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The primitive curve complex for a handlebody

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A simple closed curve in the boundary surface of a handlebody is called primitive if there exists an essential disk in the handlebody whose boundary circle intersects the curve transversely in a single point. The primitive curve complex is then defined to be the full subcomplex of the curve complex for the boundary surface, spanned by the vertices of primitive curves. Given any two primitive curves, we construct a sequence of primitive curves from one to the other one satisfying a certain property. As a consequence, we prove that the primitive curve complex for the handlebody is connected.

1 Introduction

Let V be a genus-g handlebody for $g \geq 2$, and let Σ be the boundary of V, a closed orientable surface of the same genus. A simple closed curve C in $\Sigma = \partial V$ is called a *primitive curve* if there exists an essential disk D in V such that C intersects ∂D transversely in a single point. Such a disk D is called a *dual disk* of C. Of course any primitive curve in Σ admits infinitely many nonisotopic dual disks, and conversely any nonseparating essential disk in V can be a dual disk of infinitely many nonisotopic primitive curves.

Two primitive curves C and C' are said to be *separated* if there exist dual disks D and D' of C and C', respectively, such that $C \cup D$ and $C' \cup D'$ are disjoint. If two primitive curves C and C' are separated with their dual disks D and D', then one can find quickly their common dual disk by taking a "band sum" of D and D'. On the other hand, there are infinitely many disjoint primitive curves C and C' that are not separated although they have a common dual disk (even though $C \cup C'$ is nonseparating in Σ). The main result of this work is stated as follows.

Theorem 1.1 *Let C and C' be primitive curves in Σ, the boundary of a genus-g handlebody V for $g \geq 2$. Then there exists a sequence $C = C_1, C_2, \dots, C_n = C'$ of primitive curves in Σ such that C_i and C_{i+1} are separated for each $i \in \{1, 2, \dots, n-1\}$.*

We first prove a weaker version of Theorem 1.1, in which C_i and C_{i+1} are only required to be disjoint and to have a common dual disk for each $i \in \{1, 2, \dots, n-1\}$ (Theorem 2.4), and then Theorem 1.1 will be proved for the case of $g = 2$ (Theorem 3.1) and for the case of $g \geq 3$ (Theorem 4.4). In any case, the dual disks of C and C' can be chosen arbitrarily in the first line of the proof.

The curve complex $\mathcal{C}(\Sigma)$ for a closed orientable surface Σ of genus g with $g \geq 2$ is a simplicial complex defined as follows. The vertices of $\mathcal{C}(\Sigma)$ are the isotopy classes of essential simple closed curves in Σ, and a collection of $k+1$ distinct vertices of $\mathcal{C}(\Sigma)$ spans a k-simplex if it admits a collection of representatives, all of which are pairwise disjoint. The combinatorial structure of $\mathcal{C}(\Sigma)$ has been

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widely studied for the past decades. In particular, it is well known that $\mathcal{C}(\Sigma)$ is a $(3g-4)$-dimensional connected complex, and in fact it is homotopy equivalent to a wedge sum of spheres [1]. In this work, we are interested in a special subcomplex of $\mathcal{C}(\Sigma)$ when Σ bounds a handlebody V, called the primitive curve complex.

The *primitive curve complex* $\mathcal{PC}(V)$ for V is the full subcomplex of $\mathcal{C}(\Sigma)$ spanned by vertices whose representatives are primitive curves in Σ. The following is a direct consequence of Theorem 1.1 (it is enough to see that C_i and C_{i+1} are disjoint in the theorem).

Corollary 1.2 *The primitive curve complex $\mathcal{PC}(V)$ for a genus-g handlebody V is connected for every $g \geq 2$.*

Understanding the combinatorial structure of a simplicial complex like $\mathcal{PC}(V)$ is not only an interesting problem itself, but it also has many useful applications. For example, Wajnryb [2] showed that "the cut-system complex" for a handlebody is simply connected, and found a finite presentation of the handlebody group by investigating the action of the group on the complex. The handlebody group also acts on the primitive curve complex $\mathcal{PC}(V)$ in a natural way, and so $\mathcal{PC}(V)$ could be an alternate to find a (simpler) presentation of the group.

We are also interested in a subcomplex of $\mathcal{PC}(V)$, called the primitive disk complex, when the handlebody V is standardly embedded in the 3-sphere. That is, the closure W of the complement of V in the 3-sphere is also a handlebody, which forms a Heegaard splitting of the 3-sphere together with the handlebody V. The *primitive disk complex* $\mathcal{P}(W)$ is then defined to be the full subcomplex of $\mathcal{PC}(V)$ spanned by the vertices of $\mathcal{PC}(V)$ whose representatives bound disks in the handlebody W. The vertices of $\mathcal{P}(W)$ can be considered as the isotopy classes of the disks themselves, which we call primitive disks in W. If $\mathcal{P}(W)$ is a connected complex and moreover if Theorem 1.1 is true for primitive disks in W, i.e., if any two primitive disks in W can be joined by a sequence of primitive disks in W in which any two consecutive disks in the sequence are separated, then one can show quickly that the reducing sphere complex for the splitting is also connected, which implies that the Powell conjecture is true [3].

Throughout the paper, $\overline{X}$ will denote the closure of X and $N(X)$ a regular neighborhood of X for a subspace X of a space, where the ambient space will always be clear from the context.

2 Primitive curves having a common dual disk

The goal of this section is to prove Theorem 2.4, a weaker version of our main theorem. To avoid repeating the expression of "sequence of curves" in the arguments, we simply say that two primitive curves C and C' *are c-connected* if there exists a sequence $C = C_1, C_2, \dots, C_k = C'$ of primitive curves for some k such that C_i and C_{i+1} are disjoint and have a common dual disk for each $i \in \{1, 2, \dots, k-1\}$. We define the *distance* $d(C, C')$ between C and C', concerning the c-connectedness, to be the minimum number $k-1$ over all such sequences $C = C_1, C_2, \dots, C_k = C'$. If all of $C = C_1, C_2, \dots, C_k = C'$ admit a single common dual disk D, then we say that C and C' *are c-connected with the common dual disk* D.

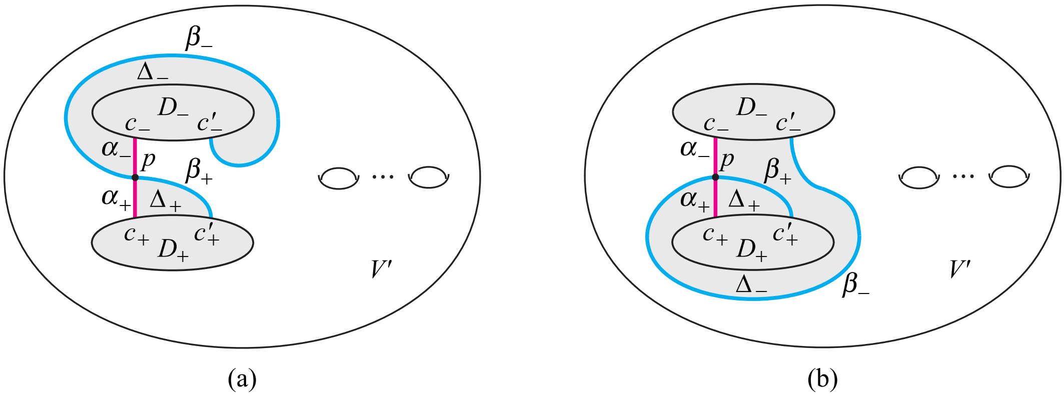

Figure 1: (a) $\Delta_+ \cap \Delta_- = p$ and (b) $\Delta_+ \subset \Delta_-$.

Lemma 2.1 (intersecting in a single point) *Let C and C' be primitive curves in Σ, the boundary of a genus-g handlebody V for $g \geq 2$. Suppose that C and C' have a common dual disk D and that $|C \cap C'| = 1$. Then C and C' are c-connected with the common dual disk D, and $d(C, C') \leq 5$.*

Proof Cutting V along D, we have a genus-$(g-1)$ handlebody V' with two copies D_+ and D_- of D on $\Sigma' = \partial V'$. The primitive curve C is then cut into an arc α with one of its endpoints c_+ in ∂D_+ and the other endpoint c_- in ∂D_-. The primitive curve C' is also cut into an arc β with its endpoints c'_+ and c'_- in ∂D_+ and ∂D_-, respectively. Let $p = \alpha \cap \beta$. Let α_+ and α_- be the subarcs of α with endpoints $\{c_+, p\}$ and $\{p, c_-\}$, respectively. Let β_+ and β_- be the subarcs of β with endpoints $\{c'_+, p\}$ and $\{p, c'_-\}$, respectively. First we consider the following special case.

Case 1 (both $\alpha_+ \cup \beta_+$ and $\alpha_- \cup \beta_-$ cut off disks Δ_+ and Δ_- from $\overline{\Sigma' - D_+}$ and $\overline{\Sigma' - D_-}$, respectively) There are two subcases (see Figure 1(a) and (b)).

(a) The two disks Δ_+ and Δ_- intersect only in the point p.

(b) One of Δ_+ and Δ_-, say Δ_-, contains Δ_+.

(a) Take a point d_+ in $\partial D_+ \cap \Delta_+$ between c_+ and c'_+. Let d_- be the point in ∂D_- that is identified with d_+. Then d_- is not in Δ_-. Take a point q in α_+ between c_+ and p. Let δ_+ be an arc in Δ_+ connecting d_+ and q. Since $D_+ \cup D_- \cup \Delta_+ \cup \Delta_-$ is homotopy equivalent to a point, we can take a nonseparating arc δ_- in $\overline{\Sigma' - (D_+ \cup D_- \cup \Delta_+ \cup \Delta_-)}$ connecting q and d_- (see Figure 2).

Let $\delta = \delta_+ \cup \delta_-$. If we identify D_+ and D_-, then δ becomes a primitive curve with a dual disk D and it is disjoint from C'. The arc δ intersects α in a single point q. Let Δ be the triangular disk determined by $\{q, c_+, d_+\}$. Since δ_- is nonseparating in $\overline{\Sigma' - (D_+ \cup \Delta \cup N(\alpha) \cup D_-)}$, we can take an arc ϵ in $\overline{\Sigma' - (D_+ \cup \Delta \cup N(\alpha) \cup D_-)}$ with endpoints e_+ and e_- in ∂D_+ and ∂D_-, respectively, with the following properties (see Figure 3).

- The arc ϵ is disjoint from α and δ.
- The two points e_+ and e_- become the same point if we identify D_+ and D_-.

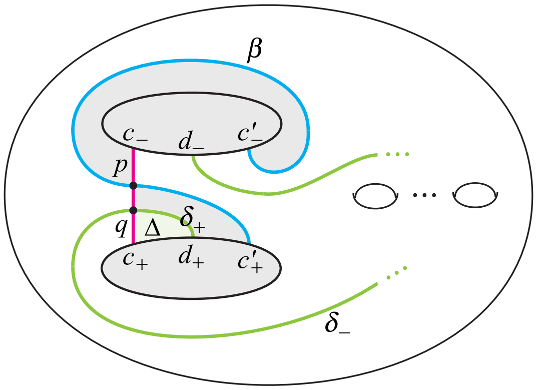

Figure 2: The arc $\delta = \delta_+ \cup \delta_-$.

If we identify D_+ and D_-, then the arc ϵ becomes a primitive curve with the dual disk D, and it is disjoint from C and δ. Hence we see that C and C' are c-connected with the common dual disk D via the curves ϵ and δ, and $d(C, C') \leq 3$ in this case.

(b) We push β_+ along Δ_+ to the opposite side of α_+. Then c'_+ is moved to the opposite side of c_+. Of course, c'_- is also moved to the opposite side of c_-. The point p and the disk Δ_+ disappear, but instead a new intersection point p' of $\alpha \cap \beta$ near c_- and a small triangular disk Δ'_- determined by $\{p', c_-, c'_-\}$ are created. The disk Δ_- disappears, but instead a disk Δ'_+ determined by $\{p', c_+, c'_+\}$ intersecting Δ'_- only in p' is created. Now the situation is like the case (a) in the above, and hence C and C' are c-connected with a common dual disk D, and we have $d(C, C') \leq 3$.

Case 2 (at least one of $\alpha_+ \cup \beta_+$ and $\alpha_- \cup \beta_-$, say $\alpha_+ \cup \beta_+$, does not cut off a disk from $\overline{\Sigma' - D_+}$) Without loss of generality, assume that β_+ is incident to p in the right side of α, and then take a point d_+ in ∂D_+ near c_+ and in the right side of c_+ as in Figure 4. Take a point q in α_+ near c_+. Let δ_+ be a short arc connecting d_+ and q such that the interior of δ_+ is disjoint from $D_+ \cup D_- \cup \alpha \cup \beta$. Let $\delta_- =$ (the subarc of α_+ from q to p) $\cup\, \beta_-$. Slide the endpoint of δ_- in ∂D_- to the point d_- that is

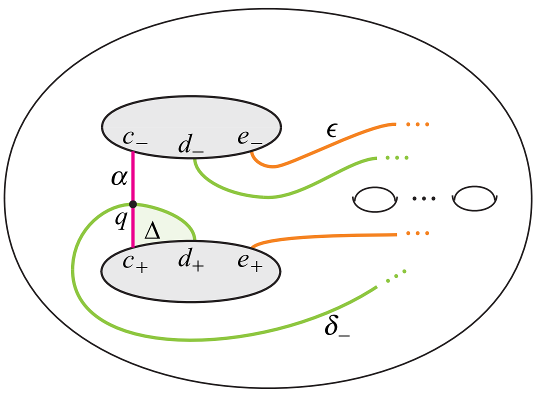

Figure 3: The arc ϵ.

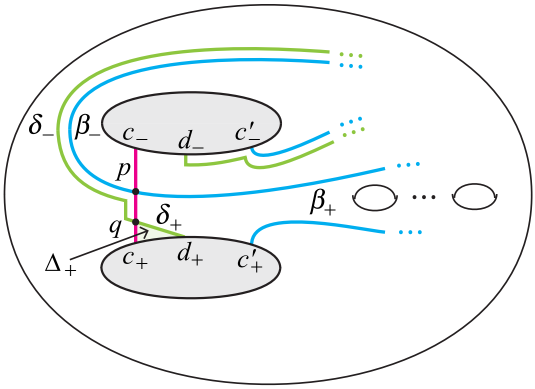

Figure 4: The arc $\delta = \delta_+ \cup \delta_-$.

identified with d_+ in Σ. Then isotope δ_- slightly so that $\delta = \delta_+ \cup \delta_-$ intersects α in a single point q, and δ is disjoint from β (see Figure 4).

Let α'_+ be the subarc of α_+ from c_+ to q, and α'_- be the subarc of α from q to c_-. The union $\alpha'_+ \cup \delta_+$ cuts off a small disk Δ_+ from $\overline{\Sigma' - D_+}$. If $\alpha'_- \cup \delta_-$ also cuts off a disk from $\overline{\Sigma' - D_-}$, then we apply case 1 to α and δ. Since the distance between curves α and δ in V is less than or equal to 3, and δ is disjoint from β, the curves C and C' are c-connected with a common dual disk D, and we have $d(C, C') \le 4$ in this case. If $\alpha'_- \cup \delta_-$ does not cut off a disk from $\overline{\Sigma' - D_-}$, then we apply case 2 again to α and δ with respect to $\alpha'_- \cup \delta_-$. More precisely, we take an arc ϵ such that ϵ is disjoint from δ, and α and ϵ give rise to two disks intersecting in a single point as in case 1(a) (see Figure 5). Since the distance between curves α and ϵ in V is less than or equal to 3, and since δ is disjoint from ϵ and β, the curves C and C' are c-connected with a common dual disk D, and we have $d(C, C') \le 5$ in this case. □

Lemma 2.2 (a stronger version of Lemma 2.1) *Let C and C' be primitive curves in Σ, the boundary of a genus-g handlebody V for $g \ge 2$. Suppose that C and C' have a common dual disk D. Then C and C' are c-connected with the common dual disk D, and $d(C, C') \le 5\,|C \cap C'|$ if $C \cap C' \ne \varnothing$.*

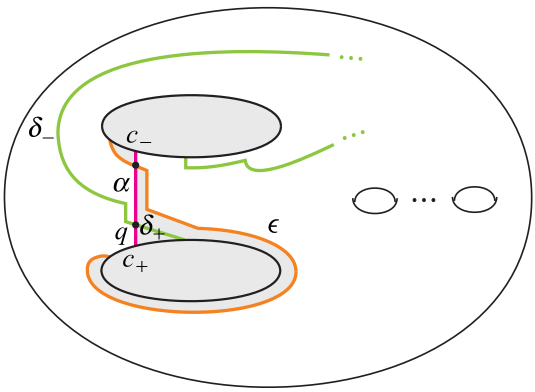

Figure 5: The arcs α and ϵ.

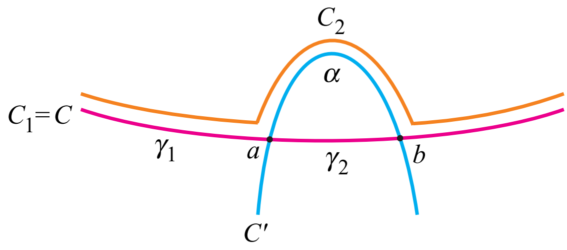

Figure 6: The primitive curve C_2 in case 1.

Proof If $C \cap C' = \varnothing$, then we are done. If $|C \cap C'| = 1$, then the result holds by Lemma 2.1. So suppose that $|C \cap C'| \geq 2$. Let $p = C \cap \partial D$ and $p' = C' \cap \partial D$.

Consider a subarc α of C' cut off by C such that the interior of α is disjoint from C. Let a and b be the two endpoints of α. The two points a and b cut C into two arcs γ_1 and γ_2. If α contains p', then let γ_1 be the arc that does not contain p. If α does not contain p', then let γ_1 be the arc that contains p. So in any case, $\gamma_1 \cup \alpha$ intersects ∂D in a single point. There are two cases according to the sides that α is incident to C at a and b.

Case 1 (α is incident to C in the same side of C at a and b) By a surgery of C along α, we obtain a new primitive curve $C_2 = \gamma_1 \cup \alpha$ with a dual disk D. After a slight isotopy, C_2 is disjoint from C_1 $(= C)$, and $|C_2 \cap C'| \leq |C_1 \cap C'| - 2$ (see Figure 6).

Case 2 (α is incident to C in the opposite sides of C at a and b) By a surgery of C along α, we obtain a new primitive curve $\Gamma = \gamma_1 \cup \alpha$ with a dual disk D. After a slight isotopy, $|\Gamma \cap C| = 1$, and $|\Gamma \cap C'| \leq |C \cap C'| - 1$ (see Figure 7). By Lemma 2.1, C and Γ are c-connected via primitive curves with a common dual disk D, and $d(C, \Gamma) \leq 5$.

By an inductive argument, we get a sequence of primitive curves with a common dual disk D, from C to C'. An upper bound of $d(C, C')$ in worst case occurs if every step of the surgery is case 2. Since the intersection number of curves decreases by only one in worst case, we have $d(C, C') \leq 5\,|C \cap C'|$. □

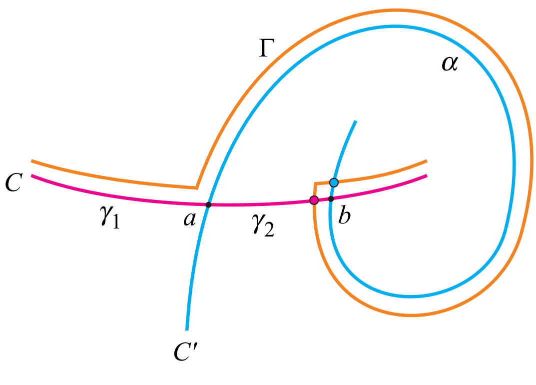

Figure 7: The primitive curve Γ in case 2.

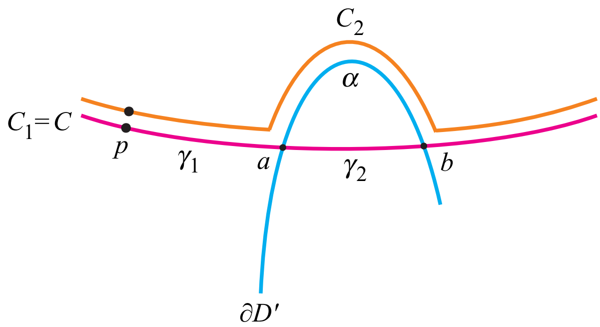

Figure 8: The primitive curve C_2 in case 1.

A key idea in the next lemma is that we do arc surgery with respect to a pair of a primitive curve and the boundary curve of an essential disk.

Lemma 2.3 (arc surgery) *Let C be a primitive curve in Σ, the boundary of a genus-g handlebody V for $g \geq 2$ with a dual disk D. Let D' be an essential disk in V disjoint from D. Then there exists a primitive curve C'' in Σ such that $|C'' \cap \partial D'| \leq 1$, and C and C'' are c-connected with the common dual disk D, and $d(C, C'') \leq 5(|C \cap \partial D'| - 1)$ if $|C \cap \partial D'| \geq 1$.*

Proof If $|C \cap \partial D'| \leq 1$, then we are done by taking $C'' = C$. So suppose that $|C \cap \partial D'| \geq 2$. Let $p = C \cap \partial D$. Consider a subarc α of $\partial D'$ cut off by C such that the interior of α is disjoint from C. Let a and b be the two endpoints of α. The two points a and b cut C into two arcs γ_1 and γ_2, and let γ_1 be the arc that contains p. There are two cases according to the sides that α is incident to C at a and b.

Case 1 (α is incident to C in the same side of C at a and b) By a surgery of C along α, we obtain a new primitive curve $C_2 = \gamma_1 \cup \alpha$ with a dual disk D. After a slight isotopy, C_2 is disjoint from C_1 $(= C)$, and $|C_2 \cap \partial D'| \leq |C_1 \cap \partial D'| - 2$ (see Figure 8).

Case 2 (α is incident to C in the opposite sides of C at a and b) By a surgery of C along α, we obtain a new primitive curve $\Gamma = \gamma_1 \cup \alpha$ with a dual disk D. After a slight isotopy, $|\Gamma \cap C| = 1$, and $|\Gamma \cap \partial D'| \leq |C \cap \partial D'| - 1$ (see Figure 9). By Lemma 2.1, C and Γ are c-connected via primitive curves with a common dual disk D.

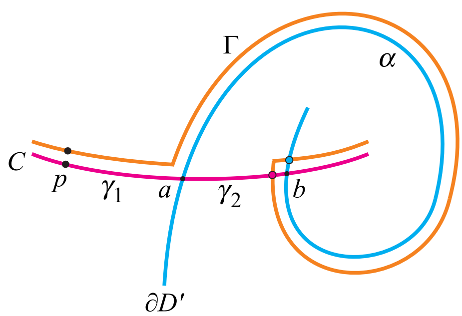

Figure 9: The primitive curve Γ in case 2.

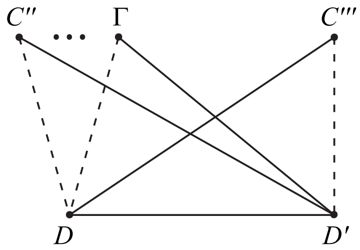

Figure 10: $|(\Gamma \cup D) \cap (C''' \cup D')| \leq 1$.

By an inductive argument, we get a primitive curve C'' with a dual disk D and $|C'' \cap \partial D'| \leq 1$. An upper bound of $d(C, C'')$ in worst case is $5(|C \cap \partial D'| - 1)$ because the surgery process finishes if $|C'' \cap \partial D'| = 1$. □

Now we are ready to prove the following theorem, a weaker version of Theorem 1.1.

Theorem 2.4 *Let C and C' be primitive curves in Σ, the boundary of a genus-g handlebody V for $g \geq 2$. Then C and C' are c-connected.*

Proof Take any dual disks D and D' of C and C', respectively. By the standard disk surgery argument, there exists a sequence $D = D_1, D_2, \ldots, D_n = D'$ of nonseparating disks in V such that D_j and D_{j+1} are disjoint for each $j \in \{1, 2, \ldots, n-1\}$. Let $C_1 = C$ and $C_n = C'$. Take a primitive curve C_j for each $j \in \{2, \ldots, n-1\}$ such that D_j is a dual disk of C_j.

To show the statement of Theorem 2.4 for C and C', it is enough to show the same statement for C_j and C_{j+1}, and each $j \in \{1, 2, \ldots, n-1\}$. So without loss of generality, we assume that C and C' admit dual disks D and D', respectively, such that D and D' are disjoint.

By Lemma 2.3, there exists a primitive curve C'' such that C and C'' are c-connected with a common dual disk D and $|C'' \cap \partial D'| \leq 1$. Similarly, there exists a primitive curve C''' such that C' and C''' are c-connected with a common dual disk D' and $|C''' \cap \partial D| \leq 1$. It remains to show that C'' and C''' are c-connected.

Suppose that one of $|C'' \cap \partial D'|$ or $|C''' \cap \partial D|$, say $|C'' \cap \partial D'|$, is 1. Then D' is a common dual disk of C'' and C'''. By Lemma 2.2, C'' and C''' are c-connected.

Now suppose that both $|C'' \cap \partial D'|$ and $|C''' \cap \partial D|$ are 0. We do an arc surgery for the pair C'' and C''' to reduce $|C'' \cap C'''|$. More precisely, we do a surgery of C'' along an arc component α of C''' cut off by C'' such that the interior of α is disjoint from C'' and that α does not contain the point $C''' \cap \partial D'$. Then we obtain a primitive curve Γ_1 satisfying $|\Gamma_1 \cap C'''| < |C'' \cap C'''|$ with a common dual disk D, and with an additional property that $|\Gamma_1 \cap \partial D'| = 0$ since $|\Gamma_1 \cap \partial D'| \leq |C'' \cap \partial D'| = 0$ by the choice of α. We iterate such an arc surgery to obtain a primitive curve Γ from C'' with a common dual disk D, and $|\Gamma \cap C'''| \leq 1$, and $|\Gamma \cap \partial D'| = 0$. The two primitive curves C'' and Γ are c-connected (using Lemma 2.1 if necessary). See Figure 10. In Figure 10, the dotted lines mean that $|C'' \cap \partial D| = |\Gamma \cap \partial D| = 1$ and $|C''' \cap \partial D'| = 1$, and the solid lines mean that $D \cap D' = \varnothing$ and $(C'' \cup \Gamma) \cap D' = \varnothing$ and $C''' \cap D = \varnothing$.

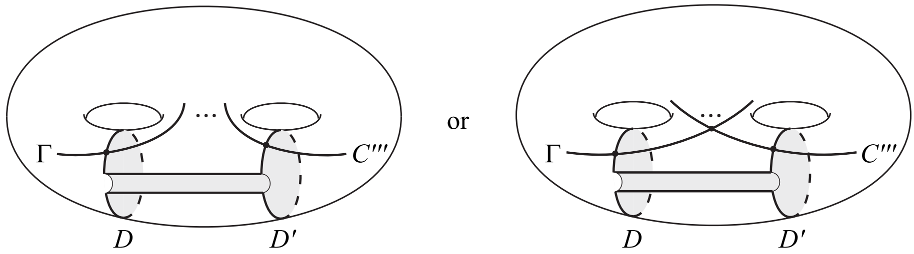

Figure 11: A band sum of D and D' is a common dual disk of Γ and C'''.

Since $|(\Gamma \cup D) \cap (C''' \cup D')| = |\Gamma \cap C'''| \le 1$, we can take a band sum of D and D' that is a common dual disk of Γ and C'''. See Figure 11.

If $|\Gamma \cap C'''| = 0$, then Γ and C''' are c-connected obviously. If $|\Gamma \cap C'''| = 1$, then Γ and C''' are again c-connected by Lemma 2.1. □

3 Proof of the main theorem in the case of $g = 2$

For convenience, we simply say that two primitive curves C and C' *are s-connected* if there exists a sequence $C = C_1, C_2, \dots, C_k = C'$ of primitive curves for some k such that C_i and C_{i+1} are separated for each $i \in \{1, 2, \dots, k-1\}$. If C and C' are s-connected, that is, C_i and C_{i+1} are separated with their disjoint dual disks D_i and D_{i+1}, respectively, then we see that C and C' are also c-connected by taking a band sum of D_i and D_{i+1} that forms a common dual disk of C_i and C_{i+1}. We will prove the following, which is Theorem 1.1 in the case of $g = 2$.

Theorem 3.1 *Let C and C' be primitive curves in Σ, the boundary of a genus-2 handlebody V. Then C and C' are s-connected.*

Proof By Theorem 2.4, C and C' are c-connected (with an arbitrary choice of dual disks of C and C'), and hence it suffices to assume that C and C' are disjoint and have a common dual disk.

Let D be a common dual disk of C and C'. Cutting V along D, we have a solid torus V' with two copies D_+ and D_- of D on $\Sigma' = \partial V'$. The primitive curve C is cut into an arc α with one of its endpoints in ∂D_+ and the other in ∂D_-. The primitive curve C' is also cut into an arc β with its endpoints in ∂D_+ and ∂D_-, respectively. If we crush D_+ and D_- to points, then $\alpha \cup \beta \cup D_+ \cup D_-$ becomes a loop, denoted by ℓ. The loop ℓ is homotopic to an inessential loop in Σ' only if C and C' are isotopic in Σ. So we may assume that ℓ is homotopic to an essential loop in the torus Σ'.

Fix a meridian m_0 and a longitude l_0 of $\Sigma' = \partial V'$. Suppose that ℓ is homotopic to a (p,q)-torus knot $K_{p,q}$. Here $K_{p,q}$ winds V' $|p|$ times in longitudinal direction and $|q|$ times in meridional direction in such a way that $|K_{p,q} \cap m_0| = |p|$ and $|K_{p,q} \cap l_0| = |q|$.

Suppose that $(p,q) = (0,1)$, i.e., ℓ is homotopic to a meridian. Take a meridian disk D_1 of V' such that $D_1 \cap (\alpha \cup \beta \cup D_+ \cup D_-) = \varnothing$. Take another meridian disk D_2 of V' such that $D_2 \cap (D_1 \cup D_+ \cup D_-) = \varnothing$ and $|\partial D_2 \cap \alpha| = |\partial D_2 \cap \beta| = 1$. See Figure 12.

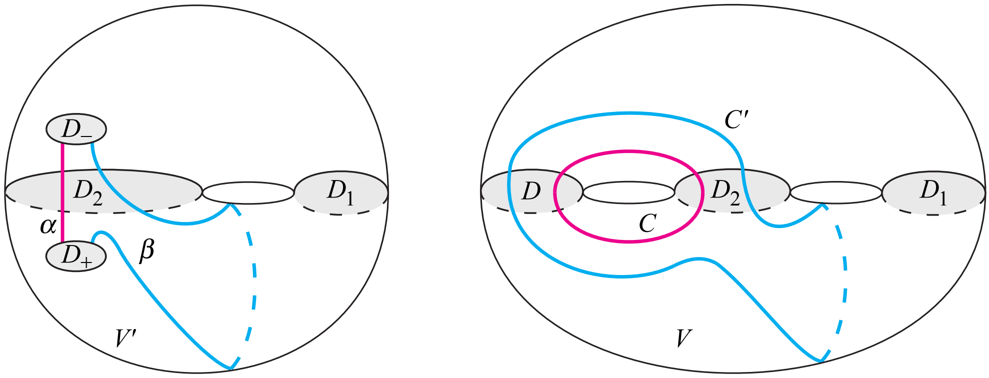

Figure 12: The loop ℓ is homotopic to a meridian.

We take a primitive curve C'' in Σ with the following properties (see Figure 13(a)).

- $C'' \cap (C \cup D) = \varnothing$.
- $|C'' \cap \partial D_1| = |C'' \cap \partial D_2| = 1$.

It is not difficult to take another primitive curve C''' in Σ with the following properties (see Figure 13(b) and (c)).

- $C''' \cap (C'' \cup D_2) = \varnothing$.
- $C''' \cap C' = \varnothing$.
- $|C''' \cap \partial D| = |C''' \cap \partial D_1| = 1$.

The curves C and C' are s-connected because $(C \cup D) \cap (C'' \cup D_1) = \varnothing$ and $(C'' \cup D_2) \cap (C''' \cup D) = \varnothing$ and $(C''' \cup D_1) \cap (C' \cup D_2) = \varnothing$. See Figure 13.

Suppose that $(p, q) = (1, 0)$, i.e., the loop ℓ is homotopic to a longitude. Figure 14 illustrates that $(C \cup D_2) \cap (C' \cup D_1) = \varnothing$. So C and C' are s-connected directly.

The signs of p and q just reflect left-handed or right-handed directions of the winding of $K_{p,q}$ with a choice of orientation. Thus without loss of generality, we only consider those p, q with $p > 0$ and $q > 0$.

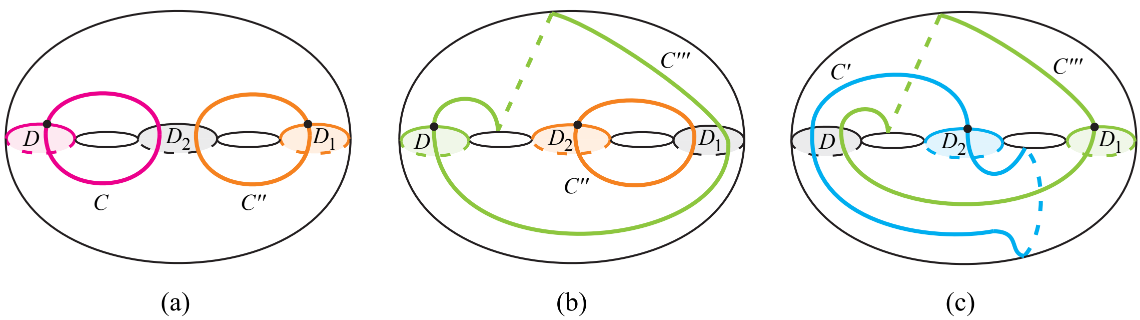

Figure 13: C and C' are s-connected via C'' and C'''.

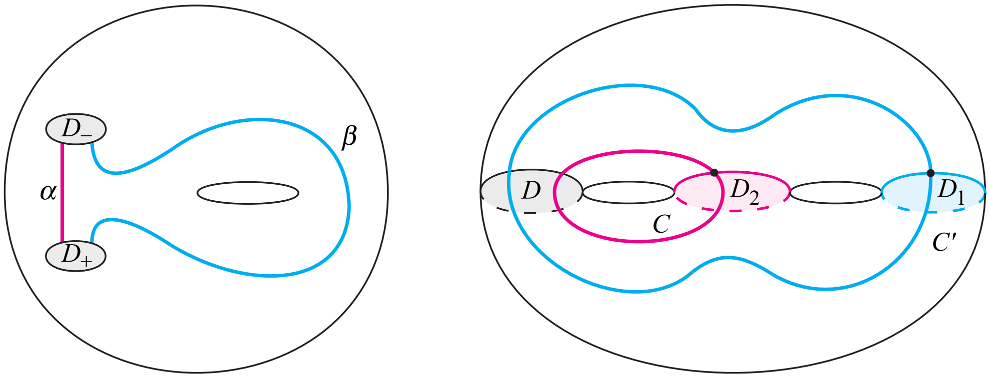

Figure 14: C and C' are separated.

Moreover, if $q \geq p > 0$, then by sufficient Dehn twists of $K_{p,q}$ along m_0, we make q is less than p. So we may assume that $p > q > 0$.

We use induction on p. For that purpose, we consider two types of arcs $\beta_{p,q}$ and $\beta'_{p,q}$ as in Figure 15(a) and Figure 15(b), respectively. Both $\beta_{p,q}$ and $\beta'_{p,q}$ have one of their endpoints in ∂D_+ and the other in ∂D_-. The arc $\beta_{p,q}$ is disjoint from α, but $\beta'_{p,q}$ intersects α in a single point, cutting off a small triangular disk together with D_-. Let $\ell_{p,q}$ and $\ell'_{p,q}$ be the loops obtained from $\beta_{p,q} \cup \alpha \cup D_+ \cup D_-$ and $\beta'_{p,q} \cup \alpha \cup D_+ \cup D_-$ by crushing D_+ and D_- to points, respectively. Both $\ell_{p,q}$ and $\ell'_{p,q}$ are homotopic to $K_{p,q}$. Figure 15 illustrates an example of the case $(p,q) = (7,5)$.

Let r and s be positive integers satisfying $ps - qr = 1$ and $r < p$ and $s \leq q$. Since $\det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = ps - qr = 1$, $|K_{p,q} \cap K_{r,s}| = 1$. For $\beta_{p,q}$ and $\beta'_{p,q}$, we take arcs $\beta_{r,s}$ and $\beta'_{r,s}$, respectively, winding r times in longitudinal direction and s times in meridional direction and $|\beta_{r,s} \cap \beta_{p,q}| = |\beta'_{r,s} \cap \beta'_{p,q}| = 1$ as in Figure 16(a) and Figure 16(b). Both $\beta_{r,s}$ and $\beta'_{r,s}$ have one of their endpoints in ∂D_+ and the other

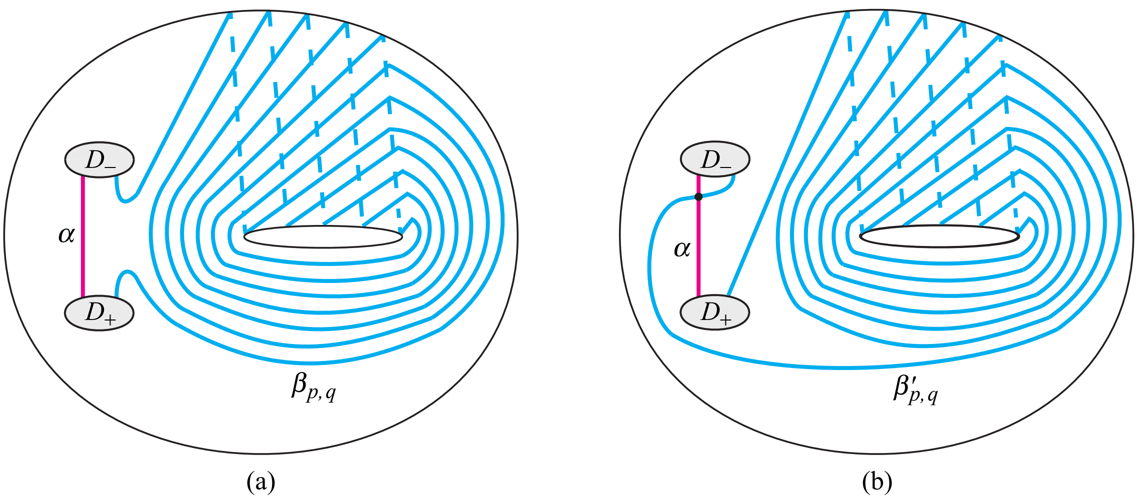

Figure 15: (a) $|\beta_{p,q} \cap \alpha| = 0$ and (b) $|\beta'_{p,q} \cap \alpha| = 1$.

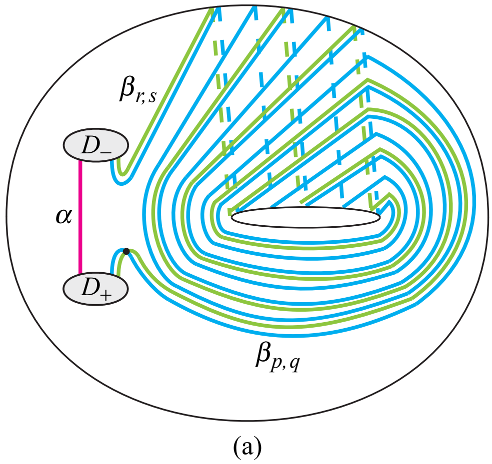

Figure 16: (a) $|\beta_{r,s} \cap \alpha| = 0$ and (b) $|\beta'_{r,s} \cap \alpha| = 1$, and $|\beta_{r,s} \cap \beta_{p,q}| = |\beta'_{r,s} \cap \beta'_{p,q}| = 1$.

in ∂D_-. Moreover, $|\beta_{r,s} \cap \alpha| = 0$ and $|\beta'_{r,s} \cap \alpha| = 1$. The arc $\beta_{r,s}$ is like a stepping stone between α and $\beta_{p,q}$, and the arc $\beta'_{r,s}$ is like a stepping stone between α and $\beta'_{p,q}$.

The disk D_- in Figure 16(a) is isotoped along two long parallel subarcs of $\beta_{r,s}$ and $\beta_{p,q}$ as in Figure 17(a) so that $\beta_{r,s}$ becomes a short arc and $\beta_{p,q}$ winds $p-r$ times in longitudinal direction. Denote the isotoped $\beta_{r,s}$ and $\beta_{p,q}$ by α_0 and $\beta'_{p-r,t}$ for some $t < q$, respectively. Similarly, D_+ in Figure 16(b) is isotoped along two long parallel subarcs of $\beta'_{r,s}$ and $\beta'_{p,q}$ as in Figure 17(b) so that $\beta'_{r,s}$ becomes a short arc and $\beta'_{p,q}$ winds $p-r$ times in longitudinal direction. Denote the isotoped $\beta'_{r,s}$ and $\beta'_{p,q}$ by α_0 and $\beta'_{p-r,t}$ for some $t < q$, respectively. In both cases, let $\ell'_{p-r,t}$ be the loop obtained from $\alpha_0 \cup \beta'_{p-r,t} \cup D_+ \cup D_-$ by crushing D_+ and D_- to points. Then $\ell'_{p-r,t}$ is homotopic to $K_{p-r,t}$. Since $K_{p-r,t}$ intersects the original $K_{p,q}$ in a single point and $\det\begin{pmatrix} p & p-r \\ q & q-s \end{pmatrix} = p(q-s) - q(p-r) = -1$, we have that $t = q-s$.

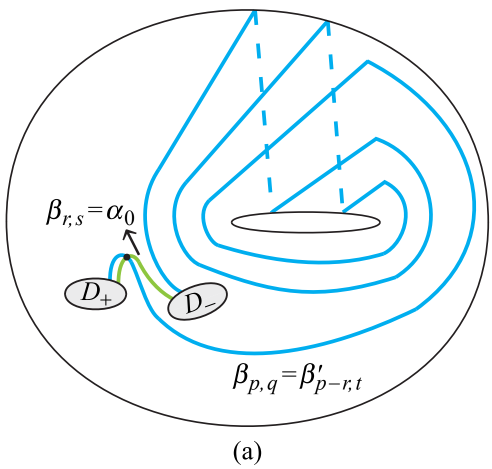

Figure 17: $\beta'_{p-r,t} = \beta'_{p-r,q-s}$.

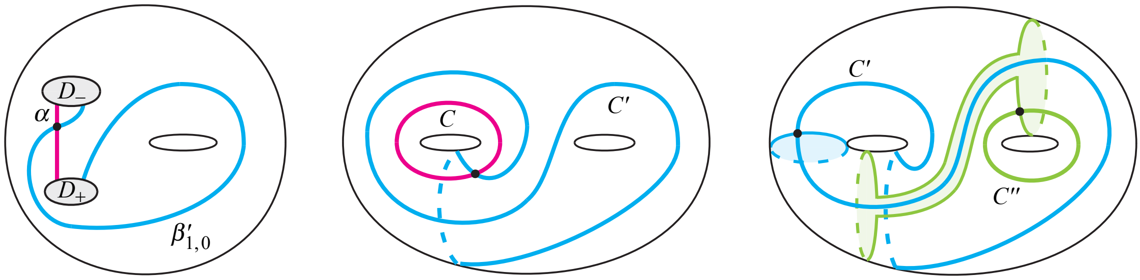

Figure 18: C and C' are s-connected via C''.

Hence the longitudinal and meridional parameters of $\beta_{p,q}$ with respect to $\beta_{r,s}$ and those of $\beta'_{p,q}$ with respect to $\beta'_{r,s}$ are $(p-r, q-s)$. Figures 15, 16, and 17 illustrate an example of $(p,q) = (7,5)$ and $(r,s) = (4,3)$ and $(p-r, q-s) = (3,2)$.

Now it remains only the case of $(p,q) = (1,0)$, the initial condition of the induction. We only need to consider $|\beta'_{1,0} \cap \alpha| = 1$ because we already dealt with the case of $|\beta_{1,0} \cap \alpha| = 0$ (see Figure 14). Figure 18 shows that C and C' are s-connected via C'', where C and C' are obtained from α and $\beta'_{1,0}$, respectively. (It is easy to see that C and C'' are separated.)

Let C, C'', C' be the primitive curves in ∂V obtained by identifying endpoints of α, $\beta_{r,s}$ (or $\beta'_{r,s}$), $\beta_{p,q}$ (or $\beta'_{p,q}$), respectively. Since $r < p$, C and C'' are s-connected by our induction hypothesis. Since $p - r < p$, C'' and C' are s-connected by our induction hypothesis. Hence C and C' are s-connected. □

4 Proof of the main theorem in the case of $g \geq 3$

Throughout the section, V will be assumed to be a genus-g handlebody for $g \geq 3$. For a primitive curve C in $\Sigma = \partial V$ with a dual disk D, we call the pair (C, D) simply a *dual pair* for V. For convenience, we simply say that two dual pairs (C, D) and (C', D') *are p-connected* if there exists a sequence $(C, D) = (C_1, D_1), (C_2, D_2), \dots, (C_k, D_k) = (C', D')$ of dual pairs for some k such that $C_i \cup D_i$ and $C_{i+1} \cup D_{i+1}$ are disjoint for each $i \in \{1, 2, \dots, k-1\}$. It is obvious that if two dual pairs (C, D) and (C', D') are p-connected then C and C' are s-connected.

Lemma 4.1 (common primitive curve) *Let (C, D) and (C, D') be dual pairs for a genus-g handlebody V with $g \geq 3$. Suppose that D and D' are disjoint. Then (C, D) and (C, D') are p-connected.*

Proof Cut V along $D \cup D'$. The resulting manifold V' is a genus-$(g-2)$ handlebody, or two handlebodies of genus g_1 and g_2, respectively, where $g_1 + g_2 = g - 1$. The primitive curve C is cut into two arcs α_1 and α_2. Whether V' is connected or not, there are two copies D_+ and D_- of D and two copies D'_+ and D'_- of D' on $\Sigma' = \partial V'$. One of α_1 and α_2, say α_1, connects D_+ and D'_- and the other α_2 connects D'_+ and D_-. Since each of $D_+ \cup \alpha_1 \cup D'_-$ and $D'_+ \cup \alpha_2 \cup D_-$ is homotopy equivalent to a point, we can take a dual pair (C'', D'') in V' disjoint from them. See Figure 19. Then (C'', D'') can be regarded as a dual pair also in V, and (C, D) and (C, D') are p-connected via (C'', D''). □

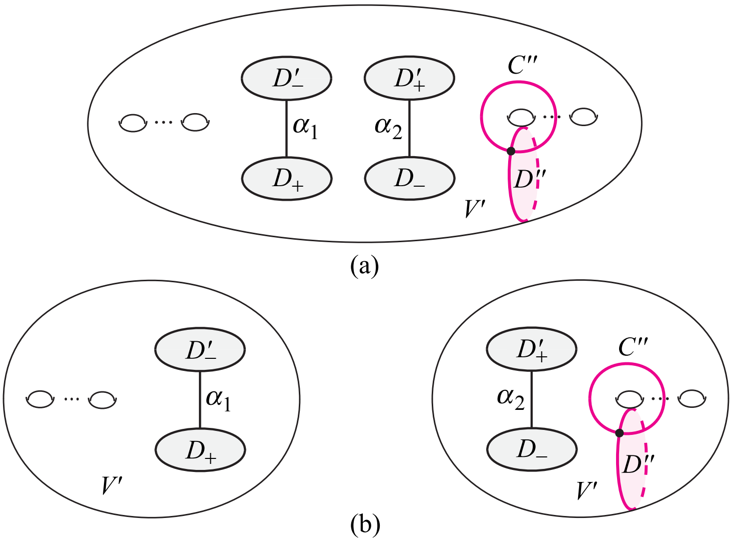

Figure 19: (a) V' is a handlebody and (b) V' is two handlebodies.

Lemma 4.2 *Let (C, D) and (C', D') be dual pairs for a genus-g handlebody V with $g \geq 3$. Suppose that $(C' \cup D') \cap D = \varnothing$. Then (C, D) and (C', D') are p-connected.*

Proof Cut V along D. Then we have a genus-$(g-1)$ handlebody V' with two copies D_+ and D_- of D on $\Sigma' = \partial V'$. The primitive curve C is cut into an arc α in Σ' such that one endpoint is in ∂D_+ and the other endpoint is in ∂D_-. Since $(C' \cup D') \cap D = \varnothing$, (C', D') is a dual pair also in V'. If $(C' \cup D') \cap \alpha = \varnothing$, then (C, D) and (C', D') are p-connected and we are done. So we may assume that $(C' \cup D') \cap \alpha \neq \varnothing$.

We use induction on $|(C' \cup D') \cap \alpha|$. Let p be a point of $(C' \cup D') \cap \alpha$ which is closest to one of the two endpoints of α, say c_+, in ∂D_+. There are two cases.

Case 1 ($p \in C'$) Let $C'_1 = C'$, and C'_2 be a parallel copy of C'_1 such that a point $q \in C'_2 \cap \alpha$ is closer to c_+ than p. Let α_+ be the subarc of α from q to c_+. Slide a small neighborhood of q in C'_2 along $\alpha_+ \cup D_+$. Then $C'_1 \cup C'_2$ bounds an annulus A containing D_+. See Figure 20.

Since (C'_1, D') is a dual pair in a genus-$(g-1)$ handlebody V', and $C'_1 \cup C'_2$ bounds an annulus, and D_- is disjoint from $D' \cup A$, we can take a new dual pair (C_3, D_3) in V' disjoint from $D' \cup A \cup D_-$. All of (C'_1, D') and (C'_2, D') and (C_3, D_3) can be regarded as dual pairs in V because they are disjoint from $D_+ \cup D_-$. The dual pairs (C'_1, D') and (C'_2, D') are p-connected via (C_3, D_3). Since $|(C'_2 \cup D') \cap \alpha| = |(C'_1 \cup D') \cap \alpha| - 1$, by an inductive argument (C'_2, D') and (C, D) are p-connected.

Case 2 ($p \in \partial D'$) Let $D'_1 = D'$, and D'_2 be a parallel copy of D'_1 such that a point $q \in \partial D'_2 \cap \alpha$ is closer to c_+ than p. Let α_+ be the subarc of α from q to c_+. Slide a small neighborhood of q in D'_2 along $\alpha_+ \cup D_+$. Then $D'_1 \cup D'_2$ bounds a region B in V' that is homeomorphic to $D' \times I$ containing D_+.

Since (C', D'_1) is a dual pair in a genus-$(g-1)$ handlebody V', and $D'_1 \cup D'_2$ bounds a 3-ball, and D_- is disjoint from $B \cup C'$, we can take a new dual pair (C_3, D_3) in V' disjoint from $B \cup C' \cup D_-$. All of (C', D'_1)

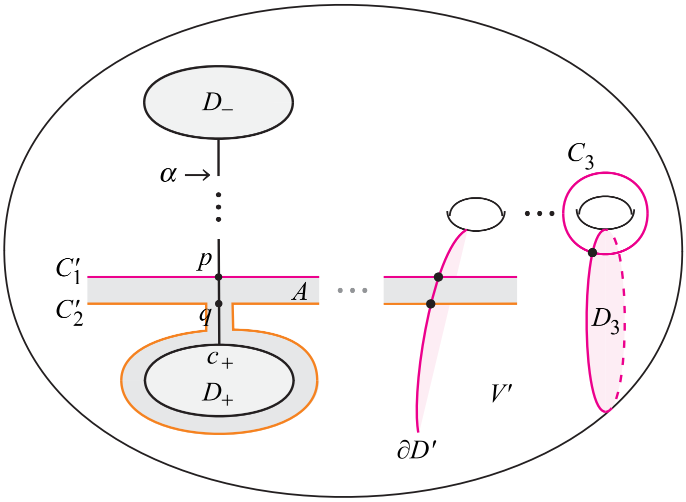

Figure 20: The dual pairs (C'_1, D') and (C'_2, D').

and (C', D'_2) and (C_3, D_3) can be regarded as dual pairs in V because they are disjoint from $D_+ \cup D_-$. The dual pairs (C', D'_1) and (C', D'_2) are p-connected via (C_3, D_3). Since $|(C' \cup D'_2) \cap \alpha| = |(C' \cup D'_1) \cap \alpha| - 1$, by an inductive argument (C', D'_2) and (C, D) are p-connected. □

Lemma 4.3 (common dual disk) *Let (C, D) and (C', D) be dual pairs for a genus-g handlebody V with $g \geq 3$. Then (C, D) and (C', D) are p-connected.*

Proof Cut V along D. Then we have a genus-$(g-1)$ handlebody V' with two copies D_+ and D_- of D on $\Sigma' = \partial V'$. The primitive curves C and C' are cut into arcs α and α' in Σ', respectively, such that for each arc one endpoint is in ∂D_+ and the other endpoint is in ∂D_-. Since $D_+ \cup \alpha' \cup D_-$ is homotopy equivalent to a point, we can take a dual pair (C'', D'') in V' disjoint from it. Then (C'', D'') can be regarded as a dual pair also in V, and (C'', D'') and (C', D) are p-connected. Since $(C'' \cup D'') \cap D = \varnothing$, by Lemma 4.2 (C, D) and (C'', D'') are p-connected. Hence (C, D) and (C', D) are p-connected. □

Now we are ready to prove Theorem 1.1 in the case of $g \geq 3$. Actually we show a slightly stronger version of it as follows.

Theorem 4.4 *Let (C, D) and (C', D') be dual pairs for a genus-g handlebody V with $g \geq 3$. Then (C, D) and (C', D') are p-connected.*

Proof As in the proof of Theorem 2.4, we may assume that D and D' are disjoint. By Lemma 2.3, there exists a sequence $C = C_1, C_2, \dots, C_n = C''$ of primitive curves with a common dual disk D such that C_i and C_{i+1} are disjoint for each $i \in \{1, 2, \dots, n-1\}$ and $|C'' \cap \partial D'| \leq 1$. Since (C_i, D) and (C_{i+1}, D) are p-connected for each $i \in \{1, 2, \dots, n-1\}$ by Lemma 4.3, (C, D) and (C'', D) are p-connected. See Figure 21. In Figure 21, the dotted lines mean that (C_i, D) for each $i \in \{1, 2, \dots, n\}$ and (C', D') are dual pairs, and the solid lines mean that $C_i \cap C_{i+1} = \varnothing$ for each $i \in \{1, 2, \dots, n-1\}$ and $D \cap D' = \varnothing$. It remains to show that (C'', D) and (C', D') are p-connected.

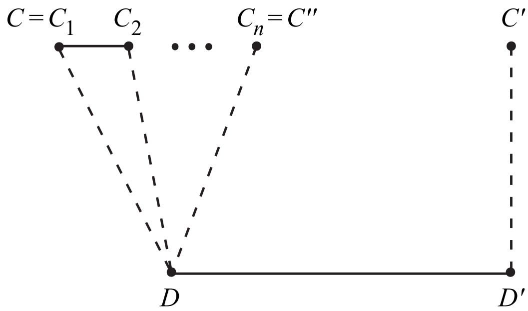

Figure 21: (C, D) and (C'', D) are p-connected and $|C'' \cap \partial D'| \leq 1$.

Suppose that $|C'' \cap \partial D'| = 1$. Then D' is a common dual disk of C'' and C'. By Lemma 4.3, (C'', D') and (C', D') are p-connected. Since (C'', D) and (C'', D') are p-connected by Lemma 4.1, (C'', D) and (C', D') are p-connected.

Now suppose that $|C'' \cap \partial D'| = 0$. Then since $(C'' \cup D) \cap D' = \varnothing$, by Lemma 4.2 (C'', D) and (C', D') are p-connected. □

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