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DIEGO SANTORO

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We prove that if M is a rational homology sphere that is Dehn surgery on a fibred hyperbolic two-bridge link, then M is not an L-space if and only if M supports a co-orientable taut foliation. As a corollary we show that if K' is obtained by a nontrivial knot K as a result of an operation called *two-bridge replacement*, then all nonmeridional surgeries on K' support co-orientable taut foliations. This operation generalises Whitehead doubling and as a particular case we deduce that all nonmeridional surgeries on Whitehead doubles of a nontrivial knot support co-orientable taut foliations.

1 Introduction

In recent years the field of low-dimensional topology has seen a growing interest in the study of the so-called L-space conjecture. This conjecture predicts that the following notions of "complexity" are all equivalent:

Conjecture 1.1 (L-space conjecture) *For an irreducible oriented rational homology* 3*-sphere* M, *the following are equivalent*:

(1) *M supports a co-oriented taut foliation.*

(2) *M is not an L-space*, *i.e.*, *its Heegaard Floer homology is not minimal.*

(3) *M is left-orderable*, *i.e.*, *$\pi_1(M)$ is left-orderable.*

The equivalence between (1) and (2) was conjectured by Juhász [27], while the equivalence between (2) and (3) was conjectured by Boyer, Gordon and Watson [4]. This conjecture predicts strong connections among geometric, dynamical, Floer homological, and algebraic properties of 3-manifolds. Despite its boldness, as a result of [3; 4; 5; 8; 14; 22; 40] it is now known that the conjecture holds for all the graph manifolds, i.e., the manifolds whose JSJ decomposition includes only Seifert fibred pieces. Moreover the results of Oszváth–Szabó [47], Bowden [2] and Kazez–Roberts [30] imply that in general manifolds supporting co-orientable taut foliations are not L-spaces.

A natural way to investigate this conjecture is by using Dehn surgery descriptions of 3-manifolds. For instance, it is known that if a nontrivial knot K in S^3 has a positive surgery that is an L-space, then K is prime [31], fibred [20; 45] and strongly quasipositive [23]. Moreover, the r-framed surgery on such a knot K is an L-space if and only if $r \geq 2g(K)-1$, where $g(K)$ denotes the genus of K [33]. Taut foliations on manifolds obtained as surgery on knots in S^3 are constructed, for example, in [11; 12; 32; 51; 52] and it is possible to prove the left-orderability of some of these manifolds by determining which of these foliations have vanishing Euler class, as done in [25]. Another approach to study the left-orderability of surgeries on knots is via representation-theoretic methods, as presented in [9; 13].

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When it comes to investigate surgeries on links, the story becomes more mysterious. For instance, there is no generalisation of the result of [33] we cited in the previous paragraph — even if it holds in some cases, see, for example, [53, Lemma 2.6] — and links admitting L-space surgeries need not to be fibred [41, Example 3.9] nor quasipositive [7, Proposition 1.5]. Concerning foliations, Kalelkar and Roberts [28] constructed co-orientable taut foliations on some fillings of 3-manifolds that fibre over the circle and in particular their methods can also be applied to surgeries on fibred links. In [53], taut foliations on all the surgeries on the Whitehead link that are not L-spaces are constructed.

In this paper we study the L-space conjecture for manifolds that can be obtained as surgery on two-bridge links. A two-bridge link is either hyperbolic or isotopic to the $(2n,2)$ torus link, for some integer n. In the latter case the exterior is a Seifert fibred manifold and since the conjecture has been proven for graph manifolds — in particular, see [4; 14; 26; 39; 44] for the case of Seifert fibred manifolds — we focus our study on hyperbolic two-bridge links. The main theorem of this paper is the following:

Theorem 1.2 *Let L be a fibred hyperbolic two-bridge link and let M be a manifold obtained as Dehn surgery on L. Then M admits a co-orientable taut foliation if and only if M is not an L-space.*

Remark 1.3 In contrast to the case of knots, the property of being fibred for a link depends on the choice of an orientation of the link. This happens, for instance, in the case of the $(2n,2)$ torus link for $n > 1$, see, for example, [1, Example 3.1]. On the other hand, changing orientations of the components of L has no effects on the study of the L-space conjecture for the surgeries on L. For this reason we will consider links as unoriented and say that a link is fibred if there exists an orientation for which it is a fibred link.

Remark 1.4 Theorem 1.2 is a result about the exterior of the links. Links are not uniquely determined by their complement, hence it can be a priori possible that many nonisotopic hyperbolic two-bridge links share the same exterior. However it follows from [43, Theorem 1.4] that the exteriors of hyperbolic two-bridge links (with two components) are not even commensurable.

We will be able to completely determine for each fibred hyperbolic two-bridge link L the set of surgeries on L that are L-spaces. We denote by $\mathcal{L}(L)$ the set of slopes on L that produce L-spaces. Recall that since L is a link in S^3 there is a canonical identification between the set of slopes on L and $\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, obtained by considering on each component of L its canonical meridian and longitude basis. Also, notice that we can reduce our study to the rational surgeries. In fact the components of two-bridge links are unknotted, so when one of the two surgery coefficients is infinite the only rational homology spheres that can be obtained are S^3 and lens spaces. We will prove the following proposition, where the link L_n is shown in Figure 1, for $n \geq 1$.

Proposition 1.5 *Let L be a fibred hyperbolic two-bridge link.*

- *If L is isotopic as an unoriented link to L_n, then $\mathcal{L}(L) \cap \mathbb{Q}^2 = \big([n,+\infty) \times [n,+\infty)\big) \cap \mathbb{Q}^2$.*
- *If L is isotopic as an unoriented link to the mirror of L_n, then $\mathcal{L}(L) \cap \mathbb{Q}^2 = \big((-\infty,-n] \times (-\infty,-n]\big) \cap \mathbb{Q}^2$.*
- *If L is not isotopic as an unoriented link to any of the links L_n or their mirrors, then $\mathcal{L}(L) \cap \mathbb{Q}^2 = \varnothing$.*

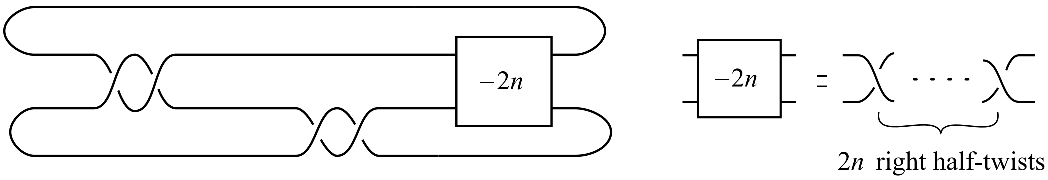


Figure 1: The link L_n.

We observe that L_1 is the Whitehead link. The L-space conjecture for surgeries on the Whitehead link was studied by Santoro [53]. As a consequence of the previous proposition we have the following Dehn surgery characterisation of the Whitehead link:

Corollary 1.6 *Let L be a fibred hyperbolic two-bridge link and suppose that the $(1,1)$-surgery on L is an L-space. Then L is isotopic, as an unoriented link, to the Whitehead link.*

We observe that all the links $\{L_n\}_{n\geq 1}$ can be obtained as surgery on a 3-component link, see Figure 33 on page 1151. On the other hand we have the following:

Proposition 1.7 *It is not possible to obtain the exteriors of all the hyperbolic fibred two-bridge links as Dehn filling on a fixed cusped hyperbolic manifold N. In particular there exists no hyperbolic link L such that every hyperbolic fibred two-bridge link is surgery on L.*

Proof It follows from the main result of [34] that there exists a family of fibred hyperbolic two-bridge links whose volumes grow to infinity (this is the family of links associated to $L(a_1,\dots,a_n) = L(2,2,\dots,2)$ in the notation introduced in Section 4). Volume decreases under hyperbolic Dehn filling [55], hence we obtain the thesis. □

As two-bridge links have tunnel number one, all surgeries on these links have at most Heegaard genus two. It has been proven by Li [37] that if a closed orientable irreducible three manifold with Heegaard genus two has left-orderable fundamental group, then it admits a co-orientable taut foliation. As a consequence of this result together with Proposition 1.5 we have:

Corollary 1.8 *Let M be obtained as (r_1,r_2)-surgery on the link L_n, with $(r_1,r_2)\in[n,+\infty)\times[n,+\infty)$, and suppose that M is irreducible. Then M is not left-orderable.*

Applications to satellites on knots and links We briefly recall the satellite operation. Suppose that P is a knot inside the standard solid torus $V=\mathbb{D}^2\times S^1$ and assume that P is not contained in a 3-ball of V. Let K be a knot in S^3 and let (μ_K,λ_K) be a meridian-longitude basis of K. Consider the orientation preserving diffeomorphism ϕ between V and a tubular neighbourhood of K mapping, as oriented curves, the meridian μ_V and longitude λ_V of V to μ_K and λ_K, respectively. The image of P under ϕ is a knot S in S^3, called a *satellite* of K. The knot K is called the *companion* of S and the knot P is called the *pattern* of S.

Let L be a fibred hyperbolic two-bridge link, denote by $\mathcal{K}$ one of its component and orient it arbitrarily. Since two-bridge links have unknotted components, the exterior of $\mathcal{K}$ in S^3 is a solid torus V and we can

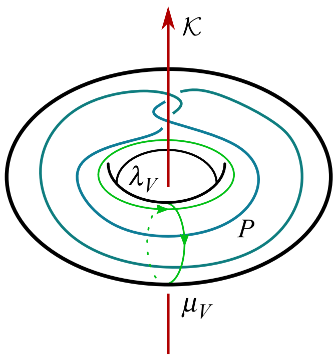


Figure 2: The (positive clasped) Whitehead pattern. The meridian μ_V is given by the longitude of the knot K_0 and the longitude λ_V by its meridian. By considering the mirror of the Whitehead link one obtains the negative clasped Whitehead pattern.

use the other component as pattern P for producing satellite knots. We also fix a meridian-longitude basis for V given by $(\mu_V, \lambda_V) = (\lambda_{\mathcal{K}}, \mu_{\mathcal{K}})$, where $\mu_{\mathcal{K}}$ and $\lambda_{\mathcal{K}}$ are the canonical meridian and longitude of $\mathcal{K}$.

Example 1.9 If L is the Whitehead link we obtain the Whitehead pattern. This is the pattern used to define Whitehead doubles of knots, see Figure 2.

We now define an operation, that we call *two-bridge replacement*, that generalises Whitehead doubling.

Definition 1.10 Let K be a knot in S^3 and let $L = \mathcal{K} \sqcup P$ be a fibred hyperbolic two-bridge link. A knot K' in S^3 is a *two-bridge replacement* of K if K' is a satellite knot of K with pattern P. More generally, if $\mathcal{L} = K_1 \sqcup \cdots \sqcup K_d$ is a link with d components, we say that $\mathcal{L}' = K_1' \sqcup \cdots \sqcup K_d'$ is a *two-bridge replacement* of $\mathcal{L}$ if each knot K_i' is a two-bridge replacement of K_i, for $i = 1, \dots, d$.

Notice that in a two-bridge replacement of a link $\mathcal{L}$ it is allowed to act on different components on $\mathcal{L}$ by two-bridge replacement with different two-bridge links. We also remark that in the definition of two-bridge replacement we ask L to be fibred and hyperbolic.

Recall that the genus of a link $\mathcal{L}$ is the minimal genus of a connected Seifert surface for $\mathcal{L}$, and that if $\mathcal{L}$ is fibred (see Definition 2.3) then a Seifert surface has minimal genus if and only if it is a fibred surface [15, Chapter 1.4]. The proofs of Theorem 1.2 and of the main theorem of [53], together with results from [28; 38], imply the following theorem.

Theorem 1.11 *Let $\mathcal{L}$ be any nontrivial knot or any fibred link with positive genus, and let $\mathcal{L}'$ denote a two-bridge replacement of $\mathcal{L}$. Then all manifolds obtained by doing surgery on each component of $\mathcal{L}'$ along a nonmeridional slope support a co-orientable taut foliation.*

Proof • We first analyse the case where $\mathcal{L}$ is a nontrivial knot. We denote it by K and we denote by K' its two-bridge replacement. We use the notation introduced above, and so we denote by $L = \mathcal{K} \sqcup P$ the fibred hyperbolic two-bridge link used in the definition of two-bridge replacement, and V is the exterior of $\mathcal{K}$ in S^3. Recall that V is a solid torus. We also denote by $E_{K'}$, E_K and E_L the exteriors in S^3 of K', K and L, respectively. Let ϕ be the map from V to a tubular neighbourhood of K given by the definition of the satellite operation. Our aim is to prove that all nontrivial surgeries on K' support a co-orientable taut foliation. The key observation is that $E_{K'}$ is obtained by gluing E_K and E_L. More precisely we have

$$E_{K'} \cong E_K \cup_\varphi E_L,$$

where φ is the restriction of ϕ to $\partial V \subset \partial E_L$.

In order to construct foliations on $E_{K'}$ we can therefore reduce ourselves to construct foliations on E_K and E_L and to study the gluing map φ. We fix the canonical meridian-longitude basis (μ_K, λ_K) for the knot K and we use it to identify slopes on K with $\overline{\mathbb{Q}}$. By definition ϕ maps the meridian μ_V of V to μ_K and the longitude λ_V of V to some longitude of K, i.e.,

$$\phi(\lambda_{\mathcal{K}}) = \phi(\mu_V) = \mu_K, \quad \phi(\mu_{\mathcal{K}}) = \phi(\lambda_V) = l\mu_K + \lambda_K$$

for some integer $l \in \mathbb{Z}$. Given two coprime integers p, q the map ϕ satisfies

$$p\mu_K + q\lambda_K = \phi((p - ql)\lambda_{\mathcal{K}} + q\mu_{\mathcal{K}})$$

and therefore its restriction φ acts on the slopes by identifying the slope $\frac{p}{q}$ on K with the slope $\left(\frac{p}{q} - l\right)^{-1}$ on $\mathcal{K}$. Since K is a nontrivial knot, it follows by [38, Theorem 1.1] that there exists an interval $(-a, b)$, where $a, b > 0$, such that for every slope $s \in (-a, b)$ there exists a co-orientable taut foliation on E_K intersecting the boundary torus in a collection of circles of slope s. The slope 0 on K corresponds, via the identification given by φ, to the slope $-\frac{1}{l}$ on $\mathcal{K}$. Hence the interval $(-a, b)$ is identified with a neighbourhood U_1 of $-\frac{1}{l} \subset \overline{\mathbb{Q}}$. The proof of [53, Theorem 1.1] when L is the Whitehead link and the proof of Theorem 1.2 when L is any other fibred hyperbolic two-bridge link imply the following fact: for every integer $l \in \mathbb{Z}$ and every neighbourhood U of $-\frac{1}{l} \subset \overline{\mathbb{Q}}$ there exists a slope $r \in U$ such that for every nonmeridional slope r' on P there is a co-orientable taut foliation on E_L intersecting the boundary tori in circles of slopes r and r', respectively.

By choosing a slope $r \in U_1$ guaranteed by the previous observation we are able to find for each nonmeridional slope r' in $E_{K'}$ taut foliations $\mathcal{F}$ on E_K and $\mathcal{F}'$ on E_L that can be glued along φ to define a co-orientable taut foliation in $E_{K'}$ intersecting the boundary in parallel curves of slope r'. By capping off with meridional discs, these foliations extend to the surgeries on K'.

- When $\mathcal{L} = K_1 \sqcup \cdots \sqcup K_d$ is a fibred link with multiple components and positive genus we can proceed in an analogous way. Let S denote the fibre surface for $\mathcal{L}$. By intersecting S with the boundaries of tubular neighbourhoods of the knots $K_1, \dots, K_d$ we obtain longitudes $\lambda_1^S, \dots, \lambda_d^S$. We use them to define meridian-longitude bases for the components of $\mathcal{L}$ and to identify slopes on the exterior of $\mathcal{L}$ with $\overline{\mathbb{Q}}^d$. It follows by [28, Theorem 1.1] that for every multislope $(r_1, \dots, r_d)$ in a neighbourhood of $0 \in \overline{\mathbb{Q}}^d$ there exists a co-orientable taut foliation in the exterior of $\mathcal{L}$ intersecting the boundary tori in parallel curves of slopes $r_1, \dots, r_d$, respectively. The statement now follows by applying to each component of $\mathcal{L}$ the same reasoning as in the previous case, where we never made use of the fact that λ_K was the canonical longitude of K. □

Two-bridge replacement generalises Whitehead doubling and we emphasise the following corollary.

Corollary 1.12 *Let K be a nontrivial knot and let K' be any Whitehead double of K. Then all nontrivial surgeries on K' support a co-orientable taut foliation.* □

Structure of the paper In Section 2 we recall some notions on two-bridge links and describe some properties of fibred two-bridge links that will be used in the subsequent sections. Section 3 is devoted to the construction of taut foliations and will take most of the paper. In Section 3.1 we introduce branched surfaces and recall some of their basic properties, together with the main result of [36]. In Section 3.2 we recall a general method of constructing branched surfaces in fibred manifolds with boundary. In Section 3.3 we briefly discuss the boundary train tracks of these branched surfaces and from Section 3.4 we start focussing our attention on surgeries on fibred hyperbolic two-bridge links: we subdivide them in four families that we analyse in Sections 3.5–3.8. In Section 4 we recall part of the main result of [50] and then we use it to study the L-space surgeries on the links $\{L_n\}_{n\geq 1}$.

2 Background on two-bridge links

In this section we briefly recall the definition of fibred link and some facts about two-bridge links that we will often use in the paper. We refer to [6] for proofs and details regarding two-bridge links.

A *two-bridge link* can be described by a rational number $\frac{p}{q}$, where p and q are coprime integers, $p > 0$, q is odd and $0 < |q| < p$, in the following way. We fix a sequence of integers $(a_1, \dots, a_n)$ such that

$$(*) \qquad \frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

and consider the link defined by the diagram in Figure 3. We denote this link by $L(a_1, \dots, a_n)$.

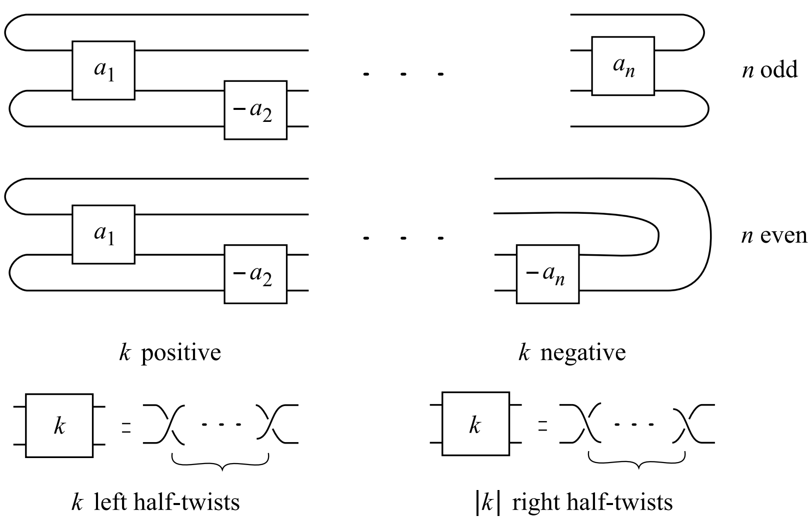


Figure 3: The two-bridge knot or link $L(a_1, \dots, a_n)$.

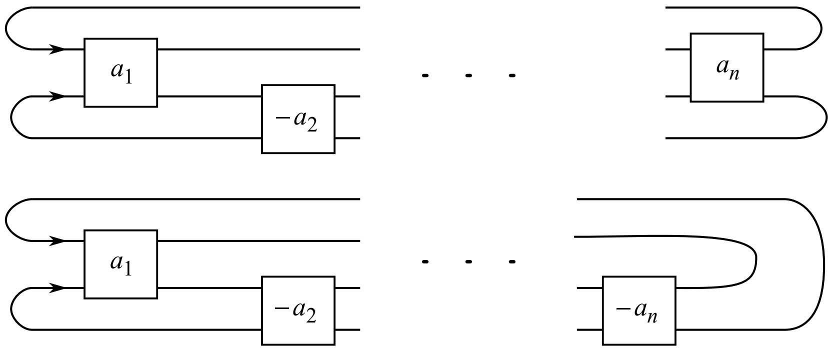


Figure 4: The oriented two-bridge link $L(a_1,\dots,a_n)$.

We are interested in the case when $L(a_1,\dots,a_n)$ has two components. This happens exactly when the fraction $\frac{p}{q}$ has numerator p even [6, Proposition 12.3]. When $L(a_1,\dots,a_n)$ is a link we orient the components as in Figure 4.

A priori it could happen that the isotopy class of the two-bridge link associated to $\frac{p}{q}$ depends on the choice of the continued fraction representation of $\frac{p}{q}$. This is not the case, by the following theorem.

Theorem 2.1 ([54], see also [6]) *Let $L = L(a_1 \dots, a_n)$ and $L' = L(b_1,\dots,b_m)$ be two oriented two-bridge links and let $\frac{p}{q}$ and $\frac{p'}{q'}$ be the rational numbers defined as in $(*)$. Then the links L and L' are isotopic if and only if $p = p'$ and $q' \equiv q^{\pm 1} \bmod 2p$. If $p = p'$ and $q' \equiv q + p \bmod 2p$ or $qq' \equiv 1 + p \bmod 2p$, then L and L' are isotopic after reversing the orientation of one of the components.*

We denote by $b(p,q)$ the two-bridge link associated to the rational number $\frac{p}{q}$. We also recall, for later reference, the following fact [29, Exercise 2.1.16].

Lemma 2.2 *Two-component two-bridge links are symmetric, i.e., there exists an ambient isotopy of S^3 interchanging their components.* □

We are interested in studying fibred hyperbolic two-bridge links. For this reason we recall the definition of fibred link.

Given an oriented surface S with boundary and an orientation-preserving homeomorphism $h: S \to S$ fixing ∂S pointwise, we denote the mapping torus of h by

$$M_h = \frac{S \times [0,1]}{(h(x),0) \sim (x,1)}.$$

We orient $S \times [0,1]$ as a product and M_h with the orientation induced by $S \times [0,1]$. We also identify S with its image in M_h via the map

$$S \to S \times \{0\} \subset M_h, \quad x \mapsto (x,0).$$

The homeomorphism h is called the *monodromy* of M_h.

Definition 2.3 Let L be an oriented link in S^3. We say that L is *fibred* if there exists a Seifert surface S for L, an orientation preserving homeomorphism h of S fixing ∂S pointwise and an orientation preserving homeomorphism

$$\chi : S^3 \setminus \mathrm{int}(N_L) \to M_h,$$

where N_L denotes a tubular neighbourhood of L in S^3, so that

- $\chi_{|S}$ is the inclusion $S \subset M_h$;
- $\chi(m_i) = \{x_i\} \times [0,1]$, where m_i is a meridian for the i-th component of L and $x_i \in \partial S$ is a point.

In the case of fibred links, we refer to the homeomorphism h as the *monodromy of the link*.

The following lemma yields a useful characterisation of fibred hyperbolic two-bridge links.

Lemma 2.4 *Let be L a two-bridge link with two components. Then L is fibred if and only if $L = L(2b_1, \dots, 2b_n)$ as an unoriented link, where $|b_i| = 1$ for all i and n is odd. Moreover L is fibred and hyperbolic if and only if it admits such a description with $(b_1, \dots, b_n) \neq \pm(1, -1, 1, \dots, (-1)^{n-1})$.*

Proof It follows from [29, Exercise 2.1.14] that any two-bridge link L with two components can be written as $L = L(2b_1, \dots, 2b_n)$, where b_i is a nonzero integer and n is odd. Moreover it follows from [18, Proposition 2][1] that L is fibred if and only if we can find such a description with $|b_i| = 1$ for all i. This proves the first part of the lemma. Since two-bridge links are nonsplit, prime, alternating links (see [6]), as a consequence of [42, Corollary 2], a two-bridge link is hyperbolic if and only if is it not a torus link. The only nontrivial torus links that are two-components two-bridge links are the $(2m, 2)$ torus links, with m a nonzero integer, and by using Theorem 2.1 one can see that L is a torus link if and only if any description of L as $L(2b_1, \dots, 2b_n)$ with all $|b_i| = 1$ satisfies $(b_1, \dots, b_n) = \pm(1, -1, 1, \dots, (-1)^{n-1})$. □

Suppose $L = L(2b_1, \dots, 2b_n)$ where $|b_i| = 1$ for all i and $n = 2k+1$ is odd. Then it is possible to draw an explicit fibre surface S for L. This surface is obtained by starting with the boundary connected sum of k Hopf bands, and then plumbing other $k+1$ Hopf bands to this surface. The sign of the Hopf bands is determined in a straightforward way from the coefficients $(b_1, \dots, b_n)$. One example is described in Figure 5. We also fix an orientation of S, so that in the figure the positive side is coloured in pink, and this induces an orientation of the link.

From this very easy description of the fibre surface of L we are able to determine the monodromy of L. More precisely, S can be described in a more abstract way as in Figure 6 and the monodromy is given by the diffeomorphism (to be read from right to left)

$$h = \tau_2^{\varepsilon_2} \tau_4^{\varepsilon_4} \dots \tau_{2k}^{\varepsilon_{2k}} \tau_1^{\varepsilon_1} \tau_3^{\varepsilon_3} \dots \tau_{2k+1}^{\varepsilon_{2k+1}}, \tag{$**$}$$

where τ_i denotes the positive (i.e., the right) Dehn twist along the curve γ_i shown in Figure 6 and

$$\varepsilon_i = \begin{cases} -\mathrm{sgn}(b_i) & \text{when } i \text{ is even,} \\ \mathrm{sgn}(b_i) & \text{when } i \text{ is odd.} \end{cases}$$

[1] The proof presented there is for knots, but the same proof works also for links.

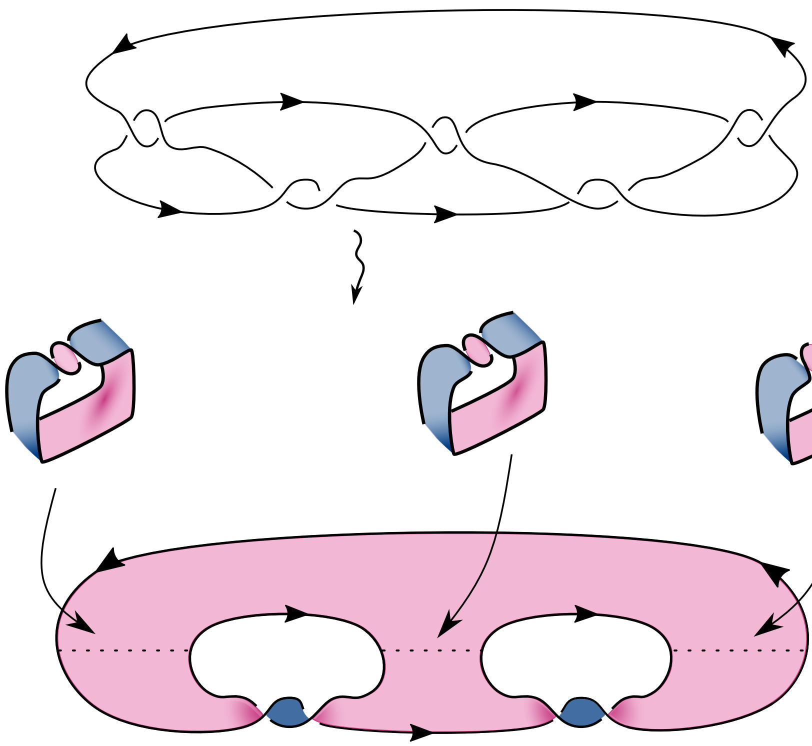


Figure 5: The fibre surface of the link $L(-2,-2,-2,2,2)$. The positive side is coloured in pink.

This follows from the fact that the monodromy of the boundary of a positive (resp. negative) Hopf band is a positive (resp. negative) Dehn twist along its core and from the way the monodromy of a plumbing or a boundary connected sum (or more generally a Murasugi sum) behaves with respect to the monodromies of the summands, see [17, Corollary 1.4].

To ease the exposition of some lemmas in the next section we fix the following notation. With reference to Figure 6 we say that a Dehn twist along one of the curves $\gamma_1, \gamma_3, \dots, \gamma_n$ is a *bridge twist*, and a Dehn twist along one of the curves $\gamma_2, \gamma_4, \dots, \gamma_{n-1}$ is a *river twist*.

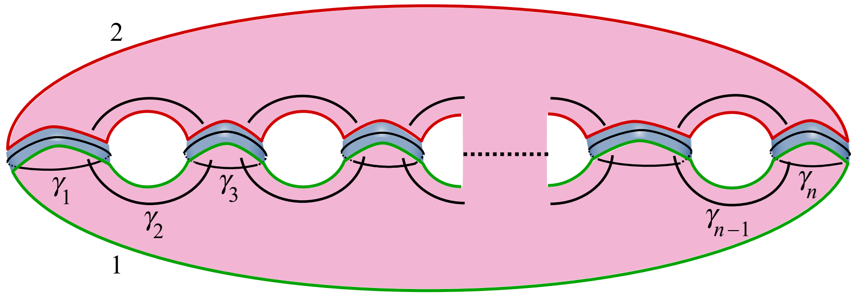


Figure 6: An abstract drawing of the fibre surface S together with the curves γ_i. We have also coloured the two boundary components of S.

3 Taut foliations

In this section we study the existence of taut foliations on the surgeries on fibred hyperbolic two-bridge links, proving the foliation part of Theorem 1.2. Branched surfaces will be our main tool. In Section 3.1 we introduce them and recall some of their basic properties, together with the main result of [36]. In Section 3.2 we recall a general method to construct branched surfaces in fibred manifolds with boundary and in Section 3.3 we discuss their boundary train tracks. In Section 3.4 we focus our attention on surgeries on fibred hyperbolic two-bridge links: we subdivide them in four families with Lemma 3.10 and then study each of these families separately in Sections 3.5–3.8.

3.1 Background

In this and in the next sections we assume familiarity with the basic notions of the theory of train tracks; see [49] for a reference. In the cases of our interest train tracks can also have bigons as complementary regions.

We now recall some basic facts about branched surfaces. We refer to [16; 46] for more details.

Definition 3.1 A *branched surface with boundary* in a 3-manifold M (possibly with boundary) is a closed subset $B \subset M$ that is locally diffeomorphic to one of the models in $\mathbb{R}^3$ of Figure 7(a) or to one of the models in the closed half space of Figure 7(b), where $\partial B := B \cap \partial M$ is represented with a bold line.

Branched surfaces generalise the concept of train tracks from surfaces to 3-manifolds. When the boundary of B is nonempty it defines a train track ∂B in ∂M.

If B is a branched surface it is possible to identify two subsets of B: the *branch locus* and the set of *triple points*. The branch locus is defined as the set of points where B is not locally homeomorphic to a surface. It is self-transverse and intersects itself in double points only. The set of triple points of B can be defined as the points where the branch locus is not locally homeomorphic to an arc. For example, the rightmost model of Figure 7(a) contains a triple point.

The complement of the branch locus in B is a union of connected surfaces. The abstract closures of these surfaces under any path metric on M are called the *branch sectors* of B. Analogously, the complement of the set of the triple points inside the branch locus is a union of 1-dimensional connected manifolds. Moreover, to each of these manifolds we can associate an arrow in B pointing in the direction of the smoothing, as in Figure 7. We call these arrows *cusp directions*.

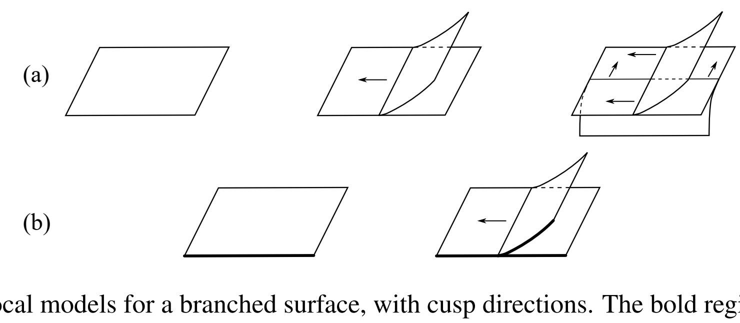


Figure 7: Local models for a branched surface, with cusp directions. The bold regions lie in ∂M.

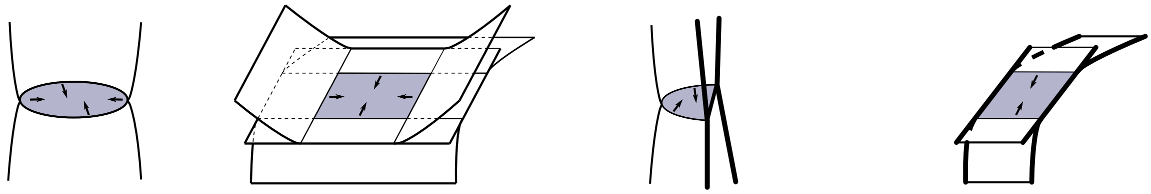


Figure 8: Examples of sink discs.

Li [35] introduced the notion of *sink disc*.

Definition 3.2 Let B be a branched surface in M and let S be a branch sector in B. We say that S is a *sink disc* if S is a disc (possibly with $S \cap \partial M \neq \varnothing$) and the branch direction of any smooth curve or arc in $\partial S \setminus \partial M$ points into S.

In Figure 8 some examples of sink discs are depicted. The bold lines represent the intersection of the branched surface with ∂M. Notice that the intersection $\partial S \cap \partial M$ can also be disconnected.

Li [35] introduced the definition of laminar branched surface and in [36] he generalised this definition to branched surfaces with boundary as follows. For the definition of trivial bubble we refer the reader to [35; 36].

Definition 3.3 [35; 36] Let B be a branched surface in a 3-manifold M. We say that B is *laminar* if B has no trivial bubbles and the following hold:

(1) $\partial_h N_B$ is incompressible and ∂-incompressible in $M \setminus \mathrm{int}(N_B)$, and no component of $\partial_h N_B$ is a sphere or a properly embedded disc in M.

(2) There is no monogon in $M \setminus \mathrm{int}(N_B)$, i.e., no disc $D \subset M \setminus \mathrm{int}(N_B)$ such that $\partial D = D \cap N_B = \alpha \cup \beta$, where α is in an interval fibre of $\partial_v N_B$ and β is an arc in $\partial_h N_B$.

(3) $M \setminus \mathrm{int}(N_B)$ is irreducible and $\partial M \setminus \mathrm{int}(N_B)$ is incompressible in $M \setminus \mathrm{int}(N_B)$.

(4) B contains no Reeb branched surfaces (see [19] for the definition).

(5) B has no sink discs.

The key property of laminar branched surfaces is that they fully carry essential laminations[2] [35] and when the ambient manifold M has torus boundary, they can be used to construct essential laminations in fillings of M, as the following theorem shows.

Theorem 3.4 [36] *Let M be an irreducible and orientable 3-manifold whose boundary is union of k incompressible tori $T_1, \dots, T_k$. Suppose that B is a laminar branched surface in M such that $\partial M \setminus \partial B$ is a union of bigons. Then for any multislope $(s_1, \dots, s_k) \in \overline{\mathbb{Q}}^k$ that is realised by the train track ∂B, if B does not carry a torus that bounds a solid torus in $M(s_1, \dots, s_k)$, there exists an essential lamination Λ in M fully carried by B that intersects ∂M in parallel simple curves of multislope $(s_1, \dots, s_k)$. Moreover this lamination extends to an essential lamination of the filled manifold $M(s_1, \dots, s_k)$.*

[2]For the definition of essential laminations see [19], but we will not need their properties for our purposes.

A train track *realises* a slope r if it can be split into finitely many copies of that slope, see [49]. An equivalent, but more computationally useful definition, is given in Section 3.3.

Remark 3.5 In [36] the statement of the theorem is given for M with connected boundary but, as already observed in [28], if M has multiple boundary components we can split B in a neighbourhood of each boundary tori T_i and the same proof of [36] works.

Remark 3.6 The statement of Theorem 3.4 is slightly more detailed than the version of [36]. The details we have added come from the proof of Theorem 3.4. In fact the idea of the proof is to split the branched surface B in a neighbourhood of ∂M so that it intersects T_i in parallel simple closed curves of slopes s_i, for $i = 1, \dots k$. In this way, when gluing the solid tori, we can glue meridional discs of these tori to B to obtain a branched surface $B(s_1, \dots, s_k)$ in $M(s_1, \dots, s_k)$ that is laminar and that by [35, Theorem 1] fully carries an essential lamination. In particular, this essential lamination is obtained by gluing the meridional discs of the solid tori to an essential lamination in M that intersects T_i in parallel simple closed curves of slopes s_i, for $i = 1, \dots, k$.

3.2 Constructing branched surfaces in fibred manifolds

In this section we recall a general method to build branched surfaces in mapping tori. This will be the starting point to construct taut foliations on surgeries on fibred two-bridge links.

Let S be a connected oriented surface with boundary, let h be an orientation preserving homeomorphism of S fixing ∂S pointwise and let M_h be the mapping torus associated to (S, h).

We consider pairwise disjoint properly embedded arcs $\alpha_1, \dots, \alpha_k$ in S and discs $\overline{D}_i = \alpha_i \times [0, 1]$ contained in $S \times [0, 1]$. Each of these discs has a "bottom" boundary, $\alpha_i \times \{0\}$, and a "top" boundary, $\alpha_i \times \{1\}$. When we consider the images of these discs in M_h under the projection map

$$S \times [0, 1] \to M_h$$

we have that the bottom and top boundaries become, respectively, $\bigcup_i \alpha_i \subset S$ and $\bigcup_i h(\alpha_i) \subset S$.

We perturb the discs $\overline{D}_i$ in a neighbourhood of $S \times \{1\} \subset S \times [0, 1]$ so that when projected to M_h their top boundaries define a family of arcs, that we still denote $h(\alpha_i)$, in S such that for each $i, j \in \{1, \dots, k\}$ the intersection between α_i and $h(\alpha_j)$ is transverse and the endpoints of α_i and $h(\alpha_i)$ are disjoint. We also denote by D_i the projected perturbed disc contained in M_h and we refer to these discs as *product discs*. If we assign (co)orientations to these discs, since S is (co)oriented, we can smoothen $S \cup D_1 \cup \dots \cup D_k$ to a branched surface B by imposing that the smoothing preserves the co-orientation of S and of the discs. In particular, each disc has two possible co-orientations and hence it can be smoothed in two different ways. This operation is demonstrated in Figure 9, where S is a torus with an open disc removed.

The following proposition shows that in this setting, under very mild hypothesis on the branched surface B, we can get taut foliations on fillings on M_h. The proof is obtained by combining Lemmas 3.16 and 3.23 of [53] and uses as a fundamental result Theorem 3.4.

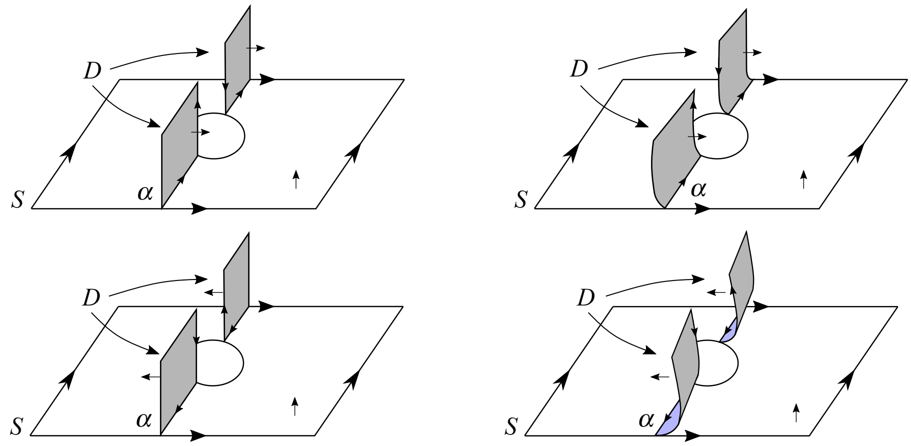


Figure 9: How to smoothen $S \cup D$ according to the co-orientations.

Proposition 3.7 *Suppose that B is a branched surface constructed as described above. Suppose also that B has no sink discs and that $S \setminus \bigcup_{i=1}^{k} \alpha_i$ has no disc components. If $(r_1, \dots, r_n)$ is a multislope realised by ∂B then $M_h(r_1 \dots, r_n)$ contains a co-orientable taut foliation. More precisely there exists a co-orientable taut foliation in M_h intersecting the boundary component T_i in a foliation by curves of slopes r_i, for $i = 1, \dots, n$.* □

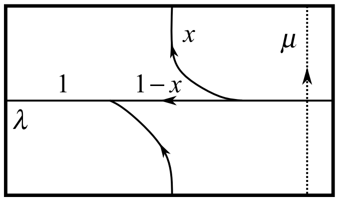
3.3 Boundary train tracks

We now briefly discuss the boundary train tracks of the branched surfaces constructed with the procedure described above. We start by describing an explicit way to compute the slopes realised by a train track.

Let τ be an oriented train track in a torus T. Fix a meridian-longitude basis μ, λ and use it to identify slopes on T with elements in $\overline{\mathbb{Q}}$. It is possible to compute the slopes realised by a train track τ by endowing it with *weight systems*. A weight system w on τ is the assignment of a positive rational number, called *weight*, to each sector of τ so that at each branch point of τ the sum of the weights of the incoming sectors is equal to the sum of the outgoing one. Given that τ is oriented, we can associate to such a weight system the rational number $\frac{w_\mu}{w_\lambda}$, where w_μ and w_λ are the *weighted* intersections of the train track with our fixed meridians μ and longitudes λ, as we would do with oriented simple closed curves. This quotient can be interpreted as a *slope* in T and it can be proved that the slopes $\frac{p}{q}$ obtained in this way are exactly those realised by the train track. For details, see [49]. Figure 10 shows an example of a train track with weight systems.

We now focus our attention on a branched surface B in a mapping torus M_h obtained by adding product discs to the fibre S. We start by fixing a meridian μ_j and a longitude λ_j for each boundary component T_j of M_h. We take λ_j to be $S \cap T_j$, with the orientation induced by S, and as meridian the curve $\mu_j = \frac{\{x_j\} \times [0,1]}{\sim_h}$, where $x_j \in S \cap T_j$, oriented in the direction of ascending $t \in [0, 1]$. Notice that when T_j is oriented as the boundary of M_h, the algebraic intersection $< \mu_j, \lambda_j >$ is negative. By using this meridian-longitude basis we can identify slopes on T_j with elements in $\overline{\mathbb{Q}}$.

train track



slopes fully carried (0, 1)

Figure 10: An example of train track τ with a weight system w_x, together with the set of slopes realised by τ. In order for the weights to be positive, x must be a rational number in (0, 1).

Given a properly embedded oriented arc α in S we denote its endpoints with $\alpha(0)$ and $\alpha(1)$, so that α goes from $\alpha(0)$ to $\alpha(1)$. Let us now consider a product disc D in the branched surface B, spanned by an arc α. The fixed orientation on D induces an orientation on α and on the two arcs of intersections $D \cap \partial M_h$. More precisely the arc in $D \cap \partial M_h$ containing $\alpha(1)$ is oriented in the direction of ascending $t \in [0, 1]$, and the one containing $\alpha(0)$ is oriented in the direction of descending $t \in [0, 1]$ (see Figure 9 for an example). For a fixed boundary component T of M_h the four possible configurations of one arc of $D \cap T$ are described in Figure 11, together with the set of slopes fully carried by the train tracks given by one component of $D \cap T$ and the longitude of T.

In the course of the paper, we will mostly encounter train tracks of the four types described in Figure 11, or those that we call of *type* $t_1 + \cdots + t_m$, where t_i is (a), (b), (c) or (d), for $i = \{1, \dots, m\}$. These are obtained by juxtaposing train tracks of type t_i, for $i = \{1, \dots, m\}$, preserving the cyclic order. See Figure 12 for an example.

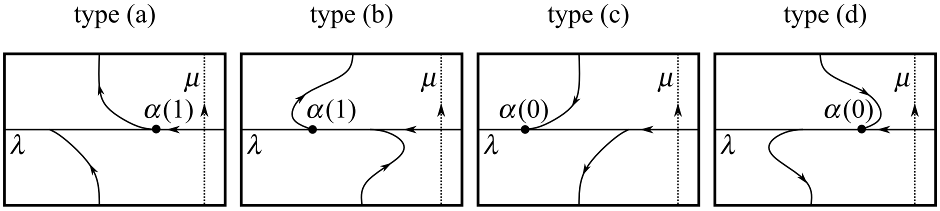
The next lemma — which follows by a direct computation with weight systems — will be useful when computing the sets of slopes realised by our train tracks.

Lemma 3.8 *Let τ be a train track of type $t_1 + \cdots + t_m$. Then, a slope r is realised by τ if and only if $r = r_1 + \cdots + r_m$, where r_i is a slope realised by the train track of type t_i.* □

In many of the cases of our interest, the boundary train tracks will satisfy the hypothesis of Lemma 3.8 and so we can use it to compute the slopes that they realise. Two examples of train tracks that do not satisfy the hypothesis of the previous lemma can be found in Figure 31 on page 1148.

type (a) type (b) type (c) type (d)

train track



slopes fully carried (0, 1) (0, +∞) (−1, 0) (−∞, 0)

Figure 11: Possible train tracks given by the longitude of T and one arc of intersection in $T \cap D$, together with the intervals of slopes they fully carry.

type (a) + (c)

train track

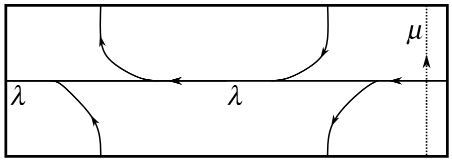


Figure 12: A train track of type (a) + (c).

3.4 Fibred hyperbolic two-bridge links

We are now ready to construct foliations on surgeries on the hyperbolic fibred two-bridge links. The general strategy is simple: we have from (∗∗) explicit descriptions of the monodromies of these links and we want to construct branched surfaces in the way described in Section 3.2. If we are able to construct these branched surfaces so that they have no sink discs, then by Proposition 3.7 we can deduce that all the surgeries corresponding to the multislopes realised by these branched surfaces contain co-orientable taut foliations. For this reason, we will have to study which multislopes are realised by the boundary train tracks.

Before we begin, we make the following remark and establish some conventions.

Remark 3.9 In [52] it is proved that all nontrivial surgeries on fibred hyperbolic knots with fractional Dehn twist coefficient zero contain a co-orientable taut foliation. A similar result does not hold for links. In fact, all fibred hyperbolic two-bridge links have fractional Dehn twist coefficient zero (on both boundary components). This can be proved by finding appropriate arcs moved to the left and to the right and by using [24, Proposition 3.1]. Nonetheless, this family contains infinitely many L-space links.

Conventions Thorough the section we will use the following conventions:

- We fix a fibre surface S for the two-bridge link $L = L(2b_1, \dots, 2b_n)$ and we fix its orientation as in Figure 5. With the induced orientation, L has linking number

$$\mathrm{lk}(L) = \sum_{i=0}^{k} b_{2i+1},$$

where $n = 2k + 1$.

- When a link is fibred there is a natural choice of meridians and longitudes that is in general different from the one induced by the ambient manifold S^3, obtained as follows. We identify $S^3 \setminus \mathrm{int}(N_L) \cong \frac{S\times[0,1]}{\sim_h}$, where N_L is a tubular neighbourhood of L. We fix a point x_i in each boundary component of S and we consider the curves $\mu_i = \frac{\{x_i\}\times[0,1]}{\sim_h}$ oriented in the direction of ascending $t \in [0,1]$ as meridians and the boundary components λ_i of S as longitudes. By definition of fibred link, the meridians defined in this way coincide with the usual meridians of the link. On the other hand these longitudes do not coincide in general with the canonical longitudes of the link components. In fact, letting l_i denote the canonical longitude of K_i, we have that

$$(\star) \qquad \lambda_i + \sum_{j \neq i} \mathrm{lk}(K_i, K_j)\mu_i = l_i$$

as elements in $H_1(\partial N_{K_i}, \mathbb{Z})$, where N_{K_i} is the connected component of N_L containing K_i.

From now on we will refer to the bases (μ_i, λ_i) as the *Seifert framing*, and to the bases (μ_i, l_i) as the *canonical framing*. Notice that it follows from $(\star)$ that the longitudes given by the Seifert framing do not depend on the choice of the Seifert surface. Unless otherwise stated we use Seifert framings.

- We will always suppose $n > 1$, because when $n = 1$ the only links obtained in this way are the Hopf links and we are interested in hyperbolic links.
- We construct branched surfaces by considering *oriented* arcs in the fibre surface S and then by attaching product discs as in Section 3.2. We will always co-orient the discs in the following way: we orient them so that the orientations on their boundaries induce the given orientation on the arcs and then we use the orientation of the ambient manifold to co-orient them. Analogously, the co-orientation of the fibre S is obtained by using the orientation of S and of the ambient manifold. A good way to keep in mind the cusps directions of branched surfaces constructed in the way is the following: looking at the positive side of S, the cusps directions is pointing right along the arcs α and β with respect to their orientations and is pointing left along the oriented arcs $h(\alpha)$ and $h(\beta)$ with respect to their orientations. See Figure 14.
- Drawings will usually show the positive side of the (abstract) fibre surface S.
- We will usually omit 1-handles from the drawings, and we draw the attaching arcs of the 1-handles in pink. Compare Figure 6 with Figure 19 on page 1136.
- The arcs $\alpha, \beta, \dots$ lie on the positive side of S and are depicted with solid lines; their images $h(\alpha), h(\beta), \dots$ via the monodromy h lie on the negative side of S and are depicted with dashed lines.
- We will usually denote the sectors of a branched surface in S with letters $\mathcal{A}, \mathcal{B}, \dots$.

With the following lemma we partition fibred hyperbolic two-bridge links in few families and study all of them separately.

Lemma 3.10 *Let $L = L(2b_1, \dots, 2b_n)$ with $|b_i| = 1$ for all i and n odd. If L is not a torus link then, up to mirroring L, $(b_1, \dots, b_n)$ satisfies:*

- Family 1: *There exist indices l and m such that $b_{2l} \neq b_{2m}$.*
- Family 2: *$(b_1, \dots, b_n) = (b_1, -1, b_3, \dots, -1, b_n)$ and there exist indices $j \neq l \neq m$ such that $b_{2j+1} = b_{2m+1} = -1$ and $b_{2l+1} = 1$.*
- Family 3: *$(b_1, \dots, b_n) = (-1, -1, -1, \dots, -1, -1)$.*
- Family 4: *$(b_1, \dots, b_n) = (b_1, -1, b_3, \dots, -1, b_n)$ where exactly one b_{2j+1} is -1.*

Proof Suppose that L does not belong to family 1. Then, up to mirroring, we can suppose that $b_{2l} = -1$ for all indices l. We now consider the integers b_{2j+1}:

- If there are at least two of them that are negative and one that is positive, then $(b_1, \dots, b_n)$ belongs to family 2.
- If none of them is positive, then $(b_1, \dots, b_n)$ belongs to family 3.
- If exactly one of them is negative, then $(b_1, \dots, b_n)$ belongs to family 4.
- If none of them is negative, then $L = (2, -2, 2, \dots, -2, 2)$ and is a torus link. □

Before beginning the study of these families, we briefly summarise what we will prove and the general strategy of the proofs. For links in families $1, 2$ and 3 we will construct taut foliations on all rational surgeries. The links in family 4 will be divided into two subfamilies, the first containing the links with $b_{2j+1} = -1$ for some j such that $2j + 1 \neq 1, n$ and the second containing those with $b_1 = -1$ or $b_n = -1$. The links in the first subfamily will have taut foliations on all rational surgeries, while those in the second will be isotopic to the links L_n in Figure 1 and we will construct taut foliations on all surgeries in $\big((-\infty, n) \times \mathbb{Q}\big) \cup \big(\mathbb{Q} \times (-\infty, n)\big)$.

For families 1 and 2, the taut foliations will be constructed by using branched surfaces obtained by adding product discs to the fibre surface. For families 3 and 4 we need a more elaborate strategy, and we will drill out one or two carefully chosen loops in the link complements to get new fibred 3-component and 4-component links. We will then study these new links, and construct branched surfaces in their exterior by adding product discs to their fibre surfaces.

3.5 Study of the links in family 1

Recall from the end of Section 2 the definition of river twist and bridge twist.

Lemma 3.11 *Let $L = L(2b_1, \dots, 2b_n)$ with n odd and $|b_i| = 1$ for all i, and let h denote its monodromy as in $(**)$. Let M denote the exterior of L.*

(1) *If there is at least one positive (resp. negative) river twist in the factorisation of the monodromy h, then the manifold $M(r_1, r_2)$ contains a co-orientable taut foliation for every multislope $(r_1, r_2) \in (-\infty, 1)^2$ (resp. for all $(r_1, r_2) \in (-1, +\infty)^2$); see Figure 13(a)–(b).*

(2) *If there are two river twists with different exponents in the factorisation of the monodromy h, then the manifold $M(r_1, r_2)$ contains a co-orientable taut foliation for every multislope $(r_1, r_2) \in \big((-1, +\infty) \times (-\infty, 1)\big) \cup \big((-\infty, 1) \times (-1 + \infty)\big)$; see Figure 13(c).*

Proof (1) Suppose that there is a positive river twist along the curve γ_i. We consider the arcs α and β as in Figure 14. The oriented arcs α and β determine a co-oriented branched surface B obtained by attaching two discs to the fibre surface S as described in Section 3.2. Since $n > 1$, S is not an annulus and therefore the complement of $\alpha \cup \beta$ in S has no disc components. Due to the fact that we have chosen α and β so that they are disjoint from γ_j for $j \neq i$ it follows that $h(\alpha) = \tau_i(\alpha)$ and $h(\beta) = \tau_i(\beta)$, as depicted in Figure 14. In Figure 14 we have also labelled the branch locus of B with the cusps directions and denoted with capital letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ the four sectors of the branched surface B in S and none of

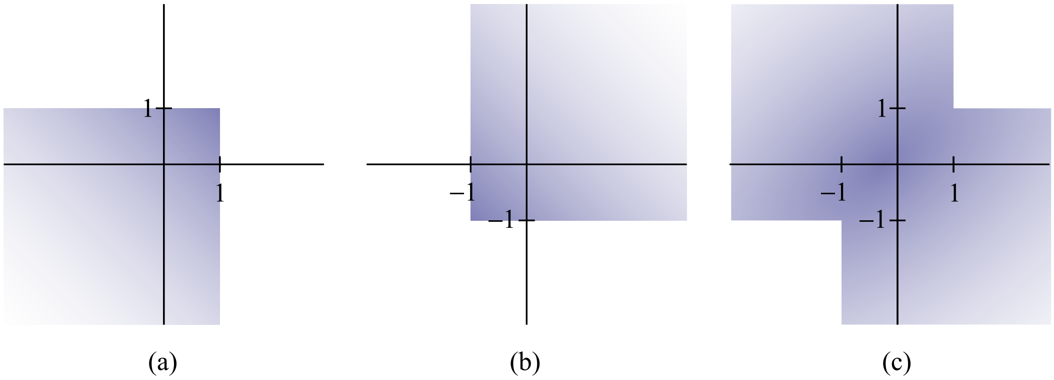


Figure 13: From left to right, the slopes (r_1, r_2) in the coloured region yield manifolds with co-orientable taut foliations in the case where there is, respectively, at least one positive river twist, at least one negative river twist, two river twists with different exponents in the factorisation of the monodromy h.

them is a sink disc. For this reason we can apply Proposition 3.7 and deduce that $M(r_1, r_2)$ supports a co-orientable taut foliation for all the multislopes (r_1, r_2) realised by ∂B. On each boundary component of M we obtain a train track, depicted in Figure 16(a), of type (a) + (d). Therefore by using Lemma 3.8 we have that the boundary train tracks of B realise all the multislopes in $(-\infty, 1)^2$ and by applying Proposition 3.7 we obtain taut foliations on $M(r_1, r_2)$ for all $(r_1, r_2) \in (-\infty, 1)^2$.

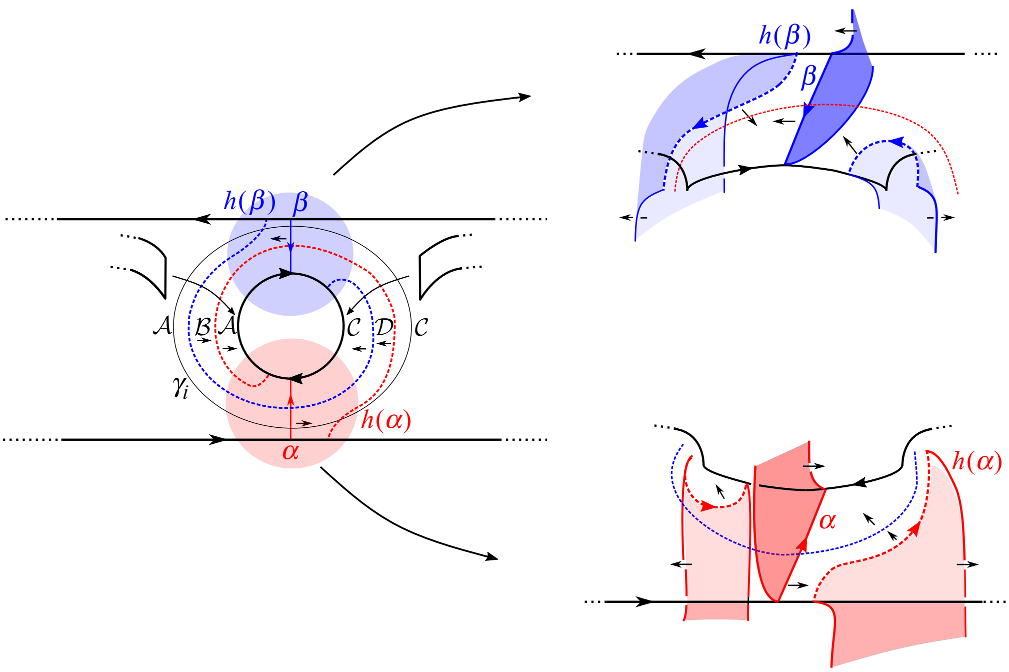


Figure 14: The arcs α and β and the co-oriented discs spanned by them. The letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ denote the sectors of B in S.

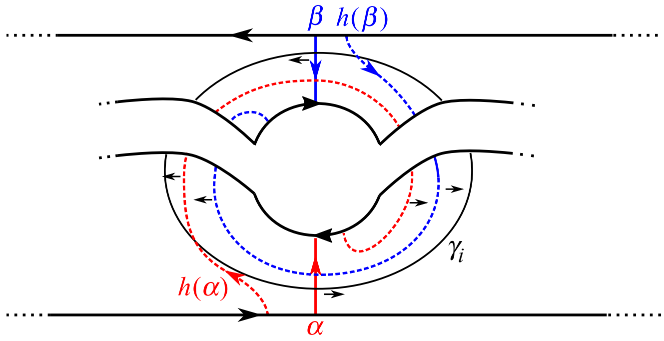


Figure 15: The arcs α and β when the river twist is negative.

If there is a negative river twist, we consider the same oriented arcs α and β (see Figure 15), and on each of the two boundary components we obtain the train track depicted in Figure 16(b), that is of type (b) + (c). This train track realises all the slopes in $(-1, +\infty)$ and so we obtain taut foliations on $M(r_1, r_2)$ for all $(r_1, r_2) \in (-1, +\infty)^2$.

(2) Suppose now that there are two river twists with different exponents in the factorisation of h and suppose that the positive one is along the curve γ_i and the negative one is along γ_j. We suppose $i < j$ but the proof does not change if $j < i$. We choose now α and β as in Figure 17 and as before we have $h(\alpha) = \tau_i(\alpha)$ and $h(\beta) = \tau_j(\beta)$. Also in this case the complement of $\alpha \cup \beta$ contains no disc components. Moreover the complement of $\alpha \cup \beta \cup h(\alpha) \cup h(\beta)$ in S is connected and this implies that there are no sink discs in the branched surface associated to α and β.

The boundary train tracks are shown in Figure 17. The one on the boundary component of M containing the boundary component of S labelled with 1 is of type (b) + (c) and by Lemma 3.8 it realises all slopes in $(-1, +\infty)$. On the other boundary component we get a train track of type (a) + (d) that hence realises all slopes in $(-\infty, 1)$. Therefore as a consequence of Proposition 3.7 we have taut foliations in $M(r_1, r_2)$ for all $(r_1, r_2) \in (-1, +\infty) \times (-\infty, 1)$. As by Lemma 2.2 two-bridge links are symmetric, we deduce that there are taut foliations also on the surgeries associated to coefficients $(r_1, r_2) \in (-\infty, 1) \times (-1, +\infty)$. $\square$

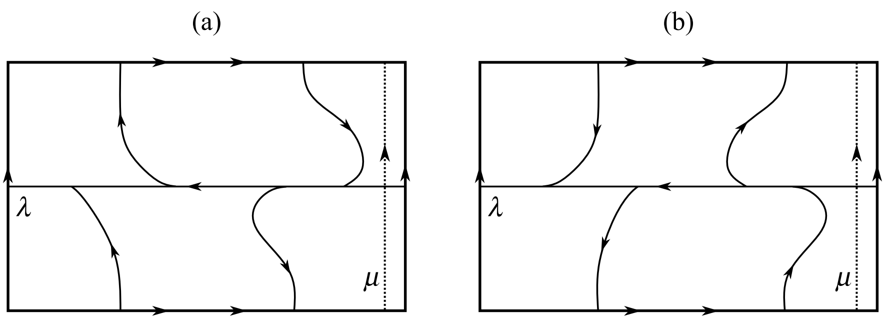


Figure 16: The boundary train tracks in the case where (a) there is a positive river twist and (b) there is a negative river twist.

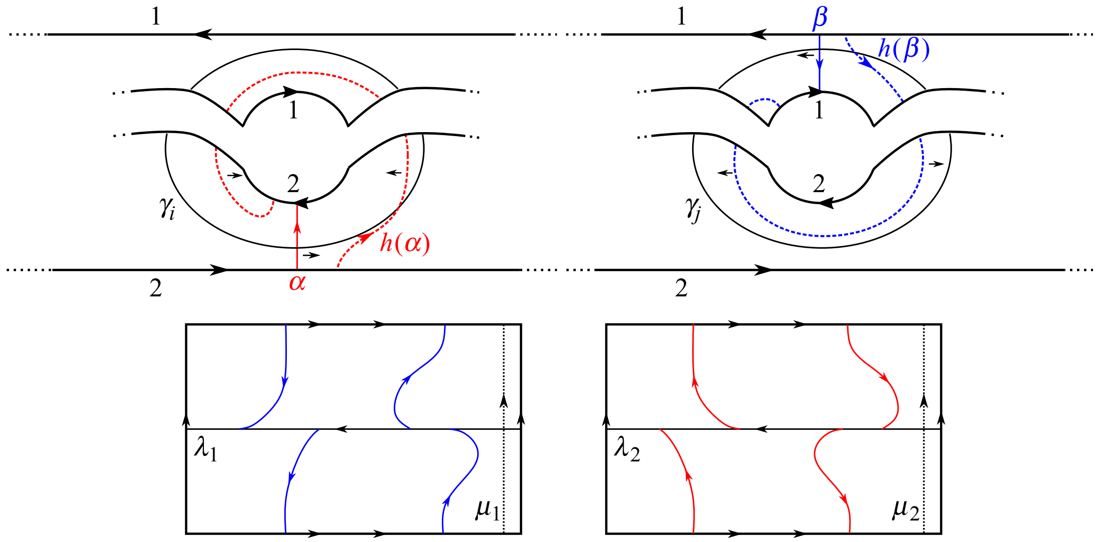


Figure 17: This picture describes the choice of the arcs α and β when the twist along the curve γ_i is positive and the one along γ_j is negative, together with the boundary train tracks of the associated branched surface.

Remark 3.12 Recall that we are working with Seifert framings. However we have already noticed that the meridians of the Seifert framing coincide with the canonical meridians of L. This implies that a surgery coefficient (r_1, r_2) on L is rational with respect to the Seifert framing if and only if it is rational with respect to the canonical framing.

Corollary 3.13 *If the factorisation of the monodromy h has two river twists with different exponents, i.e., if L belongs to family* 1*, then all the rational surgeries on the link L contain co-orientable taut foliations.*

Proof It follows from the first part of Lemma 3.11 that there are co-orientable taut foliations on $M(r_1, r_2)$ for $(r_1, r_2) \in (-\infty, 1)^2 \cup (-1, +\infty)^2$ and it follows from the second part of Lemma 3.11 that there are co-orientable taut foliations on $M(r_1, r_2)$ for $(r_1, r_2) \in \big((-1, +\infty) \times (-\infty, 1)\big) \cup \big((-\infty, 1) \times (-1, +\infty)\big)$. The union of these sets is exactly the set of all rational multislopes. □

3.6 Study of the links in family 2

As a consequence of Corollary 3.13, by taking mirrors if necessary, we can reduce our study to the case where the river twists are all positives, i.e., to links of the form $L = L(2b_1, -2, 2b_3, \dots, -2, 2b_n)$ with n odd.

Lemma 3.14 *Let $L = L(2b_1, \dots, 2b_n)$ with n odd and $|b_i| = 1$ for all i, and let h denote its monodromy as in* $(**)$. *Let M denote the exterior of L.*

(1) *If there are at least two positive (resp. negative) bridge twists in the factorisation of the monodromy h, then the manifold $M(r_1, r_2)$ contains a co-orientable taut foliation for every multislope $(r_1, r_2) \in (-\infty, 1)^2$ (resp. for all $(r_1, r_2) \in (-1, +\infty)^2$); see Figure 18*(a)–(b).

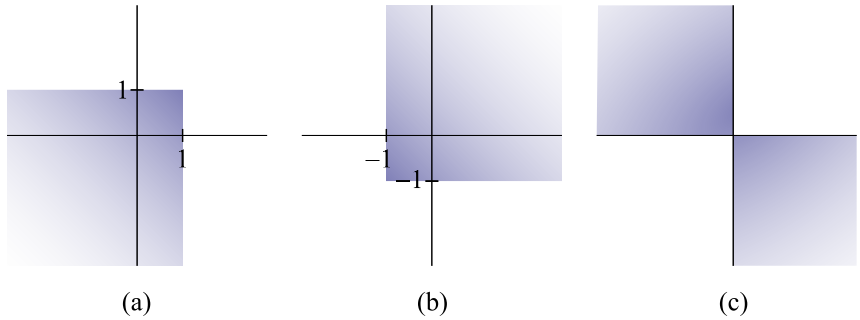


Figure 18: From left to right, the slopes (r_1, r_2) in the coloured region yield manifolds with co-orientable taut foliations in the case where there are, respectively, at least two positive bridge twists, at least two negative bridge twists, two bridge twists with different exponents in the factorisation of the monodromy h.

(2) *If there are two bridge twists with different exponents in the factorisation of the monodromy* h*, then the manifold* $M(r_1, r_2)$ *contains a co-orientable taut foliation for every multislope* $(r_1, r_2) \in \big((0, +\infty) \times (-\infty, 0)\big) \cup \big((-\infty, 0) \times (0, +\infty)\big)$*; see Figure 18*(c).

Proof (1) Suppose that the positive bridge twists are along the curves γ_i and γ_j. We consider the oriented arc α and β as in Figure 19. We have $h(\alpha) = \tau_i(\alpha)$: in fact

$$h = \underbrace{\tau_2^{\varepsilon_2} \tau_4^{\varepsilon_4} \dots \tau_{2k}^{\varepsilon_{2k}}}_{\text{river twists}} \underbrace{\tau_1^{\varepsilon_1} \tau_3^{\varepsilon_3} \dots \tau_{2k+1}^{\varepsilon_{2k+1}}}_{\text{bridge twists}}$$

and the only bridge twist that has effect on α is τ_i and the river twists have no effect on $\tau_i(\alpha)$. The same reasoning proves that $h(\beta) = \tau_j(\beta)$. Also in this case we obtain a branched surface that satisfies the hypotheses of Proposition 3.7. In fact this branched surface has at most two sectors[3] $\mathcal{A}$ and $\mathcal{B}$ in S and each of them has some cusp direction on its boundary pointing out of it, as Figure 19 shows. Therefore we just need to study the multislopes realised by the boundary train tracks of B. Both of these are of type (a) + (d) and hence the multislopes realised are the ones in $(-\infty, 1)^2$.

The case where we have two negative bridge twists is analogous: we choose α and β in the same way but now so that they turn right when they meet the curves γ_i and γ_j. Everything works in the same way but now the multislopes realised by the boundary train tracks are the ones in $(-1, +\infty)^2$.

(2) Suppose that are two bridge twists with different exponents in the factorisation of h and suppose that the positive one is along the curve γ_i and the negative one is along γ_j. We choose α and β as in Figure 20. Also in this case there are at most two sectors $\mathcal{A}$ and $\mathcal{B}$ of the resulting branched B surface in S and none of them is a sink disc. Moreover, the boundary train tracks of B realise all the slopes in $(0, +\infty)$ and $(-\infty, 0)$. In fact on one boundary component we have a train track of type (a) + (b) and on the other a train track of type (c) + (d).

Using the fact that two-bridge links are symmetric, we obtain the statement. □

[3]There are two sectors when $i = 1$ and $j = n = 3$ and there is only one sector otherwise.

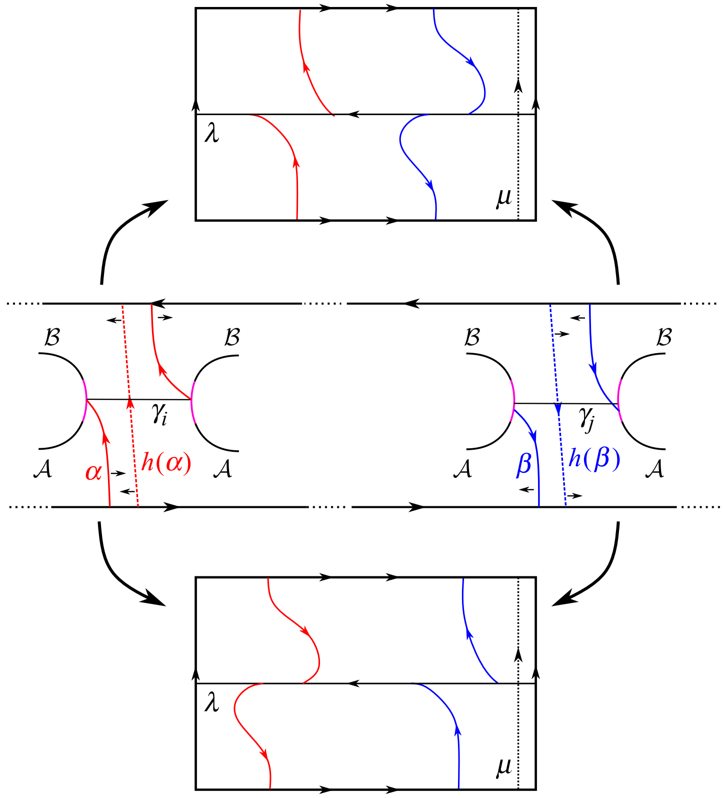


Figure 19: The arcs α and β, together with their image via the monodromy h and the cusps directions, are depicted. We also describe the train tracks obtained on the boundary of M. To simplify the picture we do not draw the 1-handles; we understand that the pink-coloured lines are pairwise identified in the obvious way.

Corollary 3.15 *Let $L = L(2b_1, -2, 2b_3, \dots, -2, 2b_n)$ with n odd and $|b_i| = 1$ for all i, and let h denote its monodromy as in $(**)$. If there are at least two negative bridge twists and one positive bridge twist in the factorisation of h, i.e., if L belongs to family 2, then all the rational surgeries on the link L contain co-orientable taut foliations.*

Proof As a consequence of the fact that the factorisation of h contains positive river twists, by Lemma 3.11 we know that $M(r_1, r_2)$ contains a taut foliation for all the multislopes $(r_1, r_2) \in (-\infty, 1)^2$. Moreover, since there are two negative bridge twists it follows from the first part of Lemma 3.14 that $M(r_1, r_2)$ contains a taut foliation for all the multislopes $(r_1, r_2) \in (-1, +\infty)^2$. As there is also at least one positive bridge twist we can apply the second part of Lemma 3.14 and deduce that $M(r_1, r_2)$ contains a taut foliation for all the multislopes $(r_1, r_2) \in \big((0, +\infty) \times (-\infty, 0)\big) \cup \big((-\infty, 0) \times (0, +\infty)\big)$. The union of these sets is exactly the set of all rational multislopes. □

3.7 Study of the links in family 3

We now focus our attention on the links composing *family* 3.

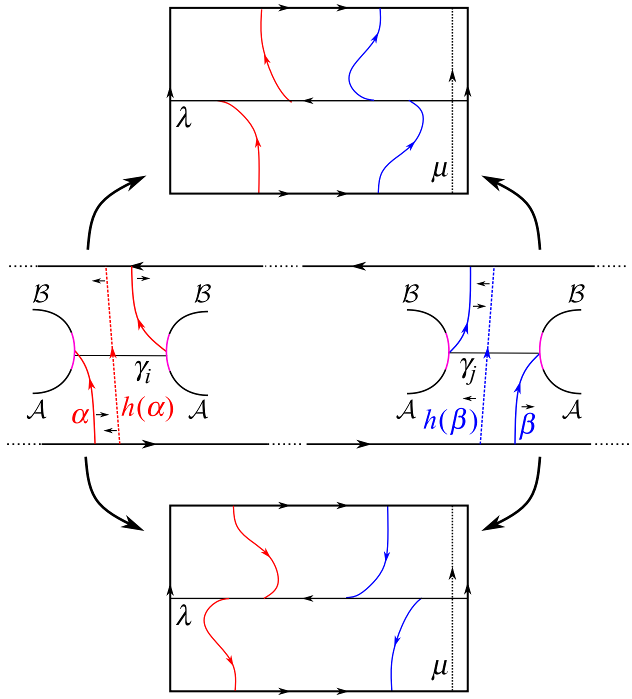


Figure 20: The arcs α and β in the case where there are two bridge twist with different exponents and the boundary train tracks realised on ∂M.

Proposition 3.16 *Let L be a two-bridge link of the form $L = L(-2, -2, -2, \dots, -2, -2)$, i.e., belonging to family* 3*. Then all the rational Dehn surgeries on L support a co-orientable taut foliation.*

Proof It follows by Lemmas 3.11 and 3.14 that, as the monodromy of L has (at least) two negative bridge twists and (at least) one positive river twist, all the surgery coefficients contained in $(-\infty, 1)^2 \cup (-1, +\infty)^2$ yield manifolds with co-orientable taut foliations. We recall that these coefficients are associated to the Seifert framing. We now consider two cases:

- $\underline{L \text{ is not the link } L(-2,-2,-2)}$: We construct a branched surface B whose boundary train tracks realise all the multislopes in $(-\infty, 1) \times (0, +\infty)$. Two-bridge links are symmetric, hence this will imply the statement. This branched surface is constructed by considering the arcs α and β in Figure 21 and satisfies the hypotheses of Proposition 3.7. In fact the complement of $\alpha \cup \beta$ in S is not a disc, and there is only one sector of B in S. Hence B has no sink discs. We can therefore use B to construct foliations on all the surgeries associated to the multislopes realised by its boundary train tracks: one of these is of type (b), and so realises all slopes in $(0, +\infty)$, and the other is of type (a) + (c) + (d) and hence realises all slopes in $(-\infty, 1)$.
- $\underline{L = L(-2,-2,-2)}$: To study this case we use an idea that will be useful also later on. We construct taut foliations on all the (r, s)-surgeries on L, where $r < 0$ or $s < 0$. This is enough because we already know from Lemma 3.14 that the surgeries associated to $(r, s) \in (-1, +\infty)^2$ contain taut foliations. Observe

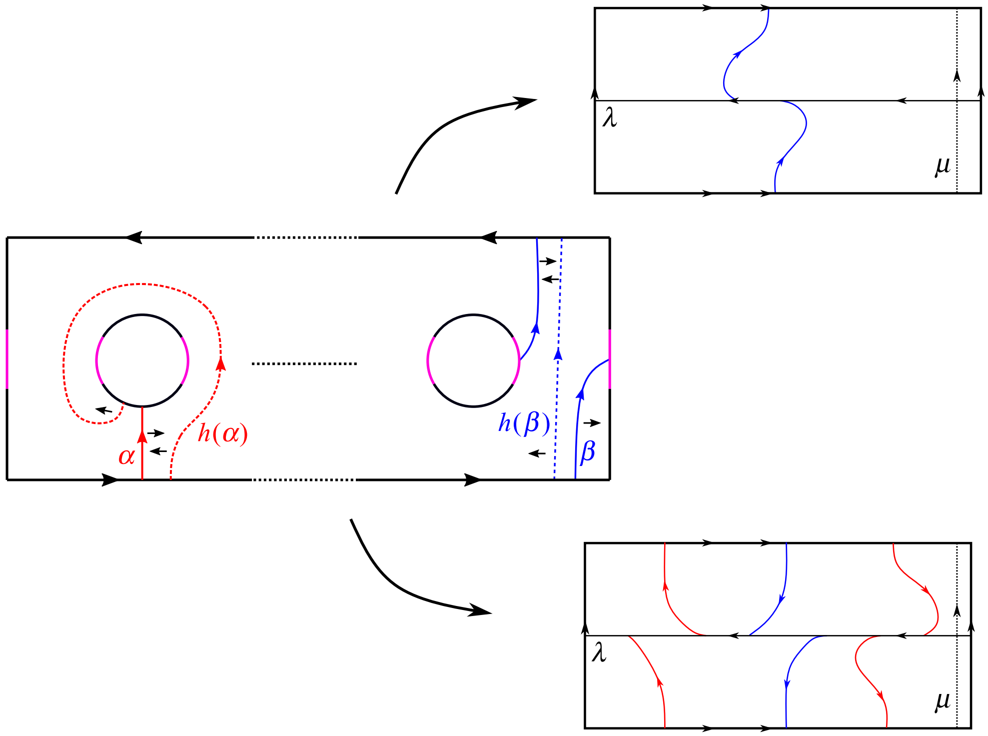


Figure 21: The arcs α and β that we consider when L is not the link $L(-2,-2,-2)$, and the boundary train track of the associated branched surface.

that L can be described as surgery on a 3-components link $\mathcal{L}$, as in Figure 22. The link $\mathcal{L}$ is also fibred, because it is boundary of a surface obtained via a sequence of Hopf plumbing, as described in Figure 22.

Moreover the monodromy of the link $\mathcal{L}$ is given by $h = \tau_4\tau_3^{-1}\tau_2\tau_1^{-1}$, where τ_i denotes the positive Dehn twist along the curve c_i shown in Figure 23.

This description of L will help us to construct the desired taut foliations. The idea is to find a branched surface in the exterior of $\mathcal{L}$ so that the boundary train tracks realise slope -1 on the boundary component

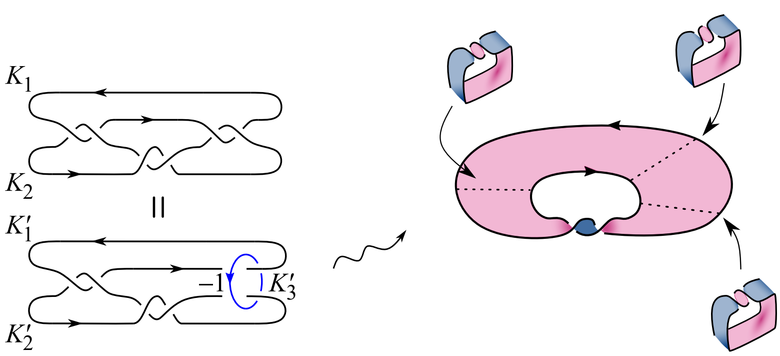


Figure 22: How to obtain the link $L(-2,-2,-2)$ as surgery on a 3-component link $\mathcal{L}$. We also describe a fibre surface for $\mathcal{L}$, obtained via a sequence of Hopf plumbings.

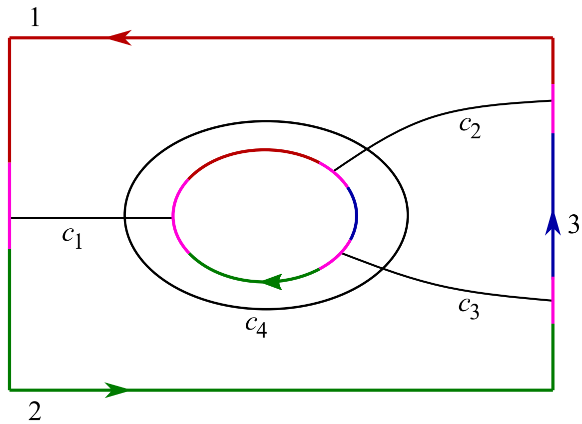


Figure 23: An abstract drawing of the fibre surface for the link $\mathcal{L}$, together with the curves c_i. We have also coloured the three boundary components of S, the boundary component labelled i corresponding to the component K_i' of the link, for $i = 1, 2, 3$.

associated to K_3'. To do this is important to pay attention to how the surgery coefficients change when passing from $\mathcal{L}$ to L. Recall that the coefficients of the slopes are written by using the identification given by the Seifert framing. The $(a, b, -1)$-surgery on $\mathcal{L}$ coincides with the $(a-1, b+1)$-surgery on L, as the following diagram suggests:

$$\begin{array}{ccc} \overbrace{(a, b, -1)}^{\text{Seifert framing for } \mathcal{L}} & \dashrightarrow & \overbrace{(a-1, b+1)}^{\text{Seifert framing for } L} \\ \downarrow & & \uparrow \\ \underbrace{(a, b+2, -1)}_{\text{canonical framing for } \mathcal{L}} & \longrightarrow & \underbrace{(a+1, b+3)}_{\text{canonical framing for } L} \end{array}$$

The changes of coefficients indicated by the vertical arrows are a consequence of formula $(\star)$ and the fact that

$$\operatorname{lk}(K_1, K_2) = -2, \quad \operatorname{lk}(K_1', K_2') = \operatorname{lk}(K_2', K_3') = -1, \quad \operatorname{lk}(K_1', K_3') = 1.$$

We construct two branched surfaces B_i in the exterior of $\mathcal{L}$, associated to the arcs α_i, β_i and γ_i, for $i = 1, 2$, as described in Figure 24. It can be checked by direct inspection that for $i = 1, 2$ the complement of $\alpha_i \cup \beta_i \cup \gamma_i$ contains no disc components in S, and that there are no sink discs. Hence we can apply Proposition 3.7 and deduce that these branched surfaces carry laminations that extend to taut foliations on the manifolds obtained by Dehn filling the boundary tori along the multislopes realised by the boundary train tracks. First, we consider the boundary train tracks of B_1. Notice that the one contained in the boundary component of M labelled with 1 does not satisfy the hypothesis of Lemma 3.8, and so we cannot use it to compute the slopes it realises. Nonetheless, a direct computation with weight systems shows that it realises all slopes in $(-\infty, 1)$. The two other boundary train tracks of B_1 are of type (b) and type (c)+(d)

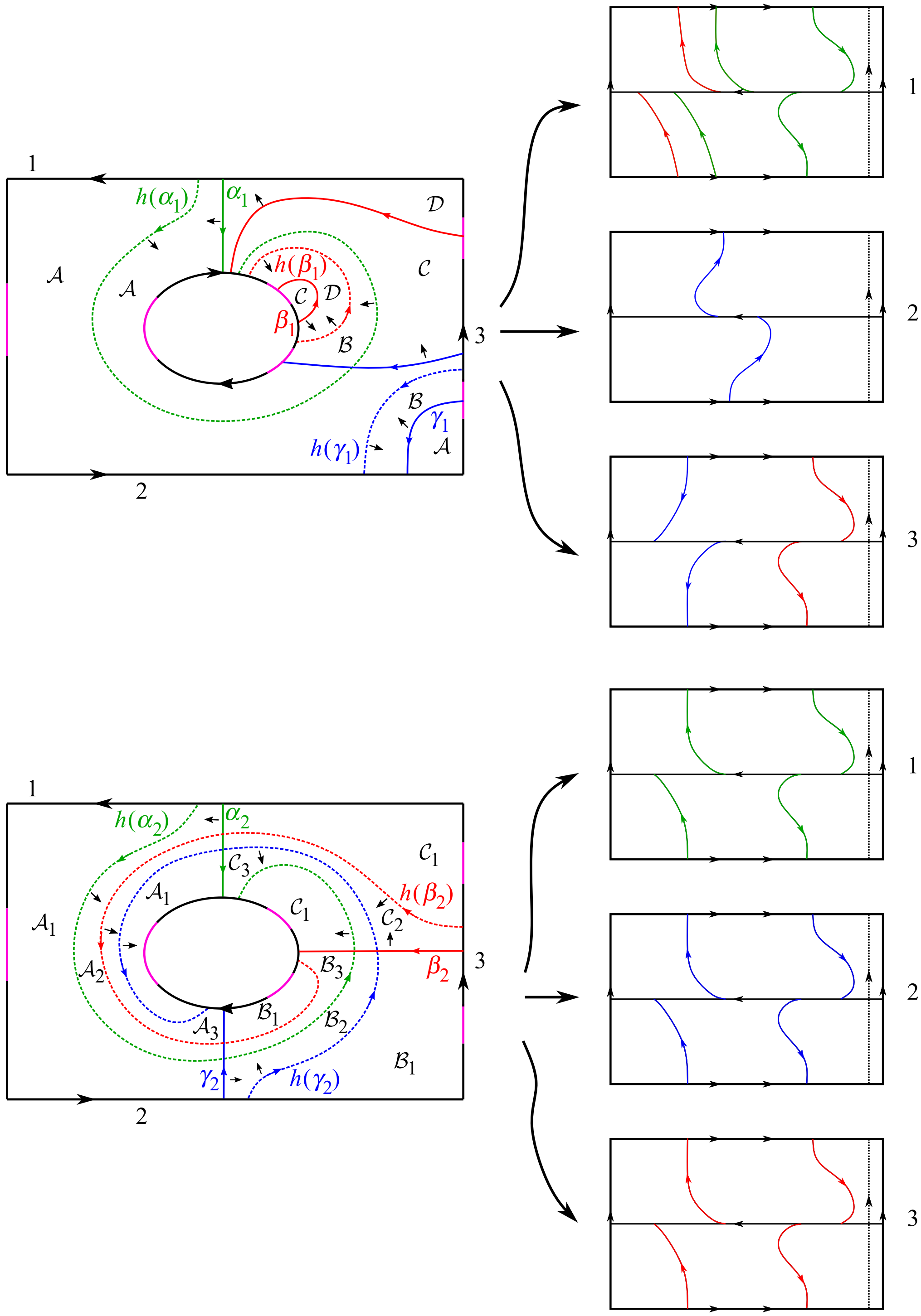


Figure 24: The arcs α_i, β_i, γ_i and their images via the monodromy h, together with the cusp directions of the associated branched surfaces and their boundary train tracks. We have also indicated the sectors of the branched surfaces in S.

and so realise all slopes in $(0,+\infty)$ and $(-\infty,0)$ on the corresponding boundary components. Summing up, the boundary train tracks of B_1 realise all the multislopes in $(-\infty,1)\times(0,+\infty)\times(-\infty,0)$. The boundary train tracks associated to B_2 are all of type (a) + (d) and hence realise all multislopes $(-\infty,1)^3$. In particular, we have taut foliations on $S^3_{r,s,-1}(\mathcal{L}) = S^3_{r-1,s+1}(L)$ for all $(r,s)\in(-\infty,1)\times\mathbb{R}$.

Since L is symmetric, and since all multislopes in $(-1,+\infty)^2$ are covered by Lemma 3.14, the statement follows. □

3.8 Study of the links in family 4

We now focus on the links of *family* 4, i.e., on the links of the form $L = L(2b_1,-2,2b_3,\dots,-2,2b_m)$ where exactly one b_i is -1 and all the others are equal to 1. We first study the case when $b_i = 1$ for some $i \neq 1, m$. We write $m = 2n+1$ for some positive integer n.

Lemma 3.17 *Let $L = L(2b_1,-2,2b_3,\dots,-2,2b_m)$ where exactly one b_{2k+1} is -1 and all the others are equal to 1 and suppose that $2k+1 \neq 1, m$. Then L is isotopic as an unoriented link to $L(-2k,-2,2,-2,-2h)$, where $h = n-k$.*

Proof We will prove this algebraically. We start by computing the fraction associated to the link $L(-2k,-2,2,-2,-2h)$. We have

$$-2k+\cfrac{1}{-2+\cfrac{1}{2+\cfrac{1}{-2-\cfrac{1}{2h}}}} = -2k+\cfrac{1}{-2+\cfrac{1}{2-\cfrac{2h}{4h+1}}} = -2k+\cfrac{1}{-2+\cfrac{4h+1}{6h+2}}$$
$$= -2k+\frac{6h+2}{-(8h+3)} = \frac{16kh+6k+6h+2}{-(8h+3)}$$

and this implies $L(-2k,-2,2,-2,-2h) = b(16kh+6k+6h+2,-(8h+3))$, where $b(p,q)$ denotes the two-bridge link associated to the rational $\frac{p}{q}$.

We now study the fraction corresponding to L. This fraction depends on k and n, or equivalently on k and $h = n-k$, and we denote its reduced representative by $\frac{\alpha_{k,h}}{\beta_{k,h}}$. Then we have

$$\frac{\alpha_{k,h}}{\beta_{k,h}} = 2+\cfrac{1}{-2+\cfrac{1}{2+\cfrac{1}{\ddots+\cfrac{1}{-2+\cfrac{1}{-2+\cfrac{q_h}{p_h}}}}}},$$

where the last -2 corresponds to $2b_{2k+1}$, and where $\frac{p_h}{q_h}$ is defined as

$$\frac{p_h}{q_h} = \overbrace{-2 + \cfrac{1}{2 + \cfrac{1}{-2 + \cfrac{1}{\ddots + \cfrac{1}{2}}}}}^{\text{length } 2h}.$$

One can check by induction on $h \geq 1$ that $\frac{p_h}{q_h} = \frac{2h+1}{-2h}$, and hence $p_h = 2h+1$ and $q_h = -2h$.

We now prove by induction on k that

$$\alpha_{k,h} = 16kh + 6k + 6h + 2,$$
$$\beta_{k,h} = 16kh - 2h + 6k - 1$$

for every h.

- Case $k = 1$: We have the equality

$$\frac{\alpha_{1,h}}{\beta_{1,h}} = 2 + \cfrac{1}{-2 + \cfrac{1}{-2 + \cfrac{q_h}{p_h}}} = 2 + \cfrac{1}{-2 - \cfrac{1+2h}{6h+2}} = 2 - \frac{6h+2}{14h+5} = \frac{22h+8}{14h+5}.$$

Since $(22h+8, 14h+5) = (q_h, p_h) = 1$ we deduce that $\alpha_{1,h} = 22h+8$ and $\beta_{1,h} = 14h+5$.

- Case $k > 1$: We can use the equality

$$\frac{\alpha_{k,h}}{\beta_{k,h}} = 2 + \cfrac{1}{-2 + \cfrac{\beta_{k-1,h}}{\alpha_{k-1,h}}} = 2 + \frac{\alpha_{k-1,h}}{-2\alpha_{k-1,h} + \beta_{k-1,h}} = \frac{3\alpha_{k-1,h} - 2\beta_{k-1,h}}{2\alpha_{k-1,h} - \beta_{k-1,h}}$$

and the fact that

$$(3\alpha_{k-1,h} - 2\beta_{k-1,h}, 2\alpha_{k-1,h} - \beta_{k-1,h}) = (\alpha_{k-1,h}, \beta_{k-1,h}) = 1$$

to deduce that $\alpha_{k,h} = 3\alpha_{k-1,h} - 2\beta_{k-1,h}$ and that $\beta_{k,h} = 2\alpha_{k-1,h} - \beta_{k-1,h}$. Therefore we have

$$\alpha_{k,h} - \beta_{k,h} = \alpha_{k-1,h} - \beta_{k-1,h} = 8h+3,$$
$$\alpha_{k,h} - \alpha_{k-1,h} = 2(\alpha_{k-1,h} - \beta_{k-1,h}) = 16h+6.$$

These equalities imply

$$\alpha_{k,h} = \alpha_{k-1,h} + 16h + 6 = 16kh + 6k + 6h + 2,$$
$$\beta_{k,h} = \alpha_{k,h} - 8h - 3 = 16kh - 2h + 6k - 1$$

and this proves the claim.

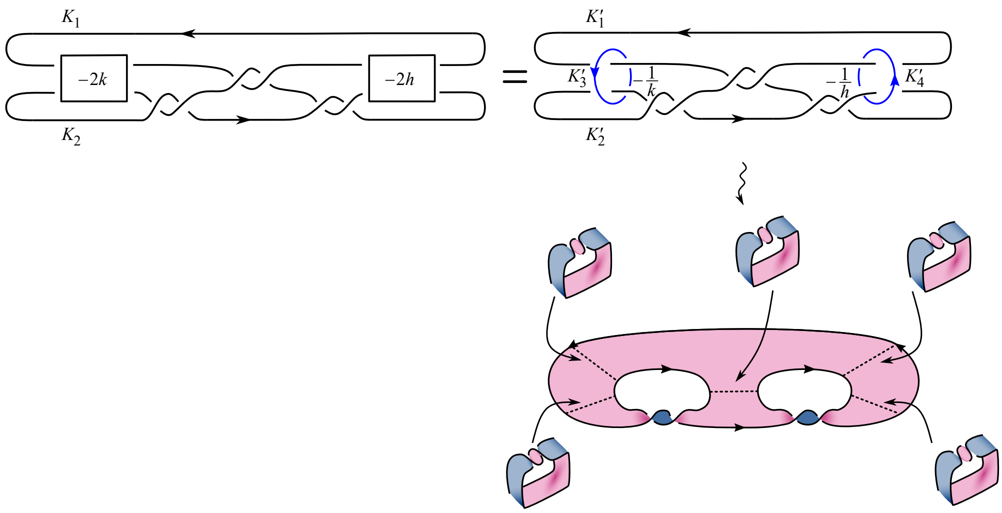


Figure 25: How to obtain the link $L(-2k,-2,2,-2,-2h)$ as surgery on a 4-component link $\mathcal{L}$. We also describe a fibre surface for $\mathcal{L}$, obtained as a sequence of Hopf plumbings.

To conclude the proof of the lemma we just have to recall from Theorem 2.1 that if $\beta' \equiv \alpha + \beta \bmod 2\alpha$ then the links $b(\alpha,\beta)$ and $b(\alpha,\beta')$ are isotopic after reversing the orientation of one of the components. In the case of our interest we have

$$\alpha_{k,h} + \beta_{k,h} \equiv -\alpha_{k,h} + \beta_{k,h} \equiv -(8h+3) \mod 2\alpha_{k,h}. \qquad \square$$

The description given by the previous lemma allows us to prove:

Proposition 3.18 *Let $L = L(2b_1,-2,2b_3,\dots,-2,2b_m)$ where exactly one b_{2k+1} is -1 and all the others are equal to 1 and suppose that $2k+1 \neq 1, m$. Then all the rational Dehn surgeries on L support co-orientable taut foliations.*

Proof By virtue of Lemma 3.17 it is equivalent to study surgeries on links of the form $L_{k,h} = L(-2k,-2,2,-2,-2h)$ where $h>0$ and $k>0$. These links can be obtained as surgeries on a 4-component fibred link $\mathcal{L}$, as described in Figure 25. Our aim now is to construct foliations on enough surgeries on $\mathcal{L}$.

The monodromy of the link $\mathcal{L}$ is given by $h = \tau_5\tau_3\tau_7\tau_6^{-1}\tau_4\tau_2\tau_1^{-1}$, where τ_i denotes the positive Dehn twist along the curve c_i shown in Figure 26. If we label the components of L and $\mathcal{L}$ as described in Figure 25, the surgery coefficients change in the following way:

$$\begin{array}{ccc}
\overbrace{\left(a,b,-\frac{1}{k},-\frac{1}{h}\right)}^{\text{Seifert framing for } \mathcal{L}} & \dashrightarrow & \overbrace{(a,b)}^{\text{Seifert framing for } L} \\
\big\downarrow & & \big\uparrow \\
\underbrace{\left(a-1,b-1,-\frac{1}{k},-\frac{1}{h}\right)}_{\text{canonical framing for } \mathcal{L}} & \longrightarrow & \underbrace{(a-1+k+h,b-1+k+h)}_{\text{canonical framing for } L}
\end{array}$$

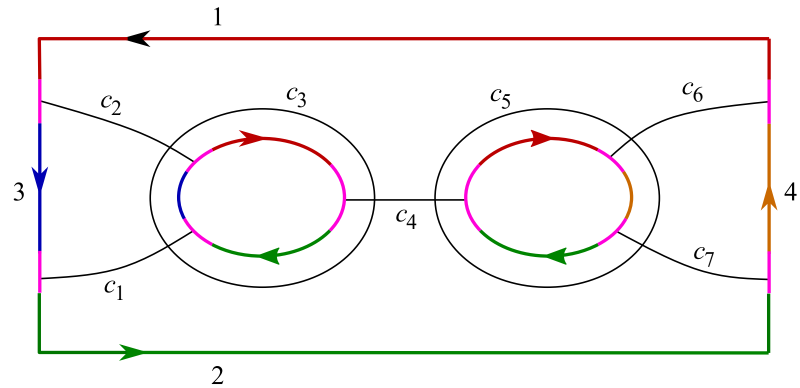


Figure 26: An abstract drawing of the fibre surface for the link $\mathcal{L}$, together with the curves c_i. We have also coloured the boundary components of S, the one labelled with i corresponding to the component K_i' of the link, for $i = 1, 2, 3, 4$.

As usual, when constructing foliations it is more natural to work with the framings given by the Seifert surfaces.

We construct two branched surfaces in the exterior of $\mathcal{L}$. The first one is associated to the arcs α, β, γ, δ, ϵ depicted in Figure 27, where we also describe the types of the boundary train tracks. The complement of these arcs in the fibre surface is not a disc (it is easier to see this by considering the complement of the images of these arcs via the diffeomorphism h) and the branched surface does not contain sink discs. In fact, there are five sectors in S, labelled with capital letters in Figure 27 and none of them is a sink disc. Therefore we can apply Proposition 3.7 and deduce that there exist taut foliations on all the surgeries on $\mathcal{L}$ corresponding to multislopes in $(0, +\infty) \times \mathbb{R} \times (-\infty, 0) \times (-\infty, 0)$.

The second branched surface is the one associated to the arcs described in Figure 28. In this case all the boundary train tracks are of type (a) + (d) and hence we are able to construct foliations on the surgeries corresponding to multislopes in $(-\infty, 1)^4$.

This implies that for every $k > 0$ and $h > 0$ all the surgeries on the link $L_{k,h}$ corresponding to multislopes in $(0, +\infty) \times \mathbb{R}$ and in $(-\infty, 1)^2$ support a co-orientable taut foliation. The conclusion follows using the fact that all these links are symmetric. □

Now we only have to study the links $L = L(2b_1, -2, 2b_3, \dots, -2, 2b_{2n+1})$ where $b_1 = -1$ and all the other b_i are 1, or where $b_{2n+1} = -1$ and all the other b_i are 1. The link $L(a_1, a_2, \dots, a_{2n+1})$ is isotopic to $L(a_{2n+1}, \dots, a_2, a_1)$, so we can reduce our study to the case when $b_{2n+1} = -1$ and we denote the corresponding link by L_n.

Lemma 3.19 *The link L_n is isotopic as unoriented link to the link $L(2, -2, -2n)$, illustrated in Figure 1.*

Proof We compute the fractions associated to these links. The one associated to $L(2, -2, -2n)$ is $\frac{6n+2}{4n+1}$. Therefore by Theorem 2.1 the link $L(2, -2, -2n)$ is isotopic, after reversing the orientation of one of the components, to the link defined by the fraction $\frac{6n+2}{-(2n+1)}$.

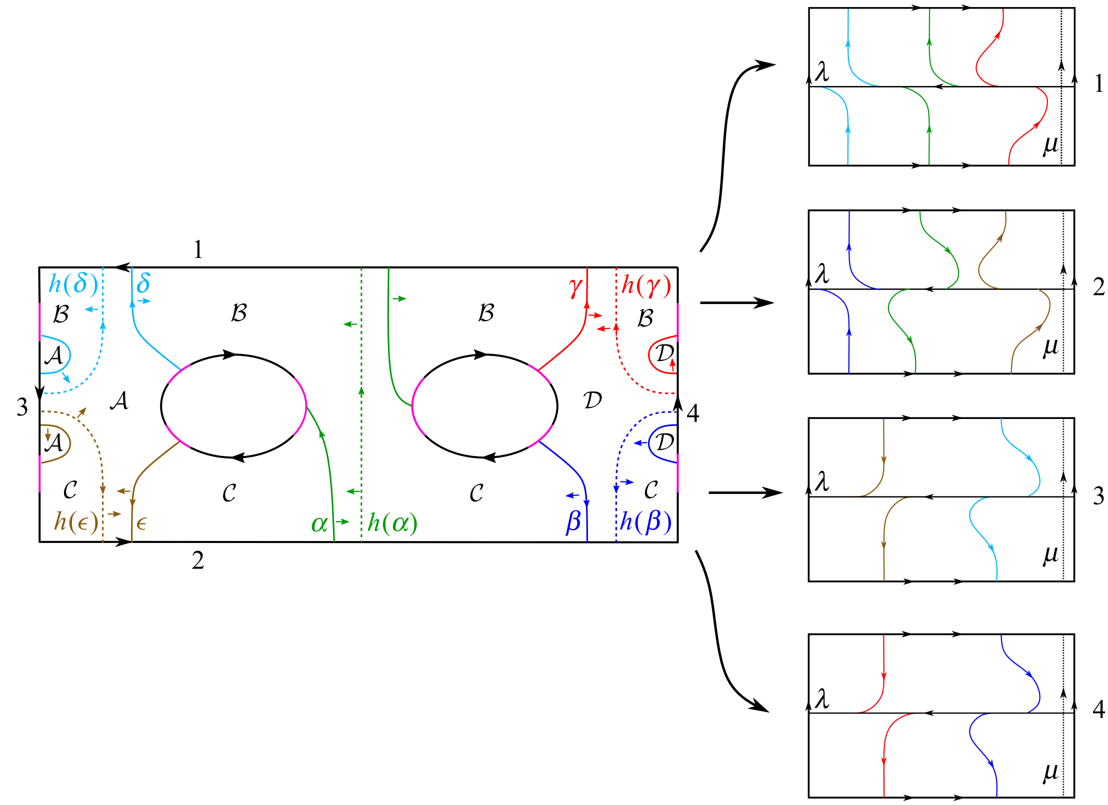


Figure 27: The arcs α, β, γ, δ, ϵ and the boundary train tracks of the associated branched surface.

The fractions $\frac{p_n}{q_n}$ associated to L_n satisfy the recursive equation

$$\frac{p_n}{q_n} = 2 + \cfrac{1}{-2 + \cfrac{q_{n-1}}{p_{n-1}}} = 2 + \frac{p_{n-1}}{-2p_{n-1} + q_{n-1}} = \frac{3p_{n-1} - 2q_{n-1}}{2p_{n-1} - q_{n-1}}. \tag{1}$$

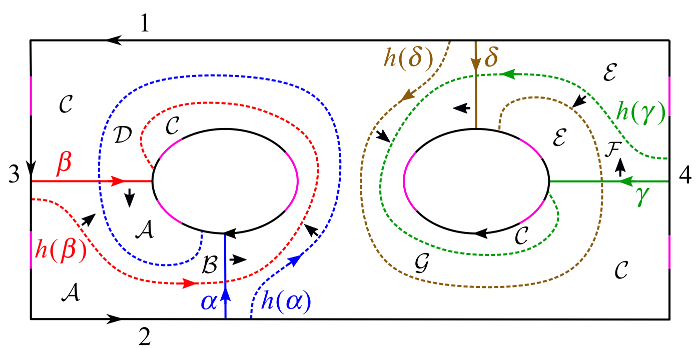


Figure 28: The arcs α, β, γ, δ used to construct the second branched surface. We have also labelled the sector contained in S and one can check that none of them is a sink disc.

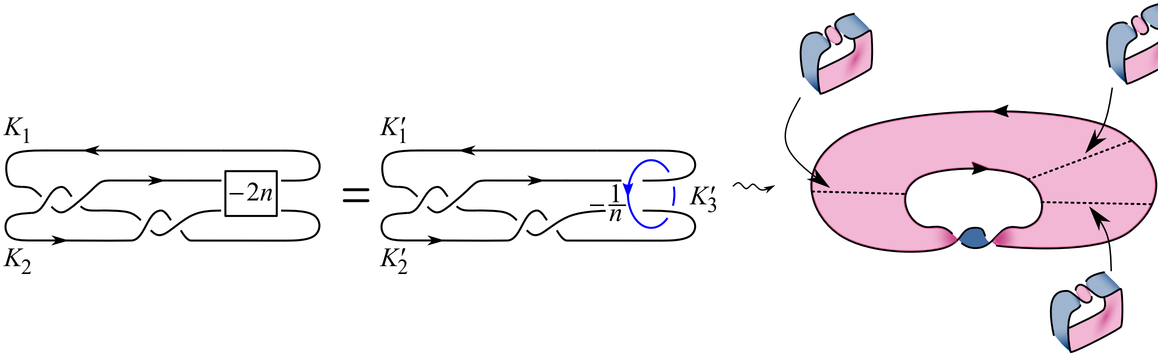


Figure 29: How to obtain the links $\{L_n\}_{n\geq 1}$ as surgery on a 3-components link $\mathcal{L}$ and a fibre surface S for $\mathcal{L}$.

Let us find an explicit formula for p_n and q_n. It follows from (1) that

$$p_n - q_n = p_{n-1} - q_{n-1}$$

and as a consequence the quantity $p_i - q_i$ does not depend on the index i. Moreover, (1) also implies

$$p_n - p_{n-1} = q_n - q_{n-1} = 2(p_{n-1} - q_{n-1})$$

and therefore also the quantity $p_i - p_{i-1} = q_i - q_{i-1}$ is constant in i. As when $n = 1$ we have $\frac{p_1}{q_1} = \frac{8}{5}$, we deduce by induction that $p_n = 8 + (n-1)6 = 6n + 2$ and $q_n = 5 + (n-1)6$. To conclude the proof is enough to observe that $-(2n+1) \equiv q_n^{-1} \bmod 2p_n$ and use again Theorem 2.1. □

We will prove in the next section (see Proposition 4.3) that when $r \geq n$, $s \geq n$ the (r,s)-surgery on L_n is an L-space, where the surgery coefficients are to be considered in the *canonical* framing. We now prove that all the other (rational) surgeries on L_n support co-orientable taut foliations.

Proposition 3.20 *Let $L_n = L(2b_1, -2, 2b_3, \dots, -2, 2b_{2n+1})$, where $b_{2n+1} = -1$ and all the other b_i are 1, and let $r < n$, $s < n$ be rational numbers. Then the (r,s)-surgery on L_n supports a co-orientable taut foliation, where the surgery coefficients are considered in the canonical framing.*

Proof We can suppose that $n \geq 2$, because L_1 is the Whitehead link, for which the corresponding result was proved in [53]. By Lemma 3.19 we have $L_n = L(2,-2,-2n)$ as unoriented links and by using this representation it is evident that $L_n = S^3_{\bullet,\bullet,-1/n}(\mathcal{L})$, where $\mathcal{L}$ is drawn in Figure 29. This figure also shows a fibre surface S for $\mathcal{L}$ obtained via a sequence of four Hopf plumbings.

We choose four triples α, β, γ of oriented arcs in S and consider the four branched surfaces in the exterior of $\mathcal{L}$ associated to these arcs, as depicted in Figures 30 and 31. Each of these triples has the property that its complement in S contains no disc components.

Moreover, the figures also illustrate the sectors of the branched surfaces contained in S and it can be checked that none of these is a sink disc. Thus, thanks to Proposition 3.7 we only need to study the boundary train tracks of these branched surfaces in order to construct the desired taut foliations. The multislopes realised by these branched surfaces in the Seifert framing of $\mathcal{L}$ are, respectively,

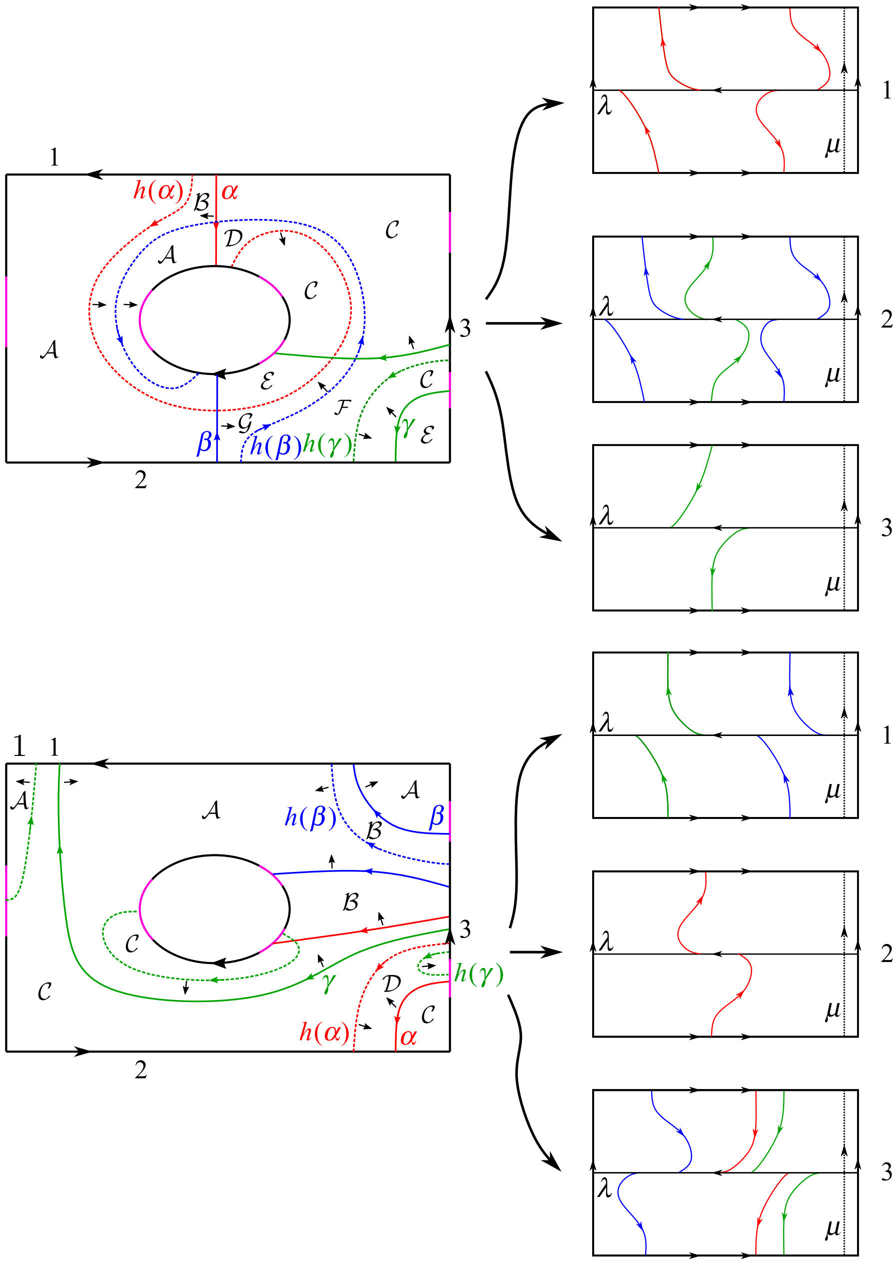


Figure 30: How to choose two of the four triples of arcs α, β, γ. The picture also represents their images via the monodromy, the boundary train tracks, and the sectors in S.

- all the multislopes in $(-\infty, 1) \times \mathbb{R} \times (-1, 0)$;
- all the multislopes in $(0, 2) \times (0, +\infty) \times (-\infty, 0)$;
- all the multislopes in $(0, 2) \times (-\infty, 0) \times (-1, 0)$;
- all the multislopes in $(-\infty, 2) \times (-1, 1) \times (-1, 0)$.

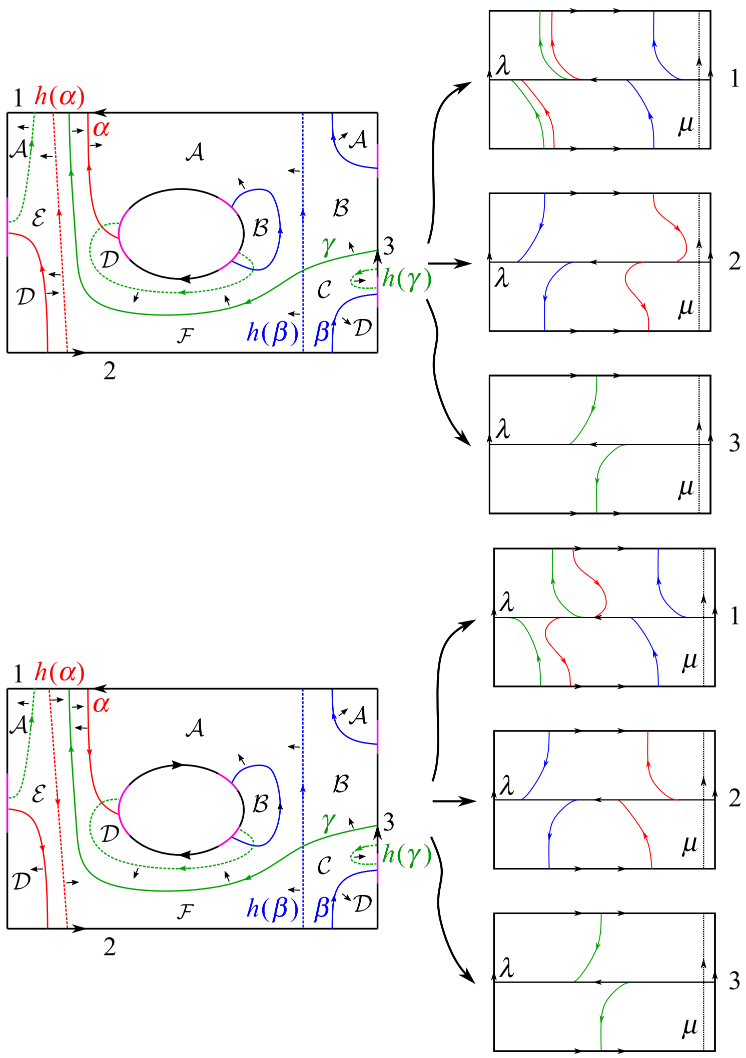


Figure 31: How to choose the two other triples of arcs α, β, γ. The picture also represents their images via the monodromy, the boundary train tracks, and the sectors in S.

We now prove that by considering L_n as $-\frac{1}{n}$ surgery on the third component of $\mathcal{L}$, we have constructed the desired foliations on the surgeries on L_n. First of all we observe that

$$\operatorname{lk}(K_1', K_2') = \operatorname{lk}(K_1', K_3') = 1, \quad \operatorname{lk}(K_2', K_3') = -1$$

and by using formula ($\star$) we deduce the change of surgery coefficients

$$\underbrace{\left(a, b, -\frac{1}{n}\right)}_{\text{Seifert framing for } \mathcal{L}} \to \overbrace{\left(a-2, b, -\frac{1}{n}\right)}^{\text{canonical framing for } \mathcal{L}} \to \overbrace{(a+n-2, b+n)}^{\text{canonical framing for } L_n}.$$

Therefore, for every $n \geq 2$, we obtain taut foliations on all the surgeries on L_n corresponding to multislopes in

- $W = (-\infty, n-1) \times \mathbb{R}$;
- $X = (n-2, n) \times (n, +\infty)$;
- $Y = (n-2, n) \times (-\infty, n)$;
- $Z = (-\infty, n) \times (n-1, n+1)$.

We now show that these four sets are enough to deduce that, for all $n \geq 2$, all the surgeries on L_n corresponding to multislopes (r_1, r_2) where $r_1 < n$ or $r_2 < n$ support a co-orientable taut foliation. In fact suppose that we have such a pair (r_1, r_2). Since L_n is symmetric we can suppose that $r_1 < n$ and we have the following cases:

- $\underline{r_1 < n-1}$: In this case the pair is contained in the set W.
- $\underline{n-1 \leq r_1 < n}$: If $r_2 > n$ the pair is contained in X, if $r_2 < n$ we conclude by using the set Y and if $r_2 = n$ we use the set Z. □

4 L-spaces

We now study the L-space surgeries on the links L_n, illustrated in Figure 1, and conclude the proof of Theorem 1.2 and Proposition 1.5. To do this we will use the main result of [50]. We recall some definitions and fix some notation.

Let Y be a rational homology solid torus, i.e., Y is a compact oriented 3-manifold with toroidal boundary such that $H_*(Y;\mathbb{Q}) \cong H_*(\mathbb{D}^2 \times S^1;\mathbb{Q})$.

We are interested in the study of the Dehn fillings on Y. We define the *set of slopes in* Y as

$$\mathrm{Sl}(Y) = \{\alpha \in H_1(\partial Y;\mathbb{Z}) \mid \alpha \text{ is primitive}\}/\pm 1.$$

We will denote with $Y(\alpha)$ the Dehn filling of Y associated to $[\alpha] \in \mathrm{Sl}(Y)$.

Notice that as a consequence of Y being a rational homology solid torus, there is a distinguished slope in $\mathrm{Sl}(Y)$ that we call the *homological longitude* of Y and that is defined in the following way. We denote with $i : H_1(\partial Y;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$ the map induced by the inclusion $\partial Y \subset Y$ and we consider a primitive element $l \in H_1(\partial Y;\mathbb{Z})$ such that $i(l)$ is torsion in $H_1(Y;\mathbb{Z})$. The element l is unique up to sign, and its equivalence class $[l] \in \mathrm{Sl}(Y)$ is the homological longitude of Y. This definition, which may seem counterintuitive, is given so that when Y is the complement of a knot in S^3, the homological longitude of Y coincides with the slope defined by the longitude of the knot.

We want to study the fillings on Y that are L-spaces. For this reason we define the set of the *L-space filling slopes* as

$$\mathcal{L}(Y) = \{[\alpha] \in \mathrm{Sl}(Y) \mid Y(\alpha) \text{ is an } L\text{-space}\}.$$

Once we fix a basis (μ, λ) for $H_1(\partial Y;\mathbb{Z})$ we can identify the set $\mathrm{Sl}(Y)$ with $\overline{\mathbb{Q}}$. The following theorem is a straightforward consequence of [50, Theorem 1.6].

Theorem 4.1 *Let Y be a rational homology solid torus and let $[\alpha] \neq [\beta]$ be two slopes in $\mathcal{L}(Y)$. Then the set $\mathcal{L}(Y)$ contains the interval in* $\mathrm{Sl}(Y)$ *between $[\alpha]$ and $[\beta]$ that does not contain the homological longitude $[l]$.* □

We want to use this result to study L-space surgeries on links. If L is a link in S^3, we denote by $\mathcal{L}(L)$ the set of slopes in the exterior of L such that the corresponding surgery is an L-space. For each component of the link we fix the canonical meridian and longitude, and in this way we can identify $\mathcal{L}(L)$ with a subset of $\overline{\mathbb{Q}}^d$, where d is the number of components of the link. We fix $d = 2$, i.e., we suppose that L has two components K_1 and K_2. Given $(r_1, r_2) \in \overline{\mathbb{Q}}^2$, we denote by

- $S^3_{r_1,r_2}(L)$ the (r_1, r_2)-surgery on L;
- $S^3_{r_1,\bullet}(L)$ the manifold obtained by drilling K_2 and performing r_1-surgery on K_1;
- $S^3_{\bullet,r_2}(L)$ the manifold obtained by drilling K_1 and performing r_2-surgery on K_2.

Recall that if L has two components, then by using Mayer–Vietoris one can see that the manifold $S^3_{r_1,r_2}(L)$ is not a rational homology sphere if and only if $\{r_1, r_2\} = \{0, \infty\}$ or $r_1 r_2 = \mathrm{lk}(L)^2$, where $\mathrm{lk}(L)$ denotes the linking number of the components of L. Hence if $r_1 \neq 0$ the manifold $S^3_{r_1,\bullet}(L)$ is a rational homology solid torus with homological longitude given by $\frac{\mathrm{lk}(L)^2}{r_1} \in \mathbb{Q}$. Analogously, if $r_2 \neq 0$ the manifold $S^3_{\bullet,r_2}(L)$ is a rational homology solid torus with homological longitude given by $\frac{\mathrm{lk}(L)^2}{r_2} \in \mathbb{Q}$.

Proposition 4.2 *Let L be a two-component link with two unknotted components. Suppose $(r_1, r_2) \in \mathcal{L}(L)$ with $r_1 r_2 > \mathrm{lk}(L)^2$ and $r_1 > 0, r_2 > 0$. Then $\big([r_1, \infty] \times [r_2, \infty]\big) \cap \overline{\mathbb{Q}}^2$ is contained in $\mathcal{L}(L)$. Analogously, if $r_1 r_2 > \mathrm{lk}(L)^2$ and $r_1 < 0, r_2 < 0$ then $\big([\infty, r_1] \times [\infty, r_2]\big) \cap \overline{\mathbb{Q}}^2$ is contained in $\mathcal{L}(L)$.*

Proof The proof is the straightforward adaptation of the proof of [53, Lemma 2.6]. We report it here for convenience of the reader. We prove the proposition in the case $r_1 r_2 > \mathrm{lk}(L)^2$ and $r_1 > 0$, $r_2 > 0$. The other case is analogous. We consider the manifold $Y = S^3_{r_1,\bullet}$. We have that $r_2 \in \mathcal{L}(Y)$ and since the components of L are unknotted it follows that also $\infty \in \mathcal{L}(L)$. In fact $S^3_{r_1,\infty}(L)$ is a lens space, and hence an L-space. Thus we can deduce, by virtue of Theorem 4.1, that the interval between r_2 and ∞ that does not contain the homological longitude is contained in $\mathcal{L}(Y)$. The homological longitude $\frac{\mathrm{lk}(L)^2}{r_1}$ is smaller than r_2, so we deduce that $[r_2, \infty] \cap \overline{\mathbb{Q}} \subset \mathcal{L}(Y)$. In other words we have proved that $S^3_{r_1,s}(L)$ is an L-space for all $s \geq r_2$. Now we fix $s \geq r_2$ and consider the manifold $Y_s = S^3_{\bullet,s}$. As a consequence of r_1 and ∞ belonging to $\mathcal{L}(Y_s)$, we can apply again Theorem 4.1 and deduce that the interval between r_1 and ∞ that does not contain the homological longitude is contained in $\mathcal{L}(Y_s)$. Since $r_1 \geq \frac{\mathrm{lk}(L)^2}{r_2} \geq \frac{\mathrm{lk}(L)^2}{s}$ and the last term in the chain of inequalities is the homological longitude of Y_s, we conclude that $[r_1, \infty] \cap \overline{\mathbb{Q}} \subset \mathcal{L}(Y_s)$ for all $s \geq r_2$. This is exactly equivalent to saying that $\big([r_1, \infty] \times [r_2, \infty]\big) \cap \overline{\mathbb{Q}}^2 \subset \mathcal{L}(L)$. A pictorial sketch of the proof is described in Figure 32. □

We are now ready to prove the main result of this section. Recall that since two-bridge links have unknotted components, the rational homology spheres associated to surgery coefficients (r, s) where at least one of r and s is ∞ are all L-spaces. For this reason, we will only consider the set $\mathcal{L}(L_n) \cap \mathbb{Q}^2$.

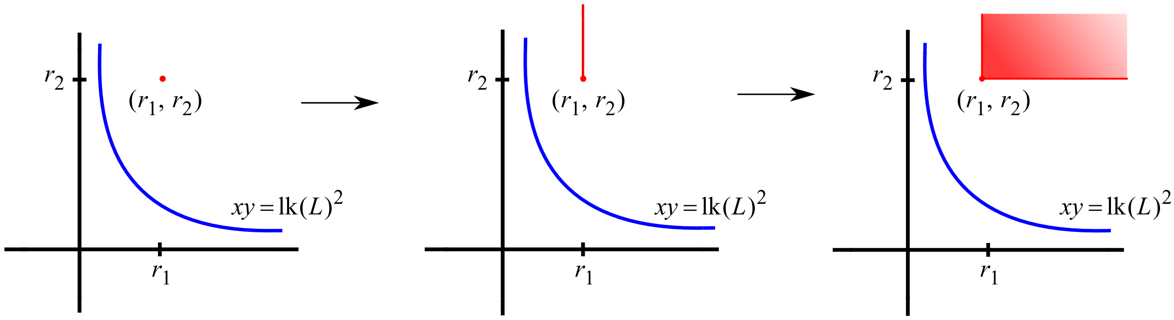


Figure 32: A pictorial sketch of the proof.

Proposition 4.3 *Let L_n be the link described in Figure 1. Then $\mathcal{L}(L_n) \cap \mathbb{Q}^2 = [n, +\infty)^2 \cap \mathbb{Q}^2$.*

Proof Since L-space do not support taut foliations, it follows from Proposition 3.20 that it suffices to prove that

$$[n, +\infty)^2 \subset \mathcal{L}(L_n).$$

The link L_n satisfies $\mathrm{lk}(L_n)^2 = (n-1)^2$ and its components are unknotted, hence by Proposition 4.2 it is enough to prove that $(n, n) \in \mathcal{L}(L_n)$. We can see the links L_n as surgeries on a three-component link $\mathscr{L}$, as represented in Figure 33. We have also fixed an orientation of this link, that we will use later in the proof.

More precisely we have that $S^3_{a,b,-1/(n-1)}(\mathscr{L}) = S^3_{a+n-1,b+n-1}(L_n)$. This implies that the statement is equivalent to proving that $S^3_{1,1,-1/(n-1)}(\mathscr{L})$ is an L-space for all $n \geq 1$ and to prove this we will apply Theorem 4.1 to the rational homology solid torus $S^3_{1,1,\bullet}(\mathscr{L})$. Denoting this manifold by Y, we have:

- $\underline{\infty \in \mathcal{L}(Y)}$: In fact $S^3_{1,1,\infty}(\mathscr{L})$ is $(1,1)$-surgery on the Whitehead link. This is the Poincaré homology sphere and manifolds with finite fundamental group are L-spaces [48, Proposition 2.2].

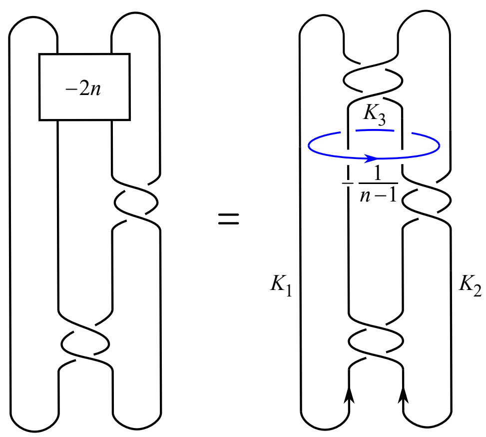


Figure 33: How to obtain the links $\{L_n\}_{n\geq 1}$ as surgeries on a 3-component link $\mathscr{L}$.

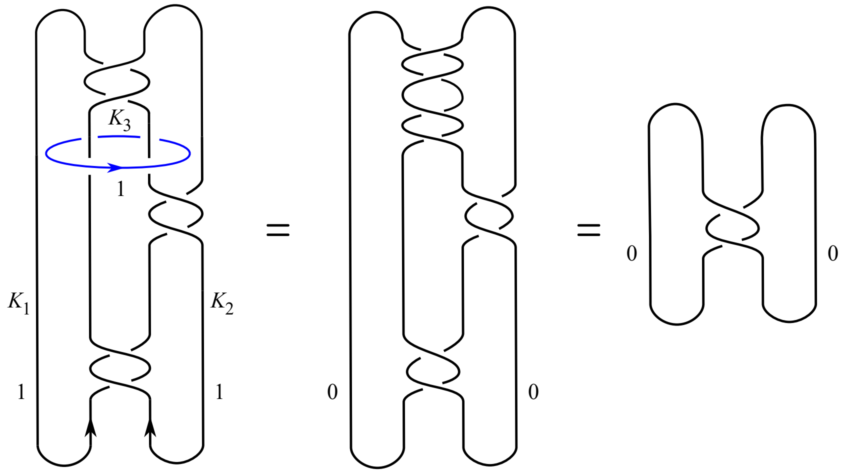


Figure 34: The $(1,1,1)$-surgery on $\mathscr{L}$ is $(0,0)$-surgery on the Hopf link.

- $1 \in \mathcal{L}(Y)$: In fact $S^3_{1,1,1}(\mathscr{L})$ is $(0,0)$-surgery on the Hopf link, see Figure 34. This manifold is S^3 and therefore an L-space.

- The homological longitude of Y is the slope 2: To prove this we have to do a simple computation. We fix an orientation for the link and we denote the components of $\mathscr{L}$ with K_1, K_2 and K_3 as in Figure 33.

We have that $\mathrm{lk}(K_1,K_3) = \mathrm{lk}(K_2,K_3) = 1$ and $\mathrm{lk}(K_1,K_2) = 0$. Consequently, a presentation matrix for $H_1(S^3_{1,1,p/q}(\mathscr{L}),\mathbb{Z})$ is given by

$$A = \begin{pmatrix} 1 & 0 & q \\ 0 & 1 & q \\ 1 & 1 & p \end{pmatrix}$$

and in particular $S^3_{1,1,p/q}(\mathscr{L})$ is not a rational homology sphere if and only if the determinant of A is zero. This happens if and only if $p = 2q$ and therefore 2 is the homological longitude of the manifold $S^3_{1,1,\bullet}(\mathscr{L})$.

What we have proved implies by Theorem 4.1 that $(-\infty,1] \cap \mathbb{Q} \subset \mathcal{L}(Y)$. In particular $S^3_{1,1,-1/(n-1)}(\mathscr{L})$ is an L-space for all $n \geq 1$ and this manifold is exactly the (n,n)-surgery on L_n. □

Remark 4.4 In the terminology of [21], the links L_n are *L-space links*. Liu [41] conjectured that a two-bridge link is an L-space link if and only if is of the form $b(pq-1,-q)$, where p and q are odd positive integers. This conjecture was proved by Dawra [10]. It is not difficult to prove that the link L_n, as an unoriented link, is isotopic to $b(6n+2,-3)$.

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DIEGO SANTORO diego.santoro95@gmail.com
University of Vienna, Vienna, Austria

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