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Stellar subdivisions, wedges and Buchstaber numbers

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The Buchstaber number of a simplicial complex K is a significant invariant in toric topology. In particular, when K is a (polytopal) PL sphere, the maximal Buchstaber number is closely connected to several important objects, such as toric manifolds, quasitoric manifolds, and topological toric manifolds. A PL sphere is called a *seed* if it cannot be obtained from another PL sphere through a wedge operation. The *toric colorable seed inequality*, established by Choi and Park in 2017, bounds the maximal number of vertices of a seed with a maximal Buchstaber number. This inequality plays a key role in characterizing PL spheres that achieve maximal Buchstaber numbers.

We prove that the inequality is tight. Specifically, we show how to construct larger seeds from existing ones using stellar subdivisions and wedges, while preserving both the maximality of Buchstaber numbers and polytopality.

1 Introduction

Let K be an $(n-1)$-dimensional simplicial complex on $[m]=\{1,2,\dots,m\}$ and (X,Y) a topological pair. The *polyhedral product* $(\underline{X},\underline{Y})^K$ is a subspace of X^m with respect to K:

$$(\underline{X},\underline{Y})^K := \bigcup_{\sigma\in K}\{(x_1,\dots,x_m)\in X^m \mid x_i\in Y \text{ when } i\notin\sigma\}.$$

One of our main considerations is the *moment-angle complex*

$$\mathcal{Z}_K := (\underline{D^2},\underline{S^1})^K$$

of K, where $D^2=\{z\in\mathbb{C}\mid |z|\le 1\}$ is the 2-dimensional disk, and $S^1=\partial D^2$ is its boundary circle. The canonical action of S^1 on D^2 induces the action of the m-dimensional torus $T^m=S^1\times\cdots\times S^1$ on $\mathcal{Z}_K$. The *Buchstaber number* $s(K)$ of K is the maximum number r such that there is an r-dimensional subtorus H of T^m freely acting on $\mathcal{Z}_K$.

It is known that the following inequality holds by Buchstaber and Panov [2] and Erokhovets [8]:

$$1\le s(K)\le m-n.$$

The case when K is a PL sphere with $s(K)=m-n$, particularly when it is a polytopal PL sphere, is of special interest because many significant toric spaces, including complete nonsingular toric varieties, simply called *toric manifolds*, and their topological generalizations such as *quasitoric manifolds*, defined by Davis and Januszkiewicz [6], and *topological toric manifolds*, defined by Ishida, Fukukawa and Masuda [14], are formed in $\mathcal{Z}_K/H$, where K is a (polytopal) PL sphere and H is a free torus action of

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rank $m-n$ on $\mathcal{Z}_K$. Hence, such PL spheres are said to be *toric colorable* and the characterization of toric colorable PL spheres is of considerable importance.

The breakthrough in addressing this problem is the approach by Choi and Park. The *wedge* is an operation on simplicial complexes that preserves the PL sphereness and polytopality, and a PL sphere K is called a *seed* if it cannot be obtained by a wedge operation from any other PL sphere. Ewald [9] and Bahri, Bendersky, Cohen and Gitler [1] showed an advantageous property of the wedge operation in toric topology that it preserves the maximality of the Buchstaber number. On the other hand, the wedge operation does not change the difference between the number of vertices and the dimension. Therefore, since every toric colorable PL sphere can be obtained from toric colorable seeds by a sequence of wedges, it is enough to consider toric colorable seeds. In [5], Choi and Park proved that for a fixed $m-n$, the number of $(n-1)$-dimensional toric colorable seeds K with m vertices is finite. More precisely, K must satisfy the inequality, which we refer to as the *toric colorable seed inequality*,

$$m \leq 2^{m-n} - 1. \tag{1-1}$$

This inequality transforms the problem of finding all PL spheres with a maximal Buchstaber number into a finite problem for each fixed $p := m-n$. An interesting question is whether this inequality is tight. If the upper bound of (1-1) can be reduced, it would further transform the problem into a smaller finite problem. It is known that the inequality is tight for $p = 3$ and 4. Denote by $C^4(7)$ the cyclic polytope of dimension 4 with seven vertices. The boundary complex of $C^4(7)$ is a seed, and, by Erokhovets [7], its Buchstaber number is maximal. Hence, the inequality is tight for $p = 3$. According to Choi, Jang and Vallée [3], there are four PL spheres that confirms the tightness for $p = 4$ as well. However, in general, it is not known whether the inequality is tight. Cyclic and stacked polytopes are often considered to provide an example due to their simplicity in construction, but they do not provide extremal examples for this inequality. When n is significantly larger than p, the former are not toric colorable by Hasui [13], and the latter are not seeds.

In this paper, we construct a new polytopal toric colorable seed from an old one using two operations, the stellar subdivision and the wedge on PL spheres, that preserve the maximality of Buchstaber numbers and polytopality. Theorem 3.1 is our main theorem.

Theorem 3.1 *Let K be a seed on $[m]$ and $J = (j_1, j_2, \dots, j_m)$ a positive integer tuple such that $j_v \leq 2$ for all $v \in [m]$. For any assembled face σ of $K(J)$, a new seed is obtained in two ways depending on the cardinality of σ.*

(1) *If $\sigma = \{v\}$ for a vertex v of K, then $Ss_{\{v,v'\}}(K(J))$ is a suspended seed.*

(2) *If $|\sigma| > 1$ and K is nonsuspended, then $Ss_\sigma(K(J))$ is a nonsuspended seed.*

By using this result, we can show that (1-1) is tight for all $p \geq 3$, even for the class of polytopal PL spheres. In other words, one can precisely determine the conditions on m and n for an $(n-1)$-dimensional seed K with m vertices to support quasitoric manifolds or topological toric manifolds.

Corollary 3.2 *Assume that $p \geq 3$. For any $m \leq 2^p - 1$ and $n \geq 2$ with $p = m - n$, there exists a polytopal $(n-1)$-dimensional seed K with m vertices such that $s(K) = p$.*

We end the introduction with a discussion on the characterization of PL spheres that support toric manifolds. To characterize them, it is sufficient to consider the class of nonsuspended seeds that support a toric manifold. Let t_p denote the sharp upper bound of m for $(n-1)$-dimensional nonsuspended seeds supporting a toric manifold with $m - n = p$. However, in this class, (1-1) is not tight. For instance, when $p = 3$, $t_3 = 5$, which is less than $2^3 - 1 = 7$, as noted by Gretenkort, Kleinschmidt and Sturmfels [12].

Since a stellar subdivision and a wedge operation are stable within this class, our main theorem provides the following meaningful lower bound of t_p:

$$3 \cdot 2^{p-2} - 1 \leq t_p \leq 2^p - 1.$$

Problem 1.1 Let K be the underlying complex of a nonsingular complete fan with m rays in $\mathbb{R}^n$. Assume that K is a seed. For a fixed $p = m - n$, find the sharp upper bound t_p of m.

2 Basic operations on simplicial complexes

Let K be a simplicial complex on $[m] = \{1, 2, \dots, m\}$. A nonempty element of K is called a *face* of K. Any singleton in K is often called a *vertex*, and a maximal face of K with respect to inclusion is called a *facet*. We often identify a vertex $\{v\}$ of K with its unique element v as well as with the simplicial complex consisting of the unique vertex v. The *dimension* of a face σ is $|\sigma| - 1$, and K is said to be *pure* if the dimensions of all facets are the same. The *dimension* of pure simplicial complex K is the dimension of its facet. Throughout this paper, every simplicial complex is assumed to be pure.

For any face σ of K, the *boundary complex* of σ is the simplicial complex $\partial\sigma = \{\tau \in K \mid \tau \subsetneq \sigma\}$, the *star* of σ in K is the simplicial complex

$$\mathrm{St}_K(\sigma) = \{\tau \in K \mid \tau \cup \sigma \in K\},$$

and the *link* of σ in K is the simplicial complex

$$\mathrm{Lk}_K(\sigma) = \{\tau \in K \mid \tau \cup \sigma \in K,\ \tau \cap \sigma = \varnothing\}.$$

For another simplicial complex L whose vertex set is disjoint to that of K, the *join* of K and L is the simplicial complex

$$K * L = \{\sigma \cup \tau \mid \sigma \in K,\ \tau \in L\}.$$

The *stellar subdivision* of K at $\sigma \in K$ is the simplicial complex

$$\mathrm{Ss}_\sigma(K) = K \setminus \mathrm{St}_K(\sigma) \cup v_\sigma * \partial\sigma * \mathrm{Lk}_K(\sigma), \tag{2-1}$$

where v_σ is a new vertex. See Figure 1(c).

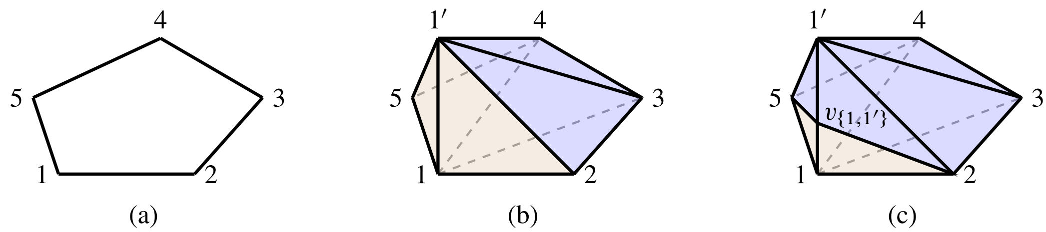

Figure 1: Operations on simplicial complexes.

Let v be a vertex of K, and I the 1-dimensional simplicial complex with two vertices v and v'. The *wedge* of K at v is the simplicial complex

$$\mathrm{Wed}_v(K) = I * \mathrm{Lk}_K(v) \cup \partial I * \{\sigma \in K \mid v \notin \sigma\}. \tag{2-2}$$

The face $\{v, v'\}$ of $\mathrm{Wed}_v(K)$ is called a *wedged edge* of $\mathrm{Wed}_v(K)$. Note that $\mathrm{Lk}_{\mathrm{Wed}_v(K)}(v') = K$, and $\mathrm{Lk}_{\mathrm{Wed}_v(K)}(v)$ is isomorphic to K by identifying v' and v. In this sense, v' can be regarded as a copy of v. See Figure 1(b).

There is another construction of a wedge using minimal nonfaces, known as the *J-construction*, established by Bahri, Bendersky, Cohen and Gitler [1]. The set of minimal nonfaces of $\mathrm{Wed}_v(K)$ is obtained by duplicating v as v and v' in each minimal nonface of K. In this construction, one can observe that wedge operations are associative and commutative up to suitable vertex identification. Then we can use the notation $K(J)$ for a positive integer tuple $J = (j_1, j_2, \dots, j_m)$ to refer the simplicial complex obtained by a sequence of wedge operations with j_v copies $v = v^{(0)}, v' = v^{(1)} \dots, v^{(j_v-1)}$ of each vertex v. Using this notation, $\mathrm{Wed}_v(K)$ can be expressed as $K(1, \dots, 1, 2, 1, \dots, 1)$, where 2 appears in the v-th position. To clarify, we note that K is a subcomplex of $K(J)$, that is, any face of K is a face of $K(J)$. Moreover, choosing $0 \le s_v \le j_v - 1$ for each $v \in [m]$, then for a face σ of $K(J)$, the set $\sigma^{(s_1,s_2,\dots,s_m)}$ obtained by replacing $v \in \sigma$ by $v^{(s_v)}$ is also a face of $K(J)$. In such a way, we denote the set $\{1^{(s_1)}, 2^{(s_2)}, \dots, m^{(s_m)}\}$ by $[m]^{(s_1,s_2,\dots,s_m)}$. The *assembled face* of $K(J)$ by $(s_1, s_2, \dots, s_m)$ denotes the set $[m]^{(s_1,s_2,\dots,s_m)} \setminus \{v \in [m] \mid j_v = 1\}$. The following lemma shows that any assembled face is literally a face of $K(J)$.

Lemma 2.1 *Let K be a simplicial complex on $[m]$, and $J = (j_1, j_2, \dots, j_m)$ a positive integer tuple. Choose $0 \le s_v \le j_v - 1$ for each $v \in [m]$. Then $[m]^{(s_1,s_2,\dots,s_m)} \setminus \{v \in [m] \mid j_v = 1\}$ is a face of $K(J)$.*

Proof If we prove the case $s_1 = s_2 = \cdots = s_m = 0$, then the statement holds by the above argument. In this case, it is enough to observe that for a vertex v and a face σ of K, the set $\sigma \cup \{v\}$ is a face of $\mathrm{Wed}_v(K)$ by (2-2). □

A simplicial complex K is called a *PL sphere* if there exists a subdivision of K such that it can be expressed as a subdivision of the boundary complex of a simplex. A PL sphere is called a *seed* if it is not a wedge of any other PL sphere. Then any PL sphere K is written as $K = K'(J)$ for some seed K' and some positive integer tuple J.

Proposition 2.2 [3; 4; 11] *If K is a PL sphere, then both $Ss_\sigma(K)$ and $K(J)$ are also PL spheres for any $\sigma \in K$ and any positive integer tuple J. In particular, if K is polytopal, then so are $Ss_\sigma(K)$ and $K(J)$.*

For an $(n-1)$-dimensional PL sphere with m vertices, the *Picard number* of the complex K is defined as $\mathrm{Pic}(K) := m - n$. Note that the dimension of $\mathrm{Ss}_\sigma(K)$ is $n-1$, and the dimension of $\mathrm{Wed}_v(K)$ of K is n, while both complexes have $m+1$ vertexes. Therefore, the stellar subdivision increases the Picard number by 1, whereas the wedge operation preserves the Picard number:

$$\mathrm{Pic}(K) = \mathrm{Pic}(K(J)) = \mathrm{Pic}(\mathrm{Ss}_\sigma(K)) - 1.$$

A PL sphere is said to be *toric colorable* if its Buchstaber number is equal to its Picard number.

Proposition 2.3 [9; 10] *Let K be a PL sphere. If K is toric colorable, then both $Ss_\sigma(K)$ and $K(J)$ are also toric colorable for any $\sigma \in K$ and any positive integer tuple J:*

$$\mathrm{Pic}(K) = s(K) = s(K(J)) = s(Ss_\sigma(K)) - 1.$$

The rest of this section is devoted to the introduction to the suspension operation. If the vertex v was a ghost vertex, then (2-2) becomes $\partial I * K$. This is called the *suspension* of K. The pair $\{v, v'\}$ of vertices of $\partial I * K$ is called a *suspended pair* of $\partial I * K$, and the vertices v and v' are called *suspended vertices*.

Proposition 2.4 [5] *Let K be a PL sphere. For vertices v and w of K, every facet of K contains v or w if and only if K is the wedge with a wedged edge $\{v, w\}$ or the suspension with a suspended pair $\{v, w\}$.*

Lemma 2.5 *Let K be a seed, J a positive integer tuple, and v, w two distinct vertices of K. If every facet of $K(J)$ contains $v^{(s_v)}$ or $w^{(s_w)}$ for some $0 \le s_v \le j_v - 1$ and $0 \le s_w \le j_w - 1$, then $\{v, w\}$ is a suspended pair of K.*

Proof Without loss of generality, we may assume that $s_v = s_w = 0$. For any vertex $\neq v, w$ of $K(J)$, every facet of the link of the vertex still contains $v = v^{(0)}$ or $w = w^{(0)}$. The repeated link operations with vertices $u^{(s_u)}$ for all $u \in [m]$ and $s_u \ge 1$ gives K, and then every facet of K contains v or w. By Proposition 2.4, $\{v, w\}$ is a suspended pair of K since K has no wedged edge. □

The set of minimal nonfaces of the join of two simplicial complexes is the union of the sets of minimal nonfaces of those. Then one can easily observe that join and wedge operations are associative and commutative. Hence we can write a PL sphere K as

$$K = \partial I_1(J_1) * \partial I_2(J_2) * \cdots * \partial I_\ell(J_\ell) * L(J_{\ell+1}), \tag{2-3}$$

where each I_k is a 1-simplex for $1 \le k \le \ell$, and L is a seed without a suspended pair. A PL sphere is said to be *nonsuspended* if it is not the suspension of any other PL sphere. The interpretation (2-3) of J-constructions involving suspensions explains the following.

Proposition 2.6 *Let K be a PL sphere.*

- *K is a suspension if and only if so is its wedge at any vertex v of K not contained in a suspended pair.*
- *K is a seed if and only if so is the suspension of K.*

3 Main construction

Let K be a seed on $[m]$ and $J = (j_1, \dots, j_m)$ a positive integer tuple. Recall that K is a subcomplex of $K(J)$. Then the wedged edge $\{v, v'\}$ is a face of $\mathrm{Wed}_v(K)$ for each vertex v of K, and each assembled face σ of $K(J)$ is a face of $K(J)$ by Lemma 2.1. Therefore, both $\mathrm{Ss}_{\{v,v'\}}(\mathrm{Wed}_v(K))$ and $\mathrm{Ss}_\sigma(K(J))$ are well defined.

Theorem 3.1 *Let K be a seed on $[m]$ and $J = (j_1, j_2, \dots, j_m)$ a positive integer tuple such that $j_v \le 2$ for all $v \in [m]$. For any assembled face σ of $K(J)$, a new seed is obtained in two ways depending on the cardinality of σ.*

(1) *If $\sigma = \{v\}$ for a vertex v of K, then $Ss_{\{v,v'\}}(K(J))$ is a suspended seed.*

(2) *If $|\sigma| > 1$ and K is nonsuspended, then $Ss_\sigma(K(J))$ is a nonsuspended seed.*

Proof Without loss of generality, we may assume $\sigma = \{v \mid j_v = 2\}$. First, let $\sigma = \{v\}$. Since $\mathrm{Ss}_{\{v,v'\}}(\mathrm{Wed}_v(K))$ is isomorphic to the suspension of K, it is a seed by Proposition 2.6.

Next, let $|\sigma| > 1$ and let K be nonsuspended. Suppose that $\mathrm{Ss}_\sigma(K(J))$ has two vertices x and y such that every facet of $\mathrm{Ss}_\sigma(K(J))$ contains x or y. By Proposition 2.4, $\{x, y\}$ is a wedged edge or a suspended pair of $\mathrm{Ss}_\sigma(K(J))$.

From the construction (2-1) of stellar subdivisions, one can observe that the facet set $\mathcal{F}$ of $\mathrm{Ss}_\sigma(K(J))$ is partitioned into two subsets

$$\mathcal{F}_1 = \mathcal{F} \cap K(J) \setminus \mathrm{St}_{K(J)}(\sigma),$$
$$\mathcal{F}_2 = \mathcal{F} \cap v_\sigma * \partial\sigma * \mathrm{Lk}_{K(J)}(\sigma).$$

The subset $\mathcal{F}_1$ consists of the facets of $K(J)$ not containing σ, and the subset $\mathcal{F}_2$ is involved in the facets of $K(J)$ that contain σ. Let $\mathcal{F}_\sigma$ be the set consisting of facets of $K(J)$ containing σ. Then the facet set of $K(J)$ is $\mathcal{F}_1 \cup \mathcal{F}_\sigma$. Since every facet in $\mathcal{F}_2$ contains x or y, there are four possibilities:

(1) $x = v_\sigma$,

(2) $x, y \in \sigma$,

(3) $x \in \sigma$ and $y \notin \sigma$, and

(4) $x, y \notin \sigma$.

We want to prove that all of these four cases lead to contradictions, so there is no such pairs of vertices x and y.

Case 1 ($x = v_\sigma$): There is no facet in $\mathcal{F}_1$ containing x, so every facet in $\mathcal{F}_1$ contains y. For any $z \in \sigma$ with $z \ne y$, every facet of $K(J)$ contains y or z since the facets in $\mathcal{F}_1$ contain y, and the facets in $\mathcal{F}_\sigma$ contain $\sigma \ni z$. If $y \in \sigma$, then every facet of $K(J)$ contains y, but it is impossible that every facet of a PL sphere shares one vertex. Thus, y is not contained in σ.

Suppose that $y \ne z'$ for any $z \in \sigma$. Then $\mathrm{Lk}_{K(J)}(\{z' \mid z \in \sigma\}) = K$ contains y or z for any $z \in \sigma$ as we discussed in the proof of Lemma 2.5. By Proposition 2.6, K is a suspension or a wedge, but we assumed that K is a nonsuspended seed. Thus, there exists $y_0 \in \sigma$ such that $y = y_0'$. By Lemma 2.5, $\{y_0, z\}$ is a suspended pair of K for any other vertices $z \in \sigma$. The assumption that $|\sigma| > 1$ ensures the existence of such z, and this contradicts the assumption that K is nonsuspended.

Case 2 ($x, y \in \sigma$): The face σ of $K(J)$ contains $\{x, y\}$. Then $\{x, y\}$ is a wedged edge of $K(J)$ since any facet in $\mathcal{F}_1$ contains x or y, and any facet in $\mathcal{F}_\sigma$ contains $\{x, y\}$. Note that σ is also a face of K. Then x and y are two distinct vertices of K, and then $\{x, y\}$ is a suspended pair of K by Lemma 2.5, but K is assumed to have no suspended pair.

Case 3 ($x \in \sigma$ and $y \notin \sigma$): Note that $\sigma \setminus \{x\}$ is a facet of the boundary complex $\partial\sigma$. Then for any facet τ of $\mathrm{Lk}_{K(J)}(\sigma)$, the set $\{v_\sigma\} \cup \sigma \setminus \{x\} \cup \tau$ is a facet in $\mathcal{F}_2$ not containing x. Hence it must contain y, but there is a facet τ of $\mathrm{Lk}_{K(J)}(\sigma)$ not containing y since it is a PL sphere, which contradicts the assumption that every facet of $\mathrm{Ss}_\sigma(K(J))$ contains x or y.

Case 4 ($x, y \notin \sigma$): Every facet in $\mathrm{Lk}_{K(J)}(\sigma)$ contains x or y. Since it is a PL sphere, both x and y have to appear in $\mathrm{Lk}_{K(J)}(\sigma)$ as its vertices. By the assumption that $j_v \leq 2$ for any vertex v of K and the definition of σ, $\mathrm{Lk}_{K(J)}(\sigma)$ is isomorphic to K. By Proposition 2.4, $\mathrm{Lk}_{K(J)}(\sigma)$ is a wedge or a suspension, which contradicts the assumption that $\mathrm{Lk}_{K(J)}(\sigma) \cong K$ is a nonsuspended seed. □

Corollary 3.2 *Assume that $p \geq 3$. For any $m \leq 2^p - 1$ and $n \geq 2$ with $p = m - n$, there exists a polytopal $(n-1)$-dimensional seed K with m vertices such that $s(K) = p$.*

Proof We prove by induction on $p \geq 3$ that there exists a polytopal toric colorable seed with m vertices for any $p+2 \leq m \leq 2^p - 1$, and the one with $m = 2^p - 1$ is nonsuspended. Note that $p+2 \leq m \leq 2^p - 1$ is equivalent to $2 \leq n \leq 2^p - p - 1$.

For $p = 3$, the face structures of a pentagon for $n = 2$, a cross-polytope for $n = 3$, and $C^4(7)$ for $n = 4$ are toric colorable seeds. These are all polytopal by Mani [15], and the one with $n = 4$ is nonsuspended.

Assume that the statement holds for some $p \geq 3$. Let K_n be an $(n-1)$-dimensional polytopal toric colorable seed of Picard number p for $2 \leq n \leq 2^p - p - 1$. Now, it is enough to show the existence of a polytopal toric colorable seed of Picard number $p+1$ for $2 \leq n \leq 2^{p+1} - p - 2$.

For $n = 2$, note that the face complex of an $(n+p+1)$-gon is a polytopal toric colorable seed of Picard number $p+1$.

For $3 \leq n \leq 2^p - p$, by (1) of Theorem 3.1, $\mathrm{Ss}_{\{v,v'\}}(\mathrm{Wed}_v(K_{n-1}))$ for any vertex $v \in K_{n-1}$ is an $(n-1)$-dimensional seed of Picard number $p+1$ that is polytopal by Proposition 2.2 and toric colorable by Proposition 2.3.

For $2^p - p + 1 \leq n \leq 2^{p+1} - p - 2$, we consider K_{2^p-p-1} that is nonsuspended by the induction hypothesis. Note that K_{2^p-p-1} has $2^p - 1$ vertices. Let $\sigma = \{1, 2, 3, \dots, k\}$ for $2 \leq k \leq 2^p - 1$, and J the positive integer tuple whose first k components are 2 and the other components are 1. Then the Picard number and the dimension of $L_k := \mathrm{Ss}_\sigma(K_{2^p-p-1}(J))$ are $p+1$ and $2^p - p + k - 2$, respectively. For each $2 \leq k \leq 2^p - 1$, L_k is indeed an example what we want for $n = 2^p - p + k - 1$ since L_k is a nonsuspended seed by (2) of Theorem 3.1, polytopal by Proposition 2.2, and toric colorable by Proposition 2.3. □

Remark 3.3 If we restrict the assumption of the previous corollary to $p \geq 4$, then all the resulting polytopal seeds can be constructed to be nonsuspended as follows. First, one can easily observe that there exists at least one $(n-1)$-dimensional polytopal toric colorable nonsuspended seed for each $n = 2, 3$, and for the dimensions, all PL spheres are toric colorable. Then for a Picard number p, if there exists an $(n-1)$-dimensional polytopal toric colorable nonsuspended seed with m vertices for each $n \geq 2$ and $m \leq 2^p - 1$ with $p = m - n$, then we can replace the suspension process in the proof of Corollary 3.2 with (2) of Theorem 3.1 by constructing an $(n+1)$-dimensional nonsuspended seed of Picard number $p+1$. The obstruction is that for $p = 3$ and $n = 3$, the only toric colorable seed is the boundary of cross-polytope, which is suspended. However, we can construct an $(n+1)$-dimensional nonsuspended seed of Picard number 4 from the boundary of a pentagon by applying (2) of Theorem 3.1 with $J = (2, 2, 2, 1, 1)$.

Remark 3.4 To achieve the optimal case where $m = 2^{m-n} - 1$, the operation $K(J)$ is applied, where all components of J are equal to 2. This operation is commonly referred to as the *doubling operation*. For more details, see work by Ustinovskii [17] or Park [16].

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