(\cdot)$ form power series generators over the integers for all K-theory operations [3, p. 128]. Secondly, if $\widetilde{E} := E - \underline{\mathbb{C}}^{\dim(E)}$, where $\underline{\mathbb{C}}^{\dim(E)}$ denotes the trivial bundle of rank E, then $\gamma^1(\widetilde{\xi_n}), \dots, \gamma^n(\widetilde{\xi_n})$ generate over the integers the K-ring of the classifying space $\mathrm{BU}(n)$ in a suitable sense [22, p. 253], where ξ_n is the tautological n-plane bundle over $\mathrm{BU}(n)$.

Let (M, τ) be a closed stably almost-complex manifold, where τ is a complex vector bundle over a closed smooth manifold M such that the underlying real vector bundle $\tau_{\mathbb{R}}$ is isomorphic to the stable tangent bundle of M. The Chern classes of M are defined to be those of τ: $c_i(M) := c_i(\tau) \in H^{2i}(M;\mathbb{Z})$. When $\dim(M) = 2n$, with respect to the canonical orientation induced from τ the set of Chern numbers $\{c_\lambda[M] \mid \lambda \text{ is an integer partition of weight } n\}$ can be defined. It is well known that two stably almost-complex manifolds are complex cobordant if and only if they have the same Chern numbers [27].

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The integrality of the linear combinations of Chern numbers

$$\int_M \mathrm{ch}\big(\gamma^{k_1}(\tilde{\tau})\big)\cdots\mathrm{ch}\big(\gamma^{k_i}(\tilde{\tau})\big)\,\mathrm{td}(M)\in\mathbb{Z},\quad \forall\, k_1,\dots,k_i\in\mathbb{Z}_{\geq 0},\ \gamma^0(\tilde{\tau}):=1, \tag{1-3}$$

was first observed by Atiyah and Hirzebruch as corollaries of their differential Riemann–Roch theorem [4; 5], where $\mathrm{ch}(\cdot)$ denotes the Chern character and $\mathrm{td}(\cdot)$ the Todd polynomial. Now (1-3) is a direct corollary of the Atiyah–Singer index theorem [6] as each $\gamma^k(\widetilde{E})$ is an *integral* linear combination of those $\wedge^i(E)$ due to (1-1) and (1-2). Conversely, they conjectured in [5] that (1-3) give all the integral relations for Chern numbers, which was proved by Hattori and Stong independently [12; 33] and is now called the Hattori–Stong theorem. More precisely, we have:

Theorem 1.1 (Atiyah–Hirzebruch, Hattori–Stong) *Given a positive integer n and a set of integers $\{c_\lambda \mid \lambda$ is a partition of weight $n\}$, they can be realized as Chern numbers of some $2n$-dimensional stably almost-complex manifold if and only if they satisfy the integral conditions* (1-3).

There are also analogies to Theorem 1.1 for smooth and spin manifolds, in which the Hirzebruch's L-polynomial and the $\hat{A}$-polynomial are involved [33; 34; 35]. In order to effectively apply Theorem 1.1 and these related results in concrete problems, we need, when expanding these quantities in terms of Chern and Pontrjagin classes, *both closed and explicit* formulas for coefficients in front of them. Recently some considerable progress towards it has been made. Berglund and Bergström gave explicit formulas in terms of multiple zeta values for the coefficients in front of Pontrjagin classes for the L-polynomial and the $\hat{A}$-polynomial [8]. Combining some ideas in [8] and other combinatorial tools, Li [24] explored the coefficients in front of Chern classes for the complex genera including the Todd polynomial. So in order to apply Theorem 1.1, we need to know an explicit formula for the coefficients $b_\lambda^{(k)}$ when expanding $\mathrm{ch}\big(\gamma^k(\widetilde{E})\big)$ in terms of the Chern monomials $c_\lambda(E)$:

$$\mathrm{ch}\big(\gamma^k(\widetilde{E})\big)=\sum_\lambda b_\lambda^{(k)}\cdot c_\lambda(E),\quad b_\lambda^{(k)}\in\mathbb{Q}. \tag{1-4}$$

Our first main result (Theorem 2.1) is to give an explicit closed formula for $b_\lambda^{(k)}$ in (1-4) by further developing the techniques in [24]. *Our second main result* (Theorem 2.4) is to give an evenness condition in terms of Chern numbers for the signature of stably almost-complex manifolds. To this end, still applying the tools in [24], we shall give an explicit formula for the coefficients in front of Chern numbers when expressing the signature of a stably almost-complex manifold in terms of them. As an application, we combine the first two main results to show that the signature of a $2n$-dimensional stably almost-complex manifold whose possibly nonzero Chern numbers being c_n and c_ic_{n-i} for some i is even (Theorem 2.6), which particularly excludes the existence of such a structure on rational projective planes. Some other related results and remarks are also discussed at places along the main line of this article.

The rest of this article is organized as follows. The main results as well as some necessary background notation are stated in Section 2. Some preliminaries on symmetric functions and Stirling numbers of the second kind are presented in Section 3. Then Sections 4, 5 and 6 are devoted to the proofs of Theorems 2.1, 2.4 and 2.6 respectively. Some remarks on integral/rational projective planes are collected in Section 7.

2 Main results

2.1 Background notation

Before stating the main results, we introduce some more notation in combinatorics.

An *integer partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is a finite sequence of positive integers in nonincreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$. Denote by $l(\lambda) := l$ and $|\lambda| := \sum_{i=1}^{l} \lambda_i$ the *length* and *weight* of the partition λ respectively. These λ_i are called *parts* of the integer partition λ. It is also convenient to use another notation which indicates the number of times each integer appears: $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots)$. This means that i appears $m_i(\lambda)$ times among $\lambda_1, \dots, \lambda_{l(\lambda)}$. For example,

$$\lambda = (6, 5, 5, 4, 2, 2, 1, 1, 1) = (1^3 2^2 3^0 4^1 5^2 6^1).$$

With this notation the Chern monomials $c_\lambda(E)$ in (1-4) are defined by

$$c_\lambda(E) := \prod_{i=1}^{l(\lambda)} c_{\lambda_i}(E).$$

Given a (nonempty) finite set S, a *partition of* S is a collection of disjoint nonempty subsets of S whose union is exactly S [31, p. 33]. Namely, a partition π of S is of the form $\pi = \{\pi_1, \dots, \pi_k\}$, where each π_i is a nonempty subset of S, $\pi_i \cap \pi_j = \varnothing$ for any $i \neq j$, and $\bigcup_i \pi_i = S$. In this case, each π_i is called a *block* of π. Let $l(\pi) = k$ and call it the *length* of π. As usually $|\pi_i|$ denotes the cardinality of π_i.

Put $[n] := \{1, \dots, n\}$ and denote by Π_n the set consisting of all partitions of $[n]$. For instance, $\big\{\{1,3,6\},\{2\},\{4,5\}\big\} \in \Pi_6$, whose length is 3.

Define $S(n,k)$ to be the number of partitions of $[n]$ into exactly k blocks. $S(n,k)$ is usually called *a Stirling number of the second kind* [31, p. 33]. Then $S(n,k) > 0$ if $1 \leq k \leq n$. By convention we put

$$S(n,k) := 0 \quad \text{if } k > n, \qquad S(n,0) := 0 \quad \text{if } n > 0, \qquad S(0,0) := 1. \tag{2-1}$$

For example, $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$, $S(n,n-1) = \binom{n}{2}$ and $S(n,n) = 1$.

Let λ be an integer partition and consider $\Pi_{l(\lambda)}$, which consists of all partitions of $[l(\lambda)] = \{1, 2, \dots, l(\lambda)\}$. If π is the partition $\{\pi_1, \dots, \pi_{l(\pi)}\} \in \Pi_{l(\lambda)}$, then define

$$\lambda_{\pi_i} := \sum_{j \in \pi_i} \lambda_j \quad \big(1 \leq i \leq l(\pi)\big).$$

Let $B_i \in \mathbb{Q}_{>0}$ ($i \geq 1$) be the *Bernoulli numbers* without sign [28, p. 281], which are defined by

$$\frac{x}{\sinh(x)} =: 1 + \sum_{i \geq 1} \frac{(-1)^i (2^{2i} - 2) B_i}{(2i)!} x^{2i} \quad \big(B_1 = \tfrac{1}{6},\ B_2 = \tfrac{1}{30},\ B_3 = \tfrac{1}{42}, \dots\big). \tag{2-2}$$

2.2 Main results

With the notation above in mind, we now state in this subsection our main results.

The first result is the following explicit closed formula for $b_\lambda^{(k)}$ in (1-4).

Theorem 2.1 *Let E be a complex vector bundle over some topological space. Fix an integer partition $\lambda=(\lambda_1,\dots,\lambda_{l(\lambda)})=(1^{m_1(\lambda)}2^{m_2(\lambda)}\dots)$. Then the coefficients $b_\lambda^{(k)}$ $(k\geq 1)$ in (1-4) are given by*

$$\sum_{k\geq 1} b_\lambda^{(k)}\cdot t^k = \frac{(-1)^{|\lambda|-l(\lambda)}}{\prod_{i\geq 1} m_i(\lambda)!}\sum_{\pi\in\Pi_{l(\lambda)}}\left\{\left[\prod_{i=1}^{l(\pi)}(|\pi_i|-1)!\right]\left[\prod_{i=1}^{l(\pi)}\sum_{j=0}^{\lambda_{\pi_i}-1}\frac{S(\lambda_{\pi_i},\lambda_{\pi_i}-j)}{\binom{\lambda_{\pi_i}-1}{j}\cdot j!}(-t)^{\lambda_{\pi_i}-1-j}\right]t^{l(\pi)}\right\}. \tag{2-3}$$

In other words, $b_\lambda^{(k)}$ is the coefficient in front of t^k on the right-hand side of (2-3).

Remark 2.2 The highest degree of t on the right-hand side of (2-3) is

$$l(\pi)+\sum_{i=1}^{l(\pi)}(\lambda_{\pi_i}-1)=|\lambda|,$$

and hence $b_\lambda^{(k)}=0$ whenever $k>|\lambda|$. More vanishing information on $b_\lambda^{(k)}$ can be seen in Lemma 4.4.

The following example is a simple illustration of Theorem 2.1 and it can be easily checked.

Example 2.3 When $l(\lambda)\leq 2$, the values $b_\lambda^{(k)}$ given by (2-3) are

$$b_{(i)}^{(k)}=\frac{(-1)^{i-k}S(i,k)\cdot(k-1)!}{(i-1)!},\qquad b_{(i,i)}^{(k)}=\frac{1}{2\cdot(2i-1)!},\qquad b_{(i,j)}^{(k)}=\frac{(-1)^{i+j}}{(i+j-1)!}\quad (i>j). \tag{2-4}$$

Let $\sigma(M)$ be the signature of a closed oriented smooth manifold M, which is zero unless $\dim(M)=4k$ for some $k\in\mathbb{Z}_{>0}$. By Hirzebruch's signature theorem [14] $\sigma(M)$ is a rational linear combination of Pontrjagin numbers. In [8, Theorem 1] an explicit and closed formula in terms of (some variant of) multiple zeta values for the coefficients in front of these Pontrjagin numbers is presented. When M admits a stably almost-complex structure, Pontrjagin numbers are determined by Chern numbers and so is $\sigma(M)$. Our following second result is an evenness condition for the $\sigma(M)$ in terms of Chern numbers via an explicit formula for the coefficients in front of them.

Theorem 2.4 *Let*

$$h_{2i}:=\frac{(-1)^i\cdot 2^{2i+1}\cdot(2^{2i-1}-1)\cdot B_i}{(2i)!}\quad\text{and}\quad h_{2i-1}:=0\qquad (i\geq 1). \tag{2-5}$$

Let M be a $4k$-dimensional closed stably almost-complex manifold with Chern numbers $c_\lambda[M]$ and

$$\sigma(M)=:\sum_{|\lambda|=2k} h_\lambda\cdot c_\lambda[M].$$

(1) *The coefficients* h_λ *are given by*

$$h_\lambda = \frac{(-1)^{l(\lambda)}}{\prod_{i\geq 1} m_i(\lambda)!} \sum_{\pi\in\Pi_{l(\lambda)}} \Big\{(-1)^{l(\pi)} \prod_{i=1}^{l(\pi)} \big[(|\pi_i|-1)!\cdot h_{\lambda_{\pi_i}}\big]\Big\}. \tag{2-6}$$

(2) *If the Chern numbers* $c_\lambda[M]$ *satisfy*

$$\{\lambda \mid c_\lambda[M]\neq 0\} \subset \{\lambda \mid m_i(\lambda) = 0 \text{ or } 1 \text{ for all } i\}, \tag{2-7}$$

then $\sigma(M)$ *is even.*

Remark 2.5 (1) The condition (2-7) means that if some Chern number $c_\lambda[M]$ is nonzero, then all the parts λ_i of this integer partition λ must be mutually distinct.

(2) By (2-6), the coefficient in front of the top Chern number $c_{2k}[M]$ is $h_{(2k)} = h_{2k}$, which was known to Hirzebruch [17, Formula (8)]. Before [8], Fowler and Su presented a *recursive* formula [11, Appendix] for the coefficients of $\sigma(M)$ in front of Pontrjagin numbers.

An n-dimensional smooth closed connected orientable manifold whose Betti numbers satisfy

$$b_0 = b_{n/2} = b_n = 1 \qquad \text{and} \qquad b_i = 0 \quad \text{for other } i$$

is called a *rational projective plane* (RPP for short). Classical such manifolds include complex, quaternionic and octonionic projective planes $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$, whose dimensions are 4, 8 and 16 respectively. Various constraints on the possibly existing dimensions on RPP were studied by Adem and Hirzebruch [2; 13; 17; 18] and more details can be found in Section 7. It turns out that such an existence problem is closely related to that of manifolds with prescribed Betti numbers (see (7-1)). Recently the existence problem of RPP was systematically treated by Zhixu Su and her coauthors [11; 23; 36; 37]. Moreover Su [38] and Hu [20] independently showed that any RPP whose dimension is larger than 4 cannot admit any almost-complex structure.

The signature of any RPP is ± 1 and hence odd. As a simple application of Theorems 2.1 and 2.4, we shall show the following result, which particularly excludes the existence of *stably almost-complex structure* on any RPP whose dimension is larger than 4.

Theorem 2.6 *Let* M *be a* $4k(k>1)$-dimensional stably almost-complex manifold whose possibly nonzero Chern numbers are $c_{2k}[M]$ and $c_i c_{2k-i}[M]$ for some $0<i<2k$. Then $\sigma(M)$ is even.

Remark 2.7 We would like to stress at this moment that the notion "stably almost-complex" is strictly weaker than that of "almost-complex". For instance, it is well known that the sphere S^n admits an almost-complex structure if and only if $n=2$ or 6 [9]. Nevertheless, *any* S^n does admit a stably almost-complex structure as its tangent bundle is stably trivial.

3 Preliminaries

3.1 Symmetric functions

In this subsection we briefly recall three bases of the vector space consisting of symmetric functions and collect several facts related to them, which we shall apply to prove Theorem 2.1. Two standard references on symmetric function theory are [25, §1; 32, §7].

Let $\mathbb{Q}[\![\boldsymbol{x}]\!] = \mathbb{Q}[\![x_1, x_2, \dots]\!]$ be the ring of formal power series over $\mathbb{Q}$ in a countably infinite set of (commuting) variables $\boldsymbol{x} = (x_1, x_2, \dots)$. An $f(\boldsymbol{x}) = f(x_1, x_2, \dots) \in \mathbb{Q}[\![\boldsymbol{x}]\!]$ is called a *symmetric function* if it satisfies

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = f(x_1, x_2, \dots) \quad \forall\, \sigma \in S_k,\ \forall\, k \in \mathbb{Z}_{>0}.$$

Here, if $\sigma \in S_k$, $\sigma(i) = i$ for $i > k$ is understood.

Let $\Lambda^k(\boldsymbol{x})$ be the vector space of symmetric functions of homogeneous degree k ($\deg(x_i) := 1$). Then the *ring* of symmetric functions $\Lambda(\boldsymbol{x}) := \bigoplus_{k=0}^{\infty} \Lambda^k(\boldsymbol{x})$ ($\Lambda^0(\boldsymbol{x}) := \mathbb{Q}$) consists of all symmetric functions with *bounded* degree.

Definition 3.1 Let $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$ be an integer partition with $|\lambda| = n$. The k-th *elementary* symmetric function $e_k(\boldsymbol{x})$ and *power sum* symmetric function $p_k(\boldsymbol{x})$ are defined respectively by

$$e_k(\boldsymbol{x}) := \sum_{1 \le i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad p_k(\boldsymbol{x}) := \sum_{i=1}^{\infty} x_i^k,$$

and

$$e_\lambda(\boldsymbol{x}) := \prod_{i=1}^{l(\lambda)} e_{\lambda_i}(\boldsymbol{x}) \in \Lambda^n(\boldsymbol{x}), \quad p_\lambda(\boldsymbol{x}) := \prod_{i=1}^{l(\lambda)} p_{\lambda_i}(\boldsymbol{x}) \in \Lambda^n(\boldsymbol{x}). \tag{3-1}$$

The *monomial* symmetric function $m_\lambda(\boldsymbol{x}) \in \Lambda^n(\boldsymbol{x})$ is defined by

$$m_\lambda(\boldsymbol{x}) := \sum_{(\alpha_1, \alpha_2, \dots)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots,$$

where the sum is over all *distinct* permutations $(\alpha_1, \alpha_2, \dots)$ of the entries of the vector $(\lambda_1, \dots, \lambda_{l(\lambda)}, 0, \dots)$. In other words, $m_\lambda(\boldsymbol{x})$ is the *smallest* symmetric function containing the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_{l(\lambda)}^{\lambda_{l(\lambda)}}$.

It is well known [25, §1.2] that the three sets

$$\{e_\lambda(\boldsymbol{x}) \mid |\lambda| = n\}, \quad \{p_\lambda(\boldsymbol{x}) \mid |\lambda| = n\}, \quad \{m_\lambda(\boldsymbol{x}) \mid |\lambda| = n\}$$

are all additive bases of the vector space $\Lambda^n(\boldsymbol{x})$, and so it is natural to ask what the transition matrices are among these bases. In [10] Doubilet applied the combinatorial Möbius inversion tools to give a unified and compact treatment on the entries of the transition matrices among these three bases as well as other bases (complete symmetric function, forgotten symmetric function and so on). The following transition relation will be used in our later proof [10, Theorem 2].

Theorem 3.2 (Doubilet) *Let* $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)}) = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \dots)$ *be an integer partition. Then*

$$m_\lambda(\boldsymbol{x}) = \frac{(-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!} \sum_{\pi \in \Pi_{l(\lambda)}} \left\{ (-1)^{l(\pi)} \prod_{i=1}^{l(\pi)} \Big[\big(|\pi_i| - 1 \big)! \cdot p_{\lambda_{\pi_i}}(\boldsymbol{x}) \Big] \right\}, \tag{3-2}$$

where the related notation involved in (3-2) *has been introduced in Section 2.1.*

Remark 3.3 This formula has intimate relation with a well-known multiple zeta value formula due to Hoffman (see [8, §3]) and this phenomenon has been extended in [24, §3].

The following fact is well known and usually attributed to Cauchy ([14, §1.4] or [24, Lemma 4.1])

Lemma 3.4 *Let* $Q(x) = 1 + \sum_{i \ge 1} a_i x^i$ *be a formal power series. If the coefficients* a_i *are viewed as* $a_i = e_i(\boldsymbol{y}) = e_i(y_1, y_2, \dots)$, *the* i*-th elementary symmetric functions of the variables* $y_1, y_2, \dots,$ *then the power sum symmetric functions* $p_i(\boldsymbol{y})$ *are determined by* $a_i = e_i(\boldsymbol{y})$ *via*

$$\sum_{i \ge 1} (-1)^{i-1} \cdot p_i(\boldsymbol{y}) \cdot x^{i-1} = \frac{Q'(x)}{Q(x)}. \tag{3-3}$$

The following identity will also be used in the sequel [32, p. 292].

Lemma 3.5 *Let* $\boldsymbol{x} = (x_1, x_2, \dots)$ *and* $\boldsymbol{y} = (y_1, y_2, \dots)$ *be two countably infinite sets of (commuting) variables* x_i *and* y_j. *Then we have*

$$\prod_{i,j \ge 1} (1 + x_i y_j) = 1 + \sum_{|\lambda| \ge 1} m_\lambda(\boldsymbol{y}) e_\lambda(\boldsymbol{x}), \tag{3-4}$$

where the sum is over all positive integer partitions.

3.2 Stirling numbers of the second kind

Recall the definition of the Stirling numbers of the second kind $S(n,k)$ and their convention in Section 2.1. For $S(n,k)$ we have the following two formulas.

Lemma 3.6 *These* $S(n,k)$ *satisfy*

$$\frac{(e^x - 1)^k}{k!} = \sum_{n \ge 0} S(n,k) \cdot \frac{x^n}{n!} \quad \forall\, k \ge 0, \tag{3-5}$$

and

$$S(n+1, k+1) = \sum_{k \le l \le n} \binom{n}{l} S(l,k) \overset{(2\text{-}1)}{=\!=} \sum_{0 \le l \le n} \binom{n}{l} S(l,k). \tag{3-6}$$

Proof The identity (3-5) is well known [31, p. 34]. For (3-6), we differentiate both sides of (3-5) with respect to x:

$$\sum_{n \ge 0} S(n,k) \cdot \frac{x^{n-1}}{(n-1)!} = e^x \cdot \frac{(e^x - 1)^{k-1}}{(k-1)!} \overset{(3\text{-}5)}{=\!=} \sum_{l,m \ge 0} S(l, k-1) \cdot \frac{x^{l+m}}{l! \cdot m!}. \tag{3-7}$$

Comparing the coefficient in front of x^{n-1} on both sides of (3-7) yields

$$S(n,k)=\sum_{k-1\le l\le n-1}\binom{n-1}{l}S(l,k-1),$$

which is (3-6). Alternatively, we can give a direct combinatorial proof for (3-6) as follows. There are $S(n+1,k+1)$ ways to partition $[n+1]$ into $k+1$ blocks. We consider the block containing the element $n+1$. Assume that this block has $l+1$ elements ($l\ge 0$). We have $\binom{n}{l}$ ways to choose this block and in each way we can partition the resulting $n-l$ elements into k blocks in $S(n-l,k)$ ways. So

$$S(n+1,k+1)=\sum_{0\le l\le n}\binom{n}{l}S(n-l,k)=\sum_{k\le l\le n}\binom{n}{l}S(l,k). \qquad \square$$

4 Proof of Theorem 2.1

Let $\boldsymbol{x}=(x_1,x_2,\dots)$ be the *formal Chern roots* of a complex vector bundle E. Usually finite variables whose number is no less than the rank of E are enough, but here in order to be consistent with symmetric function theory, we take infinite variables instead. This means that $e_k(\boldsymbol{x})=c_k(E)$. In this notation it turns out that [33, p. 268]

$$\mathrm{ch}\big(\gamma^k(\widetilde{E})\big)=e_k(e^{\boldsymbol{x}}-1):=e_k(e^{x_1}-1,e^{x_2}-1,\dots), \tag{4-1}$$

and hence determining the coefficients $b_\lambda^{(k)}$ in (1-4) can be translated into a pure combinatorial problem, i.e., the coefficients $b_\lambda^{(k)}$ determined by

$$e_k(e^{\boldsymbol{x}}-1)=:\sum_\lambda b_\lambda^{(k)}\cdot e_\lambda(\boldsymbol{x}). \tag{4-2}$$

The exponential function e^x is a monic formal power series. Thus we may broaden our scope by considering the following question, which may be of independent interest in combinatorics and related topics.

Question 4.1 Let $Q(x)=1+\sum_{i\ge 1}a_i\cdot x^i$ be a monic formal power series. Consider the symmetric function $e_k\big(Q(\boldsymbol{x})-1\big):=e_k\big(Q(x_1)-1,Q(x_2)-1,\dots\big)$ and

$$e_k\big(Q(\boldsymbol{x})-1\big)=:\sum_\lambda b_\lambda^{(k)}(Q)\cdot e_\lambda(\boldsymbol{x}). \tag{4-3}$$

What is the closed formula for $b_\lambda^{(k)}(Q)$?

Our solution to Question 4.1 is:

Theorem 4.2 *Let the notation be as above and $Q_t(x):=1+\sum_{i\ge 1}(ta_i)x^i$. For each integer partition $\lambda=(\lambda_1,\dots,\lambda_{l(\lambda)})=(1^{m_1(\lambda)}2^{m_2(\lambda)}\dots)$ we have*

$$\sum_{k\ge 1}b_\lambda^{(k)}(Q)\cdot t^k=\frac{(-1)^{|\lambda|-l(\lambda)}}{\prod_{i\ge 1}m_i(\lambda)!}\sum_{\pi\in\Pi_{l(\lambda)}}\left\{\left[\prod_{i=1}^{l(\pi)}(|\pi_i|-1)!\right]\left[\prod_{i=1}^{l(\pi)}\left(\frac{1}{2\pi\sqrt{-1}}\oint\frac{Q_t'(x)}{x^{\lambda_{\pi_i}}Q_t(x)}\,dx\right)\right]\right\}. \tag{4-4}$$

Proof Write $ta_i =: e_i(\boldsymbol{y})$, the i-th elementary symmetric function of the variables $\boldsymbol{y} = (y_1, y_2, \dots)$. Then

$$Q_t(x) = \prod_{i\geq 1}(1 + y_i x) \tag{4-5}$$

and

$$\begin{aligned} 1 + \sum_{|\lambda|\geq 1}\Big(\sum_{k\geq 1} b_\lambda^{(k)}(Q)\cdot t^k\Big)e_\lambda(\boldsymbol{x}) &= 1 + \sum_{k\geq 1} e_k\big(Q(\boldsymbol{x}) - 1\big)t^k \quad \big(\text{by (4-3)}\big)\\ &= \prod_{i\geq 1}\Big[1 + \big(Q(x_i) - 1\big)t\Big]\\ &= \prod_{i\geq 1} Q_t(x_i)\\ &= \prod_{i,j\geq 1}(1 + x_i y_j) \quad \big(\text{by (4-5)}\big)\\ &= 1 + \sum_{|\lambda|\geq 1} m_\lambda(\boldsymbol{y})e_\lambda(\boldsymbol{x}) \quad \big(\text{by (3-4)}\big). \end{aligned} \tag{4-6}$$

Comparing both sides of (4-6) yields

$$\begin{aligned} \sum_{k\geq 1} b_\lambda^{(k)}(Q)\cdot t^k &= m_\lambda(\boldsymbol{y})\\ &= \frac{(-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!}\sum_{\pi\in\Pi_{l(\lambda)}}\Big\{(-1)^{l(\pi)}\prod_{i=1}^{l(\pi)}\Big[\big(|\pi_i| - 1\big)!\cdot p_{\lambda_{\pi_i}}(\boldsymbol{y})\Big]\Big\} \quad \big(\text{by (3-2)}\big). \end{aligned} \tag{4-7}$$

By (3-3) we have

$$\begin{aligned} (-1)^{\lambda_{\pi_i}-1}p_{\lambda_{\pi_i}}(\boldsymbol{y}) &= \text{the coefficient of } x^{\lambda_{\pi_i}-1} \text{ in } \frac{Q_t'(x)}{Q_t(x)}\\ &= \frac{1}{2\pi\sqrt{-1}}\oint\frac{Q_t'(x)}{x^{\lambda_{\pi_i}}Q_t(x)}\,dx \quad \text{(the residue formula)}, \end{aligned}$$

and therefore

$$\prod_{i=1}^{l(\pi)} p_{\lambda_{\pi_i}}(\boldsymbol{y}) = (-1)^{|\lambda|-l(\pi)}\prod_{i=1}^{l(\pi)}\Big(\frac{1}{2\pi\sqrt{-1}}\oint\frac{Q_t'(x)}{x^{\lambda_{\pi_i}}Q_t(x)}\,dx\Big). \tag{4-8}$$

Putting (4-8) into (4-7) leads to the desired (4-4). □

In view of (2-3) and (4-4), Theorem 2.1 follows from the following:

Lemma 4.3 *We have*

$$\frac{1}{2\pi\sqrt{-1}}\oint\frac{Q_t'(x)}{x^k Q_t(x)}\,dx \overset{Q(x)=e^x}{=\!=\!=} t\sum_{i=0}^{k-1}\frac{S(k, k-i)}{\binom{k-1}{i}\cdot i!}(-t)^{k-1-i}.$$

Proof In this case $Q_t(x)=1+t(e^x-1)$. For simplicity we use $\langle x^i\rangle f(x)$ to denote the coefficient in front of x^i in $f(x)$. Then

$$\begin{aligned}
\frac{1}{2\pi\sqrt{-1}}\oint\frac{Q_t'(x)}{x^kQ_t(x)}\,dx &= \frac{1}{2\pi\sqrt{-1}}\oint\frac{te^x}{x^k\big[1+t(e^x-1)\big]}\,dx\\
&=\langle x^{k-1}\rangle te^x\big[1+t(e^x-1)\big]^{-1}\\
&=\langle x^{k-1}\rangle te^x\sum_{i=0}^{k-1}(-t)^{k-1-i}(e^x-1)^{k-1-i}\\
&=t\sum_{i=0}^{k-1}\big[(-t)^{k-1-i}\cdot\langle x^{k-1}\rangle e^x(e^x-1)^{k-1-i}\big]\\
&=t\sum_{i=0}^{k-1}\Big[(-t)^{k-1-i}(k-1-i)!\sum_{0\le l\le k-1}\frac{S(l,k-1-i)}{l!\cdot(k-1-l)!}\Big] && \big(\text{by (3-5)}\big)\\
&=t\sum_{i=0}^{k-1}\Big[\frac{(-t)^{k-1-i}(k-1-i)!}{(k-1)!}\sum_{0\le l\le k-1}\binom{k-1}{l}S(l,k-1-i)\Big]\\
&=t\sum_{i=0}^{k-1}\Big[\frac{(-t)^{k-1-i}(k-1-i)!}{(k-1)!}S(k,k-i)\Big] && \big(\text{by (3-6)}\big)\\
&=t\sum_{i=0}^{k-1}\frac{S(k,k-i)}{\binom{k-1}{i}\cdot i!}(-t)^{k-1-i}.
\end{aligned}$$

□

Before ending this section, we collect several general vanishing facts on $b_\lambda^{(k)}$ in the following lemma.

Lemma 4.4 *We have $b_\lambda^{(k)}=0$ unless $|\lambda|\ge k$, $b_{(k)}^{(k)}=1$, and $b_\lambda^{(k)}=0$ whenever $|\lambda|=k$ and $l(\lambda)\ge 2$.*

Proof The first one has been explained in Remark 2.2. To see the latter two cases it suffices to note that

$$e_k(e^{x_1}-1,e^{x_2}-1,\dots)=e_k(x_1,x_2,\dots)+\text{higher-degree terms}.$$

□

5 Proof of Theorem 2.4

Let (M,τ) be a closed $2n$-dimensional stably almost-complex manifold. Recall that the value of a *complex genus* φ which corresponds to a monic formal power series $Q(x)=1+\sum_{i\ge1}a_ix^i$ $(a_i\in\mathbb{Q})$ on M, $\varphi(M)$, is a rational linear combination of Chern numbers $c_\lambda[M]$ and defined as follows [19, §1.8]. Let

$$\prod_{i\ge1}Q(x_i)=:1+\sum_{j\ge1}Q_j\big(e_1(\boldsymbol{x}),\dots,e_j(\boldsymbol{x})\big),$$

where $Q_j\big((e_1(\boldsymbol{x}),\dots,e_j(\boldsymbol{x})\big)$, viewed as a polynomial of $e_1(\boldsymbol{x}),\dots,e_j(\boldsymbol{x})$, denotes the homogeneous part of degree j in $\prod_{i\ge1}Q(x_i)$ $(\deg(x_i)=1)$. Then

$$\varphi(M)=\int_M Q_n\big(c_1(M),\dots,c_n(M)\big)=:h_n(\varphi)c_n[M]+\sum_{\substack{|\lambda|=n\\ l(\lambda)\ge2}}h_\lambda(\varphi)c_\lambda[M]. \tag{5-1}$$

The coefficients $h_\lambda(\varphi)$ in (5-1) can be expressed in terms of $h_i(\varphi)$ as follows (see [24, Lemma 4.3]).

Lemma 5.1 *Let φ be a complex genus which corresponds to the formal power series $Q(x)$. Then the coefficients $h_\lambda(\varphi)$ in* (5-1) *are given by*

(5-2)
$$h_\lambda(\varphi)=\frac{(-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!}\sum_{\pi\in\Pi_{l(\lambda)}}\Big\{(-1)^{l(\pi)}\prod_{i=1}^{l(\pi)}\Big[\big(|\pi_i|-1\big)!\cdot h_{\lambda_{\pi_i}}(\varphi)\Big]\Big\},$$

where $h_i(\varphi)$ are determined by

(5-3)
$$\sum_{i\geq 1}(-1)^{i-1}\cdot h_i(\varphi)\cdot x^{i-1}=\frac{Q'(x)}{Q(x)}.$$

Now the complex genus which corresponds to the formal power series $Q(x)=x/\tanh(x)$ is nothing but the signature $\sigma(\cdot)$ due to Hirzebruch's signature theorem [14]. Therefore

(5-4)
$$\begin{aligned}\frac{Q'(x)}{Q(x)} &\overset{Q(x)=\frac{x}{\tanh(x)}}{=\!=\!=\!=} \frac{1}{x}\Big[1-\frac{2x}{\sinh(2x)}\Big]\\ &=\sum_{i\geq 1}\frac{(-1)^{i-1}\cdot 2^{2i+1}\cdot(2^{2i-1}-1)\cdot B_i}{(2i)!}x^{2i-1}\quad \big(\text{by (2-2)}\big).\end{aligned}$$

Comparing (5-3) and (5-4) leads to

$$h_{2i-1}(\sigma)=0,\quad h_{2i}(\sigma)=\frac{(-1)^i\cdot 2^{2i+1}\cdot(2^{2i-1}-1)\cdot B_i}{(2i)!},$$

which, together with (5-2), yields (2-6) and hence finishes the proof of the first part in Theorem 2.4.

To prove the second part in Theorem 2.4. We need some more notation and facts.

The 2-*adic valuation* of an integer k is defined to be

$$\nu_2(k):=\begin{cases}\max\{t\in\mathbb{Z}_{\geq 0}:2^t\text{ divides }k\} & \text{if } k\neq 0,\\ \infty & \text{if } k=0.\end{cases}$$

Clearly $\nu_2(k)=0$ if and only if k is odd, $\nu_2(\cdot)$ satisfies $\nu_2(k_1k_2)=\nu_2(k_1)+\nu_2(k_2)$ and

$$\nu_2(k_1/k_2):=\nu_2(k_1)-\nu_2(k_2)\quad (k_2\neq 0)$$

is well defined.

Lemma 5.2 *Let λ be an integer partition such that $m_i(\lambda)=0$ or 1 for **all** i, i.e., all the parts λ_i of this λ are mutually distinct. Then the* 2*-adic valuation of the rational number h_λ given in* (2-6) *satisfies $\nu_2(h_\lambda)\geq 1$.*

Proof Write the Bernoulli numbers B_i as $B_i=:N_i/D_i$ ($N_i,D_i\in\mathbb{Z}_{>0}$) in irreducible form. Since D_i is equal to the product of all primes p such that $p-1$ divides $2i$ [28, p. 284], we have that $\nu_2(D_i)=1$ and hence $\nu_2(N_i)=0$.

Let $\mathrm{wt}(i)$ denote the number of nonzero terms in the binary expansion of i. Then $\nu_2(i!)+\mathrm{wt}(i)=i$, which is indeed a special case of Legendre's formula ([29, Theorem 2.6.4]; also see [11, Lemma 2.2]).

This implies that

$$\nu_2\big((2i)!\big) = 2i - \mathrm{wt}(2i) = 2i - \mathrm{wt}(i),$$

and thus

$$\nu_2(h_{2i}) \overset{(2\text{-}5)}{=\!=} (2i+1) + \nu_2(N_i) - \nu_2\big((2i)!\big) - \nu_2(D_i) = \mathrm{wt}(i) \geq 1. \tag{5-5}$$

The condition of $m_i(\lambda) = 0$ or 1 for all i is equivalent to

$$\nu_2\Big(\prod_{i\geq 1} m_i(\lambda)!\Big) = 0,$$

which, together with (5-5), yields that for such λ the coefficient h_λ given by (2-6) satisfies $\nu_2(h_\lambda) \geq 1$. □

Now we can finish the proof of the second part of Theorem 2.4.

Proof If all nonzero Chern numbers $c_\lambda[M]$ are such that the integer partitions λ satisfy $m_i(\lambda) = 0$ or 1 for all i, as required by the assumption (2-7), every nonzero summand $h_\lambda c_\lambda[M]$ in the signature formula

$$\sigma(M) = \sum_{|\lambda|=2k} h_\lambda c_\lambda[M]$$

satisfies $\nu_2(h_\lambda c_\lambda[M]) \geq 1$ due to Lemma 5.2. This implies that $\nu_2\big(\sigma(M)\big) \geq 1$, i.e., $\sigma(M)$ is even. □

6 Proof of Theorem 2.6

The following congruence property will be used in proving Theorem 2.6 for the case of $i = k = 2$, which may be of interest on its own.

Lemma 6.1 *Let M be a $4k$-dimensional stably almost-complex manifold. The signature $\sigma(M)$ and the top Chern number $c_{2k}[M]$ satisfy*

$$\sigma(M) \equiv (-1)^k c_{2k}[M] \pmod 4. \tag{6-1}$$

Proof First we have, for any $4k$-dimensional almost-complex manifold N,

$$\sigma(N) \equiv (-1)^k \chi(N) \pmod 4, \tag{6-2}$$

where $\chi(N)$ is the Euler characteristic of N. This fact can be proved by using various symmetric properties of the Hirzebruch χ_y-genus and appears in [16, p. 777]. A detailed proof can be found in [30, (3.5), (3.7)]. Secondly, any $4k$-dimensional stably almost-complex M is complex cobordant to a $4k$-dimensional almost-complex manifold N [21, Corollary 5]. Since Chern numbers (and hence signature) are complex cobordant invariants, we have $\sigma(M) = \sigma(N)$ and $c_{2k}[M] = c_{2k}[N] = \chi(N)$. Substituting them into (6-2) yields (6-1). □

We now proceed to show Theorem 2.6 and therefore assume that the possibly nonzero Chern numbers of the $4k$-dimensional stably almost-complex manifold (M, τ) are $c_{2k}[M]$ and $c_i c_{2k-i}[M]$ for some $1 \leq i \leq k$.

If $1 \le i \le k-1$, then $\sigma(M)$ is even due to Theorem 2.4 as the assumption condition (2-7) is satisfied.

If $i = k$, the possibly nonzero Chern numbers are $c_{2k}[M]$ and $c_k^2[M]$ and so the relevant Chern classes we need to deal with are $c_{2k}(M)$ and $c_k(M)$. In this situation

$$\operatorname{ch}\big(\gamma^1(\widetilde{E})\big) \overset{\text{(1-4)}}{=\!=} b^{(1)}_{(k)} c_k(M) + b^{(1)}_{(k,k)} c_k^2(M) + b^{(1)}_{(2k)} c_{2k}(M) + \text{other terms}, \tag{6-3}$$

and thus by Theorem 1.1 we have

$$\mathbb{Z} \ni \int_M \Big[\operatorname{ch}\big(\gamma^1(\widetilde{E})\big)\operatorname{ch}\big(\gamma^1(\widetilde{E})\big)\operatorname{td}(M)\Big] \overset{\text{(6-3)}}{=\!=} \big(b^{(1)}_{(k)}\big)^2 c_k^2[M] \overset{\text{(2-4)}}{=\!=} \frac{c_k^2[M]}{\big[(k-1)!\big]^2}.$$

This implies that when $i = k \ge 3$, the Chern number $c_k^2[M]$ is divisible by 4 and therefore

$$\sigma(M^{4k}) = h_{(2k)} c_{2k}[M] + h_{(k,k)} c_k^2[M] \overset{\text{(2-6)}}{=\!=} h_{2k} \cdot c_{2k}[M] + \tfrac{1}{2}(h_k^2 - h_{2k}) c_k^2[M]$$

is also even due to the facts that $h_{2i-1} = 0$ and $\nu_2(h_{2i}) \ge 1$ from (5-5).

It suffices to show the case $i = k = 2$, i.e., the case of 8-dimensional stably almost-complex manifold M^8 with possibly nonzero Chern numbers $c_4 := c_4[M]$ and $c_2^2 := c_2^2[M]$. In this dimension

$$\sigma(M^8) = \tfrac{1}{45}(14c_4 + 3c_2^2), \qquad \int_M \operatorname{td}(M) = \tfrac{1}{720}(-c_4 + 3c_2^2) \tag{6-4}$$

and so $45\sigma(M^8) = 14c_4 + 3c_2^2$, which, together with (6-1), yields

$$c_4 \equiv c_2^2 \pmod 4. \tag{6-5}$$

If $\sigma(M^8)$ is odd, then $c_4 = 2\alpha + 1$ ($\alpha \in \mathbb{Z}$), still due to (6-1). Then by the integrality of the Todd number $\int_M \operatorname{td}(M)$ in (6-4) we have

$$\begin{aligned} \mathbb{Z} \ni \int_M \operatorname{td}(M) &= \tfrac{1}{720}(-c_4 + 3c_2^2) = \tfrac{1}{720}(-(2\alpha+1) + 3(2\alpha+1+4\beta)) \quad \big(\text{by (6-5), } \beta \in \mathbb{Z}\big) \\ &= \tfrac{1}{720}(4\alpha + 12\beta + 2), \end{aligned} \tag{6-6}$$

which is a contradiction. Thus $\sigma(M^8)$ is even and this completes the proof of Theorem 2.6.

7 Remarks on integral/rational projective planes

An n-dimensional smooth closed connected orientable manifold M whose integral cohomology groups (resp. Betti numbers) satisfy $H^i(M;\mathbb{Z}) = \mathbb{Z}$ for $i = 0, n/2, n$ and $H^i(M;\mathbb{Z}) = 0$ for other i (resp. $b_0 = b_{n/2} = b_n = 1$ and $b_i = 0$ for other i) is called an *integral projective plane* (resp. a *rational projective plane*) (IPP or RPP for short). As mentioned in Section 2.2, such (classical) examples exist when $n = 4$, 8 or 16. The study of the existence/nonexistence of higher-dimensional IPP or RPP was initiated by Adem and Hirzebruch [2; 13; 17; 18]. Now it turns out that an IPP can *only* exist when $n = 4$, 8 or 16 [15, p. 766], which is a consequence of Adams' solution to the nonexistence of Hopf invariant one [1]. Hirzebruch [18, §3, Theorem 1] showed that an n-dimensional ($n > 8$) RPP exists only if n is of the form $n = 8(2^a + 2^b)$ with $a, b \in \mathbb{Z}_{\geq 0}$, i.e., n is divisible by 8 and $\mathrm{wt}(n) \leq 2$ (recall the notation in the proof of Lemma 5.2). Moreover, a *spin* RPP can only exist when $n = 8$ or 16 [15, p. 786; 23, Theorem C].

Recently, the existence of RPP was investigated systematically by Su and her coauthors in a series of papers [11; 23; 36; 37], whose main tool is the Barge–Sullivan rational surgery realization theorem [7; 26; 39] as well as the Hattori–Stong theorems. Among other things, they showed that an RPP exists in dimension $n \leq 512$ if and only if $n \in \{4, 8, 16, 32, 128, 256\}$ [23, Theorem A; 36, Theorem 1.1]. Moreover, whether a given dimension $n = 8(2^a + 2^b)$ supports an RPP is reduced to a solvability of a quadratic residue equation [23, Theorems 6, 9].

Recall that the Betti numbers $b_i = b_i(M)$ of a *general* n-dimensional (closed connected orientable) manifold M satisfy

$$b_0 = b_n = 1, \quad b_i = b_{n-i}, \quad b_{n/2} \text{ is even whenever } n = 4k-2. \tag{7-1}$$

The first two restrictions in (7-1) are clear and the third one is due to the fact that in this case the intersection pairing on $H^{2k-1}(M; \mathbb{R})$ is both skew-symmetric and nondegenerate. Conversely, given a sequence of nonnegative integers $(b_0, b_1, \ldots, b_n)$ satisfying (7-1), can we find an n-dimensional manifold whose Betti numbers realize this sequence? The following proposition tells us that such a realization problem is deeply related to the existence of an RPP when $n = 4k$.

Proposition 7.1 *Let $(b_0, b_1, \ldots, b_n)$ be a sequence of nonnegative integers satisfying the restrictions in* (7-1).

(1) *When n is odd, $n = 4k-2$, or $n = 4k$ and b_{2k} is even, there exists an n-dimensional manifold whose Betti numbers realize this sequence.*

(2) *When $n = 4k$ and b_{2k} is odd, there exists an n-dimensional manifold whose Betti numbers realize this sequence if there exists a $4k$-dimensional RPP.*

Remark 7.2 (1) Equation (7-1) is known to some experts. For instance, we were told by Yang Su that M. Kreck has known it for many years. But it seems that it was never written down in the literature, to our best knowledge. So we present it here for the reader's convenience.

(2) As discussed above, in some dimensions RPP are known to exist, so in these dimensions the answer to the Betti number realization problem is also affirmative. For some dimension $4k$, if an RPP does not exist in this dimension, we do not know how to construct a desired manifold realizing a general prescribed Betti numbers.

Proof The idea is to apply connected sum operations to the building blocks $S^i \times S^{n-i}$ $(1 \leq i \leq \lfloor n/2 \rfloor)$.

When n is odd, we may take connected sums of b_i copies of $S^i \times S^{n-i}$ for all $1 \leq i \leq \frac{1}{2}(n-1)$.

When $n = 4k-2$ (resp. $n = 4k$ and b_{2k} is even), we may take connected sums of b_i copies of $S^i \times S^{n-i}$ for all $1 \leq i \leq 2k-2$ (resp. $1 \leq i \leq 2k-1$) and $b_{2k-1}/2$ (resp. $b_{2k}/2$) copies of $S^{2k-1} \times S^{2k-1}$ (resp. $S^{2k} \times S^{2k}$).

When $n = 4k$ and b_{2k} is odd, we may take connected sums of b_i copies of $S^i \times S^{n-i}$ for all $1 \leq i \leq b_{2k-1}$, $\frac{1}{2}(b_{2k}-1)$ copies of $S^{2k} \times S^{2k}$, and an RPP. □

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References

[1] **J F Adams**, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) 72 (1960) 20–104 MR

[2] **J Adem**, *The iteration of the Steenrod squares in algebraic topology*, Proc. Nat. Acad. Sci. U.S.A. 38 (1952) 720–726 MR

[3] **M F Atiyah**, *K-theory*, W. A. Benjamin, New York (1967) MR

[4] **M F Atiyah**, **F Hirzebruch**, *Riemann–Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. 65 (1959) 276–281 MR

[5] **M F Atiyah**, **F Hirzebruch**, *Cohomologie-Operationen und charakteristische Klassen*, Math. Z. 77 (1961) 149–187 MR

[6] **M F Atiyah**, **I M Singer**, *The index of elliptic operators, III*, Ann. of Math. (2) 87 (1968) 546–604 MR

[7] **J Barge**, *Structures différentiables sur les types d'homotopie rationnelle simplement connexes*, Ann. Sci. École Norm. Sup. (4) 9:4 (1976) 469–501 MR

[8] **A Berglund**, **J Bergström**, *Hirzebruch L-polynomials and multiple zeta values*, Math. Ann. 372:1-2 (2018) 125–137 MR

[9] **A Borel**, **J-P Serre**, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. 75 (1953) 409–448 MR

[10] **P Doubilet**, *On the foundations of combinatorial theory, VII*: *Symmetric functions through the theory of distribution and occupancy*, Studies in Appl. Math. 51 (1972) 377–396 MR

[11] **J Fowler**, **Z Su**, *Smooth manifolds with prescribed rational cohomology ring*, Geom. Dedicata 182 (2016) 215–232 MR

[12] **A Hattori**, *Integral characteristic numbers for weakly almost complex manifolds*, Topology 5 (1966) 259–280 MR

[13] **F Hirzebruch**, *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) 60 (1954) 213–236 MR

[14] **F Hirzebruch**, *Topological methods in algebraic geometry*, Grundl. Math. Wissen. 131, Springer (1966) MR

[15] **F Hirzebruch**, *Gesammelte Abhandlungen - Collected papers*, *I*: 1951–1962, Springer (2013) MR

[16] **F Hirzebruch**, *Gesammelte Abhandlungen - Collected papers*, *II*: 1963–1987, Springer (2013) MR

[17] **F Hirzebruch**, *The index of an oriented manifold and the Todd genus of an almost complex manifold*, from "Gesammelte Abhandlungen - Collected papers, I: 1951–1962", Springer, Berlin (2013) 60–80

[18] **F Hirzebruch**, *On the characteristic cohomology classes of differentiable manifolds*, from "Gesammelte Abhandlungen - Collected papers, I: 1951–1962", Springer, Berlin (2013) 145–148

[19] **F Hirzebruch**, **T Berger**, **R Jung**, *Manifolds and modular forms*, Aspects of Mathematics E20, Vieweg & Sohn, Braunschweig, Germany (1992) MR

[20] **J Hu**, *Almost complex manifolds with total Betti number three*, J. Topol. Anal. 16:6 (2024) 945–953 MR

[21] **P J Kahn**, *Obstructions to extending almost X-structures*, Illinois J. Math. 13 (1969) 336–357 MR

[22] **M Karoubi**, *K-theory*: *an introduction*, Grundl. Math. Wissen. 226, Springer (1978) MR

[23] **L Kennard**, **Z Su**, *On dimensions supporting a rational projective plane*, J. Topol. Anal. 11:3 (2019) 535–555 MR

[24] **P Li**, *The complex genera, symmetric functions and multiple zeta values*, J. Combin. Theory Ser. A 206 (2024) art. id. 105893 MR

[25] **I G Macdonald**, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford Univ. Press (1995) MR

[26] **A Milivojević**, *On the characterization of rational homotopy types and Chern classes of closed almost complex manifolds*, Complex Manifolds 9:1 (2022) 138–169 MR

[27] **J Milnor**, *On the cobordism ring Ω^* and a complex analogue, I*, Amer. J. Math. 82 (1960) 505–521 MR

[28] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Annals of Mathematics Studies 76, Princeton Univ. Press (1974) MR

[29] **V H Moll**, *Numbers and functions*: *from a classical-experimental mathematician's point of view*, Student Mathematical Library 65, Amer. Math. Soc., Providence, RI (2012) MR

[30] **C Rovi**, **S Yokura**, *Hirzebruch χ_y-genera modulo 8 of fiber bundles for odd integers y*, Pure Appl. Math. Q. 12:4 (2016) 587–602 MR

[31] **R P Stanley**, *Enumerative combinatorics, I*, Cambridge Studies in Advanced Mathematics 49, Cambridge Univ. Press (1997) MR

[32] **R P Stanley**, *Enumerative combinatorics, II*, Cambridge Studies in Advanced Mathematics 62, Cambridge Univ. Press (1999) MR

[33] **R E Stong**, *Relations among characteristic numbers, I*, Topology 4 (1965) 267–281 MR

[34] **R E Stong**, *Relations among characteristic numbers, II*, Topology 5 (1966) 133–148 MR

[35] **R E Stong**, *Notes on cobordism theory*, Princeton Univ. Press (1968) MR

[36] **Z Su**, *Rational homotopy type of manifolds*, PhD thesis, Indiana University (2009) MR https://www.proquest.com/docview/304898878

[37] **Z Su**, *Rational analogs of projective planes*, Algebr. Geom. Topol. 14:1 (2014) 421–438 MR

[38] **Z Su**, *Almost complex manifold with Betti number $b_i = 0$ except $i = 0, n/2, n$* (2022) arXiv 2204.04800

[39] **D Sullivan**, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. 47 (1977) 269–331 MR

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