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Relative bounded cohomology on groups with contracting elements

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Let G be a countable group acting properly on a metric space with contracting elements and $\{H_i : 1 \le i \le n\}$ be a finite collection of Morse subgroups in G. We prove that each H_i has infinite index in G if and only if the relative second bounded cohomology $H_b^2(G, \{H_i\}_{i=1}^n; \mathbb{R})$ is infinite-dimensional. In addition, we also prove that for any contracting element g, there exists $k > 0$ such that $H_b^2(G, \langle\langle g^k \rangle\rangle; \mathbb{R})$ is infinite-dimensional. Our results generalize a theorem of Pagliantini–Rolli for finite-rank free groups and yield new results on the (relative) second bounded cohomology of groups.

1 Introduction

1.1 (Relative) bounded cohomology

Bounded cohomology was introduced by Johnson and Trauber in the context of Banach algebra and developed into a comprehensive and rich theory by Gromov in his seminal paper "Volume and bounded cohomology" [28]. Since then, it has become a fundamental tool in several fields, most notably in the study of the geometry of manifolds. Many properties, in particular geometric ones, of a group can be characterized by its bounded cohomology. In particular, the notion of bounded cohomology allows us to retrieve certain negatively curved features of the group.

The theory of quasimorphisms has been extensively exploited to study the second bounded cohomology of a group. To be precise, a *quasimorphism* on a group G is a map $\phi : G \to \mathbb{R}$ such that

$$\sup_{g,h \in G} |\phi(gh) - \phi(g) - \phi(h)| < \infty.$$

That is to say, a quasimorphism is locally close to a genuine homomorphism from the group to $\mathbb{R}$. We denote by $\mathrm{QM}(G)$ the $\mathbb{R}$-vector space of quasimorphisms. It follows from the definitions that the coboundary of a quasimorphism is a bounded 2-cocycle and therefore there is a linear map

$$\mathrm{QM}(G) \to H_b^2(G; \mathbb{R}).$$

Moreover, the image of this map is the kernel of the comparison map $H_b^2(G; \mathbb{R}) \to H^2(G; \mathbb{R})$ induced by the inclusion of bounded cochains into ordinary cochains [12, Theorem 2.50].

The first example of this approach is using Brooks' counting quasimorphism [9] to show that the rank-2 free group F_2 has infinite-dimensional second bounded cohomology.

More generally, D. Epstein and K. Fujiwara [18] showed that a nonelementary hyperbolic group has infinite-dimensional second bounded cohomology. This was proved by using a modified version of Brooks

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counting quasimorphism. Later, Fujiwara used this generalization to obtain the same conclusion regarding the dimension of the second bounded cohomology of a group acting properly on a hyperbolic space [24].

When dealing with manifolds with boundary we naturally consider the relative homology and cohomology. Similarly, in group theory, we often investigate the subgroups in order to explore properties of the ambient group. M. Gromov provided a definition of relative bounded cohomology between pairs of topological spaces and also pairs of groups in [28]. This leads to applications in geometry, topology and other fields.

In retrospect, absolute bounded cohomology of many classes of groups are already known to be infinite-dimensional, as demonstrated in

(1) nonelementary Gromov hyperbolic groups [18];

(2) groups with infinitely many ends [25];

(3) groups admitting a nonelementary proper discontinuous action on a Gromov hyperbolic space [24];

(4) groups admitting a nonelementary weakly proper discontinuous action on a Gromov hyperbolic space [5] (see also [30]);

(5) groups admitting a nonelementary weakly proper discontinuous action on a CAT(0) space which contains a rank-one isometry [6];

(6) acylindrically hyperbolic groups [33].

Gromov developed the theory of bounded cohomology in order to compute simplicial volume of manifolds. In particular, the understanding of simplicial volume of manifolds with nonempty boundary could be increased by studying bounded cohomology of pairs of space and pairs of groups, which is the so-called *relative bounded cohomology*. However, very few results on relative bounded cohomology are known. The first computation of the relative bounded cohomology of a specific class of groups was given by C. Pagliantini and P. Rolli [42]. Besides, it was proved independently by Bucher–Burger–Frigerio–Iozzi–Pagliantini–Pozzetti [11] and Kim–Kuessner [35] that the bounded cohomology of a CW-complex X relative to an amenable subcomplex Y is isometrically isomorphic to the absolute bounded cohomology of X; in particular, in this situation, $H_b^*(X, Y; \mathbb{R})$ is infinite-dimensional whenever X fits into the list given in the previous paragraph. Another relevant result was given by Franceschini [21], who proved that if (G, H) is a relatively hyperbolic pair, then the comparison map $H_b^k(G, H; V) \to H^k(G, H; V)$ is surjective for every $k \geq 2$ and any bounded G-module V.

1.2 Contracting element

The purpose of this paper is to generalize the existing results on the second bounded cohomology mentioned above to the relative version. Note that the examples of groups described above all satisfy the contracting properties outlined below.

Let X be a geodesic metric space and A be a closed subset of X. For a constant $C \geq 0$, we denote by $N_C(A)$ the open C-neighborhood of A in X. Let

$$\pi_A : X \to 2^A, \quad x \mapsto \pi_A(x) = \{y \in A : d(x, y) = d(x, A)\},$$

be the map given by the closest point projection. We say that A is *C-contracting* for $C \geq 0$ if $\mathrm{diam}(\pi_A(\gamma)) \leq C$ for any geodesic (segment) γ with $\gamma \cap N_C(A) = \varnothing$. In fact, this notion of contracting is equivalent to the usual one: $\mathrm{diam}(\pi_A(B)) \leq C'$ for any metric ball B disjoint from A; see [6, Corollary 3.4] for a proof. For an isometric group action of a group G on X, an element $g \in G$ is called *contracting* if some (or equivalently, any) orbit of $\langle g \rangle$ is a contracting quasigeodesic.

The prototype of a contracting element is a loxodromic isometry on a Gromov hyperbolic space, but many more examples are known to be contracting:

- rank-1 elements in CAT(0) groups acting on a CAT(0) space, see [2; 6];
- hyperbolic elements in groups with nontrivial Floyd boundary (e.g., relatively hyperbolic groups) acting on their Cayley graph with respect to a generating set, see [17; 40; 47];
- certain infinite-order elements in graphical small cancellation groups acting on their Cayley graph with respect to a generating set, see [1];
- pseudo-Anosov elements in mapping class groups acting on the Teichmüller space equipped with Teichmüller metric, or on the curve complex, see [20; 19; 44].

In this paper, a group is called *nonelementary* if it is not virtually cyclic. Given an isometric group action on a metric space, a subgroup is called *Morse* if some (or equivalently, any) orbit of this subgroup is weakly quasiconvex (see Definition 3.1). Now, we state our main result.

Theorem 1.1 *Let G be a nonelementary countable group acting properly on a geodesic metric space with contracting elements. Consider a finite collection of Morse subgroups $H_1, \dots, H_n$ with infinite index. Then there is an injective $\mathbb{R}$-linear map $\omega : \ell^1 \to H_b^2(G;\mathbb{R})$ such that each coclass in the image $\omega(\ell^1)$ has a representative vanishing on H_i for each i $(1 \leq i \leq n)$.*

Moreover, the dimension of $H_b^2(G, \{H_i\}_{i=1}^n;\mathbb{R})$ as a vector space over $\mathbb{R}$ has the cardinality of the continuum.

Here we denote by ℓ^1 the Banach space of summable sequences of real numbers with the norm $\|(x_i)\| = \sum_{i=1}^{\infty} |x_i|$.

Remark 1.2 (1) If one of the H_i's is of finite index in G, then Corollary 2.24 below implies that $H_b^2(G, \{H_i\}_{i=1}^n;\mathbb{R}) = 0$. Based on this fact, we only consider the second relative bounded cohomology of G with respect to subgroups with infinite index. Since any finitely generated subgroup of a free group is Morse, Theorem 1.1 generalizes the result of Pagliantini–Rolli [42] which states that for a free group F_n $(n \geq 2)$ and a finitely generated subgroup $H \leq F_n$, the subgroup H has infinite index in F_n if and only if the dimension of the second relative bounded cohomology $H_b^2(F_n, H;\mathbb{R})$ as a vector space over $\mathbb{R}$ is infinite.

(2) In [23; 33], the authors extend a nontrivial quasimorphism on a subgroup to the ambient group. Here we take the opposite approach, by constructing a quasimorphism on the ambient group whose restriction to the subgroup is trivial.

To prove Theorem 1.1, we need the following result.

Proposition 1.3 (Proposition 3.3) *Let G be a nonelementary countable group acting properly on a geodesic metric space with contracting elements. Consider a finite collection of Morse subgroups $H_1, \dots, H_n$ with infinite index. Then there is a quasitree on which G acts and each H_i acts elliptically on it for $1 \le i \le n$.*

An interesting corollary of Theorem 1.1 is as follows. This generalizes a result of Kotschick [36, Corollary 11]. See Definition 5.5 for the definition of bounded generation.

Corollary 1.4 (Corollary 5.6) *Under the assumption of Theorem 1.1, G is not boundedly generated by $\{H_i : 1 \le i \le n\}$.*

When a subgroup is a Morse subgroup of infinite index, its limit set is always a proper subset of the limit set of the ambient group, which provides evidence for the existence of enough relative quasimorphisms. (See [31; 49] for more details about convergence boundary.) Thus, we can ask the following question.

Question 1.5 *Let G be a nonelementary countable group acting properly on a geodesic metric space X with convergence boundary. Let $\{H_1, \dots, H_n\}$ be a finite collection of subgroups with proper limit sets. Then is the dimension of $H_b^2(G, \{H_i\}_{i=1}^n; \mathbb{R})$ as a vector space over $\mathbb{R}$ infinite?*

Another natural question is that when the subgroup is taken to be a normal subgroup of infinite index, does the conclusion of Theorem 1.1 still hold? If the normal subgroup has an amenable quotient, then Proposition 2.25 gives a negative answer. If the normal subgroup is normally generated by a higher power of a contracting element, then we have the following result.

Proposition 1.6 (Proposition 6.1) *Let G be a nonelementary countable group acting properly on a geodesic metric space with contracting elements. Then for any contracting element $g \in G$, there exists $k = k(g) > 0$ such that the dimension of $H_b^2(G, \langle\langle g^k \rangle\rangle; \mathbb{R})$ as a vector space over $\mathbb{R}$ has the cardinality of the continuum.*

Remark 1.7 Proposition 1.6 is already known for nonelementary hyperbolic groups. In [15], Delzant showed that for any hyperbolic element g in a nonelementary hyperbolic group G, there exists $k \in \mathbb{N}$ such that $G/\langle\langle g^k \rangle\rangle$ is still hyperbolic. Together with [18, Theorem 1.1] and Proposition 2.22 below, one gets the result.

In general, we raise the following question.

Question 1.8 *Let G be a nonelementary countable group acting properly on a geodesic metric space X with contracting elements. Let H be a normal subgroup of G with nonamenable quotient. Is the dimension of $H_b^2(G, H; \mathbb{R})$ as a vector space over $\mathbb{R}$ infinite?*

1.3 Sketch of proof

There are two main ingredients in the proof of Theorem 1.1. The first is Proposition 1.3, namely the construction of an appropriate projection complex on which each H_i acts elliptically. This notion was introduced by Bestvina–Bromberg–Fujiwara [3]. In Section 3, we are going to make use of the Morse property of $\{H_i\}$ (see Lemma 3.9) to show that there exists a contracting element g such that every $h \in H_i$

has a uniformly bounded projection to the geodesic segment $[o, go]$. To achieve this goal, we use some techniques developed by Han–Yang–Zou [31]. Then we construct the projection complex $\mathcal{P}_K(\mathcal{F})$ whose vertices are the G-translates of the axis $\mathrm{Ax}(g)$ of g and show that each H_i acts elliptically on $\mathcal{P}_K(\mathcal{F})$ (see Lemma 3.10). The projection complex is shown to be a quasitree on which G acts acylindrically (see Section 5).

The second ingredient is Proposition 4.1, which states that if G acts WPD (a weaker notion than acylindrical action) on a δ-hyperbolic space X and each H_i acts elliptically on X, then the dimension of $H_b^2(G, \{H_i\}_{i=1}^n; \mathbb{R})$ as a vector space over $\mathbb{R}$ has the cardinality of the continuum. To prove this proposition, we need a result of Bestvina–Fujiwara [5, Proposition 2] which constructs an infinite collection of words $\{f_i\}$ in a rank-two free subgroup of G. As in [24], we can produce a corresponding collection of quasimorphisms $\{h_i\}$ on G, which satisfy some special properties stated in Proposition 4.13. Then Proposition 4.1 follows from Proposition 4.13. The bridge connecting two ingredients is a result of Bestvina–Bromberg–Fujiwara–Sisto [4, Theorem 5.6]. As a result, we get Theorem 1.1.

As for Proposition 1.6, where H is assumed to be a normal subgroup generated by a higher power of a contracting element, we first construct a quasitree of spaces $\mathcal{C}(\mathcal{F})$ for every contracting element which is a blow-up of the projection complex. The space $\mathcal{C}(\mathcal{F})$ turns out to be a quasitree on which G acts acylindrically (see Lemma 6.4). Later, we construct a hyperbolic cone-off over a scaled $\mathcal{C}(\mathcal{F})$ along $\mathcal{F}$ and obtain a very rotating family (see Lemma 6.5). With the help of the theory of rotating families developed by Dahmani–Guirardel–Osin [14], we can reduce the proof of Proposition 1.6 to the proof of Proposition 4.14, which is completed at the end of Section 4.

Structure of the paper The paper is organized as follows. Section 2 is devoted to recalling some preliminary material about Gromov-hyperbolic spaces, contracting subsets, projection complexes, (relative) bounded cohomology, and quasimorphisms. In Section 3, we prove Proposition 1.3. Specifically, we make use of the recent work of Han–Yang–Zou [31] to construct a projection complex and show that each H_i acts elliptically on it. In Section 4, we review the previous work on Epstein–Fujiwara quasimorphisms in [24] and prove Proposition 4.1. Then we complete the proof of Theorem 1.1 in Section 5. In Section 6, we first recall some facts about hyperbolic cone-offs and rotation families, and then we prove Proposition 1.6.

2 Preliminaries

We first introduce some fundamental notation and definitions that will be used throughout this paper.

2.1 Gromov-hyperbolic spaces

We only introduce some necessary knowledge about Gromov-hyperbolic spaces here. For a more detailed introduction, we refer the readers to [8, Part III. H; 16, Chapter 11]. Let (X, d) be a geodesic metric space. For $S \subset X$ and $r > 0$, we denote by $N_r(S)$ the open r-neighborhood of S. For two subsets $S, T \subset X$, we denoted by $d_H(S, T)$ the *Hausdorff distance* between A and B, which is defined by $d_H(S, T) := \inf\{r > 0 : S \subset N_r(T), T \subset N_r(S)\}$.

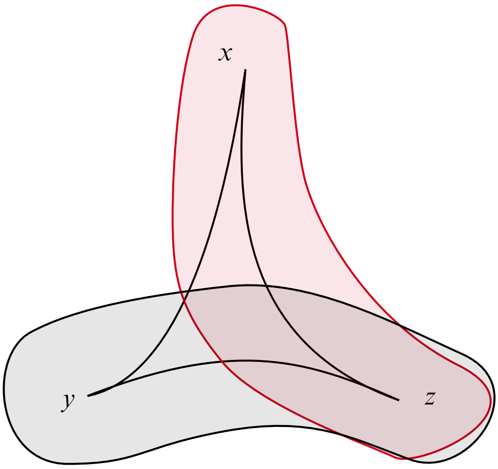

Figure 1: $[x,y] \subset N_\delta([x,z] \cup [y,z])$.

For any two points $x, y \in X$, denote by $[x,y]$ a choice of a geodesic segment between x and y. A *geodesic triangle* in X consists of three points $x, y, z \in X$ and three geodesic segments $[x,y]$, $[y,z]$, and $[z,x]$.

A geodesic metric space (X,d) is called (*Gromov*) δ*-hyperbolic* for a constant $\delta \geq 0$ if every geodesic triangle in X is δ*-thin*: each of its sides is contained in the δ-neighborhood of the union of the other two sides. See Figure 1 for an illustration.

A finitely generated group is called *Gromov-hyperbolic* if its Cayley graph with respect to some finite generating set is a δ-hyperbolic metric space for some $\delta \geq 0$.

Let (X_1,d_1) and (X_2,d_2) be two metric spaces. A (not necessarily continuous) map $f : X_1 \to X_2$ is called a (λ,ϵ)*-quasi-isometric embedding* if there exist constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that for all $x, y \in X_1$ we have

$$\frac{1}{\lambda} d_1(x,y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x,y) + \epsilon.$$

If, in addition, every point of X_2 lies in the ϵ-neighborhood of the image of f, then f is called a (λ,ϵ)*-quasi-isometry*. When such a map exists, the two spaces X_1 and X_2 are said to be *quasi-isometric*.

A (λ,ϵ)*-quasigeodesic* in a metric space X is the image of a (λ,ϵ)-quasi-isometric embedding $c : I \to X$, where I is an interval (possibly bounded or unbounded). For simplicity, a (λ,λ)-quasigeodesic will be referred to as a λ*-quasigeodesic*.

The following result is the well-known Morse lemma in geometric group theory. We refer the readers to [8, Chapter III.H, Theorem 1.7; 16, Theorem 11.72] for a proof.

Lemma 2.1 (Morse lemma) *For all $\delta \geq 0$, $\lambda \geq 1$, $\epsilon \geq 0$, there exists a constant $L = L(\lambda,\epsilon,\delta)$ such that any two (λ,ϵ)-quasigeodesics in a δ-hyperbolic space with the same endpoints are contained in an L-neighborhood of each other.*

Isometries on Gromov-hyperbolic spaces Let (X, d) be a Gromov-hyperbolic space. Two geodesic rays in X are said to be *asymptotic* if the Hausdorff distance between them is finite. Being asymptotic is an equivalence relation on geodesic rays. The *Gromov boundary* of X, denoted by ∂X, is defined to be the set of equivalence classes of geodesic rays in X. We refer the readers to [8; 16] for a detailed discussion about Gromov boundary. By Gromov [29], the isometries of a hyperbolic space X can be subdivided into three classes. A nontrivial element $g \in \mathrm{Isom}(X)$ is called *elliptic* if some $\langle g\rangle$-orbit is bounded. Otherwise, it is called *loxodromic* (resp. *parabolic*) if it has exactly two fixed points (resp. one fixed point) in the Gromov boundary of X. If g is a loxodromic element, any $\langle g\rangle$-invariant quasigeodesic between the two fixed points will be referred to as a *quasiaxis* for g, denoted by L_g.

For an isometry g on a hyperbolic space (X, d), we define the *stable translation length* $\|g\|$ of g as

$$\|g\| := \lim_{n\to\infty} \frac{d(x, g^n x)}{n}$$

for some (or any) $x \in X$. A well-known fact is that an isometry g is loxodromic if and only if $\|g\| > 0$ [13, Chapter 10, Proposition 6.3].

Lemma 2.2 *Let g be an isometry on a metric space (X, d). Then for any $n > 0$ and $x \in X$, one has $d(x, g^n x) \geq n\|g\|$.*

Proof For any $m, n > 0$, it follows from $d(x, g^{mn}x) \leq m \cdot d(x, g^n x)$ that

$$\frac{d(x, g^{mn}x)}{mn} \leq \frac{d(x, g^n x)}{n}.$$

By letting $m \to \infty$, one gets that

$$\|g\| \leq \frac{d(x, g^n x)}{n}. \qquad \square$$

2.2 Contracting subsets

Let (X, d) be a geodesic metric space. For any two $A, B \subset X$, define $d(A, B) := \inf_{x\in A, y\in B} d(x, y)$. For a given closed subset $A \subset X$, define the *closest point projection* $\pi_A : X \to 2^A$ as follows:

- For any $x \in X$, $\pi_A(x) := \{y \in A : d(x, y) = d(x, A)\}$.
- For any subset $B \subset X$, $\pi_A(B) := \bigcup_{x\in B} \pi_A(x)$.

We use the notation $\mathrm{diam}(A)$ to denote the diameter of a subset A in X.

Definition 2.3 (contracting subset) Let (X, d) be a geodesic metric space. A subset $Y \subseteq X$ is called C-*contracting* for $C \geq 0$ if for any geodesic (segment) α in X with $d(\alpha, Y) \geq C$, we have $\mathrm{diam}(\pi_Y(\alpha)) \leq C$. The subset Y is called a *contracting subset* if there exists $C \geq 0$ such that Y is C-contracting, and C is called a *contraction constant* of Y. See Figure 2 for an illustration.

Let G be a finitely generated group. A subgroup $H \leq G$ is called *contracting* if it is a contracting subset in the Cayley graph $\mathcal{G}(G, S)$ of G with respect to some finite generating set S.

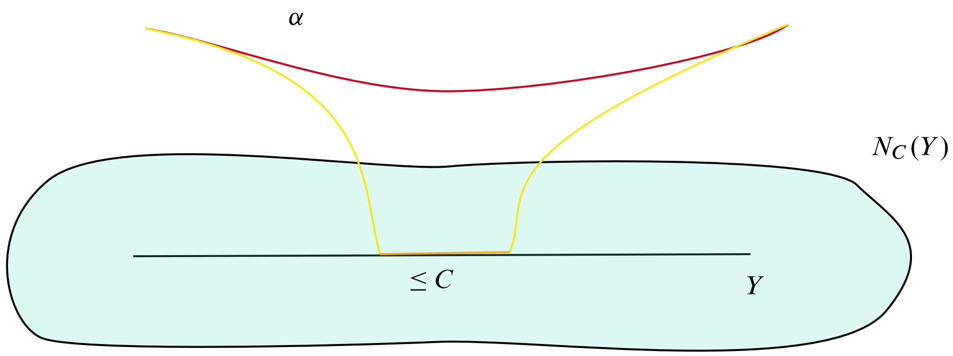

Figure 2: Y is C-contracting.

Example 2.4 The following are well-known examples of contracting subsets and contracting subgroups:

(1) bounded sets in a metric space;

(2) quasigeodesics and quasiconvex subsets in Gromov-hyperbolic spaces [27];

(3) fully quasiconvex subgroups, and maximal parabolic subgroups in particular, in relatively hyperbolic groups [26, Proposition 8.2.4];

(4) the subgroup generated by a hyperbolic element in groups whose Floyd boundary is nontrivial [47, Section 7];

(5) contracting segments and axes of rank-1 elements in CAT(0)-spaces in the sense of Bestvina and Fujiwara [6, Corollary 3.4];

(6) the axis of any pseudo-Anosov element in the Teichmüller space equipped with Teichmüller distance by a theorem of Minsky [37].

It has been proven in [6, Corollary 3.4] that the definition of a contracting subset is equivalent to the following one considered by Minsky [37]. A subset $Y \subseteq X$ is contracting if and only if there exists $C' \geq 0$ such that any open metric ball B with $B \cap Y = \varnothing$ satisfies $\operatorname{diam}(\pi_Y(B)) \leq C'$.

Despite this equivalence, we will always rely on Definition 2.3 for a contracting subset.

Definition 2.5 (contracting system) Let (X, d) be a geodesic metric space. A set $\mathbb{X} = \{X_i : X_i \subset X\}_{i \in \mathbb{N}}$ is called a *contracting system* with a contraction constant C if each X_i is a C-contracting subset in X.

Definition 2.6 (bounded intersection) Two subsets $Y, Z \subseteq X$ have *$\mathcal{R}$-bounded intersection* for a function $\mathcal{R} : [0, +\infty) \to [0, +\infty)$ if $\operatorname{diam}(N_r(Y) \cap N_r(Z)) \leq \mathcal{R}(r)$, for all $r \geq 0$. A contracting system $\mathbb{X}$ has *$\mathcal{R}$-bounded intersection* if any two elements in $\mathbb{X}$ have *$\mathcal{R}$-bounded intersection*.

Admissible paths For a finite rectifiable path p in a metric space X, we denote by $|p|$ the length of p and by p_-, p_+ the initial and terminal points of p respectively.

Definition 2.7 (admissible path) Let (X, d) be a geodesic metric space and $\mathbb{X}$ be a contracting system in X. Let $D, \tau \geq 0$ and $\mathcal{R} : [0, +\infty) \to [0, +\infty)$ be a function, which is called the *bounded intersection gauge*.

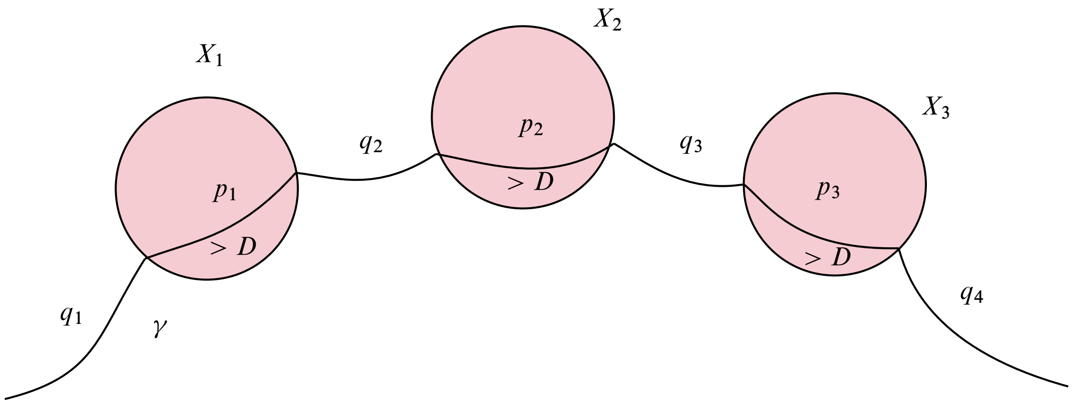

Figure 3: $\gamma = q_1 p_1 q_2 p_2 q_3 p_3 q_4$ is a (D, τ)-admissible path.

A path γ is called a (D, τ)-*admissible path* with respect to $\mathbb{X}$ if the path γ consists of a (finite, infinite, or bi-infinite) concatenation of consecutive geodesic segments $\gamma = \cdots q_i p_i q_{i+1} p_{i+1} \cdots$, satisfying the following "long local" and "bounded projection" properties:

- **(LL1)** For each p_i there exists $X_i \in \mathbb{X}$ such that the two endpoints of p_i lie in X_i, and $|p_i| > D$ unless p_i is the first or last geodesic segment in γ.
- **(BP)** For each X_i we have $\operatorname{diam}(\pi_{X_i}\{(p_i)_+, (p_{i+1})_-\}) \le \tau$, and $\operatorname{diam}(\pi_{X_i}\{(p_{i-1})_+, (p_i)_-\}) \le \tau$. Here $(p_{i+1})_- = \gamma_+$ if p_{i+1} does not exist, and $(p_{i-1})_+ = \gamma_-$ if p_{i-1} does not exist.
- **(LL2)** Either $X_i \ne X_{i+1}$ and X_i and X_{i+1} have $\mathcal{R}$-bounded intersection, or $|q_i| > D$.

Remark 2.8 See Figure 3 for an illustration of an admissible path. The collection of $X_i \in \mathbb{X}$ indexed as in (LL1), denoted by $\mathbb{X}(\gamma)$, will be referred to as *contracting subsets* for γ. The union of all $X_i \in \mathbb{X}(\gamma)$ is called the *saturation* of γ.

In the following definitions, a sequence of points x_i in a path α is called *linearly ordered* if x_{i+1} lies in the subpath of α from x_i to α_+ for each i.

Definition 2.9 (fellow travel) Let $\gamma = p_0 q_1 p_1 \cdots q_n p_n$ be a (D, τ)-admissible path, and α be a path such that $\alpha_- = \gamma_-$, $\alpha_+ = \gamma_+$. Given $\epsilon > 0$, the path α ϵ-*fellow travels* γ if there exists a sequence of linearly ordered points $z_0, w_0, z_1, w_1, \ldots, z_n, w_n$ on α such that $d(z_i, (p_i)_-) \le \epsilon$, $d(w_i, (p_i)_+) \le \epsilon$.

See Figure 4 for an illustration of ϵ-fellow travel property.

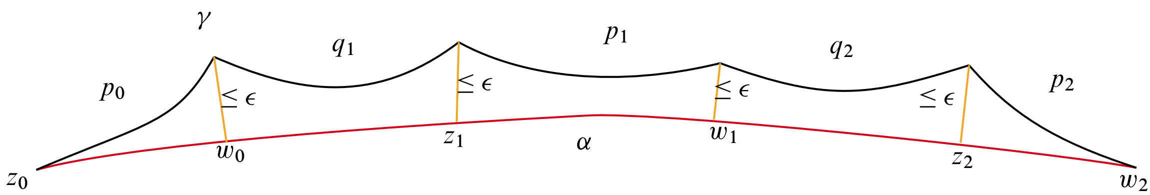

Figure 4: $\gamma = p_0 q_1 p_1 q_2 p_2$ is a (D, τ)-admissible path and α ϵ-fellow travels γ.

Proposition 2.10 [48, Proposition 2.7] *For any $\tau > 0$, and any function $\mathcal{R} : [0, +\infty) \to [0, +\infty)$, there exist $D, \lambda, \epsilon > 0$ depending on $\tau, \mathcal{R}$ such that any (D, τ)-admissible path is a λ-quasigeodesic which is ϵ-fellow traveled by any geodesic with the same endpoints.*

Group actions with contracting elements Let G be a group acting isometrically on a geodesic metric space (X, d) with a base point $o \in X$.

Definition 2.11 (contracting element) An element $h \in G$ is called a *contracting element* if $\langle h \rangle \cdot o$ is a contracting subset in X and the map $\mathbb{Z} \to X$, $n \mapsto h^n o$, is a quasi-isometric embedding.

A group action $G \curvearrowright X$ is called *(metrically) proper* if for any $D > 0$, the set $\{g \in G : d(o, go) \le D\}$ is finite. From now on, we assume that G acts properly on (X, d) with a contracting element.

Definition 2.12 (weakly independent) Suppose $g, h \in G$ are two contracting elements. They are called *weakly independent* if $\langle g \rangle \cdot o$ and $\langle h \rangle \cdot o$ have $\mathcal{R}$-bounded intersection for some $\mathcal{R} : [0, +\infty) \to [0, +\infty)$.

By [48, Lemma 2.11], each contracting element g is contained in a maximal elementary subgroup $E(g)$ defined as

$$E(g) = \{h \in G : \exists r > 0, h\langle g \rangle o \subset N_r(\langle g \rangle o) \text{ and } \langle g \rangle o \subset N_r(h\langle g \rangle o)\},$$

and the index $[E(g) : \langle g \rangle]$ is finite. Roughly speaking, the subgroup $E(g)$ is the set of elements that do not move the orbit of g too much. Hence, if G is not virtually cyclic, then there are at least two weakly independent contracting elements in G. For example, let $g \in G$ be a contracting element by assumption. Since G is not virtually cyclic, $E(g)$ is a proper subgroup of G. By selecting an element $f \in G \setminus E(g)$, one has that g and fgf^{-1} are weakly independent [46, Lemma 7.13]. Actually, by [46, Lemma 2.30], there exist infinitely many pairwise weakly independent elements in G.

For a contracting element g, the subset defined by $\mathrm{Ax}(g) := E(g)o$ is called an *extended-defined quasigeodesic*, which is also called the *axis* of g (depending on o). Compared with the quasiaxis L_g of a loxodromic element g which can be an arbitrary $\langle g \rangle$-invariant quasigeodesic, we use the term "extended-defined" here to emphasize their difference.

The following result, proved in [48, Proposition 2.9 and Lemma 2.14], gives a procedure to construct infinitely many contracting elements.

Lemma 2.13 (extension lemma) *Suppose that a nonelementary group G acts properly on (X, d) with a contracting element. Then there exist a set $F \subset G$ of three contracting elements and $D, \tau, \lambda > 0$ with the following property. For any $g \in G$, there exists $f \in F$ such that the bi-infinite concatenated path $\gamma = \bigcup_{n \in \mathbb{Z}} (gf)^n [o, gfo]$ is a (D, τ)-admissible path with contracting subsets $\{(gf)^n g \,\mathrm{Ax}(f) : n \in \mathbb{Z}\}$. In particular, γ is a C-contracting λ-quasigeodesic, where C depends on $d(o, go)$. Hence gf is a contracting element.*

2.3 Projection complexes

In the assumption of our main result, i.e., Theorem 1.1, the geodesic metric space X on which the group G acts with contracting elements is not hyperbolic. To apply the method proposed by Bestvina–Fujiwara [5], we need to construct a suitable hyperbolic space on which G acts. This construction makes use of the projection complex techniques developed by Bestvina–Bromberg–Fujiwara [3], which will be introduced in this subsection.

Definition 2.14 (projection axioms) Let $\mathcal{F}$ be a collection of metric spaces equipped with (set-valued) projection maps

$$\{\pi_U : \mathcal{F} \setminus \{U\} \to 2^U\}_{U \in \mathcal{F}}.$$

Define $d_U(V, W) := \operatorname{diam}(\pi_U(V) \cup \pi_U(W))$ for $V \neq U \neq W \in \mathcal{F}$. The pair $(\mathcal{F}, \{U\}_{U \in \mathcal{F}})$ satisfies projection axioms for a constant $\kappa > 0$ if

(1) $\operatorname{diam}(\pi_U(V)) \leq \kappa$ when $U \neq V$;

(2) if U, V, W are distinct and $d_V(U, W) > \kappa$ then $d_U(V, W) \leq \kappa$;

(3) the set $\{U \in \mathcal{F} : d_U(V, W) > \kappa\}$ is finite for $V \neq W$.

We caution the readers that the projection maps in the above definition are just abstract maps, which may not be the closest point projections defined previously. By definition, the triangle inequality

$$d_Y(V, W) \leq d_Y(V, U) + d_Y(U, W) \tag{2-1}$$

holds for any $U, V, Y, W \in \mathcal{F}$. It is well known that the projection axioms hold for a contracting system with bounded intersection (see [47, Appendix]). From now on, we assume that G acts properly on X with contracting elements. Fix a base point $o \in X$.

Lemma 2.15 [49, Lemma 2.18] *Let $g \in G$ be a contracting element. Then the set $\mathcal{F} = \{f \operatorname{Ax}(g) : f \in G\}$ with closest point projections $\pi_U(V)$ satisfies the projection axioms with a constant $\kappa = \kappa(\mathcal{F}) > 0$.*

In [3, Definition 3.1], a modified version of d_U is introduced such that it is symmetric and agrees with the original d_U up to an additive amount 2κ. Thus, the axioms (1)–(3) still hold for 3κ, and the triangle inequality in (2-1) holds up to a uniform error. In what follows, we actually need to work with this modified d_U to define the projection complex, but for the sake of simplicity, we stick to the above definition of d_U.

We consider the interval-like set, for $K > 0$ and $V, W \in \mathcal{F}$,

$$\mathcal{F}_K(V, W) := \{U \in \mathcal{F} : d_U(V, W) > K\}.$$

Define $\mathcal{F}_K[V, W] := \mathcal{F}_K(V, W) \cup \{V, W\}$. It possesses a total order described in the next lemma. Let $\mathcal{F}$ and κ be given by Lemma 2.15.

Lemma 2.16 [3, Theorem 3.3.G] *There exist constants $D = D(\kappa)$, $K = K(\kappa) > 0$ such that the set $\mathcal{F}_K[V, W]$ admits a total order "$<$" with least element V and greatest element W, such that given $U_0, U_1, U_2 \in \mathcal{F}_K[V, W]$, if $U_0 < U_1 < U_2$, then*

$$d_{U_1}(V, W) - D \leq d_{U_1}(U_0, U_2) \leq d_{U_1}(V, W) \quad \text{and} \quad d_{U_0}(U_1, U_2) \leq D \text{ and } d_{U_2}(U_0, U_1) \leq D.$$

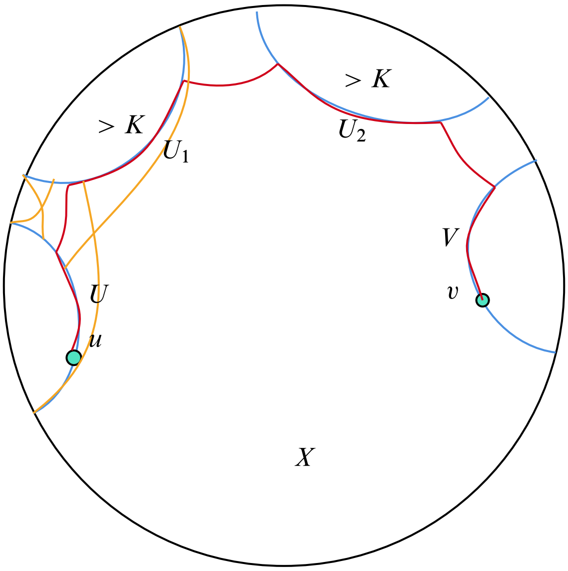

Figure 5: Each blue line is a translate of $\mathrm{Ax}(g)$; $\mathcal{F}_K[U, V] \cup \{U, V\} = \{U < U_1 < U_2 < V\}$ is a standard path in $\mathcal{P}_K(\mathcal{F})$; the yellow line represents the closest point projection between U and U_1; the red line is a lifted standard path from $u \in U$ to $v \in V$ in X.

We now give the definition of a projection complex.

Definition 2.17 The projection complex $\mathcal{P}_K(\mathcal{F})$ for K satisfying Lemma 2.16 is a graph with the vertex set consisting of the elements in $\mathcal{F}$. Two vertices U and V are connected if $\mathcal{F}_K(U, V) = \varnothing$. We equip $\mathcal{P}_K(\mathcal{F})$ with a length metric $d_{\mathcal{P}}$ induced by assigning unit length to each edge.

The projection complex $\mathcal{P}_K(\mathcal{F})$ is connected, since by [3, Proposition 3.7], the interval set $\mathcal{F}_K[U, V]$ gives a connected path between U and V in $\mathcal{P}_K(\mathcal{F})$: the consecutive elements directed by the total order are adjacent in $\mathcal{P}_K(\mathcal{F})$. The path $\mathcal{F}_K[U, V] \cup \{U, V\} = \{U < U_1 < \cdots < U_k < V\}$ is called the *standard path* from U to V. See Figure 5 for an illustration of a standard path. By [4, Corollary 3.7], standard paths are $(2, 1)$-quasigeodesics in $\mathcal{P}_K(\mathcal{F})$. A *quasitree* is a geodesic metric space quasi-isometric to a tree. The structural result about the projection complex is the following.

Lemma 2.18 [3, Corollary 3.25] *For $K \gg 0$ as in Lemma 2.16, the projection complex $\mathcal{P}_K(\mathcal{F})$ is a quasitree, on which G acts nonelementarily and coboundedly.*

For any two points $u \in U$ and $v \in V$, we often need to *lift* a standard path $\mathcal{F}_K[U, V] \cup \{U, V\} = \{U < U_1 < \cdots < U_k < V\}$ in $\mathcal{P}_K(\mathcal{F})$ to a path from u to v in X. The *lifted standard path* in X is a piecewise geodesic path (admissible path) from u to v as concatenation of the normal paths between two consecutive vertices which are G-translates of $\mathrm{Ax}(g)$ in X and geodesics contained in vertices. See Figure 5 for an illustration of a lifted standard path. This is explained by the following lemma proved in [32, Lemma 4.5].

Lemma 2.19 *For any $K > 0$, there exist a constant $D = D(K,\kappa) \ge 0$ with $D \to \infty$ as $K \to \infty$ and a uniform constant $B = B(\kappa) > 0$ with the following property. For any two points $u \in U$, $v \in V$ there exists an (D, B)-admissible path γ in X from u to v with saturation $\mathcal{F}_K[U, V]$.*

In practice, we always assume that K is sufficiently large such that by taking $\tau = B$ and $\mathcal{R}$ as the bounded intersection gauge of $\mathcal{F}$, the constant D in the above lemma satisfies Proposition 2.10, and then the path γ shall be a quasigeodesic.

2.4 (Relative) bounded cohomology

This subsection is devoted to the basic definitions and results on the bounded cohomology of a group and the relative bounded cohomology of a pair of groups. We refer the reader to [10] for full details on ordinary group cohomology theory, to [22; 28; 34] on bounded cohomology of groups, and to [28; 43] for the relative case.

Let G be a group. For a coefficient ring R, the *bar complex* $C_*(G; R)$ is the complex generated in dimension n by n-tuples $(g_1, \dots, g_n)$ with $g_i \in G$ and with boundary map ∂ defined by the formula

$$\partial(g_1, \dots, g_n) = (g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^n (g_1, \dots, g_{n-1}).$$

We let $C^*(G; R)$ denote the dual cochain complex $\mathrm{Hom}(C_*(G), R)$, and let d denote the adjoint of ∂. The homology groups of $C^*(G; R)$ are called the *group cohomology of G with coefficients in R*, and are denoted by $H^*(G; R)$.

In this paper, we take $R = \mathbb{R}$. A cochain $\alpha \in C^n(G; \mathbb{R})$ is *bounded* if

$$\sup |\alpha(g_1, \dots, g_n)| < \infty,$$

where the supremum is taken over all n-tuples $(g_1, \dots, g_n)$ with $g_i \in G$. The set of all bounded cochains forms a subcomplex $C_b^*(G; \mathbb{R})$ of $C^*(G; \mathbb{R})$, and its homology is the so-called *bounded cohomology* $H_b^*(G; \mathbb{R})$.

Amenable groups Recall that a *mean* on G is a linear functional on $C_b^1(G; \mathbb{R})$ which maps the constant function $\phi(g) = 1$ to 1, and maps nonnegative functions to nonnegative numbers.

Definition 2.20 A group G is *amenable* if there is a G-invariant mean $\pi : C_b^1(G; \mathbb{R}) \to \mathbb{R}$ where G acts on $C_b^1(G; \mathbb{R})$ by

$$g \cdot \phi(h) = \phi(g^{-1} h)$$

for all $g, h \in G$ and $\phi \in C_b^1(G; \mathbb{R})$.

Examples of amenable groups are finite groups, solvable groups, and Grigorchuk's groups of intermediate growth.

We list three important facts here for future use. Some of them are mentioned in the introduction. See [7; 12, §2.4.2] for details.

Lemma 2.21 (1) $H_b^1(G;\mathbb{R}) = 0$ *for any group* G.

(2) $H_b^*(G;\mathbb{R})$ *vanishes identically when* G *is an amenable group.*

(3) *Let* $H \to G \to K \to 1$ *be a (right) exact sequence of groups. Then the induced sequence on second bounded cohomology*

$$0 \to H_b^2(K;\mathbb{R}) \to H_b^2(G;\mathbb{R}) \to H_b^2(H;\mathbb{R})$$

is (left) exact.

Unless otherwise stated, *a pair of groups* (G, H) in this paper means that G is a group and H is a subgroup of G. Now we come to the definition of relative bounded cohomology of the pair (G, H). The kernel of the obvious restriction map $C_b^*(G,\mathbb{R}) \to C_b^*(H,\mathbb{R})$ is denoted by $C_b^*(G, H;\mathbb{R})$, and we have the short exact sequence of complexes

$$0 \to C_b^*(G, H;\mathbb{R}) \to C_b^*(G;\mathbb{R}) \to C_b^*(H;\mathbb{R}) \to 0,$$

which induces a long exact sequence in cohomology

$$\cdots \to H_b^{n-1}(H;\mathbb{R}) \to H^n(C_b^*(G, H;\mathbb{R})) \to H_b^n(G;\mathbb{R}) \to H_b^n(H;\mathbb{R}) \to \cdots. \tag{2-2}$$

The module $H^n(C_b^*(G, H;\mathbb{R}))$ is the n-th bounded cohomology of the pair (G, H). We denote it as $H_b^n(G, H;\mathbb{R})$. Let $\{H_i\}_{i=1}^m$ be a finite collection of subgroups in G. We denote by $C_b^*(G,\{H_i\}_{i=1}^m;\mathbb{R})$ the kernel of the multiple restriction map $C_b^*(G,\mathbb{R}) \to \bigoplus_{i=1}^m C_b^*(H_i,\mathbb{R})$. In the same way as above, we define $H_b^n(G,\{H_i\}_{i=1}^m;\mathbb{R})$ as the n-th bounded cohomology $H^n(C_b^*(G,\{H_i\}_{i=1}^m;\mathbb{R}))$. There are no substantial differences from the relative cohomology in ordinary cohomology group theory.

Proposition 2.22 *Let* (G, H) *be a pair of groups with* $H \lhd G$. *Then* $H_b^2(G/H;\mathbb{R}) \cong H_b^2(G, H;\mathbb{R})$.

Proof Recall that Lemma 2.21(3) gives us a left-exact sequence

$$0 \to H_b^2(G/H;\mathbb{R}) \to H_b^2(G;\mathbb{R}) \to H_b^2(H;\mathbb{R}).$$

From the definition of relative bounded cohomology (2-2), we also have the exact sequence

$$H_b^1(H;\mathbb{R}) \to H_b^2(G, H;\mathbb{R}) \to H_b^2(G;\mathbb{R}) \to H_b^2(H;\mathbb{R}).$$

As any group has a trivial first bounded cohomology, we can conclude that

$$H_b^2(G/H;\mathbb{R}) \cong \ker(H_b^2(G;\mathbb{R}) \to H_b^2(H;\mathbb{R})) \cong H_b^2(G, H;\mathbb{R}). \tag{2-3}$$ □

We remark that the second isomorphism in (2-3) always holds as long as (G, H) is a pair of groups.

Finite-index subgroups Let (G, H) be a pair of groups. The inclusion map from H to G induces the *restriction map* which is denoted by $\mathrm{res} : H^n(G,\mathbb{R}) \to H^n(H,\mathbb{R})$. A subgroup H of a topological group G is called *cocompact* (or uniform) if the quotient space $G/\overline{H}$ is compact, where $\overline{H}$ denotes the closure of H in G. For cocompact subgroups such that the quotient admits a finite invariant measure, the restriction map from the cohomology group of the ambient group to the cohomology group of the subgroup is injective. This notably encompasses the case of uniform lattices (of Lie groups) and finite index subgroups (of any group). The standard argument uses the existence of a *transfer map* which is the left inverse to the restriction map (or up to a constant multiple depending on different definitions of the transfer map). The transfer map is obtained by integration (or finite sum for finite index subgroups), so it is crucial for the subgroup to be cocompact. See [38, §8.6] for a detailed discussion of the transfer map.

Lemma 2.23 [38, Propositon 8.6.2] *Let (G, H) be a pair of groups. If H has finite index in G, then the natural restriction map $H_b^2(G;\mathbb{R}) \to H_b^2(H;\mathbb{R})$ is isometrically injective.*

Combining with the remark following Proposition 2.22, we have immediately that:

Corollary 2.24 *Let (G, H) be a pair of groups. If H has finite index in G, then $H_b^2(G, H;\mathbb{R}) = 0$.*

The corollary above gives us a better understanding of the relations between the index of a subgroup and the relative bounded cohomology group. In fact, Corollary 2.24 is the main motivation for the authors of [42] and us to consider the relative bounded cohomology of infinite index subgroups, aiming to find some nontrivial results. By combining Lemma 2.21(2) and Proposition 2.22, one gets another simple example as follows.

Proposition 2.25 *Let (G, H) be a pair of groups. If H is a normal subgroup with amenable quotient, then $H_b^2(G, H;\mathbb{R}) = 0$.*

2.5 Quasimorphisms

In this subsection, we recall the notion of quasimorphism on a group G. Typically, one proves that $H_b^2(G;\mathbb{R})$ is infinite-dimensional by demonstrating the existence of infinitely many linearly independent quasimorphisms on G.

Definition 2.26 A map $\phi : G \to \mathbb{R}$ is called a *quasimorphism* if there exists a constant $C > 0$ such that

$$|\phi(gh) - \phi(g) - \phi(h)| < C \quad \text{for all } g, h \in G.$$

The *defect* of a quasimorphism ϕ is defined to be

$$\Delta(\phi) := \sup_{g,h\in G} |\phi(gh) - \phi(g) - \phi(h)|.$$

The defect of a quasimorphism measures how far it is from a genuine homomorphism from the group to $\mathbb{R}$. A quasimorphism ϕ is called *trivial* if there exists a bounded map $b : G \to \mathbb{R}$ and a group homomorphism $\rho : G \to \mathbb{R}$ such that $\phi = b + \rho$.

We denote by $\mathrm{QM}(G)$ the $\mathbb{R}$-vector space of quasimorphisms on G and by $\mathrm{QM}_0(G)$ the subspace of $\mathrm{QM}(G)$ consisting of all trivial quasimorphisms on G, which is exactly $C_b^1(G;\mathbb{R}) \oplus \mathrm{Hom}(G,\mathbb{R})$.

For a quasimorphism ϕ on G, define the 1-coboundary of ϕ by $d^1\phi(g, h) = \phi(g) + \phi(h) - \phi(gh)$. Thus, $d^1\phi$ is a bounded 2-cocycle. We denote by $\omega_\phi := [d^1\phi]_b$ the corresponding bounded cohomology class. Then we have a linear map

$$\mathrm{QM}(G) \to H_b^2(G;\mathbb{R}), \quad \phi \mapsto \omega_\phi,$$

such that the sequence

$$\mathrm{QM}(G) \to H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$$

is exact [12, Theorem 2.50].

There is another important special kind of quasimorphism called a *homogeneous quasimorphism*.

Definition 2.27 A quasimorphism $\phi : G \to \mathbb{R}$ is homogeneous if $\phi(g^n) = n \cdot \phi(g)$ for every $g \in G$ and every $n \in \mathbb{Z}$.

A *class function* on G is a function that takes the same value on each conjugacy class.

Lemma 2.28 *A homogeneous quasimorphism is a class function.*

Proof Let ϕ be a homogeneous quasimorphism on G. For any two group elements $f, g \in G$, it follows from the definition of homogeneous quasimorphisms that

$$\begin{aligned}|\phi(g) - \phi(fgf^{-1})| = |\phi(g) + \phi(fg^{-1}f^{-1})| &= \frac{|\phi(g^n) + \phi(fg^{-n}f^{-1})|}{n} \\ &\le \frac{|\phi(g^n) + \phi(g^{-n})| + |\phi(f)| + |\phi(f^{-1})| + 2\Delta(\phi)}{n} = \frac{2|\phi(f)| + 2\Delta(\phi)}{n}.\end{aligned}$$

By letting $n \to \infty$, one has that $\phi(g) = \phi(fgf^{-1})$. Since f, g are arbitrary, we complete the proof. □

Remark 2.29 [12, Lemma 2.21, Corollary 2.59] Let ϕ be a quasimorphism on G. Then there exists a unique homogeneous quasimorphism $\bar{\phi}$ which stays at finite distance from ϕ. In fact, the corresponding homogeneous quasimorphism to ϕ is given by

$$\bar{\phi}(g) := \lim_{n\to\infty} \frac{\phi(g^n)}{n} \quad \text{for all } g \in G.$$

This limit exists because the coarse subadditive inequality $\phi(g^{n+m}) \le \phi(g^n) + \phi(g^m) + \Delta(\phi)$ holds. Moreover, the defect $\Delta(\bar{\phi})$ is related to $\Delta(\phi)$ by $\Delta(\bar{\phi}) \le 2\Delta(\phi)$.

We denote the subspace of $\mathrm{QM}(G)$ consisting of all homogeneous quasimorphisms on G as $\mathrm{QM}_h(G)$.

3 Morse subgroups of infinite index

In this section, we assume that G is a nonelementary group acting properly on a geodesic metric space (X, d) with contracting elements. Fix a base point $o \in X$. We first define what is a Morse subgroup.

Definition 3.1 (Morse property) A subset $A \subset X$ is η*-Morse* for a function $\eta : \mathbb{R} \to \mathbb{R}$ if every λ-quasigeodesic with endpoints in A is contained in the $\eta(\lambda)$-neighborhood of A. The function η is called a *Morse gauge* of A. A subgroup $H \le G$ is *Morse* if the subset $H \cdot o$ is η-Morse for some function $\eta : \mathbb{R} \to \mathbb{R}$.

Remark 3.2 A C-contracting set is η_0-Morse for some function $\eta_0 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ depending only on C [45, Lemma 2.8(1)].

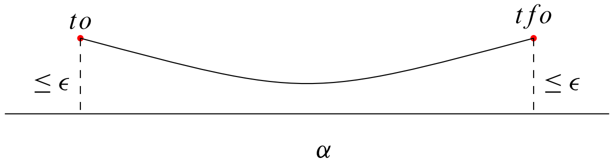

Figure 6: α contains an (ϵ, f)-barrier.

From now on, we assume that $\{H_i\}_{1\le i\le n}$ is a finite collection of Morse subgroups with infinite index in G. The purpose of this section is to prove Proposition 1.3, which is restated as:

Proposition 3.3 *There exists a quasitree on which G acts coboundedly and each H_i acts elliptically for $1 \le i \le n$.*

Proof ideas of Proposition 3.3 For a contracting element $g \in G$, Section 2.3 shows that there exists a projection complex $\mathcal{P}_K(\mathcal{F})$ where $\mathcal{F} = \{f\,\mathrm{Ax}(g) : f \in G\}$ such that $\mathcal{P}_K(\mathcal{F})$ is a quasitree on which G acts coboundedly. Hence, we only need to find an appropriate contracting element $g \in G$ such that each H_i acts elliptically on $\mathcal{P}_K(\mathcal{F})$ for $1 \le i \le n$. By analyzing the algebraic and geometric properties of $\{H_i : 1 \le i \le n\}$, we can obtain a contracting element g such that the projection of $[o, ho]$ onto $\mathrm{Ax}(g)$ is uniformly bounded for any $h \in \bigcup_{1\le i\le n} H_i$. Finally, we will show that such a contracting element g meets the requirements.

In order to characterize the magnitude of the (closest point) projection to an axis of a contracting element, we introduce a definition of barriers.

Definition 3.4 [48, Definition 4.1] Let $\epsilon \ge 0$ and $f \in G$. We say a geodesic $\alpha \subset X$ contains an (ϵ, f)-*barrier* if there exists an element $t \in G$ such that

$$d(to, \alpha) \le \epsilon, \quad d(tfo, \alpha) \le \epsilon.$$

Otherwise, α is called (ϵ, f)-*barrier-free*. An element $g \in G$ is called (ϵ, f)-*barrier-free* if some choice of geodesic from o to go is (ϵ, f)-barrier-free. See Figure 6 for an illustration.

Recall that the extension lemma, i.e., Lemma 2.13, gives a set $F \subset G$ consisting of three contracting elements. The following result of Han–Yang–Zou shows that for any contracting element $g = g_0 f$ obtained by the extension lemma, any geodesic segment with a large projection to $\mathrm{Ax}(g)$ contains an (ϵ, g_0)-barrier where ϵ depends only on F.

Lemma 3.5 [31, Lemma 2.9] *For any $g_0 \in G$, let $g = g_0 f$ be the contracting element given by Lemma 2.13. Then there exist $\epsilon = \epsilon(F)$ and $\tau = \tau(F, g)$ such that a geodesic segment α with $\mathrm{diam}(\pi_{\mathrm{Ax}(g)}(\alpha)) > \tau$ contains an (ϵ, g_0)-barrier.*

Remark 3.6 Actually, one can strengthen the conclusion of Lemma 3.5 as follows: if a geodesic segment α satisfies $\mathrm{diam}(\pi_{b\,\mathrm{Ax}(g)}(\alpha)) > \tau$ for some $b \in G$, then α contains an (ϵ, g_0)-barrier. The reason is that from the definition of barriers (see Definition 3.4), it is equivalent to say that α or $b^{-1}\alpha$ contains an (ϵ, g_0)-barrier. Thus, by substituting α with $b^{-1}\alpha$, one can apply Lemma 3.5 to obtain the stronger conclusion.

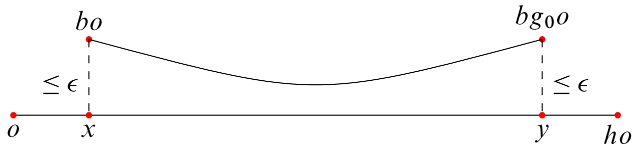

Figure 7: $[o, ho]$ contains an (ϵ, g_0)-barrier.

To get a contracting element such that the projection of each $[o, ho]$ to $\mathrm{Ax}(g)$ is uniformly bounded, we need to find an element g_0 such that h is (ϵ, g_0)-barrier-free for $h \in \bigcup_{1 \le i \le n} H_i$.

Lemma 3.7 [39, Lemma 4.1] *Let the group G be the union of finitely many cosets of subgroups $C_1, \dots, C_n$. Then the index of (at least) one of these subgroups in G does not exceed n.*

Lemma 3.8 *For any $\epsilon \ge 0$, there exists an element $g_0 \in G$ such that any $h \in H_i$ with $1 \le i \le n$ is (ϵ, g_0)-barrier-free.*

Proof At first, we claim that the set $G \setminus S\big(\bigcup_{1 \le i \le n} H_i\big)S$ is infinite for any finite subset $S \subset G$. Suppose not, by enlarging S by a finite set, we can assume that

$$G = S\Big(\bigcup_{1 \le i \le n} H_i\Big)S = \bigcup_{1 \le i \le n} (SH_iS) = \bigcup_{s \in S} \bigcup_{1 \le i \le n} (sH_is^{-1})(s \cdot S),$$

where each $s \cdot S$ is a finite set. As each conjugate of H_i is still an infinite-index subgroup of G, the above decomposition implies that G can be written as a finite union of cosets of infinite-index subgroups. This contradicts Lemma 3.7. The claim thus follows.

Since each H_i is Morse for $1 \le i \le n$, it follows from Definition 3.1 that there exists a function $\eta : \mathbb{R} \to \mathbb{R}$ such that all orbits $H_i \cdot o (1 \le i \le n)$ are η-Morse. Let $M = \eta(1)$. Define $S := \{g \in G : d(o, go) \le \epsilon + M\}$. As G acts properly on X, the set S is finite.

According to the above claim, the set $G \setminus S\big(\bigcup_{1 \le i \le n} H_i\big)S$ is infinite. We are going to show that every element $g_0 \in G \setminus S\big(\bigcup_{1 \le i \le n} H_i\big)S$ satisfies the requirement. Suppose to the contrary that there exist some $i \in \{1, \dots, n\}$ and some element $h \in H_i$ such that h is not (ϵ, g_0)-barrier-free. By definition of barriers, for any geodesic segment $[o, ho]$, there exist an element $b \in G$ and two points $x, y \in [o, ho]$ such that $d(bo, x), d(bg_0o, y) \le \epsilon$. See Figure 7 for an illustration.

Since the orbit $H_i \cdot o$ is η-Morse in X, by definition of Morse property, the geodesic segment $[o, ho]$ is contained in $N_M(H_io)$. Since $x, y \in [o, ho]$, there exist $h_1, h_2 \in H_i$ such that $d(x, h_1o), d(y, h_2o) \le M$. Combining two inequalities together, one gets that $d(bo, h_1o), d(bg_0o, h_2o) \le \epsilon + M$. By construction of S, we have $b^{-1}h_1, h_2^{-1}bg_0 \in S$. Therefore,

$$g_0 \in b^{-1}h_2S = b^{-1}h_1 \cdot h_1^{-1}h_2S \subset SH_iS,$$

which contradicts with the choice of g_0. □

By combining Remark 3.6 and Lemma 3.8, we obtain that:

Lemma 3.9 *There exist a contracting element $g \in G$ and $\tau > 0$ such that for any $b \in G$ and $h \in H_i$ with $1 \le i \le n$,*

$$\operatorname{diam}(\pi_{b\,\mathrm{Ax}(g)}([o, ho])) \le \tau.$$

Proof We caution the readers that the constant $\epsilon := \epsilon(F)$ given by Lemma 3.5 only depends on a prefixed set F. Then we choose g_0 by Lemma 3.8 such that each $h \in H_i$ with $1 \le i \le n$ is (ϵ, g_0)-barrier-free. Set $g = g_0 f$ given by Lemma 2.13 for some $f \in F$. Then $\tau = \tau(F, g)$ in Lemma 3.5 is the desired uniform projection constant. Otherwise, one gets a contradiction to Remark 3.6. □

For the contracting element given by Lemma 3.9, we next follow the process in Section 2.3 to construct a projection complex. Finally, we show that the projection complex is the quasitree which satisfies all requirements of Proposition 3.3.

Group actions on projection complex Let g be a fixed contracting element given by Lemma 3.9. Lemma 2.15 shows that $\mathcal{F} = \{f\,\mathrm{Ax}(g) : f \in G\}$ with shortest projection maps satisfies the projection axioms with constants $\kappa = \kappa(\mathcal{F}) > 0$. Hence, one can construct a projection complex $\mathcal{P}_K(\mathcal{F})$ (see Definition 2.17) for $K \gg 0$. As a result of Lemma 2.18, $\mathcal{P}_K(\mathcal{F})$ is a quasitree, on which G acts nonelementarily and coboundedly. Set $K \ge \tau + 2\kappa + 2\epsilon$ where τ is given by Lemma 3.9, κ is given by Lemma 2.15, and ϵ is the fellow travel constant (see Proposition 2.10) with respect to a (D, B)-admissible path given by Lemma 2.19.

Lemma 3.10 *For each $i \in \{1, \dots, n\}$, H_i acts elliptically on $\mathcal{P}_K(\mathcal{F})$.*

Proof Recall that the vertex set of $\mathcal{P}_K(\mathcal{F})$ is $\mathcal{F} = \{f\,\mathrm{Ax}(g) : f \in G\}$, and two vertices U, V are connected by an edge if the set $\mathcal{F}_K[U, V] = \{Z \in \mathcal{F} : d_Z(U, V) > K\}$ is empty. Let $U \in \mathcal{P}_K(\mathcal{F})$ be the point representing $\mathrm{Ax}(g)$ which by definition is the orbit $E(g) \cdot o$. It suffices for us to show that $d_{\mathcal{P}}(U, hU) \le 1$ for any $h \in H_i$ with $1 \le i \le n$.

Suppose not; then there exists at least one element in $\mathcal{F}_K[U, hU]$. Let $V \in \mathcal{F}_K[U, hU]$. It follows from the definition of $\mathcal{F}_K[U, hU]$ that $\operatorname{diam}(\pi_V(U) \cup \pi_V(hU)) > K$. Note that $o \in U$ and thus $ho \in hU$. Lemma 2.19 shows that there exists a (D, B)-admissible path in X from o to ho with saturation $\mathcal{F}_K[U, hU]$. It follows from the ϵ-fellow travel property of an admissible path that

$$\operatorname{diam}(\pi_V([o, ho])) \ge \operatorname{diam}(\pi_V(U) \cup \pi_V(hU)) - 2\kappa - 2\epsilon > K - 2\kappa - 2\epsilon \ge \tau.$$

As V represents a G-translate of $\mathrm{Ax}(g)$, one gets a contradiction to Lemma 3.9. □

As a corollary of Lemma 3.10, we get Proposition 3.3.

Remark 3.11 With Lemma 3.10, we know that the quasitree in Proposition 3.3 is actually a projection complex $\mathcal{P}_K(\mathcal{F})$. Since each vertex in $\mathcal{P}_K(\mathcal{F})$ represents a translate of $\mathrm{Ax}(g)$, any conjugate of $E(g)$ fixes a point in $\mathcal{P}_K(\mathcal{F})$. Hence, the action $G \curvearrowright \mathcal{P}_K(\mathcal{F})$ is not proper. Furthermore, it is generally difficult to obtain subgroups acting elliptically on $\mathcal{P}_K(\mathcal{F})$ other than the vertex stabilizer. However, Proposition 3.3 provides such a family of subgroups, i.e., Morse subgroups of infinite index.

4 Constructing quasimorphisms

In this section, we assume that a nonelementary countable group G acts WPD (see Definition 4.2) on a δ-hyperbolic space X and a finite collection of subgroups $\{H_i : 1 \le i \le n\}$ of G acts elliptically on X. Our goal is the following result.

Proposition 4.1 *There is an injective $\mathbb{R}$-linear map $\omega : \ell^1 \to H_b^2(G;\mathbb{R})$ such that each coclass in the image $\omega(\ell^1)$ has a representative vanishing on H_i for each $1 \le i \le n$.*

Moreover, the dimension of $H_b^2(G, \{H_i\}_{i=1}^n;\mathbb{R})$ as a vector space over $\mathbb{R}$ has the cardinality of the continuum.

Definition 4.2 [5] We say that the action of G on a hyperbolic space X satisfies *WPD* if

- G is not virtually cyclic,
- G contains at least one element that acts on X as a loxodromic isometry, and
- for every loxodromic element $g \in G$, every $x \in X$, and every $C > 0$, there exists $N > 0$ such that
$$\{h \in G : d(x, hx) \le C, d(g^N x, hg^N x) \le C\}$$
is finite.

In [5], Bestvina–Fujiwara showed that the dimension of $\mathrm{QM}(G)/(H^1(G;\mathbb{R}) \oplus C_b^1(G;\mathbb{R}))$ as a $\mathbb{R}$-vector space has the cardinality of the continuum under the assumption of WPD actions. This implies the absolute version of Proposition 4.1. However, the relative version of second bounded cohomology of a group acting WPD on a hyperbolic space has never been studied before, which is where the value of our Proposition 4.1 lies. Regarding the proof of Proposition 4.1, we generally follow the proof idea in [5], but we will utilize some new techniques (e.g., barriers) developed in this paper.

4.1 Epstein–Fujiwara quasimorphisms on groups acting on hyperbolic spaces

At first, let us recall some basic material about Epstein–Fujiwara quasimorphisms introduced in [24].

Let α be a finite path in X. We denote the length of α by $|\alpha|$. We use the action of $g \in G$ on X to define a path $g \cdot \alpha$ which is the g-translation of the path α. We say that $g \cdot \alpha$ is a *copy* of α. Let w be a finite oriented path, and let w^{-1} be the inverse path. We assume that $|w| \ge 2$ and define

$$|\alpha|_w := \{\text{the maximal number of copies of } w \text{ in } \alpha \text{ without overlapping (except at the vertices)}\}.$$

Suppose that $x, y \in X$ and that W is a number with $0 < W < |w|$. Recall that $[x, y]$ denotes some choice of a geodesic from x to y. We define

$$c_{w,W}([x, y]) = d(x, y) - \inf_{\alpha}\{|\alpha| - W|\alpha|_w\}, \tag{4-1}$$

where α ranges over all the paths from x to y. It follows from the definition that $c_{w,W}([x, y])$ does not depend on the choice of a geodesic $[x, y]$.

Remark 4.3 By choosing α to be a choice of geodesic $[x,y]$, one gets that

$$c_{w,W}([x,y]) \ge d(x,y) - (|[x,y]| - W|[x,y]|_w) = W|[x,y]|_w \ge 0.$$

Moreover, if $c_{w,W}([x,y]) = 0$, then the above inequality implies that $|[x,y]|_w = 0$ and the geodesic $[x,y]$ realizes the infimum in (4-1). However, if $c_{w,W}([x,y]) > 0$, then the realizing path (i.e., a path realizing the infimum in (4-1)) may not exist.

Lemma 4.4 [24, Lemma 3.3] *Suppose that a path α realizes the infimum above. Then the path α is a $\big(\frac{|w|}{|w|-W}, \frac{2W|w|}{|w|-W}\big)$-quasigeodesic.*

Since a realizing path does not always exist, we need another notion which is close to realizing paths in practice. A path β between x and y is called an *almost realizing path* of $c_{w,W}([x,y])$ if it satisfies that

$$|\beta| - W|\beta|_w \le \min\Big\{\inf_\alpha\{|\alpha| - W|\alpha|_w\} + W,\ \big(d(x,y) + \inf_\alpha\{|\alpha| - W|\alpha|_w\}\big)/2\Big\}, \tag{4-2}$$

where α ranges over all the paths from x to y. In other words, an almost realizing path β of $c_{w,W}([x,y])$ satisfies that

$$d(x,y) - (|\beta| - W|\beta|_w) \ge \max\big\{c_{w,W}([x,y]) - W,\ c_{w,W}([x,y])/2\big\}.$$

We remark that the requirement $d(x,y) - (|\beta| - W|\beta|_w) \ge c_{w,W}([x,y]) - W$ guarantees β is a uniform quasigeodesic (see Lemma 4.5 below) and the other requirement $d(x,y) - (|\beta| - W|\beta|_w) \ge c_{w,W}([x,y])/2$ is used to obtain $|\beta|_w > 0$ when $c_{w,W}([x,y]) > 0$. By definition and Remark 4.3, an almost realizing path of $c_{w,W}([x,y])$ always exists. Analogous to Lemma 4.4, we have that:

Lemma 4.5 *Let β be an almost realizing path of $c_{w,W}([x,y])$. Then the path β is a $\big(\frac{|w|}{|w|-W}, \frac{3W|w|}{|w|-W}\big)$-quasigeodesic.*

Proof Let $\beta : [0, |\beta|] \to X$ be an arc-length parametrization of β. Let $0 \le t < s \le |\beta|$ and set $\beta' = \beta|_{[t,s]}$. Note that $|\beta'| = s - t$. Let γ be a geodesic from $\beta(t)$ to $\beta(s)$.

Claim $$|\beta'| - W(|\beta'|_w + 3) \le |\gamma| - W|\gamma|_w.$$

Proof of claim Suppose to the contrary that $|\beta'| - W(|\beta'|_w + 3) > |\gamma| - W|\gamma|_w$. Since β is an almost realizing path, $|\beta| - W|\beta|_w \le \inf_\alpha\{|\alpha| - W|\alpha|_w\} + W$. By setting $\gamma' = \beta|_{[0,t]} \cup \gamma \cup \beta|_{[s,|\beta|]}$, one has that

$$\begin{aligned} |\gamma'| - W|\gamma'|_w &\le (t + |\gamma| + |\beta| - s) - W\big(\big|\beta|_{[0,t]}\big|_w + |\gamma|_w + \big|\beta|_{[s,|\beta|]}\big|_w\big) \\ &< |\beta| - W\big(\big|\beta|_{[0,t]}\big|_w + |\beta'|_w + \big|\beta|_{[s,|\beta|]}\big|_w + 3\big) \\ &\le |\beta| - W(|\beta|_w + 1) \le \inf_\alpha\{|\alpha| - W|\alpha|_w\}. \end{aligned}$$

This is impossible since γ' is also a path from x to y. □

Clearly $|\beta'|_w \le |\beta'|/|w|$. Therefore,

$$\begin{aligned} d(\beta(t),\beta(s)) = |\gamma| &\ge |\gamma| - W|\gamma|_w \ge |\beta'| - W|\beta'|_w - 3W \\ &\ge |\beta'| - \frac{W}{|w|}|\beta'| - 3W = \frac{|w| - W}{|w|}|\beta'| - 3W. \end{aligned}$$ □

It is clear from the definition that $c_{w,W}([x,y]) = c_{w^{-1},W}([y,x])$. We then define

$$h_{w,W}([x,y]) = c_{w,W}([x,y]) - c_{w^{-1},W}([x,y]).$$

Take $o \in X$ as a base point. We define functions $c_{w,W}$ and $h_{w,W} : G \to \mathbb{R}$ by

$$c_{w,W}(g) := c_{w,W}([o, g \cdot o]) \quad \text{and} \quad h_{w,W}(g) := h_{w,W}([o, g \cdot o]).$$

Lemma 4.6 [24, Proposition 3.10] *The map $h_{w,W} : G \to \mathbb{R}$ is a quasimorphism. Moreover, the defect $\Delta(h_{w,W}) \le 12L_0 + 6W + 48\delta$ is uniformly bounded where $L_0 = L\big(\frac{|w|}{|w|-W}, \frac{2W|w|}{|w|-W}, \delta\big)$ is given by Lemma 2.1.*

4.2 Constructing infinitely many words

From now on, we borrow some definitions and terminologies from [5]. For a base point $o \in X$ and a loxodromic element g, we denote by $L_g = \bigcup_{i \in \mathbb{Z}} g^i[o, go]$ a quasiaxis of g. Define $\mathrm{Ax}(g) := \bigcup_{k \ge 1} L_{g^k}$. By the Morse lemma, $\mathrm{Ax}(g)$ is still a quasiaxis of g. The quasiaxis $\mathrm{Ax}(g)$ of g is oriented by the requirement that g acts as a positive translation. We call this orientation the *g-orientation* of the quasiaxis. Of course, the g^{-1}-orientation is the opposite of the g-orientation. Let $\mathrm{Ax}(g)$ be a (λ, ϵ)-quasiaxis. By the Morse lemma, any two (λ, ϵ)-quasiaxes of g are within $L(\lambda, \epsilon, \delta)$ of each other. More generally, any sufficiently long path J inside the $L(\lambda, \epsilon, \delta)$-neighborhood of $\mathrm{Ax}(g)$ of g has a natural orientation given by g: a point of $\mathrm{Ax}(g)$ within $L(\lambda, \epsilon, \delta)$ of the terminal endpoint of J is ahead (with respect to the g-orientation of $\mathrm{Ax}(g)$) of a point of $\mathrm{Ax}(g)$ within $L(\lambda, \epsilon, \delta)$ of the initial endpoint of J. We call this orientation of J the *g-orientation*.

Definition 4.7 Let g_1 and g_2 be two loxodromic elements of G. We will write

$$g_1 \sim g_2$$

if there exists a constant $L' > 0$ such that an arbitrarily long segment J in $\mathrm{Ax}(g_1)$ is contained in an L'-neighborhood of $t\,\mathrm{Ax}(g_2)$ for some $t \in G$ and the map $t : J \to t(J)$ is orientation-preserving with respect to the g_1-orientation on J and the g_2-orientation on $t(J)$.

Note that $\sim$ is an equivalence relation. The following lemma gives a relation between the above definition and the definition (see Definition 3.4) of barriers which has nothing to do with the orientation.

Lemma 4.8 *If $g_1 \nsim g_2^{\pm 1}$, then for any $\epsilon' > 0$, there exists $r > 0$ such that g_2^m is (ϵ', g_1^s)-barrier-free for any $m \in \mathbb{Z}$ and $s \ge r$.*

Proof Suppose to the contrary that there exists $\epsilon' > 0$ such that for every $r \in \mathbb{N}$, there exist $m \in \mathbb{Z}$ and $s \ge r$ such that g_2^m is not (ϵ', g_1^s)-barrier-free. According to the definition of barriers, there exists $t \in G$ such that $d(to, [o, g_2^m o]) \le \epsilon'$ and $d(tg_1^s o, [o, g_2^m o]) \le \epsilon'$. See Figure 8 for an illustration.

Let $x, y \in [o, g_2^m o]$ such that $d(to, x) = d(to, [o, g_2^m o])$ and $d(tg_1^s o, y) = d(tg_1^s o, [o, g_2^m o])$. Hence, the path $\gamma = [x, to] \cup [to, tg_1^s o] \cup [tg_1^s o, y]$ is a $(1, 4\epsilon')$-quasigeodesic. By Lemma 2.1, $t[o, g_1^s o] \subset N_{L'}([o, g_2^m o])$

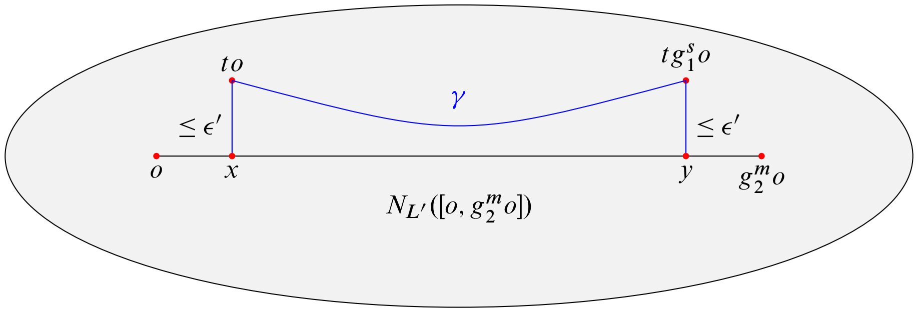

Figure 8: $[o, g_2^m o]$ contains an (ϵ', g_1^s)-barrier. The blue path represents γ. The gray area represents $N_{L'}([o, g_2^m o])$.

where $L' = L(1, 4\epsilon', \delta)$. Up to exchanging g_2 to g_2^{-1}, we can assume that the map t is orientation-preserving. Note that $s \geq r$. By Lemma 2.2, one has $d(o, g_1^s o) \geq s\|g_1\| \geq r\|g_1\|$, which implies that the length of $[o, g_1^s o] \subseteq \mathrm{Ax}(g_1)$ goes to infinity as $r \to \infty$. According to Definition 4.7, this shows that $g_1 \sim g_2$ or $g_1 \sim g_2^{-1}$ which is impossible. □

Proposition 6 in [5] shows that for every WPD action $G \curvearrowright X$, there exist two loxodromic elements g_1 and g_2 such that $g_1 \nsim g_2$. In [5, Proposition 2, Claim 1], Bestvina–Fujiwara proved that two elements f_1, f_2 are nonequivalent if they satisfy

$$f_1 = g_1^{n_1} g_2^{m_1} g_1^{k_1} g_2^{-l_1}, \quad f_2 = g_1^{n_2} g_2^{m_2} g_1^{k_2} g_2^{-l_2},$$

where $0 \ll n_1 \ll m_1 \ll k_1 \ll l_1 \ll n_2 \ll m_2 \ll k_2 \ll l_2$. But in fact, their proof only requires $0 \ll n_1, m_1, |k_1|, l_1 \ll n_2, m_2, |k_2|, l_2$, and does not require k_1, k_2 to be positive. So we can take k_1, l_1, k_2, l_2 to be $-n_1$, m_1, $-n_2$, m_2 respectively. In this case, we have $f_1 = g_1^{n_1} g_2^{m_1} g_1^{-n_1} g_2^{-m_1}$, $f_2 = g_1^{n_2} g_2^{m_2} g_1^{-n_2} g_2^{-m_2} \in [G, G]$ and the remaining proof of [5, Proposition 2, Claim 1] shows that:

Lemma 4.9 *There exist two loxodromic elements g_1 and g_2 in $[G, G]$ on X such that $g_1 \nsim g_2$.*

Since g_1 and g_2 are independent, we may replace g_1, g_2 by high positive powers of conjugates to ensure that the subgroup F of G generated by g_1, g_2 is free with basis $S = \{g_1, g_2\}$, each nontrivial element of F is loxodromic, and F is quasiconvex with respect to the action on X (see [24, Proposition 4.3]). We will call such free subgroups *Schottky groups*. Let $\mathcal{G}(F, S)$ be the Cayley graph of F with respect to the generating set $S = \{g_1, g_2\}$. Then $\mathcal{G}(F, S)$ is a tree and each oriented edge has a label $g_i^{\pm 1}$. Choose a base point $o \in X$ and construct an F-equivariant map $\Phi : \mathcal{G}(F, S) \to X$ that sends 1 to o and sends each edge to a geodesic arc. Quasiconvexity implies that Φ is a (λ_0, ϵ_0)-quasi-isometric embedding for some $\lambda_0 \geq 1, \epsilon_0 \geq 0$ and in particular for every $1 \neq f \in F$ the Φ-image of the axis of f in $\mathcal{G}(F, S)$ is a (λ_0, ϵ_0)-quasiaxis of f in X.

Choose positive constants

$$0 \ll n_1 \ll m_1 \ll k_1 \ll l_1 \ll n_2 \ll m_2 \ll \cdots$$

and define

$$f_i = g_1^{n_i} g_2^{m_i} g_1^{k_i} g_2^{-l_i}$$

for $i = 1, 2, 3, \dots$.

Proposition 4.10 [5, Proposition 2] *$\{f_i : i \geq 1\}$ is an infinite sequence of loxodromic elements in G such that*

(1) $f_i \nsim f_i^{-1}$ *for* $i \geq 1$, *and*

(2) $f_i \nsim f_j^{\pm 1}$ *for* $j < i$.

If $f \in F$ is cyclically reduced as a word in g_1, g_2 (equivalently, if its axis passes through $1 \in \mathcal{G}(F, S)$) then by the quasiconvexity of F in G we have

$$d(o, f^m(o)) \geq m\big(d(o, f(o)) - 2L_1\big), \tag{4-3}$$

where $L_1 = L(\lambda_0, \epsilon_0, \delta) > 0$ is given by Lemma 2.1 and is a constant independent of f and m.

4.3 Constructing infinitely many quasimorphisms

From now on, we fix an integer $W \geq 3L_1$ and will only consider a path w with $|w| > W$. Thus, an almost realizing path α as in Lemma 4.5 will be a quasigeodesic with constants independent of w and the endpoints. Moreover, α is contained in a uniform neighborhood, say, L_2-neighborhood, of any geodesic joining the endpoints of α. We will also omit W from the notation and write c_w and h_w for simplicity.

The next lemma is crucial for our discussion, not only in absolute bounded cohomology but also in the relative case. Recall the definition of barriers from Definition 3.4.

Lemma 4.11 *Let $w = [o, fo]$ and $g \in G$ be an (L_2, f)-barrier-free element. Then we have $c_w(g) = 0$ and $c_{w^{-1}}(g) = 0$.*

Proof Assume that $c_w(g) > 0$ and that α is an almost realizing path of $c_w(g)$. From Lemma 4.5 we know that α is a $\big(\frac{|w|}{|w|-W}, \frac{3W|w|}{|w|-W}\big)$-quasigeodesic. Thus, $\alpha \subseteq N_{L_2}([o, go])$. Additionally, we have $d(o, go) - (|\alpha| - W|\alpha|_w) \geq c_w(g)/2 > 0$. Therefore, $|\alpha|_w > (|\alpha| - d(o, go))/W > 0$. From the definition of $|\alpha|_w$, there exists some element $t \in G$ such that $t \cdot w \subseteq \alpha \subseteq N_{L_2}([o, go])$. This leads to a contradiction, as g is (L_2, f)-barrier-free. Therefore, $c_w(g) = 0$. We note that as long as there is no $t \in G$ such that $t \cdot w \subseteq N_{L_2}([o, go])$, there is also no $t \in G$ such that $t \cdot w^{-1} \subseteq N_{L_2}([o, go])$. The conclusion that $c_{w^{-1}}(g) = 0$ then follows. □

For simplicity, for any $f \in G$, we set $c_f := c_{[o, fo]}$ and $h_f := h_{[o, fo]}$. Let $\{f_i : i \geq 1\}$ be the sequence from Proposition 4.10. As the relation $\sim$ is invariant under conjugation, we assume in addition that each f_i is cyclically reduced.

Lemma 4.12 *For all $i \geq 1$, there exists $r_i > 0$ such that for all $j < i$, we have*

(1) $h_{f_i^{r_i}}(f_i^{r_i m}) \geq L_1 m$ *for any* $m \geq 0$,

(2) $h_{f_i^{r_i}}$ *is* 0 *on* $\langle f_j \rangle$.

Proof We first prove item (2). Fix $i \geq 1$. As each element in the finite set $A := \{f_1^{\pm 1}, \dots, f_{i-1}^{\pm 1}\}$ is not equivalent to f_i, for any sufficiently large $\epsilon' > 0$, Lemma 4.8 gives a constant $r_i > 0$ such that each $a \in A$ satisfies that a^m is $(\epsilon', f_i^{r_i})$-barrier-free for all $m \in \mathbb{Z}$.

Let $\epsilon' \geq L_2$. As a result of Lemma 4.11, we obtain

$$c_{f_i^{r_i}}(a^m) = c_{f_i^{-r_i}}(a^m) = 0$$

for any $a \in A$ and $n \in \mathbb{Z}$.

Hence, $h_{f_i^{r_i}}(f_j^m) = c_{f_i^{r_i}}(f_j^m) - c_{f_i^{-r_i}}(f_j^m) = 0$ for all $m \in \mathbb{Z}$.

Now, we turn to proving item (1).

Claim *For each $i \geq 1$, there exists $r_i' > 0$ such that $c_{f_i^r}(f_i^{-rm}) = 0$ for any $m \geq 0$ and $r \geq r_i'$.*

Proof of claim Suppose not. Then there exists $i \geq 1$ such that for every sufficiently large $r_i' > 0$, there exists $m \geq 0$ and $r \geq r_i'$ such that $c_{f_i^r}(f_i^{-rm}) > 0$. Define $w = [o, f_i^r o]$. Let α be an almost realizing path of $c_w(f_i^{-rm})$ in (4-2) with f_i^{-1}-orientation. Then we have $d(o, f_i^{-rm}o) - (|\alpha| - W|\alpha|_w) \geq c_w(f_i^{-rm})/2 > 0$. Thus, $|\alpha|_w > (|\alpha| - d(o, f_i^{-rm}o))/W > 0$. From the definition of $|\alpha|_w$ there exists some element $t \in G$ such that $t \cdot w \subseteq \alpha \subseteq N_{L_2}([o, f_i^{-rm}o])$ and the map t respects the orientation. As r_i' can be arbitrarily large, this implies that $f_i \sim f_i^{-1}$, which is a contradiction. □

Now we return to the proof of item (1). For each $i \geq 1$, Lemma 4.8 allows us to require $r_i \geq r_i'$ where r_i is the constant appearing in the proof of item (2). Define $w = [o, f_i^{r_i} o]$. For $n \geq 1$, let γ be the concatenated path $\bigcup_{0 \leq k \leq m-1} f_i^{r_i k} w$. According to (4-3), we have

$$|\gamma| = md(o, f_i^{r_i} o) \leq d(o, f_i^{r_i m} o) + 2L_1 m.$$

Obviously, $|\gamma|_w = m$. Recall that $W \geq 3L_1$. Then (4-1) gives that

$$c_{f_i^{r_i}}(f_i^{r_i m}) \geq d(o, f_i^{r_i m} o) - (|\gamma| - W|\gamma|_w) \geq d(o, f_i^{r_i m} o) - (|\gamma| - 3L_1 m) \geq L_1 m.$$

Therefore,

$$h_{f_i^{r_i}}(f_i^{r_i m}) = c_{f_i^{r_i}}(f_i^{r_i m}) - c_{f_i^{-r_i}}(f_i^{r_i m}) = c_{f_i^{r_i}}(f_i^{r_i m}) - c_{f_i^{r_i}}(f_i^{-r_i m}) \geq L_1 m.$$ □

Define $h_i : G \to \mathbb{R}$ as $h_i = h_{f_i^{r_i}}$, where $r_i > 0$ is chosen as in Lemma 4.12. Then we obtain the following.

Proposition 4.13 *$\{h_i : i \geq 1\}$ is an infinite sequence of quasimorphisms on G such that*

(1) *$h_i(f_j^m) = 0$ for all $i \neq j$ and for all $m \geq 0$;*

(2) *$h_i(f_i^{r_i m}) \geq L_1 m$ for all $i \geq 1$ and for all $m \geq 0$;*

(3) *$\psi(f_i) = 0$ for all homomorphisms $\psi : G \to \mathbb{R}$;*

(4) *the distance $d(o, f_i o)$ tends to infinity as i tends to infinity;*

(5) *$h_i(h) = 0$ for all $h \in H_j$ with $1 \leq j \leq n$.*

Proof The first two items follow directly from Lemma 4.12. Recall from Lemma 4.9 that $g_1, g_2 \in [G, G]$. Hence, each $f_i \in F = \langle g_1, g_2 \rangle$ is a product of finitely many commutators, which implies item (3). Recall from the paragraph following Lemma 4.9 that $F = \langle g_1, g_2 \rangle$ is a Schottky group which means that the orbital map $\Phi : \mathcal{G}(F, \{g_1, g_2\}) \to X$ is a (λ_0, ϵ_0)-quasi-isometric embedding. Since $f_i = g_1^{n_i} g_2^{m_i} g_1^{k_i} g_2^{-l_i}$ for $0 \ll n_1, m_1, k_1, l_1, \ll n_2, m_2, k_2, l_2 \ll \cdots$, one gets that $d(o, f_i o) \geq \lambda_0^{-1}(n_i + m_i + k_i + l_i) - \epsilon_0$ which implies item (4). It suffices for us to verify item (5).

As each $H_j (1 \leq j \leq n)$ acts elliptically on X, there is a $D > 0$ such that $d(o, ho) \leq D$ for all $h \in \bigcup_{1 \leq j \leq n} H_j$. Since $0 \ll n_1, m_1, k_1, l_1 \ll n_i, m_i, k_i, l_i$ for each $i \geq 2$, we can require $n_1 \gg 0$ such that $d(o, f_i^{r_i} o) \geq \lambda_0^{-1} r_i (n_i + m_i + k_i + l_i) - \epsilon_0 > 2L_2 + D$. Then it follows from the definition of barriers (see Definition 3.4) that each h is $(L_2, f_i^{r_i})$-barrier-free. Hence, as a result of Lemma 4.11, $c_{f_i^{r_i}}(h) = c_{f_i^{-r_i}}(h) = 0$. This shows that $h_i(h) = 0$ for all $h \in \bigcup_{1 \leq j \leq n} H_j$. □

At the end of this section, we prove Proposition 4.1.

Proof of Proposition 4.1 At first, we claim that for each $g \in G$ and $i \gg 0$, one has $h_i(g) = 0$. Indeed, as Proposition 4.13(4) shows, $d(o, f_i o)$ tends to infinity as $i \to \infty$. Hence, for $i \gg 0$, g is $(L_2, f_i^{r_i})$-barrier-free since the diameter of $N_{L_2}([o, go])$ is finite. Thus one gets that $h_i(g) = 0$ by Lemma 4.11. Therefore, if $(a_i)_{i=1}^{\infty} \in \ell^1$, then $\sum_{i=1}^{\infty} a_i h_i$ is well defined as an element of $C^1(G; \mathbb{R})$ since $\sum_{i=1}^{\infty} a_i h_i(g)$ is in fact a finite sum for each $g \in G$. For the same reason, $\sum_{i=1}^{\infty} a_i d^1 h_i$ is a well-defined 2-cocycle. By Lemma 4.6, all the 2-cocycles $d^1 h_i$ have a common bound, which means that there exists a constant $M > 0$ such that $\sup_{g, g' \in G} |d^1 h_i(g, g')| \leq \Delta(h_i) \leq M$ for $i \geq 1$. It follows that if a sequence $(a_i)_{i=1}^{\infty} \in \ell^1$ then $\sum_{i=1}^{\infty} a_i d^1 h_i$ is a bounded 2-cocycle. Therefore,

$$\sum_{i=1}^{\infty} a_i d^1 h_i = d^1 \left(\sum_{i=1}^{\infty} a_i h_i \right).$$

We get a real linear map $\omega : \ell^1 \to H_b^2(G; \mathbb{R})$ which sends the sequence $(a_i)_{i=1}^{\infty}$ to the cohomology class represented by $\sum_{i=1}^{\infty} a_i d^1 h_i$. From Proposition 4.13(5), we know each $d^1 h_i$ lies in $H_b^2(G, \{H_i\}_{i=1}^n; \mathbb{R})$. So the real linear map is actually $\omega : \ell^1 \to H_b^2(G, \{H_i\}_{i=1}^n; \mathbb{R})$. In order to see that ω is injective, suppose $\omega((a_i)) = 0$. Then

$$d^1 \left(\sum_{i=1}^{\infty} a_i h_i \right) = d^1 b$$

for some bounded real-valued map $b \in C_b^1(G; \mathbb{R})$. This means the function

$$\phi := \sum_{i=1}^{\infty} a_i h_i - b$$

is a homomorphism from G to $\mathbb{R}$. Applying this equality of 1-cochains to $f_i^{r_i m} \in G$, we find

$$a_i h_i(f_i^{r_i m}) - b(f_i^{r_i m}) = \phi(f_i^{r_i m}) = 0 \quad \text{for all } m \geq 0.$$

Since $h_i(f_i^{r_i m}) \geq L_1 m$ and b is a bounded map, this forces a_i to be 0. As i is arbitrary, (a_i) must be the zero vector. This shows the injectivity of $\omega : \ell^1 \to H_b^2(G, \{H_i\}_{i=1}^n; \mathbb{R})$.

Finally, as ℓ^1 has the dimension equal to the cardinality of the continuum and the space of bounded cochains has cardinality $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$, we complete the proof. □

As a consequence of Proposition 4.1, we have:

Proposition 4.14 *Let G be a countable group and H a normal subgroup of G. Suppose that G/H acts WPD on a hyperbolic space. Then the dimension of $H_b^2(G, H; \mathbb{R})$ as a vector space over $\mathbb{R}$ has the cardinality of the continuum.*

Proof Proposition 4.1 shows that the dimension of $H_b^2(G/H; \mathbb{R})$ as a vector space over $\mathbb{R}$ has the cardinality of the continuum. Since Proposition 2.22 shows that $H_b^2(G/H; \mathbb{R}) \cong H_b^2(G, H; \mathbb{R})$, the conclusion follows. □

5 The proof of Theorem 1.1

Let us recall the conditions of Theorem 1.1: G acts properly on X with contracting elements and $\{H_i\}$ $(1 \leq i \leq n)$ is a finite collection of Morse subgroups with infinite index in G.

Definition 5.1 The action of a group G on a metric space X is *acylindrical* if for all $L > 0$ there exist $D > 0$ and $B > 0$ such that if $x, y \in X$ and $d(x, y) > D$, then there are at most B elements $g \in G$ with $d(x, gx) \leq L$ and $d(y, gy) \leq L$.

An acylindrical action $G \curvearrowright X$ is called *nonelementary* if the action is unbounded and G is not virtually cyclic.

Lemma 5.2 *A nonelementary acylindrical action on a hyperbolic space must be WPD.*

Proof Let G be a group which admits a nonelementary acylindrical action on a hyperbolic space X. Since $G \curvearrowright X$ is acylindrical, for all $L > 0$, there exists $D > 0$ such that if $x, y \in X$ and $d(x, y) > D$, then the set $\{g \in G : d(x, gx) \leq L, d(y, gy) \leq L\}$ is finite.

Now we verify that $G \curvearrowright X$ is also a WPD action according to Definition 4.2. By [41, Theorem 1.1], G contains infinitely many independent loxodromic elements. Thus, it remains to verify the third item in Definition 4.2. For every loxodromic element $g \in G$, every $x \in X$, and every $L > 0$, let $D > 0$ be the constant given by the above acylindrical action and $N > 0$ be an integer depending only on g such that $d(x, g^N x) \geq N\|g\| > D$. Then the above acylindrical action implies that the set $\{h \in G : d(x, hx) \leq L, d(g^N x, hg^N x) \leq L\}$ is finite. □

Fix a contracting element g given by Lemma 3.9 and $K \gg 0$. Section 3 produces a projection complex $\mathcal{P}_K(\mathcal{F})$.

As shown in [45, Theorem 1.5], $E(g)$ is a hyperbolically embedded subgroup of G. Therefore, as a result of [4, Theorem 5.6], G acts acylindrically on $\mathcal{P}_K(\mathcal{F})$. In particular, as a result of Lemma 5.2:

Lemma 5.3 *The action $G \curvearrowright \mathcal{P}_K(\mathcal{F})$ satisfies WPD.*

Proof of Theorem 1.1 Recall that Lemma 3.10 shows that each H_i $(1 \le i \le n)$ acts elliptically on $\mathcal{P}_K(\mathcal{F})$. Hence, the action $G \curvearrowright \mathcal{P}_K(\mathcal{F})$ satisfies the setup of Section 4. Therefore, Theorem 1.1 follows from Proposition 4.1. □

A natural research direction to generalize our Theorem 1.1 is to assume instead that each subgroup H_i is a subgroup with proper limit sets on a convergence boundary of X. See [31; 49] for more details about convergence boundary. A Morse subgroup with infinite index satisfies this property. Hence, one may wonder whether the following proposition is always true:

Question 5.4 *Let G be a nonelementary countable group acting properly on a geodesic metric space X with convergence boundary. Let H be a subgroup of G with proper limit sets. Is the dimension of $H_b^2(G, H; \mathbb{R})$ as a vector space over $\mathbb{R}$ infinite?*

At the end of this section, we provide an application of Theorem 1.1.

Definition 5.5 A group G is *boundedly generated* by a finite collection of subgroups $H_1, \dots, H_k$ if for every $g \in G$ there is a number N such that all powers g^n can be written in the form

$$g^n = \prod_{i=1}^{N} h_i(n),$$

where each $h_i(n)$ is conjugate to some element in $\bigcup_{1 \le j \le k} H_j$.

This definition is a variation of [36, Definition 4]. There Kotschick required each subgroup to be cyclic.

Corollary 5.6 *Under the assumption of Theorem 1.1, G is not boundedly generated by $\{H_i : 1 \le i \le n\}$.*

Proof Suppose to the contrary that G is boundedly generated by $\{H_i : 1 \le i \le n\}$. Then for every $g \in G$, there exists an $N \in \mathbb{N}$ such that every power g^m can be written as a product of N elements in conjugations of $\bigcup_{1 \le i \le n} H_i$.

As Proposition 4.13 shows, there is at least one unbounded quasimorphism ϕ on G such that $\phi(h) = 0$ for each $h \in H_i$ with $1 \le i \le n$. Let $\bar{\phi}$ be the homogenization of ϕ given by Remark 2.29. Proposition 4.13 also gives an element $g \in G$ such that $\bar{\phi}(g) > 0$. Then there exists $N \in \mathbb{N}$ such that $g^m = h'_1 \cdots h'_N$ for any $m > 0$ and each $h'_i = g_i h_i g_i^{-1}$ with $g_i \in G, h_i \in \bigcup_{1 \le j \le n} H_j$. Note that homogeneous quasimorphisms take constant values on conjugacy classes. Hence, $\bar{\phi}(h'_i) = \bar{\phi}(h_i) = 0$ for each i. Then one has that

$$m|\bar{\phi}(g)| = |\bar{\phi}(g^m)| = |\bar{\phi}(h'_1 \cdots h'_N)| \le |\bar{\phi}(h'_1 \cdots h'_{N-1})| + \Delta(\bar{\phi}) \le \cdots \le N\Delta(\bar{\phi}),$$

where $\Delta(\bar{\phi})$ is the defect of $\bar{\phi}$. By letting $m \to \infty$, we reach a contradiction. □

6 Rotation family and relative bounded cohomology

In a group G, the normal closure of an element g is denoted as $\langle\langle g \rangle\rangle$. The goal of this section is as follows:

Proposition 6.1 *Let G be a nonelementary countable group acting properly on a geodesic metric space X with contracting elements. Then for any contracting element $g \in G$, there exists $k = k(g) > 0$ such that the dimension of $H_b^2(G, \langle\langle g^k \rangle\rangle; \mathbb{R})$ as a vector space over $\mathbb{R}$ has the cardinality of the continuum.*

Proof ideas of Proposition 6.1 Since $\langle\langle g^k \rangle\rangle$ is a normal subgroup of G, by Proposition 4.14, we only need to find a hyperbolic space such that the quotient group $G/\langle\langle g^k \rangle\rangle$ acts acylindrically on it. According to the theory of rotating family developed by Dahmani–Guirardel–Osin [14], we need to find a hyperbolic space such that G acts acylindrically on it and $(\mathcal{F} = \{f \operatorname{Ax}(g) : f \in G\}, \{f \langle g^k \rangle f^{-1} : f \in G\})$ forms a rotating family. To obtain such a hyperbolic space, we need a construction of quasitrees of spaces $\mathcal{C}(\mathcal{F})$, which can be seen as a blow-up of the projection complex $\mathcal{P}_K(\mathcal{F})$. Theorem 6.9 of [4] shows that $\mathcal{C}(\mathcal{F})$ is a quasitree on which G acts acylindrically and g is a loxodromic element. In order to get a suitable rotating family, we will consider a cone-off space $\dot{Z}_r(\mathcal{F})$ with apexes $\mathcal{F}$ over a scaled metric space $(\mathcal{C}(\mathcal{F}), l \cdot d_{\mathcal{C}})$. For $r \gg 0$, $\dot{Z}_r(\mathcal{F})$ is also hyperbolic [14, Corollary 5.39]. Moreover, [32, Lemma 5.3] shows that $(\mathcal{F}, \{f \langle g^k \rangle f^{-1} : f \in G\})$ is a suitable rotating family on $\dot{Z}_r(\mathcal{F})$. As a result of [14, Proposition 5.28], we get that $\dot{Z}_r(\mathcal{F})/\langle\langle g^k \rangle\rangle$ is still hyperbolic. Finally, we verify that both the extended action $G \curvearrowright \dot{Z}_r(\mathcal{F})$ and the quotient action $G/\langle\langle g^k \rangle\rangle \curvearrowright \dot{Z}_r(\mathcal{F})/\langle\langle g^k \rangle\rangle$ are acylindrical.

6.1 Quasitrees of spaces

Fix a contracting element $g \in G$. We define $\mathcal{F} = \{f \operatorname{Ax}(g) : f \in G\}$. Section 2.3 gives a projection complex $\mathcal{P}_K(\mathcal{F})$ whose vertex set is $\mathcal{F}$ and two vertices $U, V \in \mathcal{F}$ are connected by an edge if and only if $\mathcal{F}_K(U, V) := \{W \in \mathcal{F} : d_W(U, V) > K\} = \varnothing$. Fix a positive number L such that $1/2K \le L \le 2K$. We now define a blowup version, $\mathcal{C}(\mathcal{F})$, of the projection complex $\mathcal{P}_K(\mathcal{F})$ by preserving the geometry of each $U \in \mathcal{F}$. Namely, we replace each $U \in \mathcal{F}$, a vertex in $\mathcal{P}_K(\mathcal{F})$, with the corresponding subspace $U \subset X$, while maintaining the adjacency relation in $\mathcal{P}_K(\mathcal{F})$: if U and V are adjacent in $\mathcal{P}_K(\mathcal{F})$ (i.e., $d_{\mathcal{P}}(U, V) = 1$), then we attach an edge of length L from every point $u \in \pi_U(V)$ to $v \in \pi_V(U)$. This choice of L, as stated in [3, Lemma 4.2], ensures that $U \subset X$ is geodesically embedded in $\mathcal{C}(\mathcal{F})$ (so the index L is omitted here).

For any contracting element $g \in G$, the infinite cyclic subgroup $\langle g \rangle$ is of finite index in $E(g)$ by [48, Lemma 2.11], so $\operatorname{Ax}(g) = E(g)o$ is quasi-isometric to a line $\mathbb{R}$. Thus, the set $\mathcal{F}$ (derived from Lemma 2.15) consists of uniform quasilines. By [3, Theorem B], we have the following:

Theorem 6.2 [3] *The quasitree of spaces $\mathcal{C}(\mathcal{F})$ is a quasitree of infinite diameter, with each $U \in \mathcal{F}$ totally geodesically embedded into $\mathcal{C}(\mathcal{F})$. Moreover, the shortest projection from U to V in $\mathcal{C}(\mathcal{F})$ agrees with the projection $\pi_U(V)$ up to a uniform finite Hausdorff distance.*

Any two points $u \in U$ and $v \in V$ in $\mathcal{C}(\mathcal{F})$ are connected via a standard path obtained from the standard path α between U and V in $\mathcal{P}_K(\mathcal{F})$, which passes through each vertex space U on α via a geodesic in U (see [3, Definition 4.3] for more details). Hence, standard paths in $\mathcal{C}(\mathcal{F})$ are also uniform quasigeodesics.

6.2 Hyperbolic cone-off and rotation family

We first introduce a construction of a hyperbolic metric space by conning off a collection of Morse subsets from a given hyperbolic space.

Let Z be a hyperbolic space with a collection $\mathcal{F}$ of uniformly Morse subsets, which means that these Morse subsets have a uniform Morse gauge (see Definition 3.1). Assume that $\mathcal{F}$ has bounded intersection (see Definition 2.6). For $r \geq 0$, we first define the hyperbolic cone-off $\dot{Z}_r(\mathcal{F})$ of Z along $\mathcal{F}$.

For each $U \in \mathcal{F}$, the *hyperbolic cone* $C_r(U)$ is the quotient space of the product $U \times [0, r]$ by collapsing $U \times 0$. The collapsed point denoted by $a(U)$ is called the *apex* of the cone, and $U \times 1$ the *base* of the cone. The cone is equipped with a geodesic metric such that it is the metric completion of the universal covering of a closed hyperbolic disk punctured at the origin.

The *hyperbolic cone-off* $\dot{Z}_r(\mathcal{F})$ is the quotient space of the disjoint union

$$Z \amalg \coprod_{U \in \mathcal{F}} C_r(U)$$

by gluing U with the base of the cone $C_r(U)$, equipped with the length metric. Since $\mathcal{F}$ has bounded intersection, for $r \gg 0$, $\dot{Z}_r(\mathcal{F})$ is also hyperbolic [14, Corollary 5.39].

Assume that G acts isometrically on Z and leaves $\mathcal{F}$ invariant. The action naturally extends by isometry to the hyperbolic cone $C_r(U)$ by the rule $g(x, t) = (gx, t)$ for any $g \in G$, $x \in U$, $0 \leq t \leq r$. This is a prototype of the notion of a rotating family introduced in [14].

Definition 6.3 Assume G acts isometrically on a metric space $\dot{Z}$. Let A be a G-invariant set in $\dot{Z}$ and a collection of subgroups $\{G_a : a \in A\}$ of G such that $G_a(a) = a$, $gG_ag^{-1} = G_{ga}$ for any $a \in A$, $g \in G$. We call such a pair $(A, \{G_a : a \in A\})$ a *rotating family*.

Returning to the above cone-off construction, the apexes $A(\mathcal{F}) = \{a(U) : U \in \mathcal{F}\}$ and the stabilizers G_a for $a \in A(\mathcal{F})$ together consist of a rotating family. Moreover, we say that A is *ρ-separated* if any two distinct apexes are at distance at least ρ.

Roughly speaking, a rotating family $(A, \{G_a : a \in A\})$ is called *very rotating* if every nontrivial element in G_a rotates around a with a very large angle. This big angle is usually achieved by taking a sufficiently deep subgroup (which is generated by a higher power of some element) of G_a.

6.3 Proof of Proposition 6.1

From now on, we suppose that G is a nonelementary countable group acting properly on a geodesic metric space X with contracting elements. Fix a base point $o \in X$ and a contracting element $g \in G$. Define $\mathcal{F} = \{f\,\mathrm{Ax}(g) : f \in G\}$. Theorem 6.2 produces a quasitree of space $\mathcal{C}(\mathcal{F})$ in which each G-translate of $\mathrm{Ax}(g)$ is totally geodesically embedded.

Lemma 6.4 [4, Theorem 6.9] *For $K \gg 0$, $\mathcal{C}(\mathcal{F})$ is a quasitree on which G acts acylindrically and g is a loxodromic element on $\mathcal{C}(\mathcal{F})$.*

Denote by $d_{\mathcal{C}}$ the metric on $\mathcal{C}(\mathcal{F})$. The following result gives a way to produce a very rotating family on some cone-off of a "scaled" quasitree of spaces. Here, a scaled metric means a constant multiple of the original metric. By scaling the metric of a hyperbolic space, one can require the hyperbolicity constant to be uniform.

Lemma 6.5 [32, Lemma 5.3] *There exist universal constants $\delta_U > 0$, $r > 20\delta_U$ and $k = k(g)$, $l = l(g) > 0$ with the following property. Consider the cone-off space $\dot{Z}_r(\mathcal{F})$ with apexes $A(\mathcal{F})$ over the scaled metric space $Z_l = (\mathcal{C}(\mathcal{F}), l \cdot d_{\mathcal{C}})$. For every $n \geq 1$, set*

$$E_n = \{f\langle g^{nk}\rangle f^{-1} : f \in G\}.$$

Then $(A(\mathcal{F}), E_n)$ is a $2r$-separated very rotating family on the δ_U-hyperbolic space $\dot{Z}_r(\mathcal{F})$.

As G acts acylindrically on $(\mathcal{C}(\mathcal{F}), d_{\mathcal{C}})$, it is straightforward to verify that G also acts acylindrically on $(\mathcal{C}(\mathcal{F}), l \cdot d_{\mathcal{C}})$. Moreover, [14, Proposition 5.40] shows that acylindricity is preserved by taking (suitable) cone-off, thus one has:

Lemma 6.6 *The extended action $G \curvearrowright \dot{Z}_r(\mathcal{F})$ is acylindrical.*

Fix $r > 10^{10}\delta_U$. As [14, Proposition 5.33] also shows that acylindricity is preserved by taking quotient of a normal subgroup generated by a $2r$-separated very rotating family, one has:

Lemma 6.7 *The quotient action $G/\langle\langle g^k\rangle\rangle \curvearrowright \dot{Z}_r(\mathcal{F})/\langle\langle g^k\rangle\rangle$ is acylindrical.*

Proof In order to apply [14, Proposition 5.33] to get the conclusion, we need to verify that there exists $K > 0$ such that for all $a \in A(\mathcal{F})$ and for all x with $|x - a| = 50\delta_U$,

$$\#\{h \in G : h(a) = a, |x - h(x)| \leq 10\delta_U\} \leq K.$$

From the construction of $A(\mathcal{F})$ above, the stabilizer of each apex in $A(\mathcal{F})$ is exactly a conjugate of $E(g)$. As $[E(g) : \langle g\rangle]$ is finite, there exists $K = K(g)$ with the desired property. □

Moreover, the quotient space $\dot{Z}_r(\mathcal{F})/\langle\langle g^k\rangle\rangle$ is $60000\delta_U$-hyperbolic by [14, Proposition 5.28].

Proof of Proposition 6.1 As Lemma 6.7 implies that the quotient group acts WPD on a hyperbolic space, the conclusion follows from Proposition 4.14. □

We conclude this section by posing the following question:

Question 6.8 *Let G be a nonelementary countable group acting properly on a geodesic metric space X with contracting elements. Let H be a normal subgroup of G with a nonamenable quotient. Is the dimension of $H^2_b(G, H; \mathbb{R})$ as a vector space over $\mathbb{R}$ infinite?*

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