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# New results on tilings via cup products and Chern characters on tiling spaces

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We study the cohomology rings of tiling spaces  $\Omega$  given by cubical substitutions. While there have been many calculations before of cohomology *groups* of such tiling spaces, the innovation here is that we use computer-assisted methods to compute the cup-product structure. This leads to examples of substitution tilings with isomorphic cohomology groups but different cohomology rings. Part of the interest in studying the cup product comes from Bellissard’s *gap-labeling conjecture*, which is known to hold in dimensions  $\leq 3$ , but where a proof is known in dimensions  $\geq 4$  only when the Chern character from  $K^0(\Omega)$  to  $H^*(\Omega, \mathbb{Q})$  lands in  $H^*(\Omega, \mathbb{Z})$ . Computation of the cup product on cohomology often makes it possible to compute the Chern character. We introduce a natural generalization of the gap-labeling conjecture, called the *equivariant gap-labeling conjecture*, which applies to tilings with a finite symmetry group. Again this holds in dimensions  $\leq 3$ , but we are able to show that it *fails* in general in dimensions  $\geq 4$ . This, plus some of our cup-product calculations, makes it plausible that the gap-labeling conjecture might fail in high dimensions.

## 1 Introduction

This paper is about tiling spaces  $\Omega$  coming from primitive substitution tilings of  $\mathbb{R}^d$ . All the tilings in this paper will be self-similar and have finite local complexity. Under those hypotheses, the action of  $\mathbb{R}^d$  on  $\Omega$  by translations is minimal and uniquely ergodic [30]. We begin with a review of the basic definitions and literature on tilings in Section 2.

The problems that we will study here were motivated by the gap-labeling conjecture of Bellissard [7]. His idea was that in a quasicrystal (an almost periodic physical system modeled mathematically by a tiling), physical properties of the material are determined by the spectra of almost periodic Schrödinger operators which live in a  $C^*$ -algebra  $\mathcal{A}_p(\Omega)$  determined by the tiling, so that gaps in the spectra can be determined by computing  $K_0(\mathcal{A}_p(\Omega))$  and the map from this  $K$ -group to the reals induced by the trace. The algebra  $\mathcal{A}_p(\Omega)$  is highly noncommutative and thus complicated to deal with, but it contains a natural copy of the commutative algebra  $C(\Omega)$  of functions on the tiling space. Bellissard noticed that at least in many cases, the  $K$ -theory of  $\mathcal{A}_p(\Omega)$  all lies in the image of the  $K$ -theory of this subalgebra, which can be computed purely topologically. The gap-labeling conjecture asserts that this should always be the case. Several papers [8; 10; 16] claimed to prove the conjecture, but they all implicitly assumed that the top-dimensional piece of the Chern character  $\text{ch}_d : K^{-d}(\Omega) \rightarrow \check{H}^d(\Omega; \mathbb{Q})$  gives an isomorphism  $K^{-d}(\Omega) \xrightarrow{\cong} \check{H}^d(\Omega; \mathbb{Z})$ . (See [1, §9] for a detailed discussion.)

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Our goal in this paper is to compute the structure of the integral cohomology ring  $H^*(\Omega; \mathbb{Z})$  for several examples of cubical substitution tilings. This is computation-intensive and requires computer assistance for the calculation of the cup product. Calculations are done in Sections 3 and 4. In several cases, we are also able to determine at least an important part of the Chern character map  $\text{ch}_d : K^{-d}(\Omega) \rightarrow \check{H}^d(\Omega; \mathbb{Q})$ . Among the major results are:

- (i) There are cases of tiling spaces with isomorphic cohomology as groups but not as *rings* (Section 3.2).
- (ii) There are cases where the Chern character does *not* give an isomorphism  $K^{-d}(\Omega) \rightarrow \check{H}^d(\Omega; \mathbb{Z})$  (Section 4.1).

Unfortunately we are not able with the present techniques to produce a counterexample to the gap-labeling conjecture in high dimensions, though we would not be surprised if such a counterexample exists.

However, we show that there is a natural generalization of the gap-labeling conjecture, which we call the *equivariant gap-labeling conjecture*, that can be formulated for tiling spaces with a finite symmetry group. Roughly speaking, this involves replacing  $K$ -theory by equivariant  $K$ -theory. This conjecture is formulated and discussed in Sections 2.5 and 2.6. We show that this equivariant conjecture fails in general when  $d \geq 4$  (see Theorem 2.6 and Section 4.3), which provides additional evidence that the original gap-labeling conjecture probably fails in high dimensions.

## 2 Tiling spaces and their cohomology

This section covers the necessary background on tiling spaces and their topological invariants.

### 2.1 Basics of tilings

To define a tiling, we start by assuming we are given a finite set of polytopes in  $\mathbb{R}^d$  called *prototiles*. A *tile* is a translate  $t + \tau$ ,  $\tau \in \mathbb{R}^d$ , of a prototile  $t$ , along with a choice of color or label  $c \in \text{Col}$  chosen from a set  $\text{Col}$  of *colors*. We will always assume that  $\text{Col}$  is finite. If  $\text{Col}$  has only one element, a common scenario, then one can ignore the color. A *tiling*  $\mathcal{T}$  of  $\mathbb{R}^d$  is a decomposition of  $\mathbb{R}^d$  as a union of tiles which only intersect along their faces. We write  $\mathbb{R}^d = \bigcup_j t_j$ , where each  $t_j$  is the *support of the tile*  $(t_j, c_j)$ , and  $c_j \in \text{Col}$  is the color of the tile. In a slight abuse of notation, we will often fail to distinguish between tiles and their supports, but this should not cause confusion from context. A *patch* of  $\mathcal{T}$  is the union of finitely many tiles. Two tiles  $t_1, t_2$  of a tiling  $\mathcal{T}$  are *translation-equivalent* if there exists a  $\tau \in \mathbb{R}^d$  such that  $t_1 = t_2 - \tau$ , i.e., they are translations of the same prototile, and their labels are equal. Two patches  $P_1, P_2$  of  $\mathcal{T}$  are translation equivalent if there is a  $\tau \in \mathbb{R}^d$  such that  $P_1 = P_2 - \tau$ . The translation  $\phi_\tau$  of tiles and patches by vectors  $\tau \in \mathbb{R}^d$  extends to all of a tiling  $\mathcal{T}$ , and we denote by  $\phi_\tau(\mathcal{T})$  the tiling consisting of the union of tiles of the form  $\phi_\tau(t)$ , where  $t$  is a tile of  $\mathcal{T}$ .

Let  $R_* > 0$  be the smallest number so that any ball of radius  $R_*$  in  $\mathbb{R}^d$  contains a tile of  $\mathcal{T}$ . For any  $R \geq R_*$ , an  $R$ -patch of  $\mathcal{T}$  is the largest patch of  $\mathcal{T}$  completely contained in a fixed ball of radius  $R$ . The

tiling  $\mathcal{T}$  has *finite local complexity* or *FLC* if, given any  $R > 0$ , the set of equivalence classes (under translation) of  $R$ -patches of the family of tilings  $\{\phi_t(\mathcal{T})\}_{t \in \mathbb{R}^d}$  is finite. All tilings considered in this paper will have FLC.

Given a tiling  $\mathcal{T}$ , define the metric  $d$  on the set of all translates  $\{\phi_t(\mathcal{T})\}_{t \in \mathbb{R}^d}$  as follows. Set

$$d'(\phi_t(\mathcal{T}), \phi_s(\mathcal{T})) = \inf\{\epsilon > 0 : \text{there is a } \tau \in B_\epsilon(0) \text{ such that the } \epsilon^{-1}\text{-patch of } \phi_{t+\tau}(\mathcal{T}) \text{ is equal to the } \epsilon^{-1}\text{-patch of } \phi_s(\mathcal{T})\}$$

and define

$$d(\phi_t(\mathcal{T}), \phi_s(\mathcal{T})) = \min\{1, d'(\phi_t(\mathcal{T}), \phi_s(\mathcal{T}))\}.$$

This is a metric [30]. The (*translational*) *tiling space*  $\Omega_{\mathcal{T}}$  of  $\mathcal{T}$  is the metric completion of the set of translates of  $\mathcal{T}$  with respect to the metric above, i.e.,

$$\Omega_{\mathcal{T}} := \overline{\{\phi_t(\mathcal{T}) : t \in \mathbb{R}^d\}}.$$

If  $\mathcal{T}$  is FLC, then  $\Omega_{\mathcal{T}}$  is a compact metric space, with local product structure  $B \times C$ , where  $B$  is a Euclidean ball and  $C$  is a zero-dimensional compact space (usually, but not always, a Cantor set). The translation of  $\mathcal{T}$  by vectors of  $\mathbb{R}^d$  extends to an action of  $\mathbb{R}^d$  by translations.

Let us mention two more topological spaces associated to  $\mathcal{T}$ , where one incorporates rotations in addition to translations, that is,  $\text{SE}(d) = \mathbb{R}^d \rtimes \text{SO}(d)$ . For details, see [27, §4.1–4.5]. The metric is first relaxed to allow  $\epsilon$ -rotations, then one can define

$$\Omega_{\mathcal{T}}^{\text{rot}} = \overline{\{\phi(\mathcal{T}) : \phi \in \text{SE}(d)\}}.$$

It has a canonical transversal

$$\Omega_{\mathcal{T}}^0 = \Omega_{\mathcal{T}}^{\text{rot}} / \text{SO}(d) \cong \Omega_{\mathcal{T}} / \{\text{rotational symmetries}\},$$

which we call the *rotational tiling space* of  $\mathcal{T}$ .

## 2.2 Substitution tilings

Let  $(t_1, c_1), \dots, (t_m, c_m) \subset \mathbb{R}^d \times \text{Col}$  be compact, connected sets with nonempty interior, where  $\text{Col}$  is a finite set. If  $A \subset \text{GL}(d, \mathbb{R})$  is an expanding matrix, then a *substitution rule on  $t_1, \dots, t_m$  with expansion  $A$*  is a decomposition of  $At_i$  as

$$At_i = \bigcup_{j=1}^m \bigcup_{\tau \in \Lambda_{ij}} t_j - \tau$$

for any  $i$ , where each  $\Lambda_{ij}$  is a finite set. The *substitution matrix* corresponding to this substitution rule is the doubly indexed collection of numbers  $S$  defined by  $S_{ij} = |\Lambda_{ij}|$ . The substitution is *primitive* if there is a  $k \in \mathbb{N}$  such that the substitution matrix  $S^k$  has all positive entries. Any substitution rule can be

iterated: the set  $A^2t_i$  can be tiled as

$$A^2t_i = \bigcup_{j=1}^m \bigcup_{\tau \in \Lambda_{ij}} At_j - A\tau,$$

where  $At_j$  can be decomposed as above. A tiling  $\mathcal{T}$  is given by a substitution rule with expansion  $A$  if any patch of  $\mathcal{T}$  is translation equivalent to a subpatch of  $A^k t_i$  for some  $k \in \mathbb{N}$ ,  $1 \leq i \leq m$ , where  $A^k t_i$  is a tiled set as above. In that case, a patch which is translation-equivalent to one of the form  $A^k t_i$  as above is a *level- $k$  supertile*.

Let  $\mathcal{T}$  be a FLC tiling with prototiles  $t_1, \dots, t_m$ . For a tile  $t \in \mathcal{T}$ , let  $\mathcal{T}(t)$  be the patch consisting of all tiles of  $\mathcal{T}$  which intersect  $t$ . This type of patch is called a *collared tile* of  $\mathcal{T}$ . Now consider the product  $\Omega_{\mathcal{T}} \times \mathbb{R}^d$  with the product topology, where  $\Omega_{\mathcal{T}}$  has the discrete topology and  $\mathbb{R}^d$  the traditional topology. Let  $\sim_1$  be the equivalence relation on this product which declares  $(\mathcal{T}_1, u_1) \sim_1 (\mathcal{T}_2, u_2)$  if  $\mathcal{T}_1(t_1) - u_1 = \mathcal{T}_2(t_2) - u_2$  for some tiles  $t_1, t_2$  with  $u_i \in t_i \in \mathcal{T}_i$ . The *Anderson–Putnam complex* of  $\Omega_{\mathcal{T}}$  is the set  $\Gamma_{\mathcal{T}} := (\Omega_{\mathcal{T}} \times \mathbb{R}^d) / \sim_1$ . Let  $\pi_1 : \Omega_{\mathcal{T}} \times \mathbb{R}^d \rightarrow \Gamma_{\mathcal{T}}$  be the quotient map. The space  $\Gamma_{\mathcal{T}}$  is a flat branched manifold.

If  $\mathcal{T}$  is a primitive substitution tiling then there exists a locally expanding affine map  $\gamma : \Gamma_{\mathcal{T}} \rightarrow \Gamma_{\mathcal{T}}$  such that there is a homeomorphism of tiling spaces

$$(1) \quad \Omega_{\mathcal{T}} \cong \varprojlim (\Gamma_{\mathcal{T}}, \gamma),$$

which is the seminal result of Anderson and Putnam [2]. In this case [30], the action of  $\mathbb{R}^d$  by translations is *uniquely ergodic*: there is a unique  $\mathbb{R}^d$ -invariant ergodic probability measure on  $\Omega$ , which will be denoted by  $\mu$ . By the local product structure of  $\Omega$ , this measure is locally of the form  $\text{Leb} \times \nu$ , where  $\text{Leb}$  is the Lebesgue measure on the local Euclidean component and  $\nu$  is a measure on the local Cantor component and will be discussed further in Section 2.4.

**Remark 2.1** The reader may wonder why  $\Omega_{\mathcal{T}}$  may not be recovered as the inverse limit obtained by the substitution rule directly on tiles and not necessarily on collared tiles, as required for the construction of the Anderson–Putnam complex  $\Gamma_{\mathcal{T}}$  above. Although an inverse limit would be well defined in such case, it may not necessarily be homeomorphic to the tiling space  $\Omega_{\mathcal{T}}$ . This happens if and only if the substitution rule *forces the border* (see [2] for a proof of this result). To quote [27], “A substitution rule *forces the border* if there exists a positive integer  $n$  such that any two level- $n$  supertiles of the same type have the same pattern of neighboring tiles”. Although the concept of forcing the border will not be central in this paper, it will be mentioned in a couple of the examples below, and we direct the reader to [2; 27] to learn why in that case one can use uncollared tiles to form the desired inverse limit.

**Definition 2.2** A *cubical substitution* in dimension  $d$  with  $m$  colors with expansion  $\lambda \in \mathbb{N}$  is a substitution rule on  $[0, 1]^d \times \{1, \dots, m\}$  with expansion matrix  $A = \lambda \cdot \text{Id}$ .

Note that there are  $m^{\lambda^d + 1}$  different cubical substitution rules in dimension  $d$  with  $m$  colors and expansion  $\lambda$ .

### 2.3 Topological invariants

Suppose that  $\mathcal{T}$  is a primitive substitution tiling. The Anderson–Putnam homeomorphism (1) allows us to express the cohomology and  $K$ -theory of  $\Omega = \Omega_{\mathcal{T}}$  in digestible terms:

$$(2) \quad \check{H}^*(\Omega; \mathbb{Z}) = \varinjlim(\check{H}^*(\Gamma; \mathbb{Z}), \gamma^*) \quad \text{and} \quad K^*(\Omega) = \varinjlim(K^*(\Gamma), \gamma^*).$$

Recall that the Chern character  $\text{ch} : K^*(\Omega) \rightarrow \check{H}^*(\Omega; \mathbb{Q})$  is defined by

$$(3) \quad \text{ch}([L]) = \sum_{k \geq 0} \frac{c_1([L])^k}{k!},$$

for  $L$  a line bundle on  $\Omega$  and  $[L]$  its class in  $K$ -theory, where  $c_1([L])$  is the first Chern class of  $[L]$ , and then extended to all virtual vector bundles using the “splitting principle” (which says that one can pretend all vector bundles split into direct sums of line bundles). This is a ring homomorphism, and after tensoring the domain with  $\mathbb{Q}$ , it becomes a rational isomorphism of rings. (Strictly speaking, we have only defined the Chern character on even-degree  $K$ -theory, with values in even-degree cohomology, but it extends by suspension to odd degrees as well.) This isomorphism in fact is derived from the isomorphism  $\text{ch} : K^*(\Gamma) \otimes \mathbb{Q} \xrightarrow{\cong} \check{H}^*(\Gamma; \mathbb{Q})$  at the AP complex level and (2). When the dimension of  $\Gamma$  is at least four, then because of the denominators in (3),  $\text{ch}$  does not necessarily come from an integral isomorphism between  $K^*(\Gamma)$  and  $H^*(\Gamma; \mathbb{Z})$ . (It does give such an isomorphism for spheres and tori — see, for example, [14, Proposition 4.3] — but not for  $\mathbb{C}\mathbb{P}^k$ ,  $k \geq 2$  [14, §4.1].)

In Section 4 we give an explicit example of how the Chern character can fail to be an integral isomorphism for tiling spaces. This gives a negative answer to a question of Benamèur and Mathai [9, p. 8].

### 2.4 Frequencies, traces and the gap-labeling conjecture

Suppose  $\mathcal{T}$  is a primitive substitution tiling and  $P \subset \mathcal{T}$  is a patch. Let us make some comments about patches.

First, the patch has a *frequency*: let  $\text{freq}_R(P, \mathcal{T})$  be the number of copies of  $P$  found (up to equivalence) as subpatches of  $\mathcal{T}$  which are completely contained in a ball of radius  $R$  around the origin. Since the tiling came from a primitive substitution, the limit  $R^{-d} \text{freq}_R(P, \mathcal{T}) \rightarrow \text{freq}(P, \mathcal{T}) \geq 0$  exists as  $R \rightarrow \infty$ , and it is called the *patch frequency* of  $P$ . Since there is a unique  $\mathbb{R}^d$ -invariant measure on  $\Omega_{\mathcal{T}}$ , this limit is actually independent of  $\mathcal{T}$ , meaning that the patch frequency of the patch  $P$  is the same for any tiling in  $\Omega$ . This will be denoted by  $\text{freq}(P)$ .

Secondly, if  $P$  is a patch, then  $P$  is given by a closed subset  $\bar{P} \subset \Gamma$  and a  $k \geq 0$  in that the set  $\pi_k^{-1}(\bar{P}) \subset \Omega$  is isometric to  $P \times \mathcal{C}_P$ , where  $\mathcal{C}_P$  is a (transversal) Cantor set. It follows from Birkhoff’s ergodic theorem that  $\nu(\mathcal{C}_P) = \text{freq}(P)$ , and for this reason  $\nu$  is sometimes referred to as the *frequency measure*.

Finally, if  $P$  is a patch, then it defines a cohomology class  $[P] \in \check{H}^d(\Omega; \mathbb{Z})$  as follows. Recall that there exists a closed set  $\bar{P} \subset \Gamma$  such that  $\pi_k^{-1}(\bar{P}) \subset \Omega$  is isometric to  $P \times \mathcal{C}_P$ . Then the interior of  $\bar{P}$  in  $\Gamma$  is an

open set with a Čech cohomology class  $[\bar{P}] \in \check{H}^d(\Gamma; \mathbb{Z})$  which by (2) defines a class  $[P] \in \check{H}^d(\Omega; \mathbb{Z})$ . The map  $[P] \mapsto \text{freq}(P) = \nu(C_P)$  turns out to give a homomorphism  $C_\mu : \check{H}^d(\Omega; \mathbb{Z}) \rightarrow \mathbb{R}$  called the *Schwartzman asymptotic cycle* or the *Ruelle–Sullivan current* [18]. The image  $C_\mu(\check{H}^d(\Omega; \mathbb{Z})) \subset \mathbb{R}$  is called the *frequency module*.

Associated with any substitution tiling space there are several  $C^*$ -algebras that one can define (see [17] for details). The principal one is  $\mathcal{A}_p(\Omega)$ , the *unstable (punctured) algebra*. Another important one is  $\mathcal{A}_{\text{AF}}(\Omega)$ , the AF-algebra constructed from the substitution matrix defining the tiling space  $\Omega$ . Since the substitution is assumed to be primitive, it comes with a unique tracial state  $\tau_{\text{AF}} : K_0(\mathcal{A}_{\text{AF}}(\Omega)) \rightarrow \mathbb{R}$ . For a primitive substitution tiling, there is also a unique tracial state  $\tau : K_0(\mathcal{A}_p(\Omega)) \rightarrow \mathbb{R}$  which can be recovered from the unique  $\mathbb{R}^d$ -invariant probability measure  $\mu$  on the tiling space  $\Omega$ .

How the images  $C_\mu(\check{H}^d(\Omega; \mathbb{Z}))$  and  $\tau(K_0(\mathcal{A}_p(\Omega)))$  overlap is a delicate matter. The conjecture that they are equal was proposed by Bellissard [7], and went under the name of the *gap-labeling conjecture*. There have been three papers claiming to have proved this equality [8; 10; 16]; we now briefly review what is at stake. As pointed out in [1], several of these papers invoke arguments which rely on the commutative diagram

$$(4) \quad \begin{array}{ccc} K_0(\mathcal{A}_p(\Omega)) \otimes \mathbb{Q} & \xrightarrow{\text{ch}_d \circ \chi^{-1}} & \check{H}^d(\Omega; \mathbb{Q}) \\ \downarrow \tau & & \downarrow C_\mu \\ \mathbb{R} & \xrightarrow{\text{Id}} & \mathbb{R} \end{array}$$

where  $\chi : K^{-d}(\Omega) \rightarrow K_0(\mathcal{A}_p(\Omega))$  is an isomorphism,  $\text{ch} : K^*(\Omega) \otimes \mathbb{Q} \rightarrow \check{H}^*(\Omega; \mathbb{Z})$  is the isomorphism given by the Chern character in (3), and  $\text{ch}_d$  is its projection to  $\check{H}^d(\Omega; \mathbb{Q})$ . The arguments used in several papers have the gap that they assume that  $\text{ch}_d$  factors through  $\check{H}^d(\Omega; \mathbb{Z})$  to give the diagram

$$(5) \quad \begin{array}{ccc} K_0(\mathcal{A}_p(\Omega)) & \xrightarrow{\text{ch}_d \circ \chi^{-1}} & \check{H}^d(\Omega; \mathbb{Z}) \\ \downarrow \tau & & \downarrow C_\mu \\ \mathbb{R} & \xrightarrow{\text{Id}} & \mathbb{R} \end{array}$$

Although this may hold in low dimensions, in Section 4 we give an explicit example where this is not true, and thus show that any argument to prove the gap-labeling conjecture using this isomorphism is incomplete.

### 2.5 Equivariant gap labeling

In this subsection we mention an equivariant variant of the gap-labeling conjecture which is stronger, hence more likely to fail. In fact, we produce a counterexample, though we still show that the conjecture is true in low dimensions. The reason for mentioning this equivariant conjecture here is that our methods might be usable in some other cases to disprove this equivariant conjecture, even if they do not disprove the original conjecture. Suppose  $\mathcal{T}$  is a primitive substitution tiling as before, and furthermore assume that

there is a finite group  $G$  of symmetries that acts on the associated tiling space  $\Omega$ . (Equivariant  $K$ -theory works just as well with  $G$  compact Lie, but that case isn't relevant for substitution tilings.) This group  $G$  will of course act on  $\mathcal{A}_p(\Omega)$  and preserve both the unique invariant measure on  $\Omega$  and the unique tracial state  $\tau : K_0(\mathcal{A}_p(\Omega)) \rightarrow \mathbb{R}$ . We have a natural inclusion map on *equivariant  $K$ -theory*

$$K_G^0(\Omega) = K_0^G(C(\Omega)) \rightarrow K_0^G(\mathcal{A}_p(\Omega)).$$

The *equivariant gap-labeling conjecture* asserts that this map induces an isomorphism on all trace invariants. But there are far more of these than in the nonequivariant case. To explain this, recall that equivariant  $K$ -theory is a module over  $R(G)$ , the representation ring of  $G$ , which after tensoring with  $\mathbb{C}$  can be identified with the ring of linear combinations of characters (of irreducible representations of  $G$ ), or with the algebra  $Z(G)$  of class functions on  $G$  (this is the center of the group ring  $\mathbb{C}G$ ). If  $A$  is a (unital, for simplicity)  $C^*$ -algebra equipped with an action of  $G$ , then  $K_0^G(A)$  is the Grothendieck group of equivalence classes of  $G$ -invariant self-adjoint projections  $e \in \text{End}(V) \otimes A$ ,  $(V, \pi)$  a finite-dimensional unitary representation space for  $G$  [11, §11.3]. If  $\tau$  is a  $G$ -invariant trace on  $A$ , it defines a map  $K_0^G(A) \rightarrow Z(G) \cong R(G) \otimes \mathbb{C}$  via  $[e] \mapsto (g \mapsto \tau^V((\pi(g) \otimes 1_A)e))$ , where  $\tau^V$  is the trace induced by  $\tau$  on  $\text{End}(V) \otimes A$ . Another way of thinking about this is to observe that  $R(G) \otimes \mathbb{C}$  is a semisimple ring (a direct sum of copies of  $\mathbb{C}$ , one for each conjugacy class of  $G$ ) and so  $K_0^G(A) \otimes \mathbb{C}$ , as a module over  $R(G) \otimes \mathbb{C}$ , is a direct sum of  $\mathbb{C}$ -vector spaces indexed by the conjugacy classes of  $G$ . The  $\tau$  then gives a trace on each summand. For example, if  $A = \mathbb{C}$  with  $\tau : A \rightarrow \mathbb{C}$  the identity map,  $K_0^G(A) \cong R(G)$ , and for a finite-dimensional representation  $(V, \pi)$  of  $G$ , the class  $[V] \in R(G)$  is represented by  $e = 1_V \in V$ , and  $(\pi(g) \otimes 1_A)e = \pi(g)$ , whose trace under  $\tau^V$  is just the character of  $V$ . As another example, if  $X = G/H$  is a transitive  $G$ -space, then  $K_G^0(X) = R(H)$  (with  $R(G)$ -module structure via the restriction map  $R(G) \rightarrow R(H)$ ). If  $\tau$  is counting measure on  $G/H$  (of total mass  $[G : H]$ ) and if  $(V, \pi)$  is a finite-dimensional representation of  $H$ , consider the induced  $G$ -vector bundle  $E = G \times_H V$ , which corresponds to a finitely generated projective  $G$ - $C(X)$ -module, with a class in  $K_G^0(X)$ . When  $G$  is abelian, the trace  $\tau^V$  sends this to the function in  $Z(G)$  given by the character of  $\pi$  on  $H$  extended to be 0 off of  $H$ . When  $G$  is not abelian, one gets instead the character of the induced representation  $\text{Ind}_{H \uparrow G}(\pi)$ , which is supported on the union of the conjugates of  $H$ . For example, if  $G = S_3$ ,  $H = A_3$  (a cyclic group of order 3), and  $\pi$  is a nontrivial character of  $H$ , then  $\text{Ind}_{H \uparrow G}(\pi)$  is an irreducible representation with character 2 at the identity,  $-1$  on the 3-cycles, 0 on the 2-cycles.

Now recall some ideas from [3; 29]. First of all, by [29, Proposition 3.7], every prime ideal  $\mathfrak{p}$  of  $R(G)$  has a *support*, which is a cyclic subgroup  $H$  of  $G$  determined up to conjugation in  $G$ . This is the smallest subgroup  $H < G$  such that  $\mathfrak{p}$  is the inverse image of a prime ideal in  $R(H)$  under the restriction map  $R(G) \rightarrow R(H)$ . Segal's localization theorem [3, Theorem 1.1; 28, Proposition 4.1] says that for a compact  $G$ -space such as  $\Omega$ , the localization of  $K_G^0(\Omega)$  at  $\mathfrak{p}$  has the property that  $K_G^0(\Omega)_{\mathfrak{p}} \rightarrow K_G^0(\Omega^{(H)})_{\mathfrak{p}}$  is an isomorphism. Here  $\Omega^{(H)} = G \cdot \Omega^H$  denotes the  $G$ -saturation of the subset fixed by  $H$ . (When  $G$  is abelian, this is equal to  $\Omega^H$ .) In particular, if  $\mathfrak{p}$  is the prime ideal of characters that vanish at a point  $g \in G$  (if we choose  $g = 1$ , then this  $\mathfrak{p}$  is just the augmentation ideal of  $R(G)$ ), then the support of  $\mathfrak{p}$  is the cyclic

subgroup generated by  $g$ , and if we invert characters that don't vanish at  $g$ , which is harmless if we are evaluating characters at  $g$ , then we can restrict to  $G \cdot \Omega^g$ . The trace  $\tau$  then gives a linear functional defined on  $K_G^0(G \cdot \Omega^g)_p \cong K_G^0(\Omega)_p$ , and by varying  $g$ , we get a different functional for each conjugacy class of cyclic subgroups  $\langle g \rangle$ . There are similar functionals defined on  $K_0^G(\mathcal{A}_p(\Omega))$ , after localizing at these prime ideals (one for each element of the group). These amount to the same as the map to  $Z(G)$  defined before, followed by evaluation at a group element (or conjugacy class, in the noncommutative case).

**Conjecture 2.3** (equivariant gap-labeling conjecture) *Suppose  $\mathcal{T}$  is a primitive substitution tiling as before, and further assume that there is a finite group  $G$  of symmetries that acts on the associated tiling space  $\Omega$ . This group will of course preserve the unique tracial state  $\tau : K_0(\mathcal{A}_p(\Omega)) \rightarrow \mathbb{R}$ . But  $G$  also acts on  $\mathcal{A}_p(\Omega)$ . Then the image of  $K_G^0(\Omega)$  in  $Z(G)$  under  $\tau$  coincides with the image of  $K_0^G(\mathcal{A}_p(\Omega))$ .*

Let's examine a special case to see what this actually says.

**Example 2.4** Let  $G = \{1, -1\}$  be cyclic of order 2. Then  $R(G) = \mathbb{Z}[\chi]/(\chi^2 - 1)$ , where  $\chi$  is the sign character. After inverting 2, this splits as  $\mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{2}]$ , with the two factors corresponding to the trivial representation 1 and to  $\chi$ . (For a summary of the structure of  $R(G)$ , including the complete prime ideal structure, see [25, §2].) If  $A$  is a unital  $G$ - $C^*$ -algebra with a  $G$ -invariant trace  $\tau$ , the map  $K_0^G(A) \rightarrow Z(G)$  can be viewed as the pair consisting of the usual trace,  $\tau : K_0(A) \rightarrow \mathbb{R}$ , along with the restriction of  $\tau$  to the fixed-point algebra, giving a map  $K_0(A^G) \rightarrow \mathbb{R}$ . (This is because  $R(G)$  has exactly two minimal prime ideals, the augmentation ideal, supported at  $\{1\}$ , and the kernel of  $\chi$ , supported on all of  $G$ . The remaining prime ideals are all maximal ideals with finite residue field.) So the equivariant gap-labeling conjecture amounts to the usual gap-labeling conjecture *plus* the assertion that the image of the trace on the  $G$ -invariant subalgebra  $\mathcal{A}_p(\Omega)^G$  coincides with the image of the trace on  $K_0(C(\Omega)^G) = K^0(\Omega/G)$ .

**Remark 2.5** To the best of our knowledge, the equivariant gap-labeling conjecture hasn't been formulated in this way before. But versions of it appear in [23; 24; 32; 33]. Those sources don't deal with equivariant  $K$ -theory in full generality but deal either with the  $K$ -theory of the crossed product  $\mathcal{A}_p(\Omega) \rtimes G$  (which is the same as the equivariant  $K$ -theory by the Green–Julg theorem — see [11, §11.7]) or else with the  $K$ -theory of the fixed-point algebra  $\mathcal{A}_p(\Omega)^G$  (which corresponds to the part of the equivariant  $K$ -theory attached to the trivial representation).

## 2.6 The equivariant Chern character

We can study the equivariant gap-labeling conjecture using the equivariant Chern character, just as the usual Chern character is used to study the usual gap-labeling. The equivariant Chern character (say for the action of a finite group  $G$  on a  $G$ -CW complex like the AP complex) is defined in [20], and gives a ring homomorphism (that becomes an isomorphism after tensoring the domain with  $\mathbb{Q}$ )  $\text{ch}_G : K_G^*(X) \rightarrow H_G^*(X; R(-) \otimes \mathbb{Q})$ . Here the right-hand side is *Bredon cohomology* [12] of the functor  $R(-) \otimes \mathbb{Q} : \text{Or}(G)^{\text{op}} \rightarrow \mathbb{Q}\text{-Mod}$ , where  $\text{Or}(G)$  is the orbit category with objects the transitive  $G$ -spaces and morphisms the  $G$ -equivariant maps, defined by sending  $G/H$  to  $R(H) \otimes \mathbb{Q}$ . In the special case where  $X$  has a single orbit type, say  $X = (G/H) \times Y$  with  $G$  acting trivially on  $Y$ , we have

$K_G^*(X) = R(H) \otimes K^*(Y)$ , and the equivariant Chern character is just the usual Chern character for  $Y$ , tensored with  $R(H)$ .

Now to prove the equivariant gap-labeling conjecture for a particular tiling space  $\Omega$ , one can try to replicate the strategy based on the diagram (4). The difference would be that we use equivariant  $K$ -theory and cohomology, and the map  $\tau$  to  $\mathbb{R}$  is now replaced by the map  $\tau^g$  induced by  $\tau$  on equivariant  $K$ -theory/cohomology, along with evaluation of (virtual) characters at an element  $g \in G$ . For simplicity let's take  $G$  to be abelian. Fix  $g \in G$  and apply the Segal localization theorem with  $\mathfrak{p} = \mathfrak{p}_g$  the prime ideal of  $R(G)$  consisting of virtual representations whose characters vanish at  $g$ . The support of  $\mathfrak{p}$  is the cyclic subgroup  $H = \langle g \rangle$ , so after localizing at  $\mathfrak{p}$  (in other words, inverting characters that do not vanish at  $g$ ), we can replace  $\Omega$  by  $\Omega^g$ . So diagram (4) becomes

$$(6) \quad \begin{array}{ccc} K_0^G(\mathcal{A}_{\mathfrak{p}}(\Omega))_{\mathfrak{p}} & \xrightarrow{\text{ch}_d^G \circ \chi^{-1}} & \check{H}_G^d(\Omega^g; R(-)_{\mathfrak{p}} \otimes \mathbb{Q}) \\ \downarrow \tau^g & & \downarrow C_{\mu}^g \\ \mathbb{C} & \xrightarrow{\text{Id}} & \mathbb{C} \end{array}$$

This reduces to (4) if  $g = 1$ , but in general, if  $g \neq 1$ , we need to use  $\mathbb{C}$  as the target instead of  $\mathbb{R}$  since  $G$  can have characters with nonreal values at  $g$ . (This already happens for  $G$  cyclic of order  $> 2$ .) The restriction of  $\tau^g$  to  $K_0^G(C(\Omega)) = K_G^0(\Omega)$  via the inclusion of  $C(\Omega)$  into  $\mathcal{A}_{\mathfrak{p}}(\Omega)$  can be computed via a  $G$ -invariant transversal  $N$  to the  $\mathbb{R}$ -orbits in  $\Omega$ . Since  $N$  is totally disconnected (i.e., 0-dimensional), there are no denominators in the equivariant Chern character for  $N$ , which becomes an *integral* isomorphism. So just as in [1, Remark 9.7], we have  $K_0^G(\mathcal{A}_{\mathfrak{p}}(\Omega)) \cong K_0^G(C(N) \rtimes \mathbb{Z}^d)$  and we want to relate this to  $K_0^G(C(N)) = K_G^0(N) \cong \check{H}_G^0(N; R(-))$ . The restriction of  $\tau^g$  is  $\mathbb{Z}^d$ -invariant, so it factors through  $H_G^0(N; R(-))_{\mathbb{Z}^d} \cong H_G^d(\Omega; R(-))$ . So the equivariant gap-labeling conjecture reduces to the assertion that for each  $g \in G$ , the image of  $\tau^g$  on  $K_0^G(\mathcal{A}_{\mathfrak{p}}(\Omega))$  in (6) coincides (integrally!) with the image of  $H_G^d(\Omega^g; R(-))$ . Just as in the nonequivariant case, the conjecture follows from diagram (6) in dimensions up to 3 when there are no denominators in the Chern character.

**Theorem 2.6** *The equivariant gap-labeling conjecture, Conjecture 2.3, holds when the dimension  $d$  is  $\leq 3$ . However, it **fails** for a periodic lattice tiling with  $\Omega = \mathbb{T}^4$  (the 4-torus) and  $G$  the cyclic group of order 2 interchanging the two factors in  $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ .*

**Proof** We have already explained the proof when  $d \leq 3$ . It remains to show why the conjecture fails for  $\Omega = \mathbb{T}^2 \times \mathbb{T}^2$  and  $G$  the cyclic group of order 2 interchanging the two factors. By Example 2.4, it suffices to show that the image of the trace on  $H^4(\Omega/G; \mathbb{Z}) = H^4(\text{SP}^2(\mathbb{T}^2); \mathbb{Z})$ ,  $\text{SP}^2(\mathbb{T}^2)$  the second symmetric product of  $\mathbb{T}^2$ , is strictly smaller than the image of the trace on the fixed-point algebra  $\mathcal{A}_{\mathfrak{p}}(\Omega)^G$ . Via the analogue of (4), the issue is to show that the top-degree Chern character on  $K^0(\text{SP}^2(\mathbb{T}^2))$  has image which is strictly bigger than  $H^4(\text{SP}^2(\mathbb{T}^2); \mathbb{Z})$ . For this we can apply [21, (6.30) and (7.1)], which computes the structure of the integral cohomology ring of  $\text{SP}^2(\mathbb{T}^2)$ . (In the language of [21], this is the case where  $X$  is a curve of genus  $g = 1$  and  $n = 2$ .) Macdonald shows that the cohomology ring

is torsion-free and is generated over  $\mathbb{Z}$  by generators  $\xi_1, \xi_2$  of degree 1 (which anticommute with each other and with themselves) and a generator  $\eta$  of degree 2, commuting with  $\xi_1$  and  $\xi_2$ , with the one relation  $\eta^2 = \eta\xi_1\xi_2 =$  generator of  $H^4(\mathrm{SP}^2(\mathbb{T}^2); \mathbb{Z})$ . If  $L$  is the complex line bundle with  $c_1(L) = \eta$ , then  $\mathrm{ch}([L]) = 1 + \eta + \frac{1}{2}\eta^2$ , which is *not* integral, and its projection into  $H^4(\mathrm{SP}^2(\mathbb{T}^2); \mathbb{Q})$  generates an infinite cyclic group which properly contains  $H^4(\mathrm{SP}^2(\mathbb{T}^2); \mathbb{Z})$  (with index 2).  $\square$

An example of an aperiodic tiling space with a  $\mathbb{Z}_2$ -action which is also a counterexample is described in Section 4.3. We then have the following corollary as an immediate consequence of this counterexample.

**Corollary 2.7** *The gap-labeling conjecture fails for rotational tiling spaces when the dimension  $d$  is  $\geq 4$ .*

While patch frequencies are well defined for rotational tiling spaces, thus one can compute the associated frequency module, it is unclear if there is a Schrödinger operator whose gaps in the spectra are labeled by the  $K_0$ -group of the associated  $C^*$ -algebra.

**Remark 2.8** We should mention that there seem to be different ways to formulate the equivariant conjecture, and at first sight they don't seem to be the same. Let us illustrate this with the case of Example 2.4, a  $\mathbb{Z}_2$ -action. If  $X$  is, say, a Cantor set with a  $G$ -action, where  $G = \mathbb{Z}_2$ , equipped with a  $G$ -invariant measure  $\mu$ , then  $A = C(X)$  is generated by projections  $\chi_Y$  (corresponding to clopen subsets  $Y \subseteq X$ ), whose classes also generate  $K^0(X)$ , and the trace  $\tau$  defined by  $\mu$  sends  $\chi_Y$  to  $\mu(Y)$ . In Example 2.4, we said that the equivariant conjecture amounts to looking at  $\tau$  on  $A$  along with the induced trace on  $A^G = C(X/G)$ . If  $X^G = Z$ , then  $X$  is the union of  $Z$  and the free  $G$ -space  $X \setminus Z$ , so the (unrenormalized) trace on  $C(X/G)$  has total mass  $\mu(Z) + \frac{1}{2}\mu(X \setminus Z)$ . The formulation using (6) amounts to looking instead at the pair consisting of  $\mu$  on  $X$  and  $\mu$  restricted to  $X^G = Z$ . Knowing these, one can compute  $\mu(X \setminus Z)$ , and thus get the same set of invariants. However, because of the factor of 2 that appears in the calculation, one has to be careful in looking at the *integral* range of the trace. The reduction to fixed-point sets in (6) appeals to the Segal localization theorem, which requires inverting elements of  $R(G)$  not in the augmentation ideal  $I$ . This involves inverting 2, so there is no contradiction in relating the two versions of the equivariant conjecture, the one using the quotient space and the one using the fixed-point set.

### 3 Cubical substitutions and the cup product

In this section we describe our approach to compute the cohomology rings of tiling spaces given by cubical substitutions. With an eye of doing computations in dimension four in Section 4, we work out examples in dimension two where we explicitly compute the cohomology ring structure and other invariants. We will first start describing in detail how the type of collaring scheme used to compute the cohomology groups of cubical substitutions and the ring structure is computed by following [15]. Table 1 summarizes the properties of the different examples.<sup>1</sup>

<sup>1</sup>The table lists groups as  $\mathbb{Z}[1/16]$ ,  $\mathbb{Z}[1/8]$ ,  $\mathbb{Z}[1/4]$  and  $\mathbb{Z}[1/2]$ , even though these can all be written as  $\mathbb{Z}[1/2]$ . Following [27, §3.5], we write them like this to emphasize their different scalings under the substitution rule.

example	cohomology groups	cup product	frequency module
Example 3.2	$\check{H}^2(\Omega) = \mathbb{Z}[1/25] \oplus \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3],$ $\check{H}^1(\Omega) = \mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z},$ $\check{H}^0(\Omega) = \mathbb{Z}$	surjective	$\frac{1}{6}\mathbb{Z}[1/5]$
Example 3.3	$\check{H}^2(\Omega) = \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/8] \oplus \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2],$ $\check{H}^1(\Omega) = \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z},$ $\check{H}^0(\Omega) = \mathbb{Z}$	surjective	$\frac{1}{3}\mathbb{Z}[1/2]$
Example 3.4	see Example 3.4 for details	nonsurjective	$\frac{1}{84}\mathbb{Z}[1/5]$
Example 3.5	$\check{H}^2(\Omega) = \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/8] \oplus \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2],$ $\check{H}^1(\Omega) = \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z},$ $\check{H}^0(\Omega) = \mathbb{Z}$	nonsurjective	$\frac{1}{3}\mathbb{Z}[1/2]$

Table 1: Summary of Examples 3.2–3.5.

The four examples in this section are meant as two pairs of examples to be compared. Of particular note are Examples 3.3 and 3.5. These are two tiling spaces with isomorphic cohomology groups over  $\mathbb{Z}$  and the same frequency module. Without the cup product, they would be indistinguishable. However, the cup product  $\check{H}^1 \times \check{H}^1 \rightarrow \check{H}^2$  in one case is surjective, whereas it is not in the other. This is the first example we are aware of where the cup product is the only distinguishing feature between two tiling spaces. Examples 3.2 and 3.4 have the feature that they have indistinguishable cohomology spaces over  $\mathbb{Q}$ , but can be distinguished either through their cohomology groups over  $\mathbb{Z}$  or through the ring structure.

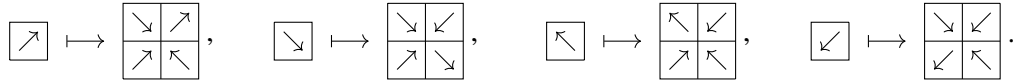
### 3.1 Computational setup

Let us first establish the notation and conventions.

Recall that a cubical substitution rule  $\zeta$  on  $m$  prototiles with (uniform) expansion matrix  $\lambda = \lambda \cdot \text{Id}_d$  has the set of prototiles  $[0, 1]^d \times \{0, \dots, m - 1\}$ . (I.e., the set of colors is  $\text{Col} = \{0, \dots, m - 1\}$ .) We will name each prototile by its second coordinate, its *label*. In addition, recall that each tile is a labeled region  $t + [0, 1]^d \subseteq \mathbb{R}^d$ . We interpret this as being formed by anchoring the point  $[0]^d \in [0, 1]^d$ , its *puncture*, in the corresponding prototile to  $t \in \mathbb{R}^d$ . We follow the convention of [15] where  $[x]$  denotes a (degenerate) interval consisting of the single point  $x$ .

We decompose the substitution rule  $\zeta$  into the following two steps: the first, denoted by  $\lambda$ , inflates each tile by  $\lambda$ ; the second, denoted by  $\sigma$ , subdivides the resulting region into tiles. If the initial tile is of the form  $(t, k)$  with  $t \in \mathbb{Z}^d$ , the resulting substituted patch will have each of the punctures of each of its tiles attached to  $\mathbb{Z}^d$ . Thus, let us put the lexicographic order on  $\{0, \dots, \lambda - 1\}^d$ , and using this order we abbreviate the substitution rule as  $\zeta(k) = (k_0, \dots, k_{\lambda^d - 1})$ , where  $0 \leq k_i \leq m - 1$  is a prototile with its puncture attached at the coordinate that is the  $i$ -th entry in  $(0, \dots, \lambda - 1)^d$ . We illustrate this convention through a well-known example.

**Example 3.1** (chair tiling) The standard chair tiling can be decomposed into squares, yielding the MLD-equivalent cubical substitution rule in  $d = 2$ , as illustrated in



Labeling the prototiles  $\{0, 1, 2, 3\} = \{\boxtimes, \boxminus, \boxplus, \boxtimes\}$ , we abbreviate this substitution rule as

$$\zeta(0) = (0, 1, 2, 0), \quad \zeta(1) = (0, 1, 1, 3), \quad \zeta(2) = (0, 2, 2, 3), \quad \zeta(3) = (3, 1, 2, 3),$$

or further abbreviate it as a list in Sage as

```
[ [0, 1, 2, 0], [0, 1, 1, 3], [0, 2, 2, 3], [3, 1, 2, 3] ]
```

For arbitrary dimension  $d > 1$ , the  $d$ -dimensional chair substitution rule is, in Sage,

```
[ [(2**d-1)-j if i==(2**d-1)-j else j for j in list(range(2**d))]
    for i in list(range(2**d))] ]
```

One immediately encounters an obstacle that is very difficult to overcome when trying to construct the AP-inverse limit and compute its cohomology. The top-dimensional cells in the complex are collared prototiles, or, for cubical substitutions, patches of size  $3^d$ , and the codimension-1 cells are obtained from adjacency information of pairs of collared tiles. The latter requires us to find all possible patches of size  $4^d$  in  $\mathcal{T}$ . In higher dimensions, e.g.,  $d = 4$ , this becomes unwieldy, since they are proportional to  $m^{4d}$ . In addition, it may require multiple substitutions before the supertiles of that level observe all such patches of size  $4^d$ . In higher dimensions, these severely restrict the substitution rules we can check. Thus, rather than the standard collaring procedure yielding the AP-complex and inverse limit, let us consider an alternative, the dual complex, introduced in [13], that is a variant of the Barge–Diamond (BD) complex from [5] for  $d = 1$  and [6] for  $d > 1$ .

For motivation, let us first consider an alternate point of view of how the AP-complex is formed. Suppose that we are given the set of all collared prototiles, viewed as patches in  $\mathcal{T}$ . They form the set of top-dimensional cells. The set of codimension-1 cells consists of unions of pairs of neighboring patches, viewed as intersections of their associated cylinder sets. Similarly, the set of codimension- $k$  cells consists of unions of patches that share a common intersection. That is, the AP-complex is the Čech complex associated to collared prototiles.

For cubical complexes, the top-dimensional cells are patches (up to translation) of size  $3^d$ , and the codimension- $k$  cells are unions of  $2^k$  patches that result in patches of  $\mathcal{T}$  and share exactly  $2^k 3^{d-k}$  tiles (so that the center tiles neighbor along a codimension- $k$  cell; analogously, so that the differences are patches of size 1 along  $k$  directions,  $3^{d-k}$  along the remaining  $d - k$  directions). Repeating this process for each level of supertiles yields the AP-inverse limit. From the Čech perspective, the fact that this inverse limit is homeomorphic to  $\Omega_{\mathcal{T}}$  is because the basic open sets of each (patches of collared supertiles for the AP point of view compared to arbitrary patches) are mutual refinements.

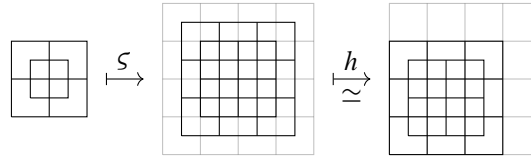


Figure 1: A substitution rule with even expansion applied to a patch of size  $2^d$  representing a top-dimensional cell (center tile shaded) returns a patch (light gray lines) whose center tiles (shaded) has collars (black lines) that are not actual patches, therefore is not cellular. One needs to compose the substitution with a homotopy  $h$  to produce a cellular map.

Rather than working with patches of size  $3^d$ , one can use patches of size  $2^d$ , then carry through the Čech construction. The resulting complex, due to [13], is the *dual complex*. More precisely, at the initial level of the inverse limit, the codimension- $k$  cells are unions of  $2^k$  patches that share exactly  $1^k 2^{d-k}$  tiles. To see that this is collaring of sorts, if a  $d$ -cell has the puncture of its first tile (in lexicographic order) situated at the origin, one pretends that the “center” tile (the actual  $d$ -cell) is the unit cube with its puncture at  $[1/2]^d$ .

To construct the complex at the supertile level, we first note that if  $\lambda$  is even, the substitution rule is not a cellular map (Figure 1). Assuming that each patch of size  $2^d$  has the puncture of its first tile located at the origin, we define a supertile to be the collection of all subpatches of size  $2^d$  (they will overlap) formed from tiles of the actual supertile with punctures located at  $(0, \dots, \lambda)^d$ . The substitution rule then takes a supertile of the dual complex and sends it to all subpatches of size  $2^d$ . Higher-level supertiles are analogous.

We observe that if we focus on the “center” tiles (and treat the surrounding patches of size  $2^d$  as decorations), this map is the ordinary substitution rule followed by a fixed homotopy, therefore the maps of the resulting inverse limit are stationary up to homotopy. Furthermore, the supertiles of this construction and the supertiles of the Anderson–Putnam construction are cofinal refinements of each other. This inverse limit is thus shape equivalent to  $\Omega_{\mathcal{T}}$ , and has the same Čech cohomology groups as those of  $\Omega_{\mathcal{T}}$  [22, Theorem 16].<sup>2</sup>

For all of the computations that are performed directly using Sage, we will use the dual complex and the resulting inverse limit.

### 3.2 Distinguishing tiling spaces through cup product

As an application of the ring structure of cohomology on cubical substitutions, let us describe pairs of tiling spaces in  $d = 2$  whose cohomology groups coincide, but the ring structure differs. We consider  $d = 2$ , since the required computations are much easier to visualize and interpret. These examples are inspired by [31, Example 6.7] due to the induced substitution matrix on  $\check{H}^1$  containing two eigenvalues that do not come from the expansion.

We will consider two classes of examples. In all of the complexes that we construct, we take as positive left-to-right and down-to-up orientations of 1-cells, and right-handed orientation of 2-cells.

<sup>2</sup>We are indebted to the referee for this observation.

**Example 3.2** (expansion 5 product) Let us consider the following two one-dimensional substitutions on two and three prototiles, respectively:

$$\zeta_1 = [[0, 0, 0, 0, 1], [1, 0, 0, 0, 0]]$$

$$\zeta_2 = [[0, 2, 1, 2, 0], [0, 1, 1, 1, 0], [0, 2, 2, 2, 0]]$$

One easily checks that both substitutions are primitive and recognizable.

To compute the first cohomology group, we use the Barge–Diamond filtration from [5], where one successively blows up lower-dimensional skeleta to become  $d$ -dimensional (here  $d = 1$ ) complexes. We will do this in detail for  $\zeta_1$ , then just state the main parts of the computation for  $\zeta_2$ .

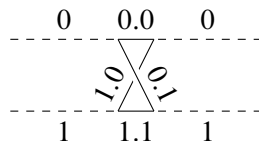
Let  $\epsilon > 0$ . We replace the prototiles themselves, assumed to be of length 1, by tiles of length  $1 - 2\epsilon$ , denoted using the original symbols, and replace the vertices that join two neighboring tiles by new tiles of length  $2\epsilon$  (called *vertex flaps* in [6]; more generally,  $k$ -flaps for new tiles of size  $1 - 2\epsilon$  in  $k$  directions,  $2\epsilon$  in the remaining  $d - k$  directions, which are blow ups of  $k$ -cells in the tiling). These new prototiles of length  $2\epsilon$  arise from the set of patches of length 2, denoted by 0.0, 0.1, 1.0, and 1.1 for this example. This new tiling is no longer self-similar, thus let us construct a homotopy that maintains the same self-similar structure as the original tiling.

To maintain the same combinatorial structure as the original substitution rule, we will require that the 0-flaps be nonexpanding, and the 1-flaps (the slightly shrunken prototiles) be fully expanding (more generally, for cubical substitutions,  $k$ -flaps are expanding in  $k$  directions, nonexpanding in the remaining  $d - k$  directions). More precisely, if  $\lambda$  is the expansion of the original substitution rule, in dimension 1, the 0-flaps are substituted, via expansion by  $\lambda$  then a homotopy, to prototiles of size  $2\epsilon$  (again 0-flaps), and the 1-flaps are sent, via expansion by  $\lambda$  then a homotopy, to patches of size  $\lambda - 2\epsilon$  (a combination of 0- and 1-flaps).

For  $\zeta_1$ , we have that the induced substitution rule, still denoted by  $\zeta_1$ , is

$$\begin{array}{c} \overline{0} \quad \mapsto \quad \overline{0} \quad \overline{0.0} \quad \overline{0} \quad \overline{0.1} \quad \overline{1} \\ \overline{1} \quad \mapsto \quad \overline{1} \quad \overline{1.0} \quad \overline{0} \quad \overline{0.0} \quad \overline{0} \\ \overline{0.0} \quad \mapsto \quad \overline{1.0} \quad , \quad \overline{1.0} \quad \mapsto \quad \overline{0.0} \\ \overline{0.1} \quad \mapsto \quad \overline{1.1} \quad , \quad \overline{1.1} \quad \mapsto \quad \overline{0.1} \end{array}$$

The Barge–Diamond filtration is a filtration of complexes  $S_0 \subseteq S_1 \subseteq \dots \subseteq S_d$ . The complex  $S_0$  is formed by taking the 0-flaps and performing identifications similar to the Anderson–Putnam construction,  $S_1$  is formed by taking  $S_0$ , then adding in the 1-flaps (then possibly performing identifications if there are 2-flaps), etc. For  $\zeta_1$ ,  $S_0$ , drawn in solid lines, is a loop



where the dashed lines indicate the 1-flaps that yield, for example, identification of the left endpoints of 0.0 and 0.1. The complex  $S_1$  is then the same picture, together with the dashed lines turned solid. We define  $\Xi_i = \varprojlim(S_i, \zeta_1)$ . Then  $\Xi_1 = \Omega_{\zeta_1}$ .

One can check that  $\zeta_1 : S_0 \rightarrow S_0$  is an orientation-reversing self-homeomorphism, thus  $\check{H}^1(\Xi_0) = \mathbb{Z}$ . For  $\Xi_1$ , we use the long exact sequence

$$0 \rightarrow \check{H}^0(\Xi_1, \Xi_0) \rightarrow \check{H}^0(\Xi_1) \rightarrow \check{H}^0(\Xi_0) \xrightarrow{\delta} \check{H}^1(\Xi_1, \Xi_0) \rightarrow \check{H}^1(\Xi_1) \rightarrow \check{H}^1(\Xi_0) \rightarrow 0$$

in relative cohomology of a pair  $(\Xi_1, \Xi_0)$ . Since  $S_0$  has a single component, over reduced cohomology,  $\check{H}^1(\Xi_1, \Xi_0) \rightarrow \check{H}^1(\Xi_1) \rightarrow \check{H}^1(\Xi_0) \rightarrow 0$  becomes a short exact sequence.

The space  $S_1/S_0$  is a wedge of two circles, with its inverse limit computed exactly from the original substitution matrix. Thus

$$\check{H}^1(\Xi_1, \Xi_0) = \varinjlim \left( \mathbb{Z}^2, \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \right) = \mathbb{Z}[1/5].$$

Since  $\check{H}^1(\Xi_0) = \mathbb{Z}$  is free, the bottom short exact sequence splits, yielding

$$\check{H}^1(\Omega_{\zeta_1}) = \mathbb{Z}[1/5] \oplus \mathbb{Z}, \quad \check{H}^0(\Omega_{\zeta_1}) = \mathbb{Z}.$$

The calculation for  $\zeta_2$  is easier. It is a proper substitution, thus forces the border (see Remark 2.1), and

$$\check{H}^1(\Omega_{\zeta_2}) = \varinjlim \left( \mathbb{Z}^3, \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \right).$$

The matrix has eigenvectors  $e_5 = (1, 1, 1)$  and  $e_3 = (0, 2, -1)$  with the subscripts their respective eigenvalues. Direct computation then shows that the direct limit splits as a direct sum, giving us

$$\check{H}^1(\Omega_{\zeta_2}) = \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \quad \text{and} \quad \check{H}^0(\Omega_{\zeta_2}) = \mathbb{Z}.$$

Let us also compute the induced substitution matrices on  $\check{H}^1(\Omega_{\zeta_1})$  using the dual complex. It is computed via direct limit of the matrix

$$\sigma_1^1 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

with eigenvalues of 3,  $-1$ . The eigenvalue of  $-1$  matches the orientation reversal of  $\zeta_1$  on its respective  $S_0$ .

We finally consider the product tiling space  $\Omega_{\zeta_1} \times \Omega_{\zeta_2}$ . A direct application of the Künneth theorem yields that

$$\begin{aligned} \check{H}^2(\Omega_{\zeta_1} \times \Omega_{\zeta_2}) &= \mathbb{Z}[1/25] \oplus \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3], \\ \check{H}^1(\Omega_{\zeta_1} \times \Omega_{\zeta_2}) &= \mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}, \\ \check{H}^0(\Omega_{\zeta_1} \times \Omega_{\zeta_2}) &= \mathbb{Z} \end{aligned}$$

with  $\check{H}^2(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$  having eigenvalues 25, 15,  $-5$ ,  $-3$  and  $\check{H}^1(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$  having eigenvalues  $5_2, 5_1, 3_2, -1_1$ , with the subscripts indicating the 1-dimensional substitution rule that gives rise to the corresponding eigenvalue.

Lifting each of the cohomology classes in  $\check{H}^1(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$  to 1-cochains, applying the cubical cup-product formula given in [15], then projecting to cohomology classes in  $\check{H}^2(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$  gives that, in the order given in the list of eigenvalues, the cup product is the resulting bilinear form

$$\check{H}^1 \times \check{H}^1 \rightarrow \check{H}^2 : (a, b) \mapsto B(a, b) := \sum_{* \in \{25, 15, -5, -3\}} B_*(a, b) \cdot e_*,$$

where  $e_n \in \check{H}^2$  is a generator with eigenvalue  $n$ , and  $B_n$  is the bilinear form given by the matrices

$$B_{25} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{15} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{-5} := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_{-3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The important observation is that, as expected, there are always cohomology classes in  $\check{H}^1(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2}; \mathbb{Z})$  cupping to any of the cohomology classes in  $\check{H}^2(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2}; \mathbb{Z})$ .

Lastly, let us compute the frequency module. The substitution matrix on the 2-cells of the dual complex is the matrix  $\sigma^2 = \sigma_{A1}$  found in the Appendix, which has Perron–Frobenius eigenvector

$$(95, 19, 25, 5, 25, 5, 5, 1, 76, 20, 20, 4, 38, 10, 10, 2, 38, 10, 10, 2, 76, 20, 20, 4, 19, 5, 5, 1, 38, 76, 10, 20, 10, 20, 2, 4).$$

Its sum is 750, giving us that the frequency module is  $\frac{1}{6}\mathbb{Z}[1/5]$ . For illustration, we also compute the frequency modules for each of the one-dimensional substitutions. The substitution matrices on the 1-cells of the corresponding dual complexes are

$$\sigma_1^1 = \begin{pmatrix} 3 & 3 & 4 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2^1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

with Perron–Frobenius eigenvectors  $(19, 5, 5, 1)$  and  $(5, 1, 4, 2, 2, 4, 1, 2, 4)$  that are duals of lifts of the generators of  $\check{H}^1(\Omega_{\varsigma_i})$  with sums 30 and 25, respectively, giving frequency modules  $\frac{1}{6}\mathbb{Z}[1/5]$  and  $\mathbb{Z}[1/5]$ , respectively. Since the Ruelle–Sullivan map is a ring homomorphism,  $\frac{1}{6}\mathbb{Z}[1/5] \cdot \mathbb{Z}[1/5] = \frac{1}{6}\mathbb{Z}[1/5]$ .

**Example 3.3** (expansion 4 product) Let us consider the following two one-dimensional substitutions on two prototiles:

$$\begin{aligned} \varsigma_1 &= [[0, 0, 0, 1], [0, 1, 1, 1]] \\ \varsigma_2 &= [[0, 1, 1, 0], [1, 0, 0, 1]] \end{aligned}$$

One, again, easily checks that both substitutions are primitive and recognizable. In particular,  $\varsigma_2$  is the (square of the) Thue–Morse substitution, whose cohomology groups were computed in [5], being

$$\check{H}^1(\Omega_{\varsigma_2}) = \mathbb{Z}[1/4] \oplus \mathbb{Z} \quad \text{and} \quad \check{H}^0(\Omega_{\varsigma_2}) = \mathbb{Z}.$$

The substitution  $\varsigma_1$  is proper, thus forces the border (see Remark 2.1), and

$$\check{H}^1(\Omega_{\varsigma_1}) = \varinjlim \left( \mathbb{Z}^2, \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \right).$$

The matrix has eigenvectors  $e_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  with the subscripts their respective eigenvalues. Direct computation then shows that the direct limit splits as a direct sum, giving us

$$\check{H}^1(\Omega_{\varsigma_1}) = \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \quad \text{and} \quad \check{H}^0(\Omega_{\varsigma_1}) = \mathbb{Z}.$$

The Künneth theorem yields that, for the product  $\Omega_{\varsigma_1} \times \Omega_{\varsigma_2}$ ,

$$\begin{aligned} \check{H}^2(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2}) &= \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/8] \oplus \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2], \\ \check{H}^1(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2}) &= \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}, \\ \check{H}^0(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2}) &= \mathbb{Z} \end{aligned}$$

with  $\check{H}^2(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$  having eigenvalues 16, 8, 4, 2 and  $\check{H}^1(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$  having eigenvalues  $4_2, 4_1, 2_1, 1_2$ , with the subscripts indicating the 1-dimensional substitution rule that gives rise to the corresponding eigenvalue.

The same procedure as the previous example gives that, in the order given in the list of eigenvalues, the cup product is the resulting bilinear form

$$\check{H}^1 \times \check{H}^1 \rightarrow \check{H}^2 : (a, b) \mapsto B(a, b) := \sum_{* \in \{16, 8, 4, 2\}} B_*(a, b) \cdot e_*,$$

where  $e_n \in \check{H}^2$  is a generator with eigenvalue  $n$ , and  $B_n$  is the bilinear form given by the matrices

$$B_{16} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_8 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_4 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad B_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Again, we observe that, as expected, there are always cohomology classes in  $\check{H}^1(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$  cupping to any of the cohomology classes in  $\check{H}^2(\Omega_{\varsigma_1} \times \Omega_{\varsigma_2})$ .

Lastly, let us compute the frequency module. The substitution matrix on the 2-cells of the dual complex is  $\sigma^2 = \sigma_{A_2}$  found in the Appendix which has Perron–Frobenius eigenvector

$$(1, 2, 1, 2, 1, 2, 1, 2, 2, 2, 2, 2, 1, 1, 1, 1).$$

Its sum is 24, giving us that the frequency module is  $\frac{1}{3}\mathbb{Z}[1/2]$ . For illustration, we also compute the frequency modules for each of the one-dimensional substitutions. The substitution matrices on the 1-cells

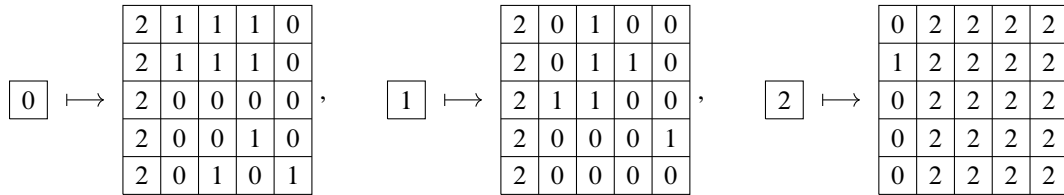


Figure 2: Substitution rule for an expansion 5 two-dimensional cubical substitution that has the cohomology groups of a product, but does not have the ring structure of a product.

of the corresponding dual complexes are

$$\sigma_1^1 = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \sigma_2^1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

with Perron–Frobenius eigenvectors that are duals of lifts of the generators of  $\check{H}^1(\Omega_{\zeta_i})$  are  $(1, 1, 1, 1)$  and  $(1, 2, 2, 1)$  with sums 4 and 6, respectively, giving frequency modules  $\mathbb{Z}[1/2]$  and  $\frac{1}{3}\mathbb{Z}[1/2]$ , respectively. As expected, since the Ruelle–Sullivan map is a ring homomorphism,  $\mathbb{Z}[1/2] \cdot \frac{1}{3}\mathbb{Z}[1/2] = \frac{1}{3}\mathbb{Z}[1/2]$ .

**Example 3.4** (expansion 5 nonproduct) We consider the following two-dimensional substitution on three prototiles (see Figure 2):

$$\zeta = [ [2, 2, 2, 2, 2, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0], \\ [2, 2, 2, 2, 2, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0], \\ [0, 0, 0, 1, 0, 2] ]$$

This does not force the border (e.g., the horizontal 1-cell containing the prototiles 0, 1 on its bottom and prototiles 0, 1 on its top stays branched regardless of the power of the substitution), but is primitive and recognizable.

Using the dual complex,  $\check{H}^2(\Omega_{\zeta})$  and  $\check{H}^1(\Omega_{\zeta})$  are computed via direct limits under the matrices

$$\sigma^2 = \begin{pmatrix} 1 & 0 & 3 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 1 & 0 & 3 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 & 3 & 2 & 2 & 2 & 0 & 1 & 1 & 20 & 2 & 1 & 2 \\ 5 & 4 & 6 & 2 & 5 & 5 & 5 & 1 & 2 & 2 & 1 & 40 & 3 & 1 & 2 \\ 1 & 0 & 3 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 6 & 3 & 9 & 3 & 5 & 5 & 5 & 1 & 3 & 2 & 2 & 0 & 4 & 1 & 2 \\ -3 & -1 & -3 & -1 & -2 & -2 & -2 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ -3 & -1 & -3 & -1 & -2 & -2 & -2 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 3 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\ 3 & 1 & 3 & 1 & 2 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 6 & 2 & 6 & 2 & 4 & 4 & 4 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 \\ 3 & 2 & 6 & 2 & 3 & 3 & 3 & 1 & 2 & 2 & 1 & 0 & 3 & 1 & 2 \\ 6 & 3 & 9 & 3 & 5 & 5 & 5 & 1 & 3 & 2 & 2 & 0 & 4 & 1 & 2 \\ 3 & 1 & 3 & 1 & 2 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

with eigenvectors

$$\begin{aligned} e_{5_1}^1 &= (1, 0, 0, 0), & e_{25}^2 &= (1, 1, 2, 4, 1, 3, -1, -1, 1, 1, 2, 1, 2, 3, 1), \\ e_{5_2}^1 &= (0, 1, 2, 1), & e_{15}^2 &= (1, 1, 0, 0, 1, 3, -1, -1, 1, 1, 2, -1, 2, 3, 1), \\ e_3^1 &= (0, 1, 0, -1), & e_{-5}^2 &= (1, 1, -4, -8, 1, 3, -1, -1, 1, 1, 2, 1, 2, 3, 1), \\ e_{-1}^1 &= (0, 1, -4, 1), & e_{-3}^2 &= \begin{pmatrix} 101, 101, 90, -72, 101, -201, 151, 151, \\ 101, -151, -302, -11, -50, -201, -151 \end{pmatrix}, \end{aligned}$$

where the subscripts indicate the corresponding eigenvalues. The cup product is the bilinear form

$$\check{H}^1 \times \check{H}^1 \rightarrow \check{H}^2 : (a, b) \mapsto B(a, b) := \sum_{* \in \{25, 15, -5, -3\}} B_*(a, b) \cdot e_*^2,$$

where  $e_n^2 \in \check{H}^2$  is a generator with eigenvalue  $n$ , and  $B_n$  is the bilinear form given by the matrices

$$B_{25} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{15} := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{-5} := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_{-3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last component of the bilinear form shows that there are no cohomology classes in  $\check{H}^1(\Omega_\zeta)$  cupping to  $e_{-3}^2$  in  $\check{H}^2(\Omega_\zeta)$ , contrasting the example coming from the product. This coincides with the fact that, up to a coboundary, the lifts of the eigenvectors  $e_1^1$  and  $e_{-1}^1$  to 1-cochains are

$$\widetilde{e}_3^1 = (0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, -1, 1), \quad \widetilde{e}_{-1}^1 = (0, 0, 0, 0, 0, 1, 1, -4, -4, 1, 0, 0, 1, 1)$$

with the first five entries vertical 1-cells, the remaining horizontal 1-cells, therefore cannot cup nontrivially (to  $e_{-3}^2$ ).

At this point, we have yet to calculate the actual cohomology groups of  $\Omega_\zeta$  over  $\mathbb{Z}$ . We follow the two-dimensional analogue of the Barge–Diamond filtration presented in [6]. For the sake of brevity, and since the calculations are in Example 3.5, here we skip most of the details and just state the conclusion.

The long exact sequence in relative cohomology of a pair  $(\mathfrak{E}_1, \mathfrak{E}_0)$  yields the short exact sequence

$$0 \longrightarrow \check{H}^1(\mathfrak{E}_1, \mathfrak{E}_0) \longrightarrow \check{H}^1(\mathfrak{E}_1) \longrightarrow \check{H}^1(\mathfrak{E}_0) \longrightarrow 0,$$

where

$$\check{H}^1(\mathfrak{E}_1, \mathfrak{E}_0) = \varinjlim \left( \mathbb{Z}^3, \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \right),$$

and of  $(\mathfrak{E}_2, \mathfrak{E}_1)$  yields

$$0 \longrightarrow \check{H}^1(\mathfrak{E}_2, \mathfrak{E}_1) \longrightarrow \check{H}^1(\mathfrak{E}_2) \longrightarrow \check{H}^1(\mathfrak{E}_1) \xrightarrow{\delta_1^1} \check{H}^2(\mathfrak{E}_2, \mathfrak{E}_1) \longrightarrow \check{H}^2(\mathfrak{E}_2) \longrightarrow \check{H}^2(\mathfrak{E}_1) \longrightarrow 0,$$

where

$$\check{H}^2(\Xi_2, \Xi_1) = \varinjlim \left( \mathbb{Z}^3, \begin{pmatrix} 11 & 9 & 5 \\ 14 & 6 & 5 \\ 4 & 1 & 20 \end{pmatrix} \right).$$

The eigenvalues of these matrices are 5, 3 and 25, 15, -3, respectively. A short calculation gives that  $\check{H}^1(\Xi_0) = \mathbb{Z}$ . Thus  $\check{H}^1(\Xi_1, \Xi_0)$  is a direct summand of  $\check{H}^1(\Xi_1)$ . By Schur's lemma,  $\delta_1^1 = 0$ . Furthermore, since  $\Xi_2$  is a wedge of 2-spheres,  $\check{H}^1(\Xi_2, \Xi_1) = 0$ . Thus  $\check{H}^1(\Omega_\zeta) = \check{H}^1(\Xi_2) = \check{H}^1(\Xi_1)$ , which contains  $\check{H}^1(\Xi_1, \Xi_0)$  as a direct summand.

However, the direct limit of the matrix

$$\sigma = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

does not split as a direct sum  $\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]$ <sup>3</sup> To see this, consider any

$$\iota : \mathbb{Z}[1/5] \rightarrow \varinjlim (\mathbb{Z}^2, \sigma).$$

Since  $1 \in \mathbb{Z}[1/5]$  is infinitely divisible by 5, its image must be as well, thus this map must be of the form  $1 \mapsto (c, c)$ , an eigenvector with eigenvalue 5, with  $c \in \mathbb{Z}$ . By writing

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\iota_0} & \mathbb{Z}^2 \\ \downarrow \cdot 5 & & \downarrow \sigma \\ \mathbb{Z} & \xrightarrow{\iota_1} & \mathbb{Z}^2 \\ \downarrow \cdot 5 & & \downarrow \sigma \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathbb{Z}[1/5] & \xrightarrow{\iota} & \varinjlim (\mathbb{Z}^2, \sigma) \end{array}$$

where each  $\iota_n : \mathbb{Z} \rightarrow \mathbb{Z}^2$  is  $1 \mapsto (c, c)$ ,

$$\text{coker } \iota = \varinjlim \text{coker } \iota_n.$$

Since  $\mathbb{Z} \leq \text{coker } \iota_n$ , without loss of generality, let us assume  $c = 1$  and  $\text{coker } \iota_n = \mathbb{Z}$ , with  $\mathbb{Z}^2 \rightarrow \text{coker } \iota_n$  given by  $(a, b) \mapsto a - b$ . Choosing a right splitting  $\text{coker } \iota_n \rightarrow \mathbb{Z}^2$  to be  $1 \mapsto (1, 0)$  gives that the induced map

$$\sigma_* : \text{coker } \iota_n \rightarrow \text{coker } \iota_{n+1}$$

<sup>3</sup>If one has the matrix  $\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  instead, by [19, Exercise 7.5.2 (a)], the direct limit does not split as a direct sum of the eigenspaces. However, it is still a direct sum  $\mathbb{Z}[1/3] \oplus \mathbb{Z}$  with  $\mathbb{Z}[1/3]$  generated by the eigenvector with eigenvalue 3, since  $(\varinjlim (\mathbb{Z}^2, \sigma))/\mathbb{Z}[1/3] = \mathbb{Z}$ , and the short exact sequence

$$0 \longrightarrow \mathbb{Z}[1/3] \longrightarrow \varinjlim (\mathbb{Z}^2, \sigma) \longrightarrow \mathbb{Z} \longrightarrow 0$$

still splits. Note that  $\mathbb{Z}$  is not generated by the eigenvector with eigenvalue 1!

is multiplication by 3, and

$$\text{coker } \iota = \mathbb{Z}[1/3].$$

To see that this right splitting does not induce a splitting in the limit, take  $1/3 \in \mathbb{Z}[1/3]$ . This maps to  $(1/3, 0)$ . By definition, if it belongs to the direct limit, there must be some sufficiently large  $k \in \mathbb{N}$  so that  $\sigma^k(1/3, 0) \in \mathbb{Z}^2$ , but

$$\sigma^k \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \frac{1}{2} \sigma^k \left( \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \right) = \frac{1}{2 \cdot 3} \left( 5^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \notin \mathbb{Z}^2.$$

Given any right splitting, since the image of  $1/3^n$  for sufficiently large  $n \in \mathbb{N}$  under the splitting can be decomposed like this to not land in  $\mathbb{Z}^2$  under any  $\sigma^k$ , any short exact sequence of the form

$$0 \longrightarrow \mathbb{Z}[1/5] \xrightarrow{\iota} \varinjlim (\mathbb{Z}^2, \sigma) \longrightarrow \mathbb{Z}[1/3] \longrightarrow 0$$

cannot be split.

Finally, applying the same argument to the map

$$\mathbb{Z}[1/3] \rightarrow \varinjlim (\mathbb{Z}^2, \sigma)$$

shows that the direct limit does not split as  $\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]$ .

Therefore, this is an example where the cohomology groups over  $\mathbb{Q}$  are indistinguishable from those of a product, and cup product can be used to differentiate the spaces there, but the cohomology groups over  $\mathbb{Z}$  *also already differ from that of a product!*

Lastly, let us compute the frequency module. The substitution matrix on the 2-cells of the dual complex is  $\sigma = \sigma_{A_3}$  found in the Appendix which has Perron–Frobenius eigenvector

$$(1913, 962, 5250, 1750, 2213, 1775, 1775, 1750, 1113, 612, 912, 813, 4575, 1750, 1750, 675, 17500, 1725, 675, 813, 375, 1350, 474).$$

Its sum is 52500, giving us that the frequency module is  $\frac{1}{84} \mathbb{Z}[1/5]$ , which is different from the frequency module of the product.

For illustration, we also compute the frequency modules for substitution on the vertical and horizontal 1-cells of the dual complex. The substitution matrices are

$$\sigma_v^1 = \begin{pmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 5 & 5 & 0 & 5 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_h^1 = \begin{pmatrix} 0 & 0 & 0 & 4 & 3 & 0 & 0 & 0 & 3 \\ 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

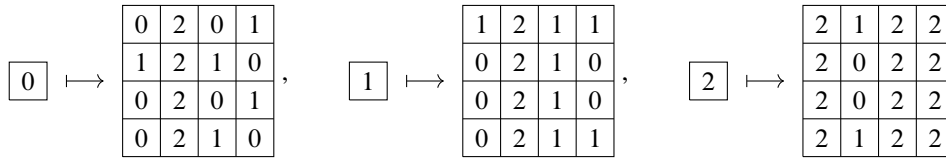


Figure 3: Substitution rule for an expansion 4 two-dimensional cubical substitution that has the cohomology groups of a product, but does not have the ring structure of a product.

The Perron–Frobenius eigenvectors that are duals of lifts of the generators of  $\check{H}^1(\Omega_\zeta)$  of eigenvalue 5 are  $(3, 1, 5, 1, 0)$  and  $(4, 6, 5, 2, 4, 7, 0, 14, 0)$ . They have sums 10 and 42, respectively, giving frequency modules  $\frac{1}{2}\mathbb{Z}[1/5]$  and  $\frac{1}{42}\mathbb{Z}[1/5]$ , respectively.

**Example 3.5** (expansion 4 nonproduct) In the final example of this section, we consider the following two-dimensional substitution on three prototiles (see Figure 3):

$$\zeta = [[0, 0, 1, 0, 2, 2, 2, 2, 1, 0, 1, 0, 0, 1, 0, 1], [0, 0, 0, 1, 2, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, 1], [2, 2, 2, 2, 1, 0, 0, 1, 2, 2, 2, 2, 2, 2, 2, 2]]$$

This, again, does not force the (see Remark 2.1) border, but is primitive and recognizable.

Using the dual complex,  $\check{H}^2(\Omega_\zeta)$  and  $\check{H}^1(\Omega_\zeta)$  are computed via direct limits under the matrices

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 2 & 1 & 3 & 0 & 0 & 0 & 2 & 0 \\ 4 & 4 & 2 & -1 & -1 & 4 & -1 & -1 & 6 & 8 & 0 & 0 & 0 \\ 4 & 2 & 5 & 0 & 0 & 3 & 0 & 0 & 3 & 4 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 3 & 0 \\ 4 & 2 & 4 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 3 & 0 \\ 3 & 2 & 4 & 1 & 1 & 3 & 1 & 2 & 0 & 0 & 0 & 2 & 0 \\ 8 & 4 & 8 & 1 & 2 & 4 & 1 & 5 & 0 & 0 & 2 & 5 & 1 \\ -2 & 0 & -4 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 2 & 2 & 1 & 0 & 0 & 2 & -1 & -1 & 3 & 4 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 8 & 0 & 0 & 0 \\ 4 & 2 & 4 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 1 & 3 & 0 \\ 6 & 2 & 8 & 1 & 2 & 3 & 1 & 2 & 0 & 0 & 1 & 2 & 1 \\ 2 & 0 & 4 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{pmatrix},$$

with eigenvectors

$$\begin{aligned} e_{4_1}^1 &= (2, 0, 1, 1), & e_{16}^2 &= (1, 2, 2, 1, 2, 2, 4, -1, 1, 1, 2, 3, 1), \\ e_{4_2}^1 &= (0, 2, -1, -1), & e_8^2 &= (1, -2, 0, 1, 2, 2, 4, -1, -1, -1, 2, 3, 1), \\ e_2^1 &= (0, 0, 1, -1), & e_4^2 &= (1, -4, -1, 1, 2, 2, 4, -1, -2, 1, 2, 3, 1), \\ e_1^1 &= (0, 1, 1, 1), & e_2^2 &= (1, 0, -1, 1, 0, 0, 0, 1, 0, 0, 0, -1, -1), \end{aligned}$$

where the subscripts indicate the corresponding eigenvalues. The cup product is the bilinear form

$$\check{H}^1 \times \check{H}^1 \rightarrow \check{H}^2 : (a, b) \mapsto B(a, b) := \sum_{* \in \{16, 8, 4, 2\}} B_*(a, b) \cdot e_*^2,$$

where  $e_n^2 \in \check{H}^2$  is a generator with eigenvalue  $n$ , and  $B_n$  is the bilinear form given by the matrices

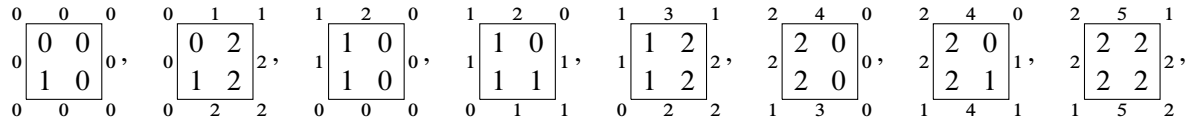
$$B_{16} := \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_8 := \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_4 := \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Like the previous example, the last component of the bilinear form shows that there are no cohomology classes in  $\check{H}^1(\Omega_\zeta)$  cupping to  $e_2^2$  in  $\check{H}^2(\Omega_\zeta)$ , contrasting the example coming from the product. This coincides with the fact that, up to a coboundary, the lifts of the eigenvectors  $e_2^1$  and  $e_1^1$  to 1-cochains are

$$\widetilde{e}_2^1 = (0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, -1), \quad \widetilde{e}_1^1 = (0, 0, 0, 0, 0, -1, 0, 0, 1, 1, -1, 1, 1, 1)$$

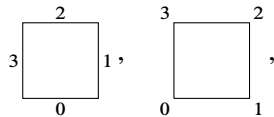
with the first five entries vertical 1-cells, the remaining horizontal 1-cells, therefore cannot cup nontrivially (to  $e_2^2$ ).

Let us now calculate the actual cohomology groups of  $\Omega_\zeta$  over  $\mathbb{Z}$  following the two-dimensional analogue of the Barge–Diamond filtration. Let us begin by drawing  $S_0$  restricted to the eventual range



where the boundary identifications are *only along the respective positions*, e.g., none of the bottom horizontal 1-cells are identified to any of the top horizontal 1-cells, despite the numbering, and none of the bottom right 0-cells are identified to any of the top right 0-cells, despite the numbering.

We arrange the 1-cells and the 0-cells on  $S_0$  in the order



then by the numbering in  $S_0$ . This gives us the coboundary matrices

$$\partial_0^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix},$$

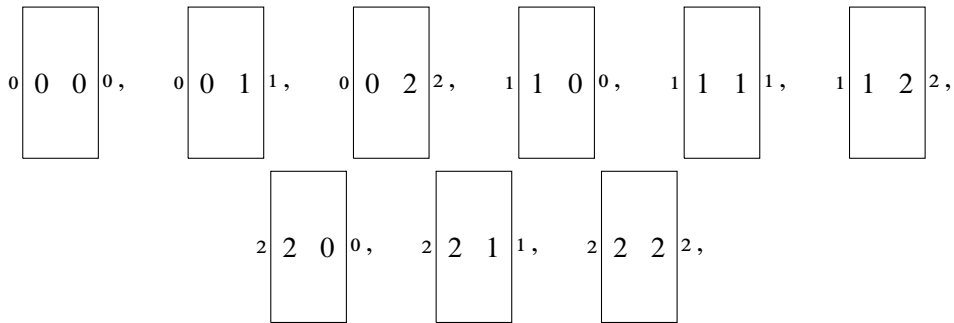
$$\partial_0^0 = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Direct calculation shows that

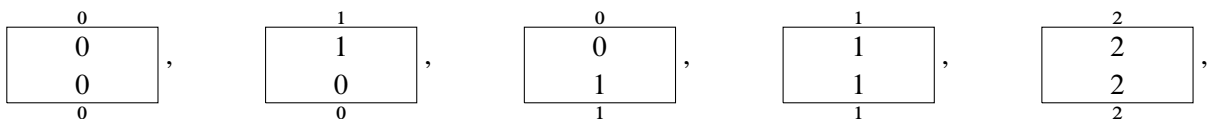
$$\check{H}^2(\Xi_0) = 0, \quad \check{H}^1(\Xi_0) = \mathbb{Z} = \langle \overline{(1, 0, 0, 0, -1, 0, 0, 1, 0, 0, -1, 0, -1, 0, 0, 1, 1, 0)} \rangle, \quad \check{H}^0(\Xi_0) = \mathbb{Z},$$

where the vector is a 1-cochain in  $S_0$ .

The extra cells that we add on to  $S_0$  to form  $S_1$  are the same as the previous example, being the vertical 1-flaps



where the top and the bottom horizontal 1-cells are identified to their appropriate bottom and top horizontal 1-cells in  $S_0$ , and the horizontal 1-flaps



where the left and the right vertical 1-cells are identified to their appropriate right and left vertical 1-cells in  $S_0$ . The space  $S_1/S_0$  has three generating loops up to homotopy, one from the vertical 1-flaps, one

from the first four horizontal 1-flaps, and one from the last horizontal 1-flap. We then compute that

$$\check{H}^1(\Xi_1, \Xi_0) = \varinjlim \left( \mathbb{Z}^3, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \right)$$

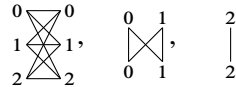
with the order of the entries of the matrix given as described. The matrix has eigenvectors

$$e_{4_1} = (1, 0, 0), \quad e_{4_2} = (0, 1, 1), \quad e_2 = (0, 1, -1),$$

with the subscripts their respective eigenvalues. Direct computation then shows that the direct limit splits as a direct sum, giving us

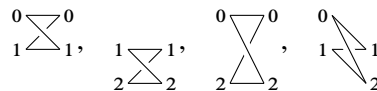
$$\check{H}^1(\Xi_1, \Xi_0) = \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2].$$

The group  $\check{H}^2(\Xi_1, \Xi_0)$  requires more effort. We observe that the generating 2-spheres of  $S_1/S_0$ , up to homotopy, occur when 1-flaps form a “tube”, e.g., the first four of the horizontal 1-flaps. Let us represent all such tubes by graphs that represents the horizontal and vertical cross sections of vertical and horizontal 1-flaps, respectively, with the vertices the 1-cells in  $S_1$  not attached to  $S_0$ , and the edges the 1-flaps themselves. In this example, they are, drawn in the orientation matching the cross sections of the respective 1-flaps,

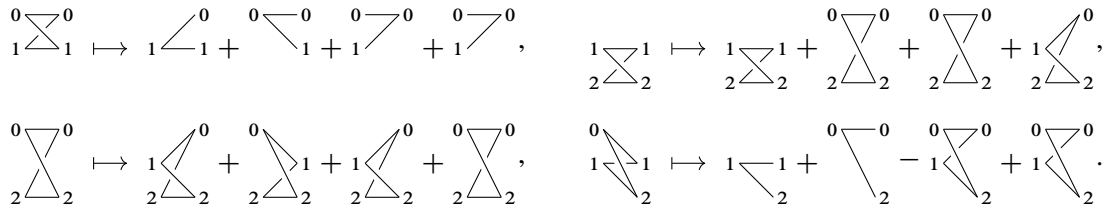


with labels of the vertices corresponding to the labels of the 1-cells marked in the set of 1-flaps.

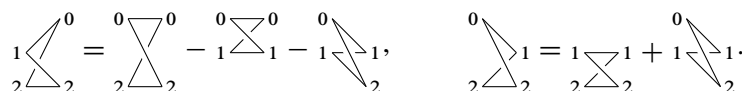
The first graph corresponding to vertical 1-flaps has a cycle basis consisting of four elements, therefore four associated generating 2-spheres in  $S_1/S_0$ , which we pick to be



oriented so that the top edge is left-to-right. In fact, let us orient for all other cycles the same way. The substitution rule then dictates that



The cycles not in our basis decompose as



A similar calculation for the second and the third graphs gives us

$$\begin{array}{c} 0 & 1 \\ \diagdown & / \\ 0 & 1 \end{array} \mapsto \begin{array}{c} 0 \\ | \\ 0 \end{array} \begin{array}{c} 1 \\ | \\ 1 \end{array} + \begin{array}{c} 2 \\ | \\ 2 \end{array} + \begin{array}{c} 1 \\ / \\ 0 \end{array} \begin{array}{c} 1 \\ | \\ 1 \end{array} + \begin{array}{c} 0 & 1 \\ \diagdown & / \\ 0 & 1 \end{array}, \quad \begin{array}{c} 2 \\ | \\ 2 \end{array} \mapsto \begin{array}{c} 2 \\ | \\ 2 \end{array} + \begin{array}{c} 1 \\ | \\ 1 \end{array} + \begin{array}{c} 2 \\ | \\ 2 \end{array} + \begin{array}{c} 2 \\ | \\ 2 \end{array}.$$

Observing that the substitution rule is trivial on the second and the third graphs indicates that we can calculate  $\check{H}^2(\Xi_1, \Xi_0)$  using only the vertical 1-flaps, which is

$$\check{H}^2(\Xi_1, \Xi_0) = \varinjlim \left( \mathbb{Z}^4, \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 3 & -1 \\ -2 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right),$$

where the entries are ordered in the cycle basis we selected. Finally,

$$\check{H}^2(\Xi_1, \Xi_0) = \mathbb{Z}[1/4] = \langle (0, 1, 1, 0) \rangle.$$

Thus the long exact sequence in relative cohomology of a pair  $(\Xi_1, \Xi_0)$  reads

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{H}^0(\Xi_1, \Xi_0) & \longrightarrow & \check{H}^0(\Xi_1) & \longrightarrow & \mathbb{Z} \\ & & & & \delta_0^0 & & \downarrow \\ & & \searrow & & \check{H}^1(\Xi_1) & \longrightarrow & \mathbb{Z} \\ & & & & \delta_0^1 & & \downarrow \\ & & \searrow & & \check{H}^2(\Xi_1) & \longrightarrow & 0 \longrightarrow 0. \\ & & & & \mathbb{Z}[1/4] & \longrightarrow & \end{array}$$

Recalling that

$$\check{H}^1(\Xi_0) = \mathbb{Z} = \overline{\langle (1, 0, 0, 0, -1, 0, 0, 1, 0, 0, -1, 0, -1, 0, 0, 1, 1, 0) \rangle}$$

we check the image of the generator under  $\delta_0^1$ , which is

$$\begin{array}{c} \boxed{\begin{array}{c} 1 \ 0 \\ 0 \end{array}} - \boxed{\begin{array}{c} 2 \ 1 \\ 4 \end{array}} + \boxed{\begin{array}{c} 0 \\ 1 \end{array}}_1 - \boxed{\begin{array}{c} 0 \ 2 \\ 1 \end{array}} - \boxed{\begin{array}{c} 1 \ 2 \\ 3 \end{array}} + \boxed{\begin{array}{c} 0 \\ 1 \end{array}}_0 + \boxed{\begin{array}{c} 1 \\ 1 \end{array}}_1 \\ \mapsto -1 \boxed{\begin{array}{c} 1 \ 0 \\ 0 \end{array}}_0 + 2 \boxed{\begin{array}{c} 2 \ 1 \\ 1 \end{array}}_1 - \boxed{\begin{array}{c} 0 \\ 1 \\ 1 \end{array}}_0 - 0 \boxed{\begin{array}{c} 0 \ 2 \\ 2 \end{array}}_2 - 1 \boxed{\begin{array}{c} 1 \ 2 \\ 2 \end{array}}_2 \\ + \boxed{\begin{array}{c} 0 \\ 1 \\ 1 \end{array}}_0 + \boxed{\begin{array}{c} 1 \\ 1 \\ 1 \end{array}}_1 \end{array}$$

and is contractible, so  $\delta_0^1 = 0$ . This is the same conclusion reached by Schur’s lemma. Thus, working over reduced cohomology,

$$\check{H}^2(\Xi_1) = \mathbb{Z}[1/4], \quad \check{H}^1(\Xi_1) = \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}.$$

Towards the relative cohomology of a pair  $(\Xi_2, \Xi_1)$ , we have that

$$\check{H}^2(\Xi_2, \Xi_1) = \varinjlim \left( \mathbb{Z}^3, \begin{pmatrix} 7 & 5 & 4 \\ 5 & 7 & 4 \\ 2 & 2 & 12 \end{pmatrix} \right)$$

with eigenvectors

$$e_{16} = (1, 1, 1), \quad e_8 = (1, 1, -1), \quad e_2 = (1, -1, 0).$$

A straightforward calculation and the observation that  $S_2/S_1$  is a wedge of 2-spheres give us

$$\check{H}^2(\Xi_2, \Xi_1) = \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/8] \oplus \mathbb{Z}[1/2], \quad \check{H}^1(\Xi_2, \Xi_1) = 0.$$

The long exact sequence in relative cohomology of a pair  $(\Xi_2, \Xi_1)$  then reads

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{H}^0(\Xi_2, \Xi_1) & \longrightarrow & \check{H}^0(\Xi_2) & \longrightarrow & \mathbb{Z} \\ & & & & \delta_1^0 & & \downarrow \\ & & & & \check{H}^1(\Xi_2) & \longrightarrow & \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z} \\ & & & & \delta_1^1 & & \downarrow \\ & & & & \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/8] \oplus \mathbb{Z}[1/2] & \longrightarrow & \check{H}^2(\Xi_2) \longrightarrow \mathbb{Z}[1/4] \longrightarrow 0. \end{array}$$

By Schur’s lemma, to show that  $\delta_1^1 = 0$ , it suffices to check that  $\delta_1^1(0, 1, -1) = 0$ , where  $(0, 1, -1)$  is the generator of  $\mathbb{Z}[1/2] \leq \check{H}^1(\Xi_1)$ , which is straightforward. In fact,  $\delta_1^1$  applied to each of the generating loops in  $S_1/S_0$  (then lifted to a 1-cochain in  $S_1$ , modulo homotopy) yields as many 2-flaps of the same type on one side of a loop as the other.

In the very last step, we observe that  $\check{H}^2(\Xi_2, \Xi_1)$  and  $\check{H}^2(\Xi_1)$  have every element 2-divisible. We first show that  $\check{H}^2(\Xi_2)$  is 2-divisible as well. Consider any  $c \in \check{H}^2(\Xi_2)$ . Let  $d \in \check{H}^2(\Xi_1)$  so that  $c \mapsto d$ . By assumption,  $d/2 \in \check{H}^2(\Xi_1)$ . Let  $c' \in \check{H}^2(\Xi_2)$  so that  $c' \mapsto d/2$ . Then  $c - 2c' \mapsto 0$ , and  $c - 2c' \in \check{H}^2(\Xi_2, \Xi_1)$ . Again, by assumption,  $(c - 2c')/2 \in \check{H}^2(\Xi_2, \Xi_1) \leq \check{H}^2(\Xi_2)$ . Then  $2((c - 2c')/2 + c') = c$ , and  $\check{H}^2(\Xi_2)$  is 2-divisible. To construct a splitting map  $\check{H}^2(\Xi_1) \rightarrow \check{H}^2(\Xi_2)$ , take any  $c \in \check{H}^2(\Xi_2)$  so that  $c \mapsto 1$ . The map is then defined by  $a/2^n \mapsto ac/2^n$ , where  $a/2^n \in \check{H}^2(\Xi_1)$ , and  $c/2^n$  exists since  $\check{H}^2(\Xi_2)$  is 2-divisible.

Thus the bottom short exact sequence splits, giving us that, over  $\mathbb{Z}$ ,

$$\check{H}^2(\Omega_\zeta) = \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/8] \oplus \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2], \quad \check{H}^1(\Omega_\zeta) = \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}, \quad \check{H}^0(\Omega_\zeta) = \mathbb{Z},$$

which are exactly the cohomology groups of a product. Therefore, this is an example over  $\mathbb{Z}$  that the cup-product structure discerns from a product space.

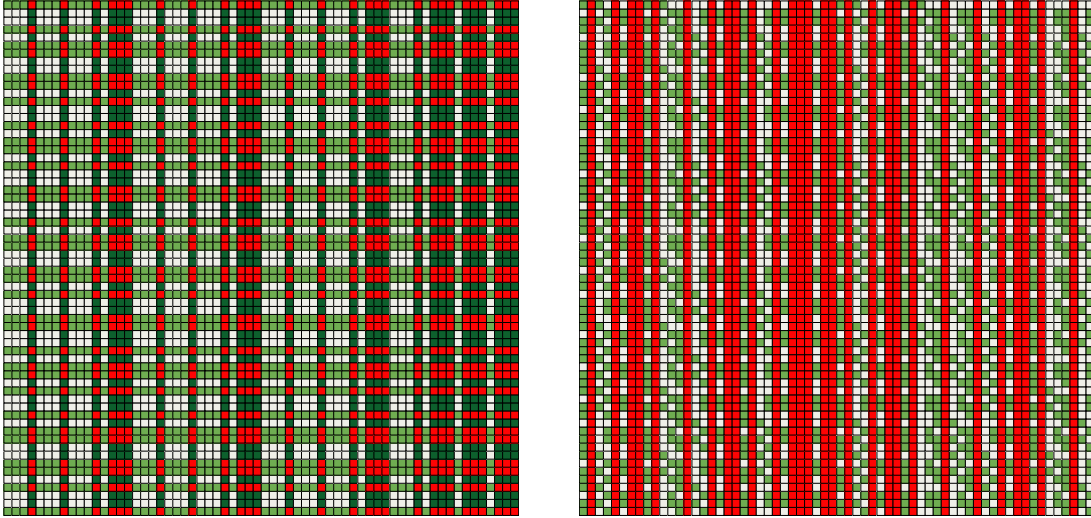


Figure 4: Patches from Examples 3.3 and 3.5, respectively. That their tiling spaces are not homeomorphic can only be detected through the cup product.

Lastly, let us compute the frequency module. The substitution matrix on the 2-cells of the dual complex is  $\sigma = \sigma_{A4}$  which is found in the Appendix and has Perron–Frobenius eigenvector

$$(2512, 2048, 2048, 2304, 55, 112, 2352, 121, 696, 2352, 1792, 4096, 80, 313, 791, 712, 448, 216, 976, 384, 168).$$

Its sum is 24576, giving us that the frequency module is  $\frac{1}{3}\mathbb{Z}[1/2]$ , making it indistinguishable from the frequency module of the product.

For illustration, we also compute the frequency modules for substitution on the vertical and horizontal 1-cells of the dual complex. There are two components to the vertical 1-cells. The substitution matrices are then

$$\sigma_{v_1}^1 = (4), \quad \sigma_{v_2}^1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \quad \sigma_h^1 = \begin{pmatrix} 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

By replacing  $e_{4_1}^1$  with  $e_{4_1}^1 + e_{4_2}^1 = (2, 2, 0, 0)$ , the Perron–Frobenius eigenvectors that are duals of lifts of the generators of  $\check{H}^1(\Omega_\zeta)$  of eigenvalue 4 are  $(1)$ ,  $(1, 1, 0, 2)$ , and  $(2, 3, 1, 0, 1, 2, 0, 1, 2)$  with the sum of the first two the lift of the new  $e_{4_1}^1$ . They have sums 1, 4, and 12, respectively, giving frequency modules  $\mathbb{Z}[1/2]$ ,  $\mathbb{Z}[1/2]$ , and  $\frac{1}{3}\mathbb{Z}[1/2]$ , respectively.

The patches from Examples 3.3 and 3.5 are shown in Figure 4.



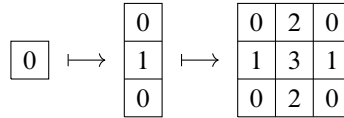


Figure 5: A two-dimensional analogue of the main example in Section 4.1. We begin with a one-dimensional checkerboard substitution on 8 prototiles (drawn vertically in the first map; only the first supertile is drawn, the rest are obtained by  $+1 \pmod 8$ ), then extending it along the remaining coordinate (horizontally) by another (different) checkerboard (second map). The resulting tiling has rotational symmetry about axis the pattern is extended along (horizontal here).

This example is built from checkerboard patterns (Figure 5). It appears that although checkerboard patterns do not necessarily force the border (see Remark 2.1), the level of supertiles required to exhaust all 1-collared<sup>4</sup> prototiles to construct the dual complex is not too large. Since we want even torsion, we create a checkerboard pattern on  $[0, 3]^3$ , then extend in the last dimension with yet another checkerboard pattern that depends on the prototile in the pattern on  $[0, 3]^3$ , taking the result to be the supertiles, so as to introduce four-fold rotational symmetry with odd expansion. For this particular example, the dual complex has 1120 4-cells, 1232 3-cells, 480 2-cells, 88 1-cells, and 8 0-cells. One can get by with fewer prototiles or simpler patterns (see the next subsection), but this is the first example we discovered where everything we desire is evident.

We only focus on  $\check{H}^2(\Omega_\zeta; \mathbb{Z})$  and  $\check{H}^4(\Omega_\zeta; \mathbb{Z})$ , since we want to identify a class  $[c] \in \check{H}^2(\Omega_\zeta; \mathbb{Z})$  so that  $[c]^2 = [c] \smile [c] \in \check{H}^4(\Omega_\zeta; \mathbb{Z})$  is not (uniquely) divisible by 2 (see (3)).  $\check{H}^2(\Omega_\zeta; \mathbb{Z})$  is the direct limit computed via the matrix  $\sigma^2 = \sigma_{A5}^2$ , which is found in the Appendix, and  $\check{H}^4(\Omega_\zeta; \mathbb{Z})$  is computed via a matrix too large to be listed here ( $354 \times 354$ ), but it suffices to know that for the dual complex,  $\check{H}^4 \cong \mathbb{Z}_4^3 \oplus F$  for some torsion-free group  $F$ . Since we only care about the even torsion terms, there being  $\mathbb{Z}_4^3$  in  $\check{H}^4$  of the dual complex, we write the substitution action on the dual complex as

$$(7) \quad \sigma^4 = \left( \begin{array}{c|c} \sigma_{\text{torsion}}^4 & * \\ \hline 0 & \sigma_{\text{torsion-free}}^4 \end{array} \right) = \left( \begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 2 & 3 & * \\ 0 & 3 & 2 & \\ \hline 0 & & & \sigma_{\text{torsion-free}}^4 \end{array} \right),$$

where  $\sigma_{\text{torsion}}^4$  is the action restricted to the torsion part of  $\check{H}^4$ .

Let us consider the eigenvector of  $\sigma^2$ , with eigenvalue 3,

$$v = (0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, -1, 1, 1, 0, 0, 0, 0, 0, 0, 0).$$

Lifting it to a 2-cochain, cupping it with itself via the cubical cup-product formula given in [15], then projecting to  $\check{H}^4$  of the dual complex gives the class

$$w = v \smile v = (2, 0, 2, \dots) \in \mathbb{Z}_4^3 \oplus F,$$

<sup>4</sup>Although the top-dimensional cells are “half-collars”, the construction of the complex requires knowledge of how they are identified in the tiling, thus another “half-collar” of the half-collared prototiles, i.e., 1-collared prototiles.







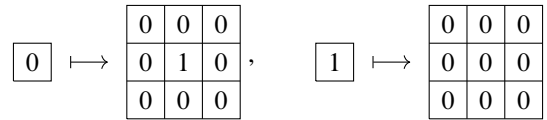


Figure 8: The four-dimensional version of this substitution rule is the same 3-skeleton as  $T^4$ , with the only difference being that there are two 4-cells (the two prototiles) attached to the 3-cells.

Its two-dimensional analogue is Figure 8.

This substitution is primitive and recognizable, and clearly forces the border (see Remark 2.1), thus we can use the *uncollared* AP-complex instead, which is the AP-construction without using collared prototiles. By the same theorem in [2], its inverse limit is homeomorphic to the tiling space.

The  $2^4$  patch of all 0’s indicates that the complex contains a copy of  $T^4$ . The prototile 1 being present in all possible positions in a  $2^4$  patch asserts that its associated 4-cell is attached to the 3-skeleton in the same exact way as the 4-cell associated to the prototile 0.

All of the coboundary maps are trivial, giving us that the cohomology groups we are interested in are

$$\check{H}^4(\Omega_\zeta; \mathbb{Z}) = \varinjlim \left( \mathbb{Z}^2, \begin{pmatrix} 80 & 1 \\ 81 & 0 \end{pmatrix} \right), \quad \check{H}^2(\Omega_\zeta; \mathbb{Z}) = \varinjlim (\mathbb{Z}^6, 9 \cdot \text{Id}),$$

where  $\mathbb{Z}^2$  is simultaneously  $C^4$  and  $H^4$  of the AP-complex, and  $\mathbb{Z}^6$  is simultaneously  $C^2$  and  $H^2$  of the AP-complex. Due to this, the cubical cup product on the cohomology ring coincides with the cubical cup product on the cochains, which is easy to describe. For example, the two generators  $c_1 = [0, 1] \times [0, 1] \times [0] \times [0]$ ,  $c_2 = [0] \times [0] \times [0, 1] \times [0, 1] \in C^2$  cup to  $(1, 1) \in C^4$ , the sum of the duals of the two prototiles. That is, the cohomology ring structure of this AP-complex only differs from that of  $T^4$  by the cup product always witnessing both 4-cochains whenever it is nontrivial.

One then follows the same exact argument provided in Theorem 2.6 and concludes that this substitution tiling space, upon quotienting by the same  $\mathbb{Z}_2$ -action, yields a counterexample to Conjecture 2.3. To be precise, let  $\Gamma$  be the AP complex of the substitution described above, and let  $\rho : \Gamma \rightarrow \mathbb{T}^4$  be the factor map which collapses the 4 cells. There is an involution on  $\Gamma$  which is equivariant (under  $\rho$ ) with the involution of  $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$  used in Theorem 2.6. This involution is defined by exchanging the 2-cells  $c_1$  and  $c_2$  described above. The induced map on cohomology  $\rho^* : H^4(\mathbb{T}^4; \mathbb{Z}) \rightarrow H^4(\Gamma; \mathbb{Z})$  sends the generator in  $H^4(\mathbb{T}^4; \mathbb{Z})$  to  $(1, 1) \in H^4(\Gamma; \mathbb{Z})$ . Let  $S_4 = \varinjlim (\mathbb{T}^4, 3 \cdot \text{Id})$  be the four dimensional solenoid constructed with maps of expansion 3, and note that it is a factor of the tiling space  $\Omega$  corresponding to the substitution above. Then the argument from Theorem 2.6 carries over to  $S_4$  in the direct limit since the expansion is by 3, and it can be pulled back to  $H^*(\Omega; \mathbb{Z})$  using the map induced by the factor map (because the expansions of both systems are by 3 and thus cannot be divided by 2 by the Chern character). Thus this example provides an aperiodic counterexample to Conjecture 2.3.

Let us remark on the mechanism that yields these counterexamples. In the original AP-complex, the nontrivial squares are of the form  $(c_1 + c_2) \smile (c_1 + c_2) = c_1 \smile c_1 + c_1 \smile c_2 + c_2 \smile c_1 + c_2 \smile c_2 = 2c_1 \smile c_2$ , which always return twice the sum of the generators in  $H^4$ , and therefore still yields integrality of







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