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Cusp-transitive 4-manifolds with every cusp section

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We realize every closed flat 3-manifold as a cusp section of a complete, finite-volume hyperbolic 4-manifold whose symmetry group acts transitively on the set of cusps. Moreover, for every such 3-manifold, a dense subset of its flat metrics can be realized as cusp sections of a cusp-transitive 4-manifold. Finally, we prove that there are many 4-manifolds with pairwise isometric cusps for any given cusp type.

1 Introduction

This work is a continuation of [22], where Rizzi constructed some new cusp-transitive hyperbolic 4-manifolds. We recall in the following the general settings and results.

We say that a complete, finite-volume hyperbolic manifold is *cusp transitive* if the action of its isometry group is transitive on its cusps. This condition is trivially satisfied by manifolds with a single cusp (also called 1*-cusped* manifolds); examples are known in dimension 2, 3 (infinitely many with bounded volume, in fact) and 4: see Kolpakov and Martelli [12]. For $n > 4$, the existence of 1-cusped hyperbolic n-manifolds is unknown, and in general there are no explicit examples of cusp-transitive manifolds in the literature. However, it is easy to obtain examples of the latter in dimension up to 8 by coloring some well-known right-angled polytopes.

The *type* of a cusp of a hyperbolic manifold is the diffeomorphism class of its section, which is a closed flat hypersurface. In this article we will look at the 4-dimensional case; the possible cusp types are closed flat 3-manifolds. Following Conway and Rossetti [4], there are ten such manifolds; six of them are orientable:

- E_1, the 3-torus;
- E_2, the $\frac{1}{2}$-twist manifold;
- E_3, the $\frac{1}{3}$-twist manifold;
- E_4, the $\frac{1}{4}$-twist manifold;
- E_5, the $\frac{1}{6}$-twist manifold;
- E_6, the Hantzsche–Wendt manifold.

The other four are nonorientable: we will call them B_1, B_2, B_3 and B_4 (respectively denoted by $+a1$, $-a1$, $+a2$ and $-a2$ in [4]). These ten manifolds can be constructed as in Figures 4 and 7 by gluing facets of polyhedral fundamental domains; the latter can be obtained, along with the associated gluing maps, by inspecting [4, Table 12].

MSC2020: 57M50.

In the case of 1-cusped orientable 4-manifolds, the cusp types E_1 (Kolpakov and Martelli [12]) and E_2 (Kolpakov and Slavich [13]) can be realized, while E_3 and E_5 cannot (Long and Reid [15]). More generally, the types E_6 (Ferrari, Kolpakov and Slavich [8]) and E_4 (Rizzi [22]) can be realized by cusp-transitive 4-manifolds. The construction in the latter paper also realizes E_1, E_2 and E_6 as cusp types. As for the nonorientable case, there exist 1-cusped 4-manifolds with cusp type B_1 (Kolpakov and Slavich [14]) and B_2 (Ratcliffe and Tschantz [21]).

We improve on the technique of [22], by requiring a less explicit construction, and we prove that the remaining cusp types can also be realized; indeed, our construction actually realizes all the cusp types. Specifically we show:

Theorem 1.1 *For each closed flat* 3*-manifold* N *there exists a cusp-transitive hyperbolic* 4*-manifold* M *with cusps of type* N*.*

By Remark 2.4, if N is orientable, we can also take M to be orientable. The proof of this theorem is split into two parts: in Section 4 we discuss the case of the manifolds E_1, E_2, E_4, E_6, B_1, B_2, B_3 and B_4, while Section 5 is dedicated to E_3 and E_5. The 4-manifolds constructed in each section are all commensurable to each other, but they are arithmetic in the first case and nonarithmetic in the second. Hence, they fall into two commensurability classes.

Using an argument inspired by a paper of Nimershiem [19], this result is strengthened in Section 6 as follows:

Theorem 6.1 *For every closed flat* 3*-manifold* N*, the set of flat metrics on* N *which can be realized as cusp sections of a cusp-transitive* 4*-manifold is dense in the space of all flat metrics of* N*.*

Moreover, with the same technique, we show an analogous result in dimension 3 (where the possible cusp types are the torus and the Klein bottle).

Finally, in Section 7, a variant of our method, combined with some arguments of Burger, Gelander, Lubotzky and Mozes [2], and Belolipetsky, Gelander, Lubotzky and Shalev [1], enables the construction of a lot of manifolds with pairwise isometric cusps:

Theorem 7.1 *For every closed flat* 3*-manifold* N*, there exists a positive constant* c *such that, for sufficiently large* $V > 0$*, there exist at least* V^{cV} *complete hyperbolic* 4*-manifolds with pairwise isometric cusps of type* N *and volume* $\leq V$*.*

By [2], there are $V^{k(n)V}$ complete hyperbolic n-manifolds without boundary of volume $\leq V$, where $k(n)$ depends only on the dimension n.

The construction used in this paper is very similar to the one of Rizzi [22]. Our manifolds are built by applying Davis' *basic construction* [5] to a so-called *developable reflectofold*, in turn obtained by gluing some copies of a hyperbolic Coxeter polytope with the following two properties:

(a) it has exactly one ideal vertex;

(b) if a bounded facet and an unbounded facet intersect, then their dihedral angle is an even submultiple of π.

We have found two hyperbolic Coxeter n-polytopes with $n \geq 4$ satisfying (a) and (b): a 4-polytope P among Im Hof's polytopes associated to Napier cycles [11] (see also Felikson and Tumarkin's web page [7]), and another 4-polytope V with 8 facets, which as far as we know does not appear in the literature. Hence, we have not been able to build manifolds of dimension greater than 4 using these techniques.

The polytope V allows us to answer Questions 1.2 and 1.3 from [22] (reported here) in the affirmative:

- Does there exist a finite-volume hyperbolic Coxeter polytope of dimension $n \geq 4$ satisfying (a) and (b)? If the dimension is $n = 4$ we require that the link of the ideal vertex is neither a parallelepiped nor a prism over a $(2, 4, 4)$-triangle.
- Does there exist a cusp-transitive hyperbolic 4-manifold with cusps of type E_3 or E_5?

Indeed, the link of the ideal vertex of V is a prism K over an equilateral triangle; the fact that E_3 and E_5 are tessellated by copies of K is crucial to the construction of the corresponding cusp-transitive manifolds. The answer to the first question naturally leads to the following.

Question 1.2 Does there exist a finite-volume hyperbolic Coxeter polytope of dimension $n \geq 5$ satisfying (a) and (b)?

The paper is organized as follows. In Section 2 we describe how to obtain a cusp-transitive manifold from a 1-cusped developable reflectofold. In Section 3 we introduce the polytope P and assemble 16 copies of it to create a bigger polytope Q. In Section 4 we show how to construct reflectofolds by gluing copies of Q according to some cubical tessellations, and consequently prove the first case of Theorem 1.1. In Section 5 we introduce the polytope V and construct reflectofolds by gluing copies of it according to certain tessellations in prisms, proving the second and final case of Theorem 1.1. In Section 6, we prove Theorem 6.1 and its 3-dimensional counterpart. Lastly, in Section 7, we prove Theorem 7.1.

2 Reflectofolds

Let N be a closed flat 3-manifold. In this section we will see that, in order to show the existence of a cusp-transitive manifold with cusp type N, it suffices to prove that there exists a 1-cusped *developable reflectofold* with cusp type N. We give some definitions in the following (compare [6, Section 5.1; 22, Section 2]).

Definition 2.1 We say that a hyperbolic manifold R with cornered boundary is a *reflectofold* if R is locally a hyperbolic Coxeter polytope.

Let R be a reflectofold. Since each local model P of R is stratified into k-dimensional faces, for $k = 0, \dots, n$, there is a naturally induced stratification of R into maximal, connected, totally geodesic submanifolds with boundary, called *k-faces*. We will call *facets* and *corners* the $(n-1)$-faces and $(n-2)$-faces of R, respectively. We can define the *dihedral angle* of a corner as the dihedral angle of the corresponding ridge of a local model.

Definition 2.2 A reflectofold is *developable* if it has the following properties:

(EF) *Embedded faces*: For each corner C there are two distinct facets F and F' such that $C \subset F \cap F'$.

(AC) *Angle consistency*: If two distinct facets F and F' intersect, then the dihedral angles of all the corners in $F \cap F'$ coincide.

Proposition 2.3 *Let R be a 1-cusped developable reflectofold with complete cusp section, and built by gluing finitely many finite-volume polytopes. Then, there exists a cusp-transitive manifold M having cusps isometric to that of R.*

Proof The desired manifold M can be constructed as in [22, Section 2] by gluing finitely many copies of R following Davis' basic construction [5, Chapter 11]. Since R has finite volume, M will also have finite volume. Note that, unlike [22], we do not require reflectofolds to be complete; however, M will be complete since it is obtained by gluing polytopes, and has complete cusp sections (see [20, Theorem 11.1.6]). □

Remark 2.4 If the cusp section of the cusp-transitive manifold M thus constructed is orientable, we can also construct an orientable cusp-transitive manifold with the same cusp section, namely the orientable double cover of M, as in [22, Corollary 2.4].

3 Some polytopes

In this section we will introduce and construct the polytopes that we are going to use in Section 4.

Following Vinberg's paper [23, Introduction], we define a *hyperbolic Coxeter polytope* as a convex polytope $P \subset \mathbb{H}^n$ with finitely many facets, with dihedral angles of the form π/m for an integer $m \geq 2$. We call *facets* and *ridges* the faces of P of codimension 1 and 2, respectively.

We associate to such a polytope its *Coxeter diagram*, a labeled graph defined as follows. We refer to the facets by progressive numbers $1, \dots, k$. The graph has a vertex for each facet, and if two facets i and j intersect with dihedral angle π/m_{ij}, then the edge joining the corresponding vertices has label m_{ij}. If $m_{ij} = 2$, the edge is conventionally omitted. If the supporting hyperplanes of i and j are tangent at infinity, we label the corresponding edge with $m_{ij} = \infty$. Finally, a dashed edge is drawn between any two vertices corresponding to ultraparallel facets i and j. The edge can be labeled by $\cosh(d_{ij})$, where d_{ij} is the distance between the supporting hyperplanes.

The *Gram matrix* G of the polytope is a real symmetric $k \times k$ matrix, defined as follows:

$$G_{ij} := \begin{cases} 1 & \text{if } i = j; \\ -\cos(\pi/m_{ij}) & \text{if the facets } i \text{ and } j \text{ intersect;} \\ -1 & \text{if the facets } i \text{ and } j \text{ are parallel;} \\ -\cosh(d_{ij}) & \text{if the facets } i \text{ and } j \text{ are ultraparallel.} \end{cases} \tag{1}$$

3A The polytope P

We now introduce a Coxeter polytope from [11].

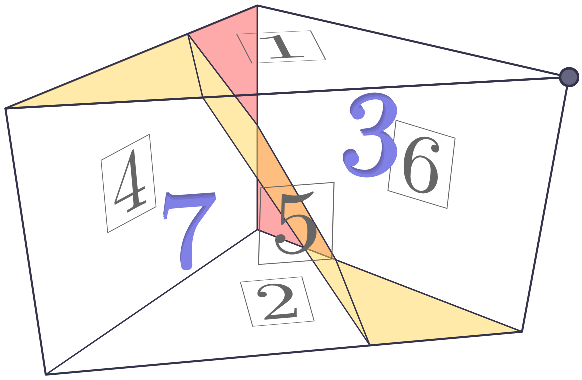
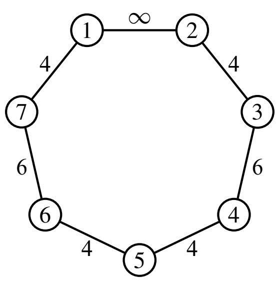


Figure 1: The projection of P onto the link L of its ideal vertex. The five faces of L and the two interior regions are labeled with the corresponding facets of P: five noncompact (**1**, **2**, **4**, **5**, **6**) and two compact (**3**, **7**).

Consider the following Coxeter diagram D_1:



This graph is the Coxeter diagram of a finite-volume arithmetic hyperbolic Coxeter 4-polytope P, introduced in [11], which satisfies (a) and (b). Indeed, by [23, Chapter 1, §3], the ideal vertices correspond to the maximal affine subdiagrams of the Coxeter diagram, and in D_1 we have exactly one of this kind (see [23, Table 2]), spanned by the vertices 1, 2, 4, 5, 6. The subdiagram spanned by these vertices is the Coxeter diagram of a Euclidean right prism over a triangle with interior angles $\pi/2$, $\pi/4$, $\pi/4$, which is the horospherical link of the ideal vertex of P. Arithmeticity can be verified using the program CoxIter [9; 10].

3B Visualizing polytopes with one ideal vertex

In order to better understand the geometry of the 4-dimensional polytope we just defined, it may prove useful to *project* the compact boundary facets onto the horospherical link of the ideal vertex as follows.

Place the polytope P in the half-space model of $\mathbb{H}^4$ with the ideal vertex at infinity, so that the noncompact facets of P become vertical, and let $\pi : \mathbb{H}^4 \to \mathbb{R}^3$ be the orthogonal projection onto a horizontal hyperplane, such as a horosphere or the ideal boundary. The image of P under π is the three-dimensional link L of the ideal vertex, and the noncompact facets are mapped to its boundary.

We can compute the face lattice of P by enumerating all spherical subdiagrams of D, and therefore determine the shape of the two regions associated to the facets **3** and **7** (Figure 1) by applying the following result.

Proposition 3.1 *Let Z be an n-polytope in $\mathbb{H}^n$ with one ideal vertex v. Let Λ be the link of v. The images of the compact facets of Z under the projection onto a horosphere centered at v partition Λ into a polyhedral complex, whose face lattice is combinatorially isomorphic to the lattice of compact faces of Z.*

Proof We consider the half-space model where v is at infinity. We say that a k-subspace of the half-space model is *vertical* if its ideal boundary contains v.

Let F be a compact k-face of Z. Then, in the half-space model, F is supported on a k-hemisphere H. Let Σ be the unique vertical $(k+1)$-space containing H. The face F is bounded by the supporting $(k-1)$-spaces of its facets $F_1, \dots, F_m$. Each F_i lies on a unique vertical k-space S_i. As such, F is the intersection of H and some hyperbolic half-$(k+1)$-spaces bounded by the S_i and contained in Σ. Let P_F be the Euclidean k-polytope obtained by intersecting the projections of these half-spaces on the horosphere; then we have $F = (P_F \times (0, +\infty)) \cap H$. Hence, the projection is a homeomorphism between F and P_F, which preserves facets. Since Z has only one ideal vertex, every compact face of Z is contained in a compact facet. Hence, by induction on the codimension of faces, the projection preserves the face lattice of the compact boundary of Z. □

Dihedral angles between facets of P are defined along ridges (codimension-2 faces), which appear in Figure 1 either as 2-faces (if the angle involves a compact facet) or as edges of L (if the angle is between two noncompact facets). We mark the former by coloring certain 2-faces of the projection, with the following rule. Every 2-face corresponds to a ridge between either two compact facets or a compact facet and a noncompact one. In the first case, we assign the dihedral angle to the 2-face, while in the second case, we assign the doubled dihedral angle. We use yellow for $\pi/2$, red for $\pi/3$, and we leave the 2-face transparent for π.

This rule keeps track of the fact that, if we glue two copies of P along a noncompact facet F, the angle along any ridge it shares with a compact facet gets doubled. For instance, the dihedral angle between facets **1** and **7** is $\pi/4$; when doubling P along **1**, the angle becomes $\pi/2$, so we color in yellow the corresponding triangle in Figure 1.

Moreover, after doubling, some ridges end up with an angle of π, meaning that the compact facet is coplanar with its mirrored copy. Leaving the 2-face transparent has the advantage of visually representing when compact facets merge together in an arbitrary gluing of copies of P.

3C The construction of the polytope Q

In this section we will take 16 copies of P and we will glue them in order to form a bigger polytope Q.

Consider the subdiagram of D spanned by the vertices $1, 5, 6$. This induces a subgroup $G \simeq \mathbb{Z}_2 \times D_4$, of cardinality 16, which is the stabilizer of a noncompact edge of P. This edge projects to the marked vertex in Figure 1.

We can define a new polytope Q by taking the orbit of P under G. Equivalently, we glue 16 copies of P along their noncompact facets **1**, **5**, **6** using the identity map. By construction, Q has exactly one ideal vertex, whose link is a right prism C with a square base, obtained by placing 16 copies of L around the marked vertex (Figure 2).

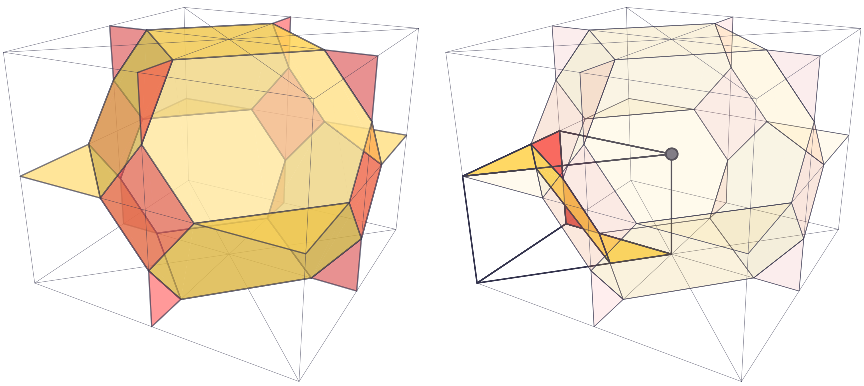


Figure 2: On the left, the projection of Q onto the link C. The truncated octahedron corresponds to a facet made of 16 copies of the facet **3** of P. The other 8 regions appear when copies of the facet **7** merge together two at a time. On the right, we emphasize one copy of L inside C.

The facets **3** merge together into a single facet of Q (which we will also call **3**), which projects to an irregular truncated octahedron in C, with 8 hexagonal and 6 quadrilateral facets. The latter correspond to ridges of Q where the facet **3** meets the noncompact facets: the dihedral angles are $\pi/6$ for the vertical quadrilaterals and $\pi/4$ for the horizontal ones.

The height, width and depth of C are in a ratio of $\cos(\pi/4) : \cos(\pi/6) : \cos(\pi/6) = \sqrt{2} : \sqrt{3} : \sqrt{3}$. Indeed, suppose that C is the cuboid $[-x_1, x_1] \times [-x_2, x_2] \times [-x_3, x_3]$. Then, in the conformal half-space model, each noncompact facet of Q is contained in the product of a facet of C and $(0, +\infty)$. The facet **3** is supported on a hemisphere, which is centered at $(0,0,0,0)$ by symmetry. It is not hard to see that, if this hemisphere has radius r, then the acute angles with the supporting hyperplanes of the noncompact facets of Q are $\theta_i := \arccos(x_i/r)$. Hence, the cosines of the θ_i are in the same ratios as the x_i.

Because of this, all symmetries of C must preserve or exchange the two horizontal faces. There are 16 such symmetries, and they are generated by reflections in the facets **1**, **5**, **6** of L, so they extend to symmetries of Q and they form a group isomorphic to G. This group can also be defined *a priori* as the group of symmetries of a *combinatorial* cube preserving or exchanging a certain pair of opposite facets.

Remark 3.2 The polytope C naturally tessellates $\mathbb{R}^3$ by translations. This gives a tessellation into truncated octahedra in the following way: some are centered and contained in the copies of C, as in Figure 2, while the others are centered at the vertices of the tessellation; one-eighth of a truncated octahedron can be seen near each vertex of C in Figure 2. The two types of truncated octahedra are actually congruent, because of the symmetry of D that exchanges 3 and 7. Indeed, passing to the dual tessellation by copies of C exchanges the two types of truncated octahedra.

4 Layered tessellations of 3-manifolds

In this section we will prove that if a flat 3-manifold N admits a tessellation in cubes with some properties, then there is a cusp-transitive hyperbolic 4-manifold with cusp type N.

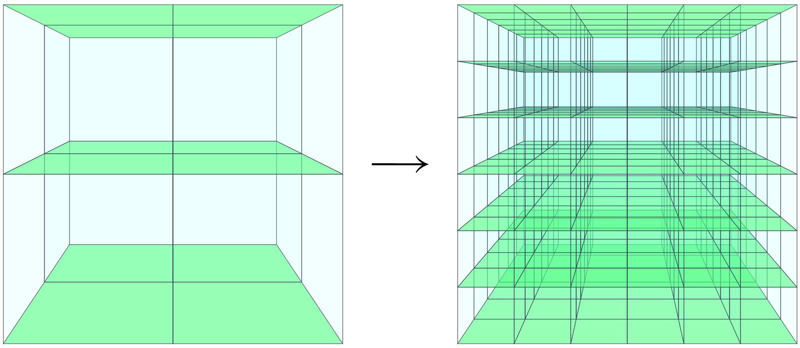


Figure 3: Subdividing a layered tessellation. Special faces are colored green.

Definition 4.1 (layered tessellation) Let T be a tessellation of a flat 3-manifold N into Euclidean cubes, with some chosen *special* 2-cells. We say that T is *layered* if each cube has two opposite special facets, and whenever two cubes C_1, C_2 share a facet F, the reflection through F sends the special facets of C_1 to those of C_2, and vice versa.

The above condition causes the special facets to fall into several embedded geodesic surfaces tessellated by squares, in a discrete analog of a foliation.

Remark 4.2 We will only make use of the combinatorial properties of layered tessellations; hence, we may define generalizations, such as layered tessellations by rectangular cuboids, in the same way.

Remark 4.3 A layered tessellation can be *subdivided* by replacing every cube with a block of $n \times n \times n$ cubes, $n \geq 1$, in which we mark as special all facets parallel to the two original special facets (see Figure 3).

Definition 4.4 We say that a layered tessellation is *proper* if

- every cube is distinct from its six neighbors, which are pairwise distinct;
- every vertex is distinct from its six neighbors, which are pairwise distinct.

Lemma 4.5 *Every layered tessellation can be subdivided into a proper one.*

Proof First, we realize each cube of the tessellation as a cube with side length 1. This gives a flat Riemannian metric on a compact 3-manifold, which has an injectivity radius $r > 0$. We then subdivide the tessellation as in Remark 4.3, in such a way that the side length of the little cubes is less than r. The resulting tessellation is proper because, for every cube, the centers of it and its neighbors are contained in an embedded ball, and a similar argument applies to the vertices. □

Theorem 4.6 *Let N be a closed 3-manifold that admits a layered tessellation. Then there exists an arithmetic cusp-transitive hyperbolic 4-manifold with cusp type N.*

Proof By Proposition 2.3, it suffices to prove that there exists a 1-cusped developable reflectofold with cusp type N, obtained by gluing copies of Q.

By Lemma 4.5, we may assume that the tessellation is proper.

We have that N is obtained by gluing some copies of C along their facets, where the gluing maps are restrictions of isometries of G: this gluing is induced by the layered tessellation, where the special facets correspond to the horizontal facets of C. Since there is a natural correspondence between facets of C and noncompact facets of Q, and every isometry in G extends to Q accordingly, we can glue copies of Q in the same pattern. This gives a 1-cusped reflectofold with cusp type N.

It remains to prove that N is developable. The facets of N are hyperbolic truncated octahedra, which arise from the merging of 16 facets **3** or **7**; we will call them *of types* **3** and **7** respectively. The former are centered at the centers of the cubes, while the latter are centered at the vertices of the tessellation.

If a facet of type **3** were adjacent to itself, it would be so along a quadrilateral corner, and so it would also occupy a neighboring cube. However, the two cubes must be distinct by properness of the tessellation. A similar argument involving vertices works for facets of type **7**.

As for angle consistency, if two facets of different type intersect, they do so in a hexagonal corner with a dihedral angle of $\pi/2$. If a facet intersects another of the same type, say **3**, it must do so at a single quadrilateral corner, since the neighbors of a given cube are pairwise distinct by properness; angle consistency follows trivially. A dual argument deals with the case of two facets of type **7**.

The 4-manifold thus constructed is arithmetic since it covers the arithmetic orbifold P. □

Corollary 4.7 *Let N be one of the manifolds E_1, E_2, E_4, E_6, B_1, B_2, B_3, B_4. Then there exists a cusp-transitive arithmetic hyperbolic 4-manifold with cusp type N.*

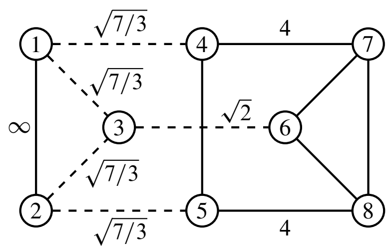
Proof As shown in Figure 4, the manifold N admits a layered tessellation, so the result follows from Theorem 4.6. □

5 The remaining manifolds

In this section we will construct cusp-transitive manifolds having the $\frac{1}{3}$-twist and $\frac{1}{6}$-twist manifolds as cusp sections.

5A The polytope V

Let D_2 be the following Coxeter diagram:



The diagram D_2 is obviously connected, and we can check that its Gram matrix has signature $(4, 1, 3)$ with nonpositive off-diagonal entries; hence, by Vinberg's theorem [23, Theorem 2.1], it defines a hyperbolic

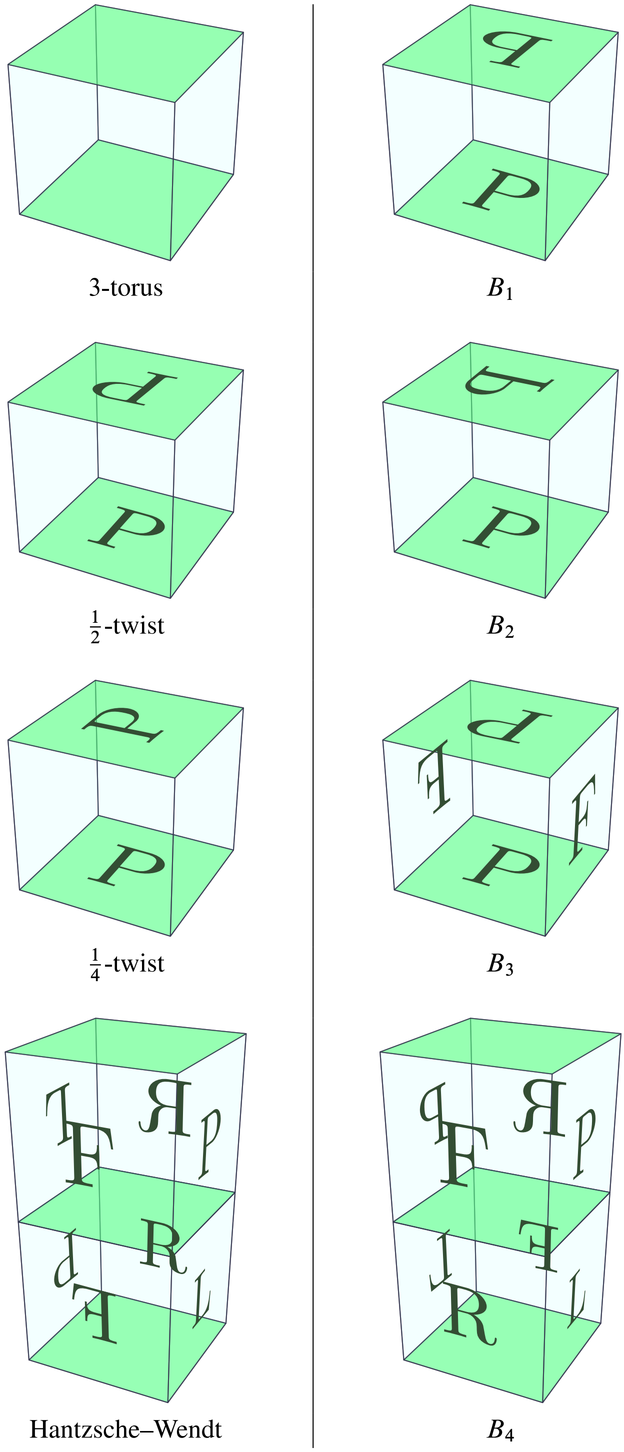


Figure 4: Layered tessellations for eight closed flat 3-manifolds. Labeled facets are glued according to their labels, while unlabeled facets are glued to the opposite facets by translation. The fundamental domains and gluing maps can be deduced from the algebraic descriptions of [4, Table 12].

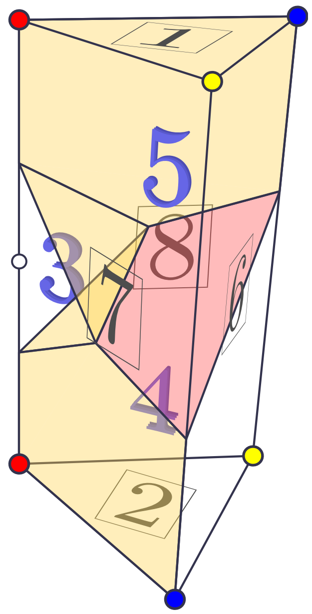


Figure 5: The projection of V onto the link K of its ideal vertex, with labels indicating the five noncompact facets ($\mathbf{1}, \mathbf{2}, \mathbf{6}, \mathbf{7}, \mathbf{8}$) and three compact facets ($\mathbf{3}, \mathbf{4}, \mathbf{5}$).

Coxeter 4-polytope V. Using CoxIter [9; 10], we find that V has finite volume and is nonarithmetic. Moreover, like the polytope P, it satisfies (a) and (b): there is exactly one ideal vertex, corresponding to the unique maximal affine subdiagram of D_2 [23, Chapter 1, §3], which is spanned by 1, 2, 6, 7, 8. Its link K is a Euclidean right prism over an equilateral triangle (Figure 5, obtained by using Proposition 3.1, and colored with the rule of Section 3B).

From the diagram D_2, we can see that the polytope V has a symmetry σ that exchanges the facets **1** and **2**, **4** and **5**, **7** and **8**. This induces a half-turn rotation of K swapping each vertex with the other one of the same color in Figure 5. This rotation is the only nontrivial color-preserving combinatorial automorphism of K.

5B Marked tessellations of 3-manifolds

Mirroring the previous sections, we will prove that if a flat 3-manifold N admits a tessellation in prisms over equilateral triangles with some properties, then there is a cusp-transitive hyperbolic 4-manifold with cusp type N.

Definition 5.1 (marked tessellation) Let T be a tessellation of a flat 3-manifold N into right prisms over equilateral triangles, such that the vertices are colored red, yellow or blue. We say that T is *marked* if the vertices of each prism are colored as in Figure 5 (up to combinatorial isomorphism), and whenever two prisms share a facet F, they are symmetrical with respect to F, including their vertex colorings.

Remark 5.2 Given a marked tessellation T and two natural numbers a, b, we can *subdivide* each prism as follows. First, note that an equilateral triangle can be subdivided into a^2 triangles (whose sides are a

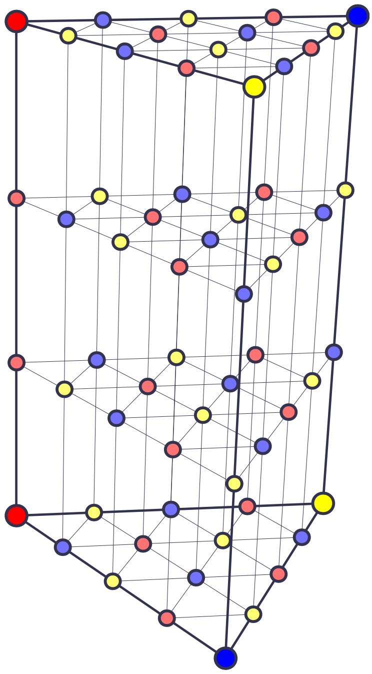


Figure 6: Subdividing a marked tessellation ($a = 4$, $b = 3$). The pattern on the bottom face of the prism works whenever $a - 1$ is a multiple of 3.

times smaller), by cutting it with three sets of equally spaced lines parallel to the sides; similarly, a prism can be subdivided into a^2 thin prisms. Each of those can then be cut, parallel to the bases, into b equal prisms. The new tessellation T' can be marked provided that b is odd and that $a - 1$ is a multiple of 3 (see Figure 6 for the case $a = 4$, $b = 3$). Indeed, for each prism t of T, we now have $b + 1$ layers of vertices of T'. It is not hard to see that, whenever $a \equiv 1 \pmod 3$, there is a unique way to 3-color the bottom layer consistently with the lower three vertices of t. Each subsequent layer will be the same as the one below it, but with yellow and blue swapped. If b is odd, then the top and bottom layers are the same with yellow and blue swapped, ensuring consistency with the coloring of t.

Remark 5.3 When gluing copies of V along noncompact facets, the compact facets merge into two kinds of new facets, which we can visualize with the help of the projection in Figure 5. The first kind arises from 6 copies of the facet **3** around the white dot, while the second comes from 12 copies of the facet **4** or **5** around the corresponding yellow dot. Both kinds of facets are bounded.

Theorem 5.4 *Let N be a closed 3-manifold that admits a marked tessellation. Then there exists a nonarithmetic cusp-transitive hyperbolic 4-manifold with cusp type N.*

Proof The proof is similar in many aspects to that of Theorem 4.6, so we will only go over the main points. Again, by Proposition 2.3, it suffices to prove that there exists a 1-cusped developable reflectofold with cusp type N, obtained by gluing copies of V.

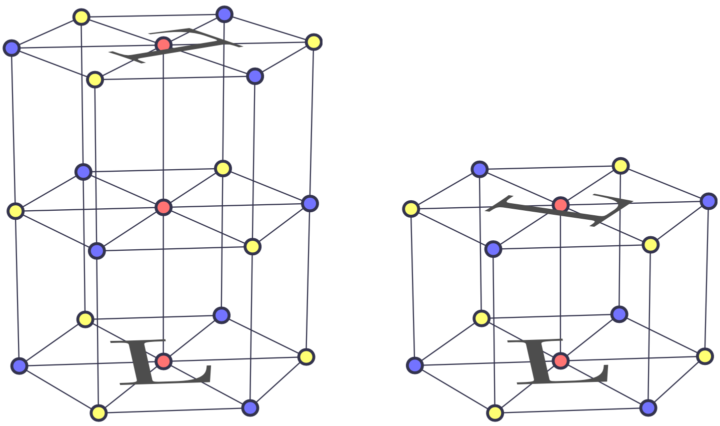


Figure 7: Marked tessellations for E_3 and E_5. Labeled facets are glued according to their labels, while unlabeled facets are glued to the opposite facets by translation. The fundamental domains and gluing maps can be deduced from the algebraic descriptions of [4, Table 12].

A marked tessellation of N induces a way to glue copies of V along their noncompact facets; a key fact is that every color-preserving combinatorial automorphism of K is induced by a symmetry of V. This gives a 1-cusped reflectofold with cusp type N.

In order to ensure (EF) and (AC), it suffices that each facet of the reflectofold, together with its adjacent facets, is inside an embedded ball. Indeed, this ensures that every facet is embedded and that the intersection of two adjacent facets is a single corner. Since the facets are bounded by Remark 5.3, this can be done by an argument involving the injectivity radius and a sufficiently fine subdivision of the marked tessellation; the reflectofold constructed from the subdivided tessellation is developable.

The 4-manifold thus constructed is nonarithmetic since it covers the nonarithmetic orbifold $V/\langle\sigma\rangle$, where σ is the order-2 isometry of V defined in Section 5A. □

Corollary 5.5 *Let N be one of the manifolds E_3, E_5. Then there exists a cusp-transitive hyperbolic 4-manifold with cusp type N.*

Proof The manifolds E_3 and E_5 have marked tessellations, as shown in Figure 7; the result follows from Theorem 5.4. □

6 Density

In this section we will strengthen the previous results by investigating the possible Euclidean structures that can be realized by the cusp sections of a cusp-transitive 4-manifold, and we show how the same techniques can be used in the 3-dimensional case. We have the following result:

Theorem 6.1 *For every closed flat 3-manifold N, the set of flat metrics on N which can be realized as cusp sections of a cusp-transitive 4-manifold is dense in the space of all flat metrics of N.*

Proof The argument is inspired by the proof of [19, Theorem 2]. We divide the proof into two cases: the manifolds E_1, E_2, E_4, E_6, B_1, B_2, B_3, B_4 (tessellated by cubes) and the manifolds E_3, E_5 (tessellated by triangular prisms).

We start with the first case. Any flat metric on N is induced by a subgroup $\Gamma < \mathrm{Isom}(\mathbb{R}^3)$. Generators for Γ are found in [19, Table 1] in the form (A, t_u), where $A \in \mathrm{O}(3)$ and t_u is a translation by the vector u, denoting the map $v \mapsto u + Av$.

The group Γ can be conjugated as in [19, pages 128–129] so that all the matrices A_i are certain fixed signed permutation matrices, which preserve the x axis, and preserve or exchange the y and z axes. In this *normalized form*, the translation vectors have some zero entries, while the k-uple of the other *free* entries can take any value in an open set of $\mathbb{R}^k$, containing $(\mathbb{R} \setminus \{0\})^k$. A dense subset of these forms can be obtained by considering only vectors of the form

$$v_i := (\sqrt{2}a_i, \sqrt{3}b_i, \sqrt{3}c_i),$$

where a_i, b_i, c_i are in $\mathbb{Q} \setminus \{0\}$ or $\{0\}$, depending on whether the corresponding entry of v_i is free or not. Let Γ' be a group with parameters in this dense subset; we shall prove that the metric resulting from Γ' can be realized by a layered tessellation by copies of C (see Remark 4.2).

The group Γ' preserves a lattice of the form

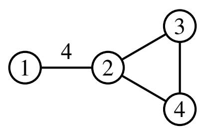
$$\left\{\frac{1}{d}(\sqrt{2}m, \sqrt{3}n, \sqrt{3}p) \mid m, n, p \in \mathbb{Z}\right\},$$

where d is a common denominator of all the a_i, b_i, c_i. Furthermore, it preserves a layered tessellation of $\mathbb{R}^3$ by copies of C — having side lengths $\sqrt{2}/d$ and $\sqrt{3}/d$ — with the special facets orthogonal to the x axis; this is because the x axis is preserved by the A_i. This tessellation descends to the quotient $\mathbb{R}^3/\Gamma'$. The result follows as in Theorem 4.6.

As for the second case, we refer to [4, Section 4]. The space of flat metrics on N has two parameters: in the hexagonal prism fundamental domains of Figure 7, they can be taken as the side length and height of the prisms. When subdividing a marked tessellation in the proof of Theorem 5.4, we can choose two parameters a and b provided that b is odd, $a-1$ is a multiple of 3, and they are sufficiently large. The effect of these parameters is to multiply the side length and height of the fundamental domains by a and b respectively. Since we can realize the tessellation with arbitrarily small copies of K, by choosing a and b in an appropriate way, we can approximate any values of the two parameters of the metric of N. The result follows as in Theorem 5.4. □

6A Dimension 3

We consider the following diagram D_3, found in [3, Appendice, Rang 4]:



The corresponding Coxeter polytope Y is a finite-volume hyperbolic tetrahedron with one ideal vertex, corresponding to the subdiagram spanned by 2, 3, 4; its link is an equilateral triangle. The polytope has the properties (a) and (b).

Theorem 6.2 *For every closed flat* 2*-manifold* S (*i.e., the torus and the Klein bottle*), *the set of flat metrics on* S *which can be realized as cusp sections of a cusp-transitive* 3*-manifold is dense in the space of all flat metrics of* S*.*

Proof We begin by gluing six copies of Y around the edge between the facets **3** and **4**. The resulting polytope Z has one ideal vertex, whose link is a regular hexagon, and one compact facet, which meets the other six facets at an angle of $\pi/4$.

If S has a tessellation by regular hexagons, such that every hexagon is embedded, then we can glue copies of Z along their noncompact facets in the same pattern. The resulting manifold with corners R is a developable reflectofold: its dihedral angles are all equal to $2 \cdot \pi/4 = \pi/2$, also implying (AC), and its facets are embedded hyperbolic hexagons (EF). Moreover, the section of its unique cusp is isometric to S up to global rescaling. As usual, by Proposition 2.3, we can construct a cusp-transitive manifold which covers R, with cusps isometric to S.

It remains to show that, up to arbitrarily small perturbations, every flat metric on the torus or the Klein bottle admits a tessellation by regular hexagons.

First, consider a parallelogram fundamental domain D_T for the torus, and overlay it onto a tessellation of small regular hexagons. We can perturb D_T by moving three vertices to the nearest hexagon center, and the fourth in such a way as to make a parallelogram; the fourth vertex will also fall into the center of a hexagon. This gives our desired tessellation.

As for the Klein bottle, the generic fundamental domain D_K is a rectangle, with two opposite sides s_1, s_2 identified with a twist. We overlay D_K onto a tessellation of small regular hexagons with some sides parallel to s_1, s_2. We can perturb D_K by translation or scaling along either axis, obtaining a rectangle with all vertices at the centers of hexagons. Note that the tessellation is symmetrical with respect to the line joining the midpoints of the perturbed s_1 and s_2. Hence, we have a tessellation of the Klein bottle. □

7 Lots of manifolds

In this section we prove the following theorem.

Theorem 7.1 *For every closed flat* 3*-manifold* N, *there exists a positive constant* c *such that*, *for sufficiently large* $V > 0$, *there exist at least* V^{cV} *complete hyperbolic* 4*-manifolds with pairwise isometric cusps of type* N *and volume* $\leq V$*.*

Proof Let R be a reflectofold constructed as in the proof of Theorem 4.6 or 5.4 (according to N). Let D be the Coxeter diagram with one vertex for each facet of R, where if two facets meet with dihedral angle of π/k (respectively do not intersect), the corresponding vertices are joined by an edge labeled k (respectively a dashed edge).

If R is constructed from a sufficiently fine subdivision of a tessellation, then the diagram D is connected. Indeed, two nonadjacent facets are connected by a dashed edge, while for any two adjacent facets F, F' there exists a third one not adjacent to them (in the projection onto the link, it can be found in a large embedded ball centered on, say, F). As a consequence, we can also assume that D has a dashed edge.

Let G be the Coxeter group associated to D. It is neither affine nor spherical, since D has a dashed edge (compare [23, Tables 1–2]). By [17, Corollary 2], G has a finite-index subgroup H with a quotient isomorphic to the free group F_2. Since H is a subgroup of a Coxeter group, it has a torsion-free subgroup H' such that $d := [G : H'] < +\infty$. The image of H' in F_2 is free of rank at least 2, so H' also has a quotient isomorphic to F_2. By [18, Theorem 2], there exists a constant $\alpha > 0$ such that F_2 has at least $\alpha r \cdot r!$ subgroups of index $\leq r$. Hence, by pulling back to H', we obtain at least $\alpha r \cdot r!$ torsion-free subgroups of G of index $\leq dr$. These correspond to manifold covers of R with degree $\leq dr$, which have pairwise isometric cusps of type N by construction. As in the proof of Proposition 2.3, we can show that these manifolds are complete.

Let v be the volume of R and let $V := vdr$. Then our manifolds have volume $\leq V$. Using Stirling's approximation, we have the following estimate (for some $k, k', c > 0$ and for r large):

$$\begin{aligned}\log(\alpha r \cdot r!) &= \log \alpha + \log r + \log r! \\ &= \log \alpha + \log r + r \log r - r + O(\log r) \\ &\geq k r \log r \\ &= k' \frac{V}{vd} \log \frac{V}{vd} \\ &\geq c V \log V \\ &= \log(V^{cV}).\end{aligned}$$

Hence, we have at least V^{cV} torsion-free subgroups of G. The associated manifolds (of volume $\leq V$) are not necessarily distinct; however, the same estimate holds on the number of isometry classes, with a smaller constant c. Indeed, if G is nonarithmetic, we conclude with the same argument of [2, "the lower bound"] using Margulis' theorem on the commensurator [16, Theorem 1, page 2], while if G is arithmetic, we conclude as in [1, Section 5.2] using the Kazhdan–Margulis theorem. □

Remark 7.2 Since we take subgroups of G which are not necessarily normal, the group G does not act on the associated covers. Hence, we do not necessarily get cusp-transitive manifolds.

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