Algebraic & Geometric Topology

Volume 26 (2026)

Issue 4 (pages 1229–1596)

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ALGEBRAIC & GEOMETRIC TOPOLOGY
msp.org/agt

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

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PUBLISHED BY

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Algebraic & Geometric Topology **26**:4 (2026) 1229–1291
DOI: 10.2140/agt.2026.26.1229
Published: April 25, 2026

The Deligne–Mumford operad as a trivialization of the circle action

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We prove that the tree-like Deligne–Mumford operad is a homotopical model for the trivialization of the circle in the higher-genus framed little discs operad. Our proof is based on a geometric argument involving nodal annuli. We use as a model for the higher-genus framed little discs an operad of Riemann surfaces with analytically parametrized boundary. We develop the formalism of topological moduli problems as a framework to accommodate the orbifold nature of the Deligne–Mumford operad.

1 Introduction

An operad is a structure that contains spaces of operations with multiple inputs and one output, and rules for composing these operations. An algebra over an operad is a given incarnation of these operations and composition rules. Associative algebras, Lie algebras, commutative algebras, Gerstenhaber algebras, Batalin–Vilkovisky algebras and hypercommutative algebras are all examples of algebras over suitably defined operads Ass, Lie, Com, Gerst, BV and HyperCom.

Many such algebraic operads can be described as homologies of topological operads, i.e., operads with topological spaces of operations. Famously, Gerst is the homology of the operad of little 2-discs, whose spaces of operations consist of Euclidean embeddings of smaller discs, seen as inputs, into the unit disc, seen as the output, and where the composition is given by rescaling the unit disc and fitting it into some other small disc. Similarly, BV is the homology of the operad of framed little 2-discs, analogous to that of little 2-discs but involving the extra data of a marked point on the boundary of each of the discs under consideration. See Figure 1. Another example is HyperCom, which is the homology of the Deligne–Mumford–Knudsen operad, whose space of k-to-one operations is the compactified moduli space $\overline{\mathcal{M}}_{0,k+1}$ of genus-0 curves with $k+1$ marked points, of which one is labeled as an output and

MSC2020: primary 18M75, 53D37; secondary 18G85, 55P48.

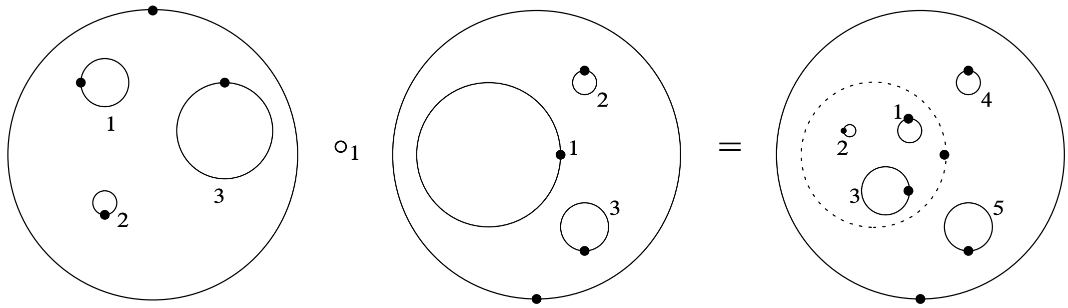

Figure 1: Composition in the operad of framed little 2-discs.

the other ones are labeled as inputs. The composition is represented by the nodal curve obtained by identifying an input of one curve with the output of another. See Figure 2.

The following result was proved by Drummond-Cole [21]. Let FLD be the operad of framed little disks and let DMK be the genus-zero Deligne–Mumford–Knudsen operad with k-to-one operations indexed by points in $\overline{\mathcal{M}}_{0,k+1}$. Let $\mathrm{FLD}_{1,1}$ be framed little disks with one input and one output (with only a space of one-to-one operations, which is up to homotopy the group S^1), and let pt be the operad with only one identity one-to-one operation. Then, in any model structure on operads with weak equivalences spanned by maps of topological operads which are levelwise weak equivalences, we have the following result.

Theorem (Drummond-Cole [21]) *The homotopy colimit of the diagram*

$$\mathrm{pt} \leftarrow \mathrm{FLD}_{1,1} \rightarrow \mathrm{FLD}$$

is related by a canonical sequence of weak equivalences to DMK.

The intuition behind this theorem is the following: to trivialize the S^1-action in FLD amounts to collapsing each small disc, as well as the boundary of the outer disc, to a point. The outcome is a genus-0 curve, i.e., an element of $\mathcal{M}_{0,k+1}$, the uncompactified moduli space of genus-0 curves with $k+1$ marked

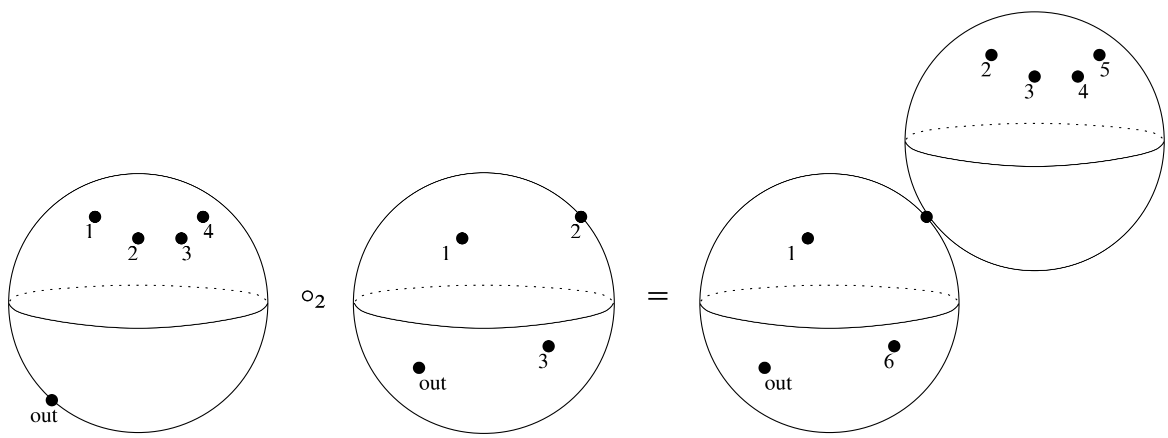

Figure 2: Composition in the Deligne–Mumford–Knudsen operad.

points. The homotopy pushout corresponds to the compactification $\overline{\mathcal{M}}_{0,k+1}$: the latter involves nodal curves whose dual graph is a tree, and trees are precisely created by the operadic bar construction, which provides cofibrant replacements.

In this paper we give a higher-genus generalization of this result. Our method of proof gives a more geometric (and indeed motivic, as seen in [56]) interpretation of Drummond-Cole's theorem.

We define the operad Fr_∂ of *framed surfaces* with spaces of operations given by the moduli spaces of complex, i.e., conformal, surfaces with analytically parametrized boundary, and with composition given by gluing boundary components along compatible parametrizations. See Section 3.1 and Figure 7 on page 1253. The operad FLD embeds in Fr_∂ by viewing the boundary of the "large" disk in which the framed little disks embed as the outgoing boundary of a conformal surface of genus zero, and the boundaries of the interior disks as incoming boundaries. This embedding establishes a homotopy equivalence between FLD and the suboperad $\mathrm{Fr}_{\partial,g=0}$ of framed surfaces of genus zero, and so Fr_∂ is a natural higher-genus generalization of FLD in the category of topological operads.

Let Ann be the suboperad of Fr_∂ consisting of annuli, i.e., genus-zero framed surfaces with one incoming and one outgoing boundary component. This operad is homotopy equivalent to $\mathrm{FLD}_{1,1}$ and to S^1, and each annulus is a homotopy unit for Fr_∂. It is convenient to enlarge Ann and Fr_∂ to strictly unital operads $\widetilde{\mathrm{Ann}}$ and $\widetilde{\mathrm{Fr}}_\partial$ by including infinitely thin annuli. See Section 3.2.

The main result of the present paper is the following theorem. As before, suppose we are working with a model structure with weak equivalences spanned by maps of topological operads which are levelwise weak equivalences. Let $\mathrm{DM}^{\mathrm{tree}}$ be the operad of "tree-like" nodal surfaces of arbitrary genus, whose spaces of operations are the partial compactifications of the moduli spaces $\mathcal{M}_{*,*} = \{\mathcal{M}_{g,k+1} : g \geq 0,\ k \geq 0\}$ of closed Riemann surfaces of genus g with $k+1$ marked points by boundary components consisting of nodal curves whose dual graph is a tree.

Theorem 1.1 *The homotopy colimit of the diagram of unital operads*

$$\mathrm{pt} \leftarrow \widetilde{\mathrm{Ann}} \rightarrow \widetilde{\mathrm{Fr}}_\partial \tag{1}$$

is related by a canonical sequence of weak equivalences to the Deligne–Mumford operad $\mathrm{DM}^{\mathrm{tree}}$.

The same statement holds for the homotopy colimit of the diagram of nonunital operads

$$\mathrm{pt} \leftarrow \mathrm{Ann} \rightarrow \mathrm{Fr}_\partial.$$

The Deligne–Mumford operad is not an operad in topological spaces, but rather an operad in topological orbifolds, or stacks. We discuss the corresponding formalism in Appendix A, where we call the relevant objects *topological moduli problems*. If one is only interested in the operad as an object of a *rational homotopy category* (e.g., by considering its chains over a field of characteristic zero), the orbifold structure can be ignored without changing the homotopy type and the homotopy colimit result holds on the level of coarse moduli spaces (note that in genus zero this distinction is irrelevant as there are no stabilizers).

Let $\mathrm{DM}^{\mathrm{tree}}_{\mathrm{coarse}}$ be the operad built out of the underlying coarse moduli spaces of the orbifold-valued operad $\mathrm{DM}^{\mathrm{tree}}$. Let k be a field of characteristic zero. In this context, the operads of chains with coefficients in k on $\mathrm{DM}^{\mathrm{tree}}$ and $\mathrm{DM}^{\mathrm{tree}}_{\mathrm{coarse}}$ are equivalent, as the homology of finite groups is trivial in characteristic zero. The following corollary, motivated by mirror symmetry considerations which we discuss below, follows from Theorem 1.1 by the universal property of colimits.

Corollary 1.2 *Let k be a field of characteristic zero. The data of an algebra over the operad of chains $C_*(\mathrm{DM}^{\mathrm{tree}})$ is equivalent to the data of a dg algebra A over $C_*(\widetilde{\mathrm{Fr}}_\partial)$ together with a derived S^1-trivialization, i.e., a chain of quasi-isomorphisms of $C_*(S^1)$-modules $\tau : A \cong V$, with V a complex of k-modules carrying a trivial S^1-action.* □

Corollary 1.2 is a higher-genus generalization of a result proved by Drummond-Cole and Vallette [22, Theorem 7.8]. The derived S^1-trivialization is equivalent to a Hodge-to-de Rham degeneration data in the sense of [22]. The characteristic zero condition can be removed at the cost of working with $\mathrm{DM}^{\mathrm{tree}}$ instead of $\mathrm{DM}^{\mathrm{tree}}_{\mathrm{coarse}}$ and considering algebras over appropriate model-theoretic replacements of the operads involved.

Motivation, history of the problem and state of the art The main motivation for Theorem 1.1 and for Drummond-Cole's theorem comes from the homological mirror symmetry conjecture of Kontsevich [37]. This conjecture postulates an equivalence between, on the symplectic side, the Fukaya category, and on the complex side, the category of coherent sheaves. In contrast, the original discovery and formulation of mirror symmetry was enumerative [9; 16], and postulated an equivalence between Gromov–Witten invariants on the symplectic side and Hodge-type numbers on the complex side. Hence the question of describing Gromov–Witten invariants, or quantum cohomology, of a closed symplectic manifold, in terms of its Fukaya category. In recent years, this has led to a flurry of activity around so-called "categorical enumerative invariants" [15; 17; 18].

Gromov–Witten invariants and the Fukaya category are related by the so-called *closed-open map*. This map induces, for sufficiently nice symplectic manifolds, an isomorphism between symplectic cohomology, which is a variant of Floer homology, and Hochschild cohomology of the Fukaya category; see [1; 25; 37]. This isomorphism intertwines the S^1-action on symplectic cohomology with the S^1-action on Hochschild cohomology. Both these actions are part of naturally defined BV-algebra structures, which can also be refined at chain level as algebra structures over the operad of chains on the framed little discs FLD. On the Floer side this was proved recently by Abouzaid–Groman–Varolgunes [2], and on the Hochschild side this is closely related to the famous problem known as the "Deligne conjecture" [6; 39; 46; 54; 59].

When the symplectic manifold is closed, we infer two different structures on its quantum cohomology. On the one hand, the fixed point map identifies it with symplectic cohomology, wherefrom a BV-algebra structure with trivial S^1-action. On the other hand, it classically carries the structure of a HyperCom algebra. (This involves genus-0 Gromov–Witten invariants, e.g., the operation in arity 2 corresponds to the quantum multiplication.) To explain this phenomenon, Kontsevich formulated in 2003 the conjecture that the framed little discs operad with a trivialization of the circle should be equivalent to the DMK operad.

This was proved in algebraic settings by Drummond-Cole and Vallette [22], as well as Khoroshkin, Markarian, Shadrin [35], and in a topological setting by Drummond-Cole [21]. The statement at the topological level is the strongest, since it implies the previous ones by passing to chains. See also Dotsenko, Shadrin, Vallette [20] for the relation between this picture and the Givental group action.

Costello [14] adopts a point of view on categorical enumerative invariants which is inspired by cohomological field theory; see also Kontsevich and Manin [38]. From that perspective, it becomes relevant to study analogues of Kontsevich's conjecture in higher genus. Our main Theorem 1.1 is the first result in that direction: we prove the conjecture of Kontsevich at the topological level, i.e., in the strongest sense, for the operadic part of the higher-genus Deligne–Mumford moduli spaces (one output).

Very recently Tu [55] proved a homological generalization of our main Theorem 1.1 in the context of modular operads. This implies the equality between Costello's categorical enumerative invariants of the ground field and the Gromov–Witten invariants of a point, and is an important step towards inferring enumerative mirror symmetry from homological mirror symmetry [15; 17; 18]. Much more in line with our topological approach, Deshmukh [19] proved a generalization of our main Theorem 1.1 from operads to input-output properads, i.e., properads that have no operations with 0 inputs and 0 outputs. The starting object in [19] is the properadic version of our operad of framed nodal surfaces Fr_∂, which we view as confirmation of the fact that this is the correct higher-genus generalization of the little 2-discs operad. The paper [19] relies on the full machinery of ∞-categories developed by Lurie [41]. In contrast, we keep technicalities to a minimum. The geometric perspective adopted in the current paper should make it appealing to both topologists and geometers.

Our paper brings into the picture a number of new ideas. We define the operad Fr_∂ of framed surfaces as a higher-genus analogue of the operad of framed little discs. Our main Theorem 1.1 extends the equivalence of operads proved by Drummond-Cole [21] to higher genus, and Corollary 1.2 extends to higher genus the algebraic formulations from [20; 35]. Our proof is geometric and makes use of certain explicit and canonical degenerations of Riemann surfaces. Remarkably, our use of Riemann surfaces with analytically parametrized boundary, which makes the gluing operation well defined, has a motivic counterpart discussed by the second author in [56].

Sketch of the proof The key technique in our proof consists in replacing the diagram

$$\mathrm{pt} \leftarrow \mathrm{Ann} \rightarrow \mathrm{Fr}_\partial$$

by the homotopy equivalent, but much more geometrically meaningful diagram (cf. Theorem 3.11)

$$\mathrm{NodAnn} \leftarrow \mathrm{Ann} \rightarrow \mathrm{Fr}_\partial. \tag{2}$$

Here NodAnn is the operad of (stable) *nodal annuli*, with only one-to-one operations consisting of a compactification of Ann by allowing the modulus to tend to ∞, which we geometrically interpret as the annulus developing a node (which is disjoint from either parametrized boundary component). The resulting operad turns out to be contractible (Lemma 3.6), hence gives rise to a diagram whose homotopy colimit is equivalent to the homotopy colimit $\mathrm{pt} \leftarrow \mathrm{Ann} \rightarrow \mathrm{Fr}_\partial$ of the theorem above. In fact, in this formulation the homotopy colimit result is visible geometrically, as the geometric pushout Theorem 3.11

from Section 3.4. In that statement we use a unital version $\widetilde{\mathrm{NodAnn}}$ of NodAnn, which corresponds to the partial compactification in the "modulus-zero" limit. (This partial compactification does not change the homotopy type.)

The proof of our main Theorem 1.1 relies on a mild homotopy enhancement of the proof of the geometric pushout Theorem 3.11.

Our proof of Proposition 6.6, stating that a certain operation of erasing seams on surfaces gives rise to a weak homotopy equivalence, has counterparts in [19, §§5.2–5.3; 21, §7], in different setups. These different perspectives complement each other, and it is instructive to compare them, including from a complexity perspective.

Topological moduli problems and operads An extra layer of technical complexity is added to our work because the "target" of our comparison, the operad $\mathrm{DM}^{\mathrm{tree}}$, is not a topological operad. Instead, it is an operad valued in topological moduli problems, because certain stable marked complex curves have automorphisms that preserve the markings. The notion of a topological moduli problem, which we discuss in Appendix A, is a variant of the notion of topological stack that is suitable for our purposes. We thus have to take care of two issues:

(1) How does one compare topological moduli problems to topological spaces?

(2) What is an operad in topological moduli problems?

We discuss the first question in Appendix A, and the second question in Appendix B. The outcome is that every topological moduli problem $\mathcal{X}$ has a "classifying space" and can thus be treated essentially as a topological space. The answer to the second question requires more formalism than the first, and the model we use is that of Segal operads, defined in [12]. In particular, the Deligne–Mumford operad $\mathrm{DM}^{\mathrm{tree}}$ is a Segal operad rather than an ordinary operad. Luckily, the two questions can be treated separately for the purposes of this paper, and neither of them needs to interfere with the model category structure.

Before comparing $\mathrm{DM}^{\mathrm{tree}}$ with the homotopy pushout, we need to replace it by a model that allows boundary. To this end, we define in Section 3.4 an operad $\mathrm{NodFr}_\partial^{\mathrm{tree}}$ of classifying spaces of nodal curves with parametrized boundary (again, an operad of topological moduli problems defined using the same framework as $\mathrm{DM}^{\mathrm{tree}}$). This operad contains both $\mathrm{DM}^{\mathrm{tree}}$ as a closed suboperad and Fr_∂ as an open suboperad, and moreover the map of operads $\mathrm{DM}^{\mathrm{tree}} \to \mathrm{NodFr}_\partial^{\mathrm{tree}}$ is a homotopy equivalence (after taking classifying spaces).

We use the homotopy equivalence

$$\mathrm{DM}^{\mathrm{tree}} \xrightarrow{\textbf{funnel}} \mathrm{NodFr}_\partial^{\mathrm{tree}}$$

(which we call the "funnel" map, as it is geometrically represented by attaching "funnels"; see Figure 8 on page 1267) as one half of a "roof" of equivalences between $\mathrm{DM}^{\mathrm{tree}}$ and the homotopy pushout. Indeed, after taking canonical resolutions in an appropriate model category, we identify the pushout operad in (2) with a certain space $\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}}$ of decorated curves which we call "Humpty-Dumpty curves", fitting into a diagram

$$\text{hocolim}(\text{diagram (2)}) \cong \mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}} \to \mathrm{NodFr}^{\mathrm{tree}}_{\partial} \leftarrow \mathrm{DM}^{\mathrm{tree}} \tag{3}$$

all of whose maps are equivalences of operads in an appropriate model category.

Model category structures We use the Berger–Moerdijk model category structure on topological operads throughout most of the paper, induced by a given model category structure on topological spaces. Note that the topological spaces we work with are not CW-complexes. This makes it inconvenient to use the standard (Quillen) model category structure on the category of topological spaces, and we replace it by the so-called *mixed model category structure* due to Cole [13]. All spaces we work with are homotopy equivalent (and not just weak homotopy equivalent) to CW-complexes, which implies that our results will also hold in the Strøm model category structure [53], where only homotopy equivalences are inverted. This is explained in Section 5.

Structure of the paper Taking advantage of the analogous nature of the proofs of the homotopical main Theorem 1.1 and of the geometric pushout Theorem 3.11, we first give in Sections 2 and 3 a self-contained statement and proof of Theorem 3.11, along with a brief introduction to topological operads and their pushouts. A reader interested in the flavor of our proof without the topological technicalities can read those sections only. In Sections 4 and 5 we introduce the formalism of model categories and the Berger–Moerdijk model category structure on topological operads, which we will be working with. We prove the main Theorem 1.1 in Section 6. For the reader's convenience we have given separately the proofs for the genus-0 case and for the higher-genus case. In the genus-0 case we only work with operads in topological spaces and there are not stacky phenomena involved, and we recover by a different, and perhaps more geometric method, the theorem of Drummond-Cole [21]. In the higher-genus case we make full use of the language of topological moduli problems and Segal operads, which we introduce and discuss in Appendices A and B, respectively.

2 Operads and topology

2.1 A brief reminder on operads

Operads were initially defined by May [43] in a topological context.

An *operad* O is a structure that specifies a class of composable operations with multiple inputs. An operad in *sets* is a collection of sets O_n, $n \geq 0$, of operations "with n inputs and one output", or operations "of arity n", together with *composition rules*

$$\gamma : O_k \times O_{n_1} \times O_{n_2} \times \cdots \times O_{n_k} \to O_{n_1+\cdots+n_k}$$

and *permutation rules*, consisting of *right* actions of the symmetric groups $\mathfrak{S}_n$ on O_n, $n \geq 0$, where $\mathfrak{S}_0 = 1$ by convention,

$$O_n \times \mathfrak{S}_n \to O_n, \quad (o, \sigma) \mapsto o\sigma.$$

The composition rules and the permutation rules are required to satisfy certain tautological relations which essentially encode the fact that they behave like composition and permutation of inputs. Generally, operads are also required to have a *unit*, $1 \in O_1$, with the property that composing $o \in O_k$, $k \geq 1$, by 1 on the left or with the tuple $(1, 1, \dots, 1)$ on the right does not change o. By default, when we use the word "operad" we will mean *unital operad*.

A *representation* of an operad O (in sets), or an *algebra over* O, is a set (S, ρ) with a collection of maps $\rho_o : S^n \to S$ indexed by $o \in O_n$, $n \geq 0$. By convention S^0 consists of a single point and therefore we interpret the collection of maps ρ_o, $o \in O_0$, as a distinguished collection of elements in S. The case in which O_0 consists of a single element is historically important; see May [43], but the operads that we will construct in this paper will have naturally richer spaces O_0. The collections of maps ρ_o, $o \in O_n$, $n \geq 1$, are interpreted as spaces of operations with n inputs and one output in S. We require these maps to satisfy the permutation rule

$$\rho_{o\sigma}(s_1, \dots, s_n) = \rho_o(s_{\sigma(1)}, \dots, s_{\sigma(n)})$$

for $\sigma \in \mathfrak{S}_n$ a permutation, and also the associativity rule

$$\rho_o \circ (\rho_{o_1} \times \cdots \times \rho_{o_k}) = \rho_{\gamma(o,(o_1,\dots,o_k))} : S^{n_1} \times \cdots \times S^{n_k} \to S.$$

Note that the only property needed in order to define operads and algebras over operads in this context is that the category Set has a symmetric monoidal structure with respect to the cartesian product and the permutation action $\mathfrak{S}_n \times S^n \to S^n$, $\sigma(s_1, \dots, s_n) = (s_{\sigma(1)}, \dots, s_{\sigma(n)})$. In particular, we can define the notion of an operad and of an algebra over an operad in any symmetric monoidal category $(\mathcal{C}, \otimes)$ with choice of unit object. The category of operads in $\mathcal{C}$ is denoted by $\mathrm{Op}_{\mathcal{C}}$.

For convenience, we shall impose a slightly stronger condition: namely, that the symmetric monoidal category $\mathcal{C}$ we work with be *closed* (see [7, §2]), which in particular implies it has (small) colimits, the colimits distribute over pushouts and there is an internal Hom functor. The cases of most interest to us are the categories Top of compactly generated weakly Hausdorff topological spaces (this is the standard category used in homotopy theory [33, Definition 2.4.21, Theorem 2.4.25]), Vect of vector spaces and $\mathrm{Vect}_{\mathrm{dg}}$ of differential graded vector spaces. Given a lax symmetric monoidal functor $\mathcal{C} \to \mathcal{D}$, we get a functor of associated operad categories $\mathrm{Op}_{\mathcal{C}} \to \mathrm{Op}_{\mathcal{D}}$.

Operads can be equivalently defined by specifying a smaller set of composition rules, the so-called *partial compositions*. More precisely, given a unital operad O one defines the partial compositions

$$- \circ_i - : O_k \times O_\ell \to O_{k+\ell-1}, \quad 1 \leq i \leq k,$$

as $u \circ_i v = \gamma(u; 1, \dots, 1, v, 1, \dots, 1)$ for $1 \leq i \leq k$. These compositions obey the tautological relations $u \circ_i (v \circ_j w) = (u \circ_i v) \circ_{i-1+j} w$ for $i, j \geq 1$ and $(u \circ_j w) \circ_i v = (u \circ_i v) \circ_{j-1+\ell} w$ for $j > i \geq 1$ and $v \in O_\ell$, called respectively *sequential composition* and *parallel composition*. These relations determine uniquely all the other composition and permutation rules for the operad O, allowing for an equivalent definition of the operad structure.

Remark 2.1 When we work with the Deligne–Mumford operad and its variations, we will need the slightly more sophisticated theory of Segal operads, which is adapted to handle stacky objects, i.e., objects with self-symmetries. This refinement is explained in Appendix B.

2.2 Free operad

This section is based on [7, §5.8; 8, §3].

Many constructions in algebra canonically output graded objects, i.e., objects of the form $\bigsqcup X_i$ for $\bigsqcup$ the coproduct operation — or $\bigoplus$ for vector spaces — and i running over some indexing set I. For example the free unital monoid on a set Γ — or a vector space V — is $\mathrm{Free}(\Gamma) = \bigsqcup_{n\in\mathbb{N}} \Gamma^n$ — or the tensor algebra $\mathrm{Free}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$. Note that the monoid structure on $\mathrm{Free}(\Gamma)$ "lives over" the standard additive monoid structure on $\mathbb{N}$. It is a common procedure to resolve a monoid, or an associative algebra, by free ones using a simplicial object (or chain complex) based on the free algebra construction of the bar complex. The analogue of the bar complex in the theory of operads is indexed not by the monoid of natural numbers but by the operad of trees, whose n-to-one operations are given by certain trees with n distinguished "input edges" and one distinguished "output edge". Some trees have automorphisms, which interact with the $\mathfrak{S}_n$-actions on spaces of operations, so properly speaking the free construction is indexed by a *groupoid* of trees.

Let $\mathcal{C}$ be any symmetric monoidal category. The *symmetric groupoid* is the category $\mathfrak{S}$ with objects the finite sets $[n] = \{1, \dots, n\}$, $n \geq 0$, and morphisms the permutations of $[n]$. We define the category of $\mathfrak{S}$*-collections* in $\mathcal{C}$ as $\mathrm{Fun}(\mathfrak{S}^{\mathrm{op}}, \mathcal{C})$, also denoted by $\mathfrak{S}\text{-}\mathrm{Mod}_{\mathcal{C}}$. Explicitly, the objects of $\mathfrak{S}\text{-}\mathrm{Mod}_{\mathcal{C}}$ are sequences $X_* := (X_0, X_1, X_2, \dots)$ with $X_n \in \mathcal{C}$ a $\mathfrak{S}_n$-module for $n \geq 0$, and the morphisms are equivariant sequences of morphisms in $\mathcal{C}$.

Let Set_f be the category with objects the finite sets and morphisms the bijections between finite sets. The categories $\mathfrak{S}$ and Set_f are canonically equivalent. As a consequence, the category of $\mathfrak{S}$-collections $\mathrm{Fun}(\mathfrak{S}^{\mathrm{op}}, \mathcal{C})$ is canonically isomorphic to the category $\mathrm{Fun}(\mathrm{Set}_f^{\mathrm{op}}, \mathcal{C})$ [42, Proposition 1.51]. When viewing a $\mathfrak{S}$-collection X_* as a functor $\mathrm{Set}_f^{\mathrm{op}} \to \mathcal{C}$, we denote by X_F the object in $\mathcal{C}$ that is associated to a finite set F.

We have a canonical forgetful functor

$$\mathrm{forg} : \mathrm{Op}_{\mathcal{C}} \to \mathfrak{S}\text{-}\mathrm{Mod}_{\mathcal{C}},$$

which associates to an operad O the sequence $(O_0, O_1, O_2, \dots)$ of its spaces of operations. This functor has a left adjoint

$$\mathrm{Free} : \mathfrak{S}\text{-}\mathrm{Mod}_{\mathcal{C}} \to \mathrm{Op}_{\mathcal{C}}$$

called the *free operad functor*. The adjunction relation reads

$$\mathrm{Hom}_{\mathrm{Op}_{\mathcal{C}}}(\mathrm{Free}(X_*), O) \cong \mathrm{Hom}_{\mathfrak{S}\text{-}\mathrm{Mod}_{\mathcal{C}}}(X_*, \mathrm{forg}(O)).$$

2.2.1 The operad of labeled rooted trees We first need to describe an important operad based on the following heuristic idea: an operation with n inputs is represented by a rooted tree with n distinguished

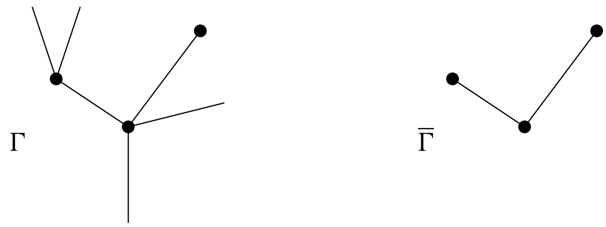

Figure 3: A tree of operations Γ and its associated graph of full edges $\overline{\Gamma}$.

leaves labeled by the set $\{1, \dots, n\}$, *up to isomorphism*. Composition of operations is represented by grafting such trees one upon another.

A *graph with half-edges* is a graph Γ with a set of vertices Vert_Γ, a set of oriented edges Edge_Γ each having one tail and one head vertex, and an additional set of oriented half-edges Half_Γ with only one end (either head or tail). We denote by Half_Γ^+ the set of incoming half-edges. We denote by $\overline{\Gamma}$ the oriented graph of full edges. We say that a graph with half-edges Γ is a *tree of operations* if $\overline{\Gamma}$ is a rooted tree and Γ has exactly one outgoing half-edge which is attached to the root of $\overline{\Gamma}$. This definition allows for an arbitrary number (including 0) of incoming half-edges for Γ, it allows for some (or all) of the leaves of $\overline{\Gamma}$ to have no incoming half-edge attached to them, and it allows for the incoming half-edges of Γ to be attached at any vertex of $\overline{\Gamma}$. Each interior vertex of $\overline{\Gamma}$ has a unique outgoing edge attached to it. See Figure 3. In addition to the above, we also introduce the *trivial tree* $|$ consisting of a unique edge and no vertex. We do not consider our trees of operations as being endowed with a planar structure.

A *labeling* of a tree of operations τ with $n \geq 1$ incoming half-edges is the data of a bijection $\lambda : \{1, \dots, n\} \xrightarrow{\sim} \mathrm{Half}_\tau^+$, which we view as assigning an element in $\{1, \dots, n\}$ to each incoming half-edge. A *labeled tree* of operations is a pair (τ, λ) consisting of a tree of operations τ and a labeling λ. Two labeled trees (τ, λ) and (τ', λ') are *equivalent* if there exists an isomorphism $\phi : \tau \xrightarrow{\sim} \tau'$ that intertwines the labelings, i.e., such that $\lambda' = \phi\lambda$. Write

$$\mathrm{Tree}_n$$

for the set of all labeled trees of operations with $n \geq 0$ incoming half-edges, and write

$$\mathit{Tree}_n$$

for the equivalence classes under the above equivalence relation. This is a $\mathfrak{S}_n$-equivariant groupoid with respect to the right action of $\mathfrak{S}_n$ on labelings.

The $\mathfrak{S}_n$-groupoids Tree_n, $n \geq 0$, form an operad in the following way:

(1) Given an equivalence class of a labeled tree $[\tau, \lambda] \in \mathit{Tree}_k$ and a collection of equivalence classes $[\tau_i, \lambda_i] \in \mathit{Tree}_{n_i}$, $1 \leq i \leq k$, we define the composition $\gamma\big([\tau, \lambda], ([\tau_1, \lambda_1], \dots, [\tau_k, \lambda_k])\big)$ as follows. We choose representatives for each of the previous equivalence classes, we build a tree T by gluing for each $i \in \{1, \dots, k\}$ the outgoing half-edge of τ_i to the i-th incoming half-edge of τ as distinguished by the labeling λ, producing thus for each i a new interior edge whose tail vertex is the root of τ_i and whose

head vertex is the same as that of the i-th incoming edge of τ. (If τ_i is the trivial tree, the gluing is innocuous.) We define a labeling ℓ of the tree T by concatenating the labelings $\lambda_1, \dots, \lambda_k$. The result of the composition is the equivalence class of the labeled tree (T, ℓ).

(2) The unit is provided by the trivial tree $|$ with its unique labeling.

The resulting operad, denoted by *Tree*, is *the operad of labeled rooted trees*.

Remark 2.2 We refer to [36] for a different description of trees.

2.2.2 The free operad functor Let X_* be a $\mathfrak{S}$-collection, which we view as a functor $\mathrm{Set}_f^{\mathrm{op}} \to \mathcal{C}$. The heuristic idea for the construction of the free operad $\mathrm{Free}(X_*)$ is the following: the space of operations in arity $n \geq 0$ consists of elements of Tree_n, decorated at each vertex v by an element of $X_{\mathrm{in}(v)}$, where $\mathrm{in}(v)$ is the set of incoming edges and half-edges at v. The composition of operations is inherited from the composition of trees.

Given a tree of operations τ, define

$$X^\tau = \prod_{v \in \mathrm{Vert}_\tau} X_{\mathrm{in}(v)}.$$

A *labeled rooted tree with vertices colored by elements of X_* and with $n \geq 0$ incoming half-edges* is a triple $(\tau, \boldsymbol{x}, \lambda)$ with $(\tau, \lambda) \in \mathrm{Tree}_n$ and $\boldsymbol{x} \in X^\tau$. Two such triples $(\tau, \boldsymbol{x}, \lambda)$ and $(\tau', \boldsymbol{x}', \lambda')$ are *equivalent* if there exists an isomorphism $\phi : (\tau, \lambda) \xrightarrow{\sim} (\tau', \lambda')$ such that the colors $\boldsymbol{x} = (x_v)_{v \in \mathrm{Vert}_\tau}$ and $\boldsymbol{x}' = (x'_w)_{w \in \mathrm{Vert}_{\tau'}}$ satisfy the condition $x'_{\phi(v)} = x_v \sigma_\phi$, where $\sigma_\phi : X_{\mathrm{in}(v)} \to X_{\mathrm{in}(\phi(v))}$ is the isomorphism determined by the bijection $\phi : \mathrm{in}(v) \to \mathrm{in}(\phi(v))$. Let

$$\mathrm{Tree}_n(X_*)$$

be the space of labeled rooted trees with vertices colored by elements of X_* and with $n \geq 0$ incoming half-edges. This carries a natural topology and splits as a disjoint union of topological spaces indexed by the elements of Tree_n. Let

$$\mathit{Tree}_n(X_*)$$

be the space of equivalence classes under the above equivalence relation, which again carries a natural topology and splits as a disjoint union of topological spaces indexed by the elements of Tree_n. This is naturally a $\mathfrak{S}_n$-space under the action of the permutation group on labelings.

Definition 2.3 The spaces of operations in the *free operad* $\mathrm{Free}(X_*)$ are

$$\mathrm{Free}(X_*)_n = \mathit{Tree}_n(X_*), \quad n \geq 0.$$

These form a topological operad with compositions, unit, and $\mathfrak{S}$-structure inherited from the operad *Tree*.

2.3 Pushout of operads

Suppose that

$$P \leftarrow A \to Q$$

is a diagram of topological operads. We define the *amalgamated product* or *pushout*,

$$P \sqcup_A Q,$$

to be the colimit of the diagram in topological operads. Explicitly, this is a quotient (interpreted as a colimit) of the free operad $\mathrm{Free}(P_* \sqcup Q_*)$ and is defined as follows. Consider the counit of the free-forgetful adjunction: this is the natural transformation between the functors $\mathrm{Free} \circ \mathrm{forg}$ and $\mathrm{Id}_{\mathrm{Op}}$ which associates to each operad O the "product" morphism of operads $\prod : \mathrm{Free}(O_*) \to O$ obtained by applying composition maps in O to a tree of elements in O recursively until the tree has a single vertex (this is independent of the order by the associativity of operations in operads). Now $P \sqcup_A Q$ is the quotient of $\mathrm{Free}(P_* \sqcup Q_*)$ by the equivalence relation generated by the relations

$$\sim_1 \sqcup \sim_2$$

described as follows:

($\sim_1$) If $o_{\mathrm{free}} \in \mathrm{Free}(P_* \sqcup Q_*)$ is a free element over a tree τ and τ has a subtree τ_0 all of whose vertices are labeled by elements of P (or Q) then o_{free} is equivalent to o'_{free} with all vertices and all full edges of τ_0 contracted to a point, and with the product $\prod(o_{\mathrm{free}}|_{\tau_0})$ written at that point.

($\sim_2$) Denote the two operad maps by $i : A \to P$ and $j : A \to Q$. If $o_{\mathrm{free}} \in \mathrm{Free}(P_* \sqcup Q_*)$ is a free element over a tree τ which on some vertex $v \in \tau$ has a label which is equal to $i(a)$ for some $a \in A$, we set $o_{\mathrm{free}} \sim o'_{\mathrm{free}}$ where o'_{free} has the label on v replaced by $j(a)$.

The amalgamated product can be defined more generally for operads in categories which do not live over the category of sets. The above relations should then be understood as coequalizer conditions in the underlying category.

3 Operads based on Riemann surfaces with boundary

3.1 The operad of framed surfaces

Definition 3.1 A *framed surface* is a compact Riemann surface Σ with boundary $\partial\Sigma$ locally analytically modeled on the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}\, z \geq 0\}$, together with an analytic parametrization $\varphi_i : S^1 \to C_i$ for each boundary component $C_i \subset \partial\Sigma$.

A component $C_i \subset \partial\Sigma$ is called an *input* or an *output* if the orientation induced by the parametrization coincides, respectively is opposite to the boundary orientation of C_i.

Write $\mathrm{Fr}_\partial^{m,n}$ for the moduli space of framed surfaces with m incoming and n outgoing boundary components.

The space of oriented analytic diffeomorphisms $S^1 \to S^1$ which preserve a basepoint $1 \in S^1$ is contractible. Indeed, this set is identified with the space of analytic functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions $f(0) = 0$, $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$, and $f' > 0$, which is convex. (The function $f(x) = x$ can be taken as a basepoint.) As such, once an orientation of each boundary component has been specified (which is the same as a labeling of the components as inputs or outputs), an analytic parametrization $\varphi_i : S^1 \to C_i$ is determined up to homotopy by the choice of a basepoint $p_i = \varphi_i(1) \in C_i$.

Framed surfaces can be glued at inputs and outputs because of the following phenomenon.

A framed surface is canonically isomorphic in the neighborhood of each of its boundary components to a closed annulus

$$A_\varepsilon = \{z \in \mathbb{C} : 1 - \varepsilon \leq |z| \leq 1\}$$

for some $\varepsilon > 0$. Indeed, given a component C_i with an analytic parametrization φ_i, the latter locally extends uniquely, and these local extensions coincide on the overlaps by uniqueness of holomorphic continuation. The original parametrization φ_i corresponds then to the restriction of the extended parametrization to the circle $|z| = 1$ if C_i is an output, respectively to the restriction to the circle $|z| = 1 - \varepsilon$ if C_i is an input. As a consequence, any two framed surfaces are uniquely locally isomorphic in the neighborhood of any of their incoming, respectively outgoing boundary components.

Given two annuli A_ε, $A_{\varepsilon'}$ (viewed as complex manifolds with canonically parametrized boundary) the incoming boundary of the first can be glued to the outgoing boundary of the second (to produce an annulus with modulus $\ln 1/(1-\varepsilon) + \ln 1/(1-\varepsilon')$; see Section 3.2 below). Since every framed surface is isomorphic in a neighborhood of each of its boundary components to such an annulus, this gives us the local data necessary for gluing two framed surfaces along boundary components of opposite orientation,

$$(\Sigma, \gamma), (\Sigma, \gamma') \mapsto \Sigma \,\#_{\gamma,\gamma'}\, \Sigma'.$$

Note that this also makes sense if Σ, Σ' are disconnected and also if γ, γ' are boundary components consisting of multiple circles, as long as the orientations are compatible.

In particular, the moduli spaces $\mathrm{Fr}_\partial^{m,n}$ form a topological PROP, and the moduli spaces $\mathrm{Fr}_\partial^{m,1}$ with one output form a topological operad. We denote this latter topological operad by Fr_∂. We call it *the operad of framed surfaces*.

Note that the moduli space $\mathrm{Fr}_\partial^{m,n}$ is a priori a stacky object, as a surface can have automorphisms. However, this can only happen when both m and n are equal to zero, as no nontrivial automorphism of a connected complex surface can fix an embedded curve or boundary component pointwise. Since we will only be interested in the operad Fr_∂, which involves the moduli spaces $\mathrm{Fr}_\partial^{m,n}$ with $n = 1$, we will never encounter any stacky phenomena involving framed surfaces.

Remark 3.2 It is understood here that the elements of Fr_∂ are *labeled* framed Riemann surfaces, meaning that, for each framed Riemann surface $\Sigma \in \mathrm{Fr}_\partial^{m,1}$, we are given a bijection λ between $\{1, \dots, m\}$ and the set of incoming boundary components of Σ. The bijection λ is called a *labeling*, and there are of course $m!$ choices of labelings. The labeling is necessary in order to define composition by gluing and hence the operad structure on Fr_∂. This additional presence of labelings is standard for operads constructed out of Riemann surfaces, similarly to the case of the Deligne–Mumford spaces $\overline{\mathcal{M}}_{g,n}$ where the n marked points are also labeled. The symmetric group $\mathfrak{S}_m$ acts on the right on the set of labelings of a framed Riemann surface Σ by composition at the source $(\lambda, \sigma) \mapsto \lambda\sigma$, $\sigma \in \mathfrak{S}_m$. For readability we will henceforth not mention explicitly the labelings of surfaces, but whenever we will write "framed surface" we will mean "labeled framed surface".

Remark 3.3 We will be interested in Fr_∂ as a topological operad and we now specify the topology on the moduli spaces involved. Given any point of $\mathrm{Fr}_\partial^{m,1}$ corresponding to a surface S, we can glue in disks (with standard parametrization of the boundary) to all the inputs and outputs of S to obtain a closed Riemann surface $\bar{S}$. This gives an identification of $\mathrm{Fr}_\partial^{m,1}$ with the moduli space of Riemann surfaces with $m+1$ parametrized loops bounding disks isomorphic to the standard disk $D \subset \mathbb{C}$ and with standard boundary parametrization. In particular, Fr_∂ is a subspace of the space of tuples $(X, \gamma_1, \dots, \gamma_{m+1})$ with X a closed Riemann surface (corresponding to a point of some $\mathcal{M}_{g,m+1}$) and the γ_i, $i = 1, \dots, m+1$, pairwise nonintersecting contractible embedded analytic loops in X. This is a bundle over $\mathcal{M}_{g,m+1}$. We topologize $\mathrm{Fr}_\partial^{m,1}$ as a locally closed subset of this bundle of tuples.

This presents Fr_∂ as a complex infinite-dimensional manifold. Its local model at a framed Riemann surface of genus g with $m+1$ boundary components is the total space of a fibration over a neighborhood of the corresponding element in $\mathcal{M}_{g,m+1}$ with fiber given by $m+1$-tuples of embeddings of the disc in $\mathbb{C}$ close to the standard one. The fact that the corresponding element in $\mathcal{M}_{g,m+1}$ may be an orbifold point is irrelevant here.

3.2 The monoid of framed annuli

The genus-0 and arity-1 part of Fr_∂ forms a topological monoid which we denote by Ann and call *the monoid of framed annuli*.

A *framed annulus* is a genus-0 Riemann surface A with two boundary components $\partial A = \partial^+ A \sqcup \partial^- A$ labeled as input and output, together with analytic parametrizations f_+ of the input $\partial^+ A$ and f_- of the output $\partial^- A$. Ignoring the parametrizations of the boundary components, such an annulus is conformally determined by its *modulus* $\alpha \in (0, \infty)$ (Schottky's theorem [51]). This is the logarithm of the ratio of the radii

$$\alpha = \ln R/r$$

of a *standard annulus* $A_{R,r} = \{z \in \mathbb{C} : r \le |z| \le R\}$, $r < R$, which is conformally equivalent to A, where the outer circle $|z| = R$ is labeled as input and the inner circle $|z| = r$ is labeled as output. The group of conformal automorphisms of the underlying Riemann surface A is canonically isomorphic to S^1: up to replacing A with a conformally equivalent standard annulus, its group of automorphisms is represented by the rotations of $\mathbb{C}$ which fix the origin. As such, the pair (f_-, f_+) is considered modulo global rotations $\theta \cdot (f_-, f_+) = (\theta + f_-, \theta + f_+)$, $\theta \in S^1$. With this understood, we write $[(A, f_-, f_+, \alpha)]$ for the equivalence class of a framed annulus (A, f_-, f_+, α).

Remark The modulus behaves additively under gluing of standard annuli. However, it *does not* behave additively under gluing of general framed annuli. This can be seen explicitly by studying configurations of nested circles in $\mathbb{C}$.

The topological monoid Ann is not unital. In order to achieve unitality, it is convenient to enlarge it to the topological monoid of *possibly degenerate framed annuli*, denoted by $\widetilde{\mathrm{Ann}}$, by including the *moduli space of framed annuli of modulus* 0, denoted by $\mathrm{Ann}^{\mathbf{0}}$.

A *framed annulus of modulus* 0 is a triple (C, f_-, f_+) consisting of a connected closed analytic 1-dimensional manifold C together with analytic diffeomorphisms $f_\pm : S^1 \to C$. We will also refer to (C, f_-, f_+) as being a *framed annulus of thickness zero*, or as being a *degenerate framed annulus*. Two such framed annuli (C, f_-, f_+) and (D, g_-, g_+) are *equivalent* if there exists an analytic diffeomorphism $\psi : C \to D$ such that $g_\pm = \psi f_\pm$. As such, the framed annulus (C, f_-, f_+) is equivalent to $(S^1, \mathrm{id}, f_-^{-1} f_+)$ and also to $(S^1, f_+^{-1} f_-, \mathrm{id})$. We choose the first expression to realize a bijection

$$\mathrm{Ann}^{\mathbf{0}} \xrightarrow{\sim} \mathrm{Aut}(S^1), \quad [(C, f_-, f_+)] \mapsto f_-^{-1} f_+.$$

The composition of the equivalence classes of two framed annuli of modulus 0 is defined by

$$[(C, f_-, f_+)] \circ [(D, g_-, g_+)] = [(C, f_-, f_+ g_-^{-1} g_+)] = [(D, g_- f_+^{-1} f_-, g_+)].$$

This makes $\mathrm{Ann}^{\mathbf{0}}$ into a group. The neutral element is the class $[(S^1, \mathrm{id}, \mathrm{id})]$, consisting of degenerate annuli (C, f_-, f_+) with $f_- = f_+$. The inverse of $[(C, f_-, f_+)]$ is $[(C, f_+, f_-)]$. As such the above bijection

$$\mathrm{Ann}^{\mathbf{0}} \xrightarrow{\sim} \mathrm{Aut}(S^1)$$

is a group isomorphism. (Had we chosen to associate to the class of an annulus $[(C, f_-, f_+)]$ the element $f_+^{-1} f_- \in \mathrm{Aut}(S^1)$, suggested by choosing as a representative the degenerate annulus $(S^1, f_+^{-1} f_-, \mathrm{id})$, we would have obtained a bijective group antihomomorphism.)

The topological monoid Ann is a trivial fiber bundle over $(0, \infty)$, which is the space of moduli of unframed annuli, with fiber $\mathrm{Aut}(S^1) \times_{S^1} \mathrm{Aut}(S^1)$, where $\mathrm{Aut}(S^1)$ stands for the group of analytic automorphisms of the circle and S^1 acts diagonally on $\mathrm{Aut}(S^1) \times \mathrm{Aut}(S^1)$ by translations in the target. We topologize $\widetilde{\mathrm{Ann}}$ by extending this trivial fiber bundle to a trivial fiber bundle over $[0, \infty)$ and collapsing the fiber at 0 via the diagonal action of $\mathrm{Aut}(S^1)$ given by $\varphi \cdot (f_-, f_+) = (\varphi f_-, \varphi f_+)$. We identify the quotient with $\mathrm{Aut}(S^1)$ via $(f_-, f_+) \mapsto f_-^{-1} f_+$ as above.

We extend the monoid structure from Ann to $\widetilde{\mathrm{Ann}}$ as described above for two elements in $\mathrm{Ann}^{\mathbf{0}}$ and by defining

$$[(A, f_-, f_+, \alpha)] \circ [(C, g_-, g_+)] = [(A, f_-, f_+ g_-^{-1} g_+, \alpha)]$$

and

$$[(D, h_-, h_+)] \circ [(A, f_-, f_+, \alpha)] = [(A, f_- h_+^{-1} h_-, f_+, \alpha)]$$

for $[(A, f_-, f_+, \alpha)] \in \mathrm{Ann}$ and $[(C, g_-, g_+)], [(D, h_-, h_+)] \in \mathrm{Ann}^{\mathbf{0}}$.

We claim that this monoid structure is compatible with the above topology, i.e., $\widetilde{\mathrm{Ann}}$ is a topological monoid. To prove the claim, let us consider sequences $[(A^\nu, f_-^\nu, f_+^\nu, \alpha^\nu)]$ and $[(B^\nu, g_-^\nu, g_+^\nu, \beta^\nu)]$, $\nu \geq 1$ with $\alpha^\nu, \beta^\nu > 1$, and such that, for $\nu \to \infty$, we have $\alpha^\nu \to \alpha$, $\beta^\nu \to \beta$ with α or β equal to 1. We can assume without loss of generality that A^ν and B^ν are standard annuli whose inner radius is equal to 1

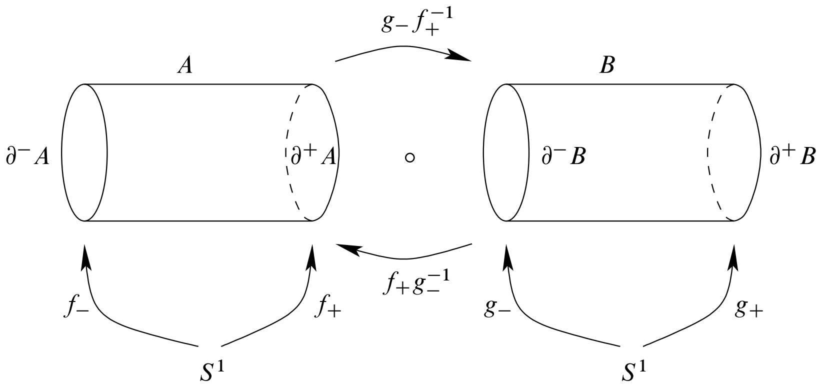

Figure 4: We depict a (framed) annulus as a horizontal cylinder of finite length, with its input boundary component to the right and its output boundary component to the left. The composition $A \circ B$ of two framed annuli is depicted by drawing A to the left of B.

and whose outer radius is equal to α^ν, respectively β^ν, and also that $f_\pm^\nu \to f_\pm$, $g_\pm^\nu \to g_\pm$, the limits being analytic parametrizations of the standard circles of corresponding radii 1, α and β.

We prove the claim in the case $\alpha > 1$ and $\beta = 1$. The glued annulus $A^\nu \# B^\nu$ has input given by the boundary component $\partial^+ B^\nu$ with parametrization g_+^ν, and output given by the boundary component $\partial^- A^\nu$ with parametrization f_-^ν. See Figure 4. As $\nu \to \infty$, the input $\partial^+ B^\nu$ of B^ν — which is the standard circle of radius β^ν in $\mathbb{C}$ — converges pointwise with respect to the standard parametrization to the standard circle of radius 1 with its standard parametrization, viewed as $\partial^- B^\nu$ for all ν. The latter is identified with $\partial^+ A^\nu$ via $f_+^\nu (g_-^\nu)^{-1}$. As such, the limit of the composition $A^\nu \# B^\nu$ is canonically identified with the limit A of the sequence A^ν, and this identification is given by $f_+ g_-^{-1}$ along the input boundary component. The input boundary component of the limit inherits the parametrization g_+, and via this identification the latter corresponds to the parametrization $f_+ g_-^{-1} g_+$ of the input boundary component of A. As far as the output boundary component of the limit is concerned, it is canonically identified with the output boundary component of A and inherits as such the parametrization f_-. This shows that

$$\begin{aligned}\lim_{\nu\to\infty} [(A^\nu, f_-^\nu, f_+^\nu, \alpha^\nu)] \circ [(B^\nu, g_-^\nu, g_+^\nu, \beta^\nu)] &= [(A, f_-, f_+ g_-^{-1} g_+, \alpha)] \\ &= [(A, f_-, f_+, \alpha)] \circ [(S^1, g_-, g_+)] \\ &= \lim_{\nu\to\infty} [(A^\nu, f_-^\nu, f_+^\nu, \alpha^\nu)] \circ \lim_{\nu\to\infty} [(B^\nu, g_-^\nu, g_+^\nu, \beta^\nu)].\end{aligned}$$

The proof of the claim in the cases $\alpha = 1$, $\beta > 1$ and $\alpha = \beta = 1$ is analogous and we omit it.

Definition 3.4 We define $\widetilde{\mathrm{Fr}}_\partial$ to be the extension of Fr_∂ by possibly degenerate framed annuli,

$$\widetilde{\mathrm{Fr}}_\partial = \mathrm{Fr}_\partial \sqcup_{\mathrm{Ann}} \widetilde{\mathrm{Ann}}.$$

3.3 Framed nodal annuli

Ordinary annuli have modulus parameter $\alpha \in (0, \infty)$. By introducing degenerate annuli, we have extended the possible parameters to $[0, \infty)$. In this section we will further extend the possible modulus parameters

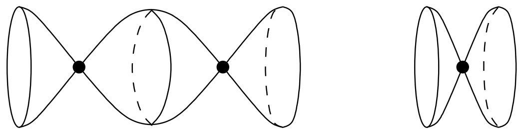

Figure 5: Unstable/stable framed nodal annuli.

from $[0,\infty)$ to $[0,\infty]$. We do this by adding a new class of annuli, called *nodal annuli*, which have modulus parameter ∞. While introducing degenerate moduli did not change the homotopy type of the topological monoid Ann, adding in nodal annuli has a strong destructive effect: it makes the monoid contractible.

Definition 3.5 We say that a complex surface with analytically parametrized boundary is a *framed nodal annulus* if it has two boundary components, genus zero, and at most nodal singularities. (In order to shorten notation, the term "nodal annuli" includes ordinary annuli with no nodes.)

We say that a framed nodal annulus is *unstable* if it has an irreducible component which contains no boundary components (equivalently, if it has a component of genus zero and infinite automorphism group), and *stable* otherwise. See Figure 5. Note that all stable framed nodal annuli either have one irreducible component containing both boundary circles (i.e., they are ordinary framed annuli), or two irreducible components of which one contains the incoming circle and the other contains the outgoing circle. The *stabilization* of an unstable nodal annulus is obtained by contracting all irreducible components which have no boundary. We will be interested in the moduli space of stable framed nodal annuli, viewed as quotients of possibly unstable framed nodal annuli by the equivalence relation induced by stabilization. We write

$$\mathrm{NodAnn}$$

for the moduli space of stable framed nodal annuli. We topologize this space similarly to our moduli space of surfaces with boundary above. Namely, given a stable framed nodal annulus, we get a point of $\overline{\mathcal{M}}_{0,4}$ by gluing in disks along both parametrized boundary components, and marking the images of $\pm 1 \subset S^1$ in both boundary components in the resulting genus-zero curve. In this way, we can view NodAnn as a subspace in the bundle over $\overline{\mathcal{M}}_{0,4}$ whose fiber consists of pairs of parametrized disjoint embedded analytic closed curves whose parametrizations map $\pm 1 \in S^1$ to the marked points.

For the next lemma, recall that we denote by $\mathrm{Aut}(S^1)$ the group of analytic automorphisms of S^1 with analytic inverse, and $\mathrm{Aut}_0(S^1) \subset \mathrm{Aut}(S^1)$ denotes the subgroup of automorphisms which fix $1 \in S^1$. As explained in Section 3.1, the group $\mathrm{Aut}_0(S^1)$ is contractible.

Lemma 3.6 *The moduli space of stable framed nodal annuli is homeomorphic to*

$$(\mathrm{Aut}_0(S^1) \times \mathrm{Aut}_0(S^1)) \times \mathbb{C}.$$

In particular, it is contractible.

Proof Consider the action of S^1 on $\mathrm{Aut}(S^1)$ by translations in the target. The moduli space of stable framed nodal annuli containing a node is identified with $\mathrm{Aut}(S^1)/S^1 \times \mathrm{Aut}(S^1)/S^1$. Indeed, each of the two irreducible components of the underlying Riemann surface is equivalent to a disk with a marked point at the origin. The group of automorphisms of the latter is S^1, given by rotations, and it acts on the analytic parametrizations of its boundary by translations in the target. Writing $f \pmod{S^1}$ for the class of an element of $\mathrm{Aut}(S^1)$ modulo the action of S^1, an arbitrary element of this moduli space can thus be written $(f_- \pmod{S^1}, f_+ \pmod{S^1})$.

With this understood, the topology on NodAnn can be alternatively described as follows. Let $[(A^\nu, f_\pm^\nu)]$, $\nu \geq 1$, be a sequence in Ann with moduli $\alpha^\nu \to \infty$, $\nu \to \infty$. Choose representatives $A^\nu = [-\alpha^\nu/2, \alpha^\nu/2] \times S^1$ and $(f_-^\nu, f_+^\nu) \in \mathrm{Aut}(S^1) \times_{S^1} \mathrm{Aut}(S^1)$ and assume that $(f_-^\nu, f_+^\nu) \to (f_-, f_+)$ as $\nu \to \infty$. We then have by definition

$$[(A^\nu, f_\pm^\nu)] \to (f_- \pmod{S^1}, f_+ \pmod{S^1}), \quad \nu \to \infty.$$

By marking the point $1 \in S^1$ we obtain homeomorphisms

$$\mathrm{Aut}(S^1)/S^1 \simeq \mathrm{Aut}_0(S^1), \quad \mathrm{Aut}(S^1) \simeq \mathrm{Aut}_0(S^1) \times S^1$$

and

$$\mathrm{Aut}(S^1) \times_{S^1} \mathrm{Aut}(S^1) \simeq \mathrm{Aut}_0(S^1) \times \mathrm{Aut}_0(S^1) \times S^1.$$

(None of these identifications preserves any group structure; see also Remark 3.7 below.)

We have already seen that the moduli space Ann of framed annuli is a trivial bundle over $(0, \infty)$ with fiber $\mathrm{Aut}(S^1) \times_{S^1} \mathrm{Aut}(S^1)$, where S^1 acts diagonally. In view of the isomorphism $S^1 \times (0, \infty) \simeq \mathbb{C}^\times$, after choosing a trivialization of the bundle $\mathrm{Ann} \to (0, \infty)$ we obtain a homeomorphism

$$\mathrm{Ann} \simeq \mathrm{Aut}_0(S^1) \times \mathrm{Aut}_0(S^1) \times \mathbb{C}^\times.$$

With respect to this identification, the projection $\mathrm{Ann} \to (0, \infty)$ corresponds to the projection $\mathbb{C}^\times \to (0, \infty)$, $z \mapsto |z|$. Also, with respect to the identification of the moduli space of stable framed nodal annuli containing a node with $\mathrm{Aut}_0(S^1) \times \mathrm{Aut}_0(S^1)$, the definition of convergence for a sequence

$$(f_-^\nu, f_+^\nu, z^\nu) \in \mathrm{Ann} \simeq \mathrm{Aut}_0(S^1) \times \mathrm{Aut}_0(S^1) \times \mathbb{C}^\times$$

such that $|z^\nu| \to \infty$ and $(f_-^\nu, f_+^\nu) \to (f_-, f_+)$ as $\nu \to \infty$ translates into the relation $(f_-^\nu, f_+^\nu, z^\nu) \to (f_-, f_+)$. In other words, we have a homeomorphism

$$\mathrm{NodAnn} \simeq \mathrm{Aut}_0(S^1) \times \mathrm{Aut}_0(S^1) \times (\mathbb{C}^\times \cup \{\infty\}).$$

Up to an inversion on the factor $\mathbb{C}^\times$, this is the statement of the lemma. □

Remark 3.7 Consider the group homomorphism with kernel $\mathrm{Aut}_0(S^1)$ given by the map $\mathrm{Aut}(S^1) \to S^1$, $f \mapsto f(1)$. This admits a section which associates to each element of S^1 the corresponding translation, and thus exhibits $\mathrm{Aut}(S^1)$ as a semidirect product $\mathrm{Aut}(S^1) \simeq \mathrm{Aut}_0(S^1) \rtimes S^1$. Although the action of S^1 on $\mathrm{Aut}_0(S^1)$ by conjugation is nontrivial, we do nevertheless have a homeomorphism at the level of the underlying topological spaces $\mathrm{Aut}(S^1) \simeq \mathrm{Aut}_0(S^1) \times S^1$. On the other hand, there is of course no canonical group structure on the quotient $\mathrm{Aut}(S^1)/S^1$.

Nodal annuli provide a partial compactification of the space of annuli "in the modulus-∞ limit", whereas in the previous section we gave a compactification of the space of annuli "in the modulus-0 limit". In particular, these two compactifications can be combined into a new separable topological space of *possibly degenerate stable framed nodal annuli*,

$$\widetilde{\mathrm{NodAnn}} = \widetilde{\mathrm{Ann}} \sqcup_{\mathrm{Ann}} \mathrm{NodAnn}.$$

Given two possibly degenerate nodal annuli we can glue them to produce a new possibly degenerate nodal annulus. Note that if both annuli have modulus ∞ (i.e., have two irreducible components), the resulting glued space will be unstable. Under our convention, we identify the resulting space with its stabilization. It is immediate to check that the resulting composition operation is associative; it is continuous by an argument analogous to the one used in the previous section for the continuity of the multiplication operation on $\widetilde{\mathrm{Ann}}$.

3.4 Tree-like nodal surfaces

We recall that all our framed surfaces are labeled; see Remark 3.2.

Definition 3.8 Define

$$\mathrm{NodFr}_\partial^{\mathrm{tree}}$$

to be the moduli space of stable nodal Riemann surfaces with nonnodal analytically parametrized boundary, with the restriction that *the dual graph of irreducible components is a tree*. Further define

$$\widetilde{\mathrm{NodFr}_\partial^{\mathrm{tree}}} = \mathrm{NodFr}_\partial^{\mathrm{tree}} \sqcup_{\mathrm{Ann}} \widetilde{\mathrm{Ann}}.$$

Note that an element of $\mathrm{NodFr}_\partial^{\mathrm{tree}}$ can have (stable) interior components which carry no boundary parametrizations, and these can have discrete automorphism groups. We view $\mathrm{NodFr}_\partial^{\mathrm{tree}}$ as a *topological moduli problem* in the sense of the next definition. We build a theory of such spaces in Appendix A.

Definition 3.9 (see Definition A.3) A *topological moduli problem* is a contravariant functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Gpd}$ from the category of topological spaces to the category of groupoids.

We write TMP for the category of such functors, with maps $\mathcal{X} \to \mathcal{Y}$ given by natural transformations. Given a map $f : S \to S'$ of topological spaces we write $f^* : \mathcal{X}(S') \to \mathcal{X}(S)$ for the (contravariantly) associated functor of groupoids. We refer to Appendix A for further details, and simply recall here that a groupoid is a category $\mathcal{C}$ all of whose morphisms are invertible and such that the isomorphism classes form a set denoted by $\pi_0(\mathcal{C})$.

We view $\mathrm{NodFr}_\partial^{\mathrm{tree}}$ as a topological moduli problem as follows. Given a topological space S, an object of the groupoid $\mathrm{NodFr}_\partial^{\mathrm{tree}}(S)$ consists of a continuously varying S-family of stable nodal framed Riemann surfaces, and a morphism in this groupoid is an isomorphism of two such families that preserves the structure. We refer to Appendix A for a more precise definition of the meaning of continuity for an S-family, which is the analogue of defining the topology for a topological moduli problem.

For the purposes of the current section we limit ourselves to considering the corresponding coarse moduli spaces

$$\mathrm{NodFr}_\partial^{\mathrm{tree,coarse}} \quad \text{and} \quad \widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree,coarse}}.$$

While any topological moduli problem has an associated coarse moduli space as described in Appendix A, in our situation the coarse moduli spaces can be obtained from $\mathrm{NodFr}_\partial^{\mathrm{tree}}$, or $\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree}}$, by topologizing isomorphism classes of points, i.e., the isomorphism classes in the groupoids obtained by applying these TMPs to pt.

Note that $\mathrm{NodAnn} \subset \mathrm{NodFr}_\partial^{\mathrm{tree,coarse}}$. Also, $\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree,coarse}}$ differs from $\mathrm{NodFr}_\partial^{\mathrm{tree,coarse}}$ in that it contains degenerate annuli. We topologize $\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree,coarse}}$ as before, by viewing it as embedded in a bundle over the (tree-like) coarse moduli space of closed nodal Riemann surfaces (possibly with some marked points).

Gluing along the boundary and possibly collapsing determines an operad structure on $\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree,coarse}}$. We call it *the operad of possibly degenerate coarse tree-like framed nodal surfaces*, with n-to-one operations given by the coarse moduli spaces of framed nodal curves with n incoming boundary components.

Remark 3.10 In contrast, the fine moduli spaces $\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree}}$ fit into the structure of a *Segal operad* in the sense of Appendix B. We call it *the Segal operad of possibly degenerate tree-like framed nodal surfaces*. This operad will play a role in the proof of our main Theorem in Section 6.

For further reference we denote by

$$\mathrm{NodFr}_\partial$$

the moduli space of stable nodal Riemann surfaces with nonnodal analytically parametrized boundary, without any restriction on the dual graph, and also

$$\widetilde{\mathrm{NodFr}}_\partial = \mathrm{NodFr}_\partial \sqcup_{\mathrm{Ann}} \widetilde{\mathrm{Ann}}.$$

Theorem 3.11 (geometric pushout theorem) *The operad*

$$\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree,coarse}}$$

of coarse moduli spaces of possibly degenerate tree-like framed nodal surfaces is canonically isomorphic to the pushout of the diagram

$$\widetilde{\mathrm{NodAnn}} \leftarrow \widetilde{\mathrm{Ann}} \to \widetilde{\mathrm{Fr}}_\partial$$

in which both arrows are inclusions and we work in the category of topological operads.

The geometric idea of the proof is that a nodal surface can be described, though not uniquely, by a successive gluing of framed nonnodal surfaces and nodal annuli. See Figure 6. When the dual graph of irreducible components is a tree, this data is equivalently encoded in the pushout construction. The equivalence relations defining the pushout construction precisely eliminate the ambiguity, i.e., nonuniqueness, of this description. The equivalence relations underlie pushouts in the topological

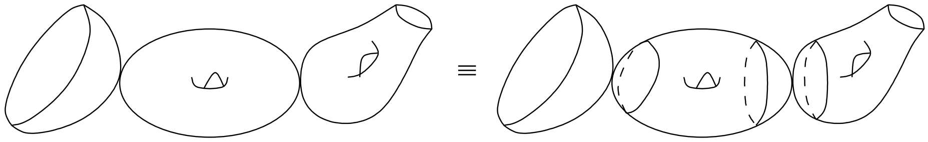

Figure 6: One possible presentation of a nodal surface by gluing.

category, and in particular (as we are not taking homotopy pushouts yet), self-equivalences are ignored, and this explains the "coarse" nature of the result.

As such the proof of Theorem 3.11 on the level of operads in sets is quite straightforward. By reinterpreting the free-forgetful adjunction on the operad of framed surfaces and its relatives, we give a proof of this theorem which also accounts for the topology on the two sides. While the theorem does not imply the homotopy-theoretic pushout result (in order to get a correct model for the homotopy pushout, the diagram of operads must be replaced by a suitable resolution), it is a good intuitive approximation for it. Indeed, the eventual homotopical proof will be based on a topologically enhanced version of exactly the argument presented in the next section.

3.5 Split surfaces and the geometric pushout theorem

The objects of interest in this section will be various moduli spaces of framed surfaces with "seams" at embedded curves, which we call "split surfaces". We again recall that all our framed surfaces are labeled; see Remark 3.2.

Definition 3.12 A *split framed surface with k interior seams* is a pair (Σ, S) consisting of a framed surface Σ with boundary, together with an analytic embedding $S : (S^1)^{\sqcup k} \hookrightarrow \Sigma$ mapping into the interior of Σ.

By definition, the seams are *parametrized* curves: the *interior seams* are the components $S_i : S^1 \to \Sigma$ of the embedding $S = \bigsqcup_{i=1}^k S_i : (S^1)^{\sqcup k} \hookrightarrow \Sigma$; the parametrized boundary components of Σ are called *exterior seams*. In the definition we allow $k = 0$, i.e., no interior seams.

Given a framed surface Σ, write

$$\mathrm{Split}^k_\Sigma$$

for the moduli space of all split surface structures on Σ with k unordered interior seams. Equivalently, Split^k_Σ is the space of analytic embeddings $(S^1)^{\sqcup k} \hookrightarrow \Sigma$ endowed with the compact-open topology. Write

$$\mathrm{Split}^k = \bigsqcup_{\Sigma \in \mathrm{Fr}_\partial} \mathrm{Split}^k_\Sigma$$

for the moduli space of all split framed surfaces with k unordered interior seams, and write

$$\mathrm{Split} = \bigsqcup_{k \geq 0} \mathrm{Split}^k$$

for the moduli space of all split framed surfaces with an arbitrary number of unordered interior seams, and

$$\mathrm{Split}_\Sigma = \bigsqcup_{k\geq 0} \mathrm{Split}_\Sigma^k$$

for the moduli space of all split surface structures on Σ with an arbitrary number of unordered interior seams.

To every split surface (Σ, S) is associated a "dual graph"

$$\Gamma_{\Sigma,S}$$

with k interior edges, which is a directed graph with half-edges. Vertices are indexed by the connected components of $\Sigma \setminus S$, internal edges are indexed by interior seams (the orientation of the normal bundle along a seam determines a direction for the corresponding edge) and half-edges are indexed by external seams (each of these belongs to the closure of a single connected component of $\Sigma \setminus S$). In particular, since the incoming external seams of Σ are labeled by definition, the dual graph inherits a labeling of its incoming half-edges. Note that two split surfaces in the same connected component of Split have the same dual graph, so given a labeled graph Γ we can write

$$\mathrm{Split}_\Gamma$$

for the union of connected components of Split with dual graph Γ. The following observation is straightforward.

Lemma 3.13 *Let Σ be connected. The dual graph $\Gamma_{\Sigma,S}$ associated to a split surface (Σ, S) is a tree if and only if the image of each interior seam is separating*, i.e., *its complement is disconnected.* □

Split framed surfaces are a convenient model for the free operad on the $\mathfrak{S}$-graded space underlying Fr_∂ (the source of the free-forgetful adjunction map), as we now explain. Write

$$\mathrm{Split}_\Sigma^{k,\mathrm{tree}} \subset \mathrm{Split}_\Sigma^k$$

for the moduli space of all split surface structures S on Σ with k unordered interior seams such that the dual graph $\Gamma_{\Sigma,S}$ is a tree. Further define

$$\begin{aligned}
\mathrm{Split}^{k,\mathrm{tree}} &= \bigsqcup_{\Sigma\in \mathrm{Fr}_\partial} \mathrm{Split}_\Sigma^{k,\mathrm{tree}} \subset \mathrm{Split}^k,\\
\mathrm{Split}^{\mathrm{tree}} &= \bigsqcup_{k\geq 0} \mathrm{Split}^{k,\mathrm{tree}} \subset \mathrm{Split},\\
\mathrm{Split}_\Sigma^{\mathrm{tree}} &= \bigsqcup_{k\geq 0} \mathrm{Split}_\Sigma^{k,\mathrm{tree}} \subset \mathrm{Split}_\Sigma,\\
\mathrm{Split}_\Gamma^{\mathrm{tree}} &= \mathrm{Split}_\Gamma
\end{aligned}$$

for any labeled tree Γ. We call these moduli spaces of split surface structures *tree-like*.

Let τ be a labeled tree of operations and let $[\tau]$ be its isomorphism class with respect to the isomorphism relation described in Section 2.2.1. Recall that, for any operad O, the free operad $\mathrm{Free}(O)$ has components $\mathrm{Free}_{[\tau]}(O)$ indexed by such isomorphism classes.

Lemma 3.14 *Let τ be a labeled tree of operations. We have a canonical homeomorphism*

$$G : \mathrm{Free}_{[\tau]}(\mathrm{Fr}_\partial) \xrightarrow{\simeq} \mathrm{Split}^{\mathrm{tree}}_\tau.$$

The fact which underlies the proof of Lemma 3.14 is that, given a framed surface Σ'', the data of an interior seam whose image is separating is equivalent to the data of a decomposition of Σ'' as a gluing of two framed surfaces along one boundary component. Obviously, such a seam determines such a decomposition of Σ''. Conversely, given two framed surfaces Σ, Σ' and a choice of boundary components $\gamma \subset \Sigma$ which is incoming and $\gamma' \subset \Sigma'$ which is outgoing, with corresponding framings $f : S^1 \to \gamma$ and $f' : S^1 \to \gamma'$, the glued surface $\Sigma'' = \Sigma \,\#_{\gamma,\gamma'}\, \Sigma'$ inherits a seam, i.e., a distinguished analytic embedding of S^1 into its interior, given with respect to the canonical inclusions $\Sigma, \Sigma' \hookrightarrow \Sigma''$ by either of the equal compositions

$$S^1 \xrightarrow{f} \gamma \hookrightarrow \Sigma \hookrightarrow \Sigma'' \quad \text{or} \quad S^1 \xrightarrow{f'} \gamma' \hookrightarrow \Sigma' \hookrightarrow \Sigma''.$$

Proof of Lemma 3.14 Let $n \geq 0$ be the number of incoming half-edges of τ. Recall from Section 2.2.2 that $\mathrm{Tree}_n(\mathrm{Fr}_\partial)$ denotes the space of labeled rooted trees with vertices colored by elements of Fr_∂ and with n incoming half-edges. Denote by $\mathrm{Tree}_{[\tau]}(\mathrm{Fr}_\partial) \subset \mathrm{Tree}_n(\mathrm{Fr}_\partial)$ the subset consisting of those elements whose underlying labeled rooted tree is isomorphic to τ. We have a canonical "gluing" map

$$G : \mathrm{Tree}_{[\tau]}(\mathrm{Fr}_\partial) \to \mathrm{Split}^{\mathrm{tree}}_\tau$$

given by gluing framed surfaces according to the underlying labeled rooted tree. Indeed, the incoming boundary components of the element of Fr_∂ that colors a vertex $v \in \tau$ are labeled by the finite set $\mathrm{in}(v)$, and this prescribes the gluing uniquely. By definition, the resulting split surface belongs to $\mathrm{Split}^{\mathrm{tree}}_\tau$.

The map is clearly continuous, surjective, and the fiber over each element of $\mathrm{Split}^{\mathrm{tree}}_\tau$ is canonically identified with an equivalence class as described in Section 2.2.2. As such, it descends to a homeomorphism

$$G : \mathrm{Free}_{[\tau]}(\mathrm{Fr}_\partial) \xrightarrow{\simeq} \mathrm{Split}^{\mathrm{tree}}_\tau,$$

where $\mathrm{Free}_{[\tau]}(\mathrm{Fr}_\partial) = \mathit{Tree}_{[\tau]}(\mathrm{Fr}_\partial)$ is the quotient of $\mathrm{Tree}_{[\tau]}(\mathrm{Fr}_\partial)$ under the equivalence relation described in Section 2.2.2. □

To extend the above result to $\widetilde{\mathrm{Fr}}_\partial$, we compactify Split by allowing interior components of thickness zero:

Definition 3.15 Let

$$\widetilde{\mathrm{Split}}$$

be the partial compactification of Split which allows two seams (internal or external) $S^1 \to \Sigma$ to intersect if and only if they have the same image with the same orientation, and which also allows Σ to be a framed degenerate annulus.

We have corresponding partial compactifications

$$\widetilde{\mathrm{Split}}^k_\Sigma, \quad \widetilde{\mathrm{Split}}^k, \quad \widetilde{\mathrm{Split}}_\Sigma$$

of the moduli spaces Split^k_Σ, Split^k, and Split_Σ respectively, and also for their tree-like and labeled tree-like counterparts, with similar notation $\widetilde{\mathrm{Split}}^{\mathrm{tree}}$ etc.

Points of $\widetilde{\mathrm{Split}}_\Sigma$ over a fixed surface Σ are indexed by maps $S : (S^1)^{\sqcup k} \to \Sigma$ which allow seams with compatible orientation to coincide as above, with the additional data of an ordering of all copies of S^1 mapping to a given closed oriented curve. The notion of dual graph $\Gamma = \Gamma_{(\Sigma,S)}$ for such an element (Σ, S) is defined as follows. The vertices of Γ are of two kinds: they correspond either to the connected components of $\Sigma \setminus S$, or to pairs of interior seams which have the same image and which are immediate successors for the given ordering. The edges correspond to interior seams. One sees that the ordering of the copies of S^1 mapping to a given closed oriented curve precisely resolves the ambiguity in the dual graph by specifying a "composition order" of the thickness-zero annuli they "bound".

Given a labeled tree Γ we have corresponding moduli spaces $\widetilde{\mathrm{Split}}_\Gamma = \widetilde{\mathrm{Split}}^{\mathrm{tree}}_\Gamma$. The proof of the following lemma is in all points similar to that of Lemma 3.14, hence we omit it.

Lemma 3.16 *Let τ be a labeled tree of operations. We have a canonical homeomorphism*

$$G : \mathrm{Free}_{[\tau]}(\widetilde{\mathrm{Fr}}_\partial) \xrightarrow{\simeq} \widetilde{\mathrm{Split}}^{\mathrm{tree}}_\tau. \qquad \square$$

In order to extend the result to $\widetilde{\mathrm{NodFr}}^{\mathrm{tree,coarse}}_\partial$ we need to further define moduli spaces of framed nodal surfaces with seams.

Definition 3.17 Let

$$\mathrm{NodSplit}$$

be the moduli space of framed nodal surfaces Σ endowed with an embedding $(S^1)^{\sqcup k} \to \mathring{\Sigma}_{\mathrm{smooth}}$ of a finite number $k \geq 0$ of parametrized seams *in the open smooth locus*. The objects classified by NodSplit are called *split framed nodal surfaces*.

The notion of dual graph for a split framed nodal surface (Σ, S) is defined as follows: its vertices are the *connected* components (not the irreducible components) of $\Sigma \setminus S$, and in particular the dual graph in this context ignores nodes. The edges correspond to interior seams as before. We can further define moduli spaces $\mathrm{NodSplit}^{\mathrm{tree}}$, $\mathrm{NodSplit}^{\mathrm{tree,coarse}}$ etc. as above.

It is again convenient for unitality purposes to extend the setup by including degenerate annuli.

Definition 3.18 Let

$$\widetilde{\mathrm{NodSplit}}$$

be the partial compactification of NodSplit obtained by allowing S to include coinciding circles bounding thickness-zero annuli, as in Definition 3.15.

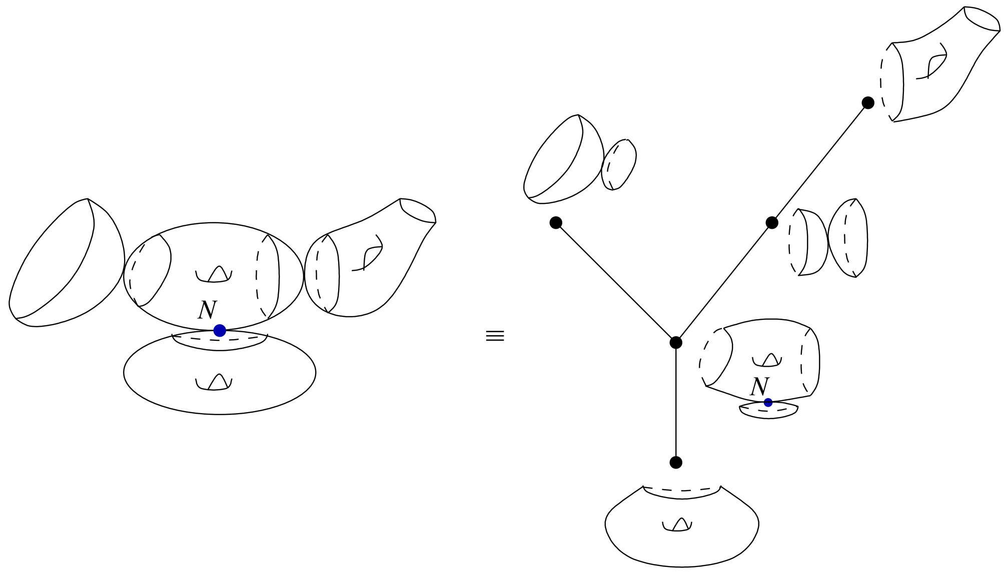

Figure 7: Tree-like split structure on a nodal surface, together with its dual graph. It becomes "protected" by adding one seam around the node N on the trivalent component.

Similarly to the nonnodal case, we consider as part of the data an ordering of the interior seams which have the same oriented image. We have the same notion of dual graph, and we can further define moduli spaces $\widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}$, $\widetilde{\mathrm{NodSplit}}^{\mathrm{tree,coarse}}$ etc. as above.

In the proof of the geometric pushout Theorem 3.11 we will encounter the following new kind of moduli space. We single out the definition before the proof, for the convenience of the reader.

Definition 3.19 Define

$$\widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}_{\mathrm{protected}}$$

to be the moduli space of split nodal surfaces with dual graph a tree (with half-edges) and such that every nodal component is a nodal annulus. We call such surfaces *tree-like and protected.*

The idea of the definition is that every node has to be "protected" on two sides by a pair of seams.

Proof of the geometric pushout Theorem 3.11 Consider the diagram

$$\widetilde{\mathrm{NodAnn}} \leftarrow \widetilde{\mathrm{Ann}} \to \widetilde{\mathrm{Fr}}_\partial.$$

Recall from Section 2.3 the definition of its pushout

$$P \simeq \mathrm{Free}(\widetilde{\mathrm{Fr}}_\partial \sqcup \widetilde{\mathrm{NodAnn}})/\sim,$$

where $\sim$ is the equivalence relation generated by relations $\sim_1 \sqcup \sim_2$.

As in Lemma 3.14 there is a tautological gluing map

$$\mathrm{Free}(\widetilde{\mathrm{Fr}}_\partial \sqcup \widetilde{\mathrm{NodAnn}}) \to \widetilde{\mathrm{NodSplit}}^{\mathrm{tree,coarse}}.$$

The preimage of this map over a given split surface (Σ, S) consists of all possible choices of decorating the vertices of the dual graph of (Σ, S) by the corresponding point of $\widetilde{\mathrm{NodAnn}}$ or $\widetilde{\mathrm{Fr}}_\partial$. Thus, in order for a split surface (Σ, S) to have nonempty preimage, connected components of $\Sigma \setminus S$ must be either smooth surfaces or nodal annuli. Components indexed by nonnodal annuli can be labeled either way and contribute to the ambiguity of the lifting. It is precisely this ambiguity that is resolved by the relation $\sim_2$, in a way that is compatible with the topology as it identifies connected components in their entirety. Thus the map to $\widetilde{\mathrm{NodSplit}}^{\mathrm{tree,coarse}}$ above factors as

$$\begin{tikzcd} \frac{\mathrm{Free}(\widetilde{\mathrm{Fr}}_\partial \sqcup \widetilde{\mathrm{NodAnn}})}{\sim_2} \arrow[r] \arrow[dr, "\simeq"'] & \widetilde{\mathrm{NodSplit}}^{\mathrm{tree,coarse}} \\ & \widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}_{\mathrm{protected}} \arrow[u, hook] \end{tikzcd} \tag{4}$$

and maps homeomorphically the partial quotient to the target $\widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}_{\mathrm{protected}}$, which is in turn a union of connected components of $\widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}$ consisting of split nodal curves with dual graph a tree, and such that each nodal component is a nodal annulus. (Note that as protected split curves are glued out of smooth framed curves and nodal annuli, neither of which have automorphisms, there is no need to take the coarse space here.)

Consider now the map

$$\widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}_{\mathrm{protected}} \to \widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree,coarse}} \tag{5}$$

defined by erasing the seams. Note that erasing a seam which is a common boundary component of two framed surfaces in $\widetilde{\mathrm{Fr}}_\partial$ corresponds precisely to gluing, i.e., composition in the operad $\widetilde{\mathrm{Fr}}_\partial$. Similarly, erasing a seam which is a common boundary component of two nodal annuli creates an unstable component which must be further discarded, and this corresponds again to gluing, i.e., composition in the operad $\widetilde{\mathrm{NodAnn}}$.

It thus follows that the above map is constant along the equivalence classes defined by relation $\sim_1$, which identifies pairs of points inside $\widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}_{\mathrm{protected}}$ which are related by removing a single seam (note that such a seam must either be between two nodal annuli or between two smooth framed surfaces). On the level of sets, it is clear that $\sim_1$ identifies any two points in $\widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}_{\mathrm{protected}}$ which correspond to splittings of the same nodal curve. We turn this intuition into a precise topological colimit argument as follows.

Given a tree-like nodal surface Σ with k nodes and given mutually disjoint neighborhoods $\mathcal{V}_i$, $i = 1, \dots, k$, of its nodes, define $\mathcal{V} = \bigsqcup_{i=1}^k \mathcal{V}_i$ and write $\mathrm{Split}_\Sigma^{\mathcal{V}} \subset \mathrm{Split}_\Sigma^{\mathrm{tree}}$ for those tree-like split surface structures on Σ whose seams lie away from $\mathcal{V}$. Since seams are not allowed to pass through nodes, these spaces filter $\mathrm{Split}_\Sigma^{\mathrm{tree}}$ as $\mathcal{V}$ runs over a neighborhood basis of the nodes of Σ. Now write $(\Sigma, S_{\mathcal{V}}) \in \mathrm{Split}_\Sigma^{\mathrm{tree}}$ for a splitting given by a collection of $2k$ circles parametrized in some analytic fashion and with images contained in $\mathcal{V}$, such that each neighborhood $\mathcal{V}_i$ of a node contains exactly two such circles, one on each irreducible component adjacent to the node. Then every element in $\mathrm{Split}_\Sigma^{\mathcal{V}}$ is identified (in a way consistent with the topology) with $(\Sigma, S_{\mathcal{V}})$ via $\sim_1$. Further, for $\mathcal{V}' \subset \mathcal{V}$ we have $(\Sigma, S_{\mathcal{V}'}) \sim (\Sigma, S_{\mathcal{V}})$: indeed, by $\sim_1$ used for $\widetilde{\mathrm{NodAnn}}$ they are both equivalent to the split surface $(\Sigma, S_{\mathcal{V}''})$ for some sufficiently small neighborhood $\mathcal{V}''$ which does not intersect $S_{\mathcal{V}} \cup S_{\mathcal{V}'}$.

We have thus proved that the fiber of the map (5) is given by the equivalence classes with respect to $\sim_1$. As a consequence, the map (5) is a bijection, and because the previous identifications can be performed continuously in a neighborhood of any given nodal surface Σ, this map is also continuous. Finally, we claim that the map is a homeomorphism. To prove the claim note that, given any tree-like nodal split surface (Σ, S), its image is Σ with the *same* analytic parametrization of the boundary. Thus, in order to prove the claim, it is enough to prove that the induced map on the moduli spaces of surfaces with one marked point on each boundary component is a homeomorphism. This holds true because it is a continuous bijection (just like (5)), with a Hausdorff source and a locally compact target. We infer that the map (5) is a homeomorphism as claimed. (The reduction to moduli spaces of surfaces with one marked point on each boundary component, which gets rid of the infinite dimensional degrees of freedom given by the analytic parametrization, was necessary precisely in order to place ourselves in a setup with locally compact target.)

Together with the homeomorphism (4), we obtain a homeomorphism

$$\widetilde{\mathrm{Fr}}_\partial \sqcup_{\widetilde{\mathrm{Ann}}} \widetilde{\mathrm{NodAnn}} \cong \widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree,coarse}}. \qquad \square$$

4 Model categories and homotopy (co)limits

Our references for this section are Lurie [40, Appendix A.2], May–Ponto [45], Hovey [33], Hirschhorn [32] and Ginot [26].

4.1 Model category theory

Suppose that $\mathcal{C}$ is a category and I is a class of morphisms in $\mathcal{C}$ "to be inverted". We say that I is a *class of weak equivalences* if the following conditions are satisfied:

(1) **Category structure** The objects of $\mathcal{C}$ with the morphisms in I form a subcategory.

(2) **Two-out-of-three** Given any commutative diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

with two of the three morphisms in I, the third is also in I.

Note that the first axiom is sometimes replaced by an identity axiom, as composition compatibility is part of the two-out-of-three axiom. Now given a class of weak equivalences, one would like to produce a "localized" category in which these are inverted, i.e., a category $\mathcal{C}_I$ with a functor $\mathcal{C} \to \mathcal{C}_I$ such that the image of any morphism in I is invertible, and which is initial — up to taking care of set-theoretic issues — among such categories. Modulo some set-theoretic difficulties such a $\mathcal{C}_I$ can be proven to exist. In fact, when $\mathcal{C}$ is an ordinary category, the localization $\mathcal{C}_I$ comes naturally as the set of connected components of morphism spaces in a simplicial category, which should be considered in the context of ∞-category theory.

The problem is that for a general class I of weak equivalences, the localization $\mathcal{C}_I$ (whether as a category or a simplicial category) is incredibly difficult to access. In particular, it is hopeless to calculate $\mathrm{Hom}_{\mathcal{C}_I}(X, Y)$ for two objects X, Y of $\mathcal{C}$. In order to turn $\mathcal{C}_I$ into a manageable object, it is necessary to endow $\mathcal{C}$ with some additional data. One remarkably elegant and versatile solution is to exhibit a so-called model category structure. A *model category structure* consists in endowing $\mathcal{C}$ with two new classes of morphisms called *fibrations*, P, and *cofibrations*, Q, such that the objects of $\mathcal{C}$ with either P or Q form subcategories of $\mathcal{C}$. We call the elements of $I \cap P$ *trivial fibrations*, and the elements of $I \cap Q$ *trivial cofibrations*. The category $\mathcal{C}$ together with the classes I, P, Q need to satisfy a collection of conditions among themselves, for which we refer the reader to [31, §3]. Some conditions that we will use here are as follows:

(1) The category $\mathcal{C}$ has an initial object, $\varnothing$, a final object, pt, and all finite limits and colimits.

(2) For any morphism $X \xrightarrow{f} Y$ of objects, there is a "fibrant factorization" $X \xrightarrow{i} X' \xrightarrow{f'} Y$ such that $i \in I \cap Q$ is a trivial cofibration and $f' \in P$ is a fibration.

(3) Similarly, for any morphism $X \xrightarrow{f} Y$ of objects, there is a "cofibrant factorization" $X \xrightarrow{f'} X' \xrightarrow{j} Y$ such that $f' \in Q$ is a cofibration and $j \in I \cap P$ is a trivial fibration.

(4) All three categories P, Q, I are closed with respect to taking retracts of morphisms.

(5) Given the subcategories I of weak equivalences and Q of cofibrations (resp. the subcategory P of fibrations), the subcategory P of fibrations (resp. Q of cofibrations) is uniquely characterized by a lifting property.

Note that neither cofibrant nor fibrant factorization is required to be functorial, though there often is a functorial choice (in fact, there is a sense in which the choice is unique up to homotopy). If a map $X \xrightarrow{f} Y$ is a fibration we write shorthand

$$X \xrightarrow{f} \!\!\!\!\!\rightarrow Y,$$

and similarly if $X \xrightarrow{f} Y$ is a cofibration we write

$$X \xhookrightarrow{f} Y.$$

If $X \xrightarrow{f} Y$ is an equivalence we write $X \xrightarrow[\sim]{f} Y$, with evident compound meanings for $X \xhookrightarrow[\sim]{f} Y$ (trivial cofibration) and $X \xrightarrow[\sim]{f} \!\!\!\!\!\rightarrow Y$ (trivial fibration).

4.2 The homotopy category

Suppose that $\mathcal{C}$ is a category with weak equivalences I and model structure P, Q.

- We say that an object X is *fibrant* if the map $X \to \mathrm{pt}$ to the terminal object is a fibration.
- We say that an object X is *cofibrant* if the map $\varnothing \to X$ is a cofibration.

Note that, by applying a suitable factorization axiom to the map $\varnothing \to X$ or $X \to \mathrm{pt}$, every object X admits a trivial cofibration to a fibrant object, $X \xhookrightarrow{\sim} X_P \twoheadrightarrow \mathrm{pt}$, and a trivial fibration from a cofibrant one, $\varnothing \hookrightarrow X_Q \xrightarrow{\sim} X$. We call X_P (resp. X_Q) a *fibrant* (*resp. cofibrant*) *replacement of* X. The fibrant (resp. cofibrant) replacements are in general not canonical, but in many situations of interest they can be chosen to be functorial. The W-construction discussed in Section 5.2 provides such a functorial cofibrant replacement for operads.

Let $X \sqcup X \xrightarrow{\mathbb{1}\sqcup\mathbb{1}} X$ be the codiagonal map, and $X \sqcup X \hookrightarrow C_X \xrightarrow{\sim} X$ a factorization. Any such object C_X is called *a cylinder object* for X. It admits a trivial fibration $C_X \xrightarrow{\sim} X$ and two cofibrations $X \xhookrightarrow{i_0, i_1} C_X$, which are also weak equivalences by the two-out-of-three axiom.

Similarly, let $\Delta : X \to X \times X$ be the diagonal map, and $X \xhookrightarrow{\sim} P_X \twoheadrightarrow X \times X$ a factorization. Any such object P_X is called *a path object* for X. It admits a trivial cofibration $X \xhookrightarrow{\sim} P_X$ and two fibrations $P_X \xrightarrow{p_0, p_1} X$, which are also weak equivalences by the two-out-of-three axiom.

Definition 4.1 Write $\mathcal{C}_P, \mathcal{C}_Q, \mathcal{C}_{QP}$ for the full subcategories of $\mathcal{C}$ consisting of fibrant, cofibrant, and fibrant-cofibrant objects, respectively.

Definition 4.2 Suppose that $f, g : X \to Y$ is a pair of maps, and choose a cylinder object C_X and a path object P_Y.

- f and g are *left homotopic* if the map $f \sqcup g : X \sqcup X \to Y$ factors through C_X as

$$X \sqcup X \xhookrightarrow{i_0 \sqcup i_1} C_X \xrightarrow{h} Y$$

for some choice of map ("homotopy") h.

- f and g are *right homotopic* if the map $X \xrightarrow{f \times g} Y \times Y$ factors through P_Y as

$$X \xrightarrow{k} P_Y \twoheadrightarrow Y \times Y$$

for some choice of map ("cohomotopy") k.

Lemma 4.3 [33, Proposition 1.2.5; 49] *If X is cofibrant* (*and Y is arbitrary*), *the relation $\sim_L$ of left homotopy equivalence on* $\mathrm{Hom}(X, Y)$ *is an equivalence relation*, *and does not depend on the choice of cylinder object C_X.*

If Y is fibrant (*and X is arbitrary*), *the relation $\sim_R$ of right homotopy equivalence on* $\mathrm{Hom}(X, Y)$ *is an equivalence relation and does not depend on choice of path object P_Y.*

If X is fibrant and Y is cofibrant, *then the two equivalence relations $\sim_L$ and $\sim_R$ on* $\mathrm{Hom}(X, Y)$ *are the same.*

Definition 4.4 The category $\mathrm{Ho}_{\mathcal{C}}$ is the category with objects $\mathcal{C}_{QP}$ and morphisms $\mathrm{Hom}_{\mathrm{Ho}_{\mathcal{C}}}(X, Y)$ defined as the quotient of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ by left (or, equivalently, right) homotopy equivalence.

Theorem 4.5 [33, Theorem 1.2.10; 49] *The homotopy category* $\mathrm{Ho}_{\mathcal{C}}$ *is canonically equivalent to the localized category* $\mathcal{C}[I^{-1}]$.

Remark 4.6 Recall that, given a ring A with a localizing set of elements I, there is a condition on I called the *left* (*resp. right*) *Ore condition* which allows one to write down the localization $A[I^{-1}]$ as the ring of fractions $i^{-1}f$ (resp. fi^{-1}) for $i \in I$. Similarly, given a category $\mathcal{C}$ there is a notion of left (resp. right) Ore condition, which is part of a so-called "calculus of fractions" on $\mathcal{C}$ [34, A.2.1.11(h)]. If the left Ore condition is satisfied then the category $\mathcal{C}[I^{-1}]$ can be expressed as the category of objects of $\mathcal{C}$ with morphisms $X \to Y$ represented by "roofs" $X \xrightarrow{f} Z \underset{\sim}{\xleftarrow{g}} Y$, with Z arbitrary and g a weak equivalence, subject to a straightforward equivalence relation determined by diagrams of maps commuting with a weak equivalence $Z' \xrightarrow{\sim} Z$. If $\mathcal{C}$ is a model category then the category of cofibrant objects and maps up to left homotopy satisfies the left Ore condition with quotient $\mathrm{Ho}_{\mathcal{C}}$, and the category of fibrant objects and maps up to right homotopy satisfies the right Ore condition with quotient $\mathrm{Ho}_{\mathcal{C}}$.

4.3 Some important model categories

We will give a few examples of model category structures on simplicial sets, topological spaces and differential complexes that will be important to us. Recall that in order to define a model structure, it suffices to specify just two classes of morphisms: either weak equivalences and fibrations, or weak equivalences and cofibrations. The third class is then determined by a lifting property.

4.3.1 Model category structure on simplicial sets Let SSet be the category of simplicial sets. A map of simplicial sets $f : X \to Y$ is called a *weak homotopy equivalence* if it induces a weak homotopy equivalence between geometric realizations $|f| : |X| \to |Y|$. We denote by WE the class of weak homotopy equivalences. We say that f is a *Kan fibration* if it has the right lifting property with respect to the inclusions of all horns $\Lambda^n_k \hookrightarrow \Delta^n$, $n \geq 0$, $0 \leq k \leq n$. Here Λ^n_k is the simplicial subset of Δ^n obtained by removing the nondegenerate n-simplex and the face opposite to the k-th vertex.

Theorem 4.7 (Quillen model structure [49, II.3, Theorem 3]) *There is a model structure on the category* SSet *with weak equivalences given by* WE *and fibrations given by Kan fibrations. The cofibrations are the maps of simplicial sets that are degreewise inclusions. In particular, any simplicial set is a cofibrant object.*

Other references for this foundational theorem are [27, Chapter I, Theorem 11.3; 30, Chapter 8, Theorem 8.19; 33, Chapter 3].

4.3.2 Model category structures on topological spaces Let Top be the category of compactly generated weakly Hausdorff topological spaces; see [4; 33, Definition 2.4.21, Theorem 2.4.25; 44, §6.4; 52]. Recall that a map $f : X \to Y$ is a homotopy equivalence if it admits a homotopy inverse, and is a weak homotopy equivalence if it is a bijection on path-connected components and, for any $x \in X$, the map $\pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for each $n \geq 1$. Any homotopy equivalence $X \to Y$ is a weak homotopy equivalence; the converse is true provided X, Y are CW-complexes, but not true in general. Both homotopy equivalences and weak homotopy equivalences clearly satisfy the conditions required to define a class of weak equivalences. We denote by WE the class of weak homotopy equivalences.

Theorem 4.8 (Quillen model structure [49, II.2, Theorem 1]) *There is a model structure on the category* Top *with weak equivalences given by* WE *and fibrations given by Serre fibrations. A space is cofibrant in this model structure if and only if it is a retract of a relative CW-complex, and any space is fibrant.* □

Theorem 4.9 (Strøm model structure [53]) *There is a model structure on the category* Top *with weak equivalences given by homotopy equivalences and with fibrations given by Hurewicz fibrations. The cofibrations are retracts of Hurewicz cofibrations with closed image, and in particular any space is cofibrant. Also, any space is fibrant.*

Theorem 4.10 (mixed model structure, Cole [13], see also [45, §17.3–4]) *There is a model structure on the category* Top, *called* ***mixed*** *model structure, with weak equivalences given by* WE *and fibrations given by Hurewicz fibrations. A space is cofibrant in the mixed model structure if and only if it is homotopy equivalent to a CW-complex, and any space is fibrant.*

4.3.3 Chain complexes Let k be a ring (e.g., $k = \mathbb{Z}$ or $k = \mathbb{Q}$). Then the categories $C(k)$ (resp. $C_+(k)$) of chain complexes of k-modules (resp. supported in nonnegative degrees) have model structures with weak equivalences given by quasi-isomorphisms and fibrations given by maps of complexes which are termwise surjective (resp. in all positive degrees). In particular all objects are fibrant. Cofibrant objects in $C_+(k)$ are termwise projective complexes of k-modules (see [33, Remark 2.3.7] for a discussion of cofibrant objects in $C(k)$). This is called the *standard* or *projective* model structure on the category of chain complexes [26, §2.3; 33, §2.3; 45, §18.4–5].

4.4 Quillen adjunction

It is a natural question to ask when a functor of model categories induces a functor of homotopy categories, and when this functor is a weak equivalence. (The functor most interesting for us will be the functor of chains from topological operads up to weak homotopy equivalence to dg operads up to quasi-isomorphism.) A convenient condition on a functor $F : \mathcal{C} \to \mathcal{D}$ of model categories that guarantees (in a functorial way) a functor on homotopy categories is the notion of so-called *Quillen adjunction*.

Definition 4.11 A functor $F : \mathcal{C} \to \mathcal{D}$ between model categories is a *left Quillen functor* if it admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$ such that F preserves cofibrations and trivial cofibrations and G preserves fibrations and trivial fibrations. In this situation we call G a *right Quillen functor*, and the adjunction (F, G) is called a *Quillen adjunction*.

Quillen adjunctions induce pairs of adjoint (in the conventional sense) functors between homotopy categories: one gets

$$\mathrm{ho}_F : \mathrm{Ho}_{\mathcal{C}} \leftrightarrows \mathrm{Ho}_{\mathcal{D}} : \mathrm{ho}_G,$$

defined by applying F, resp. G to fibrant-cofibrant representatives (in fact, it is sufficient to apply F to a fibrant representative and G to a cofibrant representative to get the correct functor on homotopy categories). These should be thought of as the left (resp. right) derived functors of F (resp. G). A Quillen adjunction is called a *Quillen equivalence* if ho_F (equivalently, ho_G) is an equivalence on homotopy categories.

The primordial Quillen adjunction, and one that will be important in this paper, is the adjunction

$$C_* : \mathrm{Top} \leftrightarrows C_+(\mathbb{Z}) : |\cdot|.$$

Here we define $C_*(X)$ for $X \in \mathrm{Top}$ to be the complex of singular chains on X. Its adjoint $|\cdot|$ is given by taking the geometric realization of an associated simplicial set. This adjunction can be written as a composition

$$\mathrm{Top} \leftrightarrows \mathrm{SSet} \leftrightarrows \mathrm{SAb} \leftrightarrows C_+(\mathbb{Z}).$$

The first one associates to a topological space its singular simplicial set, and to a simplicial set its geometric realization, and is a Quillen equivalence. The second one is the free-forgetful adjunction between simplicial sets and simplicial abelian groups, which is not a Quillen equivalence. The third one is the Dold–Kan correspondence, and is an equivalence of categories. See [26, Corollary 3.2.15, Theorem 3.4.4; 49, §II.3; 60, §8.4].

4.5 Homotopy (co)limits

Our sources for this section are Ginot [26, §2.5], Dwyer and Spalinski [24] and Hirschhorn [32, Chapter 13].

Suppose $\mathcal{C}$ is a model category which is cocomplete, i.e., it has all small colimits. Let J be a small "diagram" category, which we are interested in mapping to $\mathcal{C}$. The functor category

$$\mathcal{C}^J := \mathrm{Fun}(J, \mathcal{C})$$

inherits a natural notion of *weak equivalence*: we say that a natural transformation $F \to G$ of functors $F, G : J \to \mathcal{C}$ is a weak equivalence if $F(j) \to G(j)$ is such for each object $j \in J$. There are several natural model structures on the diagram category, one of which is the *projective* model structure, with fibrations determined objectwise on a map of diagrams. If X is an object of $\mathcal{C}$, there is a constant diagram $\underline{X}$ with every object of J sent to X and every arrow sent to the identity morphism of X. This determines a functor $\mathrm{const} : \mathcal{C} \to \mathcal{C}^J$. Its left adjoint is by definition the colimit functor[1]

$$\mathrm{colim} : \mathcal{C}^J \leftrightarrows \mathcal{C} : \mathrm{const}.$$

Assume now that J is given by a poset (more generally, assume J to be *very small* in the sense of [24, §10.13 sqq.; 26, Définition 2.5.11 sqq.]). The projective structure defines in this case a model category structure on $\mathcal{C}^J$ [24; 26]. The above adjunction is a Quillen adjunction, and thus induces a functor of associated homotopy categories, called the *homotopy colimit functor*, written

$$\mathrm{hocolim} : \mathrm{Ho}_{\mathcal{C}^J} \to \mathrm{Ho}_{\mathcal{C}}.$$

[1] The right adjoint is the limit functor, which is defined when J-indexed limits exist, e.g., if $\mathcal{C}$ is complete.

We will be primarily interested in calculating homotopy pushouts, i.e., homotopy colimits in the functor category $\mathcal{C}^J$ with $\mathcal{C}$ a model category and J the poset

$$a \leftarrow b \rightarrow c.$$

Given an object X in $\mathcal{C}^J$, the homotopy colimit $\mathrm{hocolim}(X)$ is by definition isomorphic to $\mathrm{colim}(X')$, where X' is a cofibrant object in $\mathcal{C}^J$ weakly equivalent to X. It is shown in [24, Proposition 10.6] that a pushout diagram X' is cofibrant in the projective model structure if and only if the maps $X'(a) \leftarrow X'(b) \rightarrow X'(c)$ are cofibrations and $X'(b)$ is cofibrant. (Hence the objects $X'(a)$ and $X'(c)$ are also cofibrant.)

The homotopy colimit of a diagram is well defined up to equivalence, but giving an explicit model depends on the choice of cofibrant resolution of the diagram.[2]

In certain situations it is possible to compute the homotopy pushout with fewer cofibrant replacements. The next result is stated in [40] as Proposition A.2.4.4.(i). We will use it in the proof of Proposition 5.5.

Lemma 4.12 *Let $\mathcal{C}$ be a model category. Given a diagram*

$$A \leftarrow B \hookrightarrow C$$

with $B \hookrightarrow C$ a cofibration and A, B (and hence C) cofibrant, we have

$$\mathrm{hocolim}(A \leftarrow B \hookrightarrow C) = \mathrm{colim}(A \leftarrow B \hookrightarrow C).$$

Proof Let $B \hookrightarrow A' \xrightarrow{\sim} A$ be a factorization of the map $B \rightarrow A$ into a cofibration and a trivial fibration. The diagram $A' \hookleftarrow B \hookrightarrow C$ is then a cofibrant replacement of the initial diagram (see Section 4.5)

$$\begin{tikzcd} A' \arrow[d, two heads, "\sim"] & B \arrow[l, hook'] \arrow[r, hook] \arrow[d, equal] & C \arrow[d, equal] \\ A & B \arrow[l] \arrow[r, hook] & C \end{tikzcd}$$

and therefore $\mathrm{hocolim}(A \leftarrow B \hookrightarrow C) = \mathrm{colim}(A' \hookleftarrow B \hookrightarrow C)$, also denoted by $A' \sqcup_B C$.

We claim that the canonical map $A' \sqcup_B C \rightarrow A \sqcup_B C$ is a weak equivalence. This is seen by considering the two pushout squares

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, hook] & C \arrow[d, hook] \\ A' \arrow[r, hook] \arrow[d, two heads, "\sim"] & A' \sqcup_B C \arrow[d, "\sim"] \\ A \arrow[r, hook] & A \sqcup_B C \end{tikzcd}$$

[2]When passing from the homotopy category to the richer ∞-category language, the category of such choices is contractible, and thus homotopy colimits are unique up to homotopy in a strong sense.

By assumption the maps $B \hookrightarrow A'$ and $B \hookrightarrow C$ are cofibrations. Since cofibrations are stable under pushout [26, 2.1.12], the maps $A' \hookrightarrow A' \sqcup_B C$ and $C \hookrightarrow A' \sqcup_B C$ are cofibrations. By assumption the map $A' \xrightarrow{\sim} A$ is a weak equivalence between cofibrant objects, and a result of Reedy [32, Proposition 13.1.2] states that the pushout of a weak equivalence *between cofibrant objects* along a cofibration is a weak equivalence. Therefore the map $A' \sqcup_B C \xrightarrow{\sim} A \sqcup_B C$ is a weak equivalence. □

Call a model category *left proper* if weak equivalences are preserved by pushouts along cofibrations, *right proper* if weak equivalences are preserved by pullbacks along fibrations, and *proper* if it is both left proper and right proper. The Quillen model category structures on SSet and Top are proper [32, Theorems 13.1.10 and 13.1.13].

Lemma 4.13 *Let $\mathcal{C}$ be a left proper model category. Given a diagram*

$$A \leftarrow B \hookrightarrow C$$

with $B \hookrightarrow C$ a cofibration, we have

$$\mathrm{hocolim}(A \leftarrow B \hookrightarrow C) = \mathrm{colim}(A \leftarrow B \hookrightarrow C). \qquad \square$$

As a consequence, in a left proper model category the homotopy pushout of a diagram $A \leftarrow B \to C$ is weakly equivalent to the ordinary pushout of the diagram obtained by replacing one arrow by a cofibration. Lemma 4.13 is stated in [40, Proposition A.2.4.4.(ii)] and proved in [3, Proposition 5.4].

The previous discussion of homotopy colimits and homotopy pushouts has a dual counterpart for homotopy limits and homotopy pullbacks. The previous results hold true for homotopy pullbacks by reversing the direction of the arrows and exchanging cofibrations and left properness into fibrations and right properness. We will use this in the discussion of homotopy fibers of maps of simplicial sets in Section A.2. The dual of Lemma 4.13 is stated and proved in [32, Proposition 13.3.7].

5 The Berger–Moerdijk model structure for operads

5.1 Existence of model structure

Suppose that $\mathcal{C}$ is a symmetric monoidal category with weak equivalences. Then we say that a map of operads $O \to O'$ in $\mathcal{C}$ is a *weak equivalence* if it is so objectwise, i.e., if $O_n \to O'_n$ is a weak equivalence for each n. Berger and Moerdijk [7] show that if $\mathcal{C}$ is a model category satisfying certain additional conditions, then this notion of weak equivalence is part of a model category structure on operads in $\mathcal{C}$, for which the fibrations $O \to O'$ are objectwise fibrations. In particular, they prove the following result.

Theorem 5.1 [7, Theorem 3.2] *If $\mathcal{C}$ is a* **cartesian+** *closed symmetric monoidal model category, then the category of operads in $\mathcal{C}$ has a model structure with weak equivalences and fibrations determined levelwise.*

We have not spelled out the meaning of "cartesian+". This is a shorthand notation for a cartesian category satisfying some additional properties (cofibrantly generated with cofibrant terminal object and admitting symmetric monoidal fibrant replacement functor); see the assumptions of Theorem 3.2 in [7]. For our purposes it suffices to record that this holds for all three model structures that we consider on Top.

5.2 W-construction and cofibrant replacement

The *W-construction* for operads plays the role of the familiar bar resolution for algebras. Our references here are Vogt [58] and Berger and Moerdijk [8]. We refer to Section 2.2 for notation concerning the definition of the free operad associated to a graded object.

Given a topological operad O, we denote by O_* the graded topological space $O_* = (O_1, O_2, \dots)$. We define a new operad $W(O)$ out of $\mathrm{Free}(O_*)$ as follows. For each $n \geq 1$ we define

$$W(O)_n = \coprod_{[\tau]\in \mathit{Tree}_n} O^{[\tau]} \times [0,1]^{\mathrm{Edge}_{[\tau]}} / {\sim_W}$$

for a certain equivalence relation $\sim_W$. Here $O^{[\tau]} \times [0,1]^{\mathrm{Edge}_{[\tau]}}$ is a notation for the quotient of the space $\coprod_{\tau\in[\tau]} O^{\tau} \times [0,1]^{\mathrm{Edge}_{\tau}}$, where $\tau \in \mathrm{Tree}_n$ ranges over the elements of the equivalence class $[\tau] \in \mathit{Tree}_n$, by the equivalence relation given by isomorphisms of labeled trees, which act on the first factor as in Section 2.2.2 and which act on the second factor via their action on the sets of edges of trees. Thus $O^{[\tau]} \times [0,1]^{\mathrm{Edge}_{[\tau]}}$ should be interpreted as the $[\tau]$-component of $\mathrm{Free}(O_*)$, which consists of all possible labelings of the vertices v of a tree τ by elements of $O_{|\mathrm{Child}(v)|}$, with the additional data of a length in $[0,1]$ for each internal edge. The equivalence relation $\sim_W$ consists simply in identifying two vertices v, w which are connected by an edge of length 0, and replacing their corresponding labels, which are elements of $O_{|\mathrm{Child}(v)|}$ and $O_{|\mathrm{Child}(w)|}$, by their composition in O which is an element of $O_{|\mathrm{Child}(v)|+|\mathrm{Child}(w)|-1}$.

Loosely speaking, *$W(O)$ is obtained from* $\mathrm{Free}(O_*)$ *by giving lengths to internal edges of trees and merging vertices according to the composition rules in O when the connecting edges acquire length zero.* The composition rule in $W(O)$ is inherited from that of $\mathrm{Free}(O_*)$, with the convention that each new internal edge which results from a composition by gluing two half-edges is attributed length 1.

Given a point $o \in W(O)_n$, we obtain a point of O_n by composing the operations in the corresponding tree. This results in a functorial map of operads $W(O) \to O$ which is (essentially by construction) a homotopy equivalence; see [8, Theorem 5.1].

The W-construction is useful for replacing maps of operads by cofibrations. Namely, we have the following theorem.

Theorem 5.2 [8, Proposition 6.6] *If $O \to O'$ is a map of operads which is a cofibration on the level of $\mathfrak{S}$-equivariant graded spaces, then $W(O) \to W(O')$ is a cofibration.*

A $\mathfrak{S}$-cofibration is understood levelwise with respect to the action of the symmetric groups $\mathfrak{S}_n$. While not giving the general definition, it will be enough for our purposes to record that a $\mathfrak{S}_n$-equivariant map $f : A \to B$ between $\mathfrak{S}_n$-spaces is a $\mathfrak{S}_n$-cofibration whenever the underlying nonequivariant map is a cofibration and the $\mathfrak{S}_n$-action on A and B is free. By convention the $\mathfrak{S}_n$-action on the empty set is free.

Corollary 5.3 *Let O be an operad in* Top *and assume that each space O_n is homotopy equivalent to a CW-complex and the action of $\mathfrak{S}_n$ on O_n is free. Then $W(O)$ is a cofibrant replacement in the mixed model category structure.*

Proof The conditions guarantee that $\varnothing \to O$ is a cofibration of $\mathfrak{S}$-equivariant graded spaces. By Theorem 5.2, the map $\varnothing = W(\varnothing) \to W(O)$ is a cofibration. □

In the statement of the next corollary we use the Berger–Moerdijk model structure on $\mathrm{Op}_{\mathrm{Top}}$ (Section 5.1) and the projective model structure on $\mathrm{Op}_{\mathrm{Top}}^J$ with $J = \{a \leftarrow b \to c\}$ (Section 4.5).

Corollary 5.4 *Let* Top *be endowed with the mixed model structure. Let*

$$O' \leftarrow O \to O''$$

be a diagram of topological operads which are levelwise homotopy equivalent to CW-complexes and which carry levelwise free $\mathfrak{S}_n$-actions. Assume further that each map is a levelwise cofibration. The homotopy colimit is computed as the colimit of the diagram

$$W(O') \leftarrow W(O) \to W(O'').$$

Proof The conditions imply that the maps $O \to O'$ and $O \to O''$ are cofibrations of $\mathfrak{S}$-equivariant graded spaces, so that $W(O) \to W(O')$ and $W(O) \to W(O'')$ are cofibrations (Theorem 5.2). By Corollary 5.3 applied to O, the operad $W(O)$ is cofibrant. By the discussion in Section 4.5, the diagram $W(O') \leftarrow W(O) \to W(O'')$ is a cofibrant replacement of the diagram $O' \leftarrow O \to O''$ and therefore $\mathrm{hocolim}(O' \leftarrow O \to O'') = \mathrm{colim}(W(O') \leftarrow W(O) \to W(O''))$. □

Proposition 5.5 *Let* Top *be endowed with the mixed model structure. Let*

$$O' \leftarrow O \to O''$$

be a diagram of topological operads which are levelwise homotopy equivalent to CW-complexes and which carry levelwise free $\mathfrak{S}_n$-actions. Assume that the map $O \to O''$ is a levelwise cofibration. The homotopy colimit is computed as the colimit of the diagram

$$W(O') \leftarrow W(O) \to W(O'').$$

Proof Since the homotopy colimit is defined in the homotopy category we have

$$\mathrm{hocolim}(O' \leftarrow O \to O'') = \mathrm{hocolim}(W(O') \leftarrow W(O) \to W(O'')).$$

Our assumptions ensure that the map $W(O) \to W(O'')$ is a cofibration and $W(O)$, $W(O')$ are cofibrant. By Lemma 4.12 we then have

$$\mathrm{hocolim}(W(O') \leftarrow W(O) \to W(O'')) = \mathrm{colim}(W(O') \leftarrow W(O) \to W(O'')).$$ □

Remark 5.6 In the previous statements, the assumption that the operads are levelwise homotopy equivalent to CW-complexes can be dropped provided one uses the Strøm model structure on Top.

6 Proof of the main theorem

6.1 Homotopy colimits

Let us consider the map of $\mathfrak{S}$-equivariant spaces $\widetilde{\mathrm{Ann}} \to \widetilde{\mathrm{Fr}}_\partial$. This is a map of free $\mathfrak{S}$-spaces which at the level of the underlying topological spaces is an embedding of a connected component. Moreover, the remaining connected components of $\widetilde{\mathrm{Fr}}_\partial$ are homotopy equivalent to CW-complexes, hence are cofibrant in the mixed model structure (and also in the Strøm model structure). This map is therefore a levelwise cofibration in the mixed model structure (and in the Strøm model structure).

Consider now the map of $\mathfrak{S}$-equivariant spaces $\widetilde{\mathrm{Ann}} \to \widetilde{\mathrm{NodAnn}}$. This is a map of free $\mathfrak{S}$-spaces which are homotopy equivalent to CW-complexes.

By Proposition 5.5, the homotopy colimit with respect to these model structures

$$\mathrm{hocolim}(\widetilde{\mathrm{NodAnn}} \leftarrow \widetilde{\mathrm{Ann}} \to \widetilde{\mathrm{Fr}}_\partial)$$

is computed by the operad colimit

$$\mathrm{colim}(W(\widetilde{\mathrm{NodAnn}}) \leftarrow W(\widetilde{\mathrm{Ann}}) \to W(\widetilde{\mathrm{Fr}}_\partial)).$$

Given a labeled tree of operations τ with n inputs, recall from Definition 3.15 the moduli space $\widetilde{\mathrm{Split}}_\tau$ of framed split surfaces with possibly coinciding seams and dual graph τ. Let Edge_τ be the set of edges of τ. We glue together the spaces $\widetilde{\mathrm{Split}}_\tau \times [0,1]^{\mathrm{Edge}_\tau}$, where we view the coordinate $[0,1]^{\{e\}}$ for e an edge as being attached to the seam S_e, into the full *Humpty-Dumpty space*

$$\widetilde{\mathrm{HD}}^{\mathrm{tree}} = \frac{\bigsqcup_\tau \widetilde{\mathrm{Split}}_\tau \times [0,1]^{\mathrm{Edge}_\tau}}{\sim},$$

where $\sim$ is the equivalence relation

$$(\Sigma, (S_1, \dots, S_E), (t_1, \dots, t_e = 0, \dots, t_E)) \sim (\Sigma, (S_1, \dots, \widetilde{S}_e, \dots, S_E), (t_1, \dots, \hat{t}_e, \dots t_E)).$$

Similarly, we define

$$\mathrm{HD}^{\mathrm{tree}} = \frac{\bigsqcup_\tau \mathrm{Split}_\tau \times [0,1]^{\mathrm{Edge}_\tau}}{\sim},$$

with respect to the same equivalence relation.

The spaces $\widetilde{\mathrm{HD}}^{\mathrm{tree}}$ and $\mathrm{HD}^{\mathrm{tree}}$ classify split surfaces with additional simplicial parameters that allow us to continuously "put the curve back together", hence the term "Humpty-Dumpty".

Lemma 6.1 *We have homeomorphisms*

$$\widetilde{\mathrm{HD}}^{\mathrm{tree}} \simeq W(\widetilde{\mathrm{Fr}}_\partial) \quad \textit{and} \quad \mathrm{HD}^{\mathrm{tree}} \simeq W(\mathrm{Fr}_\partial).$$

Proof This is a direct consequence of Lemma 3.16 and the description of the W-construction in Section 5.2. □

Similarly, we define the *moduli of nodal Humpty-Dumpty surfaces* as

$$\widetilde{\mathrm{NodHD}}^{\mathrm{tree}} = \frac{\bigsqcup_\tau \widetilde{\mathrm{NodSplit}}^{\mathrm{tree}}_\tau \times [0,1]^{\mathrm{Edge}_\tau}}{\sim}$$

and

$$\mathrm{NodHD}^{\mathrm{tree}} = \frac{\bigsqcup_\tau \mathrm{NodSplit}^{\mathrm{tree}}_\tau \times [0,1]^{\mathrm{Edge}_\tau}}{\sim},$$

where $\sim$ is the same equivalence relation as above.

Remark 6.2 Using the formalism of Segal operads developed in Appendix B, it is the case that $\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}$ (resp. $\mathrm{NodHD}^{\mathrm{tree}}$) is a Segal operad which is equivalent to the W-construction applied to the Segal operad in topological moduli problems $\widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_\partial$ (resp. $\mathrm{NodFr}^{\mathrm{tree}}_\partial$). We will not need this result, however, as we will be most interested in a stabilizer-free subspace of $\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}$ (resp. $\mathrm{NodFr}^{\mathrm{tree}}_\partial$).

Definition 6.3 We define

$$\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}} \quad (\text{resp. } \mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}})$$

to be the space of tuples $(\Sigma, \{S_e\}, \{t_e\})$ in $\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}$ (resp. $\mathrm{NodHD}^{\mathrm{tree}}$) such that each node of Σ is surrounded on both sides by seams which can be contracted to the node and have weight 1.

Lemma 6.4 *We have canonical isomorphisms*

$$\mathrm{colim}(W(\widetilde{\mathrm{NodAnn}}) \leftarrow W(\widetilde{\mathrm{Ann}}) \to W(\widetilde{\mathrm{Fr}}_\partial)) \cong \widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}}$$

and

$$\mathrm{colim}(W(\mathrm{NodAnn}) \leftarrow W(\mathrm{Ann}) \to W(\mathrm{Fr}_\partial)) \cong \mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}}.$$

Proof We give details only for the proof of the first isomorphism since the proof of the second one is verbatim the same after removing the symbol $\widetilde{}$ everywhere.

Using the equivalence $W(\widetilde{\mathrm{Fr}}_\partial) \simeq \widetilde{\mathrm{HD}}^{\mathrm{tree}}$ from Lemma 6.1 the statement becomes

$$\mathrm{colim}(W(\widetilde{\mathrm{NodAnn}}) \leftarrow W(\widetilde{\mathrm{Ann}}) \to \widetilde{\mathrm{HD}}^{\mathrm{tree}}) \cong \widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}}. \tag{6}$$

The operad $W(\widetilde{\mathrm{Ann}})$ can be described as consisting of standard annuli with parametrized boundary and parametrized seams given by concentric circles, with simplicial parameters in $[0,1]$ attached to them. Seams with simplicial parameter equal to 0 can be erased or added. The operad $W(\widetilde{\mathrm{NodAnn}})$ can be described in a similar way, allowing also nodal annuli.

We have a map

$$W(\widetilde{\mathrm{NodAnn}}) \sqcup_{W(\widetilde{\mathrm{Ann}})} \widetilde{\mathrm{HD}}^{\mathrm{tree}} \to \widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}}$$

given by gluing along their boundaries the framed surfaces and nodal annuli by which we decorate the vertices of a tree, while keeping track of the boundary curves by interpreting them as additional seams with weight 1. Reasoning as in the proof of the geometric pushout Theorem 3.11 we see that this map descends to a homeomorphism after imposing the pushout relations on the free product. □

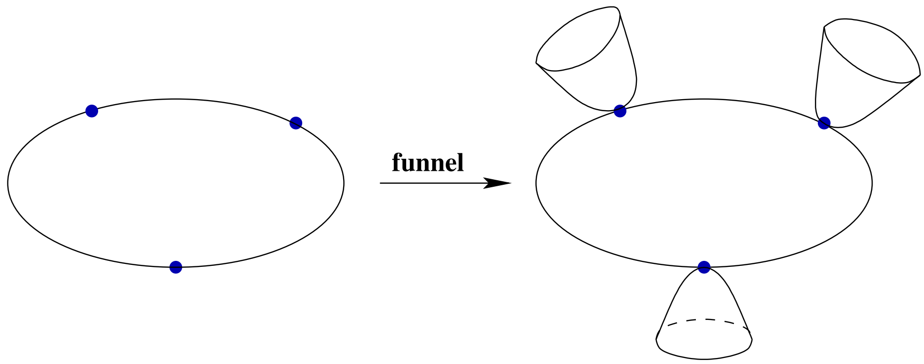

Figure 8: By attaching disks at marked points one turns a surface into a framed nodal surface.

6.2 Proof of the main theorem in genus 0

In this section we prove our main theorem in genus 0, recovering thus by a different method the theorem of Drummond-Cole [21] mentioned in the Introduction. We denote by $\overline{\mathcal{M}}_{0,*}$ the genus-0 Deligne–Mumford–Knudsen operad. Since genus-0 curves with at least 3 marked points have no automorphisms, there are no stacky phenomena to take into account. We will use the subscript notation $\mathrm{NodFr}^{\mathrm{tree}}_{g=0} \subset \mathrm{NodFr}^{\mathrm{tree}}$ etc. to denote the genus-0 suboperads of the various operads that we use.

Lemma 6.5 *We have a homotopy equivalence of operads*

$$\mathbf{funnel} : \overline{\mathcal{M}}_{0,*} \xrightarrow{\simeq} \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,g=0}.$$

Proof Let $\mathbb{D}$ be the standard unit disk, framed by the standard boundary parametrization $\theta \mapsto \exp(2\pi i\theta)$ for $\theta \in \mathbb{R}/\mathbb{Z}$ and let $0 \in \mathbb{D}$ be the origin. Let $\overline{\mathbb{D}}$ be the disk framed with the reverse boundary parametrization $\theta \mapsto \exp(-2\pi i\theta)$. Let A_α be the standard annulus of modulus $\alpha \in (0,\infty)$ framed with the standard boundary parametrizations. We then have

$$\lim_{\alpha\to\infty} A_\alpha = \mathbb{D} \cup_0 \overline{\mathbb{D}}$$

in NodAnn, compatibly with boundary parametrizations.

Given a marked nodal surface $X \in \overline{\mathcal{M}}_{0,*}$, write

$$\mathbf{funnel}(X)$$

for the framed nodal surface obtained by gluing (at $0 \in \mathbb{D}$) a copy of $\mathbb{D}$ at every input marked point of X and a copy of $\overline{\mathbb{D}}$ at the output marked point of X. See Figure 8.

By a simple stabilization argument we see that

$$\mathbf{funnel} : \overline{\mathcal{M}}_{0,*} \to \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,g=0}$$

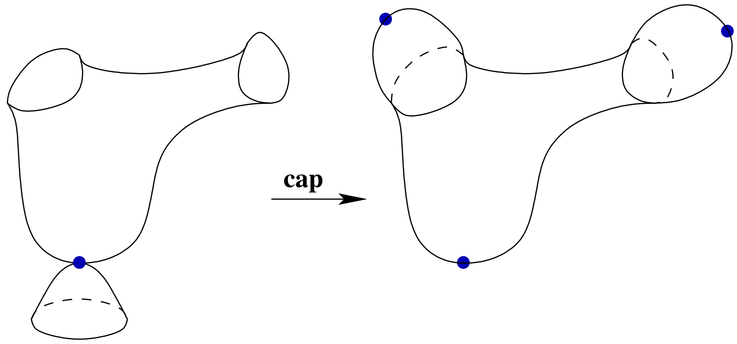

Figure 9: By attaching caps along the boundary and stabilizing one turns a framed nodal surface into a nodal surface with marked points.

is a map of topological operads. On the other hand we see that Fr_n is the embedding of a homotopy retract for each arity $n \geq 0$. Indeed, let

$$\mathbf{cap} : \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial, g=0} \to \mathrm{DM}^{\mathrm{tree}}$$

be the map (now of $\mathfrak{S}$–graded spaces, not operads) which assigns to a nodal surface with boundary the surface with marked points obtained by gluing a copy of $\overline{\mathbb{D}}$ at each input and a copy of $\mathbb{D}$ at the output, and marking all images of $0 \in \overline{\mathbb{D}}$, respectively $\mathbb{D}$. See Figure 9.

Then it is clear that

$$\mathbf{cap} \circ \mathbf{funnel} = \mathbb{1}_{\mathrm{DM}}.$$

On the other hand, consider the maps

$$\mathbf{stretch}_\alpha : \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial, g=0} \to \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial, g=0}, \quad \alpha \in [0, \infty],$$

defined by gluing the standard annulus A_α at every input and output of a framed nodal surface. This defines a homotopy equal to the identity map at $\alpha = 0$ and equal to $\mathbf{funnel} \circ \mathbf{cap}$ at $\alpha = +\infty$, which proves the homotopy retract property. □

The following result will complete the proof of our main theorem.

Proposition 6.6 *There is a weak equivalence of operads*

$$\boldsymbol{\pi} : \widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}, g=0} \to \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial, g=0},$$

where $\boldsymbol{\pi}$ forgets the simplex parameters and glues along the seams.

Proof It suffices to show that

$$\boldsymbol{\pi} : \mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}, g=0} \to \mathrm{NodFr}^{\mathrm{tree}}_{\partial, g=0} \tag{7}$$

is a trivial Serre fibration. Indeed, we can ignore thickness-zero curves because including them does not change the weak homotopy type. We do this in two steps: firstly we show that the fiber of (7) is contractible, and secondly we show that (7) is a Serre fibration.

Step 1 (we show that the fiber of (7) is contractible) Let $\Sigma \in \mathrm{NodFr}^{\mathrm{tree}}_{\partial,g=0}$ be a surface, and let $\mathrm{HD}_\Sigma := \boldsymbol{\pi}^{-1}(\Sigma)$. Given mutually disjoint neighborhoods $\mathcal{V}_i$, $i = 1, \dots, k$, of the nodes $z_1, \dots, z_k$ of Σ, define $\mathcal{V} = \bigsqcup_{i=1}^k \mathcal{V}_i$ and let $\mathrm{HD}^{\mathcal{V}}_\Sigma \subset \mathrm{HD}_\Sigma$ be the subspace consisting of those tree-like split surface structures on Σ whose seams lie away from $\overline{\mathcal{V}}$. Since seams are not allowed to pass through nodes, the spaces $\mathrm{HD}^{\mathcal{V}}_\Sigma$ filter HD_Σ by opens as $\mathcal{V}$ runs over a neighborhood basis of the nodes of Σ.

We consider now a specific neighborhood basis of the nodes: we choose a 1-parameter smooth family of open neighborhoods of the nodes, denoted by $\mathcal{V}^\varepsilon_i$, $0 < \varepsilon \le 1$, such that the $\overline{\mathcal{V}^\varepsilon_i}$ are mutually disjoint for every ε, each $\overline{\mathcal{V}^\varepsilon_i}$ is analytically diffeomorphic to a nodal annulus, $\overline{\mathcal{V}^\varepsilon_i} \subset \mathcal{V}^{\varepsilon'}_i$ for $\varepsilon < \varepsilon'$ and $\bigcap_{\varepsilon \in (0,1]} \mathcal{V}^\varepsilon_i = \{z_i\}$.

Let $\mathcal{V}^\varepsilon = \bigsqcup_{i=1}^k \mathcal{V}^\varepsilon_i$, let $\mathrm{HD}^\varepsilon_\Sigma = \mathrm{HD}^{\mathcal{V}^\varepsilon}_\Sigma$, and let

$$\mathrm{HD}^I_\Sigma = \bigsqcup_{\varepsilon \in (0,1]} \mathrm{HD}^\varepsilon_\Sigma \subset I \times \mathrm{HD}_\Sigma$$

with $I = (0, 1]$. The map

$$\mathrm{HD}^I_\Sigma \to \mathrm{HD}_\Sigma$$

is a homotopy equivalence because it is a fibration and all its fibers are nonempty intervals (with one open endpoint 0). We are thus left to prove that HD^I_Σ is contractible.

Consider the canonical map

$$\mathbf{gap} : \mathrm{HD}^I_\Sigma \to \mathrm{HD}^I_\Sigma$$

given by the tautological inclusions $\mathrm{HD}^\varepsilon_\Sigma \hookrightarrow \mathrm{HD}^{\varepsilon/2}_\Sigma$, $\varepsilon \in (0, 1]$, i.e.,

$$(w, \varepsilon) \mapsto (w, \varepsilon/2), \quad w \in \mathrm{HD}^\varepsilon_\Sigma.$$

This map is well defined because the filtration is decreasing. It is a homotopy equivalence because it is homotopic to the identity via the family of maps $\mathbf{gap}_t$, $t \in [0, 1]$, given by the tautological inclusions $\mathrm{HD}^\varepsilon_\Sigma \hookrightarrow \mathrm{HD}^{\varepsilon - t\varepsilon/2}_\Sigma$, $(w, \varepsilon) \mapsto (w, \varepsilon - t\varepsilon/2)$.

Choose a continuous family of seams S_ε, $\varepsilon \in \left(0, \tfrac{1}{2}\right]$ contained in $\mathcal{V}^{2\varepsilon} \setminus \overline{\mathcal{V}^\varepsilon}$, one on each side of a node. (We can take S_ε to be given by continuously varying parametrizations of the boundary of $\mathcal{V}^{3\varepsilon/2}$.) Define the continuous map

$$\mathbf{protect} : \left(0, \tfrac{1}{2}\right] \to \mathrm{HD}^I_\Sigma,$$

which takes ε to the pair $(\Sigma_\varepsilon, \varepsilon)$, where $\Sigma_\varepsilon = (\Sigma, S_\varepsilon)$ is the protected split surface with seams S_ε of weight 1.

We claim that $\mathbf{gap} : \mathrm{HD}^I_\Sigma \to \mathrm{HD}^I_\Sigma$ is homotopic to the map

$$\boldsymbol{\tau} : (w, \varepsilon) \mapsto \mathbf{protect}(\varepsilon/2).$$

The homotopy is constructed in two steps: starting from **gap**, we first put in the seams $S_{\varepsilon/2}$ with weight continuously changing from 0 to 1 (this is allowed because we are in the image of **gap** and there are no other seams at distance $\le \varepsilon$ from a node), and then continuously reduce all the other weights to zero (this is allowed because the presence of the $S_{\varepsilon/2}$ with weight 1 guarantees that the nodes remain protected). At the endpoint of this homotopy we read the map $\boldsymbol{\tau}$.

Since $\boldsymbol{\tau}$ factors through an interval, it is homotopic to a constant. Thus **gap** is at the same time homotopic to a constant and a homotopy equivalence, which implies that the space HD^{I}_{Σ} is contractible. This finishes the proof of Step 1.

Step 2 (we show that (7) is a Serre fibration) It is enough to check the Serre fibration property locally on the base. Our proof is based on the existence of a continuously varying thin-thick decomposition in the neighborhood of any fixed surface in $\mathrm{NodFr}^{\mathrm{tree}}_{\partial,g=0}$. In the following we designate by Σ both a point in the moduli space $\mathrm{NodFr}^{\mathrm{tree}}_{\partial,g=0}$, and the fiber at Σ of the universal curve over the moduli space.

Given $\Sigma \in \mathrm{NodFr}^{\mathrm{tree}}_{\partial,g=0}$ with nodes $z_1,\dots,z_k$, there exists a neighborhood $\mathcal{U} \subset \mathrm{NodFr}^{\mathrm{tree}}_{\partial,g=0}$ of Σ and a family $\mathcal{V}_i^{\varepsilon}(\Sigma')$, $i=1,\dots,k$, $0<\varepsilon\le 1$, $\Sigma'\in\mathcal{U}$, of open sets $\mathcal{V}_i^{\varepsilon}(\Sigma')\subset\Sigma'$ (the "thin" parts) such that:

- For each i the family $\mathcal{V}_i^{\varepsilon}(\Sigma')$ is continuous in ε and Σ'.
- For Σ', ε fixed, the closures $\overline{\mathcal{V}_i^{\varepsilon}(\Sigma')}$, $i=1,\dots,k$, are disjoint.
- For Σ', i fixed we have $\overline{\mathcal{V}_i^{\varepsilon}(\Sigma')}\subset\mathcal{V}_i^{\varepsilon'}(\Sigma')$ for $\varepsilon<\varepsilon'$.
- Each $\mathcal{V}_i^{\varepsilon}(\Sigma')$ is analytically diffeomorphic to a (possibly nodal) annulus. If $\mathcal{V}_i^{\varepsilon}(\Sigma')$ is analytically diffeomorphic to a nodal annulus for some $\varepsilon>0$, then it is analytically diffeomorphic to a nodal annulus for all $\varepsilon>0$, and in this case $\bigcap_{\varepsilon>0}\mathcal{V}_i^{\varepsilon}(\Sigma')=\{z_i'\}$, the common node of these nodal annuli.

These properties follow from the fact that a sufficiently small neighborhood of Σ in $\mathrm{NodFr}^{\mathrm{tree}}_{\partial,g=0}$ consists of surfaces Σ' that are obtained from Σ by resolving some of its nodes.

Let $I=[0,1]$. Let $\varphi: I^n\to\mathcal{U}$ be a continuous map with a lift $\tilde{\varphi}_0: I^{n-1}\times\{0\}\to\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected},g=0}$ of $\varphi|_{I^{n-1}\times\{0\}}$, i.e., $\boldsymbol{\pi}\circ\tilde{\varphi}_0=\varphi|_{I^{n-1}\times\{0\}}$. We need to construct $\tilde{\varphi}: I^n\to\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected},g=0}$ such that $\boldsymbol{\pi}\circ\tilde{\varphi}=\varphi$ and $\tilde{\varphi}|_{I^{n-1}\times\{0\}}=\tilde{\varphi}_0$. Denote by $\mathcal{D}_i\subset\mathcal{U}$, $i=1,\dots,k$ the closed set (smooth of complex codimension 1) consisting of curves Σ' such that $\mathcal{V}_i^{\varepsilon}(\Sigma')$ is analytically diffeomorphic to a nodal annulus.

Remark 6.7 Loosely said, the construction of the lift $\tilde{\varphi}$ will be done in the following steps. Firstly, we insert small protecting seams close to the nodes and close to $I^{n-1}\times\{0\}$. Secondly, we extend the already present seams close to $I^{n-1}\times\{0\}$, and then erase them. Thirdly, we extend the protecting seams throughout the family as boundaries of the thin part.

We now proceed to construct the lift.

By compactness of I^{n-1}, because the tree-like split structures encoded by $\tilde{\varphi}_0$ are protected, and because a protecting seam at a node survives in a neighborhood of the curve even if the node gets resolved, there exist $\varepsilon>0$ and, for each $i=1,\dots,k$, an open neighborhood $\mathcal{U}_i\subset\mathcal{U}$ of $\mathcal{D}_i$, such that, for $t=(t_1,\dots,t_{n-1},0)\in I^{n-1}\times\{0\}$, if $\varphi(t)\in\mathcal{U}_i$ then no seam of $\tilde{\varphi}_0(t)$ intersects $\mathcal{V}_i^{\varepsilon}(\varphi(t))$.

Let $\delta>0$ be small. We make a first extension of $\tilde{\varphi}_0$ over $I^{n-1}\times[0,\delta]$ in two steps as follows:

Step 1 Let $\rho_i:\mathcal{U}_i\to[0,1]$ be smooth cutoff functions, supported away from $\mathcal{U}\setminus\mathcal{U}_i$ and equal to 1 on $\mathcal{D}_i$. Let $\rho:[0,1]\to[0,1]$, $\rho(s)=s/\delta$ for $s\in[0,\delta]$, $\rho(s)=1$ for $s\in[\delta,1]$. We insert for each i two seams as boundaries of $\mathcal{V}_i^{\varepsilon/2}(\varphi(t))$ for $t \in I^{n-1} \times [0,\delta]$, with weight equal to $\rho(t_n)\prod_i \rho_i(\varphi(t))$. These will act as "protecting" seams in the sequel.

Step 2 We extend the other seams from $I^{n-1} \times \{0\}$ to $I^{n-1} \times [0,\delta]$ as follows. We choose a smooth trivialization of the family with fiber Σ' over $\mathcal{U} \setminus \bigcup_i \mathcal{U}_i$. This induces a trivialization of the family with fiber $\Sigma' \setminus \bigcup_i \mathcal{V}_i^{\varepsilon}(\Sigma')$ over $\mathcal{U} \setminus \bigcup_i \mathcal{U}_i$, which we extend to a trivialization of the family with the same fiber $\Sigma' \setminus \bigcup_i \mathcal{V}_i^{\varepsilon}(\Sigma')$ over $\mathcal{U}$. For every $t \in I^{n-1}$, every seam $S_{t,0}$ on $\tilde{\varphi}_0(t,0)$, and every $t_n \in [0,\delta]$, we induce via these trivializations a uniquely determined seam S_{t,t_n} on $\varphi(t,t_n)$. These seams are embedded and smoothly parametrized with continuously varying parametrizations, and they admit continuously varying analytic parametrizations. After making a choice of such continuously varying analytic parametrizations, we obtain a lift $\tilde{\varphi}_{[0,\delta]}$ of φ over $I^{n-1} \times [0,\delta]$ that extends $\tilde{\varphi}_0$.

We now modify the previous lift $\tilde{\varphi}_{[0,\delta]}$ in two steps as follows:

Step 1 We weigh each of the seams S_{t,t_n} from the previous construction by the function $1-\rho(2t_n)$ for $t_n \in [0,\delta/2]$, and erase them for $t_n \in [\delta/2,\delta]$.

As a result, for $t \in I^{n-1} \times [\delta/2,\delta]$ the seam structure is the following: for each i we have exactly two seams that are the boundaries of $\mathcal{V}_i^{\varepsilon/2}(\varphi(t))$, with weight equal to $\rho(t_n)\prod_i \rho_i(\varphi(t))$.

Step 2 For $t \in I^{n-1} \times [\delta/2,\delta]$ we modify the weight of these seams to $(1-s)\rho(t_n)\prod_i \rho_i(\varphi(t)) + s$, with $s = \frac{2}{\delta}t_n - 1$.

At this point we have obtained a lift $\tilde{\varphi}|_{[0,\delta]}$ of $\varphi|_{I^{n-1}\times[0,\delta]}$ such that, for $t \in I^{n-1} \times \{\delta\}$, the seam structure over $\varphi(t)$ consists of the boundaries of the $\mathcal{V}_i^{\varepsilon/2}(\varphi(t))$, $i = 1,\dots,k$ with weight 1. This seam structure extends tautologically for all $t \in I^{n-1} \times [\delta,1]$ and provides a lift of φ. □

Proof of the main Theorem 1.1 in genus 0 We saw in Section 6.1 that the homotopy colimit

$$\operatorname{hocolim}(\widetilde{\mathrm{NodAnn}} \leftarrow \widetilde{\mathrm{Ann}} \rightarrow \widetilde{\mathrm{Fr}}_{\partial,g=0})$$

is computed by the operad colimit

$$\operatorname{colim}(W(\widetilde{\mathrm{NodAnn}}) \leftarrow W(\widetilde{\mathrm{Ann}}) \rightarrow W(\widetilde{\mathrm{Fr}}_{\partial,g=0})).$$

In view of Lemma 6.4, Proposition 6.6 and Lemma 6.5 we obtain the sequence of weak equivalences

$$\operatorname{colim}(W(\widetilde{\mathrm{NodAnn}}) \leftarrow W(\widetilde{\mathrm{Ann}}) \rightarrow W(\widetilde{\mathrm{Fr}}_{\partial,g=0})) \cong \widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected},g=0} \xrightarrow[\boldsymbol{\pi}]{\simeq} \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,g=0} \xleftarrow[\textbf{funnel}]{\simeq} \overline{\mathcal{M}}_{0,*}. \tag{8}$$

This proves the theorem for the unital version of the involved operads.

The proof carries over verbatim to the nonunital versions of our operads Ann and Fr_∂. Alternatively, one can use the fact that the forgetful functor from operads to nonunital operads commutes with homotopy pushouts. □

6.3 Proof of the main theorem in arbitrary genus

In this section we prove our main theorem in arbitrary genus. The main new phenomenon compared to that of genus 0 is the presence of stabilizers.

We develop in Appendix A the convenient language of *topological moduli problems* (*TMP*) to deal with stabilizers. A topological moduli problem is a contravariant functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Gpd}$, and the key point is that the spaces of operations of $\mathrm{DM}^{\mathrm{tree}}$ and $\mathrm{NodFr}_\partial^{\mathrm{tree}}$ have natural structures of topological moduli problems. On the other hand, any topological space X can be viewed as a topological moduli problem via the mapping-in functor $S \mapsto \mathrm{Map}_{\mathrm{Top}}(S, X)$, so that the language of topological moduli problems provides a convenient common framework for all the spaces of operations considered in this paper.

The classical language of operads is not well adapted to deal with spaces of operations that are topological moduli problems. Indeed, in the classical language of operads the associativity relations would translate into equalities of functors, whereas in our situation we only have isomorphisms. A convenient language is that of Segal operads, which we introduce in Appendix B. Roughly speaking, whereas an operad O associates to every arity $n \geq 0$ a space of operations O_n, a Segal operad Λ associates to every *tree of operations* τ, of arbitrary arity, a space of operations Λ_τ. We explain in Section B.2 the construction of the (tree-like) *Deligne–Mumford Segal operad* $\Lambda\mathrm{DM}^{\mathrm{tree}}$ and of the *Segal operad of nodal framed surfaces* $\Lambda\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree}}$ as Segal operads in topological moduli problems.

With this terminology in place, the proof of the genus-0 case of our main theorem goes through with essentially only minor modifications.

The next lemma is the higher-genus counterpart of Lemma 6.5.

Lemma 6.8 *We have a weak equivalence of Segal operads*

$$\mathbf{funnel} : \Lambda\mathrm{DM}^{\mathrm{tree}} \xrightarrow{\simeq} \Lambda\widetilde{\mathrm{NodFr}}_\partial^{\mathrm{tree}}$$

induced by applying componentwise the map **funnel** *from Lemma 6.5.*

Proof Being a weak equivalence of Segal operads means being a homotopy equivalence on the level of simplicial chains for every space of operations, i.e., every object associated to a corolla; see Appendix B. We thus need to show that

$$\mathbf{funnel} : \mathrm{DM}_n^{\mathrm{tree}} \to \widetilde{\mathrm{NodFr}}_{\partial,n}^{\mathrm{tree}}$$

induces a homotopy equivalence on simplicial realizations.

The construction from Lemma 6.5 can be applied in order to define a left inverse **cap** for the map **funnel**, which presents **funnel** as a retract, i.e., such that **funnel** $\circ$ **cap** homotopy retracts to the identity. It is therefore enough to see that **cap** induces a homotopy equivalence on simplicial realizations.

By Proposition A.24 the map **cap** is a combinatorial fibration. The points of the fibers of **cap** over a given marked curve X correspond to framed curves which give X after gluing disks onto each boundary component and stabilizing. These fibers have no automorphisms and are representable by topological spaces (see Remark 3.3), hence **cap** is a fiberwise representable fibration. The proof of Lemma 6.5 goes through verbatim to show that the fibers of **cap** can be simultaneously contracted, which implies that **cap** is a trivial weak fibration of topological moduli problems. The conclusion then follows from Lemma A.27. □

The next proposition is the higher-genus counterpart of Proposition 6.6.

Proposition 6.9 *There is a weak equivalence of Segal operads*

$$\boldsymbol{\pi} : \Lambda\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}} \to \Lambda\widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial},$$

where $\boldsymbol{\pi}$ *forgets the simplex parameters and glues along the seams.*

Proof Being a weak equivalence of Segal operads means being a homotopy equivalence on the level of simplicial chains for every space of operations, i.e., every object associated to a corolla; see Appendix B.

We need to show that the map is a trivial fibration on the level of each space of operations in the sense of Definition A.26, from which the conclusion follows by Lemma A.27. More precisely, we need to prove that, for every corolla, i.e., for every arity $n \geq 0$, the induced map of TMPs

$$\boldsymbol{\pi} : \mathbb{M}\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected},n} \to \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,n}$$

is a trivial weak fibration.

By Proposition A.24, this map is a combinatorial fibration. It is also fiberwise representable since fibers have no automorphisms and choices of seams on a given curve form a topological space. That the fibers have no automorphisms is seen as follows: given a family σ, a point in the fiber $(\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected},n})_\sigma$ is given by a family of nodal framed curves with additional data (given by weighted seams), together with an isomorphism between this family and the family σ. Since the isomorphism is part of the data and the family σ is fixed, the only possible automorphism is the identity.

We now prove the trivial weak fibration property. For the proof, let $\sigma \in \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,n}(\Delta^m)$ be a family of curves parametrized by an m-dimensional simplex and let us write $(\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected},n})_\sigma$ for the fiber of $\boldsymbol{\pi}$ at σ. While $\widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,n}$ is a topological moduli problem, the fiber has no stabilizers as seen above. In this situation the proof of contractibility from Proposition 6.6 goes through verbatim if one works locally on the base Δ^m. This implies contractibility over the whole of Δ^m, and a fortiori also weak contractibility. □

Proof of the main Theorem 1.1 in arbitrary genus We saw in Section 6.1 that the homotopy colimit

$$\mathrm{hocolim}(\widetilde{\mathrm{NodAnn}} \leftarrow \widetilde{\mathrm{Ann}} \to \widetilde{\mathrm{Fr}}_\partial)$$

is computed by the operad colimit

$$\mathrm{colim}(W(\widetilde{\mathrm{NodAnn}}) \leftarrow W(\widetilde{\mathrm{Ann}}) \to W(\widetilde{\mathrm{Fr}}_\partial)).$$

In view of Lemmas 6.4, 6.8 and Proposition 6.9 we obtain the sequence of weak equivalences of Segal operads

$$\mathrm{colim}(W(\widetilde{\mathrm{NodAnn}}) \leftarrow W(\widetilde{\mathrm{Ann}}) \to W(\widetilde{\mathrm{Fr}}_\partial)) \cong \widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}} \xrightarrow[\boldsymbol{\pi}]{\simeq} \Lambda\widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,g=0} \xleftarrow[\mathbf{funnel}]{\simeq} \Lambda\mathrm{DM}^{\mathrm{tree}}. \tag{9}$$

This proves the theorem for the unital version of the involved operads.

The proof carries over verbatim to the nonunital versions of our operads Ann and Fr_∂. □

Appendix A Topological moduli problems

Most topological spaces we work with in this paper are *moduli spaces* of one kind or another, i.e., spaces $\mathbb{M}$ characterized by the mapping-in functor $S \mapsto \mathrm{Hom}(S, \mathbb{M})$ which classifies solutions of an appropriate moduli problem — usually, the problem of classifying certain curves with additional structure over S up to isomorphism. In some cases, such as Fr_∂, solutions to the moduli problem have no automorphisms, and thus $\mathbb{M}$ is representable in the category of topological spaces. In other cases, such as $\overline{\mathcal{M}}_{g,n}$, the solutions to the moduli problem may have automorphisms, and thus $\mathbb{M}$ is a *topological stack* of some kind. In this appendix we will describe a certain "strict" version of topological stacks (or more precisely, prestacks) that is sufficient for our purposes and that we call "topological moduli problems". On the level of homotopy categories (for the notion of weak stack homotopy equivalence relevant to this paper), any topological (pre)stack is homotopy equivalent to a topological space. For example, given a (topological) group G, the stack pt/G is homotopy equivalent to the classifying space BG, which is stabilizer-free. However, this "resolution" from topological stacks to topological spaces loses some 2-categorical information, and does not, at least naively, respect algebraic structures such as operads; thus even in order to define the operad $\mathrm{DM}^{\mathrm{tree}}$ with spaces of operations built out of the $\overline{\mathcal{M}}_{g,n}$ we must work inside a category of topological moduli problems.

A.1 Definitions

Definition A.1 A *groupoid* is a category $\mathcal{C}$ all of whose morphisms are invertible and such that the isomorphism classes of objects form a set, denoted by $\pi_0(\mathcal{C})$.

Remark A.2 (set-theoretic technicalities) We do not require for groupoids to be small categories, but rather only to be equivalent to small categories. In the constructions in the remainder of this appendix, we assume at every point that we have chosen representatives which are small categories in a consistent way. This is possible via standard small object arguments, which we assume implicitly throughout this appendix. One way to make this precise is to choose an inaccessible cardinal κ and implicitly assume that the objects we work with are κ-small — see, for example, [40, §1.2.15] for a discussion.

Definition A.3 A *topological moduli problem* (*TMP*) is a (strict) contravariant functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Gpd}$ from the category of topological spaces to the category of groupoids with (strict) groupoid functors.

We write TMP for the category of such functors, with maps $\mathcal{X} \to \mathcal{Y}$ given by natural transformations. Given a map $f : S \to S'$ of topological spaces we write $f^* : \mathcal{X}(S') \to \mathcal{X}(S)$ for the (contravariantly) associated functor of groupoids.

The notion of TMP is a "stricter" notion of a topological stack. When working with stacks one often works with nonstrict functors (defined via slice categories) and imposes a sheaf condition on these functors; we will not require either of these sophistications here (though all topological moduli problems we consider will in fact also be stacks). We only need a sufficient formalism to study homotopically our main examples by means of an associated simplicial object.

Definition A.4 Given a topological space X, define the *topological moduli problem represented by* X to be the functor

$$X : S \mapsto \mathrm{Map}_{\mathrm{Top}}(S, X)$$

(the set of continuous maps, viewed as a discrete groupoid, i.e., a groupoid with no morphisms other than Id). For readability we shall often use the same notation X for both the space and the corresponding moduli problem.

Remark A.5 If $\mathcal{X}$ is a topological moduli problem and S is a topological space, it makes sense to think of the groupoid $\mathcal{X}(S)$ as the "groupoid of maps" from S (viewed as a TMP) to $\mathcal{X}$, even when $\mathcal{X}$ does not itself come from a topological space. Note that, if $\mathcal{X}$ takes values in *discrete* groupoids, i.e., Sets, by the Yoneda lemma we have a natural isomorphism of sets

$$\mathrm{Hom}_{\mathrm{TMP}}(S, \mathcal{X}) = \mathcal{X}(S).$$

If $\mathcal{X}$ is not valued in discrete groupoids, the groupoid $\mathcal{X}(S)$ can be interpreted as a "Hom groupoid" only after passing to a higher-categorical setting, which we will not pursue. It is nevertheless a useful intuition that $\mathcal{X}(S)$ is a Hom object. In particular, we will later define for a map $\mathcal{X} \to \mathcal{Y}$ of TMPs the notion of a fiber object $\mathcal{X}_\gamma$ over an "S-object" $\gamma \in \mathcal{Y}(S)$ (Definition A.23).

Remark A.6 Associated to any topological moduli problem $\mathcal{X}$ is a so-called *coarse moduli space* $\mathcal{X}^{\mathrm{coarse}}$. This is the topological space which is the initial object in the category of pairs $(X, f : \mathcal{X} \to X)$ consisting of a topological space X with a map $f : \mathcal{X} \to X$. In certain favorable cases, e.g., orbifolds, the coarse moduli space is the set $\pi_0\mathcal{X}(\mathrm{pt})$ with a natural topology induced from $\mathcal{X}$. See [48, §4.3] for more details.

Definition A.7 (1) We say that a map of topological moduli problems $\mathcal{X} \to \mathcal{Y}$ is an *equivalence* if $\mathcal{X}(S) \to \mathcal{Y}(S)$ is an equivalence of groupoids for all topological spaces S.

(2) We say that a moduli problem $\mathcal{X}$ is *representable* if there is a topological space X and a map $\mathcal{X} \to X$ which is an equivalence.

Note that, if such a space X exists, it is unique. Namely, given a moduli problem $\mathcal{X}$, let $\mathcal{X}^{\mathrm{iso}}$ be the moduli problem defined by $\mathcal{X}^{\mathrm{iso}}(S) := \pi_0\mathcal{X}(S)$, the set of isomorphism classes of objects of $\mathcal{X}(S)$. It is clear that if $\mathcal{X}$ is representable, the map $\mathcal{X} \to \mathcal{X}^{\mathrm{iso}}$ is an equivalence. In this situation $\mathcal{X}^{\mathrm{iso}} \simeq \mathcal{X}^{\mathrm{coarse}}$.

All the spaces of operations of the operads we use or construct in this paper, and in particular the spaces involved in the sequence of homotopy equivalences (9), i.e.,

$$(\widetilde{\mathrm{Fr}}_\partial)_n, (\widetilde{\mathrm{NodFr}_\partial})_n, \mathrm{DM}_n^{\mathrm{tree}}, (\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}})_n,$$

either naturally have an interpretation as topological moduli problems, or are equivalent to such. For example, $(\mathrm{Fr}_\partial)_n$ is equivalent to the moduli problem $(\mathbb{M}\mathrm{Fr}_\partial)_n$, where $(\mathbb{M}\mathrm{Fr}_\partial)_n(S)$ is the category whose objects are all bundles of surfaces $X \to S$ with fiberwise complex structure and boundary parametrization which varies continuously (as explained in Remark 3.3), and whose morphisms are isomorphisms of such data. The resulting moduli problem is representable by a topological space, as we have seen; this is also true for the spaces of operations of $\widetilde{\mathrm{Fr}}_\partial$, $\widetilde{\mathrm{Ann}}$, $\widetilde{\mathrm{NodAnn}}$, and $\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}}$, though not for $\mathrm{DM}^{\mathrm{tree}}_n$, $\mathrm{NodFr}^{\mathrm{tree}}_\partial$, or $\mathrm{NodHD}^{\mathrm{tree}}$, which are nonrepresentable.

We will add the letter "$\mathbb{M}$" as in $\mathbb{M}\widetilde{\mathrm{Fr}}_\partial$, etc., to denote the topological moduli problem represented by the corresponding topological spaces. When working with a result that is invariant under equivalence of (nonisomorphic) topological moduli problems, we will sometimes drop the $\mathbb{M}$ (as $\mathbb{M}\mathrm{Fr}_\partial$ is equivalent to Fr_∂, etc.).

The moduli space $\mathrm{DM}^{\mathrm{tree}}_n$ is described as a TMP as follows: given a topological space S, the groupoid $\mathrm{DM}^{\mathrm{tree}}_n(S)$ is the category whose objects are all families $X \to S$ of stable nodal surfaces whose dual graph is a tree, with fiberwise complex structure and marked points which vary continuously, and whose morphisms are isomorphisms of such data. More explicitly, this can be described by viewing $\overline{\mathcal{M}}_{g,n}$ as an algebro-geometric stack which classifies stable nodal analytic curves of genus g with n marked points, and then further applying the analytification functor from algebraic varieties to topological spaces. See, for example, [28].

Denote by $\mathrm{NodFr}^{\mathrm{tree}}_{\partial,n}$ the moduli space of stable nodal framed surfaces whose dual graph is a tree and which have n boundary components. This is described as a TMP as follows: given a topological space S, the groupoid $\mathrm{NodFr}^{\mathrm{tree}}_{\partial,n}(S)$ is the category whose objects are all families $X \to S$ of stable nodal framed surfaces whose dual graph is a tree and which have n boundary components, with continuously varying fiberwise complex structure, marked points and parametrizations, and whose morphisms are isomorphisms of such data. We prescribe the local structure near the nodes by associating canonically to a stable nodal framed Riemann surface Σ with n analytically parametrized boundary components a closed stable nodal Riemann surface $\mathbf{cap}_2(\Sigma)$ with $2n$ marked points and requiring that the corresponding S-family is continuous in the previous sense. The association $\Sigma \mapsto \mathbf{cap}_2(\Sigma)$ is defined as follows: we glue a standard disc D_i to the i-th boundary component of Σ and we place one marked point x_i at the center of the disc and one other marked point y_i at $1 \in S^1 = \partial D_i$ for $i = 1, \dots, n$. The notation $\mathbf{cap}_2$ is motivated by the fact that we cap and add 2 marked points for each boundary component of Σ. The continuity of the parametrization of the boundary components is to be understood in light of the fact that, on the one hand, the boundary does not contain nodes and, on the other hand, any S-family is locally trivial away from the nodes.

The correspondence $\Sigma \mapsto \mathbf{cap}_2(\Sigma)$ defines a morphism of TMPs

$$\mathbf{cap}_2 : \mathrm{NodFr}^{\mathrm{tree}}_{\partial,n} \to \mathrm{DM}^{\mathrm{tree}}_{2n}.$$

We actually get an embedding of groupoids $\mathbf{cap}_2(S) : \mathrm{NodFr}^{\mathrm{tree}}_{\partial,n}(S) \hookrightarrow \mathrm{DM}^{\mathrm{tree}}_{2n}(S)$ for any S. It follows from the construction that this embedding has the following property: if y_i is a marked point such that, upon forgetting it, the surface becomes unstable, then y_i lives on a sphere component which contains exactly two other special points, of which one is a node and the other is a marked point denoted by x_i. Equivalently, this situation corresponds to one of the irreducible components of a framed curve being a disc.

We do not make explicit the description of $\mathrm{NodHD}^{\mathrm{tree}}$ as a TMP because it will not be needed. It can be inferred from the above.

A.2 Reminders about groupoids and simplicial sets

We give some basic definitions and results on groupoids and their connection to homotopy theory. References can be found in [23]. See also [57].

Definition A.8 (1) A groupoid is *discrete* if there are no morphisms except identity morphisms. The category of discrete groupoids is equivalent to that of sets.

(2) Given a groupoid G, its underlying discrete groupoid G_{discr} is the discrete groupoid on the set of isomorphism classes of objects.

(3) A groupoid is *quasidiscrete* if there are no automorphisms except identity morphisms. Equivalently, a groupoid G is quasidiscrete if the map $G \to G_{\mathrm{discr}}$ is an equivalence of categories.

Given a map of groupoids $G \to H$ and an object in H, we have a notion of *naive fiber* and a notion of *homotopy fiber* (directly analogous to that of fiber, respectively homotopy fiber in topology). More precisely:

Definition A.9 Let

$$\pi : G \to H$$

be a map of groupoids and let y be an object in H.

(1) The (*naive*) *fiber* of π over y is the groupoid

$$G_y$$

with objects $\{x \in G \mid \pi(x) = y\}$ and morphisms $\{f : x \to x' \mid \pi(f) = \mathrm{Id}_y\}$.

(2) The *homotopy fiber* of π over y is the groupoid $\widetilde{G}_y$ with objects pairs $\{(x, g) \mid g : \pi(x) \to y\}$. Morphisms $(x, g) \to (x', g')$ are maps $f : x \to x'$ that make the following diagram commutative:

$$\begin{tikzcd}
\pi(x) \arrow[r, "\pi(f)"] \arrow[d, "g"] & \pi(x') \arrow[dl, "g'"] \\
y &
\end{tikzcd}$$

Definition A.10 We say that a functor $G \to H$ is a *fibration* (resp. *trivial fibration*) of groupoids if the canonical inclusion functor $G_y \to \widetilde{G}_y$ is an equivalence for every y (resp. $G \to H$ is a fibration and all fibers are equivalent to the trivial groupoid $*$).

The property of being a fibration is equivalent to the following property.

Definition A.11 A functor $\pi : G \to H$ is *cartesian* if, for every two objects $x \in G$, $y \in H$ and every map $g : \pi(x) \to y$, there exists an object $x' \in G$ with $\pi(x') = y$ and a morphism $f : x \to x'$ lifting g.

A functor $\pi : G \to H$ is *trivial cartesian* if it is cartesian and an equivalence of categories.

The following result is straightforward.

Proposition A.12 *A map of groupoids $G \to H$ is cartesian* (*resp. trivial cartesian*) *if and only if it is a fibration* (*resp. trivial fibration*). □

We will deal in the sequel with fibrations with quasidiscrete fibers.

Definition A.13 A fibration of groupoids $G \to H$ is *discrete* (resp. *quasidiscrete*) if every fiber G_y is a discrete (resp. quasidiscrete) groupoid.

We will need the following proposition.

Proposition A.14 *Every quasidiscrete fibration of groupoids $G \to H$ canonically factors through a discrete fibration $G \to G' \to H$ such that $G \to G'$ is an equivalence.*

Proof We have a functor $F : H \to \mathrm{Set}$ given by $y \mapsto (G_y)_{\mathrm{discr}}$, the isomorphism classes of objects over y. Let G' be the Grothendieck construction applied to this functor: this is the category with objects $(y, \bar{x})$ for $\bar{x}$ an isomorphism class in G_y and morphisms $\bar{x} \to \bar{x}'$ corresponding to pairs $(\bar{x}, g)$ for $\bar{x}$ over y and $g : y \to y'$ such that $F(g)(\bar{x}) = \bar{x}'$. It is a direct check that the resulting functor $G' \to H$ is a fibration and that, for $G \to H$ quasidiscrete, the functor $G \to G'$ (given by $x \mapsto (\pi(x), \bar{x})$) is an equivalence. □

Remark A.15 There is a model category structure on groupoids for which fibrations are cartesian functors and in particular any arrow can be cartesian resolved — for example, by replacing G by the union $\bigsqcup_{y \in Y} \widetilde{G}_y$. We will however not use a model category structure at this level — rather, we will view groupoids as homotopy objects by taking the simplicial nerve (a.k.a., the classifying space).

We now discuss groupoids in a simplicial context. The simplicial category Δ is the category of nonempty finite totally ordered sets $[n] = \{0, \dots, n\}$. A *simplicial set* is a presheaf on Δ, i.e., a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. Given a simplicial set $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$, we write $X_n = X[n]$. More generally, a *simplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. Recall from Section 4.3.1 that simplicial sets form a model category [49, II.3, Theorem 3], which is Quillen equivalent to the Quillen model category on topological spaces (with equivalences given by weak homotopy equivalences). The equivalence is given by the pair of functors

$$|\cdot| : \mathrm{SSet} \leftrightarrows \mathrm{Top} : C_\Delta,$$

with $|\cdot|$ the topological realization functor and C_Δ the singular functor given by $C_\Delta(X)_n := \mathrm{Hom}(\Delta^n_{\mathrm{top}}, X)$, for X any topological space and Δ^n_{top} the topological n-simplex. The simplicial morphisms between the $C_\Delta(X)$ arise from the fact that the Δ^n_{top}, $n \geq 0$, form a *cosimplicial* object in Top, i.e., we have a functor $\Delta^*_{\mathrm{top}} : \Delta \to \mathrm{Top}$.

Groupoids can also be interpreted in the context of homotopy theory, via the *nerve* construction.

Definition A.16 The *nerve* of a groupoid G is the simplicial set

$$NG : [n] \mapsto \mathrm{Hom}_{\mathrm{Cat}}([n], G).$$

More concretely, NG_n is the set of composable sequences of morphisms $x_0 \to x_1 \to \cdots \to x_n$. Note that, if G is a group, the topological realization $|NG|$ of the nerve of G is a classifying space for G.

We will need the following combinatorial result.

Lemma A.17 *Suppose $G \to H$ is a discrete fibration, and let $NG \to NH$ be the associated map of nerves. A simplex $(x_0 \to \dots \to x_n) \in NG_n$ in the preimage of a simplex $\sigma = (y_0 \to \dots \to y_n) \in NH_n$ is uniquely determined by x_0.*

Proof This follows from the fact that a map $f : x \to x'$ in a discrete fibration is uniquely determined by x and a morphism $\pi(f)$ from $\pi(x)$ to $\pi(x')$. □

Simplicial sets also have a notion of homotopy fiber, which we call *simplicial preimage* and which we now explain. Write Δ^n_{simp} for the n-simplex, i.e., the simplicial set represented by $[n]$. Given a simplicial set X, an element $\sigma \in X_n$ determines canonically a map of simplicial sets $\sigma_{\mathrm{simp}} : \Delta^n_{\mathrm{simp}} \to X$.

Definition A.18 Let $f : Y \to X$ be a map of simplicial sets. The *simplicial preimage of an element $\sigma \in A_n$ under f* is the homotopy fiber product

$$Y_\sigma = \Delta^n_{\mathrm{simp}} \times^h_X Y$$

with respect to the simplicial map $\sigma_{\mathrm{simp}} : \Delta^n_{\mathrm{simp}} \to X$ determined by σ.

The homotopy fiber product is the homotopy limit of the diagram $\Delta^n_{\mathrm{simp}} \xrightarrow{\sigma_{\mathrm{simp}}} X \xrightarrow{f} Y$ for the Quillen model category structure on SSet.

We will need the following standard result from homotopy theory, usually cited as a special case of Quillen's Theorem A from [50, §1].

Proposition A.19 *A map of simplicial sets $f : Y \to X$ is a weak equivalence if and only if, for any integer $n \geq 0$ and any simplex $\sigma \in X_n$, the simplicial preimage Y_σ is weakly contractible.*

Proof The Quillen model structure on SSet is *right proper* in the sense of Section 4.5. Given a right proper model category, the homotopy limit of a diagram $Z \to X \leftarrow Y$ can be computed by replacing any of the two maps by a fibration and taking the ordinary limit of the resulting diagram [32, Proposition 13.3.7] (compare with Lemma 4.13). As a consequence, we can assume without loss of generality that the map f in the statement is a fibration for the Quillen model structure on SSet, i.e., a Kan fibration of simplicial sets. In this case the simplicial preimages are computed as ordinary pullbacks.

We now prove that a Kan fibration is a weak equivalence (also called *trivial Kan fibration*) if and only if all its simplicial preimages are contractible. This is stated in [49, II.2]. The proof relies on the equivalent characterization of trivial Kan fibrations as maps of simplicial sets that have the right lifting property with respect to all the inclusions $\partial\Delta^k \hookrightarrow \Delta^k$, $k \geq 0$ [27, Theorem 11.2; 49, II.3, Theorem 3]. We prove the direct implication: given a trivial Kan fibration $f : Y \to X$, its pullback along any map of simplicial sets is also a trivial Kan fibration because it has the right lifting property with respect to all inclusions $\partial\Delta^k \hookrightarrow \Delta^k$. In particular, any simplicial preimage of f is a trivial Kan fibration over a simplex, hence weakly contractible because the simplex is weakly contractible. We prove the converse implication: let $f : Y \to X$ be a Kan fibration and assume that all its simplicial preimages Y_σ are weakly contractible. This is equivalent to saying that the canonical map $Y_\sigma \to \Delta^n$ is a trivial Kan fibration for all $\sigma \in X_n$, $n \geq 0$. We wish to construct the dotted arrow in

$$\begin{tikzcd}
\partial\Delta^k \arrow[r] \arrow[d, hook] & Y \arrow[d, two heads] \\
\Delta^k \arrow[r, "\sigma"'] \arrow[ur, dotted] & X
\end{tikzcd}$$

Enhancing it to

$$\begin{tikzcd}
\partial\Delta^k \arrow[dr] \arrow[drrr] \arrow[dd] & & & \\
& Y_\sigma = \sigma^* Y \arrow[rr] \arrow[dd, hook] & & Y \arrow[dd, two heads, "f"] \\
\Delta^k \arrow[ur, dotted] \arrow[dr, equal] & & & \\
& \Delta^k \arrow[rr, "\sigma"'] & & X
\end{tikzcd}$$

the right lifting property for the map $Y_\sigma \to \Delta^k$ with respect to the inclusion $\partial\Delta^k \hookrightarrow \Delta^k$ determines a lift $\Delta^k \to Y_\sigma$. This lift can be postcomposed with the map $Y_\sigma \to Y$ in order to obtain the desired lift $\Delta^k \to Y$. □

In the next section we will also use the category of bisimplicial sets, defined as follows.

Definition A.20 The category of bisimplicial sets

$$\mathrm{S}^2\mathrm{Set} := \mathrm{Fun}((\Delta^{\mathrm{op}})^2, \mathrm{Set})$$

is the category of presheaves on the *bisimplicial category* $(\Delta^{\mathrm{op}})^2 = \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$.

Given a bisimplicial set X, we write $X_{m,n} := X([m] \times [n])$. Note that, by adjunction, $\mathrm{S}^2\mathrm{Set}$ is equivalent to the category

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}))$$

of simplicial objects in the category of simplicial sets. Here, if $X : \Delta^{\mathrm{op}} \to \mathrm{SSet}$ is a simplicial object in simplicial sets, the associated bisimplicial set has $X^{\mathrm{Bi}}_{m,n} := (X_m)_n$, where X_m is the simplicial set $X([m])$.

Given a bisimplicial set $X_{*,*} : (\Delta^{\mathrm{op}})^2 \to \mathrm{Set}$, we have an associated "total" simplicial set defined as follows.

Definition A.21 Define

$$X^{\mathrm{tot}} := X \circ \mathrm{Diag}_{\Delta^{\mathrm{op}}} : \Delta^{\mathrm{op}} \to \mathrm{Set},$$

where the functor $\mathrm{Diag}_{\Delta^{\mathrm{op}}} : \Delta^{\mathrm{op}} \to (\Delta^{\mathrm{op}})^2$ is the diagonal functor.

There are several different Quillen equivalent model structures on the category $\mathrm{S}^2\mathrm{Set}$ of bisimplicial sets, and all of them are Quillen equivalent to the category of simplicial sets via the totalization functor (see [27, Chapter IV]). In particular, the simplicial set X^{tot} contains the same "topological" information as the bisimplicial set X.

A.3 Simplicial realization of a topological moduli problem

We will not attempt to construct a model category (or even a homotopy category) of topological moduli problems.[3] Instead, we will associate to every topological moduli problem $\mathcal{X}$ a simplicial set[4] $C_\Delta(\mathcal{X})$ of "simplicial chains" and view $\mathcal{X}$ as represented by $C_\Delta(\mathcal{X})$ in a homotopy-theoretic sense.

We have a standard covariant *realization functor*

$$R : \Delta \to \mathrm{Top},$$

taking $[n]$ to the n-simplex Δ^n. We define the simplicial groupoid $C_\Delta^{\mathrm{Gpd}}(\mathcal{X})$ to be the strict functor

$$C_\Delta^{\mathrm{Gpd}}(\mathcal{X}) = \mathcal{X} \circ R^{\mathrm{op}} : \Delta^{\mathrm{op}} \to \mathrm{Gpd}.$$

Note that, for $\mathcal{X} = X$ the moduli problem represented by a topological space, $C_\Delta^{\mathrm{Gpd}}(X)$ is canonically equivalent to the levelwise discrete simplicial groupoid associated to the singular simplicial set of X, denoted by $C_\Delta(X)$.

Composing with the nerve functor (Definition A.16) $N : \mathrm{Gpd} \to \mathrm{SSet}$ from groupoids to simplicial sets we obtain a functor

$$N \circ C_\Delta^{\mathrm{Gpd}}(\mathcal{X}) : \Delta^{\mathrm{op}} \to \mathrm{SSet}.$$

By definition, such a functor is equivalent to a bisimplicial set. We call it *the bisimplicial realization* associated to the topological moduli problem $\mathcal{X}$, and denote it by

$$C_\Delta^{\mathrm{Bi}}(\mathcal{X}) : (\Delta^{\mathrm{op}})^2 \to \mathrm{Set}.$$

Definition A.22 For a topological moduli problem $\mathcal{X}$, we denote by

$$C_\Delta(\mathcal{X})$$

the simplicial set given by the totalization of the bisimplicial set $C_\Delta^{\mathrm{Bi}}(\mathcal{X})$. We call $C_\Delta(\mathcal{X})$ the *classifying simplicial set* of $\mathcal{X}$.

The assignment $\mathcal{X} \to C_\Delta(\mathcal{X})$ is natural and defines a covariant functor from the category of topological moduli problems to simplicial sets. To illustrate the definition we give two examples, which are the extreme cases of topological moduli problems:

[3]Though note that topological moduli problems are a full subcategory of the category of simplicial presheaves on topological spaces, which has a structure of model category Quillen equivalent to that of simplicial sets, and it would be possible to work within this framework.

[4]This is actually an ∞-groupoid.

- If $\mathcal{X} = X$ is the moduli problem represented by a topological space, the bisimplicial set $C^{\mathrm{Bi}}_{\Delta}(\mathcal{X})$ canonically decouples as a product of two functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$, namely $C_{\Delta}(X) \times \mathrm{pt}$. Thus the associated simplicial set $C_{\Delta}(\mathcal{X})$ is canonically isomorphic to the usual simplicial set of singular simplices $C_{\Delta}(X)$, further justifying our frequent abuse of notation of using the same term for the space X and the topological moduli problem $\mathcal{X}$ which it represents.
- If $\mathcal{X} = \mathrm{pt}/G$ is a point with a discrete group of automorphisms G, the bisimplicial set $C^{\mathrm{Bi}}_{\Delta}(\mathcal{X})$ canonically decouples as a product $\mathrm{pt} \times NG$, where N is the nerve functor, i.e., the simplicial classifying space functor.

Thus, the totalization $C_{\Delta}(\mathcal{X})$ combines the topological and the groupoid features of the topological moduli problem $\mathcal{X}$.

The *classifying space functor* $\mathcal{X} \mapsto |C_{\Delta}(\mathcal{X})|$ allows us to view any topological moduli problem as functorially represented by a topological space. However, as this topological space is big and difficult to work with, we avoid working with it directly. Instead we make comparisons in the category of topological moduli problems, and identify certain properties of maps of topological moduli problems which guarantee that the underlying maps of classifying spaces are homotopy equivalences.

A.4 Trivial Serre fibrations of topological moduli problems

To make sense of results such as Lemma 6.5 in higher genus, we need a notion of fiber for certain maps of topological moduli problems.

Definition A.23 A map $\mathcal{X} \to \mathcal{Y}$ of TMP is a *combinatorial fibration* if $\mathcal{X}(S) \to \mathcal{Y}(S)$ is a fibration of groupoids for each S.

Informally, if $\pi : \mathcal{X} \to \mathcal{Y}$ is a map of TMP, if $\mathcal{X}(S)$ and $\mathcal{Y}(S)$ are defined as some geometric data on S up to isomorphism, and if $\pi(S)$ is a "forgetful" map which discards some part of the geometric data, then π is a combinatorial fibration. Indeed, given an isomorphism $i : \alpha \to \beta$ between two structures $\alpha, \beta \in \mathcal{Y}(S)$, and given an object $\tilde{\beta}$ of $\mathcal{X}(S)$ over β which consists of some additional geometric data, we can pull back this geometric data under i to produce a diagram verifying the fibration property. Using this idea we can prove the following proposition.

Proposition A.24 *For any n, the maps*

$$\mathbf{cap} : \mathrm{NodFr}^{\mathrm{tree}}_{\partial,n} \to \mathrm{DM}^{\mathrm{tree}}_{n} \quad \textit{and} \quad \boldsymbol{\pi} : \mathbb{M}\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected},n} \to \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,n}$$

are combinatorial fibrations.

Proof We first discuss the map **cap**. This is the composition of the maps

$$\mathrm{NodFr}^{\mathrm{tree}}_{\partial,n} \xrightarrow{\mathbf{cap}_2} \mathrm{DM}^{\mathrm{tree}}_{2n} \xrightarrow{\mathbf{forget}_n} \mathrm{DM}^{\mathrm{tree}}_{n},$$

where $\mathbf{cap}_2$ was defined in Section A.1 and $\mathbf{forget}_n$ is the map which forgets the marked points $y_1, \dots, y_n$ (the marked points for $\mathrm{DM}^{\mathrm{tree}}_{2n}$ are denoted by $x_1, \dots, x_n, y_1, \dots, y_n$; see Section A.1). Note that the image of $\mathbf{cap_2}$ is contained in $\mathrm{DM}_{2n}^{\mathrm{tree},\prime}$, which is the TMP such that, for each S, the groupoid $\mathrm{DM}_{2n}^{\mathrm{tree},\prime}(S)$ is the full subgroupoid of $\mathrm{DM}_{2n}^{\mathrm{tree}}(S)$ consisting of families of curves with the property that, if y_i is a marked point such that, upon forgetting it, the underlying surface becomes unstable, then y_i lives on a sphere component which contains exactly two other special points, of which one is a node and the other is a marked point denoted by x_i. We claim that

$$\mathrm{NodFr}_{\partial,n}^{\mathrm{tree}} \xrightarrow{\mathbf{cap_2}} \mathrm{DM}_{2n}^{\mathrm{tree},\prime} \xrightarrow{\mathbf{forget}_n} \mathrm{DM}_n^{\mathrm{tree}}$$

is a composition of combinatorial fibrations, hence is a combinatorial fibration. In view of Proposition A.12, it is enough to show that, for any S, the maps

$$\mathrm{NodFr}_{\partial,n}^{\mathrm{tree}}(S) \xrightarrow{\mathbf{cap_2}(S)} \mathrm{DM}_{2n}^{\mathrm{tree},\prime}(S) \xrightarrow{\mathbf{forget}_n(S)} \mathrm{DM}_n^{\mathrm{tree}}(S)$$

are cartesian (Definition A.11).

- We prove that $\mathbf{cap_2}(S)$ is cartesian. Given two isomorphic S-families in $\mathrm{DM}_{2n}^{\mathrm{tree},\prime}(S)$ with a preferred isomorphism g and a lift to $\mathrm{NodFr}_{\partial,n}^{\mathrm{tree}}(S)$ of the first family, we can reinterpret that lift as the data of a continuously varying family of analytically parametrized simple curves in the fibers. The isomorphism g can then be used in order to produce such a continuously varying family of analytically parametrized simple curves in the fibers of the second family. This provides a lift of the second family, as well as a lift of g.

- We prove that $\mathbf{forget}_n(S)$ is cartesian. Consider two isomorphic S-families $E, F \in \mathrm{DM}_n^{\mathrm{tree}}(S)$, a preferred isomorphism $g : E \xrightarrow{\simeq} F$, and a lift of E to $\mathrm{DM}_{2n}^{\mathrm{tree},\prime}(S)$ denoted by $\tilde{E}$. Let $x_{1,s}, \dots, x_{n,s}$ be the marked points of the curve E_s, $s \in S$, and $\tilde{x}_{1,s}, \dots, \tilde{x}_{n,s}, y_{1,s}, \dots, y_{n,s}$ the marked points of its lift $\tilde{E}_s$. At each point $s \in S$ and for each $i = 1, \dots, n$ one of the following two things can happen: either the removal of the point $y_{i,s}$ does not destabilize the irreducible component of $\tilde{E}_s$ on which it lies, or it destabilizes it, in which case that component is a sphere which is contracted under the forgetful map, the marked point $\tilde{x}_{i,s}$ becoming $x_{i,s}$. Equivalently, the lift $\tilde{E}_s$ differs from E_s by either adding some nondestabilizing marked points $y_{i,s}$, or by blowing up some marked points $x_{i,s}$ into spheres which acquire marked points $\tilde{x}_{i,s}$ and $y_{i,s}$. We can then define a lift $\tilde{F}$ of F as follows: we mark additional points $y'_{i,s} = g(y_{i,s})$ for all indices i such that $y_{i,s}$ is not destablizing, and we blow up the points $x'_{i,s} = g(x_{i,s})$ for all indices i such that $y_{i,s}$ is destabilizing, marking two new points $\tilde{x}'_{i,s}$ and $y'_{i,s}$ on the resulting sphere bubble. Since any two spheres with three marked points are uniquely isomorphic, the isomorphism g extends uniquely to an isomorphism $\tilde{g} : \tilde{E} \xrightarrow{\simeq} \tilde{F}$.

This proves that the map $\mathbf{cap}$ is a combinatorial fibration. The proof that the map $\boldsymbol{\pi}$ is a combinatorial fibration is very similar to the proof for $\mathbf{cap_2}$, and we omit the details. □

Definition A.25 (fibers of topological moduli problems) Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a map of TMPs.

(1) Given a topological space I and an object $\gamma \in \mathcal{Y}(I)$ (to be thought of as a map $I \to \mathcal{Y}$), *the fiber* $\mathcal{X}_\gamma$ is the topological moduli problem with $\mathcal{X}_\gamma(S) := \bigsqcup_{t:S\to I} \mathcal{X}(S)_{t^*\gamma}$. Here $t^*\gamma$ is the pullback in $\mathcal{Y}(S)$ (to be thought of as the composition $\gamma \circ t$) and $\mathcal{X}(S)_{t^*\gamma}$ is the fiber for the map of groupoids $\mathcal{X}(S) \to \mathcal{Y}(S)$.

The fiber $\mathcal{X}_\gamma$ is endowed with a canonical map $\mathcal{X}_\gamma \to I$.

(2) We say that the map π is a *fiberwise representable fibration* if it is a combinatorial fibration and, for any topological space I and object $\gamma \in \mathcal{Y}(I)$, the fiber $\mathcal{X}_\gamma$ is quasidiscrete (i.e., equivalent to a topological space). We say that π is *strictly fiberwise representable* if any such fiber $\mathcal{X}_\gamma$ is discrete.

It follows from Proposition A.14 that any fiberwise representable fibration is equivalent to one which is strictly fiberwise representable.

Definition A.26 Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a map of TMP which is a fiberwise representable fibration. We say that π is a *fiberwise representable trivial weak fibration* if, for any $m \geq 0$ and any simplex $\sigma \in \mathcal{Y}(\Delta^m)$, the fiber $\mathcal{X}_\sigma$ is weakly contractible.

Lemma A.27 *Suppose that $\pi : \mathcal{E} \to \mathcal{X}$ is a fiberwise representable trivial weak fibration. Then the induced map of simplicial sets $C_\Delta(\mathcal{E}) \to C_\Delta(\mathcal{X})$ is a weak homotopy equivalence.*

Proof As noted above, we can assume without loss of generality that $\mathcal{E} \to \mathcal{X}$ is strictly fiberwise representable. In particular, the fibers of $\mathcal{E}(S) \to \mathcal{X}(S)$ are discrete for any S. Now we consider the map of bisimplicial sets $C_\Delta^{\mathrm{Bi}}(\mathcal{E})_{m,n} \to C_\Delta^{\mathrm{Bi}}(\mathcal{X})_{m,n}$, where m is the simplex degree (corresponding to the category $\mathcal{X}(\Delta^m_{\mathrm{top}})$ of maps from the topological simplex) and n is the nerve degree. It is enough to show that, at a fixed nerve degree n, the map of simplicial sets $C_\Delta^{\mathrm{Bi}}(\mathcal{E})_{*,n} \to C_\Delta^{\mathrm{Bi}}(\mathcal{X})_{*,n}$ is a weak homotopy equivalence. We do this in two steps.

Case $n = 0$ Since we set the nerve degree to zero, we are considering the simplicial set on *objects* of $\mathcal{X}(\Delta^m_{\mathrm{top}})$ (and similarly for $\mathcal{E}$). We use the following standard fact.

Proposition A.28 *Let $f : U \to V$ be a map of topological spaces. The following statements are equivalent*:

- *f is a trivial Serre fibration, i.e., a Serre fibration with weakly contractible fibers.*
- *The map of simplicial sets $C_\Delta(f) : C_\Delta(U) \to C_\Delta(V)$ is a trivial Kan fibration in the sense of Section 4.3.1.*
- *The map of simplicial sets $C_\Delta(f) : C_\Delta(U) \to C_\Delta(V)$ is a Kan fibration that has weakly contractible simplicial preimage $C_\Delta(U)_\sigma$ over any m-simplex $\sigma \in C_\Delta(V)$.*

Proof of Proposition A.28 That f is a trivial Serre fibration if and only if $C_\Delta(f)$ is a trivial Kan fibration is proved in [49, II.3, Lemma 2]; see also [30, Example 5.15].[5] On the other hand, we proved in Proposition A.19 that a Kan fibration is trivial if and only if all its simplicial preimages are weakly contractible. □

Proof of Lemma A.27 continued Given an object $\sigma_{\mathrm{top}} \in \mathcal{X}(\Delta^m_{\mathrm{top}})$, we apply Proposition A.28 to the pair of topological spaces $V = \Delta^m_{\mathrm{top}}$ and $U = \mathcal{E}_{\sigma_{\mathrm{top}}}$, the fiber of $\mathcal{E}$ over V. We are using that, by strict fiberwise representability, $\mathcal{E}_{\sigma_{\mathrm{top}}}$ is isomorphic to the (discrete) topological moduli problem represented by a space. We deduce that all simplicial preimages of simplices in $C_\Delta^{\mathrm{Bi}}(\mathcal{X})_{*,0}$ under the map $C_\Delta^{\mathrm{Bi}}(\mathcal{E})_{*,0} \to C_\Delta^{\mathrm{Bi}}(\mathcal{X})_{*,0}$ are weakly contractible, hence we get a weak homotopy equivalence of simplicial sets at $n = 0$.

[5] It is also true that f is a Serre fibration if and only if $C_\Delta(f)$ is a Serre fibration [30, Example 5.11].

Case $n \geq 0$ From Lemma A.17 we see that all simplicial preimages of the map $C_\Delta^{\mathrm{Bi}}(\mathcal{E})_{*,n} \to C_\Delta^{\mathrm{Bi}}(\mathcal{X})_{*,n}$ are isomorphic to simplicial preimages of the corresponding map at $n = 0$, i.e., $C_\Delta^{\mathrm{Bi}}(\mathcal{E})_{*,0} \to C_\Delta^{\mathrm{Bi}}(\mathcal{X})_{*,0}$. We conclude by the $n = 0$ case. □

Appendix B The dendroidal category and Segal operads

In Proposition 6.9 we prove that, for any number n of inputs, there is a map

$$\boldsymbol{\pi} : \mathbb{M}\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected},n} \to \widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial,n}$$

of topological moduli problems which induces a homotopy equivalence on underlying simplicial sets. Now the left-hand side is a model (in topological spaces) for the homotopy pushout of the diagram $\mathrm{pt} \leftarrow S^1 \to \mathrm{Fr}_\partial$, and we also know from Lemma 6.8 that the right-hand side is equivalent to $\mathrm{DM}^{\mathrm{tree}}_n$. In this appendix we explain how to "upgrade" such homotopy equivalences on the level of spaces of operations to homotopy equivalences on the level of operads. For this we need to discuss operads valued in topological moduli problems. A convenient language in this context is the formalism of dendroidal objects and Segal operads (Cisinski–Moerdijk [12]), which is a relative of Lurie's theory of ∞-operads [41]. We explain this formalism, how the operad objects and maps we have been working with, e.g., the operad $\mathrm{DM}^{\mathrm{tree}}$ and the map $\mathrm{DM}^{\mathrm{tree}} \to \mathrm{NodFr}^{\mathrm{tree}}_\partial$, fit into it, and how this formalism is compared to that of topological operads.

The reason for using Segal operads is that the classical language of operads is not well adapted to deal with spaces of operations that are topological moduli problems. Indeed, if we tried to impose an operad structure, then the associativity relations would need to be equalities of functors, whereas in our situation we only have isomorphisms. Said differently, we need to use the 2-category structure on groupoids to define operads such as $\mathrm{DM}^{\mathrm{tree}}$. (A useful analogy is symmetric monoidal groupoids vs. abelian monoids.)

B.1 The Moerdijk–Weiss tree category

Here we describe the dendroidal category Ω of Moerdijk and Weiss [47], which plays in higher operad theory a role analogous to that played in homotopy theory by the simplicial category Δ. In the same way that objects of Δ can be understood as combinatorial categories (finite ordered posets), the objects of Ω are combinatorial operads associated to trees.

Definition B.1 (the free colored operad on a tree) Let τ be a tree of operations (i.e., a finite tree with half-edges, see Section 2.2.1). The free colored operad $[\tau]$ on τ is the operad with colors $C = \mathrm{Edge}_\tau$ and operations generated by a set $\{o_v\}$ indexed by vertices $v \in \mathrm{Vert}_\tau$, where o_v has inputs $e_1^{\mathrm{in}}(v), \dots, e_{|v|}^{\mathrm{in}}(v)$, the incoming edges at v with some choice of ordering, and output $e^{\mathrm{out}}(v)$, the outgoing edge at v.

In this definition the generators are described in terms of a choice of ordering of the incoming edges at each vertex v, or equivalently in terms of a choice of planar structure for the tree. A different choice of planar structure would give different generators, but the same operad.

When τ is the linear tree $\tau_{[n]} := \to \bullet \to \cdots \to \bullet \to$ with n vertices, the colored operad only has one-to-one operations, i.e., is a category. In fact, it is the category associated to the quiver corresponding to the dual graph: it has $n+1$ objects associated to the linear tree and a unique arrow $i \to j$ for $i \le j$. In other words, it is precisely the linear poset category $[n]$.

For a general tree τ, one can check that operations in $[\tau]$ are indexed by full labeled subtrees. Namely, we say that a subgraph $\tau' \subset \tau$ is *full* if every edge or half-edge in τ with an endpoint in τ' is also an edge or a half-edge in τ' (note that a half-edge in τ' might have another endpoint in τ). The operad $[\tau]$ has a (unique) operation with inputs $e_1, \dots, e_n$ and output e^{out} if and only if there is a full labeled subtree τ' with leaves $e_1, \dots, e_n$ and root e^{out}.

Definition B.2 The *dendroidal category* Ω is the full subcategory of the category of colored operads on the objects $[\tau]$ as τ ranges over all trees.

The next definition describes objects which generate Ω in a sense that we will make precise.

Definition B.3 • Define $|$ to be the tree with one edge and no vertices. The operad $[\,|\,] \in \Omega$ is the operad with a single identity operation.

• Define the *n-corolla* κ_n to be the tree of operations with one vertex, n input edges, and one output edge. The operad $[\kappa_n]$ is the operad whose spaces of operations are empty except in arity n, where $\mathfrak{S}_n$ acts freely and transitively, i.e., $[\kappa_n]$ has a unique-up-to-relabeling operation in arity n.

Note that for a general colored operad O, maps $[\,|\,] \to O$ are in bijection with colors of O and maps $[\kappa_n] \to O$ correspond to all operations with n inputs (with arbitrary incoming and outgoing colors). In particular, if τ is a tree then $\mathrm{Hom}([\,|\,], [\tau]) = \mathrm{Edge}_\tau$ and $\mathrm{Hom}([\kappa_n], [\tau])$ is the set of full subtrees of τ with n leaves. We note two important special cases.

Definition B.4 • Given a vertex v, define $i_v : [\kappa_{|v|}] \to [\tau]$ to be the map corresponding to the subtree of edges around v (equivalently, the map classifying the operation v in $[\tau]$).

• For any tree τ with n leaves, define the *cocontraction map* $s_\tau : [\kappa_n] \to [\tau]$ to be the map classifying the operation corresponding to the tree τ itself viewed as a subtree (i.e., the composition of all operations in τ).

For any full subtree $\tau' \subset \tau$, let τ'' be the tree obtained by contracting all internal edges of τ' into a single point. Then there are obvious maps $i_{\tau,\tau'} : [\tau'] \to [\tau]$ and $s_{\tau,\tau'} : [\tau''] \to [\tau]$ defined by extending i_v for $v \in \tau'$ and $v \in s_{\tau'}$, respectively, by the identity operation for all vertices in the complement of τ'. Maps of the form $i_{\tau,\tau'}$, $s_{\tau,\tau'}$ together with automorphisms of trees generate the tree category Ω [47, §3].

Let $\mathcal{C}$ be a category with monoidal structure given by products and O a monochromatic operad in $\mathcal{C}$. This induces a contravariant representation of the dendroidal category Ω in $\mathcal{C}$, sending every tree τ to the set $\prod_{v \in \mathrm{Vert}(\tau)} O_{|v|}$ and sending the cocontraction map s_τ to the operad composition operation $\prod_{v \in \mathrm{Vert}(\tau)} O_{|v|} \to O_N$ (for N the number of leaves). In particular, this representation determines the operad O (the identity operation is the "empty" composition operation corresponding to $\tau = |$ and the action of $\sigma \in \mathfrak{S}_n$ on O_n is recovered from the morphisms $\kappa_\sigma : \kappa_n \to \kappa_n$ which permute the incoming edges of a corolla).

B.2 Segal operads and preoperads

In this subsection we follow Cisinski and Moerdijk [12] and work with models for colored operads, despite the fact that the objects we care about are monochromatic.

Suppose $\mathcal{C}$ is a category which is closed under products and coproducts and let $\mathrm{Discr}_{\mathcal{C}}$ be the essential image of the standard functor $\mathrm{Set} \to \mathcal{C}$ given by unions of the terminal object $* \in \mathcal{C}$.

Definition B.5 A *Segal preoperad* Λ in $\mathcal{C}$ is a $\mathcal{C}$-valued presheaf on Ω, i.e., a functor $\Lambda : \Omega^{\mathrm{op}} \to \mathcal{C}$ such that $\Lambda([\,|\,]) \in \mathrm{Discr}_{\mathcal{C}}$.

One can think of Segal preoperads as colored operads with nonuniquely defined compositions; $\Lambda([\,|\,])$ is the set of colors. When Λ is a monochromatic operad, it has a model with $\Lambda([\,|\,]) \cong *$.

If $\mathcal{C}$ is additionally endowed with a class of equivalences including $\mathrm{Iso}_{\mathcal{C}}$ (for example, if $\mathcal{C}$ is a model category), we make the following definition.

Definition B.6 A *Segal operad in* $\mathcal{C}$ is a Segal preoperad in $\mathcal{C}$ such that, for every tree τ, the product of the maps

$$\Lambda(i_v) : \Lambda([\tau]) \to \Lambda([\kappa_{|v|}]) \tag{10}$$

is an equivalence between $\Lambda([\tau])$ and $\prod_{v \in \mathrm{Vert}(\tau)} \Lambda([\kappa_{|v|}])$.

Note that an ordinary (strict) colored operad in $\mathcal{C}$ is precisely a Segal preoperad where this map is an equality (i.e., where $\Lambda([\tau])$ is *defined* as being $\prod_{v \in \mathrm{Vert}(\tau)} \Lambda([\kappa_{|v|}])$). Given an ordinary operad O, we write $\mathrm{Seg}(O)$ for the Segal operad associated to O, with

$$\mathrm{Seg}(O)([\tau]) := \prod_{v \in \mathrm{Vert}(\tau)} O([\kappa_{|v|}]).$$

In fact, Segal (pre)operads have a model category structure which is related to the Berger–Moerdijk model category of topological operads (and also related to Lurie's ∞-operads [41, §2]) by a chain of Quillen equivalences. This is proved by Chu, Haugseng, and Heuts [10], and their paper relies on [5; 11; 12; 29]. Since we will only be interested in the weak equivalences, we will not need the full model structure. Rather, we need the following results from [11; 12].

Theorem B.7 [11; 12] (1) *There is a "strictification" functor*

$$W_{\mathrm{Seg}} : \mathrm{PreSegOp}_{\mathcal{C}} \to \mathrm{Op}_{\mathcal{C}}$$

from Segal preoperads in $\mathcal{C}$ to strict operads in $\mathcal{C}$, which is equivalent to the W-construction when applied to strict operads. In other words, we have a canonical isomorphism of functors

$$W_{\mathrm{Seg}} \circ \mathrm{Seg} \cong W : \mathrm{Op}_{\mathcal{C}} \to \mathrm{Op}_{\mathcal{C}}.$$

(2) *There is a natural transformation $\mathrm{Seg} \circ W_{\mathrm{Seg}} \Rightarrow \mathrm{Id}$ such that, for every Segal preoperad $\Lambda \in \mathrm{PreSegOp}_{\mathcal{C}}$, the natural map $\mathrm{Seg} \circ W_{\mathrm{Seg}}(\Lambda) \to \Lambda$ is an equivalence in $\mathrm{PreSegOp}_{\mathcal{C}}$ (and, in particular, an equivalence on the level of each tree if Λ is a Segal operad).* □

B.3 Segal operads in topological moduli problems

We will be interested in the category of Segal operads in topological moduli problems. More precisely, we can define Segal operads in topological moduli problems fitting into a diagram

$$\Lambda\mathbb{M}\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}} \to \Lambda\widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial} \leftarrow \Lambda\mathrm{DM}^{\mathrm{tree}}.$$

The definition of the Segal operad $\Lambda\mathrm{DM}^{\mathrm{tree}}$ is the following. We need to define a topological moduli problem $\Lambda\mathrm{DM}^{\mathrm{tree}}([\tau])$ for every tree τ. For a test topological space S, we define $\Lambda\mathrm{DM}^{\mathrm{tree}}([\tau])(S)$ to be the collection of families $X \to S$, where X is a family over S of *possibly disconnected* stable nodal curves, mapping to the discrete set $\mathrm{Vert}(\tau)$, with marked points indexed by half-edges of τ (including two half-edges for each full edge) and such that the component X_v over a vertex $v \in \mathrm{Vert}_\tau$ has incoming marked points indexed by incoming edges $\mathrm{In}(v)$, and outgoing marked point indexed by the outgoing edge $\mathrm{Out}(v)$. Maps between two objects of $\Lambda\mathrm{DM}^{\mathrm{tree}}([\tau])(S)$ are isomorphisms over S that preserve all combinatorial data. In order to define the Segal structure we need to write down functors

$$\Lambda\mathrm{DM}^{\mathrm{tree}}(i_{\tau,\tau'})(S) : \Lambda\mathrm{DM}^{\mathrm{tree}}([\tau])(S) \to \Lambda\mathrm{DM}^{\mathrm{tree}}([\tau'])(S)$$

corresponding to restriction to a full subtree $\tau' \subset \tau$, and

$$\Lambda\mathrm{DM}^{\mathrm{tree}}(s_{\tau,\tau'})(S) : \Lambda\mathrm{DM}^{\mathrm{tree}}([\tau])(S) \to \Lambda\mathrm{DM}^{\mathrm{tree}}([\tau''])(S)$$

corresponding to contraction of a full internal subtree $\tau' \subset \tau$. The former is simply given by removing all components of X corresponding to vertices not in τ', and the latter by gluing all curves in the component τ' along their common edges to combine them into a nodal curve, then stabilizing.[6]

Proceeding in a similar fashion, we also get Segal operads

$$\Lambda\mathbb{M}\widetilde{\mathrm{Fr}}_{\partial}, \quad \Lambda\widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial}, \quad \Lambda\mathbb{M}\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}}.$$

Note that, since $\widetilde{\mathrm{Fr}}_{\partial}$ and $\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}}$ are topological spaces, we have equivalences $\Lambda\mathbb{M}\widetilde{\mathrm{Fr}}_{\partial} \simeq \mathrm{Seg}(\widetilde{\mathrm{Fr}}_{\partial})$ and similarly for $\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}}$.

We finally have the tools to give concrete statements comparing the homotopy pushout $\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}}$ and $\mathrm{DM}^{\mathrm{tree}}$.

Theorem B.8 *There is a sequence of Segal operads of topological moduli problems, with maps that are levelwise weak equivalences (on the level of simplicial chains):*

$$\Lambda\mathbb{M}\widetilde{\mathrm{NodHD}}^{\mathrm{tree}}_{\mathrm{protected}} \to \Lambda\widetilde{\mathrm{NodFr}}^{\mathrm{tree}}_{\partial} \leftarrow \Lambda\mathrm{DM}^{\mathrm{tree}}.$$

The first map is induced by applying componentwise the map $\boldsymbol{\pi}$, and the second map is induced by applying componentwise the map **funnel**.

Proof This is the combined conclusion of Lemma 6.8 and Proposition 6.9. □

[6] A note on set-theoretic issues: both the procedures of gluing and stabilizing replace the set indexing the old complex curve X by a quotient set, whose points are subsets of X, something that is possible in the κ-small world for κ an inaccessible cardinal. In order to avoid sets with repeated indexes, we require that the set of points of X must not contain an element which is a subset of X of cardinality ≥ 2.

Combining the previous theorem with Theorem B.7 we obtain the following.

Corollary B.9 *There is a sequence of (weak) homotopy equivalences in the Berger–Moerdijk category of strict topological operads*:

$$\begin{tikzcd} \mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}} & W_{\mathrm{Seg}}|C_\Delta(\mathrm{NodHd}^{\mathrm{tree}}_{\mathrm{protected}})| \arrow[l] \arrow[d] & \\ & W_{\mathrm{Seg}}|C_\Delta(\mathrm{NodHD}^{\mathrm{tree}}_{\mathrm{protected}})| & W_{\mathrm{Seg}}|C_\Delta(\mathrm{DM}^{\mathrm{tree}})| \arrow[l] \end{tikzcd}$$ □

Here the operad $W_{\mathrm{Seg}}|C_\Delta(\mathrm{DM}^{\mathrm{tree}})|$ is the "topological classifying operad" associated to the Deligne–Mumford stack, and hence represents a topological "resolution" of the operad in topological moduli problems $\mathrm{DM}^{\mathrm{tree}}$.

Acknowledgements

This paper got started in 2017 when both authors were members of the Institute for Advanced Study in Princeton within the special year on mirror symmetry. We would both like to acknowledge the inspirational role played by the seminar on Hodge theory organized by P. Seidel, where a talk by Oancea initiated this collaboration. We have had fruitful discussions with M. Abouzaid, V. Dotsenko and E. Getzler. The mixed model category structure was pointed out to us by A. Lahtinen and S. Schwede. Oancea acknowledges financial support via the ANR grants ENUMGEOM ANR-18-CE40-0009 and COSY ANR-21-CE40-0002, as well as a Fellowship of the University of Strasbourg Institute for Advanced Study (USIAS). Vaintrob acknowledges hospitality of the Institut de Mathématiques de Jussieu-Paris Rive Gauche (IMJ-PRG, Paris) and Institut de Recherche Mathématique Avancée (IRMA, Strasbourg). Finally, we would like to thank the referee and the editors for their thoughtful suggestions and for their patience.

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Received: April 8, 2020 Revised: April 11, 2025

Algebraic & Geometric Topology **26**:4 (2026) 1293–1320
DOI: 10.2140/agt.2026.26.1293
Published: April 25, 2026

On embeddings of 4-manifolds in codimension 2

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We show that every closed orientable smooth 4-manifold admits a smooth embedding in a large class of closed 6-manifolds. In particular, we show that every smooth 4-manifold admits a smooth embedding in the complex projective 3-space. Our embedding technique also provides a new proof of embeddings of 4-manifolds in $\mathbb{R}^7$.

1 Introduction

A basic question in the field of geometric topology which concerns embeddings of manifolds can be stated as follows: given a pair of manifolds M and N, how many smooth embeddings of M exist in N?

A slightly simpler and related question is the question of finding which manifolds embed in a given manifold. Detailed investigations in this regard have led to the discovery of interesting invariants of manifolds. One of the earliest seminal results in this context is due to H. Whitney who showed that every closed manifold of dimension n admits an embedding in $\mathbb{R}^{2n}$. Subsequently, this result has been extensively generalized. Most notably, M. Hirsch [19] showed that every closed orientable odd-dimensional manifold M^{2n-1} admits a smooth embedding in $\mathbb{R}^{4n-3}$. This result, together with those by C. T. C. Wall [27] and V. Rokhlin [25], implies that every closed 3-manifold (orientable or otherwise) admits an embedding in $\mathbb{R}^5$.

For closed n-dimensional manifolds, combining the results of A. Haefliger [16], A. Haefliger and M. Hirsch [17], and W. Massey and F. Peterson [23], one knows that every such n-manifold embeds in $\mathbb{R}^{2n-1}$ when $n > 4$ and n is not a power of two. For 4-manifolds it was shown by M. Hirsch [20] and C. T. C. Wall (M. Hirsch mentions in [20] that C. T. C. Wall had independently proved this result) that every orientable PL 4-manifold admits a PL embedding in $\mathbb{R}^7$.

The purpose of this article is to show that there are smooth six-dimensional manifolds with relatively simple topology in which all closed-orientable smooth manifolds of dimension four embed. Ideally one would like to embed every closed smooth 4-dimensional manifold in $\mathbb{R}^6$. However, D. Ruberman [26] has shown that a closed smooth 4-manifold admits a smooth embedding in $\mathbb{R}^6$ if and only if it admits a spin structure and its signature is zero, a result which was also stated by S. Cappell and J. Shaneson in [7]. In particular, this implies that $\mathbb{C}P^2$ does not smoothly embed in $\mathbb{R}^6$.

Dishant M. Pancholi is thankful to the Simon's Foundation and ICTP, Trieste, Italy for the Simons Associateship, which allowed him to travel to ICTP, Trieste, Italy, where a part of the work related to this article was carried out. We are thankful to Prof. Yakov Eliashberg for constant encouragement and support. We are also thankful to Prof. S. Lakshmibala for suggestions regarding the presentation.
MSC2020: 57R40.

The simplest 6-manifold in which $\mathbb{C}P^2$ embeds is $\mathbb{C}P^3$. Furthermore, the question of embeddability of two important classes of closed orientable smooth 4-manifolds, namely, symplectic 4-manifolds and smooth algebraic surfaces have been extensively examined (see, for instance, [2; 9; 10]), and the question of their embeddability in $\mathbb{C}P^3$ is very important as any such embedding corresponds to a *Lefschetz pencil* of $\mathbb{C}P^3$ with given embedded submanifold as its generic fiber. We therefore investigate embeddings of 4-manifolds in $\mathbb{C}P^3$ and establish the following:

Theorem 1.1 *Every closed orientable smooth* 4*-manifold admits a smooth embedding in* $\mathbb{C}P^3$.

To the best of our knowledge, Theorem 1.1 above and Theorem 1.2, which establishes embedding of 4-manifolds in certain 6-manifolds of the type $N \times \mathbb{C}P^1$, are the only results demonstrating the existence of closed 6-manifolds in which all closed orientable smooth 4-manifolds embed.

The central idea for the proof of Theorem 1.1 is drawn from a well-known fact that given a projective embedding of a smooth algebraic surface, the standard Lefschetz pencil of the complex projective space generically induces a Lefschetz pencil structure on the surface. It was established by R. I. Baykur and O. Saeki [5; 6] that every closed orientable smooth 4-manifold admits a *simplified broken Lefschetz fibration* (SBLF), which can be regarded as a natural generalization of the Lefschetz pencil for an arbitrary smooth 4-manifold. This decomposition allows us to express any smooth 4-manifold as a singular fiber bundle over $\mathbb{C}P^1$ with a finite number of *Lefschetz singularities* and a unique *indefinite fold circle*. The advantage of this decomposition is that we can associate with any smooth 4-manifold certain data which comprise two constituents. These are an element of the *mapping class group* of a closed orientable surface of genus g expressed as a product of (positive) *Dehn twists*, corresponding to Lefschetz singularities, and a round handle attachment [4; 13] corresponding to the fold singularity.

Let us now briefly outline the argument establishing Theorem 1.1. We need Theorem 1.2 to prove Theorem 1.1. Hence, we begin by first stating and outlining the proof of Theorem 1.2.

Consider any closed orientable 4-manifold N which admits an embedding of a Hopf link which is *separable* in the sense of Definition 4.4. Roughly speaking, by a separable Hopf link in a manifold N, we mean that N admits a handle decomposition that satisfies the following property: the boundary of a 0-handle has a Hopf link, which is slice in the complement of the 0-handle. For any 4-manifold admitting separable Hopf link, we show:

Theorem 1.2 *Let M be an orientable closed smooth* 4*-manifold. Let N be a* 4*-manifold which admits a separable Hopf link. Then there exists an embedding* $\psi : M \to N \times \mathbb{C}P^1$.

Let us now outline the proof of Theorem 1.2. Given a closed orientable smooth 4-manifold M, consider the manifold M together with any given SBLF. We need to produce an embedding f of M in $N \times \mathbb{C}P^1$, where N is a 4-manifold admitting an embedding of separable Hopf link. The embedding will be produced such that the trivial fibration $\pi_2 : N \times \mathbb{C}P^1 \to \mathbb{C}P^1$ of $N \times \mathbb{C}P^1$ induces the given SBLF.[1]

[1] Strictly speaking one will only produce embeddings satisfying such properties up to isotopies and diffeomorphisms of source and target manifolds under consideration.

The three important steps for constructing the embedding f are the following: In the first step, using an appropriate generalization of techniques from [24], and a specific local embedding model for a given Lefschetz singularity, we provide an embedding of genus $g+1$ *Lefschetz subfibration* over a disc $\mathbb{D}^2$ in $N \times \mathbb{D}^2$, which is associated with the given SBLF. This embedding is such that the trivial product fibration $\pi_2 : N \times \mathbb{D}^2 \to \mathbb{D}^2$ induces the given *Lefschetz fibration*. This is the most important step in the proof and is detailed in Section 4. In fact, in Section 4 we show how to embed any Lefschetz fibration over a disc or $\mathbb{C}P^1$ in a trivial fibration over $\mathbb{C}P^1$ with fiber N.

Next, we use a local embedding model for fold singularities to produce an embedding of a submanifold $(\widetilde{M}, \partial\widetilde{M}) \subset M$ (having two disjoint boundary components) in $N \times I \times \mathbb{S}^1$. This embedding is constructed such that it agrees with the embedding in the first step near one of the boundary components of $\widetilde{M}$, and is a trivial fibration $\Sigma_g \times S^1$ near the other boundary component of $\widetilde{M}$. Here, Σ_g denotes a closed orientable surface of genus g. This provides us with a fiber-preserving embedding of $M \setminus \Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$. Finally, we extend the embedding of $M \setminus \Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$ using an embedding of $\Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$ to obtain the embedding $f : M \hookrightarrow N \times \mathbb{C}P^1$. These two steps are discussed in Section 5. Embedding of M in $N \times \mathbb{C}P^1$ is the content of Theorem 1.2. Theorem 1.2 immediately implies Theorem 6.1 which establishes embeddings of smooth closed orientable 4-manifolds in $\mathbb{R}^7$.

Having outlined a proof of Theorem 1.2, let us now discuss how to establish embeddings of 4-manifolds in $\mathbb{C}P^3$ as claimed in Theorem 1.1. Given a smooth, orientable, closed 4-manifold, we first consider the manifold $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ together with a specific SBLF. Next, we notice that the *blow-up* of $\mathbb{C}P^3$ along $\mathbb{C}P^1$ is a fiber bundle over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^2$ with the property that the fiber bundle is trivial in the complement of the *exceptional divisor*.

We embed $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up of $\mathbb{C}P^3$ using this specific SBLF by observing that $\mathbb{C}P^2$ admits a separable Hopf link and hence a slight generalization of the argument necessary to establish Theorem 1.2 allows us to embed $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the *blow-up* of $\mathbb{C}P^3$. Further, we ensure that the embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up of $\mathbb{C}P^3$ is such that the fiber of the specific SBLF associated to $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ has certain specific intersection property with the exceptional divisor of the blow-up of $\mathbb{C}P^3$. This allows us to show that the embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up of $\mathbb{C}P^3$ is such that when we *blow-down* the blow-up of $\mathbb{C}P^3$, we get a manifold diffeomorphic to $\mathbb{C}P^3$ that has M as its embedded submanifold. The construction of the specific SBLF, blow-up and blow-down procedures, and the proof of Theorem 1.1 are discussed in the final section.

The mathematical preliminaries to carry out these steps are given in Sections 2 and 3. In particular, we discuss relevant aspects of *broken Lefschetz fibrations* in Section 2, and of mapping class groups in Section 3.

Finally, a few remarks on conventions used in this article. By a manifold we mean a smooth compact orientable manifold with or without boundary. We denote manifolds by capital letters M, N, etc. When we need to emphasize that we are working with a manifold with boundary, we use the notation $(M, \partial M)$ consisting of the pair M and the boundary ∂M of M. As usual, the notation Σ or Σ_g is used for denoting a closed orientable surface, with g indicating the genus.

2 Review of broken Lefschetz fibrations

Broken Lefschetz fibrations (BLF) were introduced by D. Auroux, S. K. Donaldson, and L. Katzarkov [1]. These are generalized Lefschetz fibrations. R. I. Baykur [3] established that every smooth orientable closed 4-manifold admits a broken Lefschetz fibration. The purpose of this section is to review a few definitions and results related to BLF. We refer to [3; 5] for a detailed discussion on BLF. Let us begin by recalling the definition of Lefschetz singularity.

Definition 2.1 (Lefschetz singularity) Let M be an oriented 4-manifold and Σ an oriented surface. Let $f : M \to \Sigma$ be a smooth map. A point $x \in M$ is said to be a Lefschetz singularity of the map f, provided there is an orientation preserving parameterization $\phi : U \subset M \to \mathbb{C}^2$, and an orientation preserving parameterization $\psi : V \subset \Sigma \to \mathbb{C}$ such that the following properties are satisfied:

(1) $x \in U$, and $\phi(x) = (0, 0) \in \mathbb{C}^2$.

(2) $f(x) \in V$, and $\psi(f(x)) = 0 \in \mathbb{C}$.

(3) For the map $g : \mathbb{C}^2 \to \mathbb{C}$ given by $g(z_1, z_2) = z_1.z_2$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{C}^2 \\ \big\downarrow f & & \big\downarrow g \\ V & \xrightarrow{\psi} & \mathbb{C} \end{array}$$

Remark 2.2 (a) Observe that both M as well as Σ can have nonempty boundary, however, it follows from Definition 2.1 that the critical point c belongs to the interior $\mathring{M}$ of M, and $f(c) \in \mathring{\Sigma}$.

(b) Let $f : M \to S$ be a map with an isolated Lefschetz singularity at $c \in M$ such that $f(c) \in S$ is an isolated critical value. It is well known that generically the fiber over $f(c)$ is obtained by pinching a simple closed curve γ on a nearby smooth fiber Σ_g to a point. The curve γ is known as a *vanishing cycle*.

Next, we recall the definition of 1-*fold singularity*.

Definition 2.3 (1-fold singularity) Let M be an oriented 4-manifold, and let Σ be an oriented surface. Let $f : M \to \Sigma$ be a smooth map. A point $x \in M$ is said to be a 1-fold singularity of the map f, provided there is an orientation preserving parameterization $\phi : U \subset M \to \mathbb{R}^4$, and an orientation preserving parameterization $\psi : V \subset \Sigma \to \mathbb{R}^2$ such that the following properties are satisfied:

(1) $x \in U$, and $\phi(x) = (0, 0, 0, 0) \in \mathbb{R}^4$.

(2) $f(x) \in V$, and $\psi(f(x)) = (0, 0) \in \mathbb{R}^2$.

(3) For the map $h : \mathbb{R}^4 \to \mathbb{R}^2$ given by $h(t, x_1, x_2, x_3) = (t, -x_1^2 + x_2^2 + x_3^2)$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{R}^4 \\ \big\downarrow f & & \big\downarrow h \\ V & \xrightarrow{\psi} & \mathbb{R}^2 \end{array}$$

Remark 2.4 (a) If a map $f : M \to \Sigma$ has a 1-fold singularity at x, then $x \in \mathring{M}$, and $f(x) \in \mathring{\Sigma}$.

(b) When the map h in the definition of 1-fold singularity is allowed to have the local model

$$(t, x_1, x_2, x_3) \to (t, \pm x_1^2 \pm x_2^2 \pm x_3^2),$$

the singularity is called a *fold singularity*. In this article, we will only need the local model around 1-fold singularity.

We are now in a position to recall the notion of a broken Lefschetz fibration (BLF).

Definition 2.5 (broken Lefschetz fibration) Let M a smooth oriented 4-manifold. By a broken Lefschetz fibration of M we mean a smooth surjective map $f : M \to \mathbb{C}P^1$ such that f has only 1-fold or Lefschetz singularities.

Remark 2.6 (a) Given a BLF $f : M \to \mathbb{C}P^1$, the inverse image $f^{-1}(y)$ for any regular value y is called a fiber of BLF.

(b) Generically, the image set of a component of 1-fold singularities on Σ is an immersed circle in $\mathring{\Sigma}$.

A BLF without 1-fold singularity is called a Lefschetz fibration. These singular fibrations are extremely useful in algebraic geometry [15] and symplectic geometry [10]. Let us now formally define a Lefschetz fibration.

Definition 2.7 (Lefschetz fibration) Let M be a smooth oriented 4-manifold. A smooth surjective map $f : M \to \Sigma$, where Σ is an oriented surface, having its singular points modeled only on Lefschetz singularities is called a Lefschetz fibration of M.

Remark 2.8 (a) Unlike a fiber bundle or Lefschetz fibration, the regular fibers of a BLF are typically not diffeomorphic. In fact, the 1-fold singularity in the definition of BLF corresponds to a round 1-handle attachment [4; 13]. Hence, if BLF has points having fold singularity, then the topology of the regular fiber changes as we cross the image of an immersed circle coming from a 1-fold singularity.

(b) The fibers of BLF need not be connected. However, it can be shown that every 4-manifold admits a BLF with connected fibers having genus at least 2. This follows from [3, Theorem 1.1].

Observe that a BLF provides us a decomposition of a smooth manifold into simple pieces. A more simplified form of this decomposition of a smooth 4-manifold is what we will need for this article. This simplification was introduced by R. I. Baykur [4], and the proof of this simplified decomposition was given by R. I. Baykur and O. Saeki [5; 6]. This decomposition is known as a simplified broken Lefschetz fibration. Let us recall the definition of this:

Definition 2.9 (simplified broken Lefschetz fibration (SBLF)) Let $f : M \to \mathbb{C}P^1$ be a BLF. We say that this BLF is a simplified broken Lefschetz fibration (SBLF) provided the function f satisfies the following additional properties:

(1) The set Z_f of all $x \in M$ admitting a 1-fold singularity model is connected.

(2) All fibers are connected.

(3) The map f is injective when restricted to Z_f as well as when restricted to the set, C_f, of Lefschetz singular points and the set C_f is contained in the connected component of $\mathbb{C}P^1 \setminus f(Z_f)$ which has regular fibers of higher genus.

Remark 2.10 (a) Throughout this article, we will assume without loss of generality that for any Lefschetz fibration (or BLF) $f : M \to \Sigma$ any regular fiber $f^{-1}(y)$ is connected, and f is injective when restricted to critical set C_f.

(b) Observe that the definition of SBLF implies that there exists a disc $\mathcal{D}$ contained in $\mathbb{C}P^1$ such that every $y \in \mathcal{D}$ is a regular value, and the genus of the fiber over y is minimum among all fibers of SBLF. We call this fiber the *lower genus fiber*.

(c) Topologically, the unique 1-fold singularity of SBLF corresponds to adding a 1-handle to a circle worth of lower genus fibers over $\partial\mathcal{D}$. This corresponds to an attachment of a round 1-handle to $f^{-1}(\mathcal{D})$ such that a generic fiber of SLBF over $\mathbb{C}P^1 \setminus \overline{\mathcal{D}}$ has genus one more than the fibers over $\mathcal{D}$.

In [5; 6], it was shown that every orientable smooth 4-manifold admits an SBLF.

Theorem 2.11 (R. I. Baykur, O. Saeki [5, Theorem 1]) *Given any generic map from a closed, connected, oriented, smooth 4-manifold X to $\mathbb{C}P^1$, there are explicit algorithms to modify it to an SBLF. In particular, every closed orientable smooth 4-manifold admits an SBLF. Furthermore, we can always construct an SBLF on M such that the genus of the lower genus figure is bigger than 1.*

We would like to point out that Theorem 2.11 is not stated as above in [5]. The statement regarding the lower bound on the genus of a lower genus fiber is not explicitly mentioned in [5, Theorem 1]. However, it follows from the application of [5, Theorem 1] followed by [5, Theorem 2]. For the sake of completeness, we discuss the proof of Theorem 2.11.

Proof To begin with, recall that by a *trisection* of a smooth orientable closed 4-manifold M one means a decomposition of M into three 4-dimensional handlebodies (thickening of a wedge of circles), meeting pairwise in 3-dimensional handlebodies, and all three 4-dimensional handlebodies intersect in a surface. A trisection corresponds to a Morse 2-function on M. If k' is the number of indefinite folds for the Morse 2-function associated to a given trisection and g' is the genus of the surface corresponding to the common intersections of three 4-dimensional handlebodies, one says that the 4-manifold has a (g', k')-trisection.

In order to produce an SBLF as stated in Theorem 2.11, we observe that given M, according to [5, Theorem 1], there exists an SBLF $f : M \to \mathbb{C}P^1$. Let g be the genus of the lower genus fiber of the SBLF. If $g > 1$, then we are through. In case, $g \leq 1$, we apply [5, Theorem 2] to produce a (g', k')-trisection from the given SBLF $f : M \to \mathbb{C}P^1$. According to [5, Theorem 2], we get a (g', k')-trisection with $g \geq 1$.

Next, we again apply the second part of [5, Theorem 2] to produce from this trisection a new SBLF. Observe that according to [5, Theorem 2], the new SBLF has a lower genus fiber having its genus $g' + 2$. Since $g' \geq 0$, the theorem follows. □

We would like to remark that the proof of the existence of SBLF with higher genus fiber also follows from the proof of Proposition 1.3 given in [3].

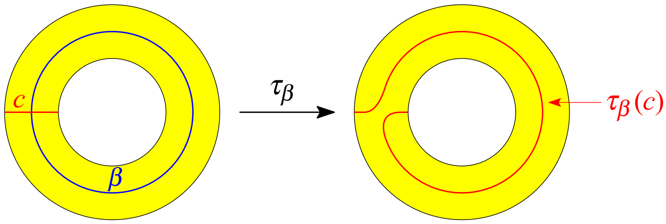

Figure 1: The figure is a pictorial description of the Dehn twist τ_β restricted to the neighborhood $\mathcal{A}(\beta) = \mathbb{S}^1 \times [0, 2\pi]$. The map τ_β is given by $\tau_\beta(\theta, t) = (\theta - t, t)$ when restricted to $\mathcal{A}(\beta)$. It sends the arc c — depicted as a red-colored arc in the picture on the left of the figure — to an arc isotopic to the arc $\tau_\beta(c)$ depicted in the picture on the right of the figure.

3 Mapping class groups of surfaces

In this section, we review some results related to mapping class groups of closed-orientable surfaces. Good references for the results discussed here are [12; 21]. Let us begin by recalling the definition of the mapping class group.

Definition 3.1 (mapping class group) Let Σ be a closed oriented surface. By the mapping class group of Σ, we mean the group of orientation preserving self diffeomorphisms of Σ up to isotopy.

We denote the mapping class group of a surface Σ by $\mathcal{MCG}(\Sigma)$. Next, let us discuss the notion of a *Dehn twist* along a simple closed curve embedded in a surface Σ. We refer to [12] for a more detailed discussion on Dehn twists.

Definition 3.2 (Dehn twist) Let Σ be an orientable surface. Let β be a simple closed curve embedded in the interior of Σ. By a Dehn twist along β, we mean a diffeomorphism which is identity outside an annulus neighborhood $\mathcal{A}(\beta)$ of β in Σ, and is given by τ_β on $\mathcal{A}(\beta)$ when restricted to $\mathcal{A}(\beta)$, where τ_β is the diffeomorphism of $\mathcal{A}(\beta)$ described in Figure 1.

M. Dehn [8] (see also [21]) established that the mapping class group of an orientable genus g surface Σ_g is generated by Dehn twists along simple closed curves embedded in Σ_g. W. Lickorish [22] further strengthened this result to show that the mapping class group of a closed orientable surface Σ_g is generated by Dehn twists along the curves a_i's, b_j's and c_k's as depicted in Figure 2. Following [24], we will refer to these curves as *Lickorish generators*.

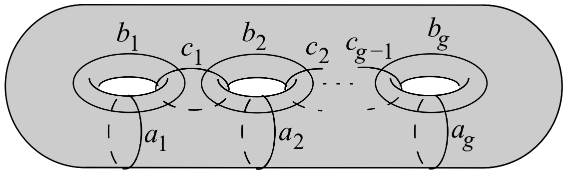

Figure 2: Dehn twists along curves a_i's, b_j's and c_k's generate the mapping class group of an orientable genus g surface.

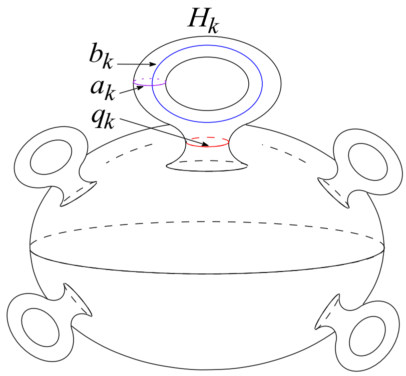

Figure 3: The figure shows a surface of genus g embedded in $\mathbb{R}^3$ as a boundary of a genus g handlebody S_g considered as a unit ball with g 1-handles attached to it.

We end this section with a proposition that is a consequence of [21, Lemma 3]. In order to state this proposition we need some terminology from [21].

Let us regard an orientable surface Σ_g of genus g as the boundary of a standard handlebody S_g. Here, a standard handlebody S_g consists of g 1-handles attached to the unit 3-ball in $\mathbb{R}^3$ as depicted in Figure 3.

Consider a typical handle H_k, as shown in Figure 3. Following [21], we say that a simple closed curve p *does not meet* the handle H_k provided it does not intersect the curve a_k depicted Figure 3.

Proposition 3.3 (Lickorish [21, Lemma 3]) *Let p be any simple closed curve on Σ_g. There exists a diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\phi(p)$ does not meet any handle of Σ_g.*

4 Lefschetz fibration embedding

Recall from Remark 2.10 that a Lefschetz fibration (LF) of a closed oriented 4-manifold is a pair $(M, \pi : M \to \Sigma)$, where $\pi : M \to \Sigma$ is an LF and Σ is either a disc or $\mathbb{C}P^1$. Furthermore, we always assume that π is injective when restricted to the critical set. Given such an LF, in this section, we show that there exists an embedding of the LF into certain manifolds of type $N^4 \times \Sigma$ which is fiber preserving in the sense of Definition 4.10 provided the genus of the regular fiber is at least 2. This result (Theorem 4.11) can be regarded as the first step towards establishing Theorem 1.1.

4.1 Flexible embedding in standard position

Let us begin this subsection by reviewing the notion of *flexible embedding*.

Definition 4.1 (flexible embedding) Let M be an orientable closed smooth manifold. A smooth embedding $\phi : \Sigma_g \hookrightarrow M$ of a closed orientable surface Σ_g is said to be flexible provided for every $f \in \mathcal{MCG}(\Sigma_g)$ there exists a diffeomorphism ψ of M isotopic to the identity which maps $\phi(\Sigma_g)$ to itself and satisfies $\phi^{-1} \circ \psi \circ \phi = f$.

Next, we state a lemma regarding a flexible embedding of any surface of genus g into a 4-manifold N, which admits a separable Hopf link. In order to state this lemma, we need to introduce the following definitions:

Definition 4.2 (embedding in standard position) An embedding $\phi : \Sigma_g \hookrightarrow N$ of a surface Σ_g is said to be in a standard position provided the following properties are satisfied:

(1) Every simple closed curve γ on $\phi(\Sigma)$ is a boundary of a 2-disc $\mathbb{D}^2$ intersecting $\phi(\Sigma_g)$ only in γ.

(2) There exists a tubular neighborhood $\mathcal{N}(\mathbb{D})$ of the disc $\mathbb{D}^2$ having the boundary γ such that $\mathcal{N}(\mathbb{D})$ is the image of a coordinate chart $\phi_\gamma : \mathbb{C}^2 \to \mathcal{N}(\mathbb{D})$ satisfying the following:

- $\phi_\gamma^{-1}(\phi(\Sigma_g) \cap \mathcal{N}(\mathbb{D}))$ is $g^{-1}(1)$, where $g : \mathbb{C}^2 \to \mathbb{C}$ is the polynomial map $g(z_1, z_2) = z_1.z_2$.

Remark 4.3 The standard position embedding $\phi : \Sigma_g \to N$ in Definition 4.2 can be thought of as an embedding of Σ_g in N such that for every simple closed curve γ on $\phi(\Sigma_g)$, there exists a compact 4-ball B^4_γ in N such that the intersection of $\phi(\Sigma_g)$ with B^4_γ is a Hopf annulus in $\partial B\gamma$.

The equivalence between Definition 4.2 and Remark 4.3 plays an important role in the constructions of embeddings throughout this article. We now outline this equivalence. Let $\pi : D^4 \to D^2$ be a Lefschetz fibration of a 4-ball D^4 whose regular fiber is an annulus, and which has a single Lefschetz singularity at the origin $0 \in D^4$, with corresponding singular value $0 \in D^2$. Here, D^2 denotes the closed 2-disc of radius 2 in $\mathbb{R}^2$, centered at the origin. One can easily see the following:

(1) For each disc $D^2_r \subset D^2$ of positive radius r, $\pi^{-1}(D^2_r)$ is diffeomorphic to a compact 4-ball and its boundary $S^3_r = \partial(\pi^{-1}(D^2_r))$ is diffeomorphic to the 3-sphere.

(2) The Lefschetz fibration $\pi : \pi^{-1}(D^2_r) \to D^2_r$ induces an open book on $\partial\pi^{-1}(D^2_r)$ with a page an annulus $A_r = \pi^{-1}(re^{i0})$ and the monodromy a Dehn twist along the central curve of A_r.

(3) The annulus A_r is a Hopf annulus in $S^3_r = \partial(\pi^{-1}(D^2_r))$.

(4) There are orientation preserving diffeomorphisms $\chi : \mathring{D}^4 \to \mathbb{C}^2$ and $\xi : \mathring{D}^2 \to \mathbb{C}^2$ such that $\xi(S^1) = S^1$ and the diagram

$$\begin{tikzcd} \mathring{D}^4 \arrow[r, "\chi"] \arrow[d, "\pi"] & \mathbb{C}^2 \arrow[d, "g"] \\ \mathring{D}^2 \arrow[r, "\xi"] & \mathbb{C} \end{tikzcd}$$

commutes, where $\mathring{D}^4$ and $\mathring{D}^2$ are the interiors of D^4 and D^2, respectively, and the map g is given by $g(z_1, z_2) = z_1.z_2$.

Hence, the 4-ball B^4_γ in Remark 4.3 can be realized as the closure of the image of the map $\phi_\gamma \circ \chi : \pi^{-1}(D^2_1) \to \mathcal{N}(\mathbb{D})$.

Conversely, suppose we are given a 4-ball B^4_γ in N such that the intersection of $\phi(\Sigma_g)$ with B^4_γ is an Hopf annulus $\mathcal{H}$ in $\partial B\gamma$. We know that the 4-ball B^4_γ admits a Lefschetz fibration $\pi' : B^4_\gamma \to D^2_1$

with only one Lefschetz singularity at $p \in B^4_\gamma$ such that $\pi'(p) = 0 \in D^2_1$ and $\pi^{-1}(e^{i0}) = \mathcal{H}'$, where $\mathcal{H}'$ is a Hopf annulus in ∂B^4_γ. Since any two positive (negative) Hopf annuli in $S^3 = \partial B^4_\gamma$ are isotopic, we can assume $\pi'^{-1}(e^{i0}) = \mathcal{H}$. Now, we first identify B^4_γ with $\pi^{-1}(D^2_1) \subset D^4$ by a diffeomorphism $h : \pi^{-1}(D^2_1) \to B^4_\gamma$ such that the diagram

$$\begin{tikzcd} N \supset B^4_\gamma \arrow[d, "\pi'"] & \pi^{-1}(D^2_1) \subset D^4 \arrow[l, "h"'] \arrow[d, "\pi"] \\ D^2_1 & D^2_1 \subset D^2 \arrow[l, "\mathrm{Id}"'] \end{tikzcd}$$

commutes. Then the desired chart $\phi_\gamma : \mathbb{C}^2 \to \mathcal{N}(\mathbb{D})$ can be defined by appropriately extending the map $h \circ \chi^{-1} : \chi(\pi^{-1}(\mathring{D}^2_1)) \subset \mathbb{C}^2 \to N$. Under this chart, we have

$$\begin{aligned} \phi_\gamma^{-1}(\phi(\Sigma_g) \cap B^4_\gamma)) &= \chi\big(h^{-1}(\phi(\Sigma_g) \cap B^4_\gamma)\big) = \chi(h^{-1}(\mathcal{H})) \\ &= \chi\big(h^{-1}(\pi'^{-1}(e^{i0}))\big) = \chi(\pi^{-1}(e^{i0})) = g^{-1}(e^{i0}), \end{aligned}$$

as desired.

Definition 4.4 (separable Hopf link) We say that a link $l_1 \sqcup l_2$ in a 4-manifold N is a separable Hopf link provided the following properties are satisfied:

(1) There exist an embedding of a 4-ball $\mathbb{D}^4 = \mathbb{D}^2 \times \mathbb{D}^2$ in N such that $\partial\mathbb{D}^2 \times \{0\} \sqcup \{0\} \times \partial\mathbb{D}^2 = l_1 \sqcup l_2$.

(2) There exists two disjoint properly embedded discs $\mathcal{D}_1$ and $\mathcal{D}_2$ in $N \setminus (\mathbb{D}^2 \times \mathbb{D}^2)^\circ$ such that $\partial\mathcal{D}_1 = l_1$ and $\partial\mathcal{D}_2 = l_2$.

Lemma 4.5 *Let N be a 4-manifold which admits a separable Hopf link. Then there exists an embedding ϕ of any closed orientable surface Σ_g of genus g in N which satisfies the following*:

(1) *The embedding is flexible.*

(2) *The embedding is in a standard position.*

Before we establish this lemma, we would like to point out that the flexible embedding of Σ_g in N was first provided by S. Hirose and A. Yasuhara [18]. Our main observation is that we can achieve the additional property of the embedding being in a standard position, provided that we use Proposition 3.3 established by Lickorish [21] in conjunction with the techniques from [18].

Proof of Lemma 4.5 We want to construct an embedding of Σ_g in N which is both flexible and in a standard position. Let $l_1 \sqcup l_2$ be a separable Hopf link in N. It follows from the definition of separable link that there exists an embedded 4-ball $\mathbb{D}^4 = \mathbb{D}^2 \times \mathbb{D}^2$ in N such that $\partial\mathbb{D}^2 \times \{0\} \sqcup \{0\} \times \partial\mathbb{D}^2 = l_1 \sqcup l_2$, and there exists two disjoint properly embedded discs $\mathcal{D}_1$ and $\mathcal{D}_2$ in $N \setminus (\mathbb{D}^2 \times \mathbb{D}^2)^\circ$ such that $\partial\mathcal{D}_1 = l_1$ and $\partial\mathcal{D}_2 = l_2$. We regard the 4-ball $\mathbb{D}^4$ as the 4-ball $B^4(0,2)$ of radius 2 in $\mathbb{C}^2$ with its center at the origin. We will also regard $\mathbb{S}^3 \times [1,2]$ as the collar $B^4(0,2) \setminus B^4(0,1)$ contained in N.

Next, observe that the link $l_1 \times \{\frac{3}{2}\} \sqcup l_2 \times \{\frac{3}{2}\}$ bounds a Hopf band say $\mathcal{H}$ in $\mathbb{S}^3 \times \{\frac{3}{2}\}$. We embed a genus g surface Σ_g in $\mathbb{S}^3 \times \{\frac{3}{2}\} \subset \mathbb{S}^3 \times [1,2] \subset N$ as the boundary of standard genus g handle body H_g

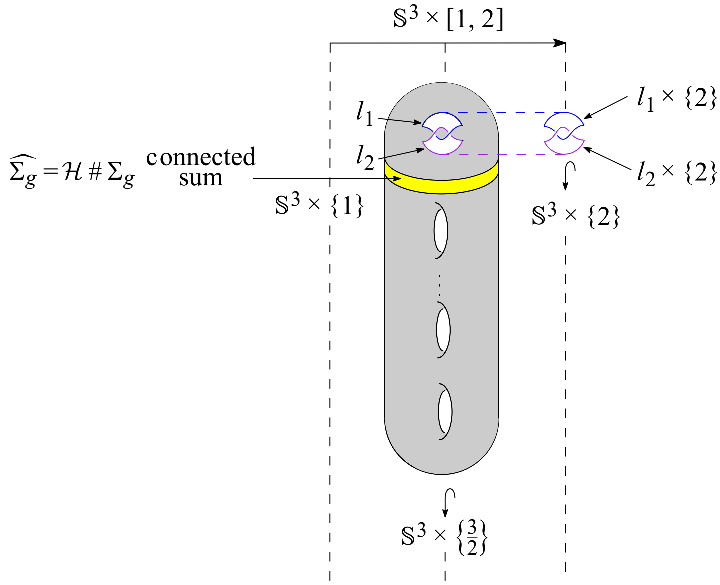

Figure 4: The figure depicts the embedding of the surface Σ_g which is flexible as well as in the standard position. The figure depicts the collar $\mathbb{S}^3 \times [1,2] \subset N$ with dashed lines representing $\mathbb{S}^3$ at levels 1, 2 and $\frac{3}{2}$.

and disjoint form $\mathcal{H}$ as depicted in Figure 3. Then we take ambient connected sum of embedded Σ_g and $\mathcal{H}$ in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ to obtain a surface $\widehat{\Sigma_g}$ with two boundary components as shown in Figure 4. Thus by adding two cylinders $l_1 \sqcup l_2 \times \left[\frac{3}{2}, 2\right]$ and two disjoint discs $\mathcal{D}_1$, $\mathcal{D}_2$ to $\widehat{\Sigma_g}$, we obtain an embedding of a closed genus g surface. Let us denote this embedding — after smoothing the corners — by ϕ. For a pictorial description of the embedding ϕ, we refer the reader to Figure 4. We claim that the embedding $\phi : \Sigma_g \hookrightarrow N$ is both flexible and in standard position. Let us now establish this claim.

The claim that the embedding is flexible is already established in [18, Theorem 3.1]. Let us briefly review the argument. First of all, notice that every Lickorish generator γ of Σ_g embedded in N via ϕ has — up to an isotopy — a Hopf annulus neighborhood which is contained in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\} \subset N$. Next, recall that the mapping class group of Σ_g is generated by Dehn twists along Lickorish generators, and in $\mathbb{S}^3$ there exists a diffeomorphism isotopic to the identity which induces a Dehn twist on a given Hopf annulus fixing its boundary pointwise. In the proof of [18, Theorem 3.1] it is shown that this implies that there exists a diffeomorphism of N isotopic to the identity which induces a Dehn twist along a Lickorish generator of $\phi(\Sigma_g)$. The claim now follows by successive application of ambient isotopies of N inducing Dehn twists on Lickorish generators. See also [24, Lemma 15] for additional details.

Let us now show that the embedding is in a standard position. First of all notice that — by the very construction, any simple closed curve on $\phi(\Sigma_g)$ can be isotoped on the surface $\phi(\Sigma_g)$ such that it is contained in $\phi(\Sigma) \cap \mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$. We claim that any simple closed curve which does not meet handles[2]

[2] Recall that a simple closed curve p does not meet the handle H_k provided it does not intersect the curve a_k depicted in Figure 3.

of $\phi(\Sigma_g)$ satisfies both the properties necessary for an embedding to be in a standard position. This is because:

(1) All curves mentioned in the claim are unknots in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ hence they bound a disc in $\mathbb{S}^3 \times \left[1, \frac{3}{2}\right]$, that meets $\phi(\Sigma)$ only in the given curve.

(2) Any curve γ mentioned in the claim is isotopic to a simple closed curve C in $\phi(\Sigma_g) \cap \mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ via an isotopy of $\phi(\Sigma_g)$ such that C admits a neighborhood $\mathcal{N}(C)$ in $\phi(\Sigma_g)$ which is a Hopf band in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$.

It follows from both the properties listed above that any simple closed curve C, which does not meet any handle, satisfies both the properties necessary for a surface to be in the standard position.

Now, according to Proposition 3.3, given any simple closed curve C, there exists a diffeomorphism of $\phi(\Sigma_g)$ which sends C to a curve which does not meet any handle. Since the embedding ϕ of Σ_g is flexible in N, given a simple closed curve C which meets some handles can be isotoped so that now it does not meet any handle. Hence, the claim that the embedding is also in a standard position follows. $\square$

Remark 4.6 (1) Any simple closed curve C in $\phi(\Sigma_g)$ that does not meet handles of $\phi(\Sigma_g)$ can be isotoped to a curve C' in $\phi(\Sigma_g) \cap \mathbb{S}^3 \times \frac{3}{2}$ such that an annular neighborhood $\mathcal{A} = \mathcal{N}(C')$ of C' in $\phi(\Sigma_g)$ is a planar annulus in $\mathbb{S}^3 \times \frac{3}{2}$. Therefore, there exists an embedded 4-ball $B^4 = B^3 \times [0,1]$ in N such that $\mathcal{A} = B^3 \times \{0\} \cap \phi(\Sigma_g)$, $\mathcal{A}$ is a planar annulus on $\partial(B^3 \times \{0\})$.

(2) Since the embedded surface $\phi(\Sigma_g)$ is flexible, for given any simple closed curve γ, there exists an embedded 4-ball $B^4_\gamma = B^3_\gamma \times [0,1]$ in N such that $B^3_\gamma \times \{0\} \cap \phi(\Sigma_g) = \mathcal{A}_\gamma$ is a annular neighborhood of γ in $\phi(\Sigma_g)$ and $\mathcal{A}_\gamma$ is a planar annulus on $\partial(B^3_\gamma \times \{0\})$.

In what follows we will work with embeddings of surfaces in N constructed using the procedure described in the proof of Lemma 4.5. We will use the term *standard embedding* for any such embedding. More precisely, we have the following:

Definition 4.7 (standard embedding) Let N be a manifold admitting a separable Hopf link. An embedding ψ of a closed orientable surface Σ_g, which is isotopic to an embedding obtained by the procedure described in the proof of Lemma 4.5, will be called a standard embedding of Σ_g

We end this subsection by establishing an embedding result regarding the embeddings of mapping tori in $N \times \mathbb{S}^1$. Recall that given a manifold Σ, the mapping torus of Σ with monodromy g, where g is a diffeomorphism of Σ, is the quotient space $\Sigma \times [0,1]/\sim$, where $(x,0) \sim (g(x),1)$. Throughout this article we will consider mapping tori up to the ambient isotopy class of g in Σ. We will denote the mapping torus by $\mathcal{MT}(\Sigma, g)$. Notice that $\mathcal{MT}(\Sigma, g)$ is a fiber bundle over $\mathbb{S}^1$. Our next lemma establishes a fiber-preserving embedding of any mapping tours of Σ into $N \times \mathbb{S}^1$. More precisely:

Lemma 4.8 *Let N be a 4-manifold admitting a separable Hopf link and let $\phi : \Sigma_g \to N$ be a standard embedding of Σ_g. Let $d_\gamma : \Sigma_g \to \Sigma_g$ be a Dehn twist along a simple closed curve γ on Σ_g. Then there* *exists an embedding Ψ of $\mathcal{M}T(\Sigma_g, d_\gamma)$ in $N \times \mathbb{S}^1$ such that the following diagram commutes*:

$$\begin{tikzcd} \mathcal{M}T(\Sigma_g, d_\gamma) \arrow[r, "\Psi"] \arrow[d, "\pi"] & N \times \mathbb{S}^1 \arrow[d, "\pi_2"] \\ \mathbb{S}^1 \arrow[r, "\mathrm{Id}"] & \mathbb{S}^1 \end{tikzcd}$$

Proof Let $\mathbb{D}^2$ be a disc in N such that it intersects $\phi(\Sigma_g)$ in only γ. Let $\mathcal{N}(\mathbb{D}^2)$ be a neighborhood in $\mathbb{D}^2$ such that $\mathcal{N}(\mathbb{D}^2)$ is the image of the coordinate chart $\phi_\gamma : \mathbb{C}^2 \to \mathcal{N}(\mathbb{D}^2)$ with $\phi_\gamma^{-1}(\phi(\Sigma_g) \cap \mathcal{N}(\mathbb{D}^2)) = g^{-1}(1)$, where the map $g : \mathbb{C}^2 \to \mathbb{C}$ is given by $g(z_1, z_2) = z_1.z_2$. Note that the monodromy of the Lefschetz fibration g over the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ is a Dehn twist along the central curve of the annulus $g^{-1}(1)$. Therefore, there is a flow $\rho_t : \mathbb{C}^2 \to \mathbb{C}^2,\ 0 \le t \le 1$, supported in $g^{-1}(\mathcal{N}(S^1))$ such that

(1) $\rho_t(g^{-1}(e^{i0})) = g^{-1}(e^{2\pi i t})$ and

(2) ρ_1 restricted to $g^{-1}(e^{i0})$ is a Dehn twist along the central curve of the annulus $g^{-1}(e^{i0})$,

where $\mathcal{N}(S^1)$ is a small annulus neighborhood of S^1 in $\mathbb{C}$. Using this flow, we can define a flow $\phi_\gamma \circ \rho_t \circ \phi_\gamma^{-1} : \mathcal{N}(\mathbb{D}^2) \to \mathcal{N}(\mathbb{D}^2),\ 0 \le t \le 1$. Since the flow ρ_t is supported in $g^{-1}(\mathcal{N}(S^1))$, the flow $\phi_\gamma \circ \rho_t \circ \phi_\gamma^{-1}$ on $\mathcal{N}(\mathbb{D}^2)$ can be extended to a flow $\xi_t : N \to N,\ 0 \le t \le 1$, by defining $\xi_t = \phi_\gamma \circ \rho_t \circ \phi_\gamma^{-1}$ on $\mathcal{N}(\mathbb{D}^2)$ and $\xi_t = \mathrm{Id}$ in the complement of $\mathcal{N}(\mathbb{D}^2)$ in N. Now, the desired embedding $\Psi : \mathcal{M}T(\Sigma_g, d_\gamma) \to N \times \mathbb{S}^1$ is given by $\Psi(x, t) = (\xi_t \circ \phi(x), e^{2\pi i t})$. □

Definition 4.9 Let N be a manifold admitting a separable Hopf link and let $\phi : \Sigma_g \to N$ be a standard embedding of Σ_g. Let $d_\gamma : \Sigma_g \to \Sigma_g$ be a Dehn twist along a simple closed curve γ on Σ_g. Then, the embedding $\Psi_\gamma : \mathcal{M}T(\Sigma_g, d_\gamma) \to N \times \mathbb{S}^1$ constructed in the proof of the above lemma will be called the standard embedding of $\mathcal{M}T(\Sigma_g, d_\gamma)$ in $N \times \mathbb{S}^1$ with respect to the standard embedding ϕ and the Dehn twist d_γ.

Before we proceed, we would like to point out that Lemma 4.8 was implicitly established in [24].

4.2 The existence of Lefschetz fibration embedding

We are now in a position to state and prove our main result regarding *Lefschetz fibration embeddings*. As usual, we denote the map $N \times \mathbb{C}P^1$ to $\mathbb{C}P^1$ corresponding to the projection on the second factor by π_2.

Definition 4.10 (Lefschetz fibration embedding) Let $(M, \pi : M \to \Sigma)$ be a Lefschetz fibration, where Σ is a 2-disc or $\mathbb{C}P^1$. An embedding $f : M \to N \times \mathbb{C}P^1$ of a manifold M into a manifold $N \times \mathbb{C}P^1$ is said to be a *Lefschetz fibration embedding* provided $\pi_2 \circ f = i \circ \pi$, where i is an inclusion of $\mathbb{D}^2$ in $\mathbb{C}P^1$ when $\partial M \neq \varnothing$, otherwise it is the identity.

Theorem 4.11 *Let M be an orientable smooth* 4*-manifold. Let N be a* 4*-manifold which admits a separable Hopf link. Let $\pi : M \to \Sigma$, where Σ is either $\mathbb{C}P^1$ or a 2-disc $\mathbb{D}^2$ embedded in $\mathbb{C}P^1$, be a Lefschetz fibration* (*LF*) *of M having genus g fibers with $g \ge 2$. If the map π is injective when restricted to the set of critical points of π, then there exists a Lefschetz fibration embedding of (M, π) in $(N \times \mathbb{C}P^1, \pi_2)$.*

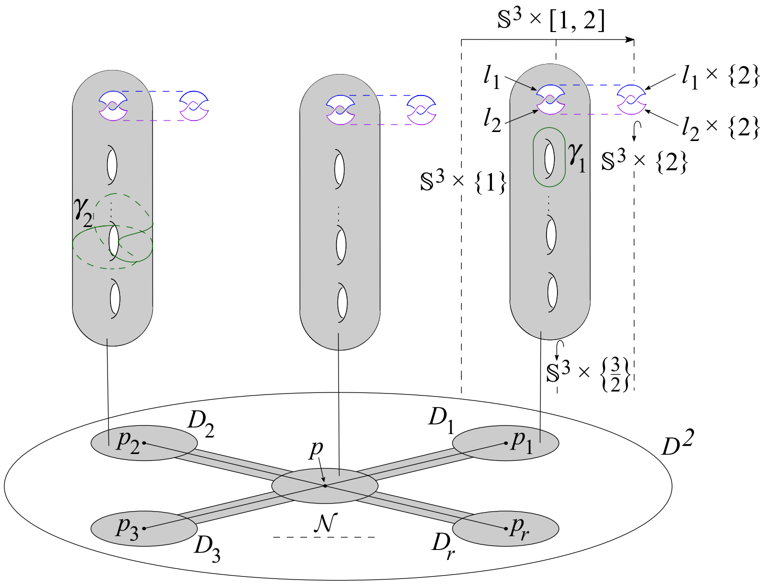

Figure 5: The figure depicts part of a Lefschetz fibration (M,π) over a disc embedded as a Lefschetz fibration in the (Lefschetz) fibration $\pi_2 : N \times D^2 \to D^2$. The embedding is such that the generic fiber of (M,π) is a flexible embedding in the standard position in N. The curves on the surface depict the vanishing cycles γ_i's.

Proof Let $c_1, c_2, \dots, c_k$ be k critical points of the Lefschetz fibration (M,π). Since the Lefschetz fibration π is injective when restricted to the set of critical points, points $\pi(c_1) = p_1, \pi(c_2) = p_2, \dots,$ and $\pi(c_k) = p_k$ are distinct points on Σ. Let γ_i be the vanishing cycle corresponding to the critical point c_i on a generic fiber Σ_g of the LF.

Let U_i be an open ball in M around c_i such that on U_i we have coordinates (z_1, z_2) such that π in this coordinates is given by $(z_1, z_2) \to z_1.z_2$. Let $\widetilde{D}_i = \pi(U_i) \subset \Sigma$. Let D_i be an open disc containing p_i with $\overline{D_i} \subset \widetilde{D}_i$.

Let $\widehat{\Sigma}$ be Σ in case $\Sigma = \mathbb{D}^2$ or $\widehat{\Sigma} = \mathbb{C}P^1 \setminus D$ where D is a small open disc in $\mathbb{C}P^1$ lying in the complement of the set $\{p_1, \dots, p_k\}$. We will first produce an embedding $\widehat{f}$ of the fibration π restricted to $\pi^{-1}(\widehat{\Sigma})$. Denote by $\widehat{M}$ the manifold $\pi^{-1}(\widehat{\Sigma})$. We have that $\widehat{M} = M$ when $\Sigma = \mathbb{D}^2$.

Embedding of $\widehat{M}$ Consider an embedding ϕ of the fiber Σ_g in N which is a standard embedding. Recall that the existence of such an embedding is the content of Lemma 4.5.

Using the flexibility of the embedding ϕ, we first produce $\widehat{f}$ restricted to $\widehat{M} \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i)$ in the manifold $N \times \left(\mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i\right)$ such that the following diagram commutes:

$$\begin{array}{ccc} M \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i) & \xrightarrow{\widehat{f}} & N \times \left(\mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i\right) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_2} \\ \mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i & \xrightarrow{\mathrm{Id}} & \mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i \end{array} \tag{1}$$

Since the embedding of Σ_g in N is standard, by Lemma 4.8, there exists a standard embedding $\Psi_i : \mathcal{M}T(\Sigma_g, d_{\gamma_i}) \to N \times \mathbb{S}^1$ for each $1 \le i \le k$ with respect to the standard embedding ϕ and the Dehn twist d_{γ_i}. Note that for each i, the embedding Ψ_i is such that the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{M}T(\Sigma_g, d_{\gamma_i}) \arrow[r, "\Psi_i"] \arrow[d, "\pi"] & N \times \mathbb{S}^1 \arrow[d, "\pi_2"] \\
\mathbb{S}^1 \arrow[r, "\mathrm{Id}"] & \mathbb{S}^1
\end{tikzcd} \tag{2}
$$

Next, considering $\partial D_i \subset \mathbb{C}P^1 = \mathbb{S}^1$ for each i, the embeddings Ψ_i's together give an embedding $\Psi : \bigsqcup_{i=1}^k \pi^{-1}(\partial D_i) \to \bigsqcup_{i=1}^k N \times \partial D_i$ such that $\pi_2 \circ \Psi = \pi$. Now for each i take an arc α_i connecting a point on ∂D_i to a fixed regular value p for the map π in $\widehat{\Sigma}$ as depicted in Figure 5. We can assume that $\widehat{\Sigma}$ is a regular neighborhood of the set $\bigsqcup D_i \cup \bigcup_i \alpha_i$. The flexibility of the embedding ϕ now implies that the embedding $\hat{f}$ restricted $\widehat{M} \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i)$ exists such that when restricted to $\partial\widehat{\Sigma}$ this embedding is an embedding of $\mathcal{M}T(\Sigma_g, F)$, where $F = \prod_{i=1}^k \tau_{\gamma_i}$ when $\widehat{\Sigma} = \Sigma$ and $F = \mathrm{id}$ when $\widehat{\Sigma} = \mathbb{C}P^1 \setminus D$.

Our next step is to show how to extend this embedding to produce a Lefschetz fibration embedding $\hat{f}$ of $\widehat{M}$ in $N \times \mathbb{C}P^1$. For this the property that the embedding ϕ of Σ_g is also in the standard position is required.

Since the embedding ϕ is in a standard position — by the definition of an embedding in a standard position given in Definition 4.2 — there exists an embedding of $\phi_{\gamma_i} : \mathbb{C}^2 \hookrightarrow N$ which satisfies the second property listed in Definition 4.2.

Next, for each critical point c_i, we claim that the diagram

$$
\begin{tikzcd}
U_i \subset M \arrow[r, "\phi_i"] \arrow[d, "\pi"] & \mathbb{C}^2 \arrow[r, "i"] \arrow[d, "g"] & \mathbb{C}^2 \times \mathbb{C} \arrow[r, "f_{c_i}"] \arrow[d, "P"] & N \times \mathbb{C}P^1 \arrow[d, "\pi_2"] \\
\widetilde{D}_i \arrow[r, "\psi_i"] & \mathbb{C} \arrow[r, "\mathrm{Id}"] & \mathbb{C} \arrow[r, "\psi_i^{-1}"] & \widetilde{D}_i
\end{tikzcd} \tag{3}
$$

commutes, where the definitions of the maps appearing in the diagram are as follows:

(1) $\phi_i : U_i \subset M \to \mathbb{C}^2$ and $\psi_i : \widetilde{D}_i \subset \mathbb{C}P^1 \to \mathbb{C}$ are orientation preserving parameterizations around critical point c_i of π and $\pi(c_i)$, respectively, such that left square commutes in the diagram above.

(2) $i : \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ are defined as $i(z_1, z_2) = (z_1, z_2, 0)$ and $g(z_1, z_2) = z_1.z_2$.

(3) $f_{c_i} : \mathbb{C}^2 \times \mathbb{C} \to N \times \mathbb{C}P^1$ and $P : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$ are defined as

$$f_{c_i}(z_1, z_2, z_3) = (\phi_{\gamma_i}(z_1, z_2), \quad \psi_i^{-1}(z_1.z_2 + z_3)), \quad P(z_1, z_2, z_3) = z_1.z_2 + z_3.$$

The commutativity of the middle square follows directly from the definitions of the maps g, i, and P. Also, the commutativity of the last square is clear by the definition of the map f_{c_i}. Next, we see that the commutative diagram (3) allows us to extend the embedding $\hat{f}$ to the embedding $\hat{f}_{c_i}$ of $\big(\widehat{M} \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i)\big) \cup U_i$. This is possible because $\hat{f}$ and $f_{c_i} \circ i \circ \phi_i$ agree on the overlapping region of the domain. Hence, $\hat{f}$ and $f_{c_i} \circ i \circ \phi_i$ together define a map $\hat{f}_{c_i}$.

Let us now notice that this allows us to extend the embedding $\widehat{f_{c_i}}$ to an embedding $\widehat{f_{c_i}}$ of the space $W_{c_i} = \widehat{M} \setminus \left(\bigcup_{l=1}^{i-1} \pi^{-1}(D_l) \cup \bigcup_{l=i+1}^{k} \pi^{-1}(D_l)\right)$ in $N \times \mathbb{C}P^1$ such that the following diagram commutes:

$$\begin{array}{ccc} W_{c_i} & \xrightarrow{\widehat{f_{c_i}}} & \widehat{f_{c_i}}(W_{c_i}) \subset N \times \mathbb{C}P^1 \\ \big\downarrow \pi & & \big\downarrow \pi_2 \\ \pi(W_{c_i}) \subset \mathbb{C}P^1 & \xrightarrow{\mathrm{Id}} & \pi_2(\widehat{f_{c_i}}(W_{c_i})) = \pi(W_{c_i}) \end{array} \tag{4}$$

Observe that by construction the embeddings $\widehat{f_{c_i}}$ and $\widehat{f_{c_j}}$ agree on $W_{c_i} \cap W_{c_j}$. Since $\widehat{M} = \bigcup_{i=1}^{k} W_{c_i}$ we get an embedding $\widehat{f}$ of $\widehat{M}$ with the required properties.

Embedding of M When $\Sigma = \mathbb{D}^2$ there is nothing to prove as in this case $\widehat{M} = M$. In the case when $\Sigma = \mathbb{C}P^1$, we recall that the embedding $\widehat{f}$ is constructed so that $\widehat{f}$ restricted to $\partial\widehat{M}$ is an embedding of $\Sigma_g \times \mathbb{S}^1$ in $N \times \mathbb{S}^1 = N \times \partial\widehat{\Sigma}$. When we regard the boundary $\partial N \times D$ as $N \times \mathbb{S}^1$, we get an embedding of $\mathcal{MT}(\Sigma_g, \mathrm{id})$ in $\mathcal{MT}(N, \mathrm{id}) = N \times \mathbb{S}^1$ via the embedding $\widehat{f}$. Hence, we get an embedding of a closed manifold $\widetilde{M}$ obtained by identifying $\partial\widehat{M}$ with $\partial\Sigma_g \times D$ along the common boundary via a diffeomorphism of $\Sigma_g \times \mathbb{S}^1$. Since the genus g of Σ_g is at least 2, it follows from the triviality of the group $\pi_1(\mathrm{Diff}_0(\Sigma_g))$ — the identity connected component of the group of diffeomorphisms of Σ_g — proved in [11, Theorem 1] that $\widetilde{M} = M$. Hence, we have the required embedding of M in $N \times \mathbb{C}P^1$. □

5 Embeddings of orientable 4-manifolds via SBLF

The purpose of this section is to establish Theorem 1.2. Recall that Theorem 1.2 claims that every closed orientable smooth 4-manifold admits an embedding in a manifold of type $N \times \mathbb{C}P^1$, where N is a 4-manifold admitting separable Hopf link. As mentioned in the introduction while outlining the proof, we will use the SBLF decomposition of a closed orientable smooth 4-manifold for constructing embeddings. We first need the following:

Definition 5.1 (1-fold simple singular fibration) Let $(M, \partial M)$ be an oriented smooth 4-manifold with boundary and let $f : M \to [-1, 1] \times \mathbb{S}^1$ be a smooth surjective map which satisfies the following:

(1) There exists a unique embedded circle Z_f in M of 1-fold singularities for f such that $f(Z_f)$ is an embedded circle in $[-1, 1] \times \mathbb{S}^1$ which is ambiently isotopic to the circle $\{0\} \times \mathbb{S}^1$.

(2) For every $x \in M \setminus Z_f$, $f(x)$ is a regular point for the map f.

(3) $\partial M = f^{-1}(\{-1\} \times \mathbb{S}^1 \sqcup \{1\} \times \mathbb{S}^1)$.

Then, we say that $f : M \to [-1, 1] \times \mathbb{S}^1$ is a 1*-fold simple singular fibration*.

Remark 5.2 (a) Since $f : M \to [-1, 1] \times \mathbb{S}^1$ has a unique embedded singular locus Z_f which projects to a circle C isotopic to $\{0\} \times \mathbb{S}^1$, the inverse image of any regular value is a closed surface Σ whose genus is either g or $g + 1$ for some $g \in \mathbb{N} \cup \{0\}$. We call a fiber with genus g as a lower genus fiber.

(b) Observe that as we cross the $f(Z_f)$, a round 1-handle is added to a manifold diffeomorphic to $\Sigma_g \times A$, where A is an annulus.

(c) We will always use the convention that fibers over $\{-1\} \times \mathbb{S}^1$ have lower genus.

Lemma 5.3 *Let $(M, \partial M)$ be an orientable smooth 4-manifold with boundary and $f : M \to [-1,1] \times \mathbb{S}^1$ be a 1-fold simple singular fibration. Let N be a 4-manifold which admits a separable Hopf link. Then, there exists an embedding $\psi : M \to N \times [-1,1] \times \mathbb{S}^1$ such that following properties are satisfied*:

(1) *The following diagram commutes*:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "f"] & N \times [-1,1] \times \mathbb{S}^1 \arrow[d, "\pi_2"] \\ {[-1,1] \times \mathbb{S}^1} \arrow[r, "\mathrm{Id}"] & {[-1,1] \times \mathbb{S}^1} \end{tikzcd} \tag{5}$$

(2) *Given a standard embedding ϕ of a surface of genus $g+1$ in N, we can ensure that ψ restricted to any higher genus fiber sends the fiber to a surface in N which is isotopic to the given embedding ϕ.*

Proof Let us define $M_0 = f^{-1}(\{-1\} \times \mathbb{S}^1)$ and $M_1 = f^{-1}(\{1\} \times \mathbb{S}^1)$. We know that $\partial M = M_0 \sqcup M_1$. Observe that M_1 is a mapping torus over $\mathbb{S}^1$ with fiber Σ_{g+1}. Recall that any mapping torus over $\mathbb{S}^1$ is determined by its monodromy — an element of $\mathcal{MCG}(\Sigma_g)$. Let ϕ be the monodromy for the fiber bundle M_1 over $\mathbb{S}^1$. Further, since $f : (M, \partial M) \to [-1,1] \times \mathbb{S}^1$ is a 1-fold simple singular fibration, we have the following: there exists a homologically nontrivial curve c in Σ_{g+1} which is mapped to itself by ϕ [5, p. 10895], and the boundary component M_0 is obtained from M_1 by the following procedure:

First cut Σ_{g+1} along c, and attach to the resulting surface a pair of discs — say D_1 and D_2. Now form the mapping torus of the resulting surface Σ_g with monodromy the map ϕ restricted to Σ_g.

This also implies that we can obtain $(M, \partial M)$ by suitably adding a round 1-handle to $\Sigma_g \times \mathbb{S}^1$ along a pair of points in Σ_g times $\mathbb{S}^1$ such that each disc $D_i \times \mathbb{S}^1$ contains a circle of the round attaching sphere.

Now, let $i : \Sigma_{g+1} \subset N$ be a standard embedding of Σ_{g+1} in N. Since the embedding is standard, we know every simple closed curve γ on Σ_{g+1} bounds a disc D in N such that the intersection of this disc with N is γ. Furthermore, recall that any simple closed curve in a standard embedding of Σ_{g+1} can be assumed to be disjoint from the separable Hopf link, and the pair of disjoint discs that the link bounds. This implies that there exist a 4-ball B^4 containing the disc D such that $\Sigma_{g+1} \cap B^4$ is an annulus A and ∂A is a pair of unlinked unknots in ∂B^4 (see Remark 4.6). We call this link $L = L_1 \sqcup L_2$.

Since the embedding is standard, from Lemma 4.8 it follows that there exist a fiber preserving embedding of M_1 in $N \times \{1\} \times \mathbb{S}^1$. Since ϕ sends c to itself $\phi(c) = \pm c$. Since the curve c bounds disc in Σ_g, without loss of generality we can assume that $\phi(c) = c$.

We know that the embedding of a surface Σ_g obtained by cutting Σ_{g+1} along a curve $\hat{c}$ obtained by pushing c slightly away from itself in a small tubular neighborhood of c agrees with Σ_{g+1} everywhere except in a ball B^4 satisfying the property that $B^4 \cap \Sigma_{g+1}$ is fixed annulus having boundary a pair of unknot. Since the ball B^4 is disjoint from the separable Hopf link and the pair of disjoint discs that the

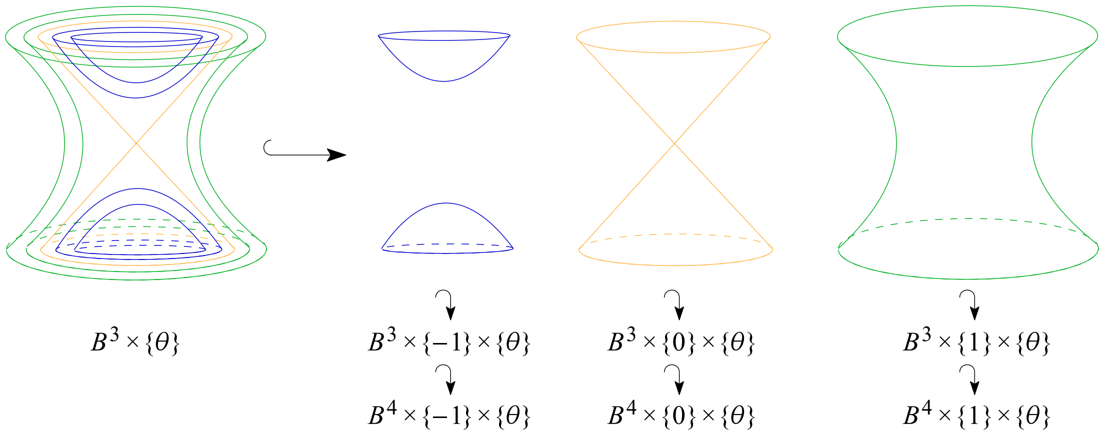

Figure 6: Simple Lefschetz fibration embedding.

link bounds, we get that the embedding of Σ_g given by cutting Σ_{g+1} is also standard. Hence, applying Lemma 4.8, we get an embedding of M_0 in $N \times \{-1\} \times \mathbb{S}^1$ which is also fiber preserving.

Observe that by very construction the embedding of $\partial M = M_0 \sqcup M_1$ can be extended to an embedding $\widehat{\psi}$ of $(M, \partial M) \setminus \mathcal{N}$ in $N \setminus B^4 \times [-1,1] \times \mathbb{S}^1$, where $\mathcal{N}$ is a neighborhood of 1-fold singularity. Furthermore, we can assume that the following diagram commutes:

$$\begin{tikzcd} M \setminus \mathcal{N} \arrow[r, "\widehat{\psi}"] \arrow[d, "f"] & N \setminus B^4 \times [-1,1] \times \mathbb{S}^1 \arrow[d, "\pi_2"] \\ {[-1,1] \times \mathbb{S}^1} \arrow[r, "\mathrm{Id}"] & {[-1,1] \times \mathbb{S}^1} \end{tikzcd} \tag{6}$$

Hence, in order to establish the lemma, we need to extend the embedding constructed so far in the region $\mathcal{N}$. We can assume that $\mathcal{N}$ is a tubular neighborhood of the 1-fold critical locus, and hence can be identified with $B^3 \times \mathbb{S}^1$.

Let (x, y, z, θ) be coordinates on a tubular neighborhood $\mathcal{N} = B^3 \times \mathbb{S}^1$ of the singular locus Z_f of f such that f sends (x, y, z, θ) to $(-x^2 + y^2 + z^2, \theta)$. Let us embed $B^3 \times \mathbb{S}^1$ in $B^4(0,1) \times [-1,1] \times \mathbb{S}^1$. The embedding $\widehat{\psi_1} : B^3 \times \mathbb{S}^1 \to B^4(0,1) \times [-1,1] \times \mathbb{S}^1$ is defined as

$$\widehat{\psi_1}(x, y, z, \theta) = (x, y, z, 0, -x^2 + y^2 + z^2, \theta).$$

For a pictorial description of the embedding $\widehat{\psi_1}$, see Figure 6. We can see $\widehat{\psi_1}$ is defined such that following diagram commutes:

$$\begin{tikzcd} B^3 \times \mathbb{S}^1 \subset M \arrow[r, "\widehat{\psi_1}"] \arrow[d, "f"] & B^4(0,1) \times [-1,1] \times \mathbb{S}^1 \subset N \times [-1,1] \times \mathbb{S}^1 \arrow[d, "\pi_2"] \\ {[-1,1] \times \mathbb{S}^1} \arrow[r, "\mathrm{Id}"] & {[-1,1] \times \mathbb{S}^1} \end{tikzcd} \tag{7}$$

Observe that the embedding $\widehat{\psi}_1$ has the property that for each (t,θ), the intersection of $\widehat{\psi}_1(f^{-1}(t,\theta))$ with $\partial B^4 \times \{(t,\theta)\}$ is a pair of unlinked unknots. Hence perturbing this embedding if necessary, it is possible to ensure that this pair is the pair $L = L_1 \sqcup L_2$ for each (t,θ).

Observe that for any $t < 0$ the embedding of $\Sigma_g \cap B^4 \times \{(t,\theta)\}$ produced by $\widehat{\psi}$ and the embedding of pair of disc bounding the unlink $L_1 \sqcup L_2$ produced by $\widehat{\psi}_1$ differ only up to bounding discs of each unknot L_i. Hence up to an isotopy, both embeddings agree. Similarly, for $t > 0$ embeddings $\widehat{\psi}$ and $\widehat{\psi}_1$ differ only up to annuli that the unlink $L = L_1 \sqcup L_2$ bound. Hence we can isotope further to ensure that for $t > 0$ they also agree. This implies that by perturbing the embedding $\widehat{\psi}$ we can assume that both embeddings agree near the boundary to produce an embedding ψ of M in $N \times [-1,1] \times \mathbb{S}^1$.

Clearly, ψ is the required embedding. This shows that we can produce an embedding of $(M, \partial M)$ in N satisfying the property (2). Since there always exists a standard embedding of Σ_{g+1}, the lemma follows. □

Let us now establish Theorem 1.2.

Proof of Theorem 1.2 Let M be a closed-oriented 4-manifold. By Theorem 2.11 there exists a smooth map $f : M \to \mathbb{C}P^1$ which defines SBLF such that the lower genus fiber Σ_g of f has genus bigger than 1.

Therefore, we can write $\mathbb{C}P^1 = D_1 \cup \mathcal{A} \cup D_2$, where D_1 is an embedded disc in $\mathbb{C}P^1$ containing all Lefschetz critical values of f, $\mathcal{A} = [-1,1] \times \mathbb{S}^1$ is an embedded annulus in $\mathbb{C}P^1$ with $\{0\} \times \mathbb{S}^1$ as the embedded image of 1-fold singularities of f, and D_2 is an embedded disc in $\mathbb{C}P^1$ containing no critical values of f such that $\partial D_1 = \{1\} \times \mathbb{S}^1$ and $\partial D_2 = \{-1\} \times \mathbb{S}^1$.

Since lower genus fiber has genus at least 2, we have a decomposition of M, $M = X_1 \sqcup X_2 \sqcup \Sigma_g \times D_2$ due to [11, Theorem 1] which satisfy the following properties:

(1) $f|_{X_1} : X_1 = f^{-1}(D_1) \to D_1$ is a Lefschetz fibration.

(2) $f|_{X_2} : X_1 = f^{-1}(\mathcal{A}) \to [-1,1] \times \mathbb{S}^1$ is a 1-fold simple singular fibration.

(3) $\Sigma_g \times D_2 = f^{-1}(D_2)$.

(4) Identifications along the boundaries of adjacent regions are always done by the identity map.

It follows from Theorem 4.11, and Lemma 5.3, that each piece of M embeds in $N \times \mathbb{C}P^1$. Also, it is clear from the second property listed in the statement of Lemma 5.3 that embeddings of each piece can be arranged such that in the overlapping region they agree. This clearly implies that we have an embedding of M in $N \times \mathbb{C}P^1$ as claimed. □

Remark 5.4 (a) The embedding $\psi : M \to N \times \mathbb{C}P^1$ produced in Theorem 1.2 satisfies $\psi \circ \pi_2 = f$, where $f : M \to \mathbb{C}P^1$ is an SBLF associated to M and $\pi_2 : N \times \mathbb{C}P^1 \to \mathbb{C}P^1$ is the projection onto the second factor of $N \times \mathbb{C}P^1$. In this case, the embedding ψ is called an *SBLF embedding*.

(b) In general, given a fiber bundle $\pi : X^6 \to \mathbb{C}P^1$ and an SBLF $f : M^4 \to \mathbb{C}P^1$, an embedding $\Psi : M^4 \to X^6$ is called an SBLF embedding if $\pi \circ \Psi = f$.

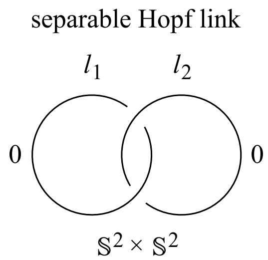

Figure 7: This figure depicts the Kirby diagram of $\mathbb{S}^2 \times \mathbb{S}^2$. Observe that attaching circles of 2-handles form a Hopf link in the boundary of the unique 0-handle, and they bound disjoint discs corresponding to attaching discs in $\mathbb{S}^2 \times \mathbb{S}^2$.

6 Embeddings in $\mathbb{R}^7$

In this section, we give a new proof of the fact that every closed smooth orientable 4-manifold admits a smooth embedding in $\mathbb{R}^7$.

Theorem 6.1 *Every closed orientable 4-manifold admits a smooth embedding in $\mathbb{R}^7$.*

Proof Consider the 4-manifold $\mathbb{S}^2 \times \mathbb{S}^2$. We observe that $\mathbb{S}^2 \times \mathbb{S}^2$ admits a separable Hopf link. This is because $\mathbb{S}^2 \times \mathbb{S}^2$ admits a handle decomposition consisting of a unique 0-handle H_0 on which a pair of two 2-handles are attached such that the attaching circles form a Hopf link in ∂H_0. For a pictorial description of this handle decomposition, we refer to Figure 7, where we have presented a Kirby diagram of $\mathbb{S}^2 \times \mathbb{S}^2$. This clearly implies that the Hopf link consisting of the pair of attaching circles is a separable Hopf link. Thus by Theorem 1.2, every 4-manifold embeds in $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{C}P^1 = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. Now as $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ embeds in $\mathbb{R}^7$, we get the required embedding of M in $\mathbb{R}^7$. □

7 Embeddings in $\mathbb{C}P^3$

Let us now establish Theorem 1.1. As mentioned in the introduction, the first step of the proof involves the construction of a specific SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. We then use this SBLF to produce an embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ of $\mathbb{C}P^3$ along $\mathbb{C}P^1$. Furthermore, we show that this embedding can be constructed such that when we blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we get an embedding of M in $\mathbb{C}P^3$. We begin by reviewing notions related to blow-up and blow-down.

7.1 Generalized Lefschetz pencil

Definition 7.1 (generalized Lefschetz pencil) Let M be an oriented smooth 4-manifold. A *generalized Lefschetz pencil* associated to M is a map $\pi : M \setminus B \to \mathbb{C}P^1$ such that the following properties are satisfied:

(1) B is finite.

(2) $\pi : M \setminus B \to \mathbb{C}P^1$ is a Lefschetz fibration.

(3) For every point $b \in B$ there is a parameterization — *not necessarily preserving orientations* — $\phi : U \subset M \to \mathbb{C}^2$ that satisfies the following:

(a) $b \in U$ and $\phi(b) = 0 \in \mathbb{C}^2$.

(b) For the map $g : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ given by $g(z_1, z_2) = [z_1 : z_2]$, the following diagram commutes:

$$\begin{tikzcd} U \setminus \{b\} \arrow[r, "\phi"] \arrow[d, "\pi"] & \mathbb{C}^2 \setminus \{0\} \arrow[d, "g"] \\ \mathbb{C}P^1 \arrow[r, "\mathrm{Id}"] & \mathbb{C}P^1 \end{tikzcd} \tag{8}$$

In this case, we call B the base locus of a generalized Lefschetz pencil associated with M.

Remark 7.2 (a) We would like to emphasize that the notion of generalized Lefschetz pencil defined above is weaker than the notion of Lefschetz pencil. Generally one demands that M and $\mathbb{C}P^1$ are oriented and the parameterization $\phi : U \subset M \to \mathbb{C}^2$ is orientation preserving in Definition 7.1.

(b) If a fibration $\pi : M \setminus B \to \mathbb{C}P^1$ is a simplified broken Lefschetz fibration, then we say that the map π is a *generalized simplified broken Lefschetz pencil* (generalized SBLP in short) of M.

(c) If the fibration $\pi : M \setminus B \to \mathbb{C}P^1$ is a simplified broken Lefschetz fibration and the parameterization $\phi : U \subset M \to \mathbb{C}^2$ is orientation preserving, then the map π is called a *simplified broken Lefschetz pencil* (SBLP).

7.2 Topological blow-up and blow-down of 4-manifolds

We begin by recalling a few standard facts from [14] about the tautological line bundle over $\mathbb{C}P^1$ and the bundle (complex) dual to this bundle.

Consider the tautological line bundle $\tau_{\mathbb{C}P^1}$ over $\mathbb{C}P^1$, and the bundle $\tau^*_{\mathbb{C}P^1}$ dual to the bundle $\tau_{\mathbb{C}P^1}$. Let $\mathcal{Z}_{\ll}$ denote the zero section of the bundle $\tau_{\mathbb{C}P^1}$, and $\mathcal{Z}_{\tau^*}$ denote the zero section of the bundle $\tau^*_{\mathbb{C}P^1}$.

We know that $\tau_{\mathbb{C}P^1} \setminus \mathcal{Z}_\tau$ and $\tau^*_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\tau^*}$ are diffeomorphic to $\mathbb{R}^4 \setminus \{0\}$ by diffeomorphisms coming from the restrictions of the projection of the second factor for the corresponding bundles. We fix this identification of the complement of zero sections with $\mathbb{R}^4 \setminus \{0\}$ for both of these bundles.

Definition 7.3 (topological blow-up) Let M a smooth 4-manifolds. Let p be a point in M. Let U be a neighborhood of p diffeomorphic to $\mathbb{R}^4$ via a diffeomorphism which sends p to $0 \in \mathbb{R}^4$. The manifold $\widehat{M}$ obtained by removing p from U and identifying $U \setminus \{p\}$ with either $\tau^*_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\ll^*}$ or with $\tau_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\ll}$ is called a topological blow-up of M along p.

Remark 7.4 (a) The operation of topological blow-up of a manifold along a point corresponds to its connected sum with $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$. While performing a topological blow-up, if we use the tautological line bundle $\tau_{\mathbb{C}P^1}$, then we get $M \# \overline{\mathbb{C}P^2}$. On the other hand, if we use the dual bundle to $\tau_{\mathbb{C}P^1}$, then we get $M \# \mathbb{C}P^2$.

(b) Topological blow-up of M along p produces a manifold $\widehat{M}$ admitting an embedded $\mathbb{C}P^1$ with self intersection number ± 1. Recall that the usual blow-up always produces an embedded $\mathbb{C}P^1$ with self intersection -1.

(c) Throughout this discussion, an embedded $\mathbb{C}P^1$ in a 4-manifold M with self intersection number ± 1 will be called an *exceptional sphere* in M.

Definition 7.5 (topological blow-down) Let $\widehat{M}$ be a smooth 4-manifold admitting an embedded $\mathbb{C}P^1$ whose normal bundle is isomorphic to $\tau_{\mathbb{C}P^1}$ or $\tau^*_{\mathbb{C}P^1}$. That is the embedded $\mathbb{C}P^1$ is an exceptional sphere in $\widehat{M}$. In this case, we can carry out the process exactly opposite of the one described in the definition of blow-up, where we remove a tubular neighborhood of $\mathbb{C}P^1$ and replace it with a 4-ball. The resulting manifold M that we obtain as a result of this process is called a topological blow-down of $\widehat{M}$.

Remark 7.6 (1) Observe that given a manifold M admitting an embedding of $\mathbb{C}P^1$ with its self intersection number ± 1, we can perform topological blow-down operation.

(2) Suppose we are given a manifold $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Let E_1 and E_{-1} be two embedded $\mathbb{C}P^1$'s corresponding to zero sections of $\tau^*_{\mathbb{C}P^1}$ and $\tau_{\mathbb{C}P^1}$, respectively. Suppose $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ is an SBLF such that the intersection number of each fiber with E_1 is 1, and the intersection number of each fiber with E_{-1} is -1. Then the two operations of blow-downs corresponding to removal of E_1 and E_{-1} produces a generalized SBLP on M. This is because the SBLF restricted to a tubular neighborhood of E_1 is isomorphic $\tau^*_{\mathbb{C}P^1}$, while a tubular neighborhood of E_{-1} is isomorphic to $\tau_{\mathbb{C}P^1}$.

7.3 Construction of SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

The purpose of this subsection is to establish an SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which satisfies the property that the intersection of each fiber with two exceptional spheres E_1 and E_{-1} corresponding to zero sections is $+1$ and -1, respectively.

Lemma 7.7 *Consider a closed oriented smooth 4-manifold $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Then, there exists an SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ which satisfies the following:*

(1) *The lower genus fiber has its genus bigger than 1.*

(2) *The fibration agrees with the standard fibration in a tubular neighborhood of both exceptional spheres E_1 and E_{-1}.*

In particular, blowing down the SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ produces a generalized SBLP on M.

In [6, Theorem 6.5], R. I. Baykur and O. Saeki established the existence of a simplified broken Lefschetz pencil for any near symplectic manifold admitting connected singular locus for near symplectic structure. It is easy to see that following the proof of [6, Theorem 6.5] — essentially verbatim — provides a proof of Lemma 7.7.

Proof To begin with, notice that there exists an embedded surface Σ in $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which satisfies the following properties:

- The self intersection of Σ is 0.
- $\Sigma \cap E_1 = +1$ and $\Sigma \cap E_{-1} = -1$.
- Σ is connected and the genus of Σ is bigger than three.

Observe that since the self-intersection number of E_1 is $+1$ and the self-intersection number of E_{-1} is -1, it is easy to construct a disconnected surface consisting of disjoint union of two spheres. By making connected sums of these two spheres with an embedded surface bounding a 3-dimensional handlebody and embedded in B^4, it is easy to construct such a surface.

Consider the map $\pi : \Sigma \times \mathbb{D}^2 \to \mathbb{D}^2$, corresponding to the projection on the second factor, and regard $\mathbb{D}^2$ as embedded in $\mathbb{C}P^1$ as a southern hemisphere. This allows us to regard π as a map from a tubular neighborhood $\mathcal{N}(\Sigma)$ of Σ to southern hemisphere. Construct a map $g : \mathcal{N}(\Sigma) \cup \mathcal{N}(E_1) \cup \mathcal{N}(E_{-1}) \to \mathbb{C}P^1$ which satisfies the following:

(1) The map g when restricted to $\mathcal{N}(E_1)$ and $\mathcal{N}(E_{-1})$ is the surjection on $\mathbb{C}P^1$ coming from the bundle projections $\pi_{E_1} : \mathcal{N}(E_1) \to E_1$ and $\pi_{E_{-1}} : \mathcal{N}(E_{-1}) \to E_{-1}$.

(2) The map g agrees with π when restricted to $\mathcal{N}(\Sigma)$.

Next, extend the map g to a generic smooth map $\hat{f} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$. According to [6, Remark 4.5], this map can be modified to produce an SBLF $\hat{f} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ such that all the modifications performed while obtaining the SBLF from g are performed away from the region where g is defined.

Next, we convert the SBLF $\hat{f} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ to an SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ whose lower genus fiber is bigger than 2 by applying a technique similar to the one which provides a proof of Theorem 2.11 or a proof of [3, Proposition 1.3]. The SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ can be ensured to satisfy the required properties because every fiber of f is homologous to the original fiber Σ and hence the intersection of fibers of f has same property that Σ had. □

Let us end this section with a convention: from now on the SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ described in the statement of Lemma 7.7 will be denoted by $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$.

7.4 Blow-up and blow-down of $\mathbb{C}P^3$ along $\mathbb{C}P^1$

Let us begin this subsection by making a convention. By a standard $\mathbb{C}P^1$ in $\overline{\mathbb{C}P^2}$, we mean a $\mathbb{C}P^1$ embedded in $\overline{\mathbb{C}P^2}$ with its normal bundle isomorphic to the dual of the tautological line bundle over $\mathbb{C}P^1$. On the other hand, by a standard $\mathbb{C}P^1$ in $\mathbb{C}P^n$, we mean $\{[z_0, z_1, \dots, z_n] \mid z_i = 0 \text{ for all } i \geq 2\}$, where $[z_0, z_1, \dots, z_n]$ denotes the homogeneous coordinates of $\mathbb{C}P^n$.

Consider $\mathbb{C}P^3$ and a standard $\mathbb{C}P^1$ embedded in it. Fix a local trivialization $\mathbb{D}^2 \times \mathbb{C}^2$ of the normal bundle $\mathcal{N}(\mathbb{C}P^1)$ of $\mathbb{C}P^1$ in $\mathbb{C}P^3$. Now consider $\mathbb{D}^2 \times \mathbb{C}P^1 \times \mathbb{C}^2$ and a subset V of $\mathbb{D}^2 \times \mathbb{C}P^1 \times \mathbb{C}^2$ given by

$$V = \{(w, l, z_1, z_2,) \mid \|z_1^2\| + \|z_2^2\| \leq 1 \text{ and } (z_1, z_2) \in l\},$$

where a point l in $\mathbb{C}P^1$ is identified with the complex linear subspace corresponding to that point.

Now, observe that the complement of $\mathbb{D}^2 \times \mathbb{C}P^1 \times \{(0,0)\}$ in V can be identified with the complement of $\mathbb{D}^2 \times \{(0,0)\}$ in $\mathbb{D}^2 \times \mathbb{C}^2$.

Choose two local trivializations $U_1 \times \mathbb{C}^2$ and $U_2 \times \mathbb{C}^2$ over open sets U_1 and U_2 such that U_1 and U_2 cover $\mathbb{C}P^1$. By the (topological) blow-up of $\mathbb{C}P^3$ along $\mathbb{C}P^1$ we mean the operation of removing $U_i \times \{(0,0)\}$ from $U_i \times \mathbb{C}^2$, for each i, and replacing it with the interior of V as discussed in the previous paragraph.

Remark 7.8 (1) An *exceptional divisor* of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ is the union of $\mathbb{D}^2 \times \mathbb{C}P^1 \times \{(0,0)\}$ over a finite collection V_s of trivializations of the bundle $\mathcal{N}(\mathbb{C}P^1)$. Again notice that the triviality of the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^3$ implies that the exceptional divisor is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

The notion of blow-up discussed above is a particular case of blow-up of a manifold along a submanifold. We refer [15, pp. 196 and 602] for a detailed discussion on blow-ups.

By a blow-down of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ we will mean the process exactly opposite to the process of blow-up. More precisely, let $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ be obtained by blowing up a $\mathbb{C}P^1$. Let E be the exceptional divisor obtained as a result of the blow-up. By blow-down of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we mean removal of a tubular neighborhood of E and replacing it by a tubular neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^3$.

We say that $\mathbb{C}P^3$ is obtained from $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ by blowing down along E. Since E is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, we sometimes do not distinguish between E and $\mathbb{C}P^1 \times \mathbb{C}P^1$ and say that $\mathbb{C}P^3$ is obtained from $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ by blowing down along $\mathbb{C}P^1 \times \mathbb{C}P^1$.

We end this subsection with the following:

Lemma 7.9 *Let $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ be a closed oriented smooth manifold. Let $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ be an SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ as in the statement of Lemma 7.7. If there exists an SBLF embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that each fiber of SBLF intersects the standard $\mathbb{C}P^1$ of the fiber $\mathbb{C}P^2$ of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ in two distinct but fixed algebraically canceling points, then there exists an embedding of M in $\mathbb{C}P^3$ such that the standard pencil of $\mathbb{C}P^3$ induces the generalized SBLP of M corresponding to the SBLF of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.*

Proof Let E_1 and E_{-1} be two exceptional divisors of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Recall the exceptional divisor of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ consists of the union of two local exceptional divisors of the type $U_i \times W$, where $W \subset \mathbb{C}P^1 \times \mathbb{C}^2$ consists of $\{(l, z_1, z_2) \mid (z_1, z_2) \in l\}$. Since by hypothesis the fiber of π_{spl} intersects the standard $\mathbb{C}P^1$ inside $\mathbb{C}P^2$ in a pair of fixed points, we can assume that the tubular neighborhoods of exceptional divisors $E_{\pm 1}$ are contained in $U_i \times W$, and since the embedding is fiber preserving it consists of $\{p_\pm\} \times W \subset U_1 \times W$.

Furthermore, by the definition of the blow-up, the fibration on $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ restricted to $U_1 \times W$ can be assumed to be given by $(u, l, z_1, z_2) \to l$. This clearly implies that when we blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ we get M embedded in $\mathbb{C}P^3$ with standard pencil of $\mathbb{C}P^3$ inducing the generalized SBLP on M associated to SBLF $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$. □

7.5 Embeddings in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$

In this subsection, we establish SBLF embedding of the special SBLF $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Proposition 7.10 *Let M be a closed-oriented smooth 4-manifold. Consider $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and let $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ be a special SBLF with the lower genus fiber having genus bigger than 1. There exists an SBLF embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that each fiber of SBLF intersects the standard $\mathbb{C}P^1$ in the fiber $\mathbb{C}P^2$ in a pair of canceling intersection points.*

Proof We will follow the line of argument we used to establish Theorem 1.2. Let us denote by π the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ obtained via blow-up of the standard pencil of $\mathbb{C}P^3$. We first consider neighborhoods of exceptional divisors E_1 and E_{-1} of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and embed them in a tubular neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that the embedding is fiber preserving. In order to produce this embedding recall that a tubular neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ is the union of two open sets $U_i \times W$, $i = 1, 2$.

Now consider a pair of points p_+, p_- in U_1, and consider spheres $\{p_\pm\} \times \mathbb{C}P^1$ embedded in $U_1 \times W$. Since tubular neighborhood of $E_{\pm 1}$ is isomorphic to tubular neighborhood of any sphere in $U_1 \times W$ of the form $\{p\} \times \mathbb{C}P^1$, where p is a point in U_1, we get that there exists an embedding of small neighborhoods of $E_{\pm 1}$ in a neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ such that π_{spl} restricted to this neighborhood agrees with restriction of π on the embedded neighborhoods.

Observe that the intersection of the embedded neighborhoods of $E_{\pm 1}$ with a fiber of the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ is a pair of discs satisfying the property that the intersection of this pair of discs with the boundary of a small tubular neighborhood of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ is a Hopf link. Furthermore, observe that since the embedding of the neighborhood of E_{-1} as a tubular neighborhood of $\{p_+\} \times \mathbb{C}P^1$ is orientation reversing, and the embedding of neighborhood of E_{+1} as a tubular neighborhood $\{p_-\} \times \mathbb{C}P^1$ is orientation preserving. This implies that if we establish that

1. $\mathbb{C}P^2$ admits a separable Hopf link,
2. there exists an embedding of any surface of genus g in $\mathbb{C}P^2$ which is standard embedding,
3. the embedded surface Σ_g intersects the standard $\mathbb{C}P^1$ contained in $\mathbb{C}P^2$ in a pair of algebraically canceling points, and $\Sigma_g \cap \partial\mathcal{N}(\mathbb{C}P^1)$ is a Hopf link in $\partial\mathcal{N}(\mathbb{C}P^1)$, where $\mathcal{N}(\mathbb{C}P^1)$ is a fixed open tubular neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^2$,

then the triviality of the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ in the complement of the exceptional divisor implies that an argument similar to the one which established Theorem 1.2 produces the required SBLF embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Hence, the task at hand is to establish an embedding of a surface satisfying the three properties listed above. To this end, we observe that it is relatively easy to verify property (1) and get an embedding satisfying property (2). In fact, in section 4 we have already shown how to achieve these for various 4-manifolds. Hence our main focus will be on proving property (3), however, for the sake of completeness,

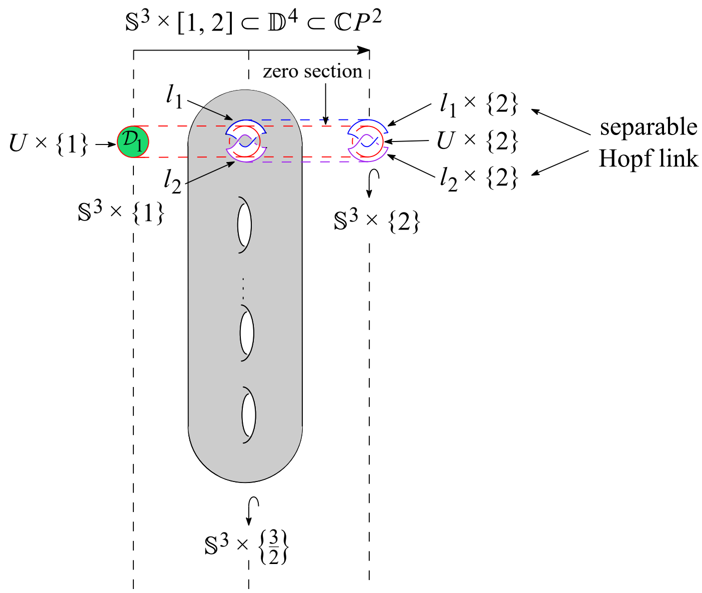

Figure 8: This figure depicts an embedded surface in $\mathbb{C}P^2$ which is flexible and in a standard position. The diagram focus on a collar $\mathbb{S}^3 \times [1,2]$ of a 4-ball $\mathbb{D}^4$ regarded as the unique zero handle H_0 of $\mathbb{C}P^2$. The circle U is the attaching circle of the unique 2-handle H_2. $U \times [1,2]$ with the core disc attached at $U \times \{2\}$ and the green disc at $U \times \{1\}$ forms the standard $\mathbb{C}P^1$ embedded in $\mathbb{C}P^2$.

we will again describe how $\mathbb{C}P^2$ satisfies property (1). After showing this we will discuss how a modification of embeddings of Σ_g discussed in section 4 produces a required embedding satisfying the remaining two properties.

To begin with, consider a handle decomposition of $\mathbb{C}P^2$ with the 0-handle H_0 corresponding to $B^4(0,2)$ — the 4-ball of radius 2 in $\mathbb{C}^2$ with its center at the origin — to which a 2-handle H_2 is attached along an unknot U with framing $+1$. Finally a 4-handle H_4 is attached to the 4-manifold, which is the union of the 0-handle $B^4(0,2)$ and the 2-handle H_2. Regarding H_0 as a ball, let $\mathbb{S}^3 \times [1,2]$ be a collar of ∂H_0. Let $U \times \{2\}$ be the attaching circle of H_2. Observe that any Hopf link consisting of a parallel copy of the attaching circle — say $l_1 \times \{2\}$ and a circle $l_2 \times \{2\}$ which links both the attaching circle and l_1 once as depicted in Figure 8 constitute a Hopf link that is separable. This is because $l_1 \times \{2\}$ bounds a parallel copy of the core of the 2-handle, and $l_2 \times \{2\}$ bounds a disc in the unique 4-handle.

Next, consider cylinders $l_i \times \left[\frac{3}{2}, 2\right]$, $i = 1, 2$. They intersect $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ in $l_i \times \left\{\frac{3}{2}\right\}$. Observe that there exists a surface Σ_g with two boundary components whose boundary is the Hopf link $l_1 \times \left\{\frac{3}{2}\right\} \sqcup l_2 \times \left\{\frac{3}{2}\right\}$; see Figure 8. It follows from an argument similar to the one used in establishing Lemma 4.5 that the embedding is both flexible and in a standard position.

Regarding the standard $\mathbb{C}P^1$ as the union of the core of 2-handle H_2 with a disc $\mathbb{D}$ that $U \times \{2\}$ bounds, we see that the embedded Σ_g intersects $\mathbb{C}P^1$ in a pair of points. This pair has to be algebraically canceling as we can push the disc $\mathbb{D}$ down to produce an isotopy of $\mathbb{C}P^1$ that sends the $\mathbb{C}P^1$ to a new $\mathbb{C}P^1$ which consists of union of core of H_2, $U \times [1,2]$, and a disc $\mathbb{D}$ that $U \times \{1\}$ bounds. The disc that $U \times \{1\}$ bounds is denoted by a green disc in Figure 8. Notice that the isotoped $\mathbb{C}P^1$ is disjoint from Σ_g implying that the algebraic intersection of Σ_g with the standard $\mathbb{C}P^1$ is zero. □

Now we have established all the results necessary to establish Theorem 1.1.

7.6 Proof of Theorem 1.1

We need to prove that every smooth orientable closed 4-manifold admits an embedding in $\mathbb{C}P^3$.

Proof of Theorem 1.1 Let M be the given closed orientable 4-manifold. Consider the manifold $\widehat{M} = M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ thought of as a blow-up of M done at two distinct points p_1 and p_2. Recall that $\widehat{M}$ admits a pair of exceptional divisors — say E_1 and E_{-1} such that $E_1 \cap E_1 = 1$ while $E_{-1} \cap E_{-1} = -1$.

Next, apply Lemma 7.7 to produce the special SBLF $\pi_{\mathrm{spl}} : \widehat{M} \to \mathbb{C}P^1$ on $\mathbb{C}P^1$. Recall that this SBLF satisfies the following:

(1) The lower genus fiber has its genus bigger than 1.

(2) The fibration agrees with the standard fibration in a tubular neighborhood of both exceptional spheres E_1 and E_{-1}.

Now, by Proposition 7.10 there exists SBLF embedding of $\widehat{M}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Also, notice that the intersection property of the embedded fiber of SBLF with standard $\mathbb{C}P^1$ contained in $\mathbb{C}P^2$ stated in Proposition 7.10 implies that the embedding is such that each fiber of the SBLF associated to $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ intersects the standard $\mathbb{C}P^1$ of a fiber $\mathbb{C}P^2$ of the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ in a pair of algebraically canceling points.

Finally, blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along its exceptional divisor. Observe that Lemma 7.9 implies that blow-down produces an embedding of M in $\mathbb{C}P^3$ such that the standard Lefschetz pencil of $\mathbb{C}P^3$ induces an SBLP on M. □

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Received: June 13, 2021 Revised: April 14, 2025

Algebraic & Geometric Topology **26**:4 (2026) 1321–1384
DOI: 10.2140/agt.2026.26.1321
Published: April 25, 2026

Recollements and stratification

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We develop various aspects of the theory of recollements of ∞-categories, including a symmetric monoidal refinement of the theory. Our main result establishes a formula for the gluing functor of a recollement on the right-lax limit of a locally cocartesian fibration determined by a sieve-cosieve decomposition of the base. As an application, we prove a reconstruction theorem for sheaves in an ∞-topos stratified over a finite poset P in the sense of Barwick, Glasman, and Haine. Combining our theorem with methods of Ayala, Mazel-Gee, and Rozenblyum, we then prove a conjecture of Barwick, Glasman, and Haine that asserts an equivalence between the ∞-category of P-stratified ∞-topoi and that of toposic locally cocartesian fibrations over P^{op}.

1 Introduction

The theory of recollements plays an important and ubiquitous role throughout topology, algebraic geometry, and representation theory. It is a common axiomatization of, on the one hand, the adjunctions

$$\begin{tikzcd} \mathbf{Shv}(U) \arrow[r, hook, shift left=2, "j_!"] \arrow[r, hook, shift right=2, "j_*"'] & \mathbf{Shv}(X) \arrow[l, "j^*" description] \arrow[r, shift left, "i^*"] & \mathbf{Shv}(Z) \arrow[l, hook', shift left, "i_*"] \end{tikzcd}$$

associated to ∞-categories of sheaves of spaces on a topological space X decomposed by an open subspace $j : U \hookrightarrow X$ and its closed complement $i : Z = X \setminus U \hookrightarrow X$, and, on the other hand, the adjunctions

$$\begin{tikzcd} \mathbf{QCoh}_Z(X) \arrow[r, hook, shift left=2, "j_!"] \arrow[r, hook, shift right=2, "j_*"'] & \mathbf{QCoh}(X) \arrow[l, "j^*" description] \arrow[r, shift left=2, "i^*"] \arrow[r, shift right=2, "i^!"'] & \mathbf{QCoh}(U) \arrow[l, hook', "i_*" description] \end{tikzcd}$$

associated to stable ∞-categories of quasicoherent complexes on a qcqs scheme X with open subscheme $i : U \hookrightarrow X$, where $\mathbf{QCoh}_Z(X)$ denotes those quasicoherent complexes set-theoretically supported on $Z = X \setminus U$. The fully faithful left adjoint $j_!$ is the definitional embedding of $\mathbf{QCoh}_Z(X)$ in $\mathbf{QCoh}(X)$, whereas the fully faithful right adjoint j_* embeds $\mathbf{QCoh}_Z(X)$ as $\mathbf{QCoh}(X^{\wedge}_Z) \subset \mathbf{QCoh}(X)$, the full subcategory of quasicoherent complexes on X complete along Z; see [4].

MSC2020: 18N60.

Recollements were introduced by Beilinson, Bernstein, and Deligne [7] in the context of derived categories of perverse sheaves and were later defined by Lurie in the ∞-categorical context in the course of his study of constructible sheaves on stratified spaces [14, §A]. The goal of this article is to continue the development of the general theory of recollements from [14, §A.8], which we recapitulate in Section 2 beginning with the basic Definition 2.1. Our first contribution is to establish a symmetric monoidal refinement of this theory:

1.1 Definition (Definition 2.20) Let $\mathcal{X}$ be a symmetric monoidal ∞-category that admits finite limits. Then a recollement

$$\begin{tikzcd} \mathcal{U} \arrow[r, hookrightarrow, shift right, "j_*"'] & \mathcal{X} \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i^*"] & \mathcal{Z} \arrow[l, hookrightarrow, shift left, "i_*"] \end{tikzcd}$$

is *symmetric monoidal* if the localization functors $j_* j^*$ and $i_* i^*$ are compatible with the symmetric monoidal structure, so that $\mathcal{U}$ and $\mathcal{Z}$ uniquely inherit symmetric monoidal structures from $\mathcal{X}$ such that the functors j^* and i^* uniquely refine to (strong) symmetric monoidal functors.

Recall that Lurie shows that given a recollement $(\mathcal{U}, \mathcal{Z})$ on $\mathcal{X}$, if we define the *gluing functor* of the recollement to be $\phi = i^* j_*$ then we may reconstruct $\mathcal{X}$ as the fiber product $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$, where $\mathrm{Ar}(\mathcal{Z}) := \mathrm{Fun}(\Delta^1, \mathcal{Z})$ is the ∞-category of arrows in $\mathcal{Z}$.[1] Now given a lax symmetric monoidal functor $\phi : \mathcal{U} \to \mathcal{Z}$ of symmetric monoidal ∞-categories, we may construct a certain *canonical* symmetric monoidal structure on $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ (Definition 2.25). We then have:

1.2 Theorem (Theorem 2.30) *Let $\mathcal{X}$ be a symmetric monoidal ∞-category decomposed by a symmetric monoidal recollement $(\mathcal{U}, \mathcal{Z})$. Then the natural equivalence $\mathcal{X} \xrightarrow{\simeq} \mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ refines to an equivalence of symmetric monoidal ∞-categories. In other words, the lax symmetric monoidal structure on the gluing functor reconstructs the symmetric monoidal structure on $\mathcal{X}$.*

1.3 Remark Although this result is a simple exercise in the theory of ∞-operads, it appears that our work was the first to give a proof, and indeed a construction of the canonical symmetric monoidal structure. The work of Ayala, Mazel-Gee, and Rozenblyum has since placed this sort of construction within the context of endowing right-lax limits with $\mathcal{O}$-monoidal structure [2, §4.4].

Our next contribution is motivated by the following problem from equivariant stable homotopy theory:

1.4 Problem *Let G be a finite group and $\mathcal{F}$ a G-family (i.e., a set of subgroups of G closed under taking subgroups and conjugation). Given a (genuine) G-spectrum $X \in \mathbf{Sp}^G$ that is $\mathcal{F}$-complete and a subgroup $H \leq G$ not in $\mathcal{F}$, give a formula for the H-geometric fixed points of X in terms of the K-geometric fixed points of X ranging over $K \in \mathcal{F}$.*

Recollement theory is relevant here because any G-family $\mathcal{F}$ defines a recollement on $\mathbf{Sp}^G$ whose open part is spanned by the $\mathcal{F}$-complete G-spectra (see [15; 20]). In fact, we may further recast this problem using the stratification theory of Ayala, Mazel-Gee, and Rozenblyum [1; 2]. In their work, they

[1] To be precise, Lurie doesn't quite formulate his result in this way. See Observation 2.9 and the discussion thereafter.

construct a certain locally cocartesian fibration $\mathbf{Sp}^G_{\phi\text{-locus}} \to P$, where P is the poset of conjugacy classes of subgroups of G and the fiber over $[H]$ is $\mathrm{Fun}(BW_G H, \mathbf{Sp})$ for $W_G H = N_G H/H$ the Weyl group, such that one has a canonical equivalence

$$\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), \mathbf{Sp}^G_{\phi\text{-locus}}) \simeq \mathbf{Sp}^G, \tag{1-1}$$

where $\mathrm{sd}(P)$ is the barycentric subdivision[2] of P regarded as a locally cocartesian fibration over P via the functor that takes the maximum, and the left-hand side denotes the full subcategory spanned by those functors $\mathrm{sd}(P) \to \mathbf{Sp}^G_{\phi\text{-locus}}$ over P preserving locally cocartesian edges. The idea is that this equivalence parametrizes a G-spectrum in terms of its geometric fixed points, and indeed given a G-spectrum X, under this equivalence X transports to a functor $\mathrm{sd}(P) \to \mathbf{Sp}^G$ that sends $[H]$ to $\Phi^H X$. Now by definition any G-family $\mathcal{F}$ defines a *sieve* (i.e., a downward closed subposet) in P, and the $\mathcal{F}$-recollement on $\mathbf{Sp}^G$ transports to a recollement on $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), \mathbf{Sp}^G_{\phi\text{-locus}})$ given by the pair

$$\bigl(\mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{F}}(\mathrm{sd}(\mathcal{F}), \mathbf{Sp}^G_{\phi\text{-locus}}|_{\mathcal{F}}), \mathrm{Fun}^{\mathrm{cocart}}_{/(P\setminus\mathcal{F})}(\mathrm{sd}(P\setminus\mathcal{F}), \mathbf{Sp}^G_{\phi\text{-locus}}|_{P\setminus\mathcal{F}})\bigr).$$

Establishing a pointwise formula for the gluing functor of this recollement would then yield a solution to Problem 1.4. In general, we prove:

1.5 Theorem (Theorem 3.26) *Let P be a poset and let P_0 be a sieve in P. Let $\mathrm{sd}(P)_0 \subset \mathrm{sd}(P)$ be the subposet on those strings that originate in P_0, and note that $\max|_{\mathrm{sd}(P)_0}$ remains a locally cocartesian fibration. Then for every locally cocartesian fibration $C \to P$, the restriction functor*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P)_0, C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C|_{P_0})$$

is a trivial fibration.

Theorem A (Theorem 3.32, Proposition 3.36, and Theorem 3.39) *Let P be a down-finite poset[3] and let $p : C \to P$ be a locally cocartesian fibration such that for every $p \in P$, the fiber C_p admits finite limits, and for every $p \le q$, the associated pushforward functor $C_p \to C_q$ preserves finite limits. Then for every sieve-cosieve decomposition $P_0, P_1 = P \setminus P_0$ of P, we obtain a recollement*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C|_{P_0}) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/P_1}(\mathrm{sd}(P_1), C|_{P_1}),$$

where j^*, i^* are given by restriction and their fully faithful right adjoints j_*, i_* are describable by the following pointwise formulas:

1. *For every $x \in P_1$, let $J_x \subset \mathrm{sd}(P)$ be the subposet on strings $[a_0 < \cdots < a_n < x]$, $n \ge 0$ with $a_i \in P_0$. Then for every $[f : \mathrm{sd}(P_0) \to C|_{P_0}]$ on the left-hand side, if we let $\bar{f}$ denote the unique extension of f over $\mathrm{sd}(P)_0$ given by Theorem 1.5, then $j_*(f)$ evaluates on $x \in P_1$ to $\lim(\bar{f}|_{J_x} : J_x \to C_x)$.*
2. *For every $[f : \mathrm{sd}(P_1) \to C|_{P_1}]$ on the right-hand side, $i_*(f)$ evaluates on $x \in P_0$ to the final object $* \in C_x$.*

[2] Recall that $\mathrm{sd}(P)$ is the poset whose objects are strings $[a_0 < \cdots < a_n]$ in P and whose morphisms are string inclusions.

[3] A poset P is *down-finite* if for every $p \in P$, the subposet $P^{\le p}$ is finite.

1.6 Remark In [20], we use Theorem A to answer Problem 1.4 in the form of [20, Theorem F].

In fact, we prove a more general theorem where we replace P and the sieve P_0 by any ∞-category S and functor $\pi : S \to \Delta^1$ determining a sieve-cosieve decomposition of S, at the possible cost of demanding more conditions on our locally cocartesian fibration $p : C \to S$.

1.7 Remark Conceptually, a locally cocartesian fibration $C \to P$ is the unstraightening of a *left-lax* diagram $P \to \mathbf{Cat}_\infty$, and the ∞-category $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ is then the *right-lax limit* of this left-lax diagram (see [2, §A]). Theorem A then amounts to an *existence theorem* for the (pointwise) right-lax Kan extension of $[C \to P]$ along a functor $\pi : P \to \Delta^1$, along with a *transitivity property* of right-lax Kan extensions with respect to the composite $P \to \Delta^1 \to *$.

Although Theorem A may appear innocuous, we can leverage it to great effect in inductive arguments that build up the right-lax limit of a locally cocartesian fibration from its strata. For example, we will use Theorem A to establish the theory of 1*-generated and extendable objects* in Section 4, which furnishes a proof of an assertion of Nikolaus and Scholze [17, Remark II.4.8] on decomposing the ∞-category of bounded-below C_{p^n}-spectra as an iterated pullback; for a precise statement, see Remark 4.19.

In this paper, our main application of Theorem A will be to prove a *reconstruction theorem* for sheaves on an ∞-topos stratified over a finite poset P that was conjectured in the work of Barwick, Glasman, and Haine [5, Remark 8.2.7]. We recall the definition of a P-stratified ∞-topos as Definition 5.5 and that of a toposic locally cocartesian fibration as Definition 5.11. The reader may want to bear in mind the example of a P-stratified ∞-topos given by $\mathbf{Shv}(X)$ for X a topological space equipped with a continuous map $\pi : X \to P$, where we endow P with the Alexandroff topology (so that its open sets are cosieves).

Theorem B (Theorems 5.13 and 5.22) *Let $\mathcal{X}$ be an ∞-topos equipped with a P-stratification $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ for a finite poset P. Then we may functorially associate to $(\mathcal{X}, \pi_*)$ a locally cocartesian fibration $\mathcal{G}(\mathcal{X}) \to P^{\mathrm{op}}$ such that we have a canonical equivalence*

$$\Theta_P : \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X})) \xrightarrow{\simeq} \mathcal{X}. \tag{1-2}$$

Moreover, Θ_P is the counit of an adjoint equivalence

$$\lim^{\mathrm{rlax}} : \mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}} \rightleftarrows \mathbf{StrTop}_{\infty,P} : \mathcal{G} \tag{1-3}$$

between the ∞-category of toposic locally cocartesian fibrations over P^{op} and the ∞-category of P-stratified ∞-topoi.

1.8 Remark We explain how to interpret Theorem B as a reconstruction theorem. Define the *p-th stratum* $\mathcal{X}_p$ to be $\mathbf{Shv}(\{p\}) \times_{\mathbf{Shv}(P),\pi_*} \mathcal{X}$, where the fiber product is formed in the ∞-category $\mathbf{Top}_\infty$ of ∞-topoi and geometric morphisms thereof. (For example, if $\mathcal{X} = \mathbf{Shv}(X)$ for a P-stratified space $\pi : X \to P$, then $\mathcal{X}_p \simeq \mathbf{Shv}(X_p)$.) Let

$$\Phi^p : \mathcal{X} \rightleftarrows \mathcal{X}_p : \rho_p$$

denote the associated geometric morphism adjunction. Then ρ_p is fully faithful, and we in fact define

$$\mathcal{G}(\mathcal{X}) := \{(x, p) \in \mathcal{X} \times P^{\mathrm{op}} : x \in \mathcal{X}_p\}$$

with respect to $\rho_p : \mathcal{X}_p \hookrightarrow \mathcal{X}$, so that $\mathcal{G}(\mathcal{X})_p \simeq \mathcal{X}_p$ (Construction 5.10). Now under the equivalence Θ_P, a sheaf (i.e., object) $x \in \mathcal{X}$ transports to a functor $f_x : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})$ whose value on $[p]$ is given by $\Phi^p(x)$ (Remark 5.16). The functor Θ_p then sends f_x to the limit of its projection into $\mathcal{X}$.

1.9 Remark The strategy of our proof of Theorem B is heavily inspired by the work of Ayala, Mazel-Gee, and Rozenblyum, who assert a similar statement in the setting of presentable stable ∞-categories [2, Theorem A]. Note however that our proof of the equivalence (1-2) (but not (1-3)) is independent of any explicit use of $(\infty, 2)$-category theory in the form of the fibrational mate correspondence for locally cocartesian fibrations (that is, [2, Lemmas A.3.6 and A.3.7]), which we recall in this paper as Theorem 5.21. Indeed, we instead use Theorem A as the basis for an inductive argument that establishes (1-2). Similarly, one can supply an alternative proof of the comparable part of [2, Theorem A] using the same strategy; as already mentioned, we implement this idea in context of equivariant stable homotopy theory in our proof of the equivalence (1-1) in the form of [20, Theorem F] (see the discussion below [20, Theorem 2.42]).

1.1 What's new in this paper

We briefly comment on the relation of this paper to [18], which we have since split up into this paper and [19; 20]. Sections 2, 3, and 4 of this paper are lightly revised versions of the corresponding sections of [18], whereas Section 5 on the application to stratified ∞-topoi is entirely new. Also, in the intervening time since we wrote [18], Ayala, Mazel-Gee, and Rozenblyum released their work on stratified noncommutative geometry [2]; this is an expansion of [1] and bears greatly on many of the topics treated in this paper. As such, we have added a few remarks throughout (in particular, Remark 3.44 and the new Section 3.2.4) explaining how our work relates to [2]. One of the main takeaways here is that one can leverage Theorem A to remove the presentability hypotheses in [2, Theorem A]. Finally, our application to the description of bounded-below C_{p^n}-spectra as given in [18] relied on some work that has now been moved into [19].

2 Recollements

In this section, we establish the basic theory of recollements, expanding upon [4; 14, §A.8]. After setting up the definitions and summarizing Lurie's results on recollements, we explain a symmetric monoidal refinement of the theory of recollements, connect the theory of stable symmetric monoidal recollements to that of smashing localizations, and record some useful projection formulas. We conclude by proving a few lemmas concerning families of recollements that we will need in [19; 20].

2.1 Definition [14, Definition A.8.1] Let $\mathcal{X}$ be an ∞-category that admits finite limits and let $\mathcal{U}, \mathcal{Z} \subset \mathcal{X}$ be full subcategories that are stable under equivalences. Then $(\mathcal{U}, \mathcal{Z})$ is a *recollement* of $\mathcal{X}$ if the inclusion functors $j_* : \mathcal{U} \subset \mathcal{X}$ and $i_* : \mathcal{Z} \subset \mathcal{X}$ admit left exact left adjoints j^* and i^* such that:

(1) j^*i_* is equivalent to the constant functor at the terminal object $*$ of $\mathcal{U}$.

(2) j^* and i^* are *jointly conservative*, i.e., if $f : x \to y$ is a morphism in $\mathcal{X}$ such that j^*f and i^*f are equivalences, then f is an equivalence.

We will call $\mathcal{U}$ the *open* part of the recollement, $\mathcal{Z}$ the *closed* part of the recollement, and i^*j_* the *gluing functor*.[4]

The main purpose of the theory of recollements is to codify the various "fracture square" decompositions that recur throughout algebra and topology. Abstractly, we have:

2.2 Proposition *Let $(\mathcal{U}, \mathcal{Z})$ be a recollement of $\mathcal{X}$ and let $\eta_j : \mathrm{id} \to j_*j^*$, $\eta_i : \mathrm{id} \to i_*i^*$ denote the unit transformations. Then we have a pullback square of functors*

$$\begin{tikzcd} \mathrm{id} \arrow[r, "\eta_i"] \arrow[d, "\eta_j"'] & i_*i^* \arrow[d, "i_*i^*\eta_j"] \\ j_*j^* \arrow[r, "\eta_i j_*j^*"] & i_*i^*j_*j^* \end{tikzcd}$$

Proof By joint conservativity of the left-exact functors j^* and i^*, it suffices to check that we have a pullback square after applying j^* and i^*, which is clear. $\square$

Next, we define morphisms of recollements.

2.3 Definition Suppose that $(\mathcal{U}_1, \mathcal{Z}_1)$ and $(\mathcal{U}_2, \mathcal{Z}_2)$ are recollements on $\mathcal{X}_1$ and $\mathcal{X}_2$. Then a functor $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a *morphism of recollements* if F sends j_1^*-equivalences to j_2^*-equivalences and i_1^*-equivalences to i_2^*-equivalences. Let **Recoll** denote the resulting ∞-category of recollements, and let $\mathbf{Recoll}^{\mathrm{lex}}$ be the full subcategory on those morphisms of recollements that are also left-exact.

2.4 Observation Suppose that $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism of recollements $(\mathcal{U}_1, \mathcal{Z}_1) \to (\mathcal{U}_2, \mathcal{Z}_2)$. Then we may define $F_U = j_2^* F j_{1*} : \mathcal{U}_1 \to \mathcal{U}_2$ and $F_Z = i_2^* F i_{1*}$ so that we have a commutative diagram

$$\begin{tikzcd} \mathcal{U}_1 \arrow[d, "F_U"] & \mathcal{X}_1 \arrow[l, "j_1^*"'] \arrow[r, "i_2^*"] \arrow[d, "F"] & \mathcal{Z}_1 \arrow[d, "F_Z"] \\ \mathcal{U}_2 & \mathcal{X}_2 \arrow[l, "j_2^*"'] \arrow[r, "i_2^*"] & \mathcal{Z}_2 \end{tikzcd}$$

such that F is left-exact if and only if F_U and F_Z are left-exact. Conversely, if we are given such a commutative diagram, then F is a morphism of recollements. Indeed, for any morphism $[f : x \to y] \in \mathcal{X}_1$ such that $j^*(f)$ (resp. $i^*(f)$) is an equivalence, $j^*F(f) \simeq Fj^*(f)$ (resp. $i^*F(f) \simeq Fi^*(f)$) is an equivalence. Moreover, since $F_U \simeq j^*Fj_*$ and $F_Z \simeq i^*Fi_*$, it follows that functors $\mathcal{U}_1 \to \mathcal{U}_2$ and $\mathcal{Z}_1 \to \mathcal{Z}_2$ induced by F as a morphism of recollements are then canonically equivalent to F_U and F_Z.

[4] Our convention on which subcategory is open and which is closed matches that for constructible sheaves, whereas other authors (e.g., [4]) use the opposite convention, which matches that for quasicoherent sheaves. Also note that in [14, Definition A.8.1], Lurie calls the open part C_1 and the closed part C_0.

2.5 Observation In the situation of Observation 2.4, by adjunction we get natural transformations $\nu : Fj_{1*} \Rightarrow j_{2*}F_U$ and $\nu' : Fi_{1*} \Rightarrow i_{2*}F_Z$. Note that if F preserves the terminal object, then ν' is an equivalence; indeed, for all $z \in \mathcal{Z}_1$ we then have

$$j_2{}^* F i_{1*}(z) \simeq F_U j_1{}^* i_{1*}(z) \simeq F_U(*) \simeq *,$$

so the unit map $Fi_{1*}(z) \to i_{2*}i_2^* Fi_{1*}(z) = i_{2*}F_Z(z)$ is an equivalence. In particular, if F is left exact, then ν' is an equivalence [14, Remark A.8.10]. On the other hand, ν is an equivalence if and only if

$$\nu'' : F_Z i_1{}^* j_{1*} \Rightarrow i_2^* j_{2*} F_U$$

is an equivalence — indeed, the "only if" direction is obvious, and for the "if" direction we may readily check that $j_2{}^*\nu$ and $i_2{}^*\nu$ are equivalences and then invoke the joint conservativity of $j_2{}^*$ and $i_2{}^*$.

2.6 Definition If ν'' in Observation 2.5 is an equivalence, then we call F a *strict* morphism of recollements. Let $\mathbf{Recoll}_{\mathrm{str}} \subset \mathbf{Recoll}$ and $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}} \subset \mathbf{Recoll}^{\mathrm{lex}}$ be the wide subcategories on the strict morphisms.

2.7 Remark If $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a strict left-exact morphism of recollements, then F is an equivalence if and only if F_U and F_Z are equivalences [14, Proposition A.8.14].

2.8 Definition Let $\pi : \mathcal{M} \to \Delta^1$ be a functor of ∞-categories with fibers $\mathcal{M}_0 = \mathcal{Z}$ and $\mathcal{M}_1 = \mathcal{U}$. Then π is a *left-exact correspondence* [14, Definition A.8.6] if

(1) π is a cartesian fibration, so determines a functor $\phi : \mathcal{U} \to \mathcal{Z}$;

(2) $\mathcal{U}$ and $\mathcal{Z}$ admit finite limits and ϕ is left-exact.

A *morphism of left-exact correspondences* is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ over Δ^1. In terms of the left-exact functors ϕ_1 and ϕ_2, this corresponds to a right-lax commutative diagram

$$\begin{tikzcd} \mathcal{U}_1 \arrow[r, "\phi_1"] \arrow[d, "F_U"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{Z}_1 \arrow[d, "F_Z"] \\ \mathcal{U}_2 \arrow[r, "\phi_2"'] & \mathcal{Z}_2 \end{tikzcd}$$

Let $\mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ denote the resulting ∞-category of left-exact correspondences as a full subcategory of $(\mathbf{Cat}_\infty)_{/\Delta^1}$, and let $\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ be the wide subcategory on those morphisms that preserve cartesian edges, so that the right-lax commutativity is actually strict. Note that under the straightening correspondence, $\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ is the full subcategory of $\mathrm{Ar}(\mathbf{Cat}_\infty)$ on left-exact functors $\phi : \mathcal{U} \to \mathcal{Z}$.

If F_U and F_Z are also left-exact, we say that the morphism F of left-exact correspondences is *left-exact*. We may then view (lax) commutative squares as residing inside the category $\mathbf{Cat}^{\mathrm{lex}}_\infty$ itself. Let $\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \subset \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ and $\mathrm{Ar}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \subset \mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ denote the resulting wide subcategories.

2.9 Observation Let $\mathcal{M} \to \Delta^1$ be a left-exact correspondence and let $\mathcal{X} = \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$ be its ∞-category of sections. Let $\mathcal{U} \subset \mathcal{X}$ be the full subcategory on the cartesian sections and let $\mathcal{Z} \subset \mathcal{X}$ be the full subcategory on those sections σ such that $\sigma(1)$ is a terminal object of $\mathcal{U}$. Then $(\mathcal{U}, \mathcal{Z})$ is a recollement of $\mathcal{X}$ [14, Proposition A.8.7]. Moreover, the formation of sections

$$\mathcal{M} \rightsquigarrow \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$$

carries morphisms of left-exact correspondences to morphisms of recollements, and thereby defines a functor[5]

$$\lim{}^{\mathrm{rlax}} : \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Recoll},$$

which is an equivalence of ∞-categories by [14, Proposition A.8.8] (which shows full faithfulness) and [14, Proposition A.8.11] (which shows that if $(\mathcal{U}, \mathcal{Z})$ is a recollement of $\mathcal{X}$, then $\mathcal{X}$ is equivalent to the right-lax limit of $i^* j_* : \mathcal{U} \to \mathcal{Z}$). Furthermore, in view of the discussion in Observation 2.5, $\lim^{\mathrm{rlax}}$ restricts to equivalences of subcategories

$$\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}_{\mathrm{str}}, \quad \mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{lex}}, \quad \mathrm{Ar}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}.$$

We next explain how to identify the ∞-category of sections of a cartesian fibration classified by the functor $\phi : \mathcal{U} \to \mathcal{Z}$ with the pullback $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$. For an efficient proof, we will use the machinery of marked simplicial sets [12, §3]. Recall that $\Delta^{n\natural}$ denotes the n-simplex with its last edge $\{n-1, n\}$ marked, and likewise for the marked horn $\Lambda^{n\natural}_n$. Moreover, given a cartesian fibration $\pi : C \to B$, we let $C^\natural$ denote the marking of all π-cartesian edges, for which $(C^\natural, \pi)$ is fibrant in the cartesian model structure on $s\mathbf{Set}^+_{/B}$.

2.10 Construction Let $\pi : \mathcal{M} \to \Delta^1$ be a cartesian fibration. By the dual of [21, Lemma 2.23], we have a trivial fibration $\mathrm{Ar}^{\mathrm{cart}}(\mathcal{M}) \to \mathrm{Ar}(\Delta^1) \times_{\mathrm{ev}_1, \Delta^1, \pi} \mathcal{M}$, which restricts to a trivial fibration $\mathrm{ev}_1 : \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathcal{M}_1$. Let χ be a section of ev_1.

Because the map $i : \Lambda^{2\natural}_2 \to \Delta^{2\natural}$ is right marked anodyne, with the structure map $\sigma^0 : \Delta^2 \to \Delta^1$, $(\sigma^0)^{-1}(0) = \{0, 1\}$ and $(\sigma^0)^{-1}(1) = \{2\}$, we have a trivial fibration

$$i^* : \mathrm{Fun}_{/\Delta^1}(\Delta^{2\natural}, \mathcal{M}^\natural) \to \mathrm{Fun}_{/\Delta^1}(\Lambda^{2\natural}_2, \mathcal{M}^\natural) \cong \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \times_{\mathrm{ev}_1, \mathcal{M}_1, \mathrm{ev}_1} \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}).$$

Let κ be a section of i^*. The section χ yields a functor

$$f = (\mathrm{id}, \chi \circ \mathrm{ev}_1) : \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \times_{\mathcal{M}_1} \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}).$$

Let $g = \kappa \circ f$. Then the various maps fit into the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \arrow[r, "g"] \arrow[d, "\mathrm{ev}_1"] & \mathrm{Fun}_{/\Delta^1}(\Delta^{2\natural}, \mathcal{M}^\natural) \arrow[r, "\mathrm{ev}_{01}"] \arrow[d, "\mathrm{ev}_{12}"] & \mathrm{Fun}(\Delta^1, \mathcal{M}_0) \arrow[d, "\mathrm{ev}_1"] \\
\mathcal{M}_1 \arrow[r, "\chi"] & \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \arrow[r, "\mathrm{ev}_0"] & \mathcal{M}_0
\end{tikzcd}$$

[5] We denote this by $\lim^{\mathrm{rlax}}$ in view of the interpretation of the sections of a cartesian fibration as defining the right-lax limit of the corresponding functor.

2.11 Lemma *The natural map* $\mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathrm{Ar}(\mathcal{M}_0) \times_{\mathrm{ev}_1, \mathcal{M}_0} \mathcal{M}_1$ *is an equivalence, so the outer square is a homotopy pullback square of* ∞*-categories.*

Proof Because the sections χ and κ are equivalences, the map g is an equivalence. Moreover, because the map $\Lambda^2_1 \to \Delta^2$ is inner anodyne, the rightmost square is a homotopy pullback square. □

2.12 Corollary *Suppose that* $(\mathcal{U}, \mathcal{Z})$ *is a recollement of* $\mathcal{X}$ *and consider the commutative*[6] *diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "i^*\eta_j"] \arrow[d, "j^*"'] & \mathrm{Ar}(\mathcal{Z}) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{U} \arrow[r, "\phi = i^* j_*"] & \mathcal{Z} \end{tikzcd}$$

where $\eta_j : \mathcal{X} \to \mathrm{Ar}(\mathcal{X})$ *is the functor that sends* x *to the unit map* $x \to j_* j^* x$*. Then the induced map*

$$\mathcal{X} \xrightarrow{\simeq} \mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$$

is an equivalence of ∞*-categories.*

Proof Combine Lemma 2.11 with the equivalence $\lim^{\mathrm{rlax}} : \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}$ of Observation 2.9. □

2.13 Remark In view of Corollary 2.12, given a recollement $(\mathcal{U}, \mathcal{Z})$ of $\mathcal{X}$ with gluing functor $\phi = i^* j_*$ we will often write objects $x \in \mathcal{X}$ as $[u, \alpha : z \to \phi(u)]$ or $[u, z, \alpha]$.

Given a left-exact functor $\phi : \mathcal{U} \to \mathcal{Z}$, we may also extract the resulting recollement from the *cocartesian* fibration classified by ϕ, even though it is difficult to encode the right-lax functoriality when working with cocartesian fibrations.

2.14 Observation Let S be an ∞-category and $C \to S$ a cocartesian fibration. Recall from [6; 21, Recollection 5.17] that the *dual cartesian fibration* $C^\vee \to S^{\mathrm{op}}$ is defined to have n-simplices[7]

$$\begin{tikzcd} {}_\natural\mathrm{TwAr}((\Delta^n)^{\mathrm{op}}) \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & {}_\natural C \arrow[d] \\ ((\Delta^n)^{\mathrm{op}})^\sharp \arrow[r] & S^\sharp \end{tikzcd}$$

where we mark the cocartesian edges in C and $\mathrm{TwAr}((\Delta^n)^{\mathrm{op}})$. In fact, because the functor $\mathrm{TwAr}'(-) : s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{/S}$ of [21, Proposition 5.18] preserves colimits, it follows that, for all simplicial sets A over S^{op},

$$\mathrm{Hom}_{/S^{\mathrm{op}}}(A, C^\vee) \cong \mathrm{Hom}_{/S}(\mathrm{TwAr}'(A^{\mathrm{op}}), {}_\natural C).$$

Consequently, we obtain an equivalence

$$\mathrm{Fun}_{/S^{\mathrm{op}}}(S^{\mathrm{op}}, C^\vee) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{TwAr}(S), C).$$

[6] We can obtain a commutative diagram of simplicial sets using standard techniques in quasicategory theory.

[7] Here, $\mathrm{TwAr}(-)$ is the *twisted arrow* ∞*-category*. We use the directionality convention of [3] instead of [14, §5.2.1], so twisted arrows are contravariant in the source and covariant in the target.

Now note that the barycentric subdivision $\mathrm{sd}(\Delta^1) = [0 \to 01 \leftarrow 1]$ is isomorphic to the twisted arrow category $\mathrm{TwAr}(\Delta^1)$. Therefore, for a cocartesian fibration $C \to \Delta^1$, we deduce that

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\mathrm{sd}(\Delta^1), C) \simeq \mathrm{Fun}_{/\Delta^1}(\Delta^1, C^\vee)$$

and hence by Lemma 2.11 we can decompose $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\mathrm{sd}(\Delta^1), C)$ as a pullback square $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ for a choice of pushforward functor $\phi : \mathcal{U} \to \mathcal{Z}$ (where $\mathcal{U} \simeq C_0$ and $\mathcal{Z} \simeq C_1$). This observation will be important for us when we discuss recollements on right-lax limits in the next section.

2.1 Stable recollements

2.15 Definition Let $\mathcal{X}$ be a stable ∞-category and let $(\mathcal{U}, \mathcal{Z})$ be a recollement of $\mathcal{X}$. Then this recollement is *stable* if $\mathcal{U}$ and $\mathcal{Z}$ are stable subcategories. Let $\mathbf{Recoll}^{\mathrm{stab}}$ (resp. $\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$) be the full subcategory of $\mathbf{Recoll}^{\mathrm{lex}}$ (resp. $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}$) whose objects are the stable recollements.

2.16 Definition If $\mathcal{M} \to \Delta^1$ is a left-exact correspondence, then $\mathcal{M}$ is *exact* if the functor $\phi : \mathcal{M}_1 \to \mathcal{M}_0$ is an exact functor of stable ∞-categories. Let $\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{stab}}_\infty)$ (resp. $\mathrm{Ar}(\mathbf{Cat}^{\mathrm{stab}}_\infty)$) be the full subcategory of $\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{lex}}_\infty)$ (resp. $\mathrm{Ar}(\mathbf{Cat}^{\mathrm{lex}}_\infty)$) on the exact correspondences.

2.17 Remark The functor $\lim^{\mathrm{rlax}}$ of Observation 2.9 restricts to equivalences

$$\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{stab}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{stab}}, \quad \mathrm{Ar}(\mathbf{Cat}^{\mathrm{stab}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}.$$

2.18 Observation Let $(\mathcal{U}, \mathcal{Z})$ be a stable recollement of $\mathcal{X}$. Then $j^* : \mathcal{X} \to \mathcal{U}$ admits a fully faithful left adjoint[8] $j_!$, i_* admits a right adjoint $i^!$, and we have norm maps $\mathrm{Nm} : j_! \to j_*$ and $\mathrm{Nm}' : i^! \to i^*$ that fit into fiber sequences

$$j_! \to j_* \to i_* i^* j_* \quad \text{and} \quad i^! \to i^* \to i^* j_* j^*,$$

where the other maps are induced by the unit transformations for $j^* \dashv j_*$ and $i^* \dashv i_*$. On objects $x = [u, z, \alpha] \in \mathcal{X}$, these amount to the fiber sequences

$$[u, 0, 0] \to [u, \phi u, \mathrm{id}] \to [0, \phi u, 0] \quad \text{and} \quad \mathrm{fib}(\alpha) \to z \xrightarrow{\alpha} \phi u.$$

Considering the various unit and counit transformations and the norm maps, we may extend the pullback square of Proposition 2.2 to a commutative diagram

$$\begin{tikzcd}
 & i_* i^! \arrow[r, "\simeq"] \arrow[d] & i_* i^! \arrow[d, "i_* \mathrm{Nm}'"] \\
j_! j^* \arrow[r] \arrow[d, "\simeq"] & \mathrm{id} \arrow[r] \arrow[d] & i_* i^* \arrow[d] \\
j_! j^* \arrow[r, "\mathrm{Nm}\, j^*"'] & j_* j^* \arrow[r] & i_* i^* j_* j^*
\end{tikzcd}$$

in which every row and column is a fiber sequence.

[8] For the existence of $j_!$, we only need that $\mathcal{Z}$ admits an initial object $\varnothing$ [14, Corollary A.8.13]. Then $j_!$ is defined by the formula $j_!(u) = [u, \varnothing \to \phi(u)]$.

2.19 Observation In the stable case, the datum of the closed part of a recollement determines the entire recollement. More precisely, if $\mathcal{Z} \subset \mathcal{X}$ is a stable reflective and coreflective subcategory of $\mathcal{X}$ and we define $\mathcal{U}$ to be the full subcategory on those objects $u \in \mathcal{X}$ such that $\mathrm{Map}_{\mathcal{X}}(z, u) \simeq *$ for all $z \in \mathcal{Z}$, then $(\mathcal{U}, \mathcal{Z})$ is a stable recollement of $\mathcal{X}$ [14, Proposition A.8.20], and conversely, if $(\mathcal{U}, \mathcal{Z})$ is a stable recollement of $\mathcal{X}$ then $j_* : \mathcal{U} \subset \mathcal{X}$ is defined as above from $\mathcal{Z}$. We may also identify $j_!(\mathcal{U})$ as given by those objects $u \in \mathcal{X}$ such that $\mathrm{Map}_{\mathcal{X}}(u, z) \simeq *$ for all $z \in \mathcal{Z}$.

Moreover, $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism of stable recollements $(\mathcal{U}_1, \mathcal{Z}_1) \to (\mathcal{U}_2, \mathcal{Z}_2)$ if and only if $F|_{\mathcal{Z}_1} \subset \mathcal{Z}_2$ and $F|_{j_!(\mathcal{U}_1)} \subset j_!(\mathcal{U}_2)$ (in particular, we then have $j_{2!} F_U \simeq F j_{1!}$). This is because $\mathcal{Z}$ coincides with the j^*-null objects and $j_!(\mathcal{U})$ with the i^*-null objects. Given this, F is then a strict morphism of stable recollements if and only if we also have that $F|_{j_*(\mathcal{U}_1)} \subset j_*(\mathcal{U}_2)$.

2.2 Symmetric monoidal recollements

We now extend the theory of recollements to the situation where $\mathcal{X}$ admits a symmetric monoidal structure $(\mathcal{X}, \otimes, \mathbb{1})$. In what follows, we will call an adjunction $F : C \rightleftarrows D : G$ between symmetric monoidal ∞-categories *symmetric monoidal* if F is (strong) symmetric monoidal.

2.20 Definition Let $\mathcal{X}$ be a symmetric monoidal ∞-category that admits finite limits. Then a recollement $(\mathcal{U}, \mathcal{Z})$ of $\mathcal{X}$ is *symmetric monoidal* if the localization functors $j_* j^*$ and $i_* i^*$ are compatible with the symmetric monoidal structure in the sense of [14, Definition 2.2.1.6], i.e., if $f : x \to x'$ is a j^*- or i^*-equivalence, then so is $f \otimes \mathrm{id} : x \otimes y \to x' \otimes y$ for any $y \in \mathcal{X}$.

A morphism $F : (\mathcal{U}, \mathcal{Z}) \to (\mathcal{U}', \mathcal{Z}')$ of recollements on $\mathcal{X}$ and $\mathcal{X}'$ is *symmetric monoidal* if the functor $F : \mathcal{X} \to \mathcal{X}'$ is symmetric monoidal. Let $\mathbf{Recoll}^{\otimes}$ denote the ∞-category of symmetric monoidal recollements and morphisms thereof.

2.21 Observation In the situation of Definition 2.20, by [14, Proposition 2.2.1.9] $\mathcal{U}$ and $\mathcal{Z}$ obtain symmetric monoidal structures such that the adjunctions $j^* \dashv j_*$ and $i^* \dashv i_*$ are symmetric monoidal. In particular, the gluing functor $i^* j_*$ is lax symmetric monoidal. Furthermore, if F is a morphism of symmetric monoidal recollements, then the induced functors F_U and F_Z of Observation 2.5 are also symmetric monoidal.

2.22 Remark Most of the results of this subsection will extend verbatim to an arbitrary reduced ∞-operad $\mathcal{O}^{\otimes}$. We leave the details to the reader.

We first show that given a lax symmetric monoidal functor $\phi : \mathcal{U} \to \mathcal{Z}$, the recollement $\lim^{\mathrm{rlax}} \phi$ is symmetric monoidal. We first recall the pointwise symmetric monoidal structure on a functor ∞-category.

2.23 Construction Let $p : C^{\otimes} \to \mathbf{Fin}_*$ be an ∞-operad, and let K be a simplicial set. We have the cotensor $p^K : (C^{\otimes})^K \to \mathbf{Fin}_*$ defined by

$$\mathrm{Hom}_{/\mathbf{Fin}_*}(A, (C^{\otimes})^K) \cong \mathrm{Hom}_{/\mathbf{Fin}_*}(A \times K, C^{\otimes}).$$

Then p^K is again an ∞-operad: this follows from the observation that for any $\mathfrak{O}$-anodyne morphism $A \to B$ of preoperads (with $\mathfrak{O}$ the defining categorical pattern for the model structure on preoperads), $A \times K \to$

$B \times K$ is again $\mathfrak{O}$-anodyne [14, Proposition B.1.9]. Moreover, if p is in addition a cocartesian fibration, then p^K is also a cocartesian fibration. The fiber of p^K over $\langle n\rangle$ is $\mathrm{Fun}(K, C^{\times n}) \simeq \prod_{i=1}^n \mathrm{Fun}(K, C)$, and for the unique active map $\langle n\rangle \to \langle 1\rangle$, if $\phi : C^{\times n} \to C$ is a choice of pushforward functor encoded by p, then the postcomposition by ϕ functor $\phi_* : \mathrm{Fun}(K, C^{\times n}) \to \mathrm{Fun}(K, C)$ is a choice of pushforward functor encoded by p^K. In other words, p^K is the pointwise symmetric monoidal structure on $\mathrm{Fun}(K, C)$.

We will also need the following lemma.

2.24 Lemma *Let $C^{\otimes}$ be a symmetric monoidal ∞-category. Then the functor*

$$e_L : (C^{\otimes})^{K \star L} \to (C^{\otimes})^L$$

induced by $L \subset K \star L$ is a cocartesian fibration of ∞-operads.

Proof Because e_L is induced by the monomorphism $L \subset K \star L$, e_L is a fibration of ∞-operads. Using the inert-active factorization system on an ∞-operad, it then suffices to prove the following two properties of e_L:

(1) For every object $\langle n\rangle \in \mathbf{Fin}_*$, $(e_L)_{\langle n\rangle}$ is a cocartesian fibration.

(2) For every active edge $\alpha : \langle n\rangle \to \langle 1\rangle$ and commutative square

$$\begin{tikzcd} f = (f_1, \dots, f_n) \arrow[r] \arrow[d, "\theta"] & f' = \bigotimes_{i=1}^n f_i \arrow[d, "\theta'"] \\ g = (g_1, \dots, g_n) \arrow[r] & g' = \bigotimes_{i=1}^n g_i \end{tikzcd}$$

in $(C^{\otimes})^{K \star L}$ with the horizontal edges as $p^{K \star L}$-cocartesian edges covering α, if θ is $(e_L)_{\langle n\rangle}$-cocartesian then θ' is $(e_L)_{\langle 1\rangle}$-cocartesian.

For (1), by [21, Lemma 4.8] we have that $(e_L)_{\langle n\rangle} : \mathrm{Fun}(K \star L, C^{\times n}) \to \mathrm{Fun}(L, C^{\times n})$ is a cocartesian fibration. Moreover, $\theta : f \to g$ is a $(e_L)_{\langle n\rangle}$-cocartesian edge if and only if its image in $\mathrm{Fun}(K, C^{\times n})$ is an equivalence. This proves (2), since the n-fold tensor product of equivalences is always an equivalence. $\square$

We are now ready to define the symmetric monoidal structure on a right-lax limit.

2.25 Definition Suppose $\phi^{\otimes} : \mathcal{U}^{\otimes} \to \mathcal{Z}^{\otimes}$ is a lax symmetric monoidal functor of symmetric monoidal ∞-categories (i.e., a map of ∞-operads). Consider the pullback square of ∞-operads

$$\begin{tikzcd} (\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes} \arrow[r] \arrow[d] & (\mathcal{Z}^{\otimes})^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{U}^{\otimes} \arrow[r, "\phi^{\otimes}"] & \mathcal{Z}^{\otimes} \end{tikzcd}$$

By Lemma 2.24, ev_1 is a cocartesian fibration, so $(\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes} \to \mathcal{U}^{\otimes} \to \mathbf{Fin}_*$ is a cocartesian fibration and therefore a symmetric monoidal ∞-category. This defines the *canonical* symmetric monoidal structure on the right-lax limit of ϕ.

2.26 Remark In Definition 2.25, at the level of objects the tensor product on $\mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$ is defined in the following way: Suppose we are given two objects $x = [u, z, \alpha]$ and $x' = [u', z', \alpha']$. Then $x \otimes x' = [u \otimes u', z \otimes z', \gamma]$, where γ is given by the composite map

$$z \otimes z' \xrightarrow{\alpha \otimes \alpha'} \phi(u) \otimes \phi(u') \to \phi(u \otimes u')$$

using the lax symmetric monoidality of ϕ for the second map.

2.27 Proposition *If $\phi : \mathcal{U} \to \mathcal{Z}$ is a lax symmetric monoidal left-exact functor, then $\lim^{\mathrm{rlax}} \phi$ is a symmetric monoidal recollement with respect to the canonical symmetric monoidal structure on $\mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$.*

Proof We only need to observe that in Definition 2.25, the two evaluation maps $j^* : \mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U} \to \mathcal{U}$ and $i^* : \mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U} \to \mathrm{Ar}(\mathcal{Z}) \xrightarrow{\mathrm{ev}_0} \mathcal{Z}$ are symmetric monoidal. $\square$

We next wish to show that given a symmetric monoidal recollement $(\mathcal{U}, \mathcal{Z})$ of $\mathcal{X}$, the symmetric monoidal structure on $\mathcal{X}$ is the canonical one of Definition 2.25. We first observe that the unit transformation of a symmetric monoidal adjunction is itself a lax symmetric monoidal functor.

2.28 Lemma *Let $C^{\otimes}$ and $D^{\otimes}$ be symmetric monoidal ∞-categories and let $F : C \rightleftarrows D : G$ be a symmetric monoidal adjunction. Then the unit transformation $\eta : C \to \mathrm{Ar}(C)$ lifts to a lax symmetric monoidal functor $\eta^{\otimes} : C^{\otimes} \to (C^{\otimes})^{\Delta^1}$ such that $\mathrm{ev}_1 \eta^{\otimes} \simeq G^{\otimes} F^{\otimes}$ and $\mathrm{ev}_0 \eta^{\otimes} \simeq \mathrm{id}$.*

Proof Let $\mathcal{M} \to \Delta^1$ be the bicartesian fibration classified by the adjunction. We may factor (or define) η as the composition

$$C \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \subset \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \simeq \mathrm{Ar}(C) \times_C D \to \mathrm{Ar}(C),$$

where we use Lemma 2.11 for the identification of the sections of $\mathcal{M}$. Let $\mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$ be equipped with its canonical symmetric monoidal structure. Because F is symmetric monoidal, the inclusion $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \subset \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$ defines a symmetric monoidal structure on $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M})$ by restriction such that the equivalence $\mathrm{ev}_0 : \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \xrightarrow{\simeq} C$ is an equivalence of symmetric monoidal ∞-categories. Also, the projection $\mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathrm{Ar}(C)$ is lax symmetric monoidal by definition. We deduce that η lifts to a lax symmetric monoidal functor $\eta^{\otimes}$ with the indicated properties. $\square$

2.29 Proposition *Let $(\mathcal{U}, \mathcal{Z})$ be a symmetric monoidal recollement of $\mathcal{X}$. Then the functor $\mathcal{X} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{X})$ realizing the pullback square of functors*

$$\begin{tikzcd} \mathrm{id} \arrow[r] \arrow[d] & i_* i^* \arrow[d] \\ j_* j^* \arrow[r] & i_* i^* j_* j^* \end{tikzcd}$$

lifts to a lax symmetric monoidal functor $\mathcal{X}^{\otimes} \to (\mathcal{X}^{\otimes})^{\Delta^1 \times \Delta^1}$. Consequently, if $A \in \mathcal{X}$ is an algebra object, then we have an equivalence of algebras

$$A \simeq (j_* j^*)(A) \times_{(i_* i^* j_* j^*)(A)} (i_* i^*)(A).$$

Proof By Lemma 2.28, the symmetric monoidal adjunction $j^* \dashv j_*$ yields a lax symmetric monoidal functor

$$(\eta_j)^{\otimes} : \mathcal{X}^{\otimes} \to (\mathcal{X}^{\otimes})^{\Delta^1}.$$

We also have the induced symmetric monoidal adjunction $\hat{i}^* : \mathrm{Ar}(\mathcal{X}) \rightleftarrows \mathrm{Ar}(\mathcal{Z}) : \hat{i}_*$ which yields a lax symmetric monoidal functor

$$(\eta_{\hat{i}})^{\otimes} : (\mathcal{X}^{\otimes})^{\Delta^1} \to (\mathcal{X}^{\otimes})^{\Delta^1 \times \Delta^1}.$$

The composite $(\eta_{\hat{i}})^{\otimes} \circ (\eta_j)^{\otimes}$ then defines the desired functor. □

2.30 Theorem *Suppose $(\mathcal{U}, \mathcal{Z})$ is a symmetric monoidal recollement of $\mathcal{X}$. Then the equivalence*

$$\mathcal{X} \xrightarrow{\simeq} \mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$$

of Corollary 2.12 refines to an equivalence of symmetric monoidal ∞-categories, where we equip $\mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$ with the canonical symmetric monoidal structure of Definition 2.25.

Proof By Lemmas 2.28 and 2.31, we have a commutative diagram of ∞-operads

$$\begin{tikzcd} \mathcal{X}^{\otimes} \arrow[r, "(i^*)^{\otimes}(\eta_j)^{\otimes}"] \arrow[d, "(j^*)^{\otimes}"'] & (\mathcal{Z}^{\otimes})^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{U}^{\otimes} \arrow[r, "(i^*)^{\otimes}(j_*)^{\otimes}"] & Z^{\otimes} \end{tikzcd}$$

such that the induced functor $\theta^{\otimes} : \mathcal{X}^{\otimes} \to (\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes}$ covers the map θ of Corollary 2.12. Since θ is an equivalence, to show that $\theta^{\otimes}$ is an equivalence it suffices to check that $\theta^{\otimes}$ is strongly symmetric monoidal. But this follows from the symmetric monoidality of the jointly conservative functors j^*, i^*. □

We include the following simple strictification result for completeness.

2.31 Lemma *Suppose we have a homotopy commutative square of ∞-operads*

$$\begin{tikzcd} A^{\otimes} \arrow[r, "F'"] \arrow[d, "G'"] & B^{\otimes} \arrow[d, "G"] \\ C^{\otimes} \arrow[r, "F"] & D^{\otimes} \end{tikzcd}$$

in the sense that there is the data of a homotopy $\theta : G \circ F' \xRightarrow{\simeq} F \circ G'$, over $\mathbf{Fin}_*$,

$$\begin{tikzcd} A^{\otimes} \times \{0\} \arrow[r, "F'"] \arrow[d] & B^{\otimes} \arrow[d, "G"] \\ A^{\otimes} \times \Delta^1 \arrow[r, "\theta"] & D^{\otimes} \\ A^{\otimes} \times \{1\} \arrow[r, "G'"] \arrow[u] & C^{\otimes} \arrow[u, "F"'] \end{tikzcd}$$

such that θ sends every edge $(a,0) \to (a,1)$ to an equivalence. Suppose also that G is a fibration of ∞-operads, i.e., a categorical fibration [14, 2.1.2.10]. *Then there exists a functor $F'' : A^{\otimes} \to B^{\otimes}$ homotopic to F' as a map of ∞-operads such that the square*

$$\begin{tikzcd} A^{\otimes} \arrow[r, "F''"] \arrow[d, "G'"] & B^{\otimes} \arrow[d, "G"] \\ C^{\otimes} \arrow[r, "F"] & D^{\otimes} \end{tikzcd}$$

strictly commutes.

Proof Given an ∞-operad $O^{\otimes}$, let $O^{\otimes,\natural}$ denote the marked simplicial set $(O^{\otimes}, \mathcal{E})$ where $\mathcal{E}$ is the collection of inert morphisms in $O^{\otimes}$ [14, 2.1.4.5]. Consider the lifting problem in marked simplicial sets

$$\begin{tikzcd} A^{\otimes,\natural} \times \{0\} \arrow[r, "F'"] \arrow[d] & B^{\otimes,\natural} \arrow[d, "G"] \\ A^{\otimes,\natural} \times (\Delta^1)^{\sharp} \arrow[r, "\theta"] \arrow[ur, dotted, "\bar{\theta}"] & D^{\otimes,\natural} \end{tikzcd}$$

Because G is assumed to be a fibration of ∞-operads, G is a fibration in the model structure on ∞-preoperads [14, 2.1.4.6]. Hence, the dotted lift $\bar{\theta}$ exists. If we then let $F'' = \bar{\theta}|_{A^{\otimes} \times \{1\}}$, the claim follows. □

We next turn to morphisms of symmetric monoidal recollements.

2.32 Observation Suppose we have a commutative diagram of symmetric monoidal ∞-categories and lax symmetric monoidal functors

$$\begin{tikzcd} \mathcal{U}^{\otimes} \arrow[r, "\phi^{\otimes}"] \arrow[d, "F_U{}^{\otimes}"'] & \mathcal{Z}^{\otimes} \arrow[d, "F_Z{}^{\otimes}"] \\ \mathcal{U}'^{\otimes} \arrow[r, "\phi'^{\otimes}"] & \mathcal{Z}'^{\otimes} \end{tikzcd}$$

Then by way of the commutative diagram

$$\begin{tikzcd} (\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes} \arrow[r] \arrow[d] & (\mathcal{Z}^{\otimes})^{\Delta^1} \arrow[r, "F_Z^{\otimes}"] \arrow[d, "\mathrm{ev}_1"] & (\mathcal{Z}'^{\otimes})^{\Delta^1} \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{U}^{\otimes} \arrow[r, "\phi^{\otimes}"] \arrow[dr, "F_U^{\otimes}"'] & \mathcal{Z}^{\otimes} \arrow[r, "F_Z^{\otimes}"] & \mathcal{Z}'^{\otimes} \\ & \mathcal{U}'^{\otimes} \arrow[ur, "\phi'^{\otimes}"'] & \end{tikzcd}$$

we obtain a lax symmetric monoidal functor $F^{\otimes} : (\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes} \to (\mathcal{Z}'^{\otimes})^{\Delta^1} \times_{\mathcal{Z}'^{\otimes}} \mathcal{U}'^{\otimes}$, which is symmetric monoidal if $F_U^{\otimes}$ and $F_Z^{\otimes}$ are symmetric monoidal.

Let $\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_{\infty}^{\otimes,\mathrm{lax}}) \subset \mathrm{Ar}(\mathbf{Cat}_{\infty}^{\otimes,\mathrm{lax}})$ be the subcategory whose objects are left-exact lax symmetric monoidal functors and whose morphisms are through symmetric monoidal functors. Then by the above construction[9] we may lift the functor $\lim^{\mathrm{rlax}} : \mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \to \mathbf{Recoll}_{\mathrm{str}}$ to

$$(\lim^{\mathrm{rlax}})^{\otimes} : \mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty^{\otimes,\mathrm{lax}}) \to \mathbf{Recoll}_{\mathrm{str}}^{\otimes}.$$

An elaboration of Theorem 2.30 shows that $(\lim^{\mathrm{rlax}})^{\otimes}$ is an equivalence; we leave the details to the reader.

One also has a lift of $\lim^{\mathrm{rlax}} : \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \to \mathbf{Recoll}$ if one considers right-lax commutative squares of ∞-operads. Since the details in this case are more involved, we leave a precise formulation to the reader.

Our next goal is to establish certain *projection formulas* satisfied by a (stable) symmetric monoidal recollement. First, we note the following about the situation in which the symmetric monoidal ∞-category $\mathcal{X}$ is in addition closed.

2.33 Observation Let $\mathcal{X}$ be a closed symmetric monoidal ∞-category and let $F(-,-)$ denote its internal hom. If $(\mathcal{U},\mathcal{Z})$ is a symmetric monoidal recollement of $\mathcal{X}$, then we define

$$F_{\mathcal{U}}(u,u') = j^*F(j_*u, j_*u') \quad \text{and} \quad F_{\mathcal{Z}}(z,z') = i^*F(i_*z, i_*z')$$

to be internal homs for $\mathcal{U}$ and $\mathcal{Z}$, so that $\mathcal{U}$ and $\mathcal{Z}$ are closed symmetric monoidal. Indeed, since $j^* \dashv j_*$ is monoidal, we have

$$\begin{aligned}&\mathrm{Map}_{\mathcal{U}}(w, j^*F(j_*u, j_*v)) \simeq \mathrm{Map}_{\mathcal{X}}(j_*w, F(j_*u, j_*v)) \simeq \mathrm{Map}_{\mathcal{X}}(j_*w \otimes j_*v, j_*v),\\ &\mathrm{Map}_{\mathcal{U}}(j^*(j_*w \otimes j_*u), v) \simeq \mathrm{Map}_{\mathcal{U}}(w \otimes u, v),\end{aligned}$$

and similarly for $F_{\mathcal{Z}}(-,-)$. Moreover we have natural equivalences

$$F(x, j_*u) \simeq j_*F_{\mathcal{U}}(j^*x, u), \quad F(x, i_*z) \simeq i_*F_{\mathcal{Z}}(i^*x, z).$$

For example, we may check

$$\begin{aligned}\mathrm{Map}_{\mathcal{X}}(x, F(y, j_*u)) &\simeq \mathrm{Map}_{\mathcal{X}}(x \otimes y, j_*u) \simeq \mathrm{Map}_{\mathcal{U}}(j^*x \otimes j^*y, u)\\ &\simeq \mathrm{Map}_{\mathcal{U}}(j^*x, F_{\mathcal{U}}(j^*y, u)) \simeq \mathrm{Map}_{\mathcal{X}}(x, j_*F_{\mathcal{U}}(j^*y, u)).\end{aligned}$$

This implies that the unit maps

$$\begin{aligned}F(j_*u, j_*u') &\to j_*j^*F(j_*u, j_*u') = j_*F_{\mathcal{U}}(u,u'),\\ F(i_*z, i_*z') &\to i_*i^*F(i_*z, i_*z') = i_*F_{\mathcal{Z}}(z,z')\end{aligned}$$

are equivalences.

2.34 Proposition (projection formulas) *Let $(\mathcal{U},\mathcal{Z})$ be a stable[10] symmetric monoidal recollement of $\mathcal{X}$.*

(1) *The natural maps $\alpha : i_*(z) \otimes x \to i_*(z \otimes i^*x)$ and $\beta : j_!(u \otimes j^*x) \to j_!(u) \otimes x$ are equivalences.*

(2) *The fiber sequence $j_!j^*x \to x \to i_*i^*x$ is equivalent to*

$$j_!(1_U) \otimes x \to x \to i_*(1_Z) \otimes x.$$

[9]Technically, to make a rigorous construction we may work at the level of preoperads and then pass to the underlying ∞-categories.

[10]We do not require stability for the $i^* \dashv i_*$ projection formula. For the assertions that only involve $j_!$, we only need that $\mathcal{X}$ be pointed.

Now suppose also that $\mathcal{X}$ is closed symmetric monoidal.

(3) *We have natural equivalences $F(j_!u, x) \simeq j_* F_{\mathcal{U}}(u, j^*x)$ and $F(i_*z, x) \simeq i_* F_{\mathcal{Z}}(z, i^!x)$.*

(4) *The fiber sequence $i_*i^!x \to x \to j_*j^*x$ is equivalent to*

$$F(i_*1_Z, x) \to x \to F(j_!1_U, x).$$

(5) *We have natural equivalences $j^*F(x, y) \simeq F_{\mathcal{U}}(j^*x, j^*y)$ and $F_{\mathcal{Z}}(i^*x, i^!y) \simeq i^!F(x, y)$.*

Proof For (1), it's easily checked that $i^*\alpha$, $j^*\alpha$ and $i^*\beta$, $j^*\beta$ are equivalences, hence α and β are equivalences. Item (2) then follows as a corollary. For (3), we have sequences of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathcal{X}}(y, F(j_!u, x)) &\simeq \mathrm{Map}_{\mathcal{X}}(y \otimes j_!u, x) \simeq \mathrm{Map}_{\mathcal{X}}(j_!(j^*y \otimes u), x) \simeq \mathrm{Map}_{\mathcal{U}}(j^*y \otimes u, j^*x)\\ &\simeq \mathrm{Map}_{\mathcal{U}}(j^*y, F_{\mathcal{U}}(u, j^*x)) \simeq \mathrm{Map}_{\mathcal{X}}(y, j_*F_{\mathcal{U}}(u, j^*x)),\\ \mathrm{Map}_{\mathcal{X}}(y, F(i_*z, x)) &\simeq \mathrm{Map}_{\mathcal{X}}(y \otimes i_*z, x) \simeq \mathrm{Map}_{\mathcal{X}}(i_*(i^*y \otimes z), x) \simeq \mathrm{Map}_{\mathcal{Z}}(i^*y \otimes z, i^!x)\\ &\simeq \mathrm{Map}_{\mathcal{Z}}(i^*y, F_{\mathcal{Z}}(z, i^!x)) \simeq \mathrm{Map}_{\mathcal{Z}}(y, i_*F_{\mathcal{Z}}(z, i^!x)).\end{aligned}$$

If we let $u = 1_U$, then $F_{\mathcal{U}}(1_U, v) \simeq v$, hence $F(j_!1_U, x) \simeq j_*F_{\mathcal{U}}(1_U, j^*x) \simeq j_*j^*x$. Item (4) then follows as a corollary. For (5), we have sequences of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathcal{U}}(u, j^*F(x, y)) &\simeq \mathrm{Map}_{\mathcal{X}}(j_!u, F(x, y)) \simeq \mathrm{Map}_{\mathcal{X}}(j_!u \otimes x, y) \simeq \mathrm{Map}_{\mathcal{X}}(j_!(u \otimes j^*x), y)\\ &\simeq \mathrm{Map}_{\mathcal{U}}(u \otimes j^*x, j^*y) \simeq \mathrm{Map}_{\mathcal{U}}(u, F_{\mathcal{U}}(j^*x, j^*y)),\\ \mathrm{Map}_{\mathcal{Z}}(z, F_{\mathcal{Z}}(i^*x, i^!y)) &\simeq \mathrm{Map}_{\mathcal{Z}}(z \otimes i^*x, i^!y) \simeq \mathrm{Map}_{\mathcal{X}}(i_*(z \otimes i^*x), y) \simeq \mathrm{Map}_{\mathcal{X}}(i_*z \otimes x, y)\\ &\simeq \mathrm{Map}_{\mathcal{X}}(i_*z, F(x, y)) \simeq \mathrm{Map}_{\mathcal{Z}}(z, i^!F(x, y)).\end{aligned}$$ □

From Proposition 2.34, we immediately deduce the fundamental decomposition formula for objects in a stable symmetric monoidal recollement.

2.35 Corollary (decomposition formula) *Suppose that $(\mathcal{U}, \mathcal{Z})$ is a stable symmetric monoidal recollement of a closed symmetric monoidal stable ∞-category $\mathcal{X}$. Then for all $x \in \mathcal{X}$, we have a commutative diagram*

$$\begin{tikzcd} x \otimes j_!(1_U) \arrow[r] \arrow[d, "\simeq"] & x \arrow[r] \arrow[d] & x \otimes i_*(1_Z) \arrow[d] \\ F(j_!(1_U), x) \otimes j_!(1_U) \arrow[r] & F(j_!(1_U), x) \arrow[r] & F(j_!(1_U), x) \otimes i_*(1_Z) \end{tikzcd}$$

in which the right-hand square is a pullback square.

For example, Corollary 2.35 abstracts the well-known fracture square decomposition of a G-spectrum with respect to a family of subgroups, and conversely can be used to deduce it (see [20, §2.2]).

Finally, we record the following relation between stable symmetric monoidal recollements and smashing localizations.

2.36 Observation Suppose $\mathcal{X}$ is a symmetric monoidal stable ∞-category and $\mathcal{Z} \subset \mathcal{X}$ is a reflective and coreflective subcategory that determines a stable recollement $(\mathcal{U}, \mathcal{Z})$ on $\mathcal{X}$. Then this recollement is symmetric monoidal if and only if $i_* i^*$ is compatible with the symmetric monoidal structure on $\mathcal{X}$ and the resulting projection formula for $i^* \dashv i_*$ holds, i.e., the natural map $i_* z \otimes x \to i_*(z \otimes i^* x)$ is an equivalence for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$. Indeed, the "only if" direction hold by Proposition 2.34, and for the "if" direction, we only need to show that for every $x \in \mathcal{X}$ such that $j^* x \simeq 0$, $j^*(x \otimes y) \simeq 0$ for every $y \in \mathcal{X}$. But $j^* x \simeq 0$ if and only if $x \simeq i_* i^* x$, and then

$$j^*(x \otimes y) \simeq j^*(i_* i^* x \otimes y) \simeq j^*(i_*(i^* x \otimes i^* y)) \simeq 0.$$

Suppose further that $\mathcal{X}$ and $\mathcal{Z}$ are presentable. In view of [15, Proposition 5.29], $\mathcal{Z}$ is a *smashing localization* of $\mathcal{X}$ in the sense that $\mathcal{Z} \simeq \mathbf{Mod}_{\mathcal{X}}(A)$ for $A = i_* i^* 1$ an idempotent E_∞-algebra in $\mathcal{X}$. We deduce that smashing localizations of $\mathcal{X}$ are in bijective correspondence with stable symmetric monoidal recollements of $\mathcal{X}$. Moreover, if $F : \mathcal{X} \to \mathcal{X}'$ is a morphism of symmetric monoidal recollements $(\mathcal{U}, \mathcal{Z}) \to (\mathcal{U}', \mathcal{Z}')$, then

$$F i_* i^* 1 \simeq i'_* i'^* F(1) \simeq i'_* i'^* 1,$$

so F preserves the defining idempotent E_∞-algebras.

2.3 Families of recollements

We conclude this section with a few extensions of recollement theory to the parametrized setting. Let S be an ∞-category, let $\mathcal{X}_\bullet : S \to \mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}$ be a functor, and let $\mathcal{X}, \mathcal{U}, \mathcal{Z} \to S$ be the cocartesian fibrations obtained via the Grothendieck construction. Then in view of Observation 2.5 and the strictness assumption, we have S-adjunctions [21, Definition 8.3][11]

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}.$$

In what follows, we use the following terminology from [21]:

(1) An *S-∞-category* is a cocartesian fibration $C \to S$.

(2) Given two S-∞-categories $C, D \to S$, the *∞-category of S-functors* $\mathrm{Fun}_S(C, D)$ is notation for $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(C, D)$.

We first show that the procedure of taking S-functor categories yields a recollement.

2.37 Lemma *For any S-∞-category K, $(\mathrm{Fun}_S(K, \mathcal{U}), \mathrm{Fun}_S(K, \mathcal{Z}))$ is a recollement of $\mathrm{Fun}_S(K, \mathcal{X})$.*

Proof By [21, Proposition 8.4], we have induced adjunctions given by postcomposition

$$\mathrm{Fun}_S(K, \mathcal{U}) \underset{\bar{j}_*}{\overset{\bar{j}^*}{\leftrightarrows}} \mathrm{Fun}_S(K, \mathcal{X}) \underset{\bar{i}_*}{\overset{\bar{i}^*}{\rightleftarrows}} \mathrm{Fun}_S(K, \mathcal{Z}),$$

[11] Recall given two cocartesian fibrations $C, D \to S$ that a relative adjunction $F : C \rightleftarrows D : G$ with respect to S in the sense of Lurie [14, Definition 7.3.2.2] is said to be an *S-adjunction* if F and G both preserve cocartesian edges.

where it is clear that $\bar{j}^* \bar{j}_* \simeq \mathrm{id}$ and $\bar{i}^* \bar{i}_* \simeq \mathrm{id}$, hence $\bar{j}_*$ and $\bar{i}_*$ are fully faithful. By [12, Proposition 5.4.7.11], the hypothesis that for all $f : s \to t$ the restriction functors $f^* : \mathcal{X}_t \to \mathcal{X}_s$ preserve finite limits ensures that $\mathrm{Fun}_S(K, \mathcal{X})$ admits finite limits (which are computed fiberwise), and similarly the induced restriction functors f_U^* and f_Z^* preserve finite limits, so $\mathrm{Fun}_S(K, \mathcal{U})$, $\mathrm{Fun}_S(K, \mathcal{Z})$ admit finite limits and $\bar{j}^*, \bar{i}^*$ preserve finite limits. Since $j^* i_* \simeq 0$ and the terminal object $0 \in \mathrm{Fun}_S(K, \mathcal{U})$ is given by $K \to S \xrightarrow{\underline{0}} \mathcal{U}$ for the cocartesian section $\underline{0} : S \to \mathcal{U}$ that selects the terminal object in each fiber, we get that $\bar{j}^* \bar{i}_* \simeq 0$. Finally, since a morphism f in $\mathrm{Fun}_S(K, \mathcal{X})$ is an equivalence if and only if $f(k)$ is an equivalence for all $k \in K$, we deduce that $\bar{j}^*$ and $\bar{i}^*$ are jointly conservative using the joint conservativity of j^* and i^*. □

2.38 Corollary *The forgetful functors* $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}} \to \mathbf{Cat}_\infty$ *and* $\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}} \to \mathbf{Cat}^{\mathrm{stab}}_\infty$ *create limits.*

Proof The first statement follows from Lemma 2.37 by taking $K = S$ and using that the ∞-category of cocartesian sections computes the limit of a diagram of ∞-categories [12, §3.3.3]. We note that the proof of Lemma 2.37 shows that the evaluation functors at any $s \in S$ are left-exact and strict morphisms of recollements, so the limit resides in $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}$. Finally, because limits in $\mathbf{Cat}^{\mathrm{stab}}_\infty$ are created in $\mathbf{Cat}_\infty$, the second statement follows. □

We can also use Lemma 2.37 to compute S-colimits in $\mathcal{X}$. For clarity, let us revert to the nonparametrized case $S = *$ for the next two results; the S-analogues will also hold by the same reasoning.

2.39 Lemma *Let* $(\mathcal{U}, \mathcal{Z})$ *be a recollement of* $\mathcal{X}$ *and suppose that* $\mathcal{U}$ *and* $\mathcal{Z}$ *admit* K*-indexed colimits. Then* $\mathcal{X}$ *admits* K*-indexed colimits.*

Proof With respect to the recollement of $\mathrm{Fun}(K, \mathcal{X})$ of Lemma 2.37, the constant diagram functor $\delta : \mathcal{X} \to \mathrm{Fun}(K, \mathcal{X})$ is obviously a morphism of recollements. Passing to left adjoints, we obtain a right-lax commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(K, \mathcal{U}) \arrow[r, "\bar{i}^* \bar{j}_*"] \arrow[d, "\mathrm{colim}"'] \arrow[dr, phantom, "\swarrow"] & \mathrm{Fun}(K, \mathcal{Z}) \arrow[d, "\mathrm{colim}"] \\ \mathcal{U} \arrow[r, "i^* j_*"'] & \mathcal{Z} \end{tikzcd}$$

which induces a morphism of recollements $\mathrm{colim} : \mathrm{Fun}(K, \mathcal{X}) \to \mathcal{X}$. We claim that colim is left adjoint to δ. In fact, if $\mathcal{M}, \mathcal{M}^K \to \Delta^1$ are the cartesian fibrations classified by $i^* j_*$ and $\bar{i}^* \bar{j}_*$ respectively, then we have a map $\delta : \mathcal{M}^K \to \mathcal{M}$ of cartesian fibrations and by [14, Proposition 7.3.2.6] a relative left adjoint $\mathrm{colim} : \mathcal{M}^K \to \mathcal{M}$. The formation of sections sends relative adjunctions to adjunctions, which proves the claim. We deduce that $\mathcal{X}$ admits K-indexed colimits. □

2.40 Corollary *Suppose* $\mathcal{U}$ *and* $\mathcal{Z}$ *are presentable* ∞*-categories and* $\phi : \mathcal{U} \to \mathcal{Z}$ *is a left-exact accessible functor. Then* $\mathcal{X} = \lim^{\mathrm{rlax}} \phi$ *is a presentable* ∞*-category.*

Proof By Lemma 2.39, $\mathcal{X}$ admits all small colimits. By [12, Corollary 5.4.7.17], $\mathcal{X}$ is accessible. We conclude that $\mathcal{X}$ is presentable. □

Finally, we describe how recollements interact with an ambidextrous adjunction (e.g., the adjunction between restriction and induction for equivariant spectra).

2.41 Lemma *Let $(\mathcal{U},\mathcal{Z})$ and $(\mathcal{U}',\mathcal{Z}')$ be stable recollements on $\mathcal{X}$ and $\mathcal{X}'$ and let $f^*:\mathcal{X}\to\mathcal{X}'$ be an exact functor such that $f^*|_{i_*(\mathcal{Z})}\subset i_*(\mathcal{Z}')$ (so f^* is not necessarily a morphism of recollements, but we still may define ${f_U}^* := j'^* f^* j_*$, ${f_Z}^* := i'^* f^* i_*$, and have ${f_U}^* j^* \simeq j'^* {f_U}^*$).*

(1) *Suppose that $f^*|_{j_!(\mathcal{U})}\subset j'_!(\mathcal{U}')$ and f^* admits a right adjoint f_*. Then:*

(a) *The essential image of $f_* j'_*$ lies in $j_*(\mathcal{U})$, so $f^*\dashv f_*$ restricts to an adjunction*

$$ {f_U}^*:\mathcal{U}\rightleftarrows\mathcal{U}':f_{U*} $$

with $j_* f_{U*}\simeq f_* j'_*$.

(b) *The natural map $j^* f_*\to f_{U*} j'^*$ is an equivalence.*

(c) *The essential image of $f_* i'_*$ lies in $i_*(\mathcal{Z})$, so $f^*\dashv f_*$ restricts to an adjunction*

$$ {f_Z}^*:\mathcal{Z}\rightleftarrows\mathcal{Z}':f_{Z*} $$

with $i_* f_{Z*}\simeq f_* i'_*$.

(2) *Suppose that $f^*|_{j_*(\mathcal{U})}\subset j'_*(\mathcal{U}')$ and f^* admits a left adjoint $f_!$. Then:*

(a) *The essential image of $f_* j'_!$ lies in $j_!(\mathcal{U})$, so $f_!\dashv f^*$ restricts to an adjunction*

$$ f_{U!}:\mathcal{U}'\rightleftarrows\mathcal{U}:{f_U}^* $$

with $j_! f_{U!}\simeq f_! j'_!$.

(b) *The natural map $f_{U!} j^*\to j'^* f_!$ is an equivalence.*

(c) *The essential image of $f_! i'_*$ lies in $i_*(\mathcal{Z})$, so $f_!\dashv f^*$ restricts to an adjunction*

$$ f_{Z!}:\mathcal{Z}'\rightleftarrows\mathcal{Z}:{f_Z}^* $$

with $i_* f_{Z!}\simeq f_! i'_*$.

(d) *The natural map $i^* f_{Z!}\to f_{Z!} i'^*$ is an equivalence.*

(3) *Suppose that $f^*\in\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$, f^* admits left and right adjoints $f_!$ and f_*, and we have the ambidexterity equivalence $f_!\simeq f_*$. Then $f_*\in\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$ and we additionally have ambidexterity equivalences $f_{U!}\simeq f_{U*}$ and $f_{Z!}\simeq f_{Z*}$.*

Proof We first prove the assertions of (1). For (1)(a), for any $u'\in\mathcal{U}'$ because we have for all $z\in\mathcal{Z}$ that

$$ \mathrm{Map}_{\mathcal{X}}(i_*z, f_* j'_* u')\simeq \mathrm{Map}_{\mathcal{U}'}(j'^* f^* i_* z, u')\simeq \mathrm{Map}_{\mathcal{U}'}(f_U^* j'^* i_* z, u')\simeq *, $$

we get $f_* j'_* u'\in j_*(\mathcal{U})$. For (1)(b), the assertion holds because the map is adjoint to the equivalence $f^* j_!\to j'_! {f_U}^*$. For (1)(c), for any $z'\in\mathcal{Z}'$ we have

$$ j^* f_* i'_* z'\simeq f_{U*} j^* i'_* z'\simeq f_{U*} 0\simeq 0, $$

hence $f_* i'_* z' \in i_*(\mathcal{Z})$. Next, the assertions of (2) hold by a dual argument; we note that the extra assertion (2)(d) holds because $f_!$ now commutes with $j_!$ instead of j_*. Finally, for (3) the functor $f_! \simeq f_*$ is in $\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$ by combining (1)(a), (1)(c), and (2)(a). For the ambidexterity assertions, the equivalence $f_{Z!} \simeq f_{Z*}$ is clear because the embedding $i_* : \mathcal{Z} \subset \mathcal{X}$ is unambiguous, whereas for $f_{U!} \simeq f_{U*}$ we note that the sequence of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathcal{U}}(u, f_{U!}u') &\simeq \mathrm{Map}_{\mathcal{X}}(j_! u, f_! j'_! u') \simeq \mathrm{Map}_{\mathcal{X}}(j_! u, f_* j'_! u') \simeq \mathrm{Map}_{\mathcal{X}'}(f^* j_! u, j'_! u') \\ &\simeq \mathrm{Map}_{\mathcal{X}'}(j'_! f_U{}^* u, j'_! u') \simeq \mathrm{Map}_{\mathcal{U}'}(f_U{}^* u, u')\end{aligned}$$

demonstrates that $f_{U!}$ is right adjoint to $f_U{}^*$ and hence $f_{U!} \simeq f_{U*}$. □

2.42 Corollary *Let G be a finite group. Suppose that $\mathcal{X}_\bullet : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$ is a functor such that the underlying G-∞-category $\mathcal{X}$ is G-stable* [16, Definition 7.1]. *Then $\mathcal{U}$ and $\mathcal{Z}$ are G-stable and all of the functors appearing in the diagram of G-adjunctions*

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook, shift right, "j_*"'] & \mathcal{X} \arrow[l, shift right, "j^*"'] \arrow[r, shift left, "i^*"] & \mathcal{Z} \arrow[l, hook', shift left, "i_*"] \end{tikzcd}$$

are G-exact.

Proof By Lemma 2.41, it only remains to check the Beck–Chevalley condition for $\mathcal{U}$ and $\mathcal{Z}$ to show the existence of finite G-products. But this follows from the same condition on $\mathcal{X}$, since the restriction and induction functors $(f_-)^*$, $(f_-)_*$ commute with the inclusion functors $(j_\bullet)_*$, $(j_\bullet)_!$, and $(i_\bullet)_*$. □

2.43 Definition In the situation of Corollary 2.42, we say that $(\mathcal{U}, \mathcal{Z})$ is a *G-stable G-recollement* of $\mathcal{X}$.

3 Recollements on lax limits of ∞-categories

Let S be an ∞-category throughout this section. Suppose $p : C \to S$ is a locally cocartesian fibration classified by a 2-functor into $\mathbf{Cat}_\infty$ [12, Definition 1.1.5.1; 13, §3], so for every 2-simplex $\Delta^2 \to S$, we have a lax commutative diagram of ∞-categories

$$\begin{tikzcd} C_0 \arrow[rr, "F_{02}"] \arrow[dr, "F_{01}"'] & \Downarrow & C_2 \\ & C_1 \arrow[ur, "F_{12}"'] & \end{tikzcd}$$

and the higher-dimensional simplices of S supply coherence data. Then the 2-functoriality of f yields two notions of lax limit corresponding to choosing two possible orientations for morphisms — informally, the *left-lax* limit of f has objects given by tuples $(x_i \in C_i, \alpha_{ij} : F_{ij}(x_i) \to x_j)$, whereas the *right-lax* limit of f has objects given by tuples $(x_i \in C_i, \alpha_{ij} : x_j \to F_{ij}(x_i))$. To give rigorous meaning to these notions, we may circumvent giving a precise formulation of the lax universal property (for instance, as carried out in [8]) and instead *define* the left-lax limit to be the ∞-category of sections

$$\lim^{\mathrm{llax}} f = \lim^{\mathrm{llax}} C := \mathrm{Fun}_{/S}(S, C)$$

and the right-lax limit to be the ∞-category

$$\lim^{\mathrm{rlax}} f = \lim^{\mathrm{rlax}} C := \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C),$$

where $\mathrm{sd}(S)$ is the *barycentric subdivision* of S (Definition 3.22) that is locally cocartesian over S via the *max* functor (Construction 3.24), and we let $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(-,-)$ be the full subcategory on those functors over S that preserve *locally cocartesian* edges. Viewing f itself as a *left-lax diagram* in $\mathbf{Cat}_\infty$, we may thereby speak of left-lax and right-lax limits of left-lax diagrams of ∞-categories; dually, we may also speak of left-lax and right-lax limits of right-lax diagrams of ∞-categories encoded as locally cartesian fibrations. We refer to [1, §1; 2, §A] for a more detailed discussion.[12]

3.1 Definition Let $S' \subset S$ be a full subcategory. Then S' is a *sieve* if for every morphism $x \to y$ in S, if $y \in S'$, then $x \in S'$. Dually, S' is a *cosieve* if $(S')^{\mathrm{op}}$ is a sieve in S^{op}.

Given a sieve $S_0 \subset S$ and cosieve $S_1 \subset S$, we say that S_0 and S_1 form a *sieve-cosieve decomposition* of S if S_0 and S_1 are disjoint and any object $x \in S$ lies either in S_0 or S_1.

3.2 Remark Sieves and cosieves are necessarily stable under equivalences. Given a sieve-cosieve decomposition (S_0, S_1) of S, we may define a functor $\pi : S \to \Delta^1$ that sends each object $x \in S$ to the integer $i \in \{0,1\}$ such that $x \in S_i$. Conversely, any functor $\pi : S \to \Delta^1$ determines a sieve-cosieve decomposition of S by taking its fibers over 0 and 1.

Our main goal in this section is to describe how sieve-cosieve decompositions of S produces recollements on right-lax limits of locally cocartesian fibrations $p : C \to S$ (Theorem 3.39).

3.3 Remark As we saw in Observation 2.9, a recollement itself is an example of a right-lax limit over Δ^1. Given a working theory of (pointwise) right-lax Kan extensions, our results should follow from the usual transitivity property of Kan extensions applied to the factorization $S \xrightarrow{\pi} \Delta^1 \to *$. However, we are not aware of such a theory that also affords the explicit description of the gluing functor given in Theorem 3.32; indeed, Theorem 3.32 should precisely amount to a pointwise formula for the right-lax Kan extension along π. We refer the interested reader to the discussion in [11, §2.2] for more on this question.

3.1 Recollements on right-lax limits of strict diagrams

Before entering into our study of left-lax diagrams, let us consider the simpler case of strict diagrams $f : S \to \mathbf{Cat}_\infty$. For this case, right-lax limits are modeled by sections of the *cartesian* fibration that classifies f. Thus suppose that $p : C \to S$ is a cartesian fibration, $\pi : S \to \Delta^1$ is a functor, and let $p_0 : C_0 \to S_0$, $p_1 : C_1 \to S_1$ denote the pullbacks of p to the fibers S_0, S_1 of π. Given a section $F : S \to C$ of p, let $j^*F : S_1 \to C_1$ be its restriction over S_1 and let $i^*F : S_0 \to C_0$ be its restriction over S_0. We obtain functors

$$j^* : \mathrm{Fun}_{/S}(S, C) \to \mathrm{Fun}_{/S_1}(S_1, C_1), \quad i^* : \mathrm{Fun}_{/S}(S, C) \to \mathrm{Fun}_{/S_0}(S_0, C_0).$$

[12] We follow [1, §1] in referring to these two types of lax limits as "left" and "right", even though lax and oplax are more standard nomenclature. The terminology is consistent with the usage of left for cocartesian-type constructions and right for cartesian-type constructions (e.g., left and right fibrations).

We first explain when j^* and i^* admit right adjoints. Suppose $G : S_1 \to C_1$ is a section of p_1. For every $x \in S$, let

$$G_x : (S_1)_{x/} := S_1 \times_S S_{x/} \to S_1 \xrightarrow{G} C_1 \subset C$$

be the composite functor and consider the commutative diagram

$$\begin{tikzcd} (S_1)_{x/} \arrow[r, "G_x"] \arrow[d] & C \arrow[d, "p"] \\ (S_1)_{x/}^{\triangleleft} \arrow[r] \arrow[ur, dotted, "\overline{G_x}"'] & S \end{tikzcd}$$

where the cone point is sent to x. By [12, Corollary 4.3.1.11], if for every $s \in S$, C_s admits $(S_1)_{x/}$-indexed limits, and for every $f : s \to t$, the pullback functor $f^* : C_t \to C_s$ preserves $(S_1)_{x/}$-indexed limits, then there exists a dotted lift $\overline{G_x}$ which is a p-limit of G_x. If this holds for all $x \in S$, then by the dual of [12, Lemma 4.3.2.13], the p-right Kan extension j_*G exists and is computed pointwise by these p-limits. Moreover, by [12, Proposition 4.3.2.17], the right adjoint j_* then exists and is computed objectwise by j_*G.

Now let $H : S_0 \to C_0$ be a section of p_0. The same results hold for computing i_*H. However, the slice ∞-categories $(S_0)_{x/}$ are empty when $x \in S_1$. Therefore, the hypotheses above amount to supposing that for all $s \in S$, C_s admits a terminal object, and for all $f : s \to t$, the pullback functor f^* preserves this terminal object.

Finally, let $\mathcal{K} = \{K_\alpha\}_{\alpha \in A}$ be a class of simplicial sets and suppose that for all $K \in \mathcal{K}$ and $s \in S$, the fiber C_s admits K-indexed limits, and for all $f : s \to t$, the pullback functor f^* preserves K-indexed limits. Then by [12, dual of Proposition 5.4.7.11 and Remark 5.4.7.13], $\mathrm{Fun}_{/S}(S, C)$ admits K-indexed limits such that the evaluation functors $\mathrm{ev}_s : \mathrm{Fun}_{/S}(S, C) \to C_s$ preserve K-indexed limits — in other words, the K-indexed limits in $\mathrm{Fun}_{/S}(S, C)$ are computed fiberwise.

3.4 Definition (standard existence assumptions, strict version) Let $p : C \to S$ be a cartesian fibration and let $\pi : S \to \Delta^1$ be a functor. We say that p satisfies the *standard recollement existence assumptions* with respect to π if:

(1) For all $s \in S$, C_s admits finite limits, and for all morphisms $f : s \to t$ in S, the pullback functor $f^* : C_t \to C_s$ preserves finite limits.

(2) For all $x \in S$, C_s admits $(S_1)_{x/}$-indexed limits, and for all morphisms $f : s \to t$ in S, the pullback functor $f^* : C_t \to C_s$ preserves $(S_1)_{x/}$-indexed limits.

Let us now suppose that we are in the situation of Definition 3.4.

3.5 Proposition *The adjunctions*

$$\mathrm{Fun}_{/S_1}(S_1, C_1) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}_{/S}(S, C) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}_{/S_0}(S_0, C_0)$$

together exhibit $\mathrm{Fun}_{/S}(S, C)$ *as a recollement of* $\mathrm{Fun}_{/S_1}(S_1, C_1)$ *and* $\mathrm{Fun}_{/S_0}(S_0, C_0)$*.*

Proof Note the functors j^* and i^* are left exact by the fiberwise computation of limits in section ∞-categories. Because $(S_0)_{x/} = \varnothing$ for all $x \in S_1$, we get that $j^* i_*$ is the constant functor at the terminal object of $\mathrm{Fun}_{/S_1}(S_1, C_1)$. Finally, i^* and j^* are jointly conservative because equivalences are detected objectwise in $\mathrm{Fun}_{/S}(S, C)$. □

3.6 Remark If the fibers of p are moreover stable ∞-categories, then the left-exact pullback functors f^* are necessarily exact and the recollement of Proposition 3.5 is stable.

3.7 Example Let $C \simeq D \times S$ and p be the projection to S. Then the recollement of Proposition 3.5 simplifies to

$$\mathrm{Fun}(S_1, D) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}(S, D) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}(S_0, D),$$

where $j : S_1 \to S$ and $i : S_0 \to S$ now denote the inclusions. Recollement theory then gives a calculational technique for computing the right Kan extension $\phi_* F$ of a functor $F : S \to D$ along $\phi : S \to T$. Namely, if we let $\phi_0 = \phi \circ i$, $\phi_1 = \phi \circ j$, $F_0 = F|_{S_0}$, and $F_1 = F|_{S_1}$, the pullback square Proposition 2.2 yields a pullback square

$$\begin{tikzcd} \phi_* F \arrow[r] \arrow[d] & (\phi_0)_* F_0 \arrow[d] \\ (\phi_1)_* F_1 \arrow[r] & (\phi_0)_*((j_* F_1)|_{S_0}) \end{tikzcd}$$

3.2 Recollements on right-lax limits of left-lax diagrams

We now seek to establish the analogue of Proposition 3.5 for right-lax limits of locally cocartesian fibrations. Although the ideas are straightforward, the categorical details turn out to be considerably more involved. We begin by proving some needed extensions to the theory of relative right Kan extensions initiated in [12, §4.1–3], which play a technical role in our construction of the recollement adjunctions. We then construct the barycentric subdivision $\mathrm{sd}(S)$ (Definition 3.22, but also see Observation 3.23), and extend the cocartesian pushforward of [21, Lemma 2.23] to the locally cocartesian situation (Theorems 3.20 and 3.26). Finally, given a sieve-cosieve decomposition of S and suitable hypotheses on the locally cocartesian fibration $p : C \to S$, we establish localizations in Theorem 3.32, Corollary 3.34, and Proposition 3.36, and show that these together constitute a recollement of the right-lax limit of p in Theorem 3.39.

3.2.1 Relative right Kan extension In [12, Proposition 4.3.1.10], Lurie gives a criterion for when a colimit diagram in a fiber of a locally cocartesian fibration is a relative colimit. In contrast, we will also need a separate understanding of when a *limit* diagram in a fiber is a relative limit. As indicated in Lemma 3.8, in this situation we can give an unconditional statement.

3.8 Lemma *Let S be an ∞-category and let $f : C \to S$ be a locally cocartesian fibration. Let $s \in S$ be an object and $\bar{p} : K^{\triangleleft} \to C_s$ a limit diagram that extends p. Then, viewed as a diagram in C, $\bar{p}$ is a*

f-limit diagram [12, 4.3.1.1], *i.e., the commutative square*

$$\begin{tikzcd} C_{/\bar{p}} \arrow[r] \arrow[d] & C_{/p} \arrow[d] \\ S_{/f\bar{p}} \arrow[r] & S_{/fp} \end{tikzcd}$$

is a homotopy pullback square.

Proof It suffices to show that

$$C_{/\bar{p}} \to C_{/p} \times_{S_{/fp}} S_{/f\bar{p}}$$

is a trivial Kan fibration. To this end, let $A \to B$ be a monomorphism of simplicial sets and consider the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C_{/\bar{p}} \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & C_{/p} \times_{S_{/fp}} S_{/f\bar{p}} \end{tikzcd}$$

This transposes to the lifting problem

$$\begin{tikzcd} A \star K^{\triangleleft} \cup_{A \star K} B \star K \arrow[r, "\beta"] \arrow[d] & C \arrow[d, "f"] \\ B \star K^{\triangleleft} \arrow[r, "\alpha"'] \arrow[ur, dotted, "\gamma"] & S \end{tikzcd}$$

Our approach will be to first pushforward to the fiber C_s using that f is a locally cocartesian fibration and then solve the lifting problem in C_s using that $\bar{p}$ is a limit diagram.

To begin, because $\bar{p}$ is a diagram in the fiber C_s, the map α factors as $B \star K^{\triangleright} \to B \star \Delta^0 \xrightarrow{\alpha'} S$ with $\alpha'|_{\Delta^0} = \{s\}$. We may define a map $r : (B \star \Delta^0) \times \Delta^1 \to B \star \Delta^0$ such that $r_0 = \mathrm{id}$ and r_1 is constant at Δ^0 in the following way: let $\pi : B \star \Delta^0 \to \Delta^1$ be the structure map of the join which sends B to $\{0\}$ and Δ^0 to $\{1\}$, and let ρ be the composite $(B \star \Delta^0) \times \Delta^1 \xrightarrow{\pi \times \mathrm{id}} \Delta^1 \times \Delta^1 \xrightarrow{\max} \Delta^1$, so the fiber of ρ over $\{0\}$ is $B \times \{0\}$. Then, recalling that maps $L \to X \star Y$ of simplicial sets over Δ^1 are equivalently specified by pairs of maps $(f_0 : L_0 \to X, f_1 : L_1 \to Y)$, r is the map over Δ^1 with respect to ρ and π given by $B \subset B \star \Delta^0$ and the constant map to Δ^0. Now let

$$h^{\alpha} : (B \star K^{\triangleleft}) \times \Delta^1 \to (B \star \Delta^0) \times \Delta^1 \xrightarrow{r} B \star \Delta^0 \xrightarrow{\alpha'} S,$$

so $h^{\alpha}_0 = \alpha$ and h^{α}_1 is constant at $\{s\}$. Also denote by h^{α} the restrictions of h^{α} to $(B \star K) \times \Delta^1$, $(A \star K^{\triangleleft}) \times \Delta^1$, and $(A \star K) \times \Delta^1$.

Let $\mathfrak{P} = (M_S, T, \varnothing)$ be the categorical pattern on $s\mathbf{Set}^+_{/S}$ that yields the locally cocartesian model structure, so M_S consists of all the edges in S, T consists of all the degenerate 2-simplices in S, and the fibrant objects are the locally cocartesian fibrations. By the criterion of [14, Lemma B.1.10] applied to $K \to B \star K$ (with the degenerate edges marked) and $\{0\} \to (\Delta^1)^\sharp$, the inclusion map of marked simplicial sets

$$(B \star K) \times \{0\} \cup_{(K \times \{0\})} K \times (\Delta^1)^\sharp \to (B \star K) \times (\Delta^1)^\sharp$$

is $\mathfrak{P}$-anodyne, and likewise replacing $K \to B \star K$ with $K^\lhd \to A \star K^\lhd$ and $K \to A \star K$. Using left properness of the locally cocartesian model structure, we deduce that the morphism

$$\begin{array}{c} (A \star K^\lhd \cup_{A \star K} B \star K) \times \{0\} \cup_{K^\lhd \times \{0\}} K^\lhd \times (\Delta^1)^\sharp \\ \downarrow \\ (A \star K^\lhd \cup_{A \star K} B \star K) \times (\Delta^1)^\sharp \end{array}$$

is $\mathfrak{P}$-anodyne. Consider the commutative square

$$\begin{tikzcd} (A \star K^\lhd \cup_{A \star K} B \star K) \times \{0\} \cup_{K^\lhd \times \{0\}} K^\lhd \times (\Delta^1)^\sharp \arrow[r] \arrow[d] & {}_\natural C \arrow[d, "f"] \\ (A \star K^\lhd \cup_{A \star K} B \star K) \times (\Delta^1)^\sharp \arrow[r, "h^\alpha"'] \arrow[ur, dotted, "h^\beta"] & S^\sharp \end{tikzcd}$$

where ${}_\natural C$ denotes the marking on C given by the f-locally cocartesian edges and the top horizontal map restricted to the first factor is β and to the second factor $K^\lhd \times (\Delta^1)^\sharp$ is the constant homotopy $K^\lhd \times \Delta^1 \xrightarrow{\mathrm{pr}} K^\lhd \xrightarrow{\bar{p}} C$. Then the dotted lift h^β exists, and the image of h^β_1 is contained in the fiber C_s.

Now consider the commutative triangle

$$\begin{tikzcd} A \star K^\lhd \cup_{A \star K} B \star K \arrow[r, "h^\beta_1"] \arrow[d] & C_s \\ B \star K^\lhd \arrow[ur, dotted, "\gamma_1"'] & \end{tikzcd}$$

Because $\bar{p} : K^\lhd \to C_s$ is a limit diagram, the map $(C_s)_{/\bar{p}} \to (C_s)_{/p}$ is a trivial Kan fibration. Therefore, the dotted lift γ_1 exists.

Next, define a map

$$\theta = (\theta', \theta'') : (B \times \Delta^1) \star K^\lhd \to (B \star K^\lhd) \times \Delta^1$$

by its factors

$$\theta' : (B \times \Delta^1) \star K^\lhd \xrightarrow{\mathrm{pr} \star \mathrm{id}} B \star K^\lhd,$$
$$\theta'' : (B \times \Delta^1) \star K^\lhd \xrightarrow{\mathrm{pr} \star \mathrm{id}} \Delta^1 \star K^\lhd \to \Delta^1 \star \Delta^0 \cong \Delta^2 \xrightarrow{\sigma^1} \Delta^1.$$

Here $\sigma^1 : \Delta^2 \to \Delta^1$ is the standard degeneracy map, so $\sigma^1(0) = 0$, $\sigma^1(1) = 1$, and $\sigma^1(2) = 1$. Also denote by θ the restriction to $(A \times \Delta^1) \star K^\lhd$, etc. Let

$$X = (A \times \Delta^1) \star K^\lhd \cup_{(A \times \Delta^1) \star K} (B \times \Delta^1) \star K \cup_{(A \times \{1\}) \star K^\lhd \cup_{(A \times \{1\}) \star K} (B \times \{1\}) \star K} B \star K^\lhd$$

and consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "(h^\beta\circ\theta)\cup\gamma_1"] \arrow[d, "\lambda"'] & C \arrow[d, "f"] \\ (B\times\Delta^1)\star K^{\lhd} \arrow[r, "h^\alpha\circ\theta"'] \arrow[ur, dotted, "h^\gamma"] & S \end{tikzcd}$$

(where for commutativity, we use that $\theta_1 : (B\times\{1\})\star K^{\lhd} \to (B\star K^{\lhd})\times\{1\}$ is an isomorphism). By the dual of [12, Lemma 2.1.2.4] applied to $A\to B$ and the right anodyne map $\{1\}\to\Delta^1$, the map

$$\lambda' : A\times\Delta^1 \cup_{A\times\{1\}} B\times\{1\} \to B\times\Delta^1$$

is right anodyne. Then by [12, Lemma 2.1.2.3] applied to λ' and the map $K\to K^{\lhd}$, λ is inner anodyne. Thus the dotted lift h^γ exists. Finally, let $\gamma = h^\gamma_0$ and observe that γ is a solution to the original lifting problem of interest. □

We briefly digress to complete the theory of Kan extensions by constructing relative Kan extensions along general functors (see Lurie's remark at the beginning of [12, §4.3.3]). Recall the relative join construction $-\star_{-}-$ of [21, Definition 4.1] along with its bifibration property [21, Lemma 4.8].

3.9 Definition Consider the commutative diagram of ∞-categories

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\phi"] & C \arrow[d, "p"] \\ Y \arrow[r, "\alpha"] & S \end{tikzcd}$$

where $p : C\to S$ is a categorical fibration. Suppose given the data of a functor $G : Y\to C$ over S and a homotopy $h : X\times\Delta^1\to C$ over S with $h_0 = G\circ\phi$ and $h_1 = F$. Let $\pi : Y\star_Y X\to Y$ be the structure map and let $\overline{G} : Y\star_Y X \xrightarrow{\pi} Y \xrightarrow{G} C$. Since $\mathrm{Fun}(Y\star_Y X, C)\to\mathrm{Fun}(Y,C)\times\mathrm{Fun}(X,C)$ is a bifibration, we may select an edge $\overline{G}\to\overline{F}$ that is cocartesian over $h : G\circ\phi\to F$ in $\mathrm{Fun}(X,C)$ with degenerate image id_G in $\mathrm{Fun}(Y,C)$. Then we say that G is a *p-right Kan extension of F* along ϕ (exhibited via h) if the commutative diagram

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\iota_X"'] & C \arrow[d, "p"] \\ Y\star_Y X \arrow[r, "\alpha\circ\pi"'] \arrow[ur, "\overline{F}"] & S \end{tikzcd}$$

exhibits $\overline{F}$ as a p-right Kan extension of F in the sense of [12, Definition 4.3.2.2].

3.10 Remark In the initial setup of Definition 3.9, given $\overline{F} : Y\star_Y X\to C$ a map over S extending $F : X\to C$, let $G = \overline{F}|_Y : Y\to C$ and let $h : X\times\Delta^1 \xrightarrow{h'} Y\star_Y X \xrightarrow{\overline{F}} C$ with h' specified by the pair (ϕ, id_Y) (see the definition [21, Definition 4.1] of $-\star_Y-$ as $j_* : s\mathbf{Set}_{/Y\times\partial\Delta^1}\to s\mathbf{Set}_{/Y\times\Delta^1}$ for the inclusion $j : Y\times\partial\Delta^1\to Y\times\Delta^1$). Then $\overline{F}$ is a p-right Kan extension in the sense of [12, Definition 4.3.2.2] if and only if G is a p-right Kan extension along ϕ in the sense of Definition 3.9. Moreover, we have an equivalence of ∞-categories $X \times_{Y \star_Y X} (Y \star_Y X)_{y/} \simeq X \times_Y Y_{y/}$ implemented by pulling back the functors $\iota_Y : Y \subset Y \star_Y X$ and $\pi : Y \star_Y X \to Y$ and the respective induced functors on the slice categories via $X \subset Y \star_Y X$. Because of this, Lurie's existence and uniqueness theorem [12, Proposition 4.3.2.15] for p-right Kan extensions applies to show that the p-right Kan extension G of F along ϕ exists if and only if for every $y \in Y$, the diagram $X \times_Y Y_{y/} \to X \xrightarrow{F} C$ extends to a p-limit diagram (which then computes the value of G on y). Moreover, there is then a contractible space of choices for G.

3.11 Remark The situation of Definition 3.9 globalizes in the following manner. Suppose every functor $F : X \to C$ admits a p-right Kan extension to $\overline{F} : Y \star_Y X \to C$. By [12, Proposition 4.3.2.17], the restriction functor $(\iota_X)^* : \mathrm{Fun}_{/S}(Y \star_Y X, C) \to \mathrm{Fun}_{/S}(X, C)$ then admits a right adjoint $(\iota_X)_*$ which is computed on objects as $F \mapsto \overline{F}$. We also have a relative adjunction [14, Definition 7.3.2.2]

$$\iota_Y : Y \rightleftarrows Y \star_Y X : \pi$$

over Y (hence over S) where ι_Y is left adjoint to π. From this, we obtain an adjunction

$$\pi^* : \mathrm{Fun}_{/S}(Y, C) \rightleftarrows \mathrm{Fun}_{/S}(Y \star_Y X, C) : (\iota_Y)^*,$$

where π^* is left adjoint to $(\iota_Y)^*$. Composing these two adjunctions, we obtain the adjunction

$$\phi^* : \mathrm{Fun}_{/S}(Y, C) \rightleftarrows \mathrm{Fun}_{/S}(X, C) : \phi_*,$$

where ϕ_* is given on objects by sending F to its p-right Kan extension along ϕ.

3.12 Corollary *Suppose we have a commutative diagram of ∞-categories*

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\phi"] & C \arrow[d, "p"] \\ Y \arrow[r, "\alpha"] & S \end{tikzcd}$$

where p is a locally cocartesian fibration and ϕ is a cartesian fibration. Suppose that for every $y \in Y$, the limit of $F|_{X_y} : X_y \to C_{\alpha(y)}$ exists. Then the p-right Kan extension $G : Y \to C$ of F along ϕ exists and $G(y) \simeq \varprojlim F|_{X_y}$. If G exists for all F, then we have an adjunction

$$\phi^* : \mathrm{Fun}_{/S}(Y, C) \rightleftarrows \mathrm{Fun}_{/S}(X, C) : \phi_*,$$

where $\phi_*(F) \simeq G$.

Proof We need to show that for every $y \in Y$, the p-limit of $F^y : X \times_Y Y_{y/} \to X \xrightarrow{F} C$ exists. By Lemma 3.8, the p-limit of $F|_{X_y}$ exists and is computed as the limit of $F|_{X_y}$ viewed as a diagram in $C_{\alpha(y)}$. Because ϕ is a cartesian fibration, we have a retraction $r : X \times_Y Y_{y/} \to X_y$ to the inclusion $i : X_y \to X \times_Y Y_{y/}$ such that r is right adjoint to i (on objects, r is given by the formula $r(x, y \xrightarrow{e} \phi(x)) = e^*(x)$, where $e^* : X_{\phi(x)} \to X_y$ is the pullback functor encoded by the lifting property of the cartesian fibration ϕ).

As a left adjoint, i is right cofinal.[13] However, since $r \circ i = \mathrm{id}$, we moreover have that r is right cofinal by the right cancellative property of right cofinal maps [12, Proposition 4.1.1.3(2)]. Hence, by [12, Proposition 4.3.1.7] applied to r and a p-limit diagram $(X_y)^{\triangleleft} \to C$, the p-limit of F^y exists and is computed as the limit of $F|_{X_y}$ in $C_{\alpha(y)}$. The claim now follows from Remark 3.10. $\square$

3.2.2 Barycentric subdivision and locally cocartesian pushforward Our main goal in this subsection is to first define the barycentric subdivision $\mathrm{sd}(S)$ (Definition 3.22) consisting of conservative functors $\sigma : [n] \to S$ (i.e., *strings* in S) along with its *maximum* functor $\max_S : \mathrm{sd}(S) \to S$, $[\sigma : [n] \to S] \mapsto \sigma(n)$, which is a locally cocartesian fibration (Lemma 3.25). This allows us to define the right-lax limit of a locally cocartesian fibration $p : C \to S$ as

$$\lim^{\mathrm{rlax}} C := \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C).$$

We will then show that for any sieve $S_0 \subset S$, if we let $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$ denote the full subcategory of strings that originate in S_0, then the inclusion $\mathrm{sd}(S_0) \hookrightarrow \mathrm{sd}(S)_0$ is a locally cocartesian equivalence over S,[14] or equivalently, for any locally cocartesian fibration $p : C \to S$, the restriction functor

$$\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S)_0, C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C|_{S_0})$$

is a trivial fibration (Theorem 3.26(2)). A choice of inverse then amounts to a choice of *locally cocartesian pushforward*. This will be the formal half of extending an object in $\lim^{\mathrm{rlax}} C|_{S_0}$ to one in $\lim^{\mathrm{rlax}} C$ itself, which we take up in the next subsection.

To set the stage for our work, we first introduce a few combinatorial constructions. Let Δ be the category with objects the finite ordinals $\{[n] = \{0 < 1 < \cdots < n\} : n \in \mathbb{N}\}$ and morphisms the order-preserving maps. Let $\xi : \mathcal{E}\Delta \to \Delta$ denote the relative nerve [12, Definition 3.2.5.2] of the canonical inclusion $i : \Delta \hookrightarrow s\mathbf{Set}$. Then ξ is a cocartesian fibration classified by i, which is an explicit model for the tautological cocartesian fibration over Δ. Explicitly, an n-simplex $\Delta^n \to \mathcal{E}\Delta$ is given by a sequence $[a_0] \xrightarrow{\alpha_0} [a_1] \xrightarrow{\alpha_1} \cdots \xrightarrow{g_{n-1}} [\alpha_n]$ of order-preserving maps in Δ together with morphisms $\kappa_i : \Delta^{\{0,\dots,i\}} \cong \Delta^i \to \Delta^{a_i}$ which fit into a commutative diagram

$$\begin{tikzcd}
\Delta^{\{0\}} \arrow[r, hook] \arrow[d, "\kappa_0"] & \Delta^{\{0,1\}} \arrow[r, hook] \arrow[d, "\kappa_1"] & \cdots \arrow[r, hook] & \Delta^{\{0,\dots,n-1\}} \arrow[r, hook] \arrow[d, "\kappa_{n-1}"] & \Delta^n \arrow[d, "\kappa_n"] \\
\Delta^{a_0} \arrow[r, "\alpha_0"] & \Delta^{a_1} \arrow[r, "\alpha_1"] & \cdots \arrow[r] & \Delta^{a_{n-1}} \arrow[r, "\alpha_{n-1}"] & \Delta^{a_n}
\end{tikzcd}$$

Let $\mathcal{E}\Delta^{\mathrm{inj}} \subset \mathcal{E}\Delta$ denote the pullback over the subcategory $\Delta^{\mathrm{inj}} \subset \Delta$ of injective order-preserving maps and also denote the structure map of $\mathcal{E}\Delta^{\mathrm{inj}}$ by ξ. Consider the span of marked simplicial sets

$$(\Delta^{\mathrm{inj}})^\sharp \xleftarrow{\xi} {}_\natural(\mathcal{E}\Delta^{\mathrm{inj}}) \xrightarrow{\xi} (\Delta^{\mathrm{inj}})^\sharp,$$

[13] We adopt Lurie's terminology in [14]: recall that a map $q : K \to L$ is right cofinal if and only if q^{op} is cofinal.

[14] Here we mark those edges that are locally cocartesian with respect to $\max_{S_0}$ (resp. $\max_S$.)

where we mark the ξ-cocartesian edges in $\mathcal{E}\Delta^{\mathrm{inj}}$. Similar to the definition in [21, Example 2.25] (which considers the source input to be instead a cartesian fibration), let

$$\widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{inj}}}(\mathcal{E}\Delta^{\mathrm{inj}}, -) := \xi_* \xi^*(-) : s\mathbf{Set}^+_{/\Delta^{\mathrm{inj}}} \to s\mathbf{Set}^+_{/\Delta^{\mathrm{inj}}}.$$

Note that with ξ a cocartesian fibration, $\xi_*\xi^*$ is right Quillen with respect to the *cartesian* model structure on $s\mathbf{Set}^+_{/\Delta^{\mathrm{inj}}}$ by the dual of [21, Theorem 2.24], hence takes cartesian fibrations to cartesian fibrations.

3.13 Recollection [12, Corollary 3.2.2.13; 21, Example 2.25] Given an ∞-category B, a cocartesian fibration $\xi : K \to B$, and a cartesian fibration $D \to B$, the *pairing construction* $\widetilde{\mathrm{Fun}}_B(K, D)$ is defined in general as $\xi_*\xi^*(D^\natural)$ and is a cartesian fibration over B whose fibers over $b \in B$ are $\mathrm{Fun}(K_b, D_b)$, and whose functoriality with respect to a morphism $\alpha : b \to b'$ is given by

$$\alpha^* : \mathrm{Fun}(K_{b'}, D_{b'}) \to \mathrm{Fun}(K_b, D_b), \quad f \mapsto \alpha^* \circ f \circ \alpha_!,$$

where $\alpha_!$ and α^* denote the pushforward functors for K and D as well.

3.14 Definition The ∞-category of *paths*[15] in an ∞-category C is

$$\widehat{\mathrm{Ar}}(C) := \widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{inj}}}(\mathcal{E}\Delta^{\mathrm{inj}}, C \times \Delta^{\mathrm{inj}}).$$

Let $\xi_C : \widehat{\mathrm{Ar}}(C) \to \Delta^{\mathrm{inj}}$ denote the structure map of the cartesian fibration and note that its fiber over $[n] \in \Delta^{\mathrm{inj}}$ is $\mathrm{Fun}(\Delta^n, C)$ and the functoriality is that of restriction in the source variable.

In addition, let $\widehat{\mathrm{Ar}}^{\simeq}(S) \subset \widehat{\mathrm{Ar}}(S)$ be the maximal subright fibration, i.e., the wide subcategory on the ξ_S-cartesian edges over Δ^{inj} (so the fiber of $\widehat{\mathrm{Ar}}^{\simeq}(S)$ over $[n]$ is $\mathrm{Map}(\Delta^n, S)$), and for a functor $p : C \to S$, let

$$\widehat{\mathrm{Ar}}^{\simeq}_S(C) := \widehat{\mathrm{Ar}}^{\simeq}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(C).$$

3.15 Remark (classifying functor for paths) By [8, Proposition 7.3], the cartesian fibration

$$\xi_C : \widehat{\mathrm{Ar}}(C) \to \Delta^{\mathrm{inj}}$$

is classified by the functor

$$(\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathbf{Cat}_\infty, \quad [n] \mapsto \mathrm{Fun}(\Delta^n, C),$$

where the functoriality is with respect to precomposition in the first variable. It follows that we have an equivalence

$$\widehat{\mathrm{Ar}}^{\simeq}(C) \simeq \Delta^{\mathrm{inj}} \times_{\mathbf{Cat}_\infty} \mathbf{Cat}_\infty^{/C}$$

of right fibrations over Δ^{inj}.

3.16 Remark If $C \to S$ is a categorical fibration, then $\widehat{\mathrm{Ar}}(C) \to \widehat{\mathrm{Ar}}(S)$ is also a categorical fibration by [14, Proposition B.2.7].

[15] For us, a path in C is any n-simplex $\Delta^n \to C$. In contrast, we reserve the term "string" for objects of the barycentric subdivision $\mathrm{sd}(C)$ (see Definition 3.22).

3.17 Remark (explicit description of simplices) By definition, the datum of an n-simplex $\Delta^n \to \widehat{\mathrm{Ar}}(C)$ is given by a map of simplicial sets

$$\Delta^n \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to C.$$

For example, suppose $n = 0$ and $\Delta^0 \to \Delta^{\mathrm{inj}}$ selects the object $[a_0]$. Then we see that 0-simplices of $\widehat{\mathrm{Ar}}(C)$ lying over $[a_0]$ correspond to maps $\Delta^{a_0} \to C$. Indeed, since the fiber $(\mathcal{E}\Delta^{\mathrm{inj}})_{[a_0]}$ is definitionally isomorphic to Δ^{a_0}, we see that the fiber $\widehat{\mathrm{Ar}}(C)_{[a_0]}$ is equivalent to $\mathrm{Fun}(\Delta^{a_0}, C)$, as promised by Remark 3.15.

Now suppose that $n = 1$ and $\Delta^1 \to \Delta^{\mathrm{inj}}$ selects the inclusion $\alpha_1 : [a_0] \subset [a_1]$. Then the data of a map $f : \Delta^1 \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to C$ is equivalent to maps $f_0 : \Delta^{a_0} \to C$, $f_1 : \Delta^{a_1} \to C$, and a natural transformation $f_{01} : f_0 \to f_1 \circ \alpha_1 = f_1|_{[a_0]}$. Moreover, this is a *cartesian* edge in $\widehat{\mathrm{Ar}}(C)$ if and only if f sends cocartesian edges to equivalences, i.e., the natural transformation f_{01} is an equivalence. This is consistent with the functoriality of $\widehat{\mathrm{Ar}}(C)$ as being given by the pullback functor

$$\alpha_1^* : \mathrm{Fun}(\Delta^{a_1}, C) \to \mathrm{Fun}(\Delta^{a_0}, C).$$

3.18 Construction (variant associated to a sieve) Let $\pi : S \to \Delta^1$ be a functor and S_0 the fiber over 0. Let $\widehat{\mathrm{Ar}}(S)_0 \subset \widehat{\mathrm{Ar}}(S)$ be the full subcategory on those objects $\sigma : \Delta^n \to S$ such that $\pi\sigma(0) = 0$ (i.e., *on those paths originating in* S_0), and let $\widehat{\mathrm{Ar}}^{\simeq}(S)_0 := \widehat{\mathrm{Ar}}(S)_0 \cap \widehat{\mathrm{Ar}}^{\simeq}(S)$. Define the "initial segment" functor

$$\lambda_S : \widehat{\mathrm{Ar}}(S)_0 \to \widehat{\mathrm{Ar}}(S_0)$$

by the following rule:

(∗) Suppose $\sigma : \Delta^n \to \widehat{\mathrm{Ar}}(S)_0$ is an n-simplex, which corresponds to a sequence of inclusions

$$\Delta^{a_0} \xhookrightarrow{\alpha_1} \Delta^{a_1} \xhookrightarrow{\alpha_2} \cdots \xhookrightarrow{\alpha_n} \Delta^{a_n}$$

determining a map $a : \Delta^n \to \Delta^{\mathrm{inj}}$ and a functor $f : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ such that for every $0 \le i \le n$, the restriction $f_i : \Delta^{a_i} \to S$ has $f_i(0) \in S_0$. Let $b_i \in \Delta^{a_i}$ be the maximum element such that $f_i(b_i) \in S_0$, and note that a restricts to yield a sequence of inclusions

$$\begin{tikzcd}
\Delta^{b_0} \arrow[r, hook, "\beta_1"] \arrow[d, hook] & \Delta^{b_1} \arrow[r, hook, "\beta_2"] \arrow[d, hook] & \cdots \arrow[r, hook, "\beta_n"] & \Delta^{b_n} \arrow[d, hook] \\
\Delta^{a_0} \arrow[r, hook, "\alpha_1"] & \Delta^{a_1} \arrow[r, hook, "\alpha_2"] & \cdots \arrow[r, hook, "\alpha_n"] & \Delta^{a_n}
\end{tikzcd}$$

because we always have that $\alpha_i(b_{i-1}) \le b_i$ as S_0 is a sieve in S stable under equivalences. Let $b : \Delta^n \to \Delta^{\mathrm{inj}}$ be the map determined by the sequence of upper horizontal inclusions. Then f restricts to yield a map f_0:

$$\begin{tikzcd}
\Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \arrow[r, "f_0"] \arrow[d, hook] & C_0 \arrow[d, hook] \\
\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \arrow[r, "f"] & C
\end{tikzcd}$$

Define $\lambda_S(\sigma) : \Delta^n \to \widehat{\mathrm{Ar}}(S_0)$ to be the n-simplex determined by f_0. Now observe that this assignment is natural in Δ^n, hence defines a map of simplicial sets.

Observe that λ_S is a retraction of the inclusion $\widehat{\mathrm{Ar}}(S_0) \to \widehat{\mathrm{Ar}}(S)_0$ induced by $S_0 \to S$.

An edge $e : \Delta^1 \to \widehat{\mathrm{Ar}}(S)_0$ is ξ_S-cartesian if and only if the corresponding functor $f : \Delta^1 \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ sends every edge $(i \in [a_0]) \to (\alpha_1(i) \in [a_1])$ to an equivalence, and similarly for ξ_{S_0}-cartesian edges in $\widehat{\mathrm{Ar}}(S_0)$. Therefore, λ_S preserves cartesian edges and restricts to a map

$$\lambda_S : \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \to \widehat{\mathrm{Ar}}^{\simeq}(S_0).$$

3.19 Construction (variant associated to a sieve, relative version) Let $p : C \to S$ be a locally cocartesian fibration and let $p_0 : C_0 \to S_0$ be its fiber over 0. Note that

$$\widehat{\mathrm{Ar}}(C)_0 \cong \widehat{\mathrm{Ar}}(S)_0 \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(C).$$

Let

$$\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0 := \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \times_{\widehat{\mathrm{Ar}}(S)_0} \widehat{\mathrm{Ar}}(C)_0 \cong \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \times_{\widehat{\mathrm{Ar}}^{\simeq}(S)} \widehat{\mathrm{Ar}}^{\simeq}_S(C),$$

so $\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0 \subset \widehat{\mathrm{Ar}}^{\simeq}_S(C)$ is the full subcategory on objects $c : \Delta^n \to C$ with $c(0) \in C_0$. The initial segment functor $\lambda_{(-)}$ fits into a commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{Ar}}^{\simeq}(S)_0 \arrow[r, hook] \arrow[d, "\lambda_S"] & \widehat{\mathrm{Ar}}(S)_0 \arrow[d, "\lambda_S"] & \widehat{\mathrm{Ar}}(C)_0 \arrow[l, "p"'] \arrow[d, "\lambda_C"] \\
\widehat{\mathrm{Ar}}^{\simeq}(S_0) \arrow[r, hook] & \widehat{\mathrm{Ar}}(S_0) & \widehat{\mathrm{Ar}}(C_0) \arrow[l, "p_0"']
\end{tikzcd}$$

and therefore defines a functor $\lambda_p : \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0 \to \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0)$.

Finally, let $\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}} \subset \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0$ be the full subcategory on those objects $c : \Delta^n \to C$ such that if $i \in \Delta^n$ is the maximum element with $c(i) \in C_0$, then c sends every edge $\{j < j+1\}$, $j \geq i$, to a locally-p cocartesian edge (i.e., a cocartesian edge over Δ^1 in the pullback $\Delta^1 \times_S C$).

The next theorem implies that we can construct a *locally cocartesian pushforward* extending from C_0 to C along paths in the base S that originate in S_0. This will amount to a section of the trivial fibration considered therein.

3.20 Theorem *The map*

$$(\lambda_p, p) : \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}} \to \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0) \times_{p_0, \widehat{\mathrm{Ar}}^{\simeq}(S_0), \lambda_S} \widehat{\mathrm{Ar}}^{\simeq}(S)_0$$

is a trivial fibration of simplicial sets.

Proof We need to solve the lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}} \arrow[d, "(\lambda_p, p)"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0) \times_{\widehat{\mathrm{Ar}}^{\simeq}(S_0)} \widehat{\mathrm{Ar}}^{\simeq}(S)_0
\end{tikzcd}$$

Let $a : \Delta^n \to \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \to \Delta^{\mathrm{inj}}$ and $b : \Delta^n \to \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0) \to \Delta^{\mathrm{inj}}$ be as discussed in the definition of λ. This lifting problem transposes to

$$\begin{tikzcd}
\Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \cup_{\partial\Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}} \partial\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \arrow[r] \arrow[d, hook, "f"] & C \arrow[d, "p"] \\
\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

Consider $\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ as a marked simplicial set where an edge $(i \in \Delta^{a_k}) \to (j \in \Delta^{a_l})$, $\alpha : \Delta^{a_k} \to \Delta^{a_l}$, $\alpha(i) \le j$, is marked if and only if $k = l$ (so $\alpha = \mathrm{id}$), $b_k \le i$ and $j = i + 1$, and let the domain of f also inherit this marking. Then it suffices to show that f is a trivial cofibration in the locally cocartesian model structure on $s\mathbf{Set}^+_{/S}$, defined by the categorical pattern $\mathfrak{P} = (M_S, T, \varnothing)$ with M_S all of the edges in S and T consisting of the 2-simplices τ in S with the edge $\tau(\{1 < 2\})$ an equivalence. Proceeding by induction on n, by a two-out-of-three argument it suffices to show that the inclusion $f' : \Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ is a trivial cofibration. We define a filtration of the poset inclusion f' as follows:

$(*)$ Let $a_n - b_n = t$. For $0 \le k \le n$, let $\alpha_k : \Delta^{a_k} \to \Delta^{a_n}$ denote the inclusion. Let $P_r \subset \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ be the subposet on those objects $(i \in \Delta^{a_k})$ such that $\alpha_k(i) - b_n \le r$. Note that $P_0 = \Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$, because if $(i \in \Delta^{a_k})$ is such that $i > b_k$, then necessarily $\alpha_k(i) > b_n$, and likewise if $i \le b_k$, then $\alpha_k(i) \le b_n$ (this follows from the definitions of the b_i and that S_0 is a sieve stable under equivalences). Then we have that f' factors as a sequence of poset sieve inclusions $\Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} = P_0 \subset P_1 \subset \cdots \subset P_t = \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$.

It now suffices to show that $P_i \subset P_{i+1}$ is a trivial cofibration for all $0 \le i < t$. For simplicity, let us suppose $i = 0$ (and $t > 0$ for nontriviality), the other cases being proved similarly. Let $k \in [n]$ be the smallest element such that $b_n + 1 \in \Delta^{a_n}$ is in the image of $\alpha_k : \Delta^{a_k} \to \Delta^{a_n}$. Note then that for all $k \le l \le n$, $\alpha_l(b_l + 1) = b_n + 1$. View the poset $\Delta^{\{k,\dots,n\}} \times \Delta^1$ as a cosieve U in P_1 via the inclusion which sends $(l, 0)$ to $(b_l \in \Delta^{a_l})$ and $(l, 1)$ to $(b_l + 1 \in \Delta^{a_l})$. Then as a marked simplicial set, we have $U = (\Delta^{\{k,\dots,n\}})^\flat \times (\Delta^1)^\sharp$. By [14, B.1.10], the inclusion

$$U \cap P_0 = (\Delta^{\{k,\dots,n\}})^\flat \times \{0\} \to U = (\Delta^{\{k,\dots,n\}})^\flat \times (\Delta^1)^\sharp$$

is $\mathfrak{P}$-anodyne. Noting that P_0 and U together cover P_1, it thus suffices to show that we have a homotopy pushout square of ∞-categories

$$\begin{tikzcd}
U \cap P_0 \arrow[r] \arrow[d] & U \arrow[d] \\
P_0 \arrow[r] & P_1
\end{tikzcd}$$

as we would then deduce the lower horizontal map to be $\mathfrak{P}$-anodyne. For this, the criterion of Lemma 3.21 is easily verified. $\square$

3.21 Lemma *Suppose P is a poset, $Z \subset P$ is a sieve and $U \subset P$ is a cosieve such that $P = Z \cup U$. Then the commutative square*

$$\begin{tikzcd} U \cap Z \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] & P \end{tikzcd}$$

is a homotopy pushout square of ∞-categories if and only if for every $a \notin U$ and $c \notin Z$ such that $a \leq c$, the subposet $P_{a/\!/c} = \{b \in U \cap Z : a \leq b \leq c\}$ is weakly contractible.

Proof Define a map $\pi : P \to \Delta^2$ by

$$\pi(x) = \begin{cases} 0, & x \notin U, \\ 2, & x \notin Z, \\ 1, & x \in U \cap Z. \end{cases}$$

Observe that $P \times_{\Delta^2} \Delta^{\{0,1\}} = Z$, $P \times_{\Delta^2} \Delta^{\{1,2\}} = U$, and $P \times_{\Delta^2} \{1\} = U \cap Z$. We may therefore apply the flatness criterion of [14, B.3.2] to π in order to deduce the criterion in question. □

We now introduce the barycentric subdivision $\mathrm{sd}(S)$.

3.22 Definition An n-simplex $\sigma : \Delta^n \to S$ is a *string* if σ is a conservative functor, i.e., if for every $0 \leq i < j \leq n$, $\sigma(\{i < j\})$ is not an equivalence.[16] The *barycentric subdivision* (or *subdivision*)

$$\mathrm{sd}(S) \subset \widehat{\mathrm{Ar}}^{\simeq}(S)$$

is the full subcategory of $\widehat{\mathrm{Ar}}^{\simeq}(S)$ on the strings in S. Note that the structure map $\xi_S : \widehat{\mathrm{Ar}}^{\simeq}(S) \to \Delta^{\mathrm{inj}}$ restricts to define a right fibration $\xi_S : \mathrm{sd}(S) \to \Delta^{\mathrm{inj}}$.

Given a functor $C \to S$, the *S-relative subdivision* $\mathrm{sd}_S(C)$ is the pullback

$$\mathrm{sd}_S(C) := \mathrm{sd}(S) \times_{\widehat{\mathrm{Ar}}^{\simeq}(S)} \widehat{\mathrm{Ar}}^{\simeq}_S(C) \cong \mathrm{sd}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(C).$$

Similarly, parallel to Constructions 3.18 and 3.19 we may define $\mathrm{sd}(S)_0$, $\mathrm{sd}_S(C)_0$, and $\mathrm{sd}_S(C)_0^{\mathrm{cocart}}$ for a locally cocartesian fibration $C \to S$ and a functor $S \to \Delta^1$. To be specific, let $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$ be the full subcategory on those strings originating in the sieve S_0, let $\mathrm{sd}_S(C)_0 := \mathrm{sd}(S)_0 \times_{\mathrm{sd}(S)} \mathrm{sd}_S(C)$, and let $\mathrm{sd}_S(C)_0^{\mathrm{cocart}} := \mathrm{sd}_S(C)_0 \times_{\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0} \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}}$.

3.23 Observation Suppose that S is the nerve of a category, which we also denote as S. Then $\mathrm{sd}(S)$ is the nerve of the category whose objects are conservative functors $\sigma : \Delta^n \to S$, and where a morphism $[\sigma : \Delta^n \to S] \to [\tau : \Delta^m \to S]$ is given by the data of a map $\alpha : [n] \hookrightarrow [m]$ in Δ^{inj} and a natural equivalence $\sigma \xrightarrow{\simeq} \alpha^* \tau$. In particular, if S is the nerve of a poset P, then $\mathrm{sd}(P)$ is the nerve of the usual barycentric subdivision of P.

On the other hand, the usual definition of the subdivision of an ∞-category [1, Definition 1.15] is as the left Kan extension of the functor $\mathrm{sd} : \Delta \to \mathbf{Cat}_\infty$ along the fully faithful inclusion $\Delta \subset \mathbf{Cat}_\infty$. By

[16] If every retract in S is an equivalence, then it suffices to check that for every $0 \leq i < n$, $\sigma(\{i < i+1\})$ is not an equivalence.

[2, Lemma A.4.8], this recovers $\mathrm{sd}(P)$ for P a poset. In fact, we may transcribe over the proof there to show that $\mathrm{sd}(S) \xleftarrow{\simeq} \mathrm{colim}_{[n]\in\Delta_{/S}} \mathrm{sd}[n]$ for any ∞-category S. Here $\Delta_{/S} := \Delta \times_{\mathbf{Cat}_\infty} (\mathbf{Cat}_\infty)^{/S}$ is the maximal subright fibration in $\widetilde{\mathrm{Fun}}_\Delta(\mathcal{E}\Delta, S \times \Delta)$ (see Remark 3.15).[17] We sketch the argument, leaving routine details to the reader:

(1) First note that for any two strings $\sigma, \tau \in \mathrm{sd}(S)$, every map $[\sigma \Rightarrow \tau] \in \Delta_{/S}$ necessarily lies over Δ^{inj}. Therefore, the inclusion $i : \mathrm{sd}(S) \subset \Delta_{/S}$ is full. Moreover, in view of the factorization system on $\mathbf{Cat}_\infty$ whose right class of maps is given by the conservative functors [9, 11.29], i admits a left adjoint. In particular, i is cofinal, so

$$\mathrm{colim}_{[n]\in\Delta_{/S}} \mathrm{sd}[n] \simeq \mathrm{colim}_{[n]\in\mathrm{sd}(S)} \mathrm{sd}[n].$$

(2) We next observe that the cocartesian fibration $\mathrm{ev}_1 : \mathrm{Ar}(\mathrm{sd}(S)) \to \mathrm{sd}(S)$ is classified by the functor $\mathrm{sd}(S) \to \Delta^{\mathrm{inj}} \subset \Delta \xrightarrow{\mathrm{sd}} \mathbf{Cat}_\infty$. Therefore, $\mathrm{colim}_{[n]\in\mathrm{sd}(S)} \mathrm{sd}[n]$ identifies with the localization of $\mathrm{Ar}(\mathrm{sd}(S))$ at the class of ev_1-cocartesian edges. But this localization also identifies with the source functor $\mathrm{ev}_0 : \mathrm{Ar}(\mathrm{sd}(S)) \to \mathrm{sd}(S)$, yielding the desired equivalence $\mathrm{colim}_{[n]\in\mathrm{sd}(S)} \mathrm{sd}[n] \to \mathrm{sd}(S)$.

We now work towards constructing the "maximum" functor $\mathrm{sd}(S) \to S$. We first define this over $\widehat{\mathrm{Ar}}(S)$:

3.24 Construction Define a *last vertex* map $\max_S : \widehat{\mathrm{Ar}}(S) \to S$ by the following rule:

$(*)$ Suppose $\sigma : \Delta^n \to \widehat{\mathrm{Ar}}(S)$ is an n-simplex, which corresponds to a sequence of inclusions

$$\Delta^{a_0} \xhookrightarrow{\alpha_1} \Delta^{a_1} \xhookrightarrow{\alpha_2} \cdots \xhookrightarrow{\alpha_n} \Delta^{a_n}$$

determining a map $a : \Delta^n \to \Delta^{\mathrm{inj}}$ and a functor $f : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$. Define a functor $\chi : \Delta^n \to \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ to be the identity on the first component and the n-simplex

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[r, hook] \arrow[d, "\kappa_0"] & \Delta^{\{0,1\}} \arrow[r, hook] \arrow[d, "\kappa_1"] & \cdots \arrow[r, hook] & \Delta^n \arrow[d, "\kappa_n"] \\ \Delta^{a_0} \arrow[r, hook, "\alpha_1"] & \Delta^{a_1} \arrow[r, hook, "\alpha_2"] & \cdots \arrow[r, hook, "\alpha_n"] & \Delta^{a_n} \end{tikzcd}$$

of $\mathcal{E}\Delta^{\mathrm{inj}}$ uniquely specified by $\kappa_i(i) = a_i$ on the second component. Then $\max_S(\sigma) = f \circ \chi : \Delta^n \to S$.

In other words, $\max_S$ is the functor induced by precomposing by the section $\Delta^{\mathrm{inj}} \to \mathcal{E}\Delta^{\mathrm{inj}}$ which selects the maximal vertex in every fiber.

The next lemma is obvious when S is a poset, so the reader only interested in that case should feel free to skip its proof.

3.25 Lemma (1) *The functor* $\max_S : \widehat{\mathrm{Ar}}(S) \to S$ *is a categorical fibration.*

(2) *The restricted functor* $\max_S : \widehat{\mathrm{Ar}}^{\simeq}(S) \to S$ *is a locally cocartesian fibration.*

(3) *The restricted functor* $\max_S : \mathrm{sd}(S) \to S$ *is a locally cocartesian fibration.*

[17] Beware that here $\Delta_{/S}$ does *not* denote the nerve of the category of simplices of S regarded as a simplicial set.

Proof (1) We first verify that $\max_S$ is an inner fibration. For this, let $n \geq 2$, $0 < k < n$, and consider the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \widehat{\mathrm{Ar}}(S) \arrow[d, "\max_S"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

Let $a : \Delta^n \to \Delta^{\mathrm{inj}}$ be the unique extension of the given $\Lambda^n_k \to \Delta^{\mathrm{inj}}$. The lifting problem then transposes to

$$\begin{tikzcd} \Delta^n \cup_{\Lambda^n_k} \Lambda^n_k \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \arrow[r] \arrow[d] & S \\ \Delta^n \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \arrow[ur, dotted] & \end{tikzcd}$$

and it suffices to show the vertical arrow is inner anodyne. Since $\mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^{\mathrm{inj}}$ is a cocartesian fibration, it is in particular a flat inner fibration, and the desired result follows.

We next show that $\max_S$ is a categorical fibration by lifting equivalences from the base. So suppose $e : \Delta^1 \to S$ is an equivalence and $\sigma : \Delta^n \to S$ is an object of $\widehat{\mathrm{Ar}}(S)$ such that $\max_S(\sigma) = \sigma(n) = e(0)$. The restriction of $\max_S$ to $\mathrm{Fun}(\Delta^n, S) \subset \widehat{\mathrm{Ar}}(S)$ is evaluation at $\{n\}$, which is a categorical fibration, so e lifts to an equivalence in $\mathrm{Fun}(\Delta^n, S)$ and hence in $\widehat{\mathrm{Ar}}(S)$.

(2) First observe that since $\widehat{\mathrm{Ar}}^{\simeq}(S) \subset \widehat{\mathrm{Ar}}(S)$ is a subcategory stable under equivalences, the restricted $\max_S$ functor is a categorical fibration by (1). To prove that $\max_S$ is a locally cocartesian fibration, it then suffices to prove that for any edge $e : s \to t$ in S that is *not* an equivalence, the pullback $\max_S(e) : \widehat{\mathrm{Ar}}^{\simeq}(S) \times_S \Delta^1 \to \Delta^1$ is a cocartesian fibration. To this end, we claim that an edge $\tilde{e} : x \to y$ lifting e is $\max_S(e)$-cocartesian if and only if the corresponding data of an inclusion $\alpha : \Delta^{a_0} \to \Delta^{a_1}$ and a functor $f : \Delta^1 \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ is such that in addition $a_1 = a_0 + 1$ and α is the inclusion of the initial segment. Note that given an object $x : \Delta^{a_0} \to S$ with $s = x(a_0)$, such a lift $\tilde{e}$ of e may be defined by "appending" e to x: indeed, let $y : \Delta^{a_0+1} \to S$ be an extension of $x \cup e : \Delta^{a_0} \cup_{a_0, \Delta^0, 0} \Delta^1 \to S$, let

$$r : \Delta^1 \times_{\alpha, \Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^{a_0+1}$$

be the retraction functor which fixes Δ^{a_0+1} and is given by α on Δ^{a_0}, and define $\tilde{e}$ as $y \circ r$. Hence, establishing the claim will complete the proof.

The "only if" direction will follow from the "if" direction together with the stability of cocartesian edges under equivalence. For the "if" direction, fix such an edge $\tilde{e}$. Recall from the definition that $\tilde{e} : x \to y$ is $\max_S(e)$-cocartesian if and only if for all objects $z \in \widehat{\mathrm{Ar}}^{\simeq}(S)$ with $\max_S(z) = t$, the commutative square

$$\begin{tikzcd} \mathrm{Map}_{\widehat{\mathrm{Ar}}^{\simeq}(S)_{\max_S = t}}(y, z) \arrow[r, "(\tilde{e})^*"] \arrow[d] & \mathrm{Map}_{\widehat{\mathrm{Ar}}^{\simeq}(S)}(x, z) \arrow[d, "\max_S"] \\ \{e\} \arrow[r] & \mathrm{Map}_S(s, t) \end{tikzcd}$$

is a homotopy pullback square. Viewing x as $x : \Delta^{a_0} \to S$, y as $y : \Delta^{a_0+1} \to S$, and z as $z : \Delta^{a_2} \to S$, and computing the mapping spaces in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ as a cartesian fibration over Δ^{inj}, we see that

$$\mathrm{Map}_{\widehat{\mathrm{Ar}}^{\simeq}(S)}(x,z) \simeq \coprod_{\gamma:[a_0]\subset[a_2]} \mathrm{Map}_{\mathrm{Map}(\Delta^{a_0},S)}(x,\gamma^*z).$$

Therefore, it suffices to show that for any *fixed* inclusion $\gamma : \Delta^{a_0} \hookrightarrow \Delta^{a_2}$ with $\gamma(a_0) < a_2$, letting $\beta : \Delta^{a_0+1} \to \Delta^{a_2}$ be the unique extension of γ with $\beta(a_0+1) = a_2$, we have that the square

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Map}(\Delta^{a_0+1},S)}(y,\beta^*z) \arrow[r, "\alpha^*"] \arrow[d] & \mathrm{Map}_{\mathrm{Map}(\Delta^{a_0},S)}(x,\gamma^*z) \arrow[d] \\
\{e\} \arrow[r] & \mathrm{Map}_{\iota S}(x(a_0), z(a_2))
\end{tikzcd}$$

is a homotopy pullback square (where the right vertical map sends $x \to \gamma^*z$ to the composite $x(a_0) \to z(\gamma(a_0)) \to z(a_2)$). (Here we implicitly use that maps in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ are natural transformations through equivalences to account for the $\max_S = t$ condition for the upper-left mapping space.) But this follows since $\mathrm{ev}_{a_0+1} : \mathrm{Fun}(\Delta^{a_0+1}, S) \to S$ is a cocartesian fibration with $\bar{x} \to y$ a cocartesian edge lifting e, where $\bar{x}$ is the degeneracy s_{a_0} applied to x (we note that $\mathrm{Map}_{\mathrm{Map}(\Delta^{a_0+1},S)}(\bar{x},\beta^*z) \simeq \mathrm{Map}_{\mathrm{Map}(\Delta^{a_0},S)}(x,\gamma^*z)$).

(3) This is clear from the description of the locally $\max_S$-cocartesian edges given in (2). □

Finally, we arrive at the main result of this subsection. Lemma 3.25 ensures that the following theorem is well formulated; also note that $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$ is a sublocally cocartesian fibration via $\max_S$ as it is the inclusion of a cosieve stable under equivalences.

3.26 Theorem *Let $p : C \to S$ be a locally cocartesian fibration and $\pi : S \to \Delta^1$ a functor. Let $p_0 : C_0 \to S_0$ be the fiber of p over 0.*

(1) *Restricting the domain and codomain of the map of Theorem 3.20 yields the map*

$$\mathrm{sd}_S(C)_0^{\mathrm{cocart}} \to \mathrm{sd}_{S_0}(C_0) \times_{\mathrm{sd}(S_0)} \mathrm{sd}(S)_0,$$

which is also a trivial fibration of simplicial sets.

(2) *Precomposition by the inclusion $\mathrm{sd}(S_0) \hookrightarrow \mathrm{sd}(S)_0$ defines a trivial fibration of simplicial sets*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S)_0, C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0).$$

For the proof, it will be convenient to introduce an auxiliary construction. Define a functor

$$\delta : \widehat{\mathrm{Ar}}(S) \to \widehat{\mathrm{Ar}}(\widehat{\mathrm{Ar}}(S))$$

by the following rule:

($*$) Suppose $\sigma : \Delta^n \to \widehat{\mathrm{Ar}}(S)$ is an n-simplex, which corresponds to a sequence of inclusions

$$\Delta^{a_0} \xhookrightarrow{\alpha_1} \Delta^{a_1} \xhookrightarrow{\alpha_2} \cdots \xhookrightarrow{\alpha_n} \Delta^{a_n}$$

determining a map $a : \Delta^n \to \Delta^{\mathrm{inj}}$ and a functor $f : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$. Define a map

$$\bar{a} : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^{\mathrm{inj}}$$

on objects by $\bar{a}(i \in \Delta^{a_k}) = \Delta^{\{0,\dots,i\}}$ and on morphisms $(i \in \Delta^{a_k}) \to (j \in \Delta^{a_l})$, $\alpha_{kl} : \Delta^{a_k} \to \Delta^{a_l}$, $\alpha_{kl}(i) \le j$, by restriction of α_{kl} to $\Delta^{\{0,\dots,i\}} \subset \Delta^{a_k}$ (which then is valued in $\Delta^{\{0,\dots,j\}} \subset \Delta^{a_l}$). Then define a functor of categories

$$\phi : (\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}) \times_{\bar{a},\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$$

by sending objects $(i \in \Delta^{a_k}, i' \le i)$ to $(i' \in \Delta^{a_k})$ and morphisms $(i \in \Delta^{a_k}, i' \le i) \to (j \in \Delta^{a_l}, j' \le j)$ (specified by the data of a map $\alpha_{kl} : \Delta^{a_k} \to \Delta^{a_l}$ such that $\alpha_{kl}(i) \le j$ and $\alpha_{kl}(i') \le j'$) to the morphism $(i' \in \Delta^{a_k}) \to (j' \in \Delta^{a_l})$ specified by the same data.

We may then specify a map

$$g : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \widehat{\mathrm{Ar}}(S)$$

defined over Δ^{inj} via $\bar{a}$ and the structure map ξ_S as adjoint to the map

$$f \circ \phi : (\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}) \times_{\bar{a},\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S.$$

The map g in turn defines the desired n-simplex $\delta(\sigma) : \Delta^n \to \widehat{\mathrm{Ar}}(\widehat{\mathrm{Ar}}(S))$.

Informally, δ sends paths $s_0 \to s_1 \to \cdots \to s_n$ to their "initial segment parametrization"

$$[s_0] \to [s_0 \to s_1] \to \cdots \to [s_0 \to s_1 \to \cdots \to s_n].$$

Next, using the functor $\max_S$ to make sense of the next statement, we may use δ to define functors

$$\begin{aligned}
&\delta : \widehat{\mathrm{Ar}}^{\simeq}(S) \to \widehat{\mathrm{Ar}}_S^{\simeq}(\widehat{\mathrm{Ar}}^{\simeq}(S)) = \widehat{\mathrm{Ar}}^{\simeq}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(\widehat{\mathrm{Ar}}^{\simeq}(S)),\\
&\delta : \mathrm{sd}(S) \to \mathrm{sd}_S(\mathrm{sd}(S)) = \mathrm{sd}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(\mathrm{sd}(S))
\end{aligned}$$

as the identity on the first factor and a restriction of δ on the second factor.

Proof of Theorem 3.26 Item (1) follows from Theorem 3.20 in view of the pullback square

$$\begin{tikzcd}
\mathrm{sd}_S(C)_0^{\mathrm{cocart}} \arrow[r] \arrow[d] & \widehat{\mathrm{Ar}}_S^{\simeq}(C)_0^{\mathrm{cocart}} \arrow[d] \\
\mathrm{sd}_{S_0}(C_0) \times_{\mathrm{sd}(S_0)} \mathrm{sd}(S)_0 \arrow[r] & \widehat{\mathrm{Ar}}_{S_0}^{\simeq}(C_0) \times_{\widehat{\mathrm{Ar}}^{\simeq}(S_0)} \widehat{\mathrm{Ar}}^{\simeq}(S)_0
\end{tikzcd}$$

For (2), we need to solve the lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & \mathrm{Fun}_{/S}^{\mathrm{cocart}}(\mathrm{sd}(S)_0, C) \arrow[d] \\
B \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}_{/S_0}^{\mathrm{cocart}}(\mathrm{sd}(S_0), C_0)
\end{tikzcd}$$

This transposes to

$$\begin{tikzcd} A \times \mathrm{sd}(S)_0 \cup_{A \times \mathrm{sd}(S_0)} B \times \mathrm{sd}(S_0) \arrow[r, "G \cup F"] \arrow[d, hook] & C \arrow[d, "p"] \\ B \times \mathrm{sd}(S)_0 \arrow[r, "\max_S"] \arrow[ur, dotted] & S \end{tikzcd}$$

The functoriality of $\mathrm{sd}_{S_0}(-)$ in its argument results in a functor

$$\mathrm{sd}_{S_0} : \mathrm{Fun}_{/S_0}(\mathrm{sd}(S_0), C_0) \to \mathrm{Fun}_{/S_0}\big(\mathrm{sd}_{S_0}(\mathrm{sd}(S_0)), \mathrm{sd}_{S_0}(C_0)\big).$$

Given $F : B \times \mathrm{sd}(S_0) \to C_0$, let $\mathrm{sd}_{S_0}(F) : B \times \mathrm{sd}_{S_0}(\mathrm{sd}(S_0)) \to \mathrm{sd}_{S_0}(C_0)$ denote the image. We then define $\overline{F}$ as the composite

$$B \times \mathrm{sd}(S_0) \xrightarrow{\mathrm{id} \times \delta} B \times \mathrm{sd}_{S_0}(\mathrm{sd}(S_0)) \xrightarrow{\mathrm{sd}_{S_0}(F)} \mathrm{sd}_{S_0}(C_0).$$

Also let $\overline{F}'$ denote $\overline{F}$ with codomain $\mathrm{sd}_S(C)_0^{\mathrm{cocart}}$ via the inclusion $\mathrm{sd}_{S_0}(C_0) \subset \mathrm{sd}_S(C)_0^{\mathrm{cocart}}$.

Similarly, given $G : A \times \mathrm{sd}(S)_0 \to C$, we may define $\overline{G}$ as the composite

$$A \times \mathrm{sd}(S)_0 \xrightarrow{\mathrm{id} \times \delta} A \times \mathrm{sd}_S(\mathrm{sd}(S)_0) \xrightarrow{\mathrm{sd}_S(G)} \mathrm{sd}_S(C)_0^{\mathrm{cocart}},$$

where we note that the codomain of $\mathrm{sd}_S(G)$ necessarily lies in $\mathrm{sd}_S(C)_0^{\mathrm{cocart}}$ by definition of the locally $\max_S$-cocartesian edges in $\mathrm{sd}(S)_0$ (here it is essential that we use $\mathrm{sd}(S)$ rather than $\widehat{\mathrm{Ar}}^{\simeq}(S)$). Clearly, $\overline{G}$ and $\overline{F}'$ are compatible on their common domain $A \times \mathrm{sd}(S_0)$ since G and F are. We thereby may factor the square above as

$$\begin{tikzcd} A \times \mathrm{sd}(S)_0 \cup_{A \times \mathrm{sd}(S_0)} B \times \mathrm{sd}(S_0) \arrow[r, "\overline{G} \cup \overline{F}'"] \arrow[d, hook] & \mathrm{sd}_S(C)_0^{\mathrm{cocart}} \arrow[r, "\max_C"] \arrow[d, "\simeq"] & C \arrow[d, "p"] \\ B \times \mathrm{sd}(S)_0 \arrow[r, "(\overline{F}\lambda, \mathrm{pr})"] \arrow[ur, dotted] & \mathrm{sd}_{S_0}(C_0) \times_{\mathrm{sd}(S_0)} \mathrm{sd}(S)_0 \arrow[r, "\max_S"] & S \end{tikzcd}$$

The dotted lift exists by (1), and postcomposition of such a lift by $\max_C$ defines the desired lift. □

3.2.3 Main results We begin by constructing a factorization system [12, Definition 5.2.8.8] on $\mathrm{sd}(S)$ associated to a sieve-cosieve decomposition of S. To do this, we need a few preparatory lemmas.

3.27 Lemma *Let $p : X \to S$ be a cartesian fibration. Given a functor $\phi : K \to X$, let*

$$\bar{p} : X^{\phi/} = \mathrm{Fun}(K^{\rhd}, X) \times_{\mathrm{Fun}(K,X)} \{\phi\} \to S^{p\phi/} = \mathrm{Fun}(K^{\rhd}, X) \times_{\mathrm{Fun}(K,X)} \{p\phi\}$$

be the functor induced by p. Then $\bar{p}$ is a cartesian fibration, and an edge $\bar{e} : \bar{x} \to \bar{y} \in X^{\phi/}$ is $\bar{p}$-cartesian if and only if the underlying edge $e : x \to y \in X$ is p-cartesian.

Proof We may mimic the proof of [12, 3.1.2.1] to prove the lemma, the essential tool being [12, 3.1.2.3]. In more detail, let E be the described collection of edges in $X^{\phi/}$ and suppose we are given a lifting problem in marked simplicial sets of the form

$$\begin{tikzcd} \Lambda^{n\natural}_n \arrow[r] \arrow[d] & (X^{\phi/}, E) \arrow[d, "\bar{p}"] \\ \Delta^{n\natural} \arrow[r] \arrow[ur, dotted] & (S^{p\phi/})^\sharp \end{tikzcd}$$

where we mark the edge $\{n-1,n\}$ of Λ^n_n (if $n > 1$) and of Δ^n. This transposes to a lifting problem of the form

$$\begin{tikzcd} \Lambda^{n\natural}_n \times K^{\triangleright} \cup_{\Lambda^{n\natural}_n \times K} \Delta^{n\natural} \times K \arrow[r, "f"] \arrow[d, "i"] & X^\natural \arrow[d, "p"] \\ \Delta^{n\natural} \times K^{\triangleright} \arrow[r] \arrow[ur, dotted] & S^\sharp \end{tikzcd}$$

where we mark the p-cartesian edges in X. Note that f is indeed a map of marked simplicial sets: this is by definition of E for f on the edge $\{n-1,n\} \times \{v\}$ ($v \in K^{\triangleright}$ the cone point), and by definition of f on $\Delta^n \times K$ as given by $\phi \circ \mathrm{pr}_K$ for the other marked edges. Applying [12, 3.1.2.3], we deduce that i is marked right anodyne, so the dotted lift exists. □

3.28 Lemma *Let $p : X \to S$ be a cartesian fibration. Suppose we have a commutative square in X*

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"] & z \arrow[d, "g"] \\ y \arrow[r, "k"] & w \end{tikzcd}$$

If the edge g is p-cartesian, then we have an equivalence

$$\mathrm{Map}_{x/\!\!/w}(y,z) \xrightarrow{\simeq} \mathrm{Map}_{px/\!\!/pw}(py,pz).$$

Proof In the statement of the lemma, the space of factorizations $\mathrm{Map}_{x/\!\!/w}(y,z)$ may be defined as the pullback

$$\begin{tikzcd} \mathrm{Map}_{x/\!\!/w}(y,z) \arrow[r] \arrow[d] & \mathrm{Map}_{x/}(y,z) \arrow[d, "g_*"] \\ \{k\} \arrow[r] & \mathrm{Map}_{x/}(y,w) \end{tikzcd}$$

and likewise for $\mathrm{Map}_{px/\!\!/pw}(py,pz)$.

Now by Lemma 3.27, $\bar{p} : X^{x/} \to S^{px/}$ is a cartesian fibration and g, viewed as an edge $h \to kf$, is a $\bar{p}$-cartesian edge. Therefore, we have a homotopy pullback square of spaces

$$\begin{tikzcd} \mathrm{Map}_{x/}(y,z) \arrow[r, "g_*"] \arrow[d, "p"] & \mathrm{Map}_{x/}(y,w) \arrow[d, "p"] \\ \mathrm{Map}_{px/}(py,pz) \arrow[r, "pg_*"] & \mathrm{Map}_{px/}(py,pw) \end{tikzcd}$$

Taking fibers over $k \in \mathrm{Map}_{x/}(y,w)$ and $pk \in \mathrm{Map}_{px/}(py,pw)$ yields the claimed equivalence. □

Fix a functor $\pi : S \to \Delta^1$ and let S_i denote the fiber over $i \in \{0,1\}$. We now define a factorization system on $\widehat{\mathrm{Ar}}^{\simeq}(S)$ that will restrict to a factorization system on the full subcategory $\mathrm{sd}(S)$. Recall that the data of a morphism $e : x \to y$ in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ is given by an inclusion $\alpha : \Delta^{a_0} \hookrightarrow \Delta^{a_1}$ and a map $f : \Delta^1 \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ that restricts to $x : \Delta^{a_0} \to S$ and $y : \Delta^{a_1} \to S$, such that f sends morphisms $(i \in \Delta^{a_0}) \to (\alpha(i) \in \Delta^{a_1})$ to equivalences in S.

3.29 Definition Let $\mathcal{L}$ be the subclass of morphisms $(\alpha, f) : x \to y$ such that for every $i \notin \operatorname{im}\alpha$, we have that $y(i) \in S_0$, and let $\mathcal{R}$ be the subclass of morphisms $(\alpha, f) : x \to y$ such that for every $i \notin \operatorname{im}\alpha$, we have that $y(i) \in S_1$.

3.30 Proposition *$(\mathcal{L}, \mathcal{R})$ defines a factorization system on $\widehat{\mathrm{Ar}}^{\simeq}(S)$ and on $\mathrm{sd}(S)$.*

Proof We will check the assertion concerning $\widehat{\mathrm{Ar}}^{\simeq}(S)$; the second assertion will then be a consequence. We first explain how to factor morphisms. Suppose that $\gamma : \Delta^{a_0} \hookrightarrow \Delta^{a_2}$, $h : \Delta^1 \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ is the data of a morphism in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ from x to z. Let $\Delta^{a_1} \subset \Delta^{a_2}$ be the subset on those $i \in \Delta^{a_2}$ such that $i \in \operatorname{im}\gamma$ or $z(i) \in S_0$. We then obtain a factorization of γ as

$$\Delta^{a_0} \xhookrightarrow{\alpha} \Delta^{a_1} \xhookrightarrow{\beta} \Delta^{a_2}.$$

Define $\bar{a} : \Delta^2 \to \Delta^{\mathrm{inj}}$, extending the given $a : \Delta^{\{0,2\}} \to \Delta^{\mathrm{inj}}$. Let $r : \Delta^2 \times_{\bar{a},\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^1 \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ be the unique retraction which is the identity on Δ^{a_0} and Δ^{a_2} and is given by β on Δ^{a_1}. Let $\bar{h} = h \circ r$. Then $\bar{h}$ is the desired factorization of h, as it corresponds to a factorization

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "h"'] & y \arrow[r, "g"] & z \end{tikzcd}$$

with $y = z \circ \beta : \Delta^{a_1} \to S$ defined so that $y(i) \in S_0$ for all $i \notin \operatorname{im}\alpha$ and $z(j) \in S_1$ for all $j \notin \operatorname{im}\beta$, hence $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

Next, observe that because S_0 and S_1 are closed under retracts, so are $\mathcal{L}$ and $\mathcal{R}$. It only remains to check that $\mathcal{L}$ is left orthogonal to $\mathcal{R}$. For this, suppose we are given a commutative square in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ on the left with $f \in \mathcal{L}$ and $g \in \mathcal{R}$ covering the square in Δ^{inj} on the right:

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"] & z \arrow[d, "g"] \\ y \arrow[r, "k"] \arrow[ur, dotted] & w \end{tikzcd} \qquad \begin{tikzcd} \Delta^a \arrow[r, "\delta"] \arrow[d, "\alpha"] & \Delta^c \arrow[d, "\beta"] \\ \Delta^b \arrow[r, "\kappa"] \arrow[ur, dotted, "\gamma"] & \Delta^d \end{tikzcd}$$

Because $\xi_S : \widehat{\mathrm{Ar}}^{\simeq}(S) \to \Delta^{\mathrm{inj}}$ is a right fibration, by Lemma 3.28 it suffices to show $\mathrm{Map}_{\Delta^a /\!\!/ \Delta^d}(\Delta^b, \Delta^c)$ is contractible. This holds if and only if $\Delta^b \subset \Delta^c$ when viewed as subsets of Δ^d, so that the mapping space is nonempty. Our hypothesis ensures that if $i \notin \operatorname{im}\beta$, then $w(i) \in S_1$, and if $i \in \Delta^b$, either $i \in \operatorname{im}\alpha$ or $y(i) \in S_0$. Therefore, we must have that for every $i \in \Delta^b$ with $i \notin \operatorname{im}\alpha$ that $w(\kappa(i)) \in S_0$, and hence $\kappa(i) \in \operatorname{im}\beta$. We conclude that the dotted lift γ exists. $\square$

Let $\mathrm{Ar}^L(\mathrm{sd}(S)) \subset \mathrm{Ar}(\mathrm{sd}(S))$ denote the full subcategory on those morphisms $x \to y$ in the class $\mathcal{L}$.

3.31 Lemma (1) *The inclusion $i : \mathrm{Ar}^L(\mathrm{sd}(S)) \subset \mathrm{Ar}(\mathrm{sd}(S))$ admits a right adjoint r that on objects sends $h : x \to y$ to $f : x \to z$ where h factors as $g \circ f$ according to the $(\mathcal{L}, \mathcal{R})$ factorization system.*

(2) *$i \dashv r$ defines a relative adjunction with respect to evaluation ev_0 at the source, and therefore for every $x \in \mathrm{sd}(S)$ we obtain an adjunction*

$$\{x\} \times_{\mathrm{sd}(S)} \mathrm{Ar}^L(\mathrm{sd}(S)) \rightleftarrows \mathrm{sd}(S)^{x/}.$$

(3) *The relative adjunction $i \dashv r$ restricts to a relative adjunction*

$$i : \mathrm{Ar}^L(\mathrm{sd}(S)) \times_{\mathrm{ev}_1, \mathrm{sd}(S)} \mathrm{sd}(S)_0 \rightleftarrows \mathrm{Ar}(\mathrm{sd}(S)) \times_{\mathrm{ev}_1, \mathrm{sd}(S)} \mathrm{sd}(S)_0 : r$$

and therefore for every $x \in \mathrm{sd}(S)$ we obtain an adjunction

$$\{x\} \times_{\mathrm{sd}(S)} \mathrm{Ar}^L(\mathrm{sd}(S)) \times_{\mathrm{sd}(S)} \mathrm{sd}(S)_0 \rightleftarrows \mathrm{sd}(S)_0^{x/}.$$

Proof Claim (1) is the dual formulation of [12, 5.2.8.19]. Claims (2) and (3) then follow by the definition of relative adjunction [14, 7.3.2.1] and its pullback property [14, 7.3.2.5]. □

We are now prepared to construct the recollement adjunctions. Note that the hypotheses of the following theorem are satisfied if S is equivalent to a finite poset and $p : C \to S$ is a locally cocartesian fibration such that the fibers admit finite limits and the pushforward functors preserve finite limits.

3.32 Theorem *Let $p : C \to S$ be a locally cocartesian fibration, let $\pi : S \to \Delta^1$ be a functor, and suppose we have a commutative diagram*

$$\begin{tikzcd} \mathrm{sd}(S)_0 \arrow[r, "F"] \arrow[d, hook, "\phi"] & C \arrow[d, "p"] \\ \mathrm{sd}(S) \arrow[r, "\max_S"] & S \end{tikzcd}$$

where F preserves locally cocartesian edges. Given $x \in \mathrm{sd}(S_1)$, let

$$J_x = \{x\} \times_{\mathrm{sd}(S)} \mathrm{Ar}^L(\mathrm{sd}(S)) \times_{\mathrm{sd}(S)} \mathrm{sd}(S)_0.$$

Note that $(\max_S \circ \mathrm{ev}_1)|_{J_x}$ is constant at $\max_S(x)$.

(1) *If for every $x \in \mathrm{sd}(S_1)$, the limit of $(F\,\mathrm{ev}_1)|_{J_x} : J_x \to C_{\max_S(x)}$ exists, then the p-right Kan extension G of F along ϕ exists and $G(x) \simeq \varprojlim F|_{J_x}$.*

(2) *If for every $f : s \to t$ in S, the pushforward functor $f_! : C_s \to C_t$ preserves all limits appearing in (1), then G preserves all locally cocartesian edges.*

(3) *If the hypotheses of (1) and (2) hold for all F, then we have an adjunction*

$$\phi^* : \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S)_0, C) : \phi_*.$$

Proof Note that $\mathrm{sd}(S_1) \subset \mathrm{sd}(S)$ is the complementary *sieve* inclusion to the *cosieve* $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$. For (1), to show existence of the p-right Kan extension it suffices for every $x \in \mathrm{sd}(S_1)$ to show that the p-limit of $F \circ \mathrm{pr}_1 : \mathrm{sd}(S)_0^{x/} \to \mathrm{sd}(S)_0 \to C$ exists. But by the argument of Corollary 3.12 applied to the adjunction $J_x \rightleftarrows \mathrm{sd}(S)_0^{x/}$ of Lemma 3.31, this follows from the given hypothesis.

For (2), first note that there are no locally $\max_S$-cocartesian edges $e : x \to y$ such that $x \in \mathrm{sd}(S_1)$ and $y \in \mathrm{sd}(S)_0$, or vice versa, so it suffices to handle the case where $e : x \to y$ is a locally $\max_S$-cocartesian edge in $\mathrm{sd}(S_1)$ only. Let $f : \max_S(x) = s \to \max_S(y) = t$ be the edge in $S_1 \subset S$. If f is an equivalence, then e is an equivalence and $G(e)$ is an equivalence, so we may suppose f is not an equivalence. Then by the description of the locally $\max_S$-cocartesian edges in Lemma 3.25, y is obtained from e by appending the edge f. Correspondingly, the functor $J_y \xrightarrow{\simeq} J_x$ defined via sending $y \to z$ to $x \to z$ by precomposing is an equivalence, using that such edges are constrained to only add objects in S_0. Examining how the functoriality of G is obtained from the pointwise existence criterion for Kan extensions, we see that the comparison morphism in C_t,

$$\psi : f_! G(x) \simeq f_!(\varprojlim F\,\mathrm{ev}_1|_{J_x}) \to G(y) \simeq \varprojlim F\,\mathrm{ev}_1|_{J_y},$$

is induced via the functoriality of limits (contravariant in the diagram, covariant in the target) from the commutative diagram

$$\begin{tikzcd} J_x \arrow[r, "F\,\mathrm{ev}_1"] & C_s \arrow[d, "f_!"] \\ J_y \arrow[u, "\simeq"] \arrow[r, "F\,\mathrm{ev}_1"] & C_t \end{tikzcd}$$

The hypothesis that $f_!$ preserve limits indexed by J_x together with $J_y \simeq J_x$ then proves that ψ is an equivalence.

Finally, for (3) it is clear that if $G : \mathrm{sd}(S) \to C$ preserves locally cocartesian edges, then the restriction $\phi^* G$ of G to $\mathrm{sd}(S)_0$ does as well. Items (1) and (2) establish the same fact for $\phi_* F$. Hence, the characteristic adjunction

$$\phi^* : \mathrm{Fun}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}_{/S}(\mathrm{sd}(S)_0, C) : \phi_*$$

of the p-right Kan extension along ϕ restricts to the full subcategories of functors preserving locally cocartesian edges in order to yield the desired adjunction. □

3.33 Remark Suppose that S is a poset and $x \in S_1 \subset \mathrm{sd}(S_1)$. Then the ∞-category J_x that appears in Theorem 3.32 is the poset whose objects are strings $[a_0 < \cdots < a_n < x]$, $n \geq 0$, with $a_i \in S_0$ and whose morphisms are string inclusions.

3.34 Corollary *Suppose the hypotheses of Theorem 3.32 are satisfied. Let $j : \mathrm{sd}(S_0) \to \mathrm{sd}(S)$ denote the inclusion. Then the functor j^* of restriction along j participates in an adjunction*

$$j^* : \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0) : j_*$$

with fully faithful right adjoint j_*.

Proof Combine Theorems 3.32 and 3.26(2). □

We have a far simpler result concerning the calculation of the left adjoint $j_!$ of j^* (but see Remark 3.41).

3.35 Proposition *Let $p : C \to S$ be a locally cocartesian fibration, let $\pi : S \to \Delta^1$ be a functor, and suppose that for every $s \in S_1$, the fiber C_s admits an initial object $\varnothing$, and for every $[f : s \to t] \in S_1$ the pushforward functors $f_!$ all preserve initial objects. Then j^* admits a fully faithful left adjoint $j_!$ such that for $F : \mathrm{sd}(S_0) \to C_0$, we have $j_! F(x) \simeq \varnothing$ for all $x \in \mathrm{sd}(S_1)$.*

Proof Suppose we have a commutative diagram

$$\begin{tikzcd} \mathrm{sd}(S)_0 \arrow[r, "F"] \arrow[d, hook, "\phi"] & C \arrow[d, "p"] \\ \mathrm{sd}(S) \arrow[r, "\max_S"] & S \end{tikzcd}$$

For all $x \in \mathrm{sd}(S_1)$, the fiber product $\mathrm{sd}(S)^{/x} \times_{\mathrm{sd}(S)} \mathrm{sd}(S)_0$ is the empty category. Therefore, under our assumption the p-left Kan extension $\phi_! F$ of F along ϕ exists and is computed by $\phi_! F(x) = \varnothing$ on $\mathrm{sd}(S_1)$. Combining this observation with Theorem 3.26(2), we obtain the desired adjunction

$$j_! : \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) : j^*. \qquad \square$$

We next turn to the cosieve inclusion $S_1 \subset S$. Note that the inclusion $i : \mathrm{sd}(S_1) \hookrightarrow \mathrm{sd}(S)$ is a sublocally cocartesian fibration with respect to $\max_S : \mathrm{sd}(S) \to S$, and is in addition a *sieve* inclusion, and hence i is a cartesian fibration. In fact, the cosieve inclusion $j : \mathrm{sd}(S)_0 \hookrightarrow \mathrm{sd}(S)$ is complementary to i.

3.36 Proposition *Let $p : C \to S$ be a locally cocartesian fibration, let $\pi : S \to \Delta^1$ be a functor, and suppose the fibers of p admit terminal objects and the pushforward functors preserve terminal objects. Then we have the adjunction*

$$i^* : \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C_1) : i_*$$

with i_* fully faithful, where i^* is given by restriction along i and i_* is p-right Kan extension along i. Moreover, for a functor $G : \mathrm{sd}(S_1) \to C_1$, we have $(i_* G)(x) \simeq * \in C_{\max_S(x)}$ for all $x \in \mathrm{sd}(S)_0$.

Proof By Corollary 3.12, using the hypothesis that the fibers of p admit terminal objects we have the adjunction

$$i^* : \mathrm{Fun}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}_{/S_1}(\mathrm{sd}(S_1), C_1) : i_*$$

with i^* and i_* as described. Then using that the pushforward functors preserve terminal objects, we see that this adjunction restricts to the one of the proposition. □

3.37 Lemma *Let $p : C \to S$ be a locally cocartesian fibration and suppose that the fibers C_s admit K-(co)limits and the pushforward functors preserve K-(co)limits. Then the category $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ admits K-indexed (co)limits, and for all $\sigma \in \mathrm{sd}(S)$ over $s = \max_S(\sigma)$, the evaluation functor $\mathrm{ev}_\sigma : \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \to C_s$ preserves K-indexed (co)limits. Moreover, if the fibers C_s are stable ∞-categories and the pushforward functors are exact, then $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ is a stable ∞-category.*

Proof Apply [12, Proposition 5.4.7.11] to the locally cocartesian fibration $\mathrm{sd}(S) \times_S C \to \mathrm{sd}(S)$, with the subcategory of $\widehat{\mathbf{Cat}}_\infty$ either taken to be those ∞-categories that admit K-indexed (co)limits and functors that preserve K-indexed (co)limits, or the subcategory $\mathbf{Cat}_\infty^{\mathrm{stab}}$ of stable ∞-categories and exact functors thereof. $\square$

We encapsulate the assumptions above on existence and preservation of various limits into the following definition (compare with Definition 3.4).

3.38 Definition (standard existence assumptions, left-lax version) Let $p : C \to S$ be a locally cocartesian fibration and let $\pi : S \to \Delta^1$ be a functor. We say that p satisfies the *standard recollement existence assumptions* with respect to π if:

(1) For all $s \in S$, C_s admits finite limits, and for all morphisms $f : s \to t$ in S, the pushforward functors $f_! : C_s \to C_t$ preserves finite limits.

(2) The hypotheses of Theorem 3.32 hold.

Finally, putting everything together, we get:

3.39 Theorem *Let $p : C \to S$ be a locally cocartesian fibration, let $\pi : S \to [1]$ be a functor, and suppose that p satisfies the standard recollement existence assumptions with respect to π. Then the two adjunctions of Corollary 3.34 and Proposition 3.36 combine to exhibit* $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ *as a recollement of* $\mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0)$ *and* $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C_1)$.

Proof We verify the conditions to be a recollement. By our assumption on p and Lemma 3.37, finite limits in $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ exist and are computed fiberwise. Therefore, the restriction functors j^* and i^* are left exact. By the formula for i_* given in Proposition 3.36, it is clear that $j^* i_*$ is constant at the terminal object. Finally, we check that j^* and i^* are jointly conservative. Suppose given a morphism $\alpha : F \to F'$ in $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ such that $j^*\alpha$ and $i^*\alpha$ are equivalences. Observe that α is an equivalence if and only if for all $x \in S$, $\alpha_x : F(x) \to F'(x)$ is an equivalence (viewing x as an object in $\mathrm{sd}(S)$). Because any object of S lies in either S_0 or S_1, we deduce that α is an equivalence. $\square$

3.40 Remark Suppose that S is a *down-finite* poset P. Let $C \to P$ be a locally cocartesian fibration such that its fiber admits finite limits and its pushforward functors preserve finite limits. Then the hypotheses of Theorem 3.32 automatically hold for every sieve-cosieve decomposition of P. Indeed, the categories J_x that appear there are all finite (see Remark 3.33).

Let us now return to the question of the existence of $j_!$.

3.41 Remark The left adjoint $j_!$ in Proposition 3.35 should exist even if we only suppose that the fibers of C admits initial objects (i.e., we need not suppose that the pushforward functors preserve initial objects). However, in that case $j_!$ will not generally be the p-left Kan extension along the inclusion ϕ, and relatedly, a direct proof of this would appear to be overly cumbersome in our framework. Rather, we can say the following (which covers most cases of practical relevance):

• Suppose that the hypotheses of Theorem 3.39 are satisfied and we have shown that $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C)$ admits an initial object. Then as in any recollement situation, the left adjoint $j_!$ exists and is computed by $j_!(u) = [u, \varnothing \to i^* j_*(u)]$.

• To exhibit the initial object of $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C)$, suppose also that S_1 is a finite poset P. Then using Theorem 3.39 in conjunction with Lemma 2.39, we may proceed by induction on the cardinality of P and repeatedly invoke our assumption that the fibers of C admit an initial object to conclude that $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C)$ admits an initial object whose evaluation at every singleton string is also initial.

We conclude this subsection by giving an application of Theorem 3.39 to the presentability of the right-lax limit $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$. First suppose that S is equivalent to a *finite* poset and write $P = S$.

3.42 Proposition *Suppose that the fibers C_s of $p : C \to P$ are presentable and the pushforward functors are left-exact and accessible. Then $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ is presentable, and for all $s \in P$, the evaluation functor $\mathrm{ev}_s : \mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C) \to C_s$ preserves (small) colimits and is accessible.*

Proof The accessibility statements follow from [12, Proposition 5.4.7.11] as in Lemma 3.37, so we only need to show the existence and preservation of small colimits. Our strategy is to proceed by induction on the cardinality of P. If $|P| \leq 1$, then the statement is clear. Suppose for the inductive hypothesis that we have established the statement for all posets Q such that $|Q| < |P|$. Let $b \in P$ be a maximal object and let $\pi : P \to \Delta^1$ be the functor determined by the sieve-cosieve decomposition $P_0 = P \setminus \{b\}$ and $P_1 = \{b\}$. Because the diagrams that appear in Theorem 3.32 are finite, we may apply Theorem 3.39 to decompose $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ as a recollement of $\mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C_0)$ and C_b. By the inductive hypothesis, both these ∞-categories admit all small colimits such that the evaluation functors at objects in P_0 and P_1 are colimit-preserving. By Lemma 2.39, we conclude that $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ admits all small colimits such that the evaluation functors for objects $s \in P$ are colimit-preserving. □

Next, we may use the equivalence (see Observation 3.23)

$$(\star) \qquad \mathrm{sd}(S) \xleftarrow{\simeq} \mathrm{colim}_{[n] \in \Delta_{/S}} \mathrm{sd}([n])$$

to promote Proposition 3.42 to a statement involving arbitrary S.

3.43 Corollary *Suppose the fibers C_s of $p : C \to S$ are presentable and the pushforward functors are left-exact and accessible. Then $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ is presentable.*

Proof We may simply copy over the proof strategy used to establish [2, Proposition 6.1.6(1)]. By ($\star$), we have that

$$\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \xrightarrow{\simeq} \lim_{[n] \in (\Delta_{/S})^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{cocart}}_{/[n]}(\mathrm{sd}[n], C|_{[n]}).$$

By Proposition 3.42 and Theorem 3.39, for every $[\sigma : [n] \to S] \in \Delta_{/S}$, $\lim^{\mathrm{rlax}} \sigma^* C$ is presentable and the evaluation functors $\{\mathrm{ev}_i : \lim^{\mathrm{rlax}} \sigma^* C \to C_{\sigma(i)}\}_{i=0}^n$ are colimit-preserving and jointly conservative. Note then that for any map $\alpha : [m] \to [n]$, the restriction functor

$$\alpha^* : \lim^{\mathrm{rlax}} \sigma^* C \to \lim^{\mathrm{rlax}} \alpha^* \sigma^* C$$

preserves colimits. Then since $\lim^{\mathrm{rlax}} C$ is a limit of presentable ∞-categories along colimit-preserving functors, it is presentable. $\square$

3.44 Remark We explain a subtle difference between our general approach and the one of [2, §6], which is adapted to the case of locally cocartesian fibrations $p : C \to P$ over a poset P whose fibers are presentable stable ∞-categories and whose pushforward functors are exact and accessible. Suppose one could prove directly that $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ is presentable (for any poset) and that the restriction functor $j^* : \mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C)$ preserves colimits, so that it admits a right adjoint j_*. Then without a pointwise formula for j_*, it is generally difficult to show that j_* is fully faithful. However, this would follow if we could also exhibit a fully faithful *left* adjoint $j_!$ to j^*, and this turns out to be easier to analyze (see Proposition 3.35). This is the strategy adopted in the proof of [2, Proposition 6.1.6].

Therefore, if we were only interested in the existence of the recollement on $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ in the stable presentable case, then we could bypass the work that goes into establishing the pointwise formula of Theorem 3.32. However, our primary motivation for undertaking this work lay precisely in having this pointwise formula. Note also that in the presentable case, the right adjoint j_* exists unconditionally even if it is not describable as a relative right Kan extension.

On the other hand, such tricks are not available in the absence of presentability (though for idempotent-complete small stable ∞-categories, one can pass to their **Ind**-completions as is done in [2, §7.2]). Over a down-finite poset P (see Remark 3.40), our Theorem 3.39 thus allows one to strengthen [2, Theorem A] by removing all of the presentability hypotheses therein.

3.2.4 Symmetric monoidal structure We briefly explain how to promote Theorem 3.39 to a statement involving symmetric monoidal recollements. First recall the notions of left-lax and right-lax morphisms of locally cocartesian fibrations from [2, §A.1 and A.3]:

3.45 Recollection Let $\lambda, \xi : \mathcal{C}, \mathcal{D} \to S$ be locally cocartesian fibrations. A *left-lax* morphism $\lambda \to \xi$ is a functor $F : \mathcal{C} \to \mathcal{D}$ over S (which need not preserve locally cocartesian edges). In contrast, a *right-lax* morphism $\lambda \to \xi$ is defined as in [2, Definition A.3.2] as the "unstraightened" counterpart to a right-lax natural transformation of left-lax functors.

The collection of locally cocartesian fibrations over S and right-lax morphisms thereof assemble into an ∞-category $\mathbf{LocCocart}_S^{\mathrm{rlax}}$ which contains $\mathbf{LocCocart}_S$ as a wide subcategory. Moreover, $\lim^{\mathrm{rlax}}$ extends to a functor over $\mathbf{LocCocart}_S^{\mathrm{rlax}}$ that is right adjoint to the constant functor $\mathrm{const} : \mathcal{E} \mapsto \mathcal{E} \times S$. See [2, Definitions A.3.2 and B.6.1].

In view of the adjunction $\mathrm{const} \dashv \lim^{\mathrm{rlax}}$, $\lim^{\mathrm{rlax}}$ sends commutative monoids in $\mathbf{LocCocart}_S^{\mathrm{rlax}}$ to symmetric monoidal ∞-categories. Moreover, a diagram chase shows that given a commutative monoid structure on $[p : C \to S]$, for any $\alpha : T \to S$ the pullback $[\alpha^* C \to T]$ is a commutative monoid in $\mathbf{LocCocart}_T^{\mathrm{rlax}}$ and the restriction functor $\lim^{\mathrm{rlax}} C \to \lim^{\mathrm{rlax}} \alpha^* C$ is symmetric monoidal. It follows that if the recollement of Theorem 3.39 exists in this situation, then it is symmetric monoidal.

3.46 Remark If $S = \Delta^1$, then a commutative monoid in $\mathbf{LocCocart}_{\Delta^1}^{\mathrm{rlax}}$ is the data of a lax symmetric monoidal functor of symmetric monoidal ∞-categories (see [11, Proposition 2.6]). In general, to endow

$p : C \to S$ with the structure of a commutative monoid entails endowing its fibers with symmetric monoidal structures and its pushforward functors and natural transformations thereof with lax symmetric monoidal structures in a coherent fashion. See [2, §4] for how to produce examples from simpler input.

4 1-generated and extendable objects

Suppose $S = \Delta^2$ and $p : C \to \Delta^2$ is a locally cocartesian fibration classified by a 2-functor

$$\begin{tikzcd} C_0 \arrow[rr, "H"] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\Downarrow"] & C_2 \\ & C_1 \arrow[ur, "G"'] & \end{tikzcd}$$

Then the data of a functor $\mathrm{sd}(\Delta^2) \to C$ over Δ^2 that preserves locally cocartesian edges can be summarized as follows:

- Objects $c_i \in C_i$ for $i = 0, 1, 2$.
- Morphisms $f : c_1 \to F(c_0)$, $g : c_2 \to G(c_1)$, and $h : c_2 \to H(c_0)$.
- A commutative square

$$\begin{tikzcd} c_2 \arrow[r, "h"] \arrow[d, "g"] & H(c_0) \arrow[d, "\mathrm{can}"] \\ G(c_1) \arrow[r, "G(f)"] & GF(c_0) \end{tikzcd}$$

Furthermore, if the map *can* is an equivalence, then the data of the commutative square and the morphism h are redundant, since then $h \simeq G(f) \circ g$ and compositions in an ∞-category are unique up to contractible choice. More precisely, if we let $\gamma_2 : \mathrm{sd}_1(\Delta^2) \subset \mathrm{sd}(\Delta^2)$ be the subposet on the set $\{[0], [1], [2], [0 < 1], [1 < 2]\}$, then the functor

$$\gamma_2^* : \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^2}(\mathrm{sd}(\Delta^2), C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^2}(\mathrm{sd}_1(\Delta^2), C)$$

is a trivial fibration onto its image when restricted to objects for which *can* is an equivalence.

Our goal in this section is to generalize this observation to the case where $S = \Delta^n$. We introduce subcategories of 1-generated and extendable objects (Definitions 4.5 and 4.12) and show their equivalence under the restriction functor γ_n^* (Theorem 4.15), given a stability hypothesis on $C \xrightarrow{p} \Delta^n$. This material will play an important role in [19].

4.1 Notation Let $\gamma_n : \mathrm{sd}_1(\Delta^n) \subset \mathrm{sd}(\Delta^n)$ be the subposet on strings $[k]$ and $[k < k+1]$.

We also introduce convenient notation for convex subposets of Δ^n.

4.2 Notation Let $[i : j] \subset \Delta^n$ denote the subposet on $i \le k \le j$.

Via its inclusion into $\mathrm{sd}(\Delta^n)$, we regard $\mathrm{sd}_1(\Delta^n)$ as a simplicial set over Δ^n (i.e., by the functor that takes the maximum) and as a marked simplicial set (so that each edge $[k] \to [k < k+1]$ is marked). We first state the analogue of Theorem 3.39 for sd_1, whose proof is far simpler.

4.3 Proposition *Let $p : C \to \Delta^n$ be a locally cocartesian fibration such that the fibers admit finite limits and the pushforward functors preserve finite limits. Let $0 \le k < n$, so the subcategories $[0:k] \cong \Delta^k$ and $[k+1:n] \cong \Delta^{n-k-1}$ of Δ^n give a sieve-cosieve decomposition. Then we have adjunctions*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:k]}\big(\mathrm{sd}_1([0:k]), C_{[0:k]}\big) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/[k+1:n]}\big(\mathrm{sd}_1([k+1:n]), C_{[k+1:n]}\big)$$

that exhibit $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)$ as a recollement.

Proof Let $j : \mathrm{sd}_1([0:k]) \to \mathrm{sd}_1(\Delta^n)$ and $i : \mathrm{sd}_1([k+1:n]) \to \mathrm{sd}_1(\Delta^n)$ be the inclusions, so j^* and i^* are defined by restriction along j and i. As in the proof of Lemma 3.37, our hypotheses on p ensure that the three ∞-categories admit finite limits and the functors j^* and i^* are left-exact. Moreover, since equivalences are detected on strings $[k]$, j^* and i^* are jointly conservative. The functor i_* is obtained by p-right Kan extension as in the proof of Proposition 3.36, and its essential image consists of functors $F : \mathrm{sd}_1(\Delta^n) \to C$ such that $F(i)$ is a terminal object in C_i for all $0 \le i \le k$, so $j^* i_*$ is the constant functor at the terminal object.

Finally, we show existence of j_*. Let $\mathrm{sd}_1([0:k])^+$ be the subposet of $\mathrm{sd}_1([0:n])$ on all objects in $\mathrm{sd}_1([0:k])$ and $\{[k < k+1]\}$, with marking inherited from $\mathrm{sd}(\Delta^n)$. Then we have a pushout square of marked simplicial sets

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d] & (\Delta^1)^\sharp \arrow[d] \\ \mathrm{sd}_1([0:k]) \arrow[r] & \mathrm{sd}_1([0:k])^+ \end{tikzcd}$$

so the inclusion $\mathrm{sd}_1([0:k]) \subset \mathrm{sd}_1([0:k])^+$ is $\mathfrak{P}$-anodyne for the categorical pattern $\mathfrak{P}$ defining the locally cocartesian model structure on $s\mathbf{Set}^+_{/\Delta^n}$. We thus obtain a trivial fibration

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:k+1]}\big(\mathrm{sd}_1([0:k])^+, C_{[0:k+1]}\big) \to \mathrm{Fun}^{\mathrm{cocart}}_{/[0:k]}\big(\mathrm{sd}_1([0:k]), C_{[0:k]}\big).$$

On the other hand, given a commutative diagram

$$\begin{tikzcd} \mathrm{sd}_1([0:k])^+ \arrow[r, "F"] \arrow[d] & C \arrow[d, "p"] \\ \mathrm{sd}_1([0:k+1]) \arrow[r] \arrow[ur, dotted, "G"] & \Delta^n \end{tikzcd}$$

since $\mathrm{sd}_1([0:k])^+ \times_{\mathrm{sd}_1([0:k+1])} \mathrm{sd}_1([0:k+1])_{[k+1]/} \cong \{[k<k+1]\}$, F admits a p-right Kan extension along $\mathrm{sd}_1([0:k])^+ \subset \mathrm{sd}_1([0:k+1])$ and G is a p-right Kan extension of F if and only if G sends the edge $[k+1] \to [k<k+1]$ to an equivalence. Therefore, we may alternate between anodyne extension

and p-right Kan extension along the filtration

$$\mathrm{sd}_1([0:k]) \subset \mathrm{sd}_1([0:k])^+ \subset \mathrm{sd}_1([0:k+1]) \subset \cdots \subset \mathrm{sd}_1([0:n-1])^+ \subset \mathrm{sd}_1(\Delta^n)$$

to define the functor j_*. Moreover, we see that the essential image of j_* consists of those functors $\mathrm{sd}_1(\Delta^n) \to C$ that send the edges $[l+1] \to [l < l+1]$ to equivalences for all $l \geq k$. □

We next wish to introduce a condition on objects of $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)$, which we term 1-*generated*, that indicates that the data of such objects is essentially determined by their restriction to $\mathrm{sd}_1(\Delta^n)$.

4.4 Notation Given a string $\sigma = [i < i+k]$ in $\mathrm{sd}(\Delta^n)$, let $Q_\sigma \subset \mathrm{sd}(\Delta^n)$ be the subposet on all strings $[i < \cdots < i+k]$. Note that Q_σ is a $(k-1)$-dimensional cube lying in the fiber $\mathrm{sd}(\Delta^n)_{\max=i+k}$ with σ as its minimal element.

4.5 Definition Let $C \to \Delta^n$ be a locally cocartesian fibration and $F : \mathrm{sd}(\Delta^n) \to C$ be a functor that preserves locally cocartesian edges. We say that F is 1-*generated* if for all strings $\sigma = [i < i+k]$ in $\mathrm{sd}(\Delta^n)$, $F|_{Q_\sigma}$ is a limit diagram in C_{i+k}.

Let $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{1\text{-gen}}$ be the full subcategory on the 1-generated objects.

4.6 Lemma *Let $C \to \Delta^n$ be a locally cocartesian fibration whose fibers are stable ∞-categories and whose pushforward functors are exact. Then $F : \mathrm{sd}(\Delta^n) \to C$ is 1-generated if and only if for all string inclusions $e : [i < i+k] \to [i < i+1 < i+k]$ in $\mathrm{sd}(\Delta^n)$, $F(e)$ is an equivalence in C_{i+k}.*

Proof We prove the stronger claim that for fixed $k \geq 2$ and all string inclusions $e_{ij} : \sigma_{ij} = [i < i+j] \to [i < i+1 < i+j]$ with $2 \leq j \leq k$, $F|_{Q_{\sigma_{ij}}}$ is a limit diagram for all $Q_{\sigma_{ij}}$ if and only if $F(e_{ij})$ is an equivalence for all e_{ij}.

We proceed by induction on k. For the base case $k = 2$, given a string inclusion $\sigma = [i < i+2] \to [i < i+1 < i+2]$, the edge is the 1-dimensional cube Q_σ, so $F|_{Q_\sigma}$ is a limit diagram if and only if $F(e)$ is an equivalence. Now let $k > 2$ and suppose we have proven the statement for all $l < k$. Note that in proving either direction of the "if and only if" statement, we may suppose that $F|_{Q_{\sigma_{ij}}}$ is a limit diagram *and* $F(e_{ij})$ for all $2 \leq j < k$, so let us do so.

Consider an edge $e : \sigma = [i < i+k] \to [i < i+1 < i+k]$. For $1 < j < k$, let $Q_{\sigma,j} \subset Q_\sigma$ be the subposet on strings excluding vertices $i+j, \dots, i+k-1$. Then we have a descending filtration of sieve inclusions

$$Q_\sigma := Q_{\sigma,k} \supset Q_{\sigma,k-1} \supset Q_{\sigma,k-2} \supset \cdots \supset Q_{\sigma,2},$$

where $Q_{\sigma,j}$ is a $(j-1)$-dimensional cube and $Q_{\sigma,2}$ consists only of the edge e. Note that if we let $Q'_{\sigma,j} = Q_{\sigma,j+1} \setminus Q_{\sigma,j}$ for $1 < j < k$, then the minimal element of $Q'_{\sigma,j}$ is given by $\sigma_j = [i < i+j < i+k]$, and if we let $\sigma'_j = [i < i+j]$, then $Q'_{\sigma,j}$ is obtained from $Q_{\sigma'_j}$ by concatenating $i+k$. By the inductive hypothesis and using that the pushforward functors are exact, we get that $F|_{Q'_{\sigma,j}}$ is a limit diagram. Taking total fibers of cubes then shows that $F|_{Q_{\sigma,j}}$ is a limit diagram if and only if $F|_{Q_{\sigma,j-1}}$ is a limit diagram. Traversing the filtration, we conclude that $F|_{Q_\sigma}$ is a limit diagram if and only if $F(e)$ is an equivalence. □

4.7 Lemma *Let $Q = \mathrm{sd}(\Delta^n)_{\max=n}$, D a stable ∞-category, and $f : Q \to D$ a functor. Suppose the following condition holds*:

($*$) *For all string inclusions $e : \sigma \to \sigma'$ in Q obtained by concatenating $[i < k] \to [i < i+1 < k]$ by a (possibly empty) suffix τ, $f(e)$ is an equivalence.*

Then f is a limit diagram if and only if $f([n] \to [n-1 < n])$ is an equivalence.

Proof The proof is similar to that of Lemma 4.6. For $0 \le j < n$, let $Q_{\ge j}$ (resp. $Q_{=j}$) be the subposet on strings σ with minimum $\ge j$ (resp. $= j$). Then $Q_{\ge j}$ is an $(n-j)$-dimensional cube, $Q_{=j} = Q_{\ge j} \setminus Q_{\ge j+1}$ is an $(n-j-1)$-dimensional cube, and we have a descending filtration

$$Q = Q_{\ge 0} \supset Q_{\ge 1} \supset Q_{\ge 2} \supset \cdots \supset Q_{\ge n-1}.$$

Observe that $Q_{=j} = Q_{[j<n]}$, so $f|_{Q_{=j}}$ is a limit diagram under our hypotheses by the proof of Lemma 4.6. Therefore, taking total fibers shows that $f|_{Q_{\ge j}}$ is a limit diagram if and only if $f|_{Q_{\ge j+1}}$ is a limit diagram. Traversing the filtration then proves the claim. $\square$

We continue to assume $C \to \Delta^n$ is a locally cocartesian fibration whose fibers are stable ∞-categories and whose pushforward functors are exact. Observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]), C_{[0:n-1]}\big) \arrow[r, "\gamma_{n-1}^*"] & \mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]), C_{[0:n-1]}\big) \\
\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C) \arrow[u, "j^*"] \arrow[r, "\gamma_n^*"] \arrow[d, "i^*"'] & \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) \arrow[u, "j^*"'] \arrow[d, "i^*"] \\
C_n \arrow[r, "\mathrm{id}"] & C_n
\end{tikzcd}$$

so in particular γ_n^* is a morphism of stable recollements. However γ_n generally fails to be a *strict* morphism of stable recollements, i.e., the natural transformation

$$i^* j_* \to i^* j_* \gamma_{n-1}^*$$

is typically not an equivalence.

4.8 Lemma *Suppose $F : \mathrm{sd}(\Delta^n) \to C$ is 1-generated. Then the comparison map*

$$i^* j_* j^* F = (j_* j^* F)(n) \to i^* j_* \gamma_{n-1}^* j^* F = (j_*(F|_{\mathrm{sd}_1([0:n-1])}))(n)$$

is an equivalence.

Proof Let $K \subset \mathrm{sd}(\Delta^n)$ be the subposet on strings σ with $\max(\sigma) = n$ and $\sigma \ne n$. By the formulas computing j_* given in Theorem 3.32 and Proposition 4.3, we see that the comparison map is given by the canonical map from the limit of $F|_K$ to $F([n-1 < n])$. Since F is 1-generated, by Lemma 4.6 the conditions of Lemma 4.7 are satisfied, so this canonical map is an equivalence. $\square$

4.9 Definition For the functor j_* defined as in Corollary 3.34 with respect to $[0:n-1]$ and $\{n\}$, we say that a functor $F : \mathrm{sd}([0:n-1]) \to C_{[0:n-1]}$ is $+$-1-*generated* if both F and j_*F are 1-generated. Let

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]), C_{[0:n-1]}\big)^{+}_{\text{1-gen}}$$

be the full subcategory on the $+$-1-generated objects.

4.10 Lemma *We have adjunctions*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]), C_{[0:n-1]}\big)^{+}_{\text{1-gen}} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{\text{1-gen}} \underset{i_*}{\overset{i^*}{\rightleftarrows}} C_n$$

that exhibit $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{\text{1-gen}}$ *as a stable recollement.*

Proof Clearly, we may define j_*, i^*, and i_* to be the restrictions of the corresponding functors for the adjunctions of Theorem 3.39. The only subtle point is that given $F : \mathrm{sd}(\Delta^n) \to C$ which is 1-generated, we require that the localization j_*j^*F is also 1-generated. But this holds, since $F \simeq j_*j^*F$ except possibly at $n \in \mathrm{sd}(\Delta^n)$ and the 1-generated condition ignores n. Therefore, we may also define j^* as the restricted functor, and the recollement conditions are then immediate. □

4.11 Corollary *The restriction* $\gamma_n^* : \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{\text{1-gen}} \to \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)$ *is a strict morphism of stable recollements with respect to Lemma 4.10 and Proposition 4.3.*

Proof This follows immediately from Lemma 4.8. □

We want to apply Corollary 4.11 to show that γ_n^* is an equivalence (in fact, a trivial fibration) onto its essential image. To understand this image as a condition on objects in the codomain, we introduce the following definition. For $0 \le i < j \le n$, let $\tau_i^j : C_i \to C_j$ denote the pushforward functor encoded by the locally cocartesian fibration.

4.12 Definition We say that a functor $f : \mathrm{sd}_1(\Delta^n) \to C$ is *extendable* if for every string $[i < i+1 < i+k]$ in $\mathrm{sd}(\Delta^n)$, the canonical map in C_{i+k}

$$\tau_i^{i+k} f(i) \to (\tau_{i+1}^{k} \circ \tau_i^{i+1}) f(i)$$

encoded by the locally cocartesian fibration is an equivalence. Let

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}}$$

denote the full subcategory on the extendable objects.

4.13 Definition For the functor j_* defined as in Proposition 4.3 with respect to $[0:n-1]$ and $\{n\}$, we say that a functor $f : \mathrm{sd}_1([0:n-1]) \to C$ is $+$-*extendable* if both f and j_*f are extendable. Let

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]), C_{[0:n-1]}\big)^{+}_{\mathrm{ext}}$$

be the full subcategory on the $+$-extendable objects.

Note that the extendability condition becomes stronger through considering the additional strings in $\mathrm{sd}(\Delta^n)$; for example, extendability is no condition on $f : \mathrm{sd}_1([0:1]) \to C_{[0:1]}$, but we acquire the condition that the map $\tau_0^2 f(0) \to \tau_1^2 \tau_0^1 f(0)$ is an equivalence upon enlarging to Δ^2. Let us first state the evident counterpart to Lemma 4.10.

4.14 Lemma *We have adjunctions*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]), C_{[0:n-1]}\big)^+_{\mathrm{ext}} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}} \underset{i_*}{\overset{i^*}{\rightleftarrows}} C_n$$

that exhibit $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}}$ *as a stable recollement.*

Proof This is immediate from restricting the recollement of Proposition 4.3. □

We have assembled all the ingredients needed to prove Theorem 4.15. Note that by Lemma 4.7, γ_n^* of a 1-generated object is extendable, so the functor of Theorem 4.15 is well defined.

4.15 Theorem *Suppose $C \to \Delta^n$ is a locally cocartesian fibration whose fibers are stable ∞-categories and whose pushforward functors are exact. Then the functor*

$$\gamma_n^* : \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{1\text{-gen}} \to \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}}$$

is an equivalence of ∞-categories.

Proof We proceed by induction on n. For the base cases $n = 0$ and $n = 1$, the result is trivial. Let $n > 1$ and suppose we have proven the theorem for all $k < n$. By the inductive hypothesis, γ_{n-1}^* is an equivalence. Observe that γ_{n-1}^* restricts to a functor

$$(\gamma_{n-1}^*)^+ : \mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]), C_{[0:n-1]}\big)^+_{1\text{-gen}} \to \mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]), C_{[0:n-1]}\big)^+_{\mathrm{ext}}.$$

If we let $(\gamma_{n-1}^*)^{-1}$ be an inverse functor, then by Lemma 4.6, if $f : \mathrm{sd}_1([0:n-1]) \to C_{[0:n-1]}$ is $+$-extendable, then $(\gamma_{n-1}^*)^{-1}(f)$ is $+$-1-generated. Therefore, $(\gamma_{n-1}^*)^+$ is also an equivalence. By Corollary 4.11 (but replacing the codomain there with the recollement of Lemma 4.14) and the two-out-of-three property of equivalences for a strict morphism of stable recollements (Remark 2.7), we deduce that γ_n^* is an equivalence. □

4.16 Observation To make better use of Theorem 4.15, let us further unpack $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)$. Note that we may write $\mathrm{sd}_1(\Delta^n)$ as the union of marked simplicial sets

$$\mathrm{sd}([0:1]) \cup_1 \mathrm{sd}([1:2]) \cup_2 \cdots \cup_n \mathrm{sd}([n-1:n]),$$

so we obtain a fiber product decomposition

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/[0:1]}\big(\mathrm{sd}([0:1]), C_{[0:1]}\big) \times_{C_1} \cdots \times_{C_{n-1}} \mathrm{Fun}^{\mathrm{cocart}}_{/[n-1:n]}\big(\mathrm{sd}([n-1:n]), C_{[n-1:n]}\big).$$

Let $\tau_i^{i+1}\colon C_i \to C_{i+1}$ be the pushforward functors as before, and with respect to the trivial fibration (induced by the inner anodyne spine inclusion $[0:1]\cup_1 \cdots \cup_{n-1} [n-1:n] \to \Delta^n$)

$$\mathrm{Fun}(\Delta^n, \mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathrm{Fun}([0:1], \mathbf{Cat}_\infty) \times_1 \cdots \times_{n-1} \mathrm{Fun}([n-1:n], \mathbf{Cat}_\infty),$$

let $\tau_\bullet\colon \Delta^n \to \mathbf{Cat}_\infty$ be a functor lifting the τ_i^{i+1}. Let $C^\vee \to (\Delta^n)^{\mathrm{op}}$ be a cartesian fibration classified by $\tau_\bullet$. Then if we let $[i+1:i] = [i:i+1]^{\mathrm{op}}$, we have that $(C^\vee)_{[i+1:i]} \simeq (C_{[i:i+1]})^\vee$ where the right-hand $(-)^\vee$ denotes the dual cartesian fibration of the cocartesian fibration $C_{[i:i+1]} \to [i:i+1]$. Then by Observation 2.14, we have an equivalences of ∞-categories

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[i:i+1]}\big(\mathrm{sd}([i:i+1]), C_{[i:i+1]}\big) \simeq \mathrm{Fun}_{/[i+1:i]}([i+1:i], C^\vee_{[i+1:i]}) \simeq \mathrm{Ar}(C_{i+1}) \times_{\mathrm{ev}_1, C_{i+1}, \tau_i^{i+1}} C_i.$$

Again using that the spine inclusion is inner anodyne, we obtain the following proposition.

4.17 Proposition *We have equivalences of ∞-categories*

$$\begin{aligned}\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) &\simeq \mathrm{Fun}_{/(\Delta^n)^{\mathrm{op}}}((\Delta^n)^{\mathrm{op}}, C^\vee)\\ &\simeq \mathrm{Ar}(C_n) \times_{C_n} \mathrm{Ar}(C_{n-1}) \times_{C_{n-1}} \cdots \times_{C_2} \mathrm{Ar}(C_1) \times_{C_1} C_0,\end{aligned}$$

where in the fiber product, the maps $\mathrm{Ar}(C_k) \to C_k$ are given by evaluation at the target, and the maps $\mathrm{Ar}(C_k) \to C_{k+1}$ are given by composing evaluation at the source with $\tau_k^{k+1}\colon C_k \to C_{k+1}$.

4.18 Notation Let $(\mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0)_{\mathrm{ext}}$ denote the full subcategory of $\mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0$ given by restricting the equivalence of Proposition 4.17 to $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}}$ on the left-hand side.

Then we can also express Theorem 4.15 as

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{1\text{-gen}} \arrow[r, "\gamma_n^*", "\simeq"'] \arrow[d, hook] & (\mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0)_{\mathrm{ext}} \arrow[d, hook] \\
\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C) \arrow[r, "\gamma_n^*"] & \mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0
\end{tikzcd}$$

which is more concrete in practice (e.g., for the example of C_{p^n}-spectra explained in Remark 4.19).

4.19 Remark The type of iterated fiber product occurring in Proposition 4.17 appears in the work of Nikolaus and Scholze when they describe the data of a (genuine) C_{p^n}-spectrum X whose geometric fixed points (except possibly $\Phi^{C_{p^n}} X$ and $\Phi^{C_{p^{n-1}}} X$) are all bounded below; see [17, Remark II.4.8].[18] In fact, Theorem 4.15 together with [2, Theorem E] applies to give a proof of [17, Remark II.4.8] that is independent of the machinery of "coalgebras for endofunctors" developed in [17, §II.5]. We will explain this in more detail in [19, §3.2] as well as prove a dihedral refinement of this assertion. For now, we give an overview of the argument:

[18] Nikolaus and Scholze elide the subtlety involving the lack of bounded-below hypotheses needed on $\Phi^{C_{p^n}} X$ and $\Phi^{C_{p^{n-1}}} X$.

By [2, Theorem E], for any finite group G with subconjugacy poset P there exists a locally cocartesian fibration $\mathbf{Sp}^G_{\phi\text{-locus}} \to P$ whose right-lax limit is canonically equivalent[19] to the ∞-category $\mathbf{Sp}^G$ of (genuine) G-spectra. Furthermore, for every subgroup $H \le G$, $(\mathbf{Sp}^G_{\phi\text{-locus}})_H \simeq \mathbf{Sp}^{hW_G H} = \mathrm{Fun}(BW_G H, \mathbf{Sp})$ where $W_G H = N_G H/H$ is the Weyl group, and the equivalence transports a G-spectrum X to its associated diagram of geometric fixed points $\{\Phi^H X \in \mathbf{Sp}^{hW_G H}\}$. If $G = C_{p^n}$, then we may identify the pushforward functor associated to $[C_{p^k} \le C_{p^m}]$ with the proper Tate construction $(-)^{\tau C_{p^{m-k}}}$ endowed with residual action; in particular, when $m = k+1$, this is the ordinary Tate construction $(-)^{tC_p}$. In addition, under the equivalence $\mathbf{Sp}^{C_{p^n}} \simeq \lim^{\mathrm{rlax}} \mathbf{Sp}^{C_{p^n}}_{\phi\text{-locus}}$ and the isomorphism $P \cong [n]$, the map γ_n^* identifies with the forgetful functor

$$\mathbf{Sp}^{C_{p^n}} \to \mathbf{Sp}^{hC_{p^n}} \times_{(-)^{tC_p}, \mathbf{Sp}^{hC_{p^{n-1}}}, \mathrm{ev}_1} \mathrm{Ar}(\mathbf{Sp}^{hC_{p^{n-1}}}) \times_{(-)^{tC_p}\mathrm{ev}_0, \mathbf{Sp}^{hC_{p^{n-2}}}, \mathrm{ev}_1} \mathrm{Ar}(\mathbf{Sp}^{hC_{p^{n-2}}}) \times \cdots \times \mathrm{Ar}(\mathbf{Sp})$$

that sends X to $[\Phi^e X, \Phi^{C_p} X \to (\Phi^e X)^{tC_p}, \dots, \Phi^{C_{p^n}} X \to (\Phi^{C_{p^{n-1}}} X)^{tC_p}]$ where the maps are the usual ones. The assertion made in [17, Remark II.4.8] is that γ_n^* restricts to an equivalence

$$\mathbf{Sp}_+^{C_{p^n}} \xrightarrow{\simeq} \mathbf{Sp}_+^{hC_{p^n}} \times_{(-)^{tC_p}, \mathbf{Sp}^{hC_{p^{n-1}}}, \mathrm{ev}_1} \mathrm{Ar}'(\mathbf{Sp}^{hC_{p^{n-1}}}) \times_{(-)^{tC_p}\mathrm{ev}_0, \mathbf{Sp}^{hC_{p^{n-2}}}, \mathrm{ev}_1} \mathrm{Ar}'(\mathbf{Sp}^{hC_{p^{n-2}}}) \times \cdots \times \mathrm{Ar}(\mathbf{Sp}),$$

where:

- $\mathbf{Sp}_+^{C_{p^n}} \subset \mathbf{Sp}^{C_{p^n}}$ denotes the full subcategory of C_{p^n}-spectra spanned by those objects whose geometric fixed points (except possibly $\Phi^{C_{p^n}}$ and $\Phi^{C_{p^{n-1}}}$) are all bounded below.
- $\mathbf{Sp}_+^{hC_{p^n}} \subset \mathbf{Sp}^{hC_{p^n}}$ denotes the full subcategory of Borel C_{p^n}-spectra spanned by those objects whose underlying spectrum is bounded below.
- Ar' denotes the full subcategory on arrows whose source is bounded below.

To invoke Theorem 4.15 to deduce this, we need to show that for every $X \in \mathbf{Sp}_+^{C_{p^n}}$, X is 1-generated as an object in $\lim^{\mathrm{rlax}} \mathbf{Sp}^{C_{p^n}}_{\phi\text{-locus}}$. If $n = 2$, this is the content of the *Tate orbit lemma* of [17, Lemma I.2.1] once one identifies the fiber of the natural transformation $\mathrm{can} : (-)^{\tau C_{p^2}} \Rightarrow ((-)^{tC_p})^{tC_{p^2}/C_p}$ encoded by $\mathbf{Sp}^{C_{p^2}}_{\phi\text{-locus}}$ with $((-)_{hC_p})^{tC_{p^2}/C_p}$. Proceeding by induction on n, it is then not difficult to verify that the condition of Lemma 4.6 holds for all $X \in \mathbf{Sp}_+^{C_{p^n}}$; we record this as [19, Corollary 3.40].

5 Reconstruction of sheaves on stratified ∞-topoi

We explain how to apply Theorem 3.39 to prove a reconstruction theorem (Theorem 5.13) for sheaves in an ∞-topos stratified by a finite poset P in the sense of Barwick, Glasman, and Haine (Definition 5.5). We then prove a conjecture of Barwick, Glasman, and Haine by establishing an equivalence (Theorem 5.22) between the ∞-category of P-stratified ∞-topoi and that of *toposic* locally cocartesian fibrations over P^{op} (Definition 5.11).

In this section, we will regard the poset P as a topological space via the Alexandroff topology. To begin with, we recall the basic structure theory of recollements of ∞-topoi.

[19] The comparison functor is defined analogously to the functor (1-2) in Theorem B; see [20, Construction 2.43].

5.1 Example Let $\mathcal{X}$ be an ∞-topos and U a (-1)-truncated object. The slice ∞-topos $\mathcal{X}_{/U}$ is said to be an *open subtopos* of $\mathcal{X}$ [12, §6.3.5].[20] Let $\mathcal{X}_{\setminus U} = \{x \in \mathcal{X} : x \times U \xrightarrow{\simeq} U\} \subset \mathcal{X}$. The set $\mathcal{X}_{\setminus U}$ is the *closed subtopos of $\mathcal{X}$ complementary to U* [12, Definition 7.3.2.6]. We then have a diagram of adjunctions

$$\begin{tikzcd} \mathcal{X}_{/U} \arrow[r, hook, shift left=2, "j_!"] \arrow[r, hook, shift right=2, "j_*"'] & \mathcal{X} \arrow[l, "j^*" description] \arrow[r, shift left, "i^*"] & \mathcal{X}_{\setminus U} \arrow[l, hook, shift left, "i_*"] \end{tikzcd}$$

that exhibits $(\mathcal{X}_{/U}, \mathcal{X}_{\setminus U})$ as a recollement of $\mathcal{X}$. Conversely, by [14, Proposition A.8.15], given a left-exact accessible functor $\phi : \mathcal{U} \to \mathcal{Z}$ between ∞-topoi, the fiber product $\mathcal{X} := \mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ is an ∞-topos and there exists a uniquely determined (-1)-truncated object U such that $\mathcal{U} \simeq \mathcal{X}_{/U}$ and $\mathcal{Z} \simeq \mathcal{X}_{\setminus U}$ compatibly with the adjunctions to $\mathcal{X}$.

In what follows, we will generically use the notation $j_! \dashv j^* \dashv j_*$ and $i^* \dashv i_*$ for these functors arising from a recollement on an ∞-topos.

5.2 Definition A *locale* is a 0-topos, i.e., a poset L such that L admits infinite joins $\bigvee_\alpha x_\alpha$ (so that L is presentable) and infinite joins distribute over finite meets.

5.3 Example Let $\mathcal{X}$ be an ∞-topos. Then its full subcategory $\mathbf{Open}(\mathcal{X})$ of (-1)-truncated objects is a locale. Note that $\mathbf{Open}(\mathcal{X})$ is isomorphic to the poset of open subtopoi of $\mathcal{X}$ (embedded in $\mathcal{X}$ via $j_!$) via the assignment $U \mapsto \mathcal{X}_{/U}$. Also, if X is a topological space, then $\mathbf{Open}(\mathbf{Shv}(X))$ is isomorphic to the poset $\mathbf{Open}(X)$ of open sets in X. If P is a poset equipped with the Alexandroff topology, then these are precisely the cosieves in P.

5.4 Example Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category and suppose there is some regular cardinal κ such that the unit and tensor product restrict to define a symmetric monoidal structure on the full subcategory $\mathcal{C}^\kappa$ of κ-compact objects in $\mathcal{C}$. Then the set of radical thick $\otimes$-ideals in $\mathcal{C}^\kappa$ forms a coherent locale [10, Theorem 3.1.9].

5.5 Definition [5, Definition 8.2.1] Let P be a poset and $\mathcal{X}$ an ∞-topos. A *P-stratification of* $\mathcal{X}$ is a geometric morphism $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ of ∞-topoi, or equivalently a geometric morphism $\pi_* : \mathbf{Open}(\mathcal{X}) \to \mathbf{Open}(P)$ of locales. We also say that the data $(\mathcal{X}, \pi_*)$ comprises that of a *P-stratified ∞-topos*.

In the next remark, we consider $P^{\mathrm{op}} \subset \mathbf{Open}(P)$ as a subposet via the map $p \mapsto P^{\geq p}$.

5.6 Remark Via the assignment $\pi_* \mapsto \pi^*|_{P^{\mathrm{op}}}$, geometric morphisms $\pi_* : \mathbf{Open}(\mathcal{X}) \to \mathbf{Open}(P)$ are in bijective correspondence with maps of posets $f : P^{\mathrm{op}} \to \mathbf{Open}(\mathcal{X})$ such that:

(1) $\bigvee_{p \in P} f(p) = \mathbb{1}$.

(2) For every $p, q \in P$, $\bigvee_{r \geq p,q} f(r) \xrightarrow{\cong} f(p) \times f(q)$.

Indeed, given any map of posets $f : P^{\mathrm{op}} \to \mathbf{Open}(\mathcal{X})$, its left Kan extension $F : \mathbf{Open}(P) \to \mathbf{Open}(\mathcal{X})$ admits a right adjoint G defined by $G(U) = \{p \in P : f(p) \leq U\}$, and F is then left-exact if and only if f satisfies conditions (1) and (2).

[20] Lurie uses the terminology "étale geometric morphism".

Furthermore, (2) is equivalent to the following factorization property: for every $p, q \in P$, the square

$$\begin{tikzcd} \mathcal{X}_{/\bigvee_{r \geq p,q} f(r)} \arrow[r, hook, "j_!"] & \mathcal{X}_{/f(p)} \\ \mathcal{X}_{/f(q)} \arrow[u, "j^*"] \arrow[r, hook, "j_!"] & \mathcal{X} \arrow[u, "j^*"'] \end{tikzcd}$$

commutes. We thus see that the notion of a P-stratification of $\mathcal{X}$ is the evident toposic analogue of the notion of a P-stratification of a presentable stable ∞-category in the sense of [2, Definition 2.4.3]. Conversely, in view of Example 5.4 one can sometimes give a "localic" reformulation of [2, Definition 2.4.3] (or rather, its symmetric monoidal refinement [2, Definition 4.3.2]).

We now proceed to notate various subtopoi associated to a P-stratified ∞-topos.

5.7 Notation [5, Notation 8.2.3] Let $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ be a P-stratification of $\mathcal{X}$. In what follows, all fiber products are computed in $\mathbf{Top}_\infty$. For any open subset $O \subset P$, we let

$$\mathcal{X}_O := \mathcal{X}_{/\pi^* O} \simeq \mathcal{X} \times_{\mathbf{Shv}(P)} \mathbf{Shv}(O).$$

Dually, for any closed subset $Z \subset P$, we let

$$\mathcal{X}_Z := \mathcal{X}_{\backslash \pi^*(P \backslash Z)} \simeq \mathcal{X} \times_{\mathbf{Shv}(P)} \mathbf{Shv}(Z).$$

For any $p \in P$, we define the *p-th stratum* of $(\mathcal{X}, \pi_*)$ to be

$$\mathcal{X}_p := \mathcal{X} \times_{\mathbf{Shv}(P)} \mathbf{Shv}(\{p\}).$$

5.8 Notation In Notation 5.7, the p-th stratum $\mathcal{X}_p$ is the closed complement of $\mathcal{X}_{P^{>p}}$ in $\mathcal{X}_{P^{\geq p}} = \mathcal{X}_{/\pi^*(p)}$, or alternatively the open complement of $\mathcal{X}_{P^{<p}}$ in $\mathcal{X}_{P^{\leq p}}$. We then have the adjunction

$$\Phi^p : \mathcal{X} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{X}_{/\pi^*(p)} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{X}_p : \rho_p,$$

in which ρ_p is a geometric morphism.

5.9 Remark Let $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ be a P-stratification of $\mathcal{X}$ and suppose $p, q \in P$ such that $p \not\geq q$. Then $\Phi^q \rho_p$ is homotopic to the constant map at the final object. Indeed, by Remark 5.6 we have a factorization of $\Phi^q \rho_p$ as

$$\begin{tikzcd} \mathcal{X}_p \arrow[r, hook, "i_*"] & \mathcal{X}_{/\pi^*(p)} \arrow[r, hook, "j_*"] \arrow[d, "j^*"] & \mathcal{X} \arrow[d, "j^*"] \\ & \mathcal{X}_{/\pi^*(P^{\geq p,q})} \arrow[r, hook, "j_*"] & \mathcal{X}_{/\pi^*(q)} \arrow[d, "i^*"] \\ & & \mathcal{X}_q \end{tikzcd}$$

and since $p \notin P^{\geq p,q}$, the composite $j^* i_* : \mathcal{X}_p \to \mathcal{X}_{/\pi^*(P^{\geq p,q})}$ is homotopic to the constant map at the final object.

Given a P-stratified ∞-topos $(\mathcal{X}, \pi_*)$, we may construct its associated *gluing diagram* in the same manner as [2, Definition 2.5.7].

5.10 Construction Let $\mathcal{G}(\mathcal{X}) = \{(x, p) : x \in \mathcal{X}_p\} \subset \mathcal{X} \times P^{\mathrm{op}}$, where $\mathcal{X}_p \subset \mathcal{X}$ via ρ_p. The projection

$$\lambda : \mathcal{G}(\mathcal{X}) \to P^{\mathrm{op}}$$

is then a locally cocartesian fibration with fibers $\mathcal{X}_p$ such that for all $q \leq p$, the corresponding pushforward functor $\Gamma^q_p : \mathcal{X}_p \to \mathcal{X}_q$ is given by $\Phi^q \circ \rho_p$ (see [2, Observation 2.5.6]).

We codify the structure of $\lambda : \mathcal{G}(\mathcal{X}) \to P^{\mathrm{op}}$ by means of the following definition.

5.11 Definition We call a locally cocartesian fibration $\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}$ *toposic* if its fibers are ∞-topoi and its pushforward functors are left-exact and accessible.

If P is finite, we will show that taking the limit in $\mathcal{X}$ furnishes an equivalence $\Theta_P : \lim^{\mathrm{rlax}} \mathcal{G}(\mathcal{X}) \xrightarrow{\simeq} \mathcal{X}$, thereby proving a *reconstruction theorem* for $(\mathcal{X}, \pi_*)$. First, we note:

5.12 Lemma *Let P be a finite poset and $\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}$ a toposic locally cocartesian fibration. Then the right-lax limit $\mathcal{X} = \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \widehat{\mathcal{X}})$ is an ∞-topos. Moreover, any cosieve $O \subset P$ determines a recollement of $\mathcal{X}$ with open subtopos given by the right-lax limit of $\lambda|_{O^{\mathrm{op}}}$ and complementary closed subtopos given by the right-lax limit of $\lambda|_{(P \setminus O)^{\mathrm{op}}}$.*

Proof Given Theorem 3.39 and proceeding by induction on the cardinality of P, the first part follows from the known statement for recollements of ∞-topoi recalled in Example 5.1. The second statement then follows by Theorem 3.39 again. □

Consider now the functor $\Theta_P : \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X})) \to \mathcal{X}$ that sends a functor $f : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})$ to $\lim_{\mathrm{sd}(P^{\mathrm{op}})}(\mathrm{pr}_{\mathcal{X}} \circ f)$.

5.13 Theorem *Suppose P is a finite poset and let $(\mathcal{X}, \pi_*)$ be a P-stratified ∞-topos. Then*

$$\Theta_P : \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X})) \to \mathcal{X}$$

is an equivalence.

Proof To ease notation, let $\mathcal{X}' := \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X}))$. We proceed by induction on the cardinality of P. We may suppose that P is nonempty. Choose a minimal element $b \in P$ and let $O = P \setminus \{b\}$. Let

$$(\pi_O)_* : \mathbf{Open}(\mathcal{X}_{/\pi^*(O)}) \to \mathbf{Open}(O)$$

denote the O-stratification of the open subtopos $\mathcal{X}_{/\pi^*(O)}$ restricted from that of $\mathcal{X}$. Note that $\mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}} \simeq \mathcal{G}(\mathcal{X}_{/\pi^*(O)})$ as locally cocartesian fibrations over O^{op}. Indeed, one observes that for all $p \in O$, the fully faithful inclusion $\rho_p : \mathcal{X}_p \hookrightarrow \mathcal{X}$ factors through $\mathcal{X}_{/\pi^*(O)}$ and identifies $\mathcal{X}_p$ with $(\mathcal{X}_{/\pi^*(O)})_p$ embedded via $(\rho_O)_p$, so the inclusion $\mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}} \subset \mathcal{X} \times O^{\mathrm{op}}$ factors through $\mathcal{X}_{/\pi^*(O)}$ (embedded via j_* in $\mathcal{X}$) and identifies with $\mathcal{G}(\mathcal{X}_{/\pi^*(O)})$.

Let $(\mathcal{X}_{/\pi^*(O)})' := \mathrm{Fun}^{\mathrm{cocart}}_{/O^{\mathrm{op}}}(\mathrm{sd}(O^{\mathrm{op}}), \mathcal{G}(\mathcal{X}_{/\pi^*(O)}))$ and write

$$\Theta_O : (\mathcal{X}_{/\pi^*(O)})' \xrightarrow{\mathrm{pr}_*} \mathrm{Fun}(\mathrm{sd}(O^{\mathrm{op}}), \mathcal{X}_{/\pi^*(O)}) \xrightarrow{\lim} \mathcal{X}_{/\pi^*(O)}.$$

We now show that $\Theta_P : \mathcal{X}' \to \mathcal{X}$ is a morphism of recollements from $((\mathcal{X}_{/\pi^*(O)})', \mathcal{X}_b)$ to $(\mathcal{X}_{/\pi^*(O)}, \mathcal{X}_b)$:

(1) We have a distinguished homotopy making the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "j^*=\mathrm{res}"] \arrow[d, "\Theta_P"] & (\mathcal{X}_{/\pi^*(O)})' \arrow[d, "\Theta_O"] \\ \mathcal{X} \arrow[r, "j^*"] & \mathcal{X}_{/\pi^*(O)} \end{tikzcd}$$

commute as follows: given $[f : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})] \in \mathcal{X}'$, consider the composite

$$g : \mathrm{sd}(P^{\mathrm{op}}) \xrightarrow{f} \mathcal{G}(\mathcal{X}) \xrightarrow{\mathrm{pr}} \mathcal{X} \xrightarrow{j^*} \mathcal{X}_{/\pi^*(O)},$$

whose limit is $j^*\Theta_P(f)$. Then since $\mathcal{X}_b \xrightarrow{i_*=\rho_b} \mathcal{X} \xrightarrow{j^*} \mathcal{X}_{/\pi^*(O)}$ is homotopic to the constant map at the final object, g is a right Kan extension of its restriction g_0 to $\mathrm{sd}(O^{\mathrm{op}})$. But since the limit of g_0 is $\Theta_O j^*(f)$, this supplies an equivalence $j^*\Theta_P(f) \simeq \Theta_O j^*(f)$ that is natural in f.

(2) Likewise, we may construct an equivalence

$$i^*\Theta_P = \Phi^b\Theta_P \simeq \mathrm{ev}_b : \mathcal{X}' \to \mathcal{X}_b$$

as follows: Let $[f : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})] \in \mathcal{X}'$ and consider the composite

$$g : \mathrm{sd}(P^{\mathrm{op}}) \xrightarrow{f} \mathcal{G}(\mathcal{X}) \xrightarrow{\mathrm{pr}} \mathcal{X} \xrightarrow{i^*} \mathcal{X}_b.$$

If $a \not\geq b$, then the composite $\mathcal{X}_a \overset{\rho_a}{\hookrightarrow} \mathcal{X} \overset{\Phi^b}{\twoheadrightarrow} \mathcal{X}_b$ is homotopic to the constant map at the final object by Remark 5.9. Consequently, g is the right Kan extension of its restriction to $\mathrm{sd}((P^{\geq b})^{\mathrm{op}})$. Let $\mathrm{sd}^+((P^{>b})^{\mathrm{op}})$ be the subposet on strings ending at b (in P^{op}) and note that $\mathrm{sd}((P^{>b})^{\mathrm{op}}) \cong \mathrm{sd}^+((P^{>b})^{\mathrm{op}})$ via the "append b" map. We then have a pullback square

$$\begin{tikzcd} \lim g|_{\mathrm{sd}((P^{\geq b})^{\mathrm{op}})} \arrow[r] \arrow[d, "\gamma'"] & \lim g|_{\mathrm{sd}((P^{>b})^{\mathrm{op}})} \arrow[d, "\gamma"] \\ g(b) \arrow[r] & \lim g|_{\mathrm{sd}^+((P^{>b})^{\mathrm{op}})} \end{tikzcd}$$

in which γ is induced by the "append b" homotopy $\mathrm{sd}((P^{>b})^{\mathrm{op}}) \times [1] \hookrightarrow \mathrm{sd}((P^{\geq b})^{\mathrm{op}})$. For all strings $\sigma = [p_1 > \cdots > p_n]$ in $(P^{>b})^{\mathrm{op}}$, letting $\sigma^+ := [p_1 > \cdots > p_n > b]$ we note that $g(\sigma \subset \sigma^+)$ is an equivalence. Therefore, γ and hence γ' is an equivalence, and this is clearly natural in the input f.

We conclude that we have a morphism of recollements

$$\begin{tikzcd} (\mathcal{X}_{/\pi^*(O)})' \arrow[d, "\Theta_O"] & \mathcal{X}' \arrow[l, "j^*=\mathrm{res}"'] \arrow[r, "i^*=\mathrm{ev}_b"] \arrow[d, "\Theta_P"] & \mathcal{X}_b \arrow[d, "="] \\ \mathcal{X}_{/\pi^*(O)} & \mathcal{X} \arrow[l, "j^*"'] \arrow[r, "i^*=\Phi^b"] & \mathcal{X}_b \end{tikzcd}$$

By the inductive hypothesis, Θ_O is an equivalence. To then deduce that Θ_P is an equivalence, by Remark 2.7 it remains to observe that we have a *strict* morphism of recollements, i.e., that the adjoint square

$$\begin{tikzcd} (\mathcal{X}_{/\pi^*(O)})' \arrow[r, "i^* j_*"] \arrow[d, "\Theta_O"] & \mathcal{X}_b \arrow[d, "="] \\ \mathcal{X}_{/\pi^*(O)} \arrow[r, "i^* j_*"] & \mathcal{X}_b \end{tikzcd}$$

commutes. But using that the lower $i^* j_* : \mathcal{X}_{/\pi^*(O)} \to \mathcal{X}_b$ is left-exact, this amounts to our formula for the gluing functor $i^* j_* : (\mathcal{X}_{/\pi^*(O)})' \to \mathcal{X}_b$ of the recollement on $\mathcal{X}'$ that we gave in Theorem 3.32. $\square$

In fact, we can elaborate upon Theorem 5.13 to also reconstruct the P-stratification of $\mathcal{X}$.

5.14 Construction Let P be a finite poset, $\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}$ a toposic locally cocartesian fibration, and $\mathcal{X} = \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \widehat{\mathcal{X}})$ its right-lax limit, which is an ∞-topos by Lemma 5.12. Given a cosieve $O \subset P$, let $\pi^*(O) \in \mathcal{X}$ be the uniquely determined (-1)-truncated object such that $\mathrm{Fun}^{\mathrm{cocart}}_{/O^{\mathrm{op}}}(\mathrm{sd}(O^{\mathrm{op}}), \widehat{\mathcal{X}}|_{O^{\mathrm{op}}}) \simeq \mathcal{X}_{/\pi^*(O)}$. Then we may define a P-stratification of $\mathcal{X}$ by the map of posets

$$\pi^* : \mathbf{Open}(P) \to \mathbf{Open}(\mathcal{X}),$$

as it is clear that π^* preserves joins and meets (e.g., in view of Remark 5.6).

5.15 Corollary *Let P be a finite poset and $(\mathcal{X}, \pi_*)$ a P-stratified ∞-topos. The P-stratification of $\mathcal{X}' := \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X}))$ given by Construction 5.14 coincides with that of $\mathcal{X}$ under the equivalence Θ_P of Theorem 5.13.*

Proof For every cosieve $O \subset P$, let $(\mathcal{X}_{/\pi^*(O)})' := \mathrm{Fun}^{\mathrm{cocart}}_{/O^{\mathrm{op}}}(\mathrm{sd}(O^{\mathrm{op}}), \mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}})$ and note that as in the proof of Theorem 5.13 that $\mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}} \simeq \mathcal{G}(\mathcal{X}_{/\pi^*(O)})$. By Theorem 5.13, $\Theta_O : (\mathcal{X}_{/\pi^*(O)})' \to \mathcal{X}_{/\pi^*(O)}$ is an equivalence. To then see that $(\mathcal{X}_{/\pi^*(O)})'$ identifies with the open subtopos $\mathcal{X}_{/\pi^*(O)}$ under the equivalence Θ_P, it remains to observe that the square

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "j^*"] \arrow[d, "\Theta_P"] & (\mathcal{X}_{/\pi^*(O)})' \arrow[d, "\Theta_O"] \\ \mathcal{X} \arrow[r, "j^*"] & \mathcal{X}_{/\pi^*(O)} \end{tikzcd}$$

commutes. We may proceed by induction on the cardinality of $P \setminus O$.[21] If $O = P$ or $O = P \setminus \{b\}$, we are

[21] Of course, we could also adapt the proof of Theorem 5.13 to show this directly.

done by the proof of Theorem 5.13. If not, let $b \in P \setminus O$ be a minimal element. We have a factorization

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "j^*"] \arrow[d, "\Theta_P"] & (\mathcal{X}_{/\pi^*(P\setminus\{b\})})' \arrow[r, "j^*"] \arrow[d, "\Theta_{P\setminus\{b\}}"] & (\mathcal{X}_{/\pi^*(O)})' \arrow[d, "\Theta_O"] \\ \mathcal{X} \arrow[r, "j^*"] & \mathcal{X}_{/\pi^*(P\setminus\{b\})} \arrow[r, "j^*"] & \mathcal{X}_{/\pi^*(O)} \end{tikzcd}$$

By the inductive hypothesis, both the inner squares commute, hence the outer square commutes. □

5.16 Remark By Corollary 5.15, it follows that given a sheaf $x \in \mathcal{X}$, under the equivalence of Theorem 5.13 x corresponds to a functor $f_x : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})$ that sends $[p]$ to $\Phi^p(x)$. The equivalence $x \simeq \Theta_P(f_x)$ then "reconstructs" x from its stratumwise values $\Phi^p(x)$ and gluing data thereof.

We next turn to questions of functoriality in the P-stratified ∞-topos.

5.17 Observation Continuing from Example 5.1, we explain how recollements of topoi are functorial in geometric morphisms. In one direction, suppose we are given a commutative square

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\phi"] \arrow[d, "(f_U)_*"'] & \mathcal{Z} \arrow[d, "(f_Z)_*"] \\ \mathcal{U}' \arrow[r, "\phi'"] & \mathcal{Z}' \end{tikzcd}$$

of ∞-topoi, where $(f_U)_*$, $(f_Z)_*$ are geometric morphisms and ϕ, ϕ' are left-exact accessible functors. Let $\mathcal{X}$ and $\mathcal{X}'$ be the ∞-topoi $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1,\mathcal{Z},\phi} \mathcal{U}$ and $\mathrm{Ar}(\mathcal{Z}') \times_{\mathrm{ev}_1,\mathcal{Z}',\phi'} \mathcal{U}'$. Then the induced functor $f_* : \mathcal{X} \to \mathcal{X}'$ admits a left adjoint f^* induced by the mate $(f_Z)^*\phi' \Rightarrow \phi(f_U)^*$; explicitly,

$$f^*[u', z' \to \phi'(u')] = [(f_U)^*(u'), (f_Z)^*(z') \to (f_Z)^*\phi'(u') \to \phi(f_U)^*(u')].$$

Moreover, since $(f_U)^*$, $(f_Z)^*$, ϕ, ϕ' are left-exact and $(j^*, i^*) : \mathcal{X} \to \mathcal{U} \times \mathcal{Z}$ creates finite limits, we see that f^* is left-exact. We conclude that f_* is a geometric morphism. Moreover, f_* is a strict morphism of recollements whose left adjoint f^* is a (not necessarily strict) morphism of recollements. Note also that if we identify $\mathcal{U} \simeq \mathcal{X}_{/U}$ and $\mathcal{U}' \simeq \mathcal{X}_{/U'}$ for (-1)-truncated objects U, U', then $f^*(U') \simeq U$.

Conversely, let $\mathcal{X}$ and $\mathcal{X}'$ be ∞-topoi decomposed by recollements $(\mathcal{U}, \mathcal{Z})$ and $(\mathcal{U}', \mathcal{Z}')$ with gluing functors ϕ and ϕ', and suppose $f_* : \mathcal{X} \to \mathcal{X}'$ is a geometric morphism such that both f^* and f_* are morphisms of recollements. Then f_* is necessarily a strict morphism of recollements, and we obtain a commutative square $(f_Z)_*\phi \simeq \phi'(f_U)_*$ as above.

Finally, the theory of recollements implies that these constructions are mutually inverse.

5.18 Definition [5, 8.2.2] A *geometric morphism of P-stratified ∞-topoi* $(\mathcal{X}, \pi_*) \to (\mathcal{Y}, \rho_*)$ is a geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ subject to the condition that the induced diagram of posets

$$\begin{tikzcd} \mathbf{Open}(\mathcal{X}) \arrow[rr, "f_*"] \arrow[dr, "\pi_*"'] & & \mathbf{Open}(\mathcal{Y}) \arrow[dl, "\rho_*"] \\ & \mathbf{Open}(P) & \end{tikzcd}$$

commutes, i.e., for all cosieves $O \subset P$, $f^*\rho^*(O) \cong \pi^*(O)$.

The collection of P-stratified ∞-topoi and geometric morphisms thereof assembles into an ∞-category $\mathbf{StrTop}_{\infty,P}$. Note also that $\mathbf{StrTop}_{\infty,P} \simeq \mathbf{Top}_\infty \times_{\mathbf{Top}_0} (\mathbf{Top}_0)_{/\mathbf{Open}(P)}$.

5.19 Definition [5, Remark 8.2.7] A *geometric morphism of toposic locally cocartesian fibrations* from $[\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}]$ to $[\xi : \widehat{\mathcal{Y}} \to P^{\mathrm{op}}]$ is a functor $F : \widehat{\mathcal{X}} \to \widehat{\mathcal{Y}}$ over P^{op} such that:

(1) F preserves locally cocartesian edges.

(2) For all $p \in P$, the fiber $F_p : \widehat{\mathcal{X}}_p \to \widehat{\mathcal{Y}}_p$ is a geometric morphism of ∞-topoi.

The collection of toposic locally cocartesian fibrations and geometric morphisms thereof assembles into an ∞-category $\mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}}$ (Barwick, Glasman, and Haine label this ∞-category as $\mathbf{LocCocart}^{\mathrm{lex,top}}_{P^{\mathrm{op}}}$).

5.20 Observation Let $f_* : (\mathcal{X}, \pi_*) \to (\mathcal{Y}, \rho_*)$ be a geometric morphism of P-stratified ∞-topoi. Then for all cosieves $O \subset P$, f_* is a strict morphism of recollements with respect to $(\mathcal{X}_{/\pi^*(O)}, \mathcal{X}_{\backslash \pi^*(O)})$ and $(\mathcal{Y}_{/\rho^*(O)}, \mathcal{Y}_{\backslash \rho^*(O)})$. Moreover, for all maps of posets $Q \to P$, restriction along $\mathbf{Shv}(Q) \to \mathbf{Shv}(P)$ (in $\mathbf{Top}_\infty$) defines a geometric morphism $f'_* : \mathbf{Shv}(Q) \times_{\mathbf{Shv}(P)} \mathcal{X} \to \mathbf{Shv}(Q) \times_{\mathbf{Shv}(P)} \mathcal{Y}$ of Q-stratified ∞-topoi. Consequently, for all $p \in P$, f_* sends the stratum $\mathcal{X}_p$ into $\mathcal{Y}_p$ (with respect to the embeddings ρ_p of Notation 5.8) and we may thus restrict $f_* \times \mathrm{id} : \mathcal{X} \times P^{\mathrm{op}} \to \mathcal{Y} \times P^{\mathrm{op}}$ to obtain a functor

$$\mathcal{G}(f_*) : \mathcal{G}(\mathcal{X}) \to \mathcal{G}(\mathcal{Y})$$

over P^{op} that preserves locally cocartesian edges. We may thereby promote Construction 5.10 to a functor

$$\mathcal{G} : \mathbf{StrTop}_{\infty,P} \to \mathbf{LocCocart}^{\mathrm{lex,top}}_{P^{\mathrm{op}}}.$$

Conversely, suppose P is a finite poset and let $F : \widehat{\mathcal{X}} \to \widehat{\mathcal{Y}}$ be a geometric morphism of toposic locally cocartesian fibrations. Let $\mathcal{X} = \lim^{\mathrm{rlax}} \widehat{\mathcal{X}}$ and $\mathcal{Y} = \lim^{\mathrm{rlax}} \widehat{\mathcal{Y}}$. Let $f_* : \mathcal{X} \to \mathcal{Y}$ denote the functor induced by F. Then by Observation 5.17, Theorem 3.39, and proceeding by induction on the cardinality of P, we see that f_* is a geometric morphism such that for every cosieve $O \subset P$, f_* is a strict morphism of recollements from $(\lim^{\mathrm{rlax}} \widehat{\mathcal{X}}|_{O^{\mathrm{op}}}, \lim^{\mathrm{rlax}} \widehat{\mathcal{X}}|_{(P \backslash O)^{\mathrm{op}}})$ to $(\lim^{\mathrm{rlax}} \widehat{\mathcal{Y}}|_{O^{\mathrm{op}}}, \lim^{\mathrm{rlax}} \widehat{\mathcal{Y}}|_{(P \backslash O)^{\mathrm{op}}})$. It follows that f_* is a geometric morphism of P-stratified ∞-topoi with respect to the P-stratifications of Construction 5.14. Therefore, $\lim^{\mathrm{rlax}}$ promotes to a functor

$$\lim{}^{\mathrm{rlax}} : \mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}} \to \mathbf{StrTop}_{\infty,P}.$$

Our remaining goal is to prove that $\mathcal{G}$ and $\lim^{\mathrm{rlax}}$ define an adjoint equivalence of ∞-categories. For the proof, we will need to use the following deep result in $(\infty, 2)$-category theory:

5.21 Theorem [2, Lemma B.5.7] *Let $\mathcal{C}, \mathcal{D} \to P^{\mathrm{op}}$ be locally cocartesian fibrations. Then the space $\mathrm{Map}^{\mathrm{llax},R}_{/P^{\mathrm{op}}}(\mathcal{C}, \mathcal{D})$ of left-lax morphisms whose fibers are right adjoints is naturally equivalent to the space $\mathrm{Map}^{\mathrm{rlax},L}_{/P^{\mathrm{op}}}(\mathcal{D}, \mathcal{C})$ of right-lax morphisms whose fibers are left adjoints, with the equivalence implemented fiberwise by passage to adjoints.*

5.22 Theorem *Let P be a finite poset. Then $\mathcal{G}$ and $\lim^{\mathrm{rlax}}$ participate in an adjoint equivalence*

$$\lim{}^{\mathrm{rlax}} : \mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}} \rightleftarrows \mathbf{StrTop}_{\infty,P} : \mathcal{G}.$$

Proof We proceed as in the proof of [2, Theorem 6.2.6]. Suppose $[\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}]$ is a toposic locally cocartesian fibration and $(\mathcal{Y}, \rho_*)$ is a P-stratified ∞-topos. In view of the adjunction $\mathrm{const} \dashv \lim^{\mathrm{rlax}}$, we first note that we have a natural equivalence[22]

$$\psi : \mathrm{Map}_{\mathbf{Cat}}(\mathcal{Y}, \lim^{\mathrm{rlax}} \widehat{\mathcal{X}}) \xrightarrow{\simeq} \mathrm{Map}^{\mathrm{rlax}}_{/P^{\mathrm{op}}}(\mathcal{Y} \times P^{\mathrm{op}}, \widehat{\mathcal{X}}).$$

Since the evaluation functors $\lim^{\mathrm{rlax}} \widehat{\mathcal{X}} \to \widehat{\mathcal{X}}_p$ at each $p \in P$ are all left adjoints, ψ restricts to the equivalence ψ' in the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathbf{Pr}^L}(\mathcal{Y}, \lim^{\mathrm{rlax}} \widehat{\mathcal{X}}) \arrow[r, "\psi'", "\simeq"'] \arrow[d, "\simeq"] & \mathrm{Map}^{\mathrm{rlax},L}_{/P^{\mathrm{op}}}(\mathcal{Y} \times P^{\mathrm{op}}, \widehat{\mathcal{X}}) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathbf{Pr}^R}(\lim^{\mathrm{rlax}} \widehat{\mathcal{X}}, \mathcal{Y}) \arrow[r, "\psi''", "\simeq"'] & \mathrm{Map}^{\mathrm{llax},R}_{/P^{\mathrm{op}}}(\widehat{\mathcal{X}}, \mathcal{Y} \times P^{\mathrm{op}})
\end{tikzcd}$$

We then have the vertical equivalences (with the right-hand one given by Theorem 5.21), yielding the equivalence ψ'' in which a right-adjoint functor $f_* : \lim^{\mathrm{rlax}} \widehat{\mathcal{X}} \to \mathcal{Y}$ transports to a functor $F : \widehat{\mathcal{X}} \to \mathcal{Y} \times P^{\mathrm{op}}$ such that for all $p \in P$, the fiber $F_p : \widehat{\mathcal{X}}_p \to \mathcal{Y}$ is the right adjoint to the composite

$$\mathcal{Y} \xrightarrow{f^*} \lim^{\mathrm{rlax}} \widehat{\mathcal{X}} \xrightarrow{\mathrm{ev}_p} \widehat{\mathcal{X}}_p.$$

We now observe that f_* is a geometric morphism of P-stratified ∞-topoi if and only if for all $p \in P$, F_p is a geometric morphism, F_p factors through $\mathcal{Y}_p$, and the resulting map $F : \widehat{\mathcal{X}} \to \mathcal{G}(\mathcal{Y})$ preserves locally cocartesian edges. Indeed, the "only if" implication follows from the first half of Observation 5.20, while for the "if" implication, we note that f_* factors as the composite

$$\lim^{\mathrm{rlax}} \widehat{\mathcal{X}} \xrightarrow{\lim^{\mathrm{rlax}} F} \lim^{\mathrm{rlax}} \mathcal{G}(\mathcal{Y}) \xrightarrow{\Theta_P} \mathcal{Y},$$

which respect P-stratifications by the second half of Observation 5.20 and Corollary 5.15, respectively. Therefore, ψ'' restricts to the desired natural equivalence

$$\psi''' : \mathrm{Map}_{\mathbf{StrTop}_{\infty,P}}(\lim^{\mathrm{rlax}} \widehat{\mathcal{X}}, \mathcal{Y}) \simeq \mathrm{Map}_{\mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}}}(\widehat{\mathcal{X}}, \mathcal{G}(\mathcal{Y})).$$

We conclude that $\lim^{\mathrm{rlax}} \dashv \mathcal{G}$. Furthermore, unpacking this equivalence of mapping spaces shows that Θ_P is the counit of the adjunction. Since Θ_P is an equivalence by Theorem 5.13, it remains to show that the unit η is an equivalence. But the compatibility of the equivalence ψ''' with restriction in the base P shows that η_p is homotopic to the identity for all $p \in P$, hence η is an equivalence. □

5.23 Remark Theorem 5.22 should be viewed as the unstable counterpart to [2, Theorem A], which sets up a similar equivalence between P-stratified stable presentable ∞-categories [2, Definition 2.4.3] and locally cocartesian fibrations fibered in such with exact accessible pushforward functors.

[22] Here, **Cat** refers to the ∞-category of large ∞-categories, so that $\mathbf{Pr}^L$ and $\mathbf{Pr}^R$ are subcategories of **Cat**.

Acknowledgements

I would like to thank J. D. Quigley for an inspiring collaboration that led to the genesis of this work. The author was supported first by NSF grant DMS-1547292 and later by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure.

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Received: January 17, 2022 Revised: October 19, 2024

Algebraic & Geometric Topology 26:4 (2026) 1385–1393
DOI: 10.2140/agt.2026.26.1385
Published: April 25, 2026

The guts of nearly fibered knots

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The guts of a knot is an invariant defined for the knot complement by Agol–Zhang. Nearly fibered knots, which are defined as knots whose Floer homology has dimension two in the top Alexander grading, were introduced by Baldwin–Sivek. We provide three models for the guts of nearly fibered knots in the 3-sphere. As a corollary, the nearly fibered condition can be purely topologically characterized and is independent of the specific version of Floer theory.

1 Introduction

By work of Ghiggini [5], Ni [14] and Kronheimer–Mrowka [9], a knot $K \subset S^3$ is fibered if and only if its knot homology in any branch of Floer theory is 1-dimensional in the top Alexander grading. Hence it is natural to ask what happens if the top grading summand of the knot homology is 2-dimensional. Recently, Baldwin–Sivek in [4] introduced the following definition.

Definition 1.1 A knot $K \subset S^3$ is said to be *nearly fibered* (in the Heegaard Floer sense) if

$$\widehat{\mathrm{HFK}}(S^3, K, g(K); \mathbb{Q}) \cong \mathbb{Q}^2.$$

Their definition is stated with Heegaard Floer theory, but we can also define nearly fibered knots in the instanton sense by requiring

$$\mathrm{KHI}(S^3, K, g(K)) \cong \mathbb{C}^2,$$

where KHI denotes the instanton knot homology [9] of $K \subset S^3$.

In this note, we show that the nearly fibered condition has a purely topological characterization, and is independent of the branches of Floer theory. To better describe this criterion, we use the notion of guts of knots recently introduced by Agol–Zhang [1].

Given a knot $K \subset S^3$, we can view its complement $S^3 \backslash N(K)$ as a sutured manifold with its whole boundary being the suture. We can pick a maximal collection of pairwise disjoint and pairwise nonparallel minimal-genus Seifert surfaces S of K, and perform a sutured manifold decomposition

$$S^3 \backslash N(K) \overset{S}{\rightsquigarrow} (M', \gamma'). \tag{1-1}$$

MSC2020: 57K10, 57K18.

We can then pick a maximal collection of pairwise disjoint and pairwise nonparallel nontrivial product annuli A inside (M', γ') and perform a second sutured manifold decomposition[1]

$$(M', \gamma') \overset{A}{\rightsquigarrow} (M, \gamma) \sqcup (M_1, \gamma_1). \tag{1-2}$$

Here (M_1, γ_1) is a product sutured manifold, and no component of (M, γ) is a product.

Definition 1.2 [1] The guts of a knot $K \subset S^3$ is defined to be the sutured manifold (M, γ).

Theorem 1.3 [1, Theorem 1.1] *The guts of a knot $K \subset S^3$ is well defined, i.e., independent of the choices of maximal collections of Seifert surfaces and product annuli in the construction.*

In this note, we prove that a knot $K \subset S^3$ is nearly fibered if and only if its guts falls into one of the three basic models described below. We only state and prove the theorem in instanton theory, but a similar argument applies to Heegaard Floer theory as well.

Theorem 1.4 *Suppose $K \subset S^3$ is a knot of genus g. Let (M, γ) be its guts. Then we have*

$$\mathrm{KHI}(S^3, K, g) \cong \mathbb{C}^2$$

if and only if its guts (M, γ) falls into one of the following three models up to orientation reversal of the ambient 3-manifold:

(M1) *M is a solid torus and γ consists of four longitudes.*

(M2) *M is a solid torus and γ consists of two curves of slope 2.*

(M3) *M is the complement of the right-handed trefoil and γ consists of two curves of slope 2.*

We have the following corollary.

Corollary 1.5 *A knot is nearly fibered in the instanton sense if and only if it is nearly fibered in the Heegaard Floer sense.*

2 Proofs and comments

Proof of Theorem 1.4 We first prove the necessary condition. Suppose $K \subset S^3$ is a genus-g nearly fibered knot. Let (M, γ) be its guts. We first study the sutured manifold decomposition in (1-1).

Claim 1 Any maximal collection of pairwise disjoint and pairwise nonparallel minimal-genus Seifert surfaces contains only one Seifert surface.

[1] Here, note that in Agol and Zhang's paper [1], they also require to decompose along nontrivial product disks to obtain the guts. In this paper we drop the step of decomposing along possible product disks because of [7, Lemma 2.13]: the only two taut balanced sutured manifolds that admit no nontrivial product annulus but admit nontrivial product disks are both product sutured manifolds, and hence are actually the components to be dropped when obtaining guts.

Proof of Claim 1 Suppose S is a minimal-genus Seifert surface of K. We can perform a sutured manifold decomposition of $S^3\backslash N(K)$ along S:

$$S^3\backslash N(K) \overset{S}{\rightsquigarrow} (S^3\backslash[-1,1]\times S, \{0\}\times\partial S).$$

By the proof of [9, Proposition 7.16], we know that there is an isomorphism

$$\mathrm{SHI}(S^3\backslash[-1,1]\times S, \{0\}\times\partial S) \cong \mathrm{KHI}(S^3, K, g) \cong \mathbb{C}^2. \tag{2-1}$$

If there is another minimal-genus Seifert surface S' that is disjoint from S and is not parallel to S, then S' also induces a nonboundary parallel surface in $(S^3\backslash[-1,1]\times S, \{0\}\times\partial S)$, which implies the sutured manifold is not horizontally prime. From the instanton version of [9, Propositions 6.5 and 6.6], we know that one of the two pieces obtained from $(S^3\backslash[-1,1]\times S, \{0\}\times\partial S)$ by cutting along S' must have 1-dimensional sutured homology, because 2 is a prime number. From [9, Theorem 7.18], that piece is a product sutured manifold, which contradicts the assumption that S' is not parallel to S. □

Now Claim 1 above and Theorem 1.3 imply that the sutured manifold (M', γ') in (1-1) can be taken to be simply the complement of S:

$$(M', \gamma') = (S^3\backslash[-1,1]\times S, \{0\}\times\partial S).$$

Next, we study the sutured manifold decomposition (1-2). Note that by construction (M_1, γ_1) is a product sutured manifold, so from [9, Theorem 7.18] and the instanton version of [9, Propositions 6.5 and 6.7], we know that

$$\mathrm{SHI}(M, \gamma) \cong \mathrm{SHI}(S^3\backslash[-1,1]\times S, \{0\}\times\partial S) \cong \mathbb{C}^2.$$

Also, the same argument as above shows that (M, γ) is horizontally prime. Thus we conclude that (M, γ) is reduced in the sense of [7, Definition 2.12]. Then [6, Corollary 1.16] applies and we conclude that

$$b_1(M) = b^1(M) \le 2 - 1 = 1.$$

We claim that $g(\partial M) = b_1(M)$. Indeed, we know that (M, γ) is obtained from the knot complement by decomposition. So [7, Lemma 5.1] implies that $H_2(M) = 0$. As a result, by the universal coefficient theorem and the Poincaré duality, we have

$$H_1(M, \partial M; \mathbb{Q}) \cong H^2(M; \mathbb{Q}) \cong H_2(M; \mathbb{Q}) = 0.$$

Hence the long exact sequence of the pair $(M, \partial M)$ implies that the map

$$i_* : H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q})$$

is surjective. Hence the "half lives and half dies" theorem in 3-dimensional topology implies that

$$g(\partial M) = \tfrac{1}{2} b_1(\partial M) = b_1(M).$$

Now $g(\partial M) = b_1(M) \leq 1$. If $g(\partial M) = 0$, since M is irreducible, we know $M = B^3$. Then for any possible γ on ∂M, we cannot have $\mathrm{SHI}(M,\gamma) \cong \mathbb{C}^2$. Hence we must have $\partial M \cong T^2$. It is well known that any (smooth) torus in S^3 bounds a solid torus. Hence we have two cases.

Case 1 (the manifold M is a solid torus) The instanton Floer homology of any sutured solid torus can be found in [11, Section 4.3]. So the only two models are the ones as stated in (M1) and (M2).

Case 2 (the manifold $S^3 \backslash M$ is a solid torus, i.e., there is a knot $J \subset S^3$ so that $M \cong S^3 \backslash N(J)$) Suppose γ has $2n$ components. Let γ_2 be the union of two adjacent components of γ, which are necessarily oppositely oriented. Next, we make the following claim.

Claim 2 Suppose (M,γ) is a balanced sutured manifold and assume that three components of γ are parallel disregarding the orientation. Write γ_3 to be the disjoint union of these three copies. Note that two components of γ_3 are coherently oriented and are opposite to the third. Let γ_1 be either of the two coherently oriented components and write $\gamma' = (\gamma \backslash \gamma_3) \cup \gamma_1$. Then we have

$$\mathrm{SHI}(M,\gamma) = \mathrm{SHI}(M,\gamma') \otimes \mathbb{C}^2.$$

Proof of Claim 2 The proof essentially follows from the proof of [8, Theorem 3.1]. There exists an embedded annulus $A \subset \partial M$ such that A contains γ_3 and each component of γ_3 is a core of A. Push the interior of A into the interior of M to produce a properly embedded annulus. Fix any orientation of A. Then there is a product annulus decomposition

$$(M,\gamma) \stackrel{A}{\rightsquigarrow} (V,\gamma^4) \sqcup (M,\gamma'),$$

where V is a solid torus and γ^4 consists of four longitudes (there is a unique way, up to isotopy, to make (V,γ^4) a balanced sutured manifold). Now an instanton version of [9, Proposition 6.7] implies that

$$\mathrm{SHI}(M,\gamma) = \mathrm{SHI}(M,\gamma') \otimes \mathrm{SHI}(V,\gamma^4).$$

In the proof of [8, Theorem 3.1], Kronheimer and Mrowka already computed that

$$\mathrm{SHI}(V,\gamma^4) \cong \mathbb{C}^2. \qquad \square$$

Applying Claim 2 repetitively, we conclude that

$$\mathrm{SHI}(M,\gamma) \cong \mathbb{C}^{2^{n-1}} \otimes \mathrm{SHI}(M,\gamma_2).$$

Since

$$\mathrm{SHI}(M,\gamma) \cong \mathbb{C}^2,$$

either $n = 2$ and $\mathrm{SHI}(M,\gamma_2) \cong \mathbb{C}$, or $n = 1$. For the former case, from [9, Theorem 7.18] we know M must also be a solid torus which reduces to Case 1. For the latter case, we further divide it into two subcases.

Case 2.1 (each component of γ represents a generator of $\ker i_* \subset H_1(\partial M) \cong \mathbb{Z}^2$, where

$$i_* : H_1(\partial M) \to H_1(M)$$

is the map induced by the natural inclusion

$$i : \partial M \hookrightarrow M)$$

In this case, first recall that M is a knot complement $S^3 \backslash N(J)$ and hence $H_2(M, \partial M)$ is generated by a minimal-genus Seifert surface T of the knot J. The assumption of Case 2.1 is equivalent to that γ is parallel to $\partial T \subset \partial M$. We can assume that $\partial T \cap \gamma = \varnothing$. If T is a disk, then $R(\gamma)$ is compressible and $\mathrm{SHI}(M, \gamma) = 0$ by the adjunction inequality (see [9, Proposition 7.5]). From now on we assume that T has genus at least 1. We know from [6, Lemma 6.2] that we have two taut decompositions

$$(M, \gamma) \overset{\pm T}{\rightsquigarrow} (M_\pm, \gamma_\pm).$$

We make the following claim.

Claim 3 We have an inclusion

$$\mathrm{SHI}(M_+, \gamma_+) \oplus \mathrm{SHI}(M_-, \gamma_-) \hookrightarrow \mathrm{SHI}(M, \gamma).$$

Proof of Claim 3 We adopt the idea in [11, Section 3]. We isotope T to $T^\pm$ such that the decomposition of (M, γ) along T^+ is (M_+, γ_+) and the decomposition of (M, γ) along $-T^-$ is (M_-, γ_-). $T^\pm$ are called positive and negative stabilizations of T as in [11, Definition 3.1], and we know that $-(T^-) = T^+$. By [11, Theorem 3.4], each $T^\pm$ induces a $\mathbb{Z}$-grading on $\mathrm{SHI}(M, \gamma)$. Then [11, Lemma 4.2] implies that

$$\mathrm{SHI}(M, \gamma, T^+, g(T)) \cong \mathrm{SHI}(M_+, \gamma_+)$$

and

$$\begin{aligned} \mathrm{SHI}(M, \gamma, T^-, -g(T)) &= \mathrm{SHI}(M, \gamma, -T^-, g(T)) \\ &\cong \mathrm{SHI}(M_-, \gamma_-). \end{aligned}$$

Proposition 4.1 of [11] implies that[2]

$$\mathrm{SHI}(M, \gamma, T^-, -g(T)) \subset \mathrm{SHI}(M, \gamma, T^+, 1 - g(T)).$$

Hence we are done since $g(T) \neq 1 - g(T)$. □

Observe that both γ_+ and γ_- contain at least three components that are parallel to each other. Let $\gamma'_\pm$ be the suture obtained from $\gamma_\pm$ by replacing three copies with one copy. Applying Claim 2, we know that

$$\mathrm{SHI}(M_\pm, \gamma_\pm) \cong \mathrm{SHI}(M_\pm, \gamma'_\pm) \otimes \mathbb{C}^2.$$

Tautness together with [9, Theorem 7.12] then implies

$$\dim \mathrm{SHI}(M_\pm, \gamma_\pm) \geq 2.$$

[2] When reversing the orientation of the manifold and the suture, positive and negative stabilizations of T are also switched.

As a result, we have

$$\dim \mathrm{SHI}(M,\gamma) \geq 4,$$

which leads to a contradiction in this case.

Case 2.2 (components of γ do not represent generators of $\ker i_*$) Let Y be the Dehn filling of M along a component of γ. We make the following claim.

Claim 4 We have $\dim I^\sharp(Y) = 2$.

Proof of Claim 4 In order to prove Claim 4, we need the following three facts.

(1) We have $\dim I^\sharp(Y) \neq 0$.

(2) We have $\dim I^\sharp(Y) = \dim \mathrm{SHI}(Y(1)) \leq \dim \mathrm{SHI}(M,\gamma) = 2$.

(3) We have $\dim I^\sharp(Y) \equiv \dim \mathrm{SHI}(M,\gamma) \bmod 2$.

To show (1), note that the fact $[\gamma] \notin \ker i_*$ implies that Y is a rational homology sphere. Hence by [15, Corollary 1.4], we know $\dim I^\sharp(Y) \neq 0$.

To show (2), recall that $M = S^3 \backslash N(J)$ is the knot complement and γ has two components. Let $\gamma_0 \subset \gamma$ be any component. We can attach a 3-dimensional 2-handle along γ_0. The resulting manifold is $Y \backslash B^3$. Hence we have a balanced sutured manifold

$$Y(1) = (Y \backslash B^3, \gamma \backslash \gamma_0).$$

Now let T be the cocore arc of the 2-handle. This arc T is a vertical tangle inside $Y(1)$ as in [16, Definition 1.1]. Now observe that (M,γ) can be obtained from $Y(1)$ by removing T, i.e.,

$$Y(1)_T = (Y(1) \backslash N(T), (\gamma \backslash \gamma_0) \cup \mu_T) \cong (M,\gamma),$$

where μ_T is a meridian of T, and the assumption of Case 2.2 implies $[T] = 0 \in H_1(Y \backslash B^3, \partial(Y \backslash B^3); \mathbb{Q})$. Then by [12, Proposition 1.4] we conclude that

$$\dim I^\sharp(Y) = \dim \mathrm{SHI}(Y(1)) \leq \dim \mathrm{SHI}(M,\gamma) = 2.$$

To show (3), we need to unpack the proof of [12, Proposition 1.4], which is ultimately the proof of [12, Proposition 3.14]. We view [12, Proposition 1.4] as a special case of [12, Proposition 3.14] when $T_0 = \varnothing$. Equation (3.2) of [12] implies that there are sutures Γ_{n-1} and Γ_n such that we have an exact triangle

$$\begin{tikzcd} \mathrm{SHI}(-M,-\Gamma_{n-1}) \arrow[rr] & & \mathrm{SHI}(-M,-\Gamma_n) \arrow[dl] \\ & \mathrm{SHI}(-M,-\gamma) \arrow[ul] & \end{tikzcd} \tag{2-2}$$

And [12, Lemma 3.21] can be rewritten (by replacing n in the original equation by $n-1$) as

$$\begin{tikzcd} \mathrm{SHI}(-M,-\Gamma_{n-1}) \arrow[rr] & & \mathrm{SHI}(-M,-\Gamma_n) \arrow[dl] \\ & I^{\sharp}(-Y)\cong \mathrm{SHI}(-Y(1)) \arrow[ul] & \end{tikzcd} \tag{2-3}$$

Hence some basic linear algebra together with (2-2) and (2-3) implies that

$$\dim I^{\sharp}(-Y) \equiv \dim \mathrm{SHI}(-M,-\gamma) \mod 2.$$

As in [10, Theorem 1.2], we know $\mathrm{SHI}(M,\gamma)$ and $\mathrm{SHI}(-M,\gamma)$ are naturally dual to each other. Since $\partial M \cong T^2$, we know γ and $-\gamma$ are isotopic, we conclude that

$$\dim \mathrm{SHI}(-M,-\gamma) = \dim \mathrm{SHI}(M,\gamma).$$

A similar argument applies to $\mathrm{SHI}(-Y(1))$. □

Recall $M = S^3\backslash N(J)$ is a knot complement and Y is obtained from M by filling along a component of γ, and hence Y can be viewed as a Dehn surgery along J. The assumption of Case 2.2 implies that the surgery slope is nonzero. By passing to the mirror of J, which corresponds to reversing the orientation of M, we can assume that the surgery slope is positive. By Claim 4, we know that

$$\dim I^{\sharp}(Y) = 2 = |H_1(Y)|.$$

Note that by [3, Theorem 1.15], the unknot and the right-handed trefoil are the only two knots on which the positive Dehn surgeries induce instanton L-spaces Y with $|H_1(Y)| = 2$. (According to the theorem, such a knot must be fibered and has genus at most 1 and thus must be either the unknot, the trefoil, or the figure eight. Note the last knot is not strongly quasipositive.) The case of unknot still reduces to Case 1. The case of the right-handed trefoil is a new one. By [2, Theorem 1.1, Table 1], the surgery slope must be 2. Hence γ consists of curves of slope 2, which concludes the proof of the necessary condition.

Finally, the sufficient condition follows immediately from the first isomorphism in (2-1) and the fact that gluing a product sutured manifold other than a 3-ball to an arbitrary sutured manifold via identification of a suture does not change the sutured instanton Floer homology (see [9, Proposition 6.7]). □

Remark 2.1 We have the following comments which strengthen the description of the guts in Theorem 1.4.

(1) We can compute the Euler characteristic in each of the three models. From [8], for the first model,

$$\chi\big(\mathrm{KHI}(S^3,K,g(K))\big) = \chi(\mathrm{SHI}(M,\gamma)) = 0.$$

As a result, we know that the symmetrized Alexander polynomial $\Delta_K(t)$ of K has degree at most $g(K)-1$. On the other hand, if (M,γ) is one of the other two models, we can compute as in [13] that

$$\chi(\mathrm{SHI}(M,\gamma)) = \pm 2.$$

As a result, we know that $\Delta_K(t)$ has degree $g(K)$ and the top nonzero coefficients are ± 2.

(2) Let S be a minimal genus Seifert surface of the knot $K \subset S^3$. Recall as in (1-1) and (1-2), we have a decomposition

$$S^3 \backslash N(K) \overset{S}{\rightsquigarrow} (M', \gamma') \overset{A}{\rightsquigarrow} (M, \gamma) \sqcup (M_1, \gamma_1),$$

where (M_1, γ_1) is a product sutured manifold and (M, γ) is the guts. We write

$$(M_1, \gamma_1) = ([-1, 1] \times F, \{0\} \times \partial F).$$

The proof of [4, Lemma 3.4] implies that the Seifert surface complement $(S^3 \backslash [-1, 1] \times S, \{0\} \times \partial S)$ admits no product annuli whose boundary has a component that is parallel to the suture $\{0\} \times \partial S$ on $\partial(S^3 \backslash [-1, 1] \times S)$. As a result, we can further conclude that ∂F must have one more component than γ, and all but one components of ∂F are glued to all of γ. This actually rules out one model in the case $g(K) = 1$ as in the following example.

Example 2.2 We keep the notation as in Remark 2.1. When $g(K) = 1$, we know that

$$S \cong (R_+(\gamma) \cup \{1\} \times F).$$

Since in all three models we have $\chi(R_+(\gamma)) = 0$, we know that

$$\chi(F) = -1.$$

From part (2) of the Remark 2.1, we know that ∂F has one more component than γ. Then $\chi(F)$ rules out the model in which γ has four components. As a result, we only have two models:

- M is the complement of the unknot and γ consists of two curves of slope 2.
- M is the complement of the right-handed trefoil and γ consists of two curves of slope 2.

Furthermore, in this case, the surface F must be a pair of pants. Yet gluing such a thickened pair of pants to (M, γ) along two of the three boundary components is equivalent to gluing a product 1-handle to (M, γ). Turning this around, we know that (M, γ) being one of the above two models is obtained from the complement of the Seifert surface by a disk decomposition. This coincides with the discussion in [4, Section 1.2] right above [4, Theorem 5.1]. Note that these two models do exist: for example, they give rise to the knot 5_2 in Rolfsen's table and the 2-twisted Whitehead double of the right-handed trefoil with positive clasp.

Acknowledgement

The authors would like to thank John A. Baldwin and Steven Sivek for helpful conversations and for telling us about their work on classifying the genus-one nearly fibered knots.

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Received: August 11, 2022 Revised: December 29, 2024

Algebraic & Geometric Topology **26**:4 (2026) 1395–1449
DOI: 10.2140/agt.2026.26.1395
Published: April 25, 2026

Finiteness properties of some groups of piecewise projective homeomorphisms

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The Lodha–Moore group G is an F_∞ counterexample to von Neumann's conjecture. The group G acts on the real line via piecewise projective homeomorphisms.

We will describe groups $F(S_i)$, $F(S_i')$, $T(S_i)$, $V(S_i)$, and $V(S_i')$ for $i = 2$ and 3. All of these are groups of piecewise projective homeomorphisms that are modelled on Thompson's groups F, T, and V (respectively); each is "locally determined" by one of four inverse semigroups, which we denote by S_i or S_i' ($i = 2, 3$). Following a method developed by Hughes and the author, we will show that all ten groups have type F_∞.

The Lodha–Moore group G is an ascending HNN extension of $F(S_2')$, and thus our results give a new proof that G has type F_∞.

1 Introduction

Monod [11] produced a large family of counterexamples to von Neumann's conjecture; i.e., nonamenable groups with no free subgroups. Corollary 3 from [11] further noted the existence of finitely generated nonamenable groups with no free subgroups, although the method of proof was nonconstructive. Lodha and Moore [9] considered a subgroup G of one of Monod's groups. Their group G, the *Lodha–Moore group*, could be generated by three elements, and was shown in [9] to be finitely presented; indeed, G admits a presentation with three generators and nine relators. In later work, Lodha [8] showed that G has type F_∞. The Lodha–Moore group is also nonamenable, making it an especially economical

MSC2020: primary 20F65, 20J05; secondary 20M18.

finitely presented counterexample to von Neumann's conjecture, and the first F_∞ counterexample to von Neumann's conjecture. (The paper [13] provided the first finitely presented nonamenable groups with no free subgroups, but the groups in question had many more generators and relations. The higher finiteness properties of the groups from [13] remain unknown to the best of this author's knowledge.)

In [6], Hughes and the author described a general approach to establishing finiteness properties for generalised Thompson groups with "piecewise" definitions. The basic theory of [6] was produced in the hope that it would unify the existing proofs of finiteness properties for such groups. The Lodha–Moore group G offers a useful test case.

We now briefly recall the methods of [6]. In the setting of [6], an inverse semigroup S is a set of partial bijections of some set X that is closed under compositions and inverses. By a *partial bijection* of X, we mean a bijection between two subsets of X. We define a group Γ_S, the *group locally determined by* S, to be the collection of all bijections of X that are finite unions of partial bijections from S. (I.e., $\gamma \in \Gamma_S$ if, for some $n \in \mathbb{N}$, there are elements $s_1, s_2, \dots, s_n \in S$ such that the domains of the s_i, denoted by D_i, form a partition of X, the images $s_i(D_i)$ are also a partition of X, and $\gamma_{|D_i} = s_i$, for $i = 1, \dots, n$.)

The construction of classifying spaces for the groups Γ_S depends upon a sequence of choices. The choice of S determines a collection of *domains* $\mathcal{D}_S$, which are simply the domains of the elements $s \in S$. The set of nonempty domains is denoted by $\mathcal{D}_S^+$. The second choice is that of an S*-structure*, which is a function $\mathbb{S} : \mathcal{D}_S^+ \times \mathcal{D}_S^+ \to \mathcal{P}(S)$ assigning a (possibly empty) collection of transformations from S to each pair (D_1, D_2) of nonempty domains. The sets $\mathbb{S}(D_1, D_2)$ are required to satisfy various "groupoid-like" properties. The specific properties that we need are summarised in Proposition 4.4. The S-structure $\mathbb{S}$ determines a set $\mathcal{V}_{\mathbb{S}}$ with a partial order $\leq$ called "expansion" (Definition 4.7). The expansion partial order is, roughly speaking, determined by the subdivision of a given domain into (finitely many) smaller domains. Under appropriate hypotheses, the partially ordered set $(\mathcal{V}_{\mathbb{S}}, \leq)$ becomes a directed Γ_S-set. The simplicial realisation $\Delta_{\mathbb{S}}$ of $\mathcal{V}_{\mathbb{S}}$ is therefore a contractible simplicial complex upon which Γ_S acts simplicially. The construction of $\mathcal{V}_{\mathbb{S}}$ is very much like the one introduced by Brown [3], where he proved the F_∞ property for a wide variety of generalised Thompson groups.

The simplicial complex $\Delta_{\mathbb{S}}$ often has undesirable properties, however. For instance, it almost always fails to be locally finite. It can be helpful to replace $\Delta_{\mathbb{S}}$ with something smaller. A (third) choice of an *expansion scheme* $\mathcal{E}$ (Definition 6.15) determines a subcomplex $\Delta_{\mathbb{S}}^{\mathcal{E}} \subset \Delta_{\mathbb{S}}$. (The complex $\Delta_{\mathbb{S}}^{\mathcal{E}}$ should be thought of as a generalisation of the complexes introduced by Stein [15], who created locally finite models for various groups of piecewise linear homeomorphisms of the line, including Thompson's group F.) The complex $\Delta_{\mathbb{S}}^{\mathcal{E}}$ can be anything from a discrete set of points to the complex $\Delta_{\mathbb{S}}$ itself, depending on the size of the expansion scheme. Given an appropriate choice of $\mathcal{E}$, the main results of [6] show how to deduce the F_∞ property for Γ_S, by applying Brown's finiteness criterion to the complex $\Delta_{\mathbb{S}}^{\mathcal{E}}$.

The main goal of this paper is to describe a family of ten groups of piecewise projective homeomorphisms of $[0, 1)$, S^1, and $[0, \infty)$, and prove that each group has type F_∞. We will also describe the Lodha–Moore group G as an ascending HNN extension of one of the groups, and therefore obtain a new proof that G has type F_∞. Our approach follows the method from [6]:

(1) First, we consider four inverse semigroups: S_2, S_3, S_2', and S_3'. The generators of S_i, for $i = 2, 3$, are A, B, and C_i, where the domain of each transformation is the interval $I = [0, 1)$, and

$$A(x) = \frac{x}{x+1}; \quad B(x) = \frac{1}{2-x}; \quad C_2(x) = \frac{2x}{x+1}; \quad C_3(x) = \frac{3x}{2x+1}.$$

The inverse semigroups S_2' and S_3' have the additional generator $T : [0, \infty) \to [1, \infty)$, defined by the rule $T(x) = x + 1$. The groups

$$F(S_i), \quad F(S_i'), \quad T(S_i), \quad V(S_i), \quad V(S_i')$$

are then defined to be the groups that are "locally determined" (in the sense of [6]) by the semigroup S_i or S_i'. The "F" groups are homeomorphisms of the line or interval, the "T" groups are homeomorphisms of the circle, and the "V" groups are groups of right-continuous bijections. (The notation is intended to recall the definitions of Thompson's group F, T, and V, as presented in (for instance) [5].)

(2) The domains of S_i and S_i' are not very tractable for our purposes. We will therefore restrict the domains under consideration to what we call a set of "generating domains" $\mathcal{D}^+_{\text{gen}}$ (Definition 3.1), which, for us, are simply the forward iterates of $I = [0, 1)$ under the transformations A and B. (The decision to work with a proper subset of $\mathcal{D}^+_S$ represents the most important departure from [6].)

For every pair of domains (D_1, D_2), we then define $\mathbb{S}(D_1, D_2)$ as the set of all transformations from S_i or S_i' having D_1 as the domain and D_2 as the range, where D_1 and D_2 are arbitrary members of $\mathcal{D}^+_{\text{gen}}$. The sets $\mathbb{S}(D_1, D_2)$ enable us to define an "expansion" operation. The expansions (Definition 4.7) from the pair $[\mathrm{id}_I, I]$ (Definition 4.5) can be usefully described by numbered binary trees, exactly as was done in [9].

(3) The directed sets from (2) are too large. In search of more tractable complexes, we define the expansion schemes $\mathcal{E}_i$ and $\mathcal{E}_i'$ as in Example 6.22. The expansion schemes systematically restrict the types of expansions that are allowed in the complexes $\Delta^{\mathcal{E}_i}_{\mathbb{S}}$ and $\Delta^{\mathcal{E}_i'}_{\mathbb{S}}$. The ascending stars in the resulting complexes are isomorphic to products of a simplicial cone on a cellulated line — see Figure 5 on page 1420. The burden of the rest of the argument is to show that $\mathcal{E}_i$ and $\mathcal{E}_i'$ are "n-connected expansion schemes" (Definition 6.18). The proof occupies Sections 7 and 8, and represents the technical heart of the paper. The material from Sections 7 and 8 is heavily indebted to the argument from [9], but generalises that work in what we consider to be interesting ways. For instance, our argument also shows that the monoid generated by the linear fractional transformations $\{A, B, C_2, c_2\}$ (even without the above restrictions on their domains) admits a finite complete rewrite system.

With (1)–(3) complete, the proof that the groups $F(S_i)$, $F(S_i')$, $T(S_i)$, $V(S_i)$, and $V(S_i')$ ($i = 2, 3$) all have type F_∞ follows by a standard argument (a variant of the main argument of [6]). This standard argument is summarised in Section 9. The relationship between $F(S_2')$ and G is described in Section 10; specifically, we show that G is an ascending HNN extension of $F(S_2')$, which directly implies that G has type F_∞.

The author had originally wanted to offer a proof of F_∞ for an infinite family of generalisations of the Lodha–Moore group G. One approach to producing an infinite family of similar groups is, for a given n, to replace the transformation C_2 with a transformation $C_n : I \to I$, defined as

$$C_n(x) = \frac{nx}{(n-1)x+1}.$$

We can then define S_n to be the inverse semigroup generated by $\{A, B, C_n\}$ (and their inverses), and define the groups $F(S_n)$, $F(S_n')$, $T(S_n)$, $V(S_n)$, and $V(S_n')$ for arbitrary n. We find, however, that our method fails for $n \geq 4$; in fact, we are not even able to build useful directed sets (along the lines sketched for $n = 2, 3$). Our difficulties are summarised in Section 5.

Let us briefly describe the structure of the paper. In Section 2, we define the inverse semigroups S_i and S_i' ($i = 2, 3$) and the groups that are locally defined by these semigroups. In Section 3, we define the "generating domains" that we need. This section also includes a proof of the "eventual invariance" property, which will be used to construct directed sets later. In Section 4, we build directed sets with Γ actions (for Γ as above) and compute vertex stabilisers. We show, in particular, that the vertex stabilisers in all of our complexes are virtually free abelian of finite rank. In Section 5, we describe an algorithm that analyses various potential generalisations of our piecewise projective homeomorphism groups. The conclusion of the section is that such generalisations are surprisingly very thin on the ground. In Section 6, we review expansion schemes, introduce expansion schemes $\mathcal{E}_i$ and $\mathcal{E}_i'$ ($i = 2, 3$), and also introduce subdivision trees, which are used to describe expansions.This section also describes an equivalence relation on subdivision trees. In Section 7, we compute finite complete semigroup presentations for the inverse semigroups S_2 and S_3. These presentations are vital in understanding the equivalence relation on subdivision trees. In Section 8, we prove an "intermediate value theorem", which is what we need in order to show that the expansion schemes $\mathcal{E}_i$ and $\mathcal{E}_i'$ define contractible complexes. The proof of the latter is in Section 9, which also assembles all of the other ingredients of the proof that the groups have type F_∞. Section 10 establishes the connection between G and $F(S_2')$.

2 A family of inverse semigroups

We consider the usual action of $\mathrm{PSL}_2(\mathbb{R})$ on the upper half-space model of the hyperbolic plane $\mathbb{H}^2 \subseteq \mathbb{C}$. A 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts as a linear fractional transformation $f_M : \mathbb{H}^2 \to \mathbb{H}^2$, where

$$f_M(z) = \frac{az+b}{cz+d} \quad (z \in \mathbb{C}; \mathrm{Im} z > 0).$$

It is well-known [14] that the assignment $M \mapsto f_M$ induces an isomorphism between $\mathrm{PSL}_2(\mathbb{R})$ and the group of all orientation-preserving isometries of $\mathbb{H}^2$, denoted by $\mathrm{Isom}^+(\mathbb{H}^2)$. In what follows, we will make no distinction between M and f_M, referring to either one by the matrix M.

In practice, we will be concerned primarily with the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\partial\mathbb{H}^2$, which we identify with $\mathbb{R}\cup\{\infty\}$. The inverse semigroups alluded to in this section's title act as partial bijections of $\partial\mathbb{H}^2$ via (restrictions of) linear fractional transformations.

Definition 2.1 (partial bijections; inverse semigroups; domains) Let X be a set. A *partial bijection* of X is a bijection $h: A_h \to B_h$ between subsets A_h and B_h of X. The composition of two partial bijections is defined on "overlaps": if $g: A_g \to B_g$ and $h: A_h \to B_h$ are partial bijections of X, then $g\circ h$ is a bijection from $h^{-1}(A_g)$ to $g(B_h\cap A_g)$.

A collection S of partial bijections of X is called an *inverse semigroup* if S is closed under inverses and compositions. We may also refer to such an S as an *inverse semigroup acting on S*.

If S is an inverse semigroup and $h: A_h\to B_h$, then we refer to A_h as a *domain* of S. Note that B_h is also a domain, since S is closed under inverses. We let $\mathcal{D}_S$ denote the set of all domains A_h, as h ranges over all $h\in S$. We let $\mathcal{D}_S^+ = \mathcal{D}_S - \{\varnothing\}$ (i.e., the set of all nonempty domains).

Remark 2.2 Let S be an inverse semigroup. We note two basic properties:

(1) If $s\in S$ and D is a domain of S that is contained in the domain of s, then $s_{|D}\in S$. Indeed, let D be the domain of $t\in S$. Then $s_{|D} = st^{-1}t$.

(2) If D is a domain of S, then $\mathrm{id}_D\in S$. Indeed, if $t\in S$ has D as its domain, then $t^{-1}t = \mathrm{id}_D$.

(3) If D_1 and D_2 are domains of S, then $D_1\cap D_2$ is also a domain of S. Indeed, letting D_1 be the domain of $t_1\in S$, and D_2 be the domain of $t_2\in S$, we find that the domain of $t_1^{-1}t_1t_2^{-1}t_2$ is $D_1\cap D_2$.

Remark 2.3 Inverse semigroups can also be defined abstractly (see [7]). The Preston–Wagner theorem states that any inverse semigroup can be realized as a collection of partial bijections (in the above sense). The proof parallels that of Cayley's theorem, which states that every group can be realized as a group of permutations.

Remark 2.4 Some readers may be familiar with the theory of étale groupoids. Inverse semigroups seem to be closely related (see, for instance, [4], where a form of equivalence between étale groupoids and inverse semigroups is established). The precise nature of the relationship between the methods of [6] (and this paper) and the broader literature of étale groupoids is unclear to the author, who has no expertise in the latter area.

Remark 2.5 In Definition 2.1, the function with empty domain and codomain plays the role of a 0.

Definition 2.6 (the inverse semigroups S_n and S_n') Let $A:[0,1)\to[0,1/2)$ be the restriction of the linear fractional transformation

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $B:[0,1)\to[1/2,1)$ be the restriction of

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

For $n = 2, 3$, let $C_n : [0, 1) \to [0, 1)$ be the restriction of

$$C_n = \begin{pmatrix} n & 0 \\ n-1 & 1 \end{pmatrix}.$$

Let $T : [0, \infty) \to [1, \infty)$ be the restriction of

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We will use lower-case letters to denote the inverses of the above transformations ($a = A^{-1}$, etc.).

For $n = 2, 3$, we let

$$S_n = \langle A, B, C_n \rangle; \quad S'_n = \langle A, B, C_n, T \rangle,$$

where the brackets indicate the inverse semigroup generated by the bracketed transformations; i.e., S_n and S'_n are closed under compositions and inverses.

Definition 2.7 (locally determined by S; the inverse semigroup $\widehat{S}$) Let S be an inverse semigroup acting on a set X. Let A and B be subsets of X. A bijective function $f : A \to B$ is *locally determined by* S if there is a finite partition $\mathcal{P} = \{D_1, \dots, D_m\}$ of A into domains (i.e., $\mathcal{P} \subseteq \mathcal{D}^+_S$) such that $f_{|D_i} \in S$, for each i.

We let $\widehat{S}$ denote the collection of all functions that are locally determined by S. The set $\widehat{S}$ is an inverse semigroup under the operation of composition.

Definition 2.8 Let $n = 2$ or 3. Let

- $F(S_n)$ be the group of homeomorphisms of $[0, 1)$ that are locally determined by S_n;
- $F(S'_n)$ be the group of homeomorphisms of $[0, \infty)$ that are locally determined by S'_n;
- $T(S_n)$ be the group of homeomorphisms of the circle $[0, 1]/\sim$ that are locally determined by S_n;
- $V(S_n)$ be the group of right-continuous bijections of $[0, 1)$ that are locally determined by S_n;
- $V(S'_n)$ be the group of right-continuous bijections of $[0, \infty)$ that are locally determined by S'_n.

Remark 2.9 The group $T(S_n)$ can also be described as the subgroup of $V(S_n)$ that preserves a cyclic ordering on $[0, 1)$.

3 A generating set of domains for S_i and S'_i

The set $\mathcal{D}^+_S$ (Definition 2.1) will be far too big when $S = S_i$ or $S = S'_i$. In this section, we define a subcollection $\mathcal{D}^+_{S,\mathrm{gen}} \subseteq \mathcal{D}^+_S$, which will be sufficient for the constructions of later sections. We note that this is in contrast with [6], which always uses the full set $\mathcal{D}^+_S$.

Definition 3.1 (generating sets of domains) Let $\{A, B\}^*$ denote the set of all positive words in the alphabet $\{A, B\}$, including the empty word. Let

$$\mathcal{D}^+_{S',\mathrm{gen}} = \{T^\alpha \omega \cdot [0, 1) \mid \omega \in \{A, B\}^*; \alpha \geq 0\} \cup \{T^\alpha \cdot [0, \infty) \mid \alpha \geq 0\}$$

and

$$\mathcal{D}^+_{S,\mathrm{gen}} = \{\omega \cdot [0,1) \mid \omega \in \{A,B\}^*\}.$$

We will often refer to $\mathcal{D}^+_{S,\mathrm{gen}}$ or $\mathcal{D}^+_{S',\mathrm{gen}}$ by the notation $\mathcal{D}^+_{\mathrm{gen}}$ if doing so should cause no ambiguity.

We may sometimes refer to the members of $\mathcal{D}^+_{S,\mathrm{gen}}$ and $\mathcal{D}^+_{S',\mathrm{gen}}$ as *generating domains*.

Remark 3.2 It will be convenient to write I in place of $[0,1)$, and to write ωI in place of $\omega \cdot I$.

The half-open intervals ωI of $\mathcal{D}^+_{S,\mathrm{gen}}$ are in one-to-one correspondence with the vertices of an infinite binary tree. The intervals $\omega A I$ and $\omega B I$ correspond to the left and right children (respectively) of ωI. In particular, $\omega' I$ contains ωI if and only if ω' is a prefix of ω, and the intervals are disjoint if neither ω' nor ω is a prefix of the other.

Note, however, that the intervals ωI are very far from being the standard dyadic intervals when the length of ω is two or more. For instance, $ABAI = [1/3, 2/5)$ and $BAI = [1/2, 2/3)$. It appears that ωI, for $\omega \in \{A,B\}^*$, is always an interval between consecutive Farey fractions (as noted in [9]), although we will not need to use this fact. The intervals $T^\alpha \omega I$ are simply the translates of the intervals ωI by nonnegative integers.

It will be useful to keep in mind that the products aB and bA are 0 in what follows.

Lemma 3.3 (an eventual invariance property) *Let $s \in S_i$, where $i = 2$ or 3. Let $D \in \mathcal{D}^+_{S,\mathrm{gen}}$ be such that D is contained in the domain of s. There is a finite partition $\mathcal{P} \subseteq \mathcal{D}^+_{S,\mathrm{gen}}$ of D such that $sP \in \mathcal{D}^+_{S,\mathrm{gen}}$, for each $P \in \mathcal{P}$.*

The analogous statement also holds true for S'_n, $i = 2, 3$.

Proof Let $D = \omega I$, where $\omega \in \{A,B\}^*$. By induction on the length of s, it suffices to prove the lemma in the case $s \in \{A, B, C_2, C_3, a, b, c_2, c_3\}$.

Suppose first that $s = A$. It follows directly that $sD = A\omega I$, so $sD \in \mathcal{D}^+_{S,\mathrm{gen}}$ and we may set $\mathcal{P} = \{D\}$. If $s = a$, it must be that $\omega = A\omega'$, for some $\omega' \in \{A,B\}^*$, and therefore $sD = \omega' I$. We can therefore again let $\mathcal{P} = \{D\}$.

If $s = B$ or $s = b$, the proof is very similar.

Let $s = C_2$. A straightforward check shows that

$$\begin{aligned} C_2AA &= AC_2; \\ C_2AB &= BAc_2; \\ C_2B &= BBC_2. \end{aligned}$$

We use these identities to "push" C_2 as close to the end of the word ω as possible. (The inverse c_2 can appear during this process. This poses no problems, since we can use the same identities to push c_2 forward as well.) In doing so, we can arrange that

$$C_2\omega = \omega' C_2^\epsilon \omega'',$$

where $\omega', \omega'' \in \{A,B\}^*$, $\epsilon = \pm 1$, and

(1) ω'' is an empty word, or

(2) $\omega'' = A$ if $\epsilon = 1$, or

(3) $\omega'' = B$ if $\epsilon = -1$.

If ω'' is not the empty word, we then let $\mathcal{P} = \{\omega AI, \omega BI\}$; if ω'' is empty, we set $\mathcal{P} = \{D\}$; these are the required partitions. (For instance, if $\epsilon = 1$ and $\omega'' = A$, we have

$$C_2 \omega AI = \omega' C_2 AAI = \omega' AC_2 I = \omega' AI \in \mathcal{D}^+_{\text{gen}}$$

and

$$C_2 \omega BI = \omega' C_2 ABI = \omega' BAc_2 I = \omega' BAI \in \mathcal{D}^+_{\text{gen}}.$$

Similar checking handles the remaining case.)

The case in which $s = c_2$ is very similar, and features the same identities, suitably rewritten so that $c_2 A$, $c_2 BA$, and $c_2 BB$ appear on the left-hand sides of the equations.

Now suppose that $s = C_3$. We have the matrix identities

$$\begin{aligned} C_3 AAA &= AC_3; \\ C_3 AAB &= BAAc_3; \\ C_3 ABA &= BABC_3; \\ C_3 ABB &= BBAc_3; \\ C_3 B &= BBBC_3. \end{aligned}$$

We can then follow the same strategy as we did in the case $s = C_2$. "Push" C_3 as close to the end of ω as possible. The result is $\omega' C_3^{\epsilon} \omega''$, where $\omega' \in \{A, B\}^*$ and

(1) ω'' is empty, or

(2) $\epsilon = 1$ and $\omega'' \in \{A, AA, AB\}$, or

(3) $\epsilon = -1$ and $\omega'' \in \{B, BA, BB\}$.

If ω'' is empty, then $\mathcal{P} = \{D\}$ is the required partition of D. If $\epsilon = 1$ and $\omega'' = A$, then we set $\mathcal{P} = \{\omega AAI, \omega ABI, \omega BAI, \omega BBI\}$. This is the required partition; indeed,

$$C_3 \omega AAI = \omega' C_3 AAAI = \omega' AC_3 I = \omega' AI \in \mathcal{D}^+_{\text{gen}},$$

and similar calculations show that $C_3(\mathcal{P}) \subseteq \mathcal{D}^+_{\text{gen}}$. If $\omega'' \in \{AA, AB\}$, then the required partition is $\mathcal{P} = \{\omega AI, \omega BI\}$. If $\epsilon = -1$ and $\omega'' \in \{B, BA, BB\}$, then one proceeds similarly. The required partitions are $\{\omega AAI, \omega ABI, \omega BAI, \omega BBI\}$ (in the first case, when $\omega'' = B$) and $\{\omega AI, \omega BI\}$ (when $\omega'' = BA$ or BB).

The extension of these arguments to S'_i is straightforward, but we consider the case of S'_2 by way of example. In this case, the domain D has one of the forms

$$T^{\alpha} \omega \cdot [0, 1) \quad \text{or} \quad T^{\alpha} \cdot [0, \infty),$$

where $\omega \in \{A, B, C, a, b, c\}^*$ and $\alpha \geq 0$. We note that, in the former case, $D \subseteq [\alpha, \alpha+1)$, while $D = [\alpha, \infty)$ in the latter case. The semigroup element $s \in S_2'$ has the form

$$T^m \widehat{\omega} T^{-n},$$

for some word $\widehat{\omega} \in \{A, B, C, a, b, c\}^*$, where m and n are nonnegative. (The proof is as follows. We know that s is not the zero element by hypothesis. The products XT and $T^{-1}X$ are always zero when $X \in \{A, B, C, a, b, c\}$. We also have the identity $T^{-1}T = \mathrm{id}_{[0,\infty)}$. It follows that, after suitable reductions, no occurrence of T^{-1} can occur immediately before a different generator, and no occurrence of T can occur immediately after a different generator. Thus, the given form describes the only possibilities.)

Now we consider cases. If $D = [\alpha, \infty)$, then s necessarily has the form $T^m T^{-n}$ by domain considerations. (If $\widehat{\omega} \neq 1$, then the domain in question is an interval of finite length. This is impossible, since D must be contained in the domain of s.) It follows that we can simply let $\mathcal{P} = \{D\}$. Suppose that $D = T^\alpha \omega \cdot [0, 1)$; we must consider the possible cases for n. It is not possible for n to be greater than α, since this would mean that the domain of s is contained in $[n, \infty)$, and thus result in D not being a subset of the domain of s. If $n < \alpha$, then s must take the form $T^m T^{-n}$ (otherwise, if $\widehat{\omega} \neq 1$, we would conclude that the domain of s is contained in $[n, n+1)$, which is disjoint from D). We apply the lemma to the case of S_2, temporarily letting $s = \mathrm{id}_{[0,1)}$ and $D = \omega \cdot [0, 1)$, to find that there is partition $\widehat{\mathcal{P}}$ of $\omega \cdot [0, 1)$ into generating domains. We can then let $\mathcal{P} = T^\alpha \cdot \widehat{\mathcal{P}}$. Noting that the property of being a generating domain is unchanged after an application of T, we see that this is the required partition. If $n = \alpha$, then we apply the lemma with $s = \widehat{\omega}$ and $D = \omega \cdot [0, 1)$ to get a partition $\widehat{\mathcal{P}}$ of D into generating domains with the additional property that $\widehat{\omega} \cdot P$ is a generating domain, for each $P \in \widehat{\mathcal{P}}$. It then follows that $\mathcal{P} = T^\alpha \cdot \widehat{\mathcal{P}}$ is the required partition. □

Remark 3.4 A further application of Lemma 3.3 is that every domain $D \in \mathcal{D}_S^+$ or $\mathcal{D}_{S'}^+$ can be partitioned into finitely many generating domains. Indeed, each such D is the domain of some word ω in the generators of S_i or S_i'. We can then show that D can be partitioned into generating domains by induction on the length of ω. The base case (in which the length of ω is 1) is trivial, since all of the domains in question are necessarily generating domains. The inductive step is then handled by applying Lemma 3.3.

4 A directed set construction

In this section, we will specify an S-structure $\mathbb{S}$ for $S \in \{S_2, S_3, S_2', S_3'\}$, in essentially the sense of [6]. In fact, the only difference is that we will define our S-structure using the domains $\mathcal{D}_{S,\mathrm{gen}}^+$ and $\mathcal{D}_{S',\mathrm{gen}}^+$, rather than the entire collection $\mathcal{D}^+$, as required in [6]. The S-structure leads to a directed set construction of a contractible simplicial complex, exactly as in [6]. We will first consider these directed set constructions for $V(S_i)$ and $V(S_i')$ $(i = 2, 3)$ in Section 4.1. The simplicial complexes for the related groups can then be obtained as subcomplexes; this is spelled out in Section 4.3.

It will be useful to let $\mathcal{D}_{\mathrm{gen}}^+$ denote either $\mathcal{D}_{S,\mathrm{gen}}^+$ or $\mathcal{D}_{S',\mathrm{gen}}^+$, depending on the context.

4.1 The directed set constructions for $V(S_i)$ and $V(S_i')$

We will first show how to make $V(S_i)$ and $V(S_i')$ act on directed sets, and (therefore) on contractible simplicial complexes. The basic approach follows [6], but we are able to use a simplified version of the basic theory, with suitable modifications. All of the results in this subsection work in the same way for all $\Gamma \in \{V(S) \mid S \in \{S_2, S_3, S_2', S_3'\}\}$, so we will use the generic notation Γ to refer to any group from the latter collection.

Definition 4.1 (structure sets; domain types) Let $S \in \{S_2, S_3, S_2', S_3'\}$. Let $D_1, D_2 \in \mathcal{D}^+_{\mathrm{gen}}$. We set

$$\mathbb{S}(D_1, D_2) = \{s \in S \mid \text{the domain of } s \text{ is } D_1 \text{ and the range is } D_2\}.$$

Two domains D_1 and D_2 have the *same type* if $\mathbb{S}(D_1, D_2) \neq \varnothing$; i.e., if there is some $s \in S$ such that D_1 is the domain of s and D_2 is the image of s.

Remark 4.2 (description of domain types) There are one or two domain types, depending on whether $S \in \{S_2, S_3\}$, on the one hand, or $S \in \{S_2', S_3'\}$, on the other. The first of the domain types consists of those sets of the form ωI, where $\omega \in \{A, B\}^*$. This domain type occurs in both S_i and S_i', and it is the only type if $S \in \{S_2, S_3\}$. The second of the domain types (present only when $S \in \{S_2', S_3'\}$) consists of domains of the form $[n, \infty)$, where n is a nonnegative integer.

Theorem 4.3 (explicit description of structure sets) *Let $S = S_2$ or S_3. Given ωI and $\omega' I \in \mathcal{D}^+_{\mathrm{gen}}$ $(\omega, \omega' \in \{A, B\}^*)$, the associated structure set takes the form*

$$\mathbb{S}(\omega I, \omega' I) = \{\omega' C^k \omega^{-1} \mid k \in \mathbb{Z}\}.$$

Let $S = S_2'$ or S_3'. Given ωI and $\omega' I \in \mathcal{D}^+_{\mathrm{gen}}$ $(\omega, \omega' \in \{A, B, T\}^*)$, the associated structure set takes the form

$$\mathbb{S}(\omega I, \omega' I) = \{\omega' C^k \omega^{-1} \mid k \in \mathbb{Z}\}.$$

The set $\mathbb{S}([m, \infty), [n, \infty))$ is $\{T^{n-m}\}$, when m, n are nonnegative integers.

Proof We first consider the case of $\mathbb{S}(I, I)$ when $S = S_2$.

Let $\omega \in \mathbb{S}(I, I)$. Thus, $\omega I = I$. Let G be the group generated by the linear fractional transformations A, B, and C, each viewed as a transformation of $\mathbb{H}^2$ or the projective line $\mathbb{R} \cup \{\infty\}$. We note that the inverses of A, B, and C may be represented by the matrices

$$a = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}; \quad c = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}.$$

It follows that, if ω is expressed as a product of matrices, then $\det(\omega) = 2^n$, for some nonnegative integer n. We note also that ω fixes the points 0 and 1 on the projective line, by our assumptions.

We let

$$\omega = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where α, β, γ, and δ are integers. The equality $\omega(0) = 0$ directly implies that $\beta = 0$. The equality $\omega(1) = 1$ then implies that $\alpha = \gamma + \delta$. Computing determinants, we find that

$$(\gamma + \delta)\delta = 2^n.$$

It follows that $\gamma + \delta = 2^k$ and $\delta = 2^\ell$, where $k + \ell = n$ and k and ℓ are nonnegative integers. Either $k \le \ell$ or $\ell \le k$; in the first case,

$$\begin{pmatrix} 2^k & 0 \\ 2^k - 2^\ell & 2^\ell \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 1 - 2^{\ell-k} & 2^{\ell-k} \end{pmatrix} = c^{\ell-k}.$$

Similarly, if $\ell \le k$, then $\omega = C^{k-\ell}$. In either case, $\omega = C^\alpha$, for appropriate α.

If $S = S_3$, then the set $\mathbb{S}(I, I)$ takes the same form. Here

$$C = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}; \quad c = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix},$$

and therefore $\det(\omega) = 3^n$, for some $n \in \mathbb{Z}$. The remainder of the argument differs from the case of S_2 primarily in the fact that it involves powers of 3, rather than powers of 2.

Now we consider a general structure set $\mathbb{S}(\omega I, \omega' I)$, where $\omega, \omega' \in \{A, B\}^*$ and S may be any of the semigroups S_2, S_3, S_2', or S_3'. Let $\sigma \in \mathbb{S}(\omega I, \omega' I)$. It follows that $(\omega')^{-1}\sigma\omega \in \mathbb{S}(I, I)$, so $(\omega')^{-1}\sigma\omega = C^k$, for some $k \in \mathbb{Z}$. Thus, $\sigma = \omega' C^k \omega^{-1}$, as claimed. Conversely, it is clear that any transformation of the form $\omega' C^k \omega^{-1}$ is in $\mathbb{S}(\omega I, \omega' I)$.

The final statement, about $\mathbb{S}([m, \infty), [n, \infty))$ follows from the fact that T^{n-m} is the only inverse semigroup element of S_i' with the given domain and codomain. $\square$

Proposition 4.4 (closure properties of $\mathbb{S}$) *Let $D_1, D_2 \in \mathcal{D}^+_{\text{gen}}$.*

(1) ***Compositions*** *If $h \in \mathbb{S}(D_1, D_2)$ and $g \in \mathbb{S}(D_2, D_3)$, then $gh \in \mathbb{S}(D_1, D_3)$.*

(2) ***Inverses*** *If $h \in \mathbb{S}(D_1, D_2)$, then $h^{-1} \in \mathbb{S}(D_2, D_1)$.*

(3) ***Identities*** $\mathrm{id}_{D_1} \in \mathbb{S}(D_1, D_1)$.

Proof All of these properties follow directly from the definition of the set $\mathbb{S}(D_1, D_2)$. $\square$

Definition 4.5 (the set $\mathcal{B}$) Let

$$\mathcal{A} = \{(f, D) \mid f \in \widehat{S};\ D \in \mathcal{D}^+_{\text{gen}};\ D \text{ is contained in the domain of } f\}.$$

(Recall that $\widehat{S}$ is the inverse semigroup of functions that are locally determined by S (Definition 2.7).) We write $(f_1, D_1) \sim (f_2, D_2)$ if there is some $h \in \mathbb{S}(D_1, D_2)$ such that $f_1 = f_2 h$. It is easily checked that $\sim$ is an equivalence relation on $\mathcal{A}$, using Proposition 4.4. We let $\mathcal{B}$ denote the set of all equivalence classes. The equivalence class of (f, D) will be denoted by $[f, D]$.

Definition 4.6 (vertices; the type of a vertex) A finite subset

$$\{[f_1, D_1], \dots, [f_m, D_m]\} \subseteq \mathcal{B}$$

is a *vertex* if

$$\bigcup_{i=1}^{m} f_i(D_i) = \mathbb{R}^+ \quad \text{or} \quad \bigcup_{i=1}^{m} f_i(D_i) = [0, 1),$$

depending upon whether the underlying semigroup is S_n' or S_n, respectively. (Here $\mathbb{R}^+$ is the set of nonnegative real numbers and m may be any natural number.)

We let $\mathcal{V}_S$ denote the set of all vertices, where $S \in \{S_2, S_3, S_2', S_3'\}$. We may sometimes write $\mathcal{V}$ in place of $\mathcal{V}_S$ if this will result in no ambiguity.

Two vertices $\{[f_1, D_1], \dots, [f_m, D_m]\}$ and $\{[g_1, E_1], \dots, [g_n, E_n]\}$ have the *same type* if the multisets $\{[D_1], \dots, [D_m]\}$ and $\{[E_1], \dots, [E_n]\}$ are identical; i.e., $m = n$ and $[D_j] = [E_j]$, for $j = 1, \dots, m$.

Definition 4.7 (expansion; contraction) Let $v = \{[f_1, D_1], \dots, [f_n, D_n]\}$ be a vertex. We say that a vertex v' is obtained from v by *expansion at* $[f_i, D_i]$ if there is some $h \in \mathbb{S}(D_i, D_i)$ and a finite partition $\mathcal{P} \subseteq \mathcal{D}^+_{\text{gen}}$ of D_i into domains such that

$$v' = (v - \{[f_i, D_i]\}) \cup \{[f_i h, P] \mid P \in \mathcal{P}\}.$$

We write $v \nearrow v'$. We also say that v is the result of *contraction* from v'.

We let $\leq$ be the reflexive, transitive closure of $\nearrow$.

Remark 4.8 (an explicit description of expansion) Consider $[f, \omega I]$, where $\omega \in \{A, B\}^*$. We note that $[f, \omega I] = [f\omega, I]$ by the definition of $\mathcal{B}$ (Definition 4.5) and because $\omega \in \mathbb{S}(I, \omega I)$. An arbitrary partition of I into generating domains takes the form

$$\{\tau I \mid \tau \in \mathcal{C}\},$$

where $\mathcal{C} \subseteq \{A, B\}^*$ is a cut set (in the sense of Section 5). It follows directly that an expansion at $[f, \omega I]$ (equivalently, $[f\omega, I]$) involves replacing $[f, \omega I]$ by the members of

$$\{[f\omega C^k \tau, I] \mid \tau \in \mathcal{C}\},$$

for some $k \in \mathbb{Z}$ and some cut set $\mathcal{C}$. (This is by Definition 4.7 and Theorem 4.3.)

The above description is particularly simple when $\mathcal{C}$ is the cut set $\{A, B\}$. It then follows that the expansion replaces $[f, \omega I]$ with the pairs

$$[f\omega C^k A, I] \quad \text{and} \quad [f\omega C^k B, I],$$

for some $k \in \mathbb{Z}$. Moreover, this is essentially the general case, since any expansion can be realized as a sequence of such expansions.

An expansion at a pair $[f, D]$, where $D = [m, \infty)$, is much more straightforward to describe: such an expansion simply replaces $[f, D]$ with the members of

$$\{[f, P] \mid P \in \mathcal{P}\},$$

where $\mathcal{P}$ is a finite partition of D into domains from $\mathcal{D}^+_{\text{gen}}$. This is because $\mathbb{S}(D, D)$ is trivial.

Proposition 4.9 *Expansion is well-defined and* Γ*-invariant*:

(1) *If* $v=\{[f_1,D_1],\dots,[f_m,D_m]\}$, $\hat{v}=\{[g_1,E_1],\dots,[g_m,E_m]\}$, $v=\hat{v}$, *and* v' *is the result of expansion from* v *at* $[f_i,D_i]$, *then* v' *is also the result of expansion from* $\hat{v}$ *at some* $[g_j,E_j]$.

(2) *If* $v\nearrow v'$ (*where* v *and* v' *are as above*) *and* $\gamma\in\Gamma$, *then* $\gamma\cdot v\nearrow\gamma\cdot v'$.

Proof We prove (1). Assume, without loss of generality, that $[f_k,D_k]=[g_k,E_k]$ for $k=1,\dots,m$. We suppose that v' is the result of expansion from v at $[f_i,D_i]$; thus, there is some $h\in\mathbb{S}(D_i,D_i)$ and a finite partition $\mathcal{P}\subseteq\mathcal{D}^+_{\mathrm{gen}}$ of D_i such that

$$v'=(v-\{[f_i,D_i]\})\cup\{[f_ih,P]\mid P\in\mathcal{P}\}.$$

Choose $j\in\mathbb{S}(D_i,E_i)$ such that $j(\mathcal{P})$ is a finite partition of E_i by members of $\mathcal{D}^+_{\mathrm{gen}}$. (For instance, if $D_i=\omega_iI$ and $E_i=\omega_i'I$, for some $\omega_i,\omega_i'\in\{A,B\}^*$, then we can set $j=\omega_i'\omega_i^{-1}$. The only remaining case is when D_i and E_i are both rays. In this case, there is only one member j of $\mathbb{S}(D_i,E_i)$, and this j satisfies the required property.) Since $[f_i,D_i]=[g_i,E_i]$, there is also some $j_1\in\mathbb{S}(D_i,E_i)$ such that $g_ij_1=f_i$, by Definition 4.5.

We claim that

$$\{[f_ih,P]\mid P\in\mathcal{P}\}=\{[g_ij_1hj^{-1},j(P)]\mid P\in\mathcal{P}\}.$$

Indeed, for each i, $(g_ij_1h^{-1}j^{-1})\circ j=f_ih$, so $[f_ih,P]=[g_ij_1hj^{-1},j(P)]$ by Definition 4.5. This proves (1).

The proof of (2) is straightforward. Indeed,

$$\gamma\cdot v=\{[\gamma f_1,D_1],\dots,[\gamma f_m,D_m]\}$$

and

$$\gamma\cdot v'=(\gamma\cdot v-\{[\gamma f_i,D_i]\})\cup\{[\gamma f_ih,P]\mid P\in\mathcal{P}\},$$

from which it directly follows that $\gamma\cdot v'$ is obtained from $\gamma\cdot v$ via expansion at $[\gamma f_i,D_i]$ (with respect to the same choices of h and $\mathcal{P}$). □

Corollary 4.10 (the partial order on vertices) *The relation* $\le$ *is a partial order on* $\mathcal{V}$. *The group* Γ *acts on* $(\mathcal{V},\le)$ *in an order-preserving fashion.*

Proof This follows directly from Proposition 4.9. □

Definition 4.11 (simplicial realisation; the complexes $\Delta(S_n)$ and $\Delta(S_n')$) Let $\mathcal{P}$ be a partially ordered set. The *simplicial realisation* of P is the simplicial complex whose vertex set is P and whose simplices are finite ascending chains in P.

We let $\Delta(S_n)$ and $\Delta(S_n')$ denote the simplicial realisations of $V(S_n)$ and $V(S_n')$, respectively.

Theorem 4.12 (the directed Γ-set of vertices) *The relation* $\le$ *is a partial order on* $\mathcal{V}$, *and* $\mathcal{V}$ *is a directed set with respect to* $\le$. *The group* Γ *acts on* $(\mathcal{V},\le)$ *in an order-preserving fashion.*

In particular, the simplicial realisations $\Delta(S_i)$ *and* $\Delta(S_i')$ *are contractible* Γ*-complexes.*

Proof It is already clear from Proposition 4.9 and Definition 4.7 that $(\mathcal{V}, \leq)$ is a partially ordered set on which Γ acts in an order-preserving fashion.

Let $S = S_i$ or S_i', for $i = 2$ or 3. We must show that $(\mathcal{V}, \leq)$ is a directed set. The main step is to show that any vertex

$$v = \{[f_1, D_1], \dots, [f_m, D_m]\}$$

can be expanded into a vertex of the form

$$\hat{v} = \{[\mathrm{id}_{E_1}, E_1], \dots, [\mathrm{id}_{E_n}, E_n]\}.$$

Note that each D_i can be partitioned into finitely many elements of $\mathcal{D}^+_{\mathrm{gen}}$ in such a way that the restriction of f_i to each piece acts as a member of S (see Definition 2.7). Thus, we may assume, possibly after expansion, that v already has this property. Consider the pair $[f_1, D_1]$. By Lemma 3.3, there is a finite partition $\mathcal{P} \subseteq \mathcal{D}^+_{\mathrm{gen}}$ of D_1 such that $f_1(P) \in \mathcal{D}^+_{\mathrm{gen}}$, for each $P \in \mathcal{P}$. We note that $f_{1|P} \in S$ by Remark 2.2. It follows that $f_1 \in \mathbb{S}(P, f_1(P))$, for each $P \in \mathcal{P}$, so

$$\{[f_1, P] \mid P \in \mathcal{P}\} = \{[\mathrm{id}, f_1(P)] \mid P \in \mathcal{P}\},$$

by the definition of the equivalence relation on pairs (see Definition 4.5). Note that the act of replacing $[f_1, D_1]$ by the collection $\{[f_i, P] \mid P \in \mathcal{P}\}$ is an expansion at $[f_1, D_1]$. By performing similar expansions at the remaining $[f_i, D_i]$ $(i = 2, \dots, m)$, we arrive at the required $\hat{v}$.

Now suppose that v_1 and v_2 are any two vertices. By the argument of the previous paragraph, we can find $\hat{v}_1$ and $\hat{v}_2$ such that $v_1 \leq \hat{v}_1$ and $v_2 \leq \hat{v}_2$, and both $\hat{v}_1$ and $\hat{v}_2$ have the general form of the vertex $\hat{v}$; i.e., each pair in $\hat{v}_i$ $(i = 1, 2)$ has the form $[\mathrm{id}, E]$, where id denotes the identity function on E and $E \in \mathcal{D}^+_{\mathrm{gen}}$. Thus, we can identify $\hat{v}_i$ $(i = 1, 2)$ with a partition of the nonnegative real numbers. (Under this identification, $\hat{v}$ would correspond to the partition $\{E_1, \dots, E_n\}$.)

Finally, we observe that the partitions determined by the $\hat{v}_i$ have a common finite refinement $\mathcal{P}'$ that is also a subset of $\mathcal{D}^+_{\mathrm{gen}}$. Letting $\tilde{v}$ denote the vertex corresponding to $\mathcal{P}'$, we find that $\hat{v}_i \leq \tilde{v}$, for $i = 1, 2$. Thus, $v_1, v_2 \leq \tilde{v}$, from which it follows that $(\mathcal{V}, \leq)$ is a directed set.

The final statement is standard. □

4.2 Vertex stabilisers

In this subsection, we consider the stabiliser Γ_v, where v is a vertex and Γ is one of the groups $V(S_i)$ or $V(S_i')$ $(i = 2$ or $3)$. We will largely follow the proof of Proposition 5.3 in [6]. We include the proof for the reader's convenience.

Proposition 4.13 (virtually free abelian vertex stabilisers) *Let*

$$v = \{[f_1, D_1], \dots, [f_m, D_m]\},$$

where $v \in \Delta(S_n)$ or $v \in \Delta(S_n')$ $(n = 2$ or $3)$. Let $\Gamma = V(S_n)$ or $V(S_n')$ (respectively).

The stabiliser group Γ_v is virtually free abelian of rank at most m.

Proof The elements of the group Γ_v permute the elements of v. That is, for each $\gamma \in \Gamma_v$, there is a permutation $\sigma_\gamma \in S_m$ such that

$$\gamma \cdot [f_j, D_j] = [\gamma \circ f_j, D_j] = [f_{\sigma_\gamma(j)}, D_{\sigma_\gamma(j)}].$$

The assignment $\gamma \mapsto \sigma_\gamma$ is a homomorphism from Γ_v to S_m, the symmetric group on m symbols. The kernel K of the latter homomorphism thus has finite index in Γ_v. Each $\gamma \in K$ fixes the members of v pointwise; i.e., $\gamma \cdot [f_j, D_j] = [\gamma \circ f_j, D_j] = [f_j, D_j]$, for $j = 1, \dots, m$. It follows, from the definition of the equivalence relation, that there are $h_j \in \mathbb{S}(D_j, D_j)$ such that $\gamma_{|f_j(D_j)} = f_j h_j f_j^{-1}$, for $j = 1, \dots, m$. The latter equalities determine an injective homomorphism

$$\Phi : K \to \prod_{j=1}^{m} \mathbb{S}(D_j, D_j)$$

defined by the rule $\gamma \mapsto (h_1, \dots, h_m)$. Since each of the groups $\mathbb{S}(D_j, D_j)$ is either infinite cyclic or trivial by Theorem 4.3, the proposition follows. □

4.3 The directed set constructions for "F" and "T" groups

The "F" and "T" groups act on a subcomplex of the complexes for $\Delta(S_i)$ and $\Delta(S_i')$.

Definition 4.14 Let $\Gamma \in \{F(S_i), F(S_i'), T(S_i)\}$. We consider the smallest subcomplex of $\Delta(S_i)$ (or $\Delta(S_i')$ if $\Gamma = F(S_i')$) that contains the vertices $[\gamma, X]$ ($X = [0, 1)$ or $[0, \infty)$, respectively), for all $\gamma \in \Gamma$, and is closed under expansion.

We denote this complex by $\Delta_F(S_i)$, $\Delta_F(S_i')$, or $\Delta_T(S_i)$, respectively.

Proposition 4.15 *The vertices of $\Delta_F(S_i)$, $\Delta_F(S_i')$, and $\Delta_T(S_i)$ form directed sets under expansion.*

In particular, the complexes $\Delta_F(S_i)$, $\Delta_F(S_i')$, and $\Delta_T(S_i)$ are contractible Γ-simplicial complexes, where $\Gamma = F(S_i)$, $F(S_i')$, or $T(S_i)$, respectively.

In all of the above cases, the vertex stabiliser groups are virtually free abelian.

Proof The Γ-equivariance of the complexes in question follows from the fact that the expansion relation is Γ-equivariant. The contractibility of these complexes follows from the fact that the vertex sets are still directed, since the vertex sets in question are closed under expansion.

The proof that the vertex stabiliser groups are free abelian follows the general idea of Proposition 4.13. □

5 An algorithm

In this section, we will describe a simple algorithm. The input is a linear fractional transformation C_n of the interval $[0, 1)$. The algorithm attempts to derive a collection of equations like those from Lemma 3.3, which are so essential to our main argument.

We first need to set some conventions. Let $A : [0, 1) \to [0, 1/2)$ and $B : [0, 1) \to [1/2, 1)$ be defined as in Definition 2.6. The vertices of a rooted infinite binary tree can be labelled by words in the monoid

$\{A, B\}^*$, as follows: The root is labelled by the empty word. If a given vertex v is labelled by $\omega \in \{A, B\}^*$, then the left and right children of v are labelled by ωA and ωB, respectively. Let us denote the label of v by $L(v)$. We can then assign a half-open interval $I(v)$ to each vertex by the rule

$$I(v) = L(v) \cdot I,$$

where I denotes the interval $[0, 1)$. Note that $I(v_1) \subseteq I(v_2)$ if and only if $L(v_1)$ is a prefix of $L(v_2)$.

By a *cut set* of a rooted infinite binary tree, we mean a set C of vertices such that every embedded geodesic ray issuing from the root passes through exactly one member of C. We may also refer to a set of words in $\{A, B\}^*$ as a cut set if the corresponding set of vertices is a cut set in the above sense.

We define, as in the introduction,

$$C_n(x) = \frac{nx}{(n-1)x+1},$$

where C_n is defined only on the interval $[0, 1)$. The (hoped-for) output is a collection of matrix identities, of the general form

$$\begin{aligned} C_n\omega_1 &= \omega_1' C_n^{\pm}; \\ C_n\omega_2 &= \omega_2' C_n^{\pm}; \\ &\vdots \\ C_n\omega_k &= \omega_k' C_n^{\pm}, \end{aligned}$$

where $\omega_i, \omega_i' \in \{A, B\}^*$ for $i = 1, \ldots, k$, and the sets $\{\omega_1, \ldots, \omega_k\}$ and $\{\omega_1', \ldots, \omega_k'\}$ are cut sets. (Collections of such equations figured prominently in the proof of Lemma 3.3.) Given the above identities and the corresponding cut sets, we can then define directed sets just as we did in Section 4. The groups that are locally determined by $\{A, B, C_n, a, b, c_n\}^*$ would then act on these directed sets exactly as before.

The algorithm works in the following way. Each vertex of the tree is assigned a type. Initially, this type is "u" for all vertices, indicating a vertex of unknown type. (Actually, the program creates new vertex objects during its run time, although we can ignore this detail for the sake of the current discussion.) Each vertex is also assigned a toggle that is initially set to "0". When the program encounters a vertex v, it performs an action depending on the type of the vertex, which is one of n (for "(internal) node"), l (for "leaf"), or u (for "unknown"), and the value of its toggle, which is either 0 or 1. A value of "0" indicates that the program still needs to do some work at or beneath a given vertex, while a "1" indicates the opposite.

If the vertex is of unknown type ("u"), the program runs the following test:

(1) It first appends C_n to the beginning of the string $L(v)$. This initialises the *matrix string product* of v, which we will here denote by $M(v)$. It is a string over the alphabet $\{A, B, C_n, a, b, c_n\}$.

(2) The program interprets $M(v)$ as a product of matrices and computes the interval $M(v) \cdot I$:

(a) If $M(v) \cdot I \subseteq [0, 1/2)$, then the program appends a to the front of $M(v)$; the result is defined to be the new $M(v)$. The program then returns to step (2).

(b) If $M(v) \cdot I \subseteq [1/2, 1)$, then the program appends b to the front of $M(v)$; the result is defined to be the new $M(v)$. The program then returns to step (2).

(c) If $M(v) \cdot I = I$, then $M(v) \in \mathbb{S}(I, I)$, so $M(v)$ is equivalent (as a linear fractional transformation) to a power of C_n by Theorem 4.3 (or by a variant thereof, if $n \neq 2$ or 3). Let us suppose that $M(v) = C_n^k$. In this case, the program appends c_n^k to the front of $M(v)$ (creating a new $M(v)$). The program now classifies the current vertex as a leaf (changing the unknown "*u*" designation to "*l*"). The toggle of the current vertex is also set to "1" (changed from "0").

(d) If $M(v) \cdot I$ satisfies none of the above (i.e., $1/2 \in M(v) \cdot I$, but $M(v) \cdot I \neq I$), then v is reclassified as an (internal) node "*n*". The toggle stays at 0.

At the end of the above process, the vertex v has been reclassified as an internal node ("*n*") or a leaf ("*l*"). In the latter case, the toggle has been set to 1 and a certain matrix string product has been produced. By construction, the (final) matrix string product of a leaf necessarily evaluates to the identity matrix when interpreted as a product of matrices.

If the current vertex v is an internal node (i.e., designated by "*n*") and its toggle value is 0, then the program determines the toggle value of the left child of v. If this value is 0, it moves to this left child. If the toggle value of the left child is 1, but the toggle value of the right child is 0, then the program moves to the right child. If both children have toggle value 1, then the program flips the toggle of v itself to 1.

If the toggle value of v is 1, then the program moves to the parent of v. If there is no such parent (i.e., v is the root), then the program terminates, and records the matrix string products for each leaf. The latter matrix string products, which take the form

$$c_n^k \omega_1 C_n \omega_2,$$

where $m \in \mathbb{Z}$, $\omega_1 \in \{a, b\}^*$, and $\omega_2 \in \{A, B\}^*$, are readily interpretable as a collection of identities having the desired form, indicated above, if $m = \pm 1$. The leaves determine a (finite) cut set. This completes the description of the algorithm.

We omit the proof of the validity of the algorithm — i.e., the proof that the program finds appropriate cut sets and associated matrix identities, if such things exist.

The author's experience of running the program has led to unexpected results. If $n = 2$, then the program finds a cut set with three elements, and returns the three matrix equations ($C_2 AA = AC_2$; etc.) displayed in the proof of Lemma 3.3. If $n = 3$, then the program finds a cut set with five elements, and the five matrix equations associated to C_3, as described in the proof of Lemma 3.3. If $n = 4$, the program fails to terminate, although it finds many leaves during its run time. The same is true for all values of $n \geq 4$ that the author has tried. (It may be worth noting here that the program computes using only integer values, not floating-point numbers, so round-off errors are apparently not a source for the problems that are encountered here.) It follows from this that an analysis of the groups $V(S_n)$ and $V(S_n')$ for $n \geq 4$ (and, indeed, the corresponding "F" and "T" versions of these groups) lies beyond the techniques described in this paper.

It is also possible to run similar tests for different transformations. One might change not only C_n, but also the transformations A and B. The author has run such tests in a few cases, but with no success to date.

6 The expansion schemes $\mathcal{E}_i$ and $\mathcal{E}'_i$

In this section, we will introduce subdivision trees as a device for diagramming expansions, and describe how subdivision trees represent partitions of $[0,1)$ into subintervals. Similar trees were considered in [9].

We will then describe expansion schemes $\mathcal{E}_i$ and $\mathcal{E}'_i$, which will eventually be used to simplify the directed set constructions from Section 4. In order to establish the required properties of $\mathcal{E}_i$ and $\mathcal{E}'_i$, we will need to understand when two subdivision trees define the same partition. The latter will be the project of Sections 7 and 8.

6.1 Subdivision trees, equivalence, and elementary equivalence

Definition 6.1 (subdivision trees) Let T be a finite rooted binary tree. The vertices of degree one are *leaves*; all other vertices are *nodes*. The topmost node is the *root*. We say that T is a *subdivision tree* if each node is labelled by an integer.

We let T_ℓ and T_r denote the left and right branches of the subdivision tree T.

Remark 6.2 (the subdivision represented by a subdivision tree) Each leaf in a subdivision tree is labelled by a word in the alphabet $\{A,B,C,c\}$. The labelling is obtained as follows. Trace the (unique) path p from the root to a given leaf ℓ. Suppose that

$$v_1, e_1, v_2, e_2, \dots, e_k, v_{k+1}$$

is a complete list of the vertices and edges encountered along the path p, written in the order that they are encountered. Thus, in particular, v_1 is the root of the tree and $v_{k+1}=\ell$. Let $n_1,n_2,\dots,n_k$ be the integers labelling the nodes $v_1,\dots,v_k$; for $i=1,\dots,k$, let X_i be A if e_i points downward and to the left, and let X_i be B if e_i points downward and to the right. The labelling of the leaf ℓ is then

$$C^{n_1}X_1C^{n_2}X_2\dots C^{n_k}X_k.$$

For instance, the leaves of the left tree in Figure 1 are labelled by the words $ACAcA$, $ACAcB$, ACB, and B. The leaves of the right tree are labelled by C^2ACA, C^2ACBC^3A, C^2ACBC^3B, C^2BcA, and C^2BcB. This labelling scheme works the same way, no matter whether we are letting C represent C_2 or C_3.

We obtain a partition of $[0,1)$ by applying these words to the interval $[0,1)$. Thus, $ACAcA$ determines the interval $ACAcA\cdot[0,1)$, and so forth.

The partition of $[0,1)$ determined by the trivial subdivision tree is $\{[0,1)\}$.

Remark 6.3 (subdivision trees over M_2 and M_3) For $i=2,3$, we let $M_i=\{A,B,C_i,c_i\}^*$; i.e., M_i is the monoid consisting of positive (possibly empty) words in the alphabet $\{A,B,C_i,c_i\}$. For $i=2,3$, we

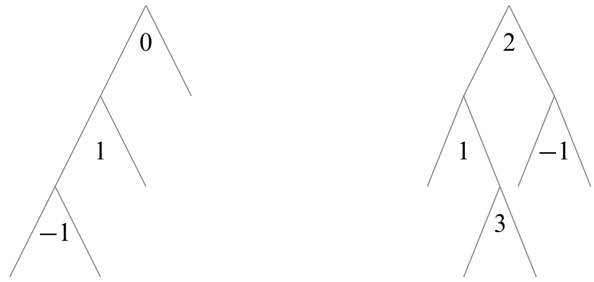

Figure 1: Subdivision trees.

have the proper inclusions $M_i \subseteq S_i$. In Section 7, we will obtain finite complete presentations of M_2 and M_3, which will aid in analysing subdivision trees.

A subdivision tree T represents one of two subdivisions of the interval $[0, 1)$, depending upon whether the "C" is interpreted as C_2 or C_3. In most contexts, it should be clear which is intended, but, in cases of possible ambiguity, we may refer to T as *a subdivision tree over* M_2 or *over* M_3, as the case may be.

Definition 6.4 (equivalent subdivision trees; the functions n and N) Two subdivision trees T_1 and T_2 (both over either M_2 or M_3) are *equivalent* if they represent the same collection of intervals. We write $T_1 \approx T_2$.

If T is a nontrivial subdivision tree, then $n(T)$ denotes the label of the root. Let

$$N(T) = \{n(T') \mid T' \approx T\}.$$

The set $N(T)$ is empty if T is the trivial subdivision tree.

Lemma 6.5 (finiteness of $N(T)$) *If T is a subdivision tree, then $N(T)$ is a finite set.*

Proof We prove the lemma in the case of M_2, the argument for the case of M_3 being similar.

Note that, if $T' \approx T$ and $n(T') = k$, then the collection of intervals $\mathcal{C}$ determined by T refines $\{[0, 2^k/(2^k+1)), [2^k/(2^k+1), 1)\}$. Thus, if $N(T)$ were infinite, $\mathcal{C}$ would refine an infinite partition of $[0, 1)$, which would force $\mathcal{C}$ to be infinite. This is impossible, since T has only finitely many leaves. □

Lemma 6.6 *Let T and T' be nontrivial subdivision trees (both over M_2 or M_3), and assume that $n(T) = n(T')$. Then $T \approx T'$ if and only if $T_\ell \approx T'_\ell$ and $T_r \approx T'_r$.*

Proof We prove the lemma in the case that T and T' are subdivision trees over M_2; the case of M_3 differs in only minor ways. Assume that T and T' are subdivision trees, and that $n(T) = n(T') = t$.

Assume that $T \approx T'$. Let $\ell_1, \dots, \ell_k$ label the leaves of T_ℓ and let $\ell_{k+1}, \dots, \ell_m$ label the leaves of T_r. (Here, and throughout the proof, the labels of the leaves are read from left to right, so ℓ_1 is the label of the leftmost leaf of T_ℓ, etc.) Let $\ell'_1, \dots, \ell'_{\hat{k}}$ label the leaves of T'_ℓ and let $\ell'_{\hat{k}+1}, \dots, \ell'_m$ label the leaves of T'_r. It follows that

$$C^t A\ell_1, \dots, C^t A\ell_k, C^t B\ell_{k+1}, \dots, C^t B\ell_m$$

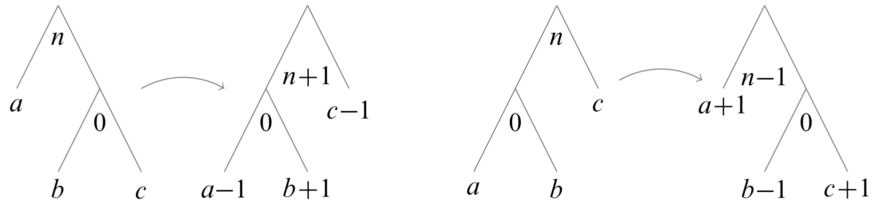

Figure 2: The relations that define elementary equivalence between subdivision trees over M_2.

label the leaves of T and

$$C^t A\ell'_1, \dots, C^t A\ell'_{\hat{k}}, C^t B\ell_{\hat{k}+1}, \dots, C^t B\ell_m$$

label the leaves of T'. It follows that the above labels pairwise determine equal intervals, in the given order: $C^t A\ell_1 \cdot [0,1) = C^t A\ell'_1 \cdot [0,1)$, etc. Since ℓ_k and $\ell'_{\hat{k}}$ label rightmost leaves (of the trees T_ℓ and T'_ℓ, respectively), $C^t A\ell_k \cdot [0,1)$ and $C^t A\ell'_{\hat{k}} \cdot [0,1)$ have the same supremum, namely $2^t/(2^t+1)$ (since $\ell_k \cdot [0,1)$ and $\ell'_{\hat{k}} \cdot [0,1)$ have the supremum 1). It follows directly that $C^t A\ell_k$ and $C^t A\ell'_{\hat{k}}$ determine the same interval; thus, $k = \hat{k}$.

It follows easily that ℓ_j and ℓ'_j determine the same interval, for $j = 1, \dots, k$ (simply cancel $C^t A$ in the relevant products). Thus, $T_\ell \approx T'_\ell$. By similar reasoning, $T_r \approx T'_r$.

Conversely, assuming that $T_\ell \approx T'_\ell$ and $T_r \approx T'_r$, we easily conclude that $T \approx T'$. □

Proposition 6.7 (equality of leaves) *Let $\omega, \omega' \in \{A, B, C, c\}^*$, where $C = C_2$ or C_3. The intervals $\omega \cdot [0,1)$ and $\omega' \cdot [0,1)$ are equal if and only if $\omega = \omega' C^k$, for some $k \in \mathbb{Z}$.*

Proof If $\omega = \omega' C^k$, then

$$\omega \cdot [0,1) = \omega' C^k \cdot [0,1) = \omega' \cdot [0,1),$$

where the final equality follows from the fact that $C \cdot [0,1) = [0,1)$.

Conversely, suppose $\omega I = \omega' I$. It follows that $(\omega')^{-1}\omega I = I$, so $(\omega')^{-1}\omega \in \mathbb{S}(I, I)$, so $(\omega')^{-1}\omega = C^k$, for some $k \in \mathbb{Z}$, by Theorem 4.3. It follows that $\omega = \omega' C^k$. □

Definition 6.8 (elementary equivalence) The two transformations in Figure 2 define *elementary equivalence* between subdivision trees over M_2.

To apply one of the transformations from Figure 2 to a subdivision tree T over M_2 is to replace a subtree of the form on the left with a subtree of the form on the right. Here the labels a, b, c represent the integer labels of the nodes of T that are attached at the leaves labelled by a, b, c (respectively). An application of the given transformation changes the integer labels of these nodes, as indicated on the right-hand tree. If one of the integers a, b, c labels a leaf in T, then that integer is ignored (since leaves of subdivision trees are never labelled by integers). We also say that two subdivision trees T_1 and T_2 are elementary equivalent over M_2 if one can be transformed into the other by a sequence of such transformations.

Elementary equivalence over M_3 is defined by the tree pairs in Figure 3.

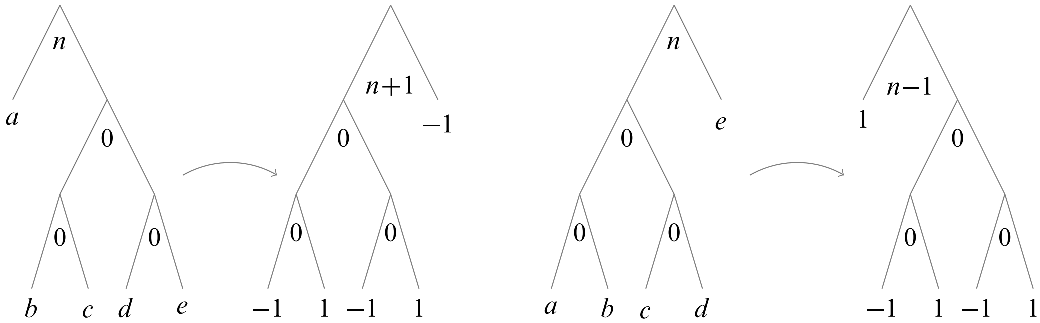

Figure 3: The relations that define elementary equivalence between subdivision trees over M_3.

In Figure 3, the labels on the leaves of the right-hand trees have been abbreviated to avoid creating an over-crowded figure. The leftmost "−1" on the second tree from the left represents "$a-1$", and so on.

Remark 6.9 The transformations in Figure 2 are inverses of each other; similarly for Figure 3.

Example 6.10 Figure 4 depicts an elementary equivalence between two subdivision trees over M_2. The right-hand tree is the result of applying the second relation to the left-hand tree at the node labelled by "4".

One easily checks that the two trees are indeed equivalent.

Lemma 6.11 *If two subdivision trees T_1, T_2 (over M_2 or M_3) are elementary equivalent, then they are equivalent.*

Proof The proof that the left-hand transformation in Figure 2 preserves equivalence relies on the system of equalities

$$\begin{aligned} C^n A C^a &= C^{n+1} A A C^{a-1}; \\ C^n B A C^b &= C^{n+1} B A C^{b+1}; \\ C^n B B C^c &= C^{n+1} B C^{c-1}, \end{aligned}$$

all of which are easily verified, and from the interpretation of subdivision trees (Remark 6.2). The other three verifications follow similarly. □

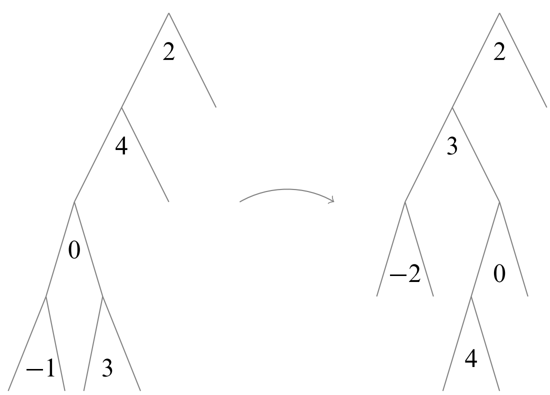

Figure 4: Elementary equivalence between two subdivision trees over M_2.

6.2 The correspondence between subdivision trees and expansions

Theorem 6.12 (subdivision trees and expansions) *We let C denote either C_2 or C_3. Let $v \in \mathcal{V}$ be the result of a sequence of expansions from $\{[\mathrm{id}_I, I]\}$; i.e., $\{[\mathrm{id}, I]\} \le v$. It follows that there is some subdivision tree T such that the set $\mathcal{L}$ of labels on the leaves satisfies*

$$v = \{[\omega, I] \mid \omega \in \mathcal{L}\}.$$

Conversely, any subdivision tree T determines a vertex v by the above equality, and $\{[\mathrm{id}_I, I]\} \le v$ for this v.

If $\{[\mathrm{id}_I, I]\} \le v, v'$ and T and T' are the subdivision trees corresponding to v and v', then $v = v'$ if and only if T and T' are equivalent.

Proof The correspondence between subdivision trees T and vertices v satisfying $\{[\mathrm{id}_I, I]\} \le v$ is straightforward, in view of the discussion in Remark 4.8.

We will now show that $v = v'$ if and only if T is equivalent to T'. Assume first that T and T' are equivalent. Thus,

$$v = \{[\omega_1, I], \dots, [\omega_n, I]\} \quad \text{and} \quad v' = \{[\omega_1', I], \dots, [\omega_n', I]\},$$

where the ω_i are the labels of the leaves of T (listed from left to right) and, similarly, ω_i' are the labels of the leaves of T' (also listed from left to right). Since T and T' are equivalent, we have

$$\omega_i = \omega_i' C^{k_i},$$

for $i = 1, \dots, n$ and for some $k_i \in \mathbb{Z}$, by Definition 6.4 and Proposition 6.7. It follows directly that, for all i, $[\omega_i, I] = [\omega_i', I]$, by Definition 4.5, letting $h = C^{k_i}$ (since $C^{k_i} \in \mathbb{S}(I, I)$). Thus, $v = v'$.

If we carry over the notation from above, the converse essentially follows from the fact that the equality $[\omega_i, I] = [\omega_i', I]$ implies the equality $\omega_i = \omega_i' C^{k_i}$ (for appropriate k_i); this is a direct consequence of Definition 4.5 and the description of $\mathbb{S}(I, I)$ from Theorem 4.3. □

6.3 A discussion of expansion schemes; the expansion schemes $\mathcal{E}_i$ and $\mathcal{E}_i'$

The directed set construction from Section 4, and the generalisations considered in [6], lead to complexes that are often too difficult to analyse when (for instance) attempting to establish finiteness properties for the acting group. One device for simplifying the complexes appeared in [6] under the name of "expansion schemes". An expansion scheme $\mathcal{E}$ assigns to each pair $[f, D] \in \mathcal{B}$ a collection of expansions. This assignment determines a simplicial complex $\Delta^{\mathcal{E}}$ in which the simplices are chains

$$v_1 < v_2 < \cdots < v_n$$

such that the vertices v_i ($i = 2, \dots, n$) are all the result of expansions from v_1 that are allowed by $\mathcal{E}$. Thus, for instance, the trivial expansion scheme, which allows no expansions, results in a discrete set of vertices. At the opposite extreme, an expansion scheme may impose no restraint at all, resulting in the original directed set construction. Since the topology of $\Delta^{\mathcal{E}}$ depends significantly on the choice of $\mathcal{E}$, it would be useful to have a criterion that recognises when the complex $\Delta^{\mathcal{E}}$ is n-connected. The idea of an "n-connected expansion scheme" offers such a criterion. The necessary definitions follow.

We will begin with a general discussion of expansion schemes; the definitions of $\mathcal{E}_i$ and $\mathcal{E}_i'$ are in Example 6.22. The definition of "pseudovertex" is from Section 4 of [6], while the other definitions and theorems in this subsection are from Section 6 of [6].

Definition 6.13 (pseudovertices) Let $v = \{[f_1, D_1], \dots, [f_m, D_m]\} \subseteq \mathcal{B}$. We say that v is a *pseudovertex* if the sets $f_i(D_i)$ $(i = 1 \dots, m)$ are pairwise disjoint.

Remark 6.14 (the partial order on pseudovertices; the action of $\widehat{S}$ on pseudovertices) The pseudovertices are partially ordered by expansion, which can be defined exactly as it was for vertices (Definition 4.7). The pseudovertices do not form a directed set, since the *support* of a given pseudovertex

$$(f_1(D_1) \cup \cdots \cup f_m(D_m))$$

is invariant under expansion. The proof of Theorem 4.12 still shows that any two pseudovertices with the same support have an upper bound. Thus, the simplicial realisation of the set of all pseudovertices is a disjoint union of contractible sets.

There is a (partial) action of $\widehat{S}$ (Definition 2.7) on $\mathcal{B}$, defined by

$$\hat{s} \cdot [f, D] = [\hat{s} f, D].$$

This action is defined for suitable $[f, D]$ and $\hat{s}$; i.e., for all pairs $[f, D]$ and $\hat{s} \in \widehat{S}$ such that $f(D)$ is a subset of the domain of $\hat{s}$.

Definition 6.15 ($\mathcal{E}$-expansion; expansion scheme) Let $\mathcal{PV}$ denote the collection of all pseudovertices. Assume that $\mathcal{E} : \mathcal{B} \to 2^{\mathcal{PV}}$ satisfies (1)–(3), for each $[f, D] \in \mathcal{B}$ (we let b, rather than $[f, D]$, denote a typical member of $\mathcal{B}$ in order to simplify notation):

(1) Each $w \in \mathcal{E}(b)$ is the result of a sequence of expansions from $\{b\}$; i.e., for each $w \in \mathcal{E}([f, D])$, we have $\{[f, D]\} \leq w$.

(2) $\{b\} \in \mathcal{E}(b)$.

(3) **$\widehat{S}$-invariance** For each $\hat{s} \in \widehat{S}$, and each $b \in \mathcal{B}$ for which $\hat{s} \cdot b$ is defined, $\hat{s} \cdot \mathcal{E}(b) = \mathcal{E}(\hat{s} \cdot b)$.

Let $v \in \mathcal{PV}$; we write $v = \{b_1, \dots, b_m\}$, where $b_1, \dots, b_m \in \mathcal{B}$. We say that v' is a result of *$\mathcal{E}$-expansion from v* if there are $v_i' \in \mathcal{E}(b_i)$, for $i = 1, \dots, m$, such that

$$v' = \bigcup_{i=1}^{m} v_i'.$$

We say that $\mathcal{E}$ is an *expansion scheme* if

(4) for every $[f, D] \in \mathcal{B}$ and every $w_1, w_2 \in \mathcal{E}([f, D])$ such that $w_1 \leq w_2$, w_2 is the result of $\mathcal{E}$-expansion from w_1.

Definition 6.16 (the complex $\Delta^{\mathcal{E}}$) Let $\mathcal{E}$ be an expansion scheme. We let $\Delta^{\mathcal{E}}$ be the subcomplex of the directed set construction made up of *$\mathcal{E}$-simplices*; i.e., simplices

$$v_1 < v_2 < \cdots < v_m$$

such that the vertices v_j $(j \in \{2, \ldots, m\})$ are obtained from v_1 by $\mathcal{E}$-expansion.

Definition 6.17 (interval subcomplexes; relative ascending links) Let v' and v'' be pseudovertices such that $v' \leq v''$ (in the sense of the expansion partial order; see Remark 6.14 and Definition 4.7). We let $\Delta^{\mathcal{E}}_{[v',v'']}$ denote the set of all $\mathcal{E}$-simplices

$$v_1 < \cdots < v_m$$

such that $v' \leq v_1 < v_m \leq v''$. This is the *interval subcomplex* determined by v', v'', and $\mathcal{E}$.

The *ascending link of v' relative to v''* is the link of v' in the complex $\Delta^{\mathcal{E}}_{[v',v'']}$.

Definition 6.18 (n-connected expansion schemes) Let $\mathcal{E}$ be an expansion scheme. We say that $\mathcal{E}$ is *n-connected* if, for each $b \in \mathcal{B}$ and each pseudovertex v such that $\{b\} < v$, the ascending link of $\{b\}$ relative to v is $(n-1)$-connected.

Theorem 6.19 (n-connectedness of $\Delta^{\mathcal{E}}$) *If $\mathcal{E}$ is an n-connected expansion scheme, then the complex $\Delta^{\mathcal{E}}$ is n-connected.*

Remark 6.20 (n-connectedness of complexes determined by pseudovertices) Theorem 6.19's conclusion carries over to complexes determined by pseudovertices in a component-by-component fashion; i.e., each connected component is n-connected.

Example 6.21 (the case of Thompson's group V) We consider a basic example of an expansion scheme. Let

$$\mathcal{C} = \prod_{n=1}^{\infty} \{0, 1\}$$

denote the usual binary Cantor set. The elements of $\mathcal{C}$ are infinite binary strings. We let $\mathcal{C}^{\mathrm{fin}}$ denote the set of all finite binary strings. For each $\omega \in \mathcal{C}^{\mathrm{fin}}$, we let D_ω denote the set of all infinite binary strings that begin with the prefix ω. For $\omega_1, \omega_2 \in \mathcal{C}^{\mathrm{fin}}$, the transformation $\sigma_{\omega_1,\omega_2} : D_{\omega_1} \to D_{\omega_2}$ removes the prefix ω_1 from the input and adds the prefix ω_2 in its place. We let

$$S_V = \{\sigma_{\omega_1,\omega_2} \mid \omega_1, \omega_2 \in \mathcal{C}^{\mathrm{fin}}\} \cup \{0\},$$

where 0 represents the empty function. The set S_V is an inverse monoid under composition. The associated set of domains $\mathcal{D}^{+}_{S_V}$ consists of all of the sets D_ω, where $\omega \in \mathcal{C}^{\mathrm{fin}}$.

The set of all bijections $\gamma : \mathcal{C} \to \mathcal{C}$ that are locally determined by S_V make up a group, which we denote by V. This is Thompson's well-known group V, as described in [5]. We define an S_V structure as follows. For each pair $(D_{\omega_1}, D_{\omega_2})$, we define

$$\mathbb{S}(D_{\omega_1}, D_{\omega_2}) = \{\sigma_{\omega_1,\omega_2}\}.$$

The verification that this assignment does, indeed, define an S_V-structure is routine. (For the sake of this discussion, we can use the properties from Proposition 4.4 as the definition of S-structure. The reader is referred to [6] for a more complete definition. We note, however, that the longer definition from the latter source is designed to address numerous complications that do not arise in the case of V.)

We now define an expansion scheme $\mathcal{E}$. For each $[f, D_\omega]$, let

$$\mathcal{E}([f, D_\omega]) = \{\{[f, D_\omega]\}, \{[f, D_{\omega 0}], [f, D_{\omega 1}]\}\}.$$

Thus, the set $\mathcal{E}([f, D_\omega])$ consists of two pseudovertices: the base pseudovertex $\{[f, D_\omega]\}$, and the pseudovertex obtained by performing the simplest possible expansion at $[f, D_\omega]$, namely the expansion that subdivides D_ω into left and right halves ($D_{\omega 0}$ and $D_{\omega 1}$, respectively). It is straightforward to check that the assignment $\mathcal{E}$ satisfies the conditions of Definition 6.15.

A simplex in $\Delta^{\mathcal{E}}$ is a chain

$$v_1 < v_2 < \cdots < v_m,$$

where $v_1 = \{[f_1, D_{\omega_1}], \dots, [f_n, D_{\omega_n}]\}$, and each vertex v_j $(2 \le j \le m)$ can be obtained from v_1 by, for a given $i \in \{1, \dots, n\}$, either replacing $[f_i, D_{\omega_i}]$ with its left and right halves (in the sense described above), or leaving $[f_i, D_{\omega_i}]$ unchanged.

It is also straightforward to check that the expansion scheme $\mathcal{E}$ is n-connected, for all n. Indeed, let $b \in \mathcal{B}$ and let $\{b\} < v$. There is a unique $\mathcal{E}$-expansion from b, and thus the 1-simplex connecting $\{b\}$ to $\{b_\ell, b_r\}$ is the star of $\{b\}$ in $\Delta^{\mathcal{E}}_{[\{b\},v]}$. The ascending link of $\{b\}$ relative to v is therefore always a point. It follows from Theorem 6.19 that $\Delta^{\mathcal{E}}$ is contractible.

Example 6.22 We now return to the main examples of this paper. Let $S = S_2$, S_3, S_2', or S_3', and let the S-structure $\mathbb{S}$ be defined as in Definition 4.1. We will define expansion schemes $\mathcal{E}_i$ and $\mathcal{E}_i'$, for $i = 2, 3$. In order to do so, we must first introduce some useful notation.

For each $k \in \mathbb{Z}$, we let u_k denote the vertex that corresponds to the subdivision tree consisting of a single caret in which the root is numbered k. (The correspondence in question is that of Theorem 6.12.) Thus,

$$u_k = \{[C^k A, I], [C^k B, I]\},$$

where $C = C_2$ or C_3, depending on the semigroup S in question. If $S = S_2$ or S_2', we let $u_{k-\frac{1}{2}}$ be the vertex corresponding to the subdivision tree consisting of two carets: a top caret (with root labelled k), and a second caret, attached to the left child of the root, labelled 0. Thus,

$$u_{k-\frac{1}{2}} = \{[C^k AA, I], [C^k AB, I], [C^k B, I]\}.$$

If $S = S_3$ or S_3', then $u_{k-\frac{1}{2}}$ is the vertex represented by the subdivision tree consisting of a top caret (with root labelled k), and a complete depth-two binary tree, attached at the left child of the root, in which each node is labelled 0. Thus,

$$u_{k-\frac{1}{2}} = \{[C^k AAA, I], [C^k AAB, I], [C^k ABA, I], [C^k ABB, I], [C^k B, I]\}.$$

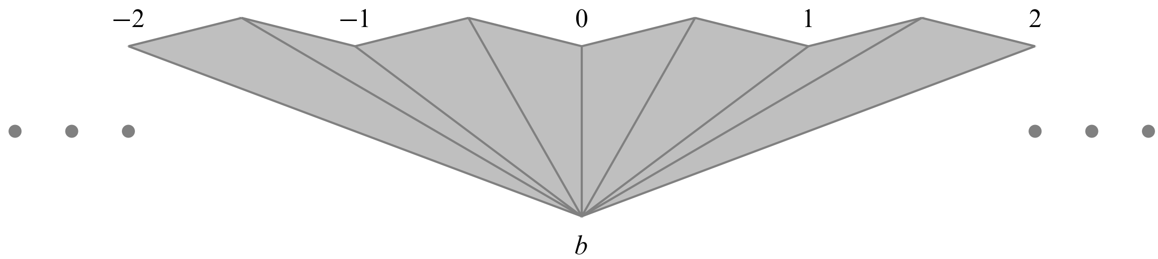

Figure 5: Above we have depicted the simplicial complex $\mathcal{E}(b)$ associated to $b = [\mathrm{id}_I, I]$ by the expansion schemes $\mathcal{E}_i$ and $\mathcal{E}_i'$. An integer k refers to the vertex u_k.

With the above conventions, we can set

$$\mathcal{E}_i([\mathrm{id}_I, I]) = \mathcal{E}_i'([\mathrm{id}_I, I]) = \{\{[\mathrm{id}_I, I]\}\} \cup \{u_{k/2} \mid k \in \mathbb{Z}\},$$

for $i = 2$ or 3. By extending $\widehat{S}$-equivariantly, we arrive at a definition of $\mathcal{E}_i(b) = \mathcal{E}_i'(b)$, for any $b = [f, I]$ and for $i = 2$ or 3, where I is contained in the domain of $f \in \widehat{S}$:

$$\mathcal{E}_i([f, I]) = \mathcal{E}_i'([f, I]) = \{\{[f, I]\}\} \cup \{f \cdot u_{k/2} \mid k \in \mathbb{Z}\}.$$

The well-definedness of this assignment is easy to check.

It is straightforward to check that $u_k \le u_{k-\frac{1}{2}}$ and $u_{k-1} \le u_{k-\frac{1}{2}}$, for each integer k. (The first inequality is clear; the second inequality follows directly after applying an elementary equivalence.) Moreover, no two of the vertices u_{k_1} and u_{k_2} are comparable and no two of the vertices $u_{k_1-\frac{1}{2}}$ and $u_{k_2-\frac{1}{2}}$ are comparable (if $k_1 \neq k_2$). It follows that the simplicial realisations of $\mathcal{E}_i(b)$ and $\mathcal{E}_i'(b)$ take the form indicated in Figure 5.

We recall that $[f, \omega I] = [f\omega, I]$ when $\omega \in \{A, B\}^*$ (see the beginning of Remark 4.8). It follows that the description of $\mathcal{E}_i$ is complete for $i = 2$ and 3.

We next define

$$\mathcal{E}_i'([f, [m, \infty)) = \{\{[f, [m, \infty)]\}, \{[f, [m, m+1)], [f, [m+1, \infty)]\}\}.$$

This completes the definition of $\mathcal{E}_i'$, for $i = 2$ and 3.

Observe that, if $\{[f, [m, \infty)]\} < v$, then

$$v = \{[f, D] \mid D \in \mathcal{P}\},$$

where $\mathcal{P} \subseteq \mathcal{D}^+_{\mathrm{gen}}$ is a finite partition of $[m, \infty)$ into generating domains (see Remark 4.8). The partition $\mathcal{P}$ is necessarily a proper refinement of $\{[m, m+1), [m+1, \infty)\}$ if $\{[f, [m, \infty)]\} < v$. It follows that the ascending link of $\{[f, [m, \infty)]\}$ relative to such v is always a point, and thus contractible. Thus, the expansion scheme $\mathcal{E}_i'$ is n-connected for a given n if and only if $\mathcal{E}_i$ is. We may therefore concentrate on $\mathcal{E}_i$ in what follows.

7 Finite complete presentations of semigroups

In order to understand equivalence between subdivision trees, we will need a full analysis of the monoids M_2 and M_3, which, by definition, are generated by the linear fractional transformations that we have denoted by A, B, C_i, and c_i (for $i = 2, 3$).

In this section, in contrast to our usual practice, the letters A, B, and C will be used as formal symbols. We will define abstract monoid presentations $\mathcal{P}_i$ and $\widehat{\mathcal{P}}_i$ $(i = 2, 3)$, with the ultimate goal of proving that the abstract monoid $M(\mathcal{P}_i)$ defined by $\mathcal{P}_i$ is isomorphic to M_i. (The monoids $M(\widehat{\mathcal{P}}_i)$ represent a necessary intermediate device.)

The arguments in this section parallel those from Section 5 of [9].

7.1 Monoid presentations and string-rewriting systems

Definition 7.1 (monoid presentations) Let Σ be a set. The *free monoid on* Σ, denoted by Σ^*, is the set of all positive (possibly empty) words in Σ, with the operation of concatenation. The empty word is denoted by 1. We write $\omega_1 \equiv \omega_2$ if $\omega_1, \omega_2 \in \Sigma^*$ are identical as words.

Let $\mathcal{R}$ be a set of ordered pairs $(r_1, r_2) \in \Sigma^* \times \Sigma^*$. We view such a pair as an equality between words in Σ^*, writing $r_1 = r_2$ if either (r_1, r_2) or (r_2, r_1) is in $\mathcal{R}$. The pair $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is called a *monoid presentation*; the set $\mathcal{R}$ is the set of *relations*. These relations determine an equivalence relation on Σ^* in the following way. If $\omega_1, \omega_2 \in \Sigma^*$, then we write $\omega_1 \approx \omega_2$ if $\omega_1 \equiv \alpha r_1 \beta$ and $\omega_2 \equiv \alpha r_2 \beta$ for some words $\alpha, \beta \in \Sigma^*$, and $(r_1, r_2) \in \mathcal{R}$. The symmetric, transitive closure of $\approx$, denoted by $=$, is an equivalence relation on Σ^*. We sometimes denote the equivalence class of a word ω by $[\omega]$.

The concatenation operation on Σ^* determines a well-defined associative operation on the set of equivalence classes $\Sigma^*/{=}$. We let $M(\mathcal{P})$ denote the set of these equivalence classes, with the operation induced by concatenation. The set $M(\mathcal{P})$ is a monoid with respect to this operation, called the *monoid determined by* $\mathcal{P}$.

Definition 7.2 (rewrite systems; string-rewriting systems) A *rewrite system* is a directed graph Γ. We allow loops and multiple edges. If v_1 and v_2 are vertices of Γ, we write $v_1 \to v_2$ if there is a directed edge issuing from v_1 and terminating at v_2. We write $v_1 \stackrel{\cdot}{\longrightarrow} v_2$ if there is a directed edge path from v_1 to v_2. Equivalently, $\stackrel{\cdot}{\longrightarrow}$ is the transitive closure of $\to$.

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a monoid presentation. We define a rewrite system $\Gamma(\mathcal{P})$ as follows. The vertex set of $\Gamma(\mathcal{P})$ is Σ^*. There is a directed edge from ω_1 to ω_2 if $\omega_1 \equiv \alpha r_1 \beta$ and $\omega_2 \equiv \alpha r_2 \beta$, where $\alpha, \beta \in \Sigma^*$ and $(r_1, r_2) \in \mathcal{R}$. The directed graph $\Gamma(\mathcal{P})$ is called the *string-rewriting system* associated to the monoid presentation $\mathcal{P}$.

Remark 7.3 Let $\stackrel{\cdot}{\longleftrightarrow}$ denote the symmetric, transitive closure of $\to$. Thus, $\stackrel{\cdot}{\longleftrightarrow}$ is an equivalence relation on the vertices of $\Gamma(\mathcal{P})$. The above definitions easily show that the relation $\stackrel{\cdot}{\longleftrightarrow}$ coincides with $=$ on Σ^*. In other words, equivalence classes of words in Σ^* modulo $=$ are in one-to-one correspondence with (undirected) path components of $\Gamma(\mathcal{P})$.

$C_2AA \to AC_2$	$C_2B \to BBC_2$	$C_2AB \to BAc_2$
$c_2A \to AAc_2$	$c_2BB \to Bc_2$	$c_2BA \to ABC_2$
$C_2c_2 \to 1$	$c_2C_2 \to 1$	
$C_2aa \to aC_2$	$C_2b \to bbC_2$	$C_2ab \to bac_2$
$c_2a \to aac_2$	$c_2bb \to bc_2$	$bA \to 0$
$aB \to 0$	$bB \to 1$	$aA \to 1$
$0X \to 0$	$X0 \to 0$	

Table 1: The relations of $\widehat{\mathcal{R}}_2$. The relations in the top box are $\mathcal{R}_2$. The "X" stands for any of the generators.

In view of this close identification between the monoid $M(\mathcal{P})$ and the string-rewriting system $\Gamma(\mathcal{P})$, it causes no harm to write $r_1 \to r_2$ for a relation $(r_1, r_2) \in \mathcal{R}$.

Definition 7.4 (terminating; confluent; locally confluent; reduced) A rewrite system Γ is *terminating* if every sequence of vertices $v_1 \to v_2 \to v_3 \to \cdots$ is finite. We say Γ is *confluent* if whenever $v_1 \dot{\longrightarrow} w_1$ and $v_1 \dot{\longrightarrow} w_2$, there is some v_2 such that $w_1 \dot{\longrightarrow} v_2$ and $w_2 \dot{\longrightarrow} v_2$. We say Γ is *locally confluent* if whenever $v_1 \to w_1$ and $v_1 \to w_2$, there is some v_2 such that $w_1 \dot{\longrightarrow} v_2$ and $w_2 \dot{\longrightarrow} v_2$.

A vertex v of Γ is called *reduced* if there is no directed edge issuing from v.

A rewrite system is *complete* if it is terminating and confluent. We say that a monoid presentation $\mathcal{P}$ is complete if the associated string-rewriting system $\Gamma(\mathcal{P})$ is complete.

Theorem 7.5 [12] *If the rewrite system Γ is terminating and locally confluent, then Γ is confluent.* $\square$

Corollary 7.6 (unique reduced forms) *If the rewrite system Γ is terminating and locally confluent, then each connected component of Γ contains a unique reduced vertex.*

In particular, if $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is a complete monoid presentation, then any connected component of $\Gamma(\mathcal{P})$ contains a unique reduced word, and any word $\omega \in \Sigma^*$ is equivalent to a unique reduced word modulo $=$. $\square$

7.2 Basic definitions of the rewrite systems

Definition 7.7 We define monoid presentations, $\mathcal{P}_2$, $\widehat{\mathcal{P}}_2$, $\mathcal{P}_3$, and $\widehat{\mathcal{P}}_3$, as follows.

(1) $\mathcal{P}_2 = \langle A, B, C_2, c_2 \mid \mathcal{R}_2 \rangle$, where $\mathcal{R}_2$ consists of the relations appearing in the top box of Table 1.

(2) $\widehat{\mathcal{P}}_2 = \langle A, B, C_2, a, b, c_2, 0 \mid \widehat{\mathcal{R}}_2 \rangle$, where $\widehat{\mathcal{R}}_2$ consists of the relations that appear in Table 1.

(3) $\mathcal{P}_3 = \langle A, B, C_3, c_3 \mid \mathcal{R}_3 \rangle$, where $\mathcal{R}_3$ consists of the relations appearing in the top box of Table 2.

(4) $\widehat{\mathcal{P}}_3 = \langle A, B, C_3, a, b, c_3, 0 \mid \widehat{\mathcal{R}}_3 \rangle$, where $\widehat{\mathcal{R}}_3$ consists of the relations that appear in Table 2.

We note that the occurrences of "X" in the tables represent arbitrary generators.

Remark 7.8 (the modified rewrite systems $\Gamma^*(\widehat{\mathcal{P}}_i)$) The string-rewriting systems $\Gamma(\widehat{\mathcal{P}}_i)$ $(i = 2, 3)$ both fail to be locally confluent. For instance, the directed edges $c_2bbB \to bc_2B$ and $c_2bbB \to c_2b$ show that $\Gamma(\widehat{\mathcal{P}}_2)$ is not locally confluent, since the words bc_2B and c_2b are both reduced. The directed edges $c_3bbbB \to c_3bb$ and $c_3bbbB \to bc_3B$ similarly show that $\Gamma(\widehat{\mathcal{P}}_3)$ is not locally confluent.

$C_3AAA \to AC_3$	$C_3AAB \to BAAc_3$	$C_3ABA \to BABC_3$
$C_3ABB \to BBAc_3$	$C_3B \to BBBC_3$	$c_3A \to AAAc_3$
$c_3BAA \to AABC_3$	$c_3BAB \to ABAc_3$	$c_3BBA \to ABBC_3$
$c_3BBB \to Bc_3$	$C_3c_3 \to 1$	$c_3C_3 \to 1$
$c_3a \to aaac_3$	$C_3aab \to baac_3$	$c_3bab \to abac_3$
$C_3abb \to bbac_3$	$c_3bbb \to bc_3$	$C_3aaa \to aC_3$
$c_3baa \to aabC_3$	$C_3aba \to babC_3$	$c_3bba \to abbC_3$
$C_3b \to bbbC_3$	$aB \to 0$	$bA \to 0$
$0X \to 0$	$X0 \to 0$	$bB \to 1$
$aA \to 1$		

Table 2: The relations of $\widehat{\mathcal{R}}_3$. The relations in the top box are $\mathcal{R}_3$.

In order to apply Theorem 7.5 (and thus establish the uniqueness of reduced forms via Corollary 7.6), we will create modified rewrite systems $\Gamma^*(\widehat{\mathcal{P}}_i)$ as follows. The modified rewrite systems have the same vertex sets as the original string-rewriting systems (i.e., vertices are words in the alphabets specified in Definition 7.7). If a word ω contains no occurrences of the subwords 0, aB, bA, bB,aA, c_2C_2, c_3C_3, C_2c_2, or C_3c_3, then the directed edges leading from ω are unchanged. If, however, one or more of the above occurs in ω as a subword, then we repeatedly apply the eight rewriting rules (from either Table 1 or 2, as appropriate) that have "1" or "0" on the right side, until the resulting word, denoted by $R(\omega)$, contains no occurrences of the above subwords, or is 0. The sole directed edge leading from ω then connects to $R(\omega)$.

It is straightforward to check that $R(\omega)$ is indeed uniquely defined; one considers the string-rewriting system that uses only the eight relations specified above. The latter is easily seen to be terminating and locally confluent, so applications of Theorem 7.5 and Corollary 7.6 establish uniqueness.

Proposition 7.9 *The rewrite systems $\Gamma(\mathcal{P}_2)$, $\Gamma(\mathcal{P}_3)$, $\Gamma^*(\widehat{\mathcal{P}}_2)$, and $\Gamma^*(\widehat{\mathcal{P}}_3)$ are locally confluent and terminating. In particular, each word ω has a unique reduced form $r(\omega)$.*

Proof We first consider local confluence in the case of the rewrite systems $\Gamma^*(\widehat{\mathcal{P}}_i)$. Indeed, there is nothing to prove here if only a single directed edge leads away from ω. It therefore suffices to consider only words ω containing no occurrences of 0, aB, bA, bB,aA, c_2C_2, c_3C_3, C_2c_2, or C_3c_3 as subwords. However, in this case local confluence is essentially trivial, since there can be no overlaps between the left sides of the relations that appear as subwords of ω. Thus, assuming that $\ell' \to r'$ and $\ell'' \to r''$ are rewriting rules that are applicable to ω, we can factor ω in the form $\omega_1\ell'\omega_2\ell''\omega_3$ (without loss of generality), where any one of the ω_i could be trivial. The result of applying the first reduction, $\omega_1r'\omega_2\ell''\omega_3$, and the result of applying the second reduction, $\omega_1\ell'\omega_2r''\omega_3$, then both flow to $\omega_1r'\omega_2r''\omega_3$, proving local confluence.

In the case of the string-rewriting systems $\Gamma(\mathcal{P}_i)$, the only overlaps between applications of rewrite rules $\ell' \to r'$ and $\ell'' \to r''$ occur when ℓ' or ℓ'' (or both) take the form cC or Cc. In all of these cases, checking local confluence is straightforward.

The "terminating" condition follows from the fact that every relation either "moves" C closer to the end of the word (possibly changing the occurrence of C to c in the process), or shortens the word. $\square$

Remark 7.10 In the case of the rewrite systems $\Gamma^*(\widehat{\mathcal{P}}_i)$, we can have two (or more) distinct reduced words that are equivalent modulo the monoid presentation $\widehat{\mathcal{P}}_i$. Indeed, bc_2B and c_2b are two such words.

There are, however, no such pairs of words in the case of the string-rewriting systems $\Gamma(\mathcal{P}_i)$, by Remark 7.3.

Definition 7.11 (the monoids M_2 and M_3) Let $T_A : [0,1) \to [0,1/2)$ be the transformation defined by the rule

$$T_A(x) = \frac{x}{x+1}.$$

Thus, T_A is exactly the transformation denoted by A in Section 2. We similarly define T_B, T_{C_2}, and T_{C_3} as B, C_2, and C_3 were defined in Section 2.

For $i = 2, 3$, we let M_i be the monoid generated by the transformations T_A, T_B, T_{C_i}; i.e., the collection of functions generated by these transformations under the operation of concatenation.

For $i = 2, 3$, we let $\widehat{M}_i$ be the inverse monoid generated by the transformations T_A, T_B, T_{C_i}; i.e., the collection of functions generated by these transformations and their inverses, under the operation of composition.

Definition 7.12 (the maps π_2, $\hat{\pi}_2$, π_3, $\hat{\pi}_3$, and π) For each $X \in \{A, B, C_2, C_3\}$, we set $\pi(X) = T_X$. We extend this map to the lower-case letters a, b, c_2, c_3 by sending each to the relevant inverses; i.e., $\pi(a) = T_A^{-1}$, $\pi(b) = T_B^{-1}$, etc. We define $\pi(0)$ to be the empty function (with empty domain and codomain).

For $i = 2, 3$, we define monoid homomorphisms $\pi_i : M(\mathcal{P}_i) \to M_i$ by letting π_i agree with π on the relevant generating sets. We similarly define monoid homomorphisms $\hat{\pi}_i : M(\widehat{\mathcal{P}}_i) \to \widehat{M}_i$ (for $i = 2, 3$) by letting $\hat{\pi}_i$ agree with π on the relevant generating sets.

The homomorphisms $\hat{\pi}_i$ restrict to π_i, for $i = 2, 3$ (and, indeed, $M(\mathcal{P}_i)$ is a submonoid of $M(\widehat{\mathcal{P}}_i)$, as the latter remark implies).

Remark 7.13 The proof that π_i and $\hat{\pi}_i$ $(i = 2, 3)$ are monoid homomorphisms depends on showing that the defining relations of $M(\mathcal{P}_i)$ and $M(\widehat{\mathcal{P}}_i)$ are satisfied by their images in M_i and $\widehat{M}_i$. This verification is routine, and is left to the reader.

It is clear that the maps π_i and $\hat{\pi}_i$ are surjective.

Remark 7.14 Note that, although $T_B^{-1}T_B = 1$ (where "1" here denotes the identity function on $[0,1]$), $T_BT_B^{-1} = \mathrm{id}_{[1/2,1)} \neq 1$. Similarly, $T_AT_A^{-1} = \mathrm{id}_{[0,1/2)}$.

Remark 7.15 The rewrite rules $1X \to X$ and $X1 \to X$ are implicit in the definitions of $M(\mathcal{P}_i)$ and $M(\widehat{\mathcal{P}}_i)$. It is technically unnecessary to include them, since "1" is simply notation for the empty string.

7.3 The "no potential cancellations" condition

Throughout this subsection, we will write "C" in place of C_2 or C_3, and similarly write "c" in place of c_2 or c_3.

Definition 7.16 (C-tracks) A subword ω' of $\omega \in \{A, B, C, a, b, c, 0\}^*$ is called a *C-track* if

(1) ω' contains at most one occurrence of C or c (not both);

(2) any occurrence of C or c is at the beginning of the word ω';

(3) ω' is a maximal subword with respect to properties (1) and (2).

Remark 7.17 Any word $\omega \in \{A, B, C, a, b, c, 0\}^*$ has a unique decomposition

$$\omega \equiv \omega_1 \dots \omega_n$$

as a product of C-tracks. For instance, the decomposition of the word

$$CCabCABC$$

into C-tracks is $\omega_1\omega_2\omega_3\omega_4$, where

$$\omega_1 = C, \quad \omega_2 = Cab, \quad \omega_3 = CAB, \quad \omega_4 = C.$$

Definition 7.18 ([9, Definition 5.7], advancing an occurrence of C or c) To *advance* an occurrence of C (or c) is to apply one of the relations from Definition 7.7, other than those of the form $Cc \to 1$, $cC \to 1$, $0X \to 0$, and $X0 \to 0$, to a subword containing that occurrence of C or c.

Definition 7.19 ([9, Definition 5.8], no potential cancellations) Assume that $\omega \in \{A, B, C, c\}^*$. Let

$$\omega \equiv \omega_1 \dots \omega_n$$

be the unique decomposition into C-tracks. We say that ω has *no potential cancellations* if the words

$$r(\omega_1)\omega_2 \dots \omega_n, \quad \omega_1 r(\omega_2) \dots \omega_n, \quad \dots, \quad \omega_1\omega_2 \dots r(\omega_n)$$

contain no occurrences of cC or Cc as subwords. Here $r(\omega)$ (for $\omega \in \{A, B, C, c\}^*$) denotes the reduced form of ω relative to $\Gamma(\mathcal{P}_i)$, for $i = 2$ or 3 (see Definition 7.7 and Proposition 7.9).

Remark 7.20 If ω has no potential cancellations, then ω contains no occurrences of cC or Cc as subwords. This follows directly from the observation that c and C are their own reduced forms; i.e., $r(c) = c$ and $r(C) = C$.

Proposition 7.21 [9, Lemma 5.9] *If $\omega \in \{A, B, C, c\}^*$ has no potential cancellations and ω' is the result of advancing a c or C exactly once, then ω' has no potential cancellations.*

Proof Let $\omega \equiv \omega_1\omega_2 \dots \omega_n$, where the right side of the equation is the unique decomposition of ω into C-tracks. Suppose that ω' is the result of advancing an occurrence of C (or c) exactly once; suppose that

$\ell \to r$	$x \equiv C$ or c	$x \equiv CA$	$x \equiv cB$
$CAA \to AC$	$CAC \dashrightarrow CAC$	$CAAC \dashrightarrow ACC$	$cBAC \dashrightarrow ABCC$
$CB \to BBC$	$CBBC \dashrightarrow B^4C^2$	$CAB^2C \dashrightarrow BAcBC$	$cB^3C \dashrightarrow cB^3C$
$CAB \to BAc$	$CBAc \dashrightarrow BBCCAc$	$CABAc \dashrightarrow BAAAcc$	$cBBAc \dashrightarrow BAAcc$
$cA \to AAc$	$cAAc \dashrightarrow A^4cc$	$CA^3c \dashrightarrow ACAc$	$cBAAc \dashrightarrow ABCAc$
$cBB \to Bc$	$cBc \dashrightarrow cBc$	$CABc \dashrightarrow BAcc$	$cBBc \dashrightarrow Bcc$
$cBA \to ABC$	$cABC \dashrightarrow AAcBC$	$CAABC \dashrightarrow AB^2C^2$	$cBABC \dashrightarrow AB^3C^2$

Table 3: The proof of Proposition 7.21 in the case of $M(\mathcal{P}_2)$.

the advanced occurrence of C or c appears in ω_i, and let $\ell \to r$ be the relation that advances this C or c. Thus $\omega_i \equiv \ell\beta$ for some word β. Let

$$\omega' \equiv \omega'_1 \omega'_2 \dots \omega'_n$$

be the unique decomposition of ω' into C-tracks. It follows directly that $\omega'_{i-1}\omega'_i \equiv \omega_{i-1} r \beta$, while $\omega'_j \equiv \omega_j$ if $j \in \{1, \dots, n\} - \{i-1, i\}$. Note that the subword $\omega_{i-1} r$ consists of the C-track of the $(i-1)$-st occurrence of a C (or c) in ω', followed by a C (or c); note also that the only chance of an occurrence of Cc or cC in the words $\omega'_1 \dots r(\omega'_{j-1})\omega'_j \dots \omega'_n$ might occur when $j = i$.

To prove the proposition, it therefore suffices to prove that, during the reduction of the subword $\omega_{i-1}r$, no occurrence of Cc or cC can arise. Note that $\omega_{i-1}r$ begins and ends with occurrences of C (or c), while all intermediate letters are A or B. Thus, after reducing ω_{i-1}, it suffices to show that an occurrence of Cc or cC cannot arise in (further) reducing $r(\omega_{i-1})r$. Finally, we note that the reduced word $r(\omega_{i-1})$ ends in a reduced word x that begins with a C or c. There are only finitely many possibilities for x: indeed, $x \in \{c, C, CA, cB\}$ in the case of $M(\mathcal{P}_2)$, while $x \in \{c, C, CA, CAA, CAB, cB, cBA, cBB\}$ in the case of $M(\mathcal{P}_3)$. Furthermore, in either case, one of the cases $x \equiv C$ or $x \equiv c$ can be ruled out, since $x\ell$ contains no occurrence of Cc or cC by hypothesis.

Thus, in summary, it suffices to show that, for each rewriting rule $\ell \to r$, no occurrence of cC or Cc can appear when reducing the word xr, where x runs over the above possibilities and further satisfies the condition that $x\ell$ itself contains neither cC nor Cc.

The relevant calculations are summarised in Tables 3 and 4. □

Definition 7.22 (negative-to-positive words) Let ω be a word in the alphabet $\{A, B, C, a, b, c, 0, 1\}$. We say that ω is *negative-to-positive* if all occurrences of a and b (if any) occur before any occurrence of either A or B.

Remark 7.23 If $\omega \neq 1, 0$ is a negative-to-positive word containing no occurrences of bB or aA or 1, then each C-track in ω is a word in either $\{a, b, C, c\}$ or $\{A, B, C, c\}$. We call a C-track in the former alphabet *negative*, while a C-track in the latter alphabet is *positive*.

A C-track consisting only of the single symbol C or c can be freely considered negative or positive.

$\ell \to r$	$x \in \{c, C, CA, CAA\}$	$x \in \{CAB, cB, cBA, cBB\}$
$CA^3 \to AC$	$CAC \overset{\cdot}{\longrightarrow} CAC$ $CAAC \overset{\cdot}{\longrightarrow} CAAC$ $CAAAC \overset{\cdot}{\longrightarrow} ACC$	$CABAC \overset{\cdot}{\longrightarrow} BABCC$ $cBAC \overset{\cdot}{\longrightarrow} cBAC$ $cBAAC \overset{\cdot}{\longrightarrow} AABCC$ $cBBAC \overset{\cdot}{\longrightarrow} ABBCC$
$CABA \to BABC$	$CBABC \overset{\cdot}{\longrightarrow} B^3CABC$ $CABABC \overset{\cdot}{\longrightarrow} BAB^4C^2$ $CAABABC \overset{\cdot}{\longrightarrow} BA^5cBC$	$CABBABC \overset{\cdot}{\longrightarrow} B^2A^4cBC$ $cBBABC \overset{\cdot}{\longrightarrow} AB^5C^2$ $cBABABC \overset{\cdot}{\longrightarrow} ABA^4cBC$ $cB^3ABC \overset{\cdot}{\longrightarrow} BA^3cBC$
$CAAB \to BAAc$	$CBAAc \overset{\cdot}{\longrightarrow} B^3CAAc$ $CABAAc \overset{\cdot}{\longrightarrow} BABCAc$ $CAABAAc \overset{\cdot}{\longrightarrow} BA^8cc$	$CABBAAc \overset{\cdot}{\longrightarrow} A^4B^2CAc$ $cBBAAc \overset{\cdot}{\longrightarrow} ABBCAc$ $cBABAAc \overset{\cdot}{\longrightarrow} ABA^7cc$ $cB^3AAc \overset{\cdot}{\longrightarrow} BA^6cc$
$CABB \to BBAc$	$CBBAc \overset{\cdot}{\longrightarrow} B^6CAc$ $CABBAc \overset{\cdot}{\longrightarrow} B^2A^4cc$ $CAABBAc \overset{\cdot}{\longrightarrow} BA^2cBAc$	$CABBBAc \overset{\cdot}{\longrightarrow} BBAcBAc$ $cBBBAc \overset{\cdot}{\longrightarrow} BA^3cc$ $cBABBAc \overset{\cdot}{\longrightarrow} ABAcBAc$ $cB^4Ac \overset{\cdot}{\longrightarrow} BcBAc$
$CB \to BBBC$	$CBBBC \overset{\cdot}{\longrightarrow} B^9C^2$ $CABBBC \overset{\cdot}{\longrightarrow} BBAcBC$ $CAABBBC \overset{\cdot}{\longrightarrow} BA^2cBBC$	$CABBBBC \overset{\cdot}{\longrightarrow} BBAcBBC$ $cBBBBC \overset{\cdot}{\longrightarrow} BcBC$ $cBABBBC \overset{\cdot}{\longrightarrow} ABAcBBC$ $cB^5C \overset{\cdot}{\longrightarrow} BcBBC$
$cA \to AAAc$	$cAAAc \overset{\cdot}{\longrightarrow} A^9c^2$ $CA^4c \overset{\cdot}{\longrightarrow} ACAc$ $CA^5c \overset{\cdot}{\longrightarrow} ACAAc$	$CABAAAc \overset{\cdot}{\longrightarrow} BABCAAc$ $cBAAAc \overset{\cdot}{\longrightarrow} AABCAc$ $cBAAAAc \overset{\cdot}{\longrightarrow} AABCAAc$ $cBBAAAc \overset{\cdot}{\longrightarrow} ABBCAAc$
$cBAA \to AABC$	$cAABC \overset{\cdot}{\longrightarrow} A^6cBC$ $CAAABC \overset{\cdot}{\longrightarrow} AB^3C^2$ $CA^4BC \overset{\cdot}{\longrightarrow} ACABC$	$CABAABC \overset{\cdot}{\longrightarrow} BABCABC$ $cBAABC \overset{\cdot}{\longrightarrow} A^2B^4C^2$ $cBA^3BC \overset{\cdot}{\longrightarrow} AABCABC$ $cBBAABC \overset{\cdot}{\longrightarrow} ABBCABC$
$cBAB \to ABAc$	$cABAc \overset{\cdot}{\longrightarrow} A^3cBAc$ $CAABAc \overset{\cdot}{\longrightarrow} BA^5cc$ $CAAABAc \overset{\cdot}{\longrightarrow} AB^3CAc$	$CABABAc \overset{\cdot}{\longrightarrow} BAB^4CAc$ $cBABAc \overset{\cdot}{\longrightarrow} ABA^4cc$ $cBAABAc \overset{\cdot}{\longrightarrow} A^2B^4CAc$ $cBBABAc \overset{\cdot}{\longrightarrow} AB^5CAc$
$cBBA \to ABBC$	$cABBC \overset{\cdot}{\longrightarrow} A^3cBBC$ $CAABBC \overset{\cdot}{\longrightarrow} BAAcBC$ $CA^3BBC \overset{\cdot}{\longrightarrow} AB^6CC$	$CABABBC \overset{\cdot}{\longrightarrow} BAB^7CC$ $cBABBC \overset{\cdot}{\longrightarrow} ABAcBC$ $cBAABBC \overset{\cdot}{\longrightarrow} AAB^7CC$ $cBBABBC \overset{\cdot}{\longrightarrow} AB^8C^2$
$cB^3 \to Bc$	$cBc \overset{\cdot}{\longrightarrow} cBc$ $CABc \overset{\cdot}{\longrightarrow} CABc$ $CAABc \overset{\cdot}{\longrightarrow} BAAcc$	$CABBc \overset{\cdot}{\longrightarrow} BBAcc$ $cBBc \overset{\cdot}{\longrightarrow} cBBc$ $cBABc \overset{\cdot}{\longrightarrow} ABAcc$ $cBBBc \overset{\cdot}{\longrightarrow} Bcc$

Table 4: The proof of Proposition 7.21 in the case of $M(\mathcal{P}_3)$.

Definition 7.24 (no potential cancellations in negative-to-positive words) Let $\omega \in \{A, B, C, a, b, c\}^*$. Assume that the reduced form of ω is not 0, and that ω also contains no occurrences of bB or aA.

Let

$$\omega \equiv \omega_1 \dots \omega_n$$

be the unique decomposition into C-tracks. We say that ω has *no potential cancellations* if the words

$$r(\omega_1)\omega_2 \dots \omega_n, \quad \omega_1 r(\omega_2) \dots \omega_n, \quad \dots, \quad \omega_1\omega_2 \dots r(\omega_n)$$

contain no occurrences of cC or Cc. Here $r(\omega)$ denotes the reduced form of ω relative to the rewrite system $\Gamma^*(\widehat{\mathcal{P}}_i)$ (see Remark 7.8 and Proposition 7.9).

Remark 7.25 If $\omega \equiv \omega_1 \dots \omega_n$ is the decomposition of the negative-to-positive word ω into C-tracks, then the words $\omega_1 \dots r(\omega_j) \dots \omega_n$ (for $j = 1, \dots, n$) need not be accessible from ω by a directed edge-path in $\Gamma^*(\widehat{\mathcal{P}}_i)$. This contrasts with the case of words in the generators $\{A, B, C, c\}$; i.e., words consisting only of positive C-tracks.

Consider the word c_2bC_2BB. We have

$$c_2br(C_2BB) \equiv c_2bBBBBC_2,$$

and there is no directed edge-path from c_2bC_2BB to $c_2bBBBBC_2$, since the application of the relation $C_2B \to BBC_2$ must be followed by an application of the cancellation $bB \to 1$:

$$c_2bC_2BB \to c_2bBBC_2B \to c_2BC_2B \to c_2BBBC_2 \to Bc_2BC_2.$$

Proposition 7.26 *Assume that*

(1) ω *is negative-to-positive*;

(2) ω *has no potential cancellations*;

(3) ω *has no subword of the form aA or bB*;

(4) *the reduced form of ω (in the sense of the rewrite system $\Gamma^*(\widehat{\mathcal{P}}_i)$) is not* 0.

Let ω' be the result of advancing a c or C exactly once, and then removing all occurrences of aA or bB, along with all occurrences of "1". The word ω' is also a negative-to-positive word with no potential cancellations.

Proof The proof is like that of Proposition 7.21. We assume that ω' is the result of advancing the i-th occurrence of C or c in ω exactly once. Let $\omega_1\omega_2 \dots \omega_n$ be the C-track decomposition of ω. There are three cases: ω_{i-1} and ω_i are both positive C-tracks, or ω_{i-1} and ω_i are both negative, or ω_{i-1} is negative and ω_i is positive.

We note that the first case (in which both C-tracks are positive) is already handled by the proof of Proposition 7.21. The second case is also handled by the proof of Proposition 7.21. This follows from the observation that each rewriting rule between words in the alphabet $\{a, b, C, c\}$ corresponds to a rewriting rule in between words in the alphabet $\{A, B, C, c\}$. One need only replace a C with c (or the reverse), an A with a b, and a B with an a. Thus, for instance, the rewriting rule $CAA \to AC$ corresponds to $cbb \to bc$. Using this substitution, we can transform Tables 3 and 4 into tables that prove the negative-to-negative case.

It remains to consider the case in which ω_{i-1} is negative and ω_i is positive. Since ω_{i-1} is a negative C-track, we have $\omega_{i-1} \equiv C^{\pm}u$, where u is a nonempty word in the generators a and b. Let $\ell \to r$ be the rewriting rule that advances the occurrence of C or c in the subword ω_i. There are nonempty words r_1, r_2 such that $r \equiv r_1 r_2$, where $r_1 \in \{A, B\}^*$ and $r_2 = c$ or C. After applying the rule $\ell \to r$ to the subword $\omega_{i-1}\ell$, and before any cancellation, we arrive at a word of the form $C^{\pm}ur_1r_2$. Since $\omega \neq 0$, the subword $C^{\pm}ur_1r_2$ cannot contain any occurrence of aB or bA. It follows that one of the rewriting rules $aA \to 1$ or $bB \to 1$ can be applied at least once to ur_1, and, indeed, that such rules can be applied to ur_1 until an entirely positive word (i.e., a word in the generators $\{A, B\}$) or an entirely negative word remains. Note that, in either case, the word ω' described in the proposition is still negative-to-positive. In fact, only the "no potential cancellations" condition remains to be proved.

The proof involves an analysis of various subcases. We first assume that, after cancellation, a negative word remains. In this subcase, $u \equiv \hat{u}r_1^{-1}$, where $\hat{u}$ is a possibly empty word in the generators $\{a, b\}$. It suffices to show that the word $r(C^{\pm}\hat{u})r_2$ contains no occurrence of the subwords cC or Cc. Suppose, for a contradiction, that there is such an occurrence. The occurrence must be at the end of the word $r(C^{\pm}\hat{u})r_2$, from which it follows that $r(C^{\pm}\hat{u}) \equiv \tilde{u}r_2^{-1}$, for some $\tilde{u} \in \{a, b\}^*$. Next, we consider again the subword $C^{\pm}u\ell$ of ω. We have

$$C^{\pm}u\ell \equiv C^{\pm}\hat{u}r_1^{-1}\ell \overset{.}{\longrightarrow} \tilde{u}r_2^{-1}r_1^{-1}\ell.$$

Since, for each rewrite rule $\ell \to r$ in $\mathcal{P}_i$, the rule $r^{-1} \to \ell^{-1}$ is a rewrite rule in $\widehat{\mathcal{P}}_i$ (see Tables 1 and 2), it follows that

$$\tilde{u}r_2^{-1}r_1^{-1}\ell \overset{.}{\longrightarrow} \tilde{u}\ell^{-1}\ell,$$

which implies that $\omega_1 \dots r(\omega_{i-1})\omega_i \dots \omega_n$ contains an occurrence of cC or Cc (which appears in the subword $\ell^{-1}\ell$). This is a contradiction of the "no potential cancellations" hypothesis.

Now we consider the subcases in which a nonempty positive word remains after cancelling within ur_1. In this subcase, we can list all of the possibilities for the word $\omega_{i-1}r \equiv C^{\pm}ur$, which arises from the subword $\omega_{i-1}\ell$ of ω after advancing the initial "c" or "C" in ω_i via the rewriting rule $\ell \to r$. Indeed, in the case of $i = 2$, the only possibilities are

$$\begin{array}{cccc} C_2bBBC_2, & c_2bBBC_2, & C_2bBAc_2, & c_2bBAc_2, \\ C_2aAAc_2, & c_2aAAc_2, & C_2aABC_2, & c_2aABC_2. \end{array}$$

Each case is easily handled. For instance, in the first case, C_2bBBC_2 becomes C_2BC_2 after cancelling bB. The word C_2B is the $(i-1)$-st C-track in ω'. Rewriting, we find that $r(C_2B) = BBC_2$, which shows that ω' still has no potential cancellations.

The case of $i=3$ involves many more cases, but we can make some useful general observations. We consider the possible forms of the word $R(C_3^{\pm}ur)$, which is the subword of ω' that occurs after advancing the leading "C" symbol in ω_i and after performing any reductions of the form $aA \to 1$ or $bB \to 1$. We note that $R(C_3^{\pm}ur)$ necessarily takes the form $C_3^{\pm}\widetilde{u}C_3^{\pm}$, where $\widetilde{u} \in \{A,B\}^*$ has length either 1 or 2, and the first and last "C" symbol may have the same or opposite exponents. An examination of Table 2 shows that the only way that the initial "C" symbol in $C_3^{\pm}\widetilde{u}C_3^{\pm}$ can be advanced to the end of the word (and thus create a potential cancellation) is if $\widetilde{u}$ does not contain both of the symbols A and B. We can therefore assume that $\widetilde{u} = A$, AA, B, or BB. Next, we note that, if $\widetilde{u}$ ends with A, then r necessarily ended with c_3, while if $\widetilde{u}$ ends with B, then r necessarily ended with C_3. (This again follows from Table 2.) Finally, we note that, if there is to be cancellation in $C_3^{\pm}\widetilde{u}C_3^{\pm}$ as the result of advancing the initial "C" symbol, then the first and last exponents must be opposite; this now follows because advancing a "C" symbol past a word of the form A, AA, B, or BB never changes the exponent. Thus, the only cases left are

$$C_3Ac_3, \quad C_3AAc_3, \quad c_3BC_3, \quad c_3BBC_3.$$

These words are all reduced, completing the proof. □

Corollary 7.27 *If ω satisfies the hypotheses of Proposition 7.26 and $\omega \dashrightarrow \omega'$ in $\Gamma^*(\widehat{\mathcal{P}}_i)$, then ω' also satisfies the hypotheses of Proposition 7.26, after we remove all occurrences of aA and bB.*

Proof This follows by repeatedly applying Proposition 7.26 to ω, and from the fact that any application of a rewriting rule to ω (under the hypotheses on ω) necessarily entails advancing a "C" symbol. □

7.4 Presentations and normal forms for the monoids M_2 and M_3

Proposition 7.28 [9, Lemma 5.10] *Let ω be a word in the generators $\{A,B,C,c\}$. Assume that ω has no potential cancellations.*

There is a word $\tau \in \{A,B\}^*$ such that $r(\omega\tau) \equiv \widehat{\omega}C^{\epsilon}$, where $\widehat{\omega}$ is a word in $\{A,B\}$ and $\epsilon \geq 0$ is the total exponent of C and c in ω.

Proof The proof is by induction on the (combined) exponent ϵ of C and c in ω. We note that, due to the "no potential cancellations" condition, it is not possible to reduce (or, indeed, increase) the exponent ϵ by applying any of the monoid relations.

Our proof will use the fact that, if $\omega \in \{A,B,C,c\}^*$ has no potential cancellations, then any word of the form $\omega\tau$ ($\tau \in \{A,B\}^*$) also has no potential cancellations. This is an easy consequence of Definition 7.19.

We first consider the case of $M(\mathcal{P}_2)$; assume $\epsilon = 1$, the case $\epsilon = 0$ being trivial. We note that $r(\omega)$ ends with one of the strings C, c, CA, or cB (and the only occurrences of C or c occur in these strings). In the case of C, there is nothing to prove. If $r(\omega)$ ends with c, we can let $\tau \equiv BA$ and then reduce the result. If $r(\omega)$ ends with either CA or cB, we can let $\tau \equiv A$ and then reduce the result. This proves the base case.

Now let $\epsilon > 1$. We can express ω as a product $\omega_1\omega_2$, where the total combined exponent of C and c in ω_2 is $\epsilon - 1$, and ω_1 contains a single occurrence of C or c. By induction, we can find $\tau_1 \in \{A,B\}^*$ such that $r(\omega_2\tau_1) \equiv \widehat{\omega}_2C^{\epsilon-1}$, where $\widehat{\omega}_2 \in \{A,B\}^*$. Thus, after reducing the word $\omega_1\omega_2\tau_1$, we obtain a word $\omega' \in \{A, B, C, c\}^*$ that ends with C^ϵ, $CAC^{\epsilon-1}$, or $cBC^{\epsilon-1}$. (Note that the case $cC^{\epsilon-1}$ is ruled out by the "no potential cancellations" hypothesis.) In the first case, we are finished; set $\tau_2 \equiv 1$. If ω' ends with $CAC^{\epsilon-1}$ or $cBC^{\epsilon-1}$, we can set $\tau_2 \equiv A^{2^{\epsilon-1}}$. After reducing the word $\omega'\tau_2$, we have a string of the required form, so the required τ is $\tau_1\tau_2$.

Now we consider $M(\mathcal{P}_3)$. Let $\omega \in \{A, B, C, c\}$ and define ϵ as before. We first consider the case $\epsilon = 1$. The word $r(\omega)$ ends with c, C, CA, CAB, CAA, cB, cBA, or cBB. If $r(\omega)$ ends with c, we can let $\tau \equiv BBA$ and apply the relation $cBBA \to ABBC$. If $r(\omega)$ ends with C, there is nothing to prove ($\tau \equiv 1$). In the remaining cases, we let $\tau \equiv AA, A, A, AA, A$, or A (respectively).

Now suppose $\epsilon > 1$. We can write ω as the product $\omega_1\omega_2$, where the total combined exponent of C and c in ω_2 is $\epsilon - 1$, and ω_1 contains a single occurrence of either C or c. Proceeding as in the case of $M(\mathcal{P}_2)$, we can right multiply by some $\tau_1 \in \{A, B\}^*$ and reduce to arrive at a word ω' that ends with one of the following strings: C^ϵ, $CAC^{\epsilon-1}$, $CABC^{\epsilon-1}$, $CAAC^{\epsilon-1}$, $cBC^{\epsilon-1}$, $cBAC^{\epsilon-1}$, $cBBC^{\epsilon-1}$. In the first case, there is nothing to prove; let $\tau_2 \equiv 1$. In the remaining cases, we multiply by $\tau_2 \equiv A^{2\cdot 3^{\epsilon-1}}$, $A^{3^{\epsilon-1}}$, $A^{3^{\epsilon-1}}$, $A^{2\cdot 3^{\epsilon-1}}$, $A^{3^{\epsilon-1}}$, or $A^{3^{\epsilon-1}}$, respectively. Thus, the required τ is $\tau_1\tau_2$. □

Proposition 7.29 *Let $\omega \in \{A, B, C, a, b, c\}^*$ be a negative-to-positive word with no potential cancellations. Assume that there is no $\tau \in \{A, B\}^*$ such that $r(\omega\tau) \equiv 0$.*

There is some $\tau' \in \{A, B\}^*$ such that $r(\omega\tau') \equiv \widehat{\omega}C^\epsilon$, where $\widehat{\omega} \in \{A, B\}^*$ and ϵ is the total combined exponent of C and c in ω.

Proof We prove this by induction on the sum k of the combined exponents of a and b in ω. The case $k = 0$ is handled by Proposition 7.28. We will use the fact that, if $\omega \in \{A, B, a, b, C, c\}^*$ is a negative-to-positive word with no potential cancellations, then so is the word $\omega\tau$, where τ is any word in $\{A, B\}^*$. This fact is easily verified from Definition 7.24.

Let $\omega \in \{A, B, C, a, b, c\}^*$, let k be defined as above, and suppose that the proposition is known to be true for smaller k. We can write $\omega \equiv \omega_1\omega_2$, where ω_1 involves no occurrences of A or B, and ω_2 involves no occurrences of a or b. We may further assume that ω_1 ends with an occurrence of a or b, since any occurrence of C or c may be subsumed by ω_2.

By Proposition 7.28, we can find a word $\tau_1 \in \{A, B\}^*$ such that $r(\omega_2\tau_1) \equiv \widehat{\omega}C^{\epsilon_2}$, where ϵ_2 is the total exponent sum of C and c in ω_2 and $\widehat{\omega} \in \{A, B\}^*$. If $\widehat{\omega}$ is not the empty word, then it must be that the initial letter of $\widehat{\omega}$ cancels with the terminal letter of ω_1 in $\omega_1\widehat{\omega}C^{\epsilon_2}$. (This is because occurrences of aB and bA cannot arise, by the hypothesis that $r(\omega\tau)$ is never 0.) After performing all cancellations of the form $aA \to 1$ and $bB \to 1$, we can call the inductive hypothesis, to find τ_2 such that $r(\omega_1\widehat{\omega}C^{\epsilon_2}\tau_2) \equiv \widetilde{\omega}C^{\epsilon_1+\epsilon_2}$, where $\widetilde{\omega} \in \{A, B\}^*$ and ϵ_1 is the total exponent of c and C in ω_1. This completes the induction, under the assumption that $\widehat{\omega}$ is not the empty word. (We note that, to apply the inductive hypothesis, we are implicitly calling Corollary 7.27, and using the completeness of the rewrite system $\Gamma^*(\widehat{\mathcal{P}}_i)$.)

If $\widehat{\omega} \equiv 1$, we simply multiply by a suitable word $\tau_{3/2}$: either $A^{2^{\epsilon_2}}$ or $A^{3^{\epsilon_2}}$ (depending on whether we are considering $M(\mathcal{P}_2)$ or $M(\mathcal{P}_3)$). After reducing, we find that $r(\omega_2\tau_1\tau_{3/2}) \equiv \widehat{\omega}'C^{\epsilon_2}$, where $\widehat{\omega}'$ is nonempty. This reduces us to the previous case, completing the induction and the proof. □

Proposition 7.30 (normal forms in $M(\mathcal{P}_i)$) *The reduced words modulo the presentation $\mathcal{P}_2$ take the form*

$$\omega_1\omega_2\omega_3,$$

where $\omega_1 \in \{A, B\}^*$, $\omega_2 \in \{C^n A, c^m B \mid m, n \in \mathbb{N}\}^*$, and $\omega_3 \in \{C, c\}^*$.

The reduced words modulo the presentation $\mathcal{P}_3$ take the form

$$\omega_1\omega_2\omega_3,$$

where $\omega_1 \in \{A, B\}^*$, $\omega_2 \in \{C^{n_1} A, C^{n_2} AA, C^{n_3} AB, c^{n_4} B, c^{n_5} BA, c^{n_6} BB \mid n_i \in \mathbb{N}\}^*$, and $\omega_3 \in \{C, c\}^*$.

Proof It is clear that the words in question are reduced. Thus, the main point is to show that every word in the generators can be reduced to a word of the given type. This is easily done by induction on the length of the word. □

Theorem 7.31 (monoid presentations for M_2 and M_3) *The monoid homomorphisms $\pi_i : M(\mathcal{P}_i) \to M_i$ are isomorphisms, for $i = 2, 3$.*

In particular, $\mathcal{P}_i$ is a presentation for M_i, for $i = 2, 3$.

Proof In view of Remark 7.13, it suffices to show that π_i is injective, for $i = 2, 3$. We suppose, for a contradiction, that π_2 is not injective. Let

$$S' = \{\{\omega_1, \omega_2\} \mid \omega_1 \not\equiv \omega_2;\ \pi_2(\omega_1) = \pi_2(\omega_2);\ \omega_1 \text{ and } \omega_2 \text{ are reduced}\}.$$

We let

$$S'' = \{\{\omega_1, \omega_2\} \mid \{\omega_1, \omega_2\} \in S';\ \omega_1, \omega_2 \text{ begin with different letters}\}.$$

We note that S' is nonempty by hypothesis, and it follows easily that S'' is also nonempty. (It suffices to cancel the maximal common prefix of the words ω_1, ω_2, where $\{\omega_1, \omega_2\} \in S'$.) Next, we note that if $\{\omega_1, \omega_2\} \in S''$, then one of ω_1 or ω_2 begins with C or c, or is trivial. (The case in which ω_1 begins with "A" and ω_2 begins with "B" (or the reverse) can be ruled out, since $\pi_2(\omega_1)$ cannot be equal to $\pi_2(\omega_2)$ under these conditions.) We will assume (without loss of generality) that it is ω_1 that begins with C or c, or is trivial.

Consider $\{\omega_1, \omega_2\} \in S''$ such that the total exponent of C and c in ω_1 and ω_2 is a minimum. We first assume that ω_1 begins with either C or c. (The case in which ω_1 is trivial is easier, and will be handled in the course of the more difficult argument.) It follows that $\omega_1 \equiv \omega_1' C^k$, where $\omega_1' \in \{C^n A, c^m B \mid n, m \in \mathbb{N}\}^*$ and $k \in \mathbb{Z}$, by Proposition 7.30. Indeed, we can assume that $k = 0$, for if $k \neq 0$, then we simply multiply both ω_1 and ω_2 on the right by c^k. The word $\omega_2' :\equiv \omega_2 c^k$ is then necessarily reduced, by the hypothesis that the total exponent of C and c in ω_1 and ω_2 is a minimum in S''. We can then replace the pair $\{\omega_1, \omega_2\}$ by $\{\omega_1', \omega_2'\}$, where the latter is still in S''.

Thus, we can assume that $\omega_1 \in \{C^n A, c^m B \mid n, m \in \mathbb{N}\}^*$, $\{\omega_1, \omega_2\} \in S''$, and the total combined exponent of C and c in ω_1 and ω_2 is a minimum within S''. We claim that the words ω_1^{-1} and ω_2 have no potential cancellations. This is obvious in the case of ω_2, since it is reduced. In ω_1^{-1}, every occurrence of a is followed by c, every occurrence of b is followed by C, and there are no occurrences of

A, B, Cc, or cC. Now note that no occurrences of Cc or cC can occur when reducing subwords of the form $C^{\pm 1}ac$ or $C^{\pm 1}bC$; from this it follows that ω_1^{-1} has no potential cancellations.

Next we claim that $\omega_1^{-1}\omega_2$ has no potential cancellations. (Here the claim is obvious if ω_1 is the trivial word; thus, the argument from this point is the general case.) Indeed, the only C-track of $\omega_1^{-1}\omega_2$ that could cause a problem is the one that begins with the terminal letter (c or C) of ω_1^{-1}. Assume that $\omega_1 \equiv C\widehat{\omega}_1$ (without loss of generality), and suppose that $\omega_1^{-1}\omega_2$ has a potential cancellation. Replacing $\{\omega_1, \omega_2\}$ by $\{\widehat{\omega}_1, r(c\omega_2)\}$, we find (possibly after cancelling common prefixes) that the latter is a pair in S'' of smaller total exponent in C and c. This contradicts the choice of $\{\omega_1, \omega_2\}$, which proves the claim.

Since $\pi_2(\omega_1) = \pi_2(\omega_2)$, we have $\widehat{\pi}_2(\omega_1^{-1}\omega_2) = \widehat{\pi}(1) = \mathrm{id}_{[0,1)}$. In particular, this means that $\omega_1^{-1}\omega_2$ satisfies the hypotheses of Proposition 7.29. We can therefore find a word $\tau \in \{A, B\}^*$ such that $r(\omega_1^{-1}\omega_2\tau) \equiv \widehat{\omega}C^k$, where $k \geq 0$ is the total combined exponent of C and c in the word $\omega_1^{-1}\omega_2$ and $\widehat{\omega} \in \{A, B\}^*$. Thus, we have $\pi_2(\widehat{\omega}C^k) = \pi_2(\tau)$. We now cancel the maximal common prefix of $\widehat{\omega}C^k$ and τ. We continue to denote the resulting strings by $\widehat{\omega}C^k$ and τ, respectively, but now either $\widehat{\omega}$ or τ is trivial.

If $\widehat{\omega}$ is trivial, but τ is not, then $\pi_2(\widehat{\omega}C^k) = C^k$, while $\pi_2(\tau)$ is a transformation whose range is a proper subinterval of $[0, 1)$. This is a contradiction. If τ is trivial, but not $\widehat{\omega}$, we find that $\pi_2(\tau)$ has the image $[0, 1)$, but $\pi_2(\widehat{\omega}C^k)$ does not, which is also a contradiction. Finally, if both $\widehat{\omega}$ and τ are trivial, we find that $\pi_2(C^k) = \mathrm{id}_{[0,1)}$, which is possible only if $k = 0$. The latter implies that ω_1 is the trivial word, $\omega_2 \in \{A, B\}^*$, and ω_2 is not the trivial word. This leads us to conclude that $\pi_2(\omega_2) = \mathrm{id}_{[0,1)}$, which is impossible since the image of $\pi_2(\omega_2)$ is a proper subinterval of $[0, 1)$. Thus, π_2 is injective.

The case of π_3 is similar. □

Remark 7.32 (presentations for certain submonoids of $\mathrm{Isom}(\mathbb{H}^2)$) Let $\tilde{A}$ denote the transformation of the projective line $\mathbb{P}_1$ $(= \partial\mathbb{H}^2)$ that agrees with A (as defined in Definition 2.6) on the $[0, 1)$. Similarly define $\widetilde{B}$, $\widetilde{C}_2$, and so forth. The transformations $\tilde{A}$, $\widetilde{B}$, $\widetilde{C}_2$, $\widetilde{C}_3$, and their inverses may equivalently be considered isometries of $\mathbb{H}^2$. Let $\widetilde{S}_i = \{\tilde{A}, \widetilde{B}, \widetilde{C}_i, \tilde{c}_i\}^*$, for $i = 2, 3$. There are obvious homomorphisms $\phi_i : \widetilde{S}_i \to M_i$ for $i = 2, 3$. It is just as clear that these homomorphisms are surjective. If $\phi_i(\alpha) = \phi_i(\beta)$, but $\alpha \neq \beta$, then α and β are transformations of $\partial\mathbb{H}^2$ that agree on $[0, 1)$. This is impossible, however, since any isometry of $\mathbb{H}^2$ is determined by its effect on any three boundary points. Thus, ϕ_i is injective, for $i = 2, 3$.

It follows from all of this that $\widetilde{S}_2$ and $\widetilde{S}_3$ admit the same presentations and normal forms as do M_2 and M_3.

8 An intermediate value theorem for the expansion scheme $\mathcal{E}_i$

In this section, we will argue that $N(T)$ is always a set of consecutive integers if T is a nontrivial subdivision tree. This will be the main ingredient to our proof that the expansion schemes $\mathcal{E}_i$ and $\mathcal{E}_i'$ are n-connected, for all n. The proof of the latter fact will be assembled in Section 9.

The essential idea, that of a "sufficiently expanded subdivision tree", is drawn from [9, Definition 5.5].

8.1 The case of M_2

In this subsection, we will argue that $N(T)$ is always a set of consecutive integers in the case that T is a subdivision tree over M_2. All of the subdivision trees in question will be subdivision trees over M_2; "C" will refer to C_2, and so forth.

We remind the reader that "node" means "interior node" (see Definition 6.1).

Definition 8.1 (sufficiently expanded subdivision trees) A subdivision tree is *sufficiently expanded* if there is a directed arc p in T from the root ϵ to a leaf ℓ such that

(1) each nonroot node on the arc has a nonzero label;

(2) if p passes through a nonroot node v and the label of v is positive, then p also passes through the left child of v;

(3) if p passes through a nonroot node v and the label of v is negative, then p also passes through the right child of v.

A sufficiently expanded subdivision tree is *left sufficiently expanded* (respectively, *right sufficiently expanded*) if some directed arc p as described above passes through the left (respectively, right) child of the root.

Lemma 8.2 *Let T be a subdivision tree.*

(1) *If $n(T)=k$ and $k-1 \notin N(T)$, then there is $T' \approx T$ such that $n(T')=k$ and T' is left sufficiently expanded.*

(2) *If $n(T)=k$ and $k+1 \notin N(T)$, then there is $T' \approx T$ such that $n(T')=k$ and T' is right sufficiently expanded.*

Proof We prove both parts simultaneously by induction on the number of carets in the subdivision tree T. The induction begins trivially, since a subdivision tree with a single caret is necessarily both left and right sufficiently expanded.

Now we consider a subdivision tree T, and assume that the lemma is true of all subdivision trees containing fewer carets. We will argue for (1); the argument proving (2) is similar. Thus, we let $n(T)=k$ and assume that $k-1 \notin N(T)$. We note that $0 \notin N(T_\ell)$; otherwise (up to equivalence) T takes the form in Figure 6. This allows us to apply an elementary equivalence from Definition 6.8, resulting in a $T' \approx T$ such that $n(T')=k-1$. However, this implies that $k-1 \in N(T)$, a contradiction. Thus, $0 \notin N(T_\ell)$, as claimed.

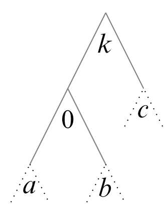

Figure 6: The case in which $0 \in N(T_\ell)$.

We let k_1 be either

(i) the smallest positive member of $N(T_\ell)$, or

(ii) the largest negative member of $N(T_\ell)$.

We can assume that $n(T_\ell) = k_1$ (possibly after replacing T_ℓ with an equivalent tree and applying Lemma 6.6). We note that $k_1 - 1 \notin N(T_\ell)$ (in case (i)), or $k_1 + 1 \notin N(T_\ell)$ (in case (ii)); therefore the inductive hypothesis applies, and we conclude that T_ℓ is equivalent to a left sufficiently expanded tree T'_ℓ (in case (i)), or to a right sufficiently expanded tree T'_ℓ (in case (ii)). In either case, we replace T_ℓ by T'_ℓ. We let T' denote the result of replacing T_ℓ by T'_ℓ in the tree T. We note that $n(T') = n(T) = k$ and $T' \approx T$ by Lemma 6.6.

Assume that we are in case (i); case (ii) is similar. Since T'_ℓ is left sufficiently expanded, there is a path p' from the root of T'_ℓ to a leaf of T'_ℓ satisfying the properties in Definition 8.1, such that p' also passes through the left child of the root of T'_ℓ. Let p be the concatenation of e and p', where e is the edge connecting the root of T' to its left child, the root of T'_ℓ. The path p satisfies all of the properties from Definition 8.1 and passes through the left child of the root in T', so T' is left sufficiently expanded. This completes the induction. $\square$

Proposition 8.3 *Let T be a nontrivial subdivision tree.*

(1) *If T is left sufficiently expanded and T' satisfies $n(T') < n(T)$ then $T' \not\approx T$.*

(2) *If T is right sufficiently expanded and T' satisfies $n(T') > n(T)$ then $T' \not\approx T$.*

Proof We first prove (1). Assume that T is left sufficiently expanded and T' is such that $n(T') < n(T)$ and $T' \approx T$. After letting a suitable power of C act at the roots of T and T', we can assume that $n(T) > 0$ and $n(T') = 0$. Since T is left sufficiently expanded, there is a directed arc p from the root of T to a leaf ℓ satisfying the conditions of Definition 8.1; the label of ℓ is a reduced word ω. There is a leaf ℓ' of T' that corresponds to ℓ; let ω' be the label of ℓ'. We have $\omega = \omega' C^k$, for some $k \in \mathbb{Z}$, by Proposition 6.7. After reducing, we find

$$\omega \equiv r(\omega' C^k).$$

However, these words cannot be equal letter-by-letter, since ω necessarily begins with an occurrence of C, but $\omega' C^k$ (and, thus, $r(\omega' C^k)$) begins with either A or B. This is a contradiction to Theorem 7.31.

The proof of (2) is similar. One can reduce to the case in which $n(T) < 0$ and $n(T') = 0$, and then argue that ω begins with a c, while $r(\omega' C^k)$ begins with either A or B. $\square$

Proposition 8.4 *Let T be a nontrivial subdivision tree.*

(1) *If $0 \notin N(T_\ell)$, then $n(T) = \min(N(T))$.*

(2) *If $0 \notin N(T_r)$, then $n(T) = \max(N(T))$.*

Proof We prove (1), the proof of (2) being similar.

We can find a subdivision tree $T'_\ell \approx T_\ell$ such that $n(T'_\ell)$ is either the smallest positive number in $N(T_\ell)$ or the largest negative number in $N(T_\ell)$. In either case, the hypothesis of Lemma 8.2 applies, and we can replace T'_ℓ by T''_ℓ, where T''_ℓ is sufficiently expanded. We can then find a directed arc p'' from the root of T''_ℓ to a leaf ℓ, where p'' satisfies the conditions from Definition 8.1. We can then replace the tree T_ℓ by T''_ℓ within the tree T, to create a new T' such that $n(T') = n(T)$, $T' \approx T$, and T''_ℓ is the left branch of the tree T'. Now let $p' = ep''$, where e is the edge connecting the root of T' to the root of T''_ℓ. The path p' satisfies all of the conditions of Definition 8.1, and shows that T' is left sufficiently expanded.

If $T'' \approx T$ and $n(T'') < n(T)$, then $n(T'') < n(T')$ and $T'' \approx T'$, which contradicts Proposition 8.3(1). It follows that $n(T) = \min(N(T))$. □

Theorem 8.5 (the intermediate value theorem for M_2) *If T is a nontrivial subdivision tree, then*

$$N(T) = [m, M] \cap \mathbb{Z},$$

where $m = \min(N(T))$ and $M = \max(N(T))$.

Proof Suppose that $k \in (m, M) \cap \mathbb{Z}$ but $k \notin N(T)$. Assume further that k is the minimal such integer.

There is a subdivision tree $T' \approx T$ such that $n(T') = k - 1$. It must be that $0 \notin N(T'_r)$ (otherwise, we can apply an elementary equivalence to produce a tree $T'' \approx T'$ such that $n(T'') = k$). Thus, $k - 1 = n(T') = \max(N(T')) = \max(N(T)) = M$, a contradiction. □

8.2 The case of M_3

In this subsection, we will argue that $N(T)$ is always a set of consecutive integers in the case that T is a subdivision tree over M_3. All of the subdivision trees in question will be subdivision trees over M_3; "C" will refer to C_3, and so forth.

Definition 8.6 (blocking trees) Let T be a subdivision tree. We say that T is a *blocking tree* if either

(1) both of the children of the root of T are nodes, and the three vertices (the root and its children) are not all labelled by 0, or

(2) one of these three vertices is a leaf, or T is trivial.

Definition 8.7 (sufficiently expanded in M_3) A subdivision tree T over M_3 is *sufficiently expanded* if there is a directed arc p from the root ϵ to some leaf ℓ such that

(1) each nonroot node on the arc p is the root of a blocking (sub)tree;

(2) if a nonroot node v on the arc p has a positive label, then the arc p passes through the left child of v;

(3) if a nonroot node v on the arc p has a negative label, then the arc p passes through the right child of v;

(4) if the arc p passes through a nonroot node v labelled by "0", then the next node along p (if any) has a nonzero label.

A sufficiently expanded subdivision tree is *left sufficiently expanded* (respectively, *right sufficiently expanded*) if p passes through the left (respectively, the right) child of ϵ.

Lemma 8.8 *Let T be a subdivision tree.*

(1) *If $n(T) = k$ and $k-1 \notin N(T)$, then there is $T' \approx T$ such that $n(T') = k$ and T' is left sufficiently expanded.*

(2) *If $n(T) = k$ and $k+1 \notin N(T)$, then there is $T' \approx T$ such that $n(T') = k$ and T' is right sufficiently expanded.*

Proof The proof resembles that of Lemma 8.2. We argue by induction on the number of carets in the subdivision tree T. If T consists of a single caret, then it is necessarily both left sufficiently expanded and right sufficiently expanded; thus, the base case is satisfied.

Now consider an arbitrary subdivision tree T, and suppose that the lemma has been proved for all subdivision trees having fewer carets. We assume that T satisfies (1); the case of (2) is similar. Since $k-1 \notin N(T)$, the left branch T_ℓ of T is a blocking tree. (Indeed, all trees in the equivalence class of T_ℓ are blocking trees, by Lemma 6.6.) There are two possibilities for T_ℓ: either $0 \in N(T_\ell)$ or $0 \notin N(T_\ell)$. In the latter case, we can proceed essentially as in the proof of Lemma 8.2. We therefore assume that $0 \in N(T_\ell)$; indeed, we can assume that $n(T_\ell) = 0$ without loss of generality. Since all trees in the equivalence class of T_ℓ are blocking trees, it must be that either $0 \notin N(T_{\ell\ell})$ or $0 \notin N(T_{\ell r})$. We assume that $0 \notin N(T_{\ell\ell})$. Thus, by induction, we can replace $T_{\ell\ell}$ by a subdivision tree $T'_{\ell\ell} \approx T_{\ell\ell}$ such that there is a path $\hat{p}$ from the root of $T'_{\ell\ell}$ to a leaf of $T'_{\ell\ell}$ that satisfies the conditions of Definition 8.7. We let T' be the result of replacing $T_{\ell\ell}$ with $T'_{\ell\ell}$ in T. Now, letting p_1 denote the path from the root of T' to the root of $T'_{\ell\ell}$ and $p = p_1\hat{p}$, the path p shows that T' is left sufficiently expanded, completing the induction. □

Proposition 8.9 *Let T be a subdivision tree.*

(1) *If T is left sufficiently expanded and T' satisfies $n(T') < n(T)$ then $T' \not\approx T$.*

(2) *If T is right sufficiently expanded and T' satisfies $n(T') > n(T)$ then $T' \not\approx T$.*

Proof The proof is no different from that of Proposition 8.3; again the crucial observation is that the left sufficiently expanded tree T has a leaf ℓ whose label is a reduced word, and the corresponding leaf ℓ' in T' has a leaf whose label, after reduction, cannot be equivalent to that of ℓ. □

Proposition 8.10 *Let T be a subdivision tree.*

(1) *If, whenever $T' \approx T_\ell$, T' is a blocking tree, then $n(T) = \min(N(T))$.*

(2) *If, whenever $T' \approx T_r$, T' is a blocking tree, then $n(T) = \max(N(T))$.*

Proof We prove (1), the proof of (2) being similar.

The proof of Lemma 8.8 allows us to replace T with an equivalent $\widehat{T}$ such that $\widehat{T}$ is left sufficiently expanded and $n(T) = n(\widehat{T})$.

Let $\widetilde{T} \approx T$. Thus, $\widetilde{T} \approx \widehat{T}$, so $n(\widetilde{T}) \geq n(\widehat{T})$, by Proposition 8.9. Thus, $n(\widetilde{T}) \geq n(T)$, which implies $n(T) = \min(N(T))$. □

Theorem 8.11 (the intermediate value theorem for M_3) *If T is a subdivision tree over M_3, then*

$$N(T) = [m, M] \cap \mathbb{Z}.$$

Proof Suppose that $k \in (m, M) \cap \mathbb{Z}$ but $k \notin N(T)$. Assume further that k is the minimal such integer.

There is a subdivision tree $T' \approx T$ such that $n(T') = k-1$. It must be that all subdivision trees that are equivalent to T'_r are blocking trees (otherwise, we can first replace T'_r by a nonblocking equivalent tree T''_r, and then apply an elementary equivalence to produce a $T'' \approx T'$ such that $n(T'') = k$). Thus,

$$k-1 = n(T') = \max(N(T')) = \max(N(T)) = M,$$

a contradiction. □

9 The proof of the F_∞ property

In this section, we will complete the proof that the expansion schemes $\mathcal{E}_i$ and $\mathcal{E}'_i$ ($i = 2, 3$) are n-connected for all n. This involves assembling a few pieces from Section 8.

We will also complete the proofs that the groups $F(S_i)$, $F(S'_i)$, $T(S_i)$, $V(S_i)$, and $V(S'_i)$ have type F_∞, for $i = 2, 3$. These proofs are almost entirely like the ones from [6].

Recall that the approach in this paper departed from that of [6] in using a proper subset $\mathcal{D}^+_{\mathrm{gen}}$ of the domains $\mathcal{D}^+$ as the foundation for the original directed set construction. This makes little difference in the final arguments, but rather than simply referring the reader to [6] (which runs to over sixty pages), we will sketch the necessary changes when it seems appropriate to do so.

9.1 Brown's finiteness criterion

Here we briefly recall Brown's finiteness criterion for the reader's convenience.

Theorem 9.1 ([3] Brown's finiteness criterion) *Let X be a CW-complex. Let G be a group acting on X. If*

(1) *X is $(n-1)$-connected,*

(2) *G acts cellularly on X, and*

(3) *there is a filtration $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k \subseteq \cdots \subseteq X$ such that*

(a) $X = \bigcup_{k=1}^{\infty} X_k$,

(b) *G leaves each $X_k^{(n)}$ invariant and acts cocompactly on each $X_k^{(n)}$,*

(c) *each p-cell stabiliser has type F_{n-p}, and*

(d) *for sufficiently large k, X_k is $(n-1)$-connected,*

then G is of type F_n. □

9.2 Contractibility of the complexes $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$

In this subsection, we will prove that the complexes $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$ are contractible, for $i = 2, 3$. This completes a line of argument that was begun at the end of Section 6, and extended through Sections 7 and 8.

Recall that the directed set constructions of the classifying spaces for the groups $F(S)$, $T(S)$, $V(S)$ differed in details (see Section 4). We will use the same notation, $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$, to denote the subcomplexes determined by $\mathcal{E}$- (or $\mathcal{E}'$-) expansions in all cases, trusting that the precise meaning will always be clear from the context.

Theorem 9.2 (contractibility of the complexes $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$) *The complexes $\Delta^{\mathcal{E}_i}$ are contractible, for each of the groups $F(S)$, $T(S)$, and $V(S)$ ($S \in \{S_2, S_3\}$).*

The complexes $\Delta^{\mathcal{E}'_i}$ are contractible, for each of the groups $F(S)$ and $V(S)$ ($S \in \{S'_2, S'_3\}$).

Proof By Theorem 6.19, it suffices to show that the expansion schemes $\mathcal{E}_i$ and $\mathcal{E}'_i$ ($i = 2, 3$) are n-connected for all n. By the discussion at the end of Example 6.22, it suffices to consider the expansion scheme $\mathcal{E}$.

In the groups $F(S_i)$, $T(S_i)$, $V(S_i)$ ($i = 2, 3$), there is just one domain type, namely $[I]$. By equivariance of $\mathcal{E}_i$, it suffices to show that, whenever $\{[\mathrm{id}_I, I]\} < v$, the ascending link of $\{[\mathrm{id}_I, I]\}$ relative to v is contractible.

Since $\{[\mathrm{id}_I, I]\} < v$, v can be represented by a subdivision tree T (by Theorem 6.12). By Theorem 8.5 or 8.11, $N(T) = [m, M] \cap \mathbb{Z}$, for some integers m and M. Thus, $v \geq u_k$ for an integer k if and only if $k \in [m, M]$, where u_k is as defined in Example 6.22).

Now we must determine precisely when $u_{k-1/2} \leq v$. Note first that, if $k - 1/2 \notin [m, M] \subseteq \mathbb{R}$, then $u_{k-1/2} \not\leq v$, since, if it were, we would conclude that $u_{k-1} \leq v$ and $u_k \leq v$ (since $u_{k-1}, u_k \leq u_{k-1/2}$ in the expansion partial order). This contradicts our hypothesis, since at least one of $k - 1$ and k is not in $[m, M]$. Now assume that $k - 1/2 \in [m, M]$. It follows from this that $k - 1, k \in N(T)$, since $u_{k-1} \leq u_{k-1/2}$ and $u_k \leq u_{k-1/2}$ in the expansion partial order. It follows that $m < k \leq M$ (in the linear order on $\mathbb{R}$). If we are in the case $S = S_2$ or S'_2, then Proposition 8.4 and the inequality $u_k \leq v$ show that $0 \in N(T_\ell)$ (since $k - 1 \in N(T)$. Thus, there is some T', $T' \approx T$, such that the root of T' is labelled by k and the left child of the root is labelled by 0. Since T' represents the vertex v, it follows that $u_{k-1/2} \leq v$. If we are in the case $S = S_3$ or S'_3, then Proposition 8.10 and the inequality $u_k \leq v$ show that there is some T', $T' \approx T$, such that the root of T' is labelled by k and the left branch of T' is not a blocking tree. It now follows directly that $u_{k-1/2} \leq v$ in this case, as well.

Thus, the ascending link of $\{[\mathrm{id}_I, I]\}$ relative to v corresponds exactly to the portion of the cellulated line ℓ between the vertices u_m and u_M, where ℓ is as depicted in Figure 5. The ascending link in question is therefore contractible. □

Remark 9.3 We review some of the relevant ideas from [6].

The basic approach to proving n-connectedness is laid out in Lemma 2.6 from [6]. Let $\widehat{\Delta}$ be a simplicial complex whose vertices are a directed set, which we denote by X. Let h be a height function defined on the vertex set, such that $h(v_1) < h(v_2)$ when $v_1 < v_2$. Assume further that $\widehat{\Delta}$ is a subcomplex of the simplicial realisation of the directed set X. Lemma 2.6 from [6] says that $\widehat{\Delta}$ is n-connected if, for every two vertices v_1, v_2 in $\widehat{\Delta}$ such that $v_1 < v_2$, the ascending link of v_1 relative to v_2 (Definition 6.17) is always $(n-1)$-connected.

Lemma 2.6 from [6] applies to our complexes $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$ directly, where the height function h sends a vertex to its cardinality (as in Definition 9.4). It therefore suffices to show that the relative ascending link is n-connected for all n.

We can argue the latter point directly as follows. If

$$v_1 = \{[f_1, D_1], \dots, [f_m, D_m]\}$$

is a vertex of either $\Delta^{\mathcal{E}_i}$ or $\Delta^{\mathcal{E}'_i}$, and $v_1 < v_2$, then we can write

$$v_2 = \bigcup_{k=1}^{m} p_k,$$

where, for $k = 1, \dots, m$, p_k is a pseudovertex having the same support as $\{[f_k, D_k]\}$. The ascending link of v_1 relative to v_2 is homeomorphic to the joins of the ascending links of $\{[f_k, D_k]\}$ relative to p_k, for $k = 1, \dots, m$. (This can be argued exactly as in the proof of Theorem 6.9 from [6].) All of the latter ascending links are contractible if they are nonempty, by the proof of Theorem 9.2 given above. At least one of the latter ascending links is nonempty (since $v_1 \neq v_2$), so the ascending link of v_1 relative to v_2 is contractible, as claimed.

9.3 Γ-finite filtrations of $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$

In this subsection, we will describe natural filtrations of the complexes $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$. We will denote the acting group by Γ; here Γ might be any of the groups $\{F(S), T(S), V(S)\}$, where $S \in \{S_2, S_3, S'_2, S'_3\}$.

Definition 9.4 (Γ-finite filtrations of the complexes $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}'_i}$) Let v be a vertex in $\Delta^{\mathcal{E}_i}$ or $\Delta^{\mathcal{E}'_i}$, or a pseudovertex. We let $|v|$ denote the cardinality of v, which we will call the *height* of v. For $n \geq 1$, we let $\Delta_n^{\mathcal{E}_i}$ denote the subcomplex of $\Delta^{\mathcal{E}_i}$ spanned by vertices of height n or less. Similarly define the subcomplexes $\Delta_n^{\mathcal{E}'_i}$ of $\Delta^{\mathcal{E}'_i}$.

Proposition 9.5 *The group Γ acts on each $\Delta_n^{\mathcal{E}_i}$ (or $\Delta_n^{\mathcal{E}'_i}$, as the case may be) cocompactly, and*

$$\Delta^{\mathcal{E}_i} = \bigcup_{n=1}^{\infty} \Delta_n^{\mathcal{E}_i}.$$

A similar equality is true of $\Delta^{\mathcal{E}'_i}$ and the subcomplexes $\Delta_n^{\mathcal{E}'_i}$.

Proof Let us first note that the Γ-action preserves height, and therefore acts on $\Delta_n^{\mathcal{E}_i}$ (or $\Delta_n^{\mathcal{E}'_i}$). It is easy to see that $\Delta^{\mathcal{E}_i}$ is the union of the subcomplexes in the filtration.

We temporarily let $\mathcal{E}$ denote an arbitrary expansion scheme. Definition 6.12 from [6] describes an action $\star$ of $\mathbb{S}(D, D)$ ($D \in \mathcal{D}_S^+$, or $D \in \mathcal{D}_{\mathrm{gen}}^+$, as in our case) on the set $\mathcal{E}([f, D])$ as

$$h \star v = (fhf^{-1}) \cdot v,$$

where $h \in \mathbb{S}(D, D)$. If the action of $\mathbb{S}(D, D)$ on the simplicial realisation of $\mathcal{E}([f, D])$ is always cocompact, for all D, then $\mathcal{E}$ is said to be $\mathbb{S}$-*finite*.

In the current situation, the group $\mathbb{S}(D, D)$ is isomorphic either to $\mathbb{Z}$ or to the trivial group (when $[D] = [I]$ or $[D] = [[0, \infty)]$, respectively). In either case, the action of $\mathbb{S}(D, D)$ is cocompact. Indeed, the action of $\mathbb{Z}$ on $\mathcal{E}([\mathrm{id}_I, I])$ is by translation (i.e., the integer n moves a vertex u_k to u_{n+k}, where the vertices u_j are as described in Example 6.22). This is clearly cocompact; see Figure 5. This reasoning applies equally to all $[f, D]$ such that $[D] = [I]$ due to the equivariance of the expansion scheme $\mathcal{E}_i$ (or $\mathcal{E}_i'$). If $[D] = [[0, \infty)]$, there is nothing to prove, since the set $\mathcal{E}_i'([f, D])$ is compact. It follows that both $\mathcal{E}_i$ and $\mathcal{E}_i'$ are $\mathbb{S}$-finite.

We can now apply Proposition 6.13 from [6], which says that when an expansion scheme $\mathcal{E}$ is $\mathbb{S}$-finite and $\mathbb{S}$ has finitely many domain types, then the action of Γ on each subcomplex $\Delta_n^{\mathcal{E}}$ is cocompact; this proves that the action of Γ on the filtration is cocompact.

The final equality in the proposition is clear. □

Remark 9.6 We sketch a more direct proof that Γ acts cocompactly.

We assume that $\Gamma = V(S_i)$, the proofs for the other groups being similar. Two vertices v_1 and v_2 are in the same Γ-orbit if and only if they have the same type (Definition 4.6). In the current context, the latter condition is equivalent to having the same height.

Now assume that Γ fails to act cocompactly on $\Delta_n^{\mathcal{E}_i}$, for some n. We note that the dimension of $\Delta_n^{\mathcal{E}_i}$ is no more than $n-1$, since a simplex in $\Delta_n^{\mathcal{E}_i}$ is an ascending chain

$$v_0 < v_1 < v_2 < \cdots < v_k,$$

and the height function strictly increases along such chains. Thus, assuming that the action of Γ is not cocompact, there are infinitely many Γ-orbits of k-simplices, for some k. Since there are only finitely many Γ-orbits of vertices, this implies that there is a vertex $v' = \{b_1, \dots, b_\ell\}$ such that infinitely many Γ-orbits of k-simplices have v' as their minimal vertex. This, however, sets up the contradiction, since all of the k-simplices in question are obtained by $\mathcal{E}_i$-expansion from v', and there are only finitely many such $\mathcal{E}_i$-expansions modulo the action $\star$.

The details of the remainder of the argument follow that of the proof of Proposition 6.13 from [6].

9.4 The F_∞ property for $V(S_n)$ and $V(S_n')$

Definition 9.7 (contracting pseudovertices) Let $\mathcal{E}$ be an arbitrary expansion scheme. We say that a pseudovertex v is *contracting relative to* $\mathcal{E}$ if v has the same type as some $w \in \mathcal{E}(b)$, where $b \in \mathcal{B}$. (Recall that "same type" was defined in Definition 4.6.)

Definition 9.8 (rich in contractions) Let $\mathcal{E}$ be an expansion scheme. We say that $\mathcal{E}$ is *rich in contractions* if there is some constant C such that, if v is a pseudovertex of height at least C, then there is some contracting pseudovertex v' such that $v' \subseteq v$.

Theorem 9.9 ([6, Theorem 8.2], groups of type F_∞) *Let $\mathbb{S}$ be an S-structure with finitely many domain types, such that the group $\mathbb{S}(D, D)$ has type F_∞ for $D \in \mathcal{D}^+$. Let $\mathcal{E}$ be an expansion scheme such that*

(1) *$\mathcal{E}$ is n-connected for all n;*

(2) *$\mathcal{E}$ is rich in contractions;*

(3) *each set $\mathcal{E}(b)$ ($b \in \mathcal{B}$) is finite.*

The group Γ_S has type F_∞.

Theorem 9.10 *The groups $V(S_i)$ and $V(S_i')$ are of type F_∞, for $i = 2, 3$.*

Proof Our strategy is to apply the proof of Theorem 9.9 (Theorem 8.2 from [6]) to the groups Γ. (We note that the groups Γ_S under consideration in Theorem 9.9 are analogous to Thompson's group V, in that there is no assumption that Γ_S preserves a linear or cyclic order.) Let us note that condition (3) is violated, since the sets $\mathcal{E}_i(b)$ and $\mathcal{E}_i'(b)$ are not finite when $b = [f, D]$ and $[D] = [I]$, so the statement does not apply directly.

We have already seen that $\mathcal{E}_i$ and $\mathcal{E}_i'$ are n-connected expansion schemes for all n.

We claim that the expansion scheme $\mathcal{E}_i$ is rich in contractions with constant $C = 2$ when $i = 2$ or 3. Let $\{[f_1, D_1], [f_2, D_2]\} \subseteq \mathcal{B}$ be a pseudovertex. Since $D_1, D_2 \in \mathcal{D}^+_{\mathrm{gen}}$, we have $D_1 = \omega_1 I$ and $D_2 = \omega_2 I$, for some words $\omega_1, \omega_2 \in \{A, B\}^*$. Thus,

$$[f_n, D_n] = [f_n, \omega_n I] = [f_n \omega_n, I],$$

for $n = 1, 2$. Define g on $[0, 1)$ by the rule

$$g(x) = \begin{cases} f_1\omega_1 a(x) & \text{if } x \in [0, 1/2), \\ f_2\omega_2 b(x) & \text{if } x \in [1/2, 1). \end{cases}$$

The pseudovertex $\{[g, I]\}$ expands to

$$\begin{aligned} \{[g, AI], [g, BI]\} &= \{[f_1\omega_1 a, AI], [f_2\omega_2 b, BI]\} \\ &= \{[f_1\omega_1, I], [f_2\omega_2, I]\} \\ &= \{[f_1, D_1], [f_2, D_2]\}. \end{aligned}$$

This proves the claim.

The expansion scheme $\mathcal{E}_i'$ is also rich in contractions with constant $C = 2$. If $\{[f_1, D_1], [f_2, D_2]\} \subseteq \mathcal{B}$ is a pseudovertex and D_1, D_2 have the same domain type as I, then the proof of the previous paragraph shows that a contraction can be performed on $\{[f_1, D_1], [f_2, D_2]\}$. The only remaining case to consider is when $[D_1] = [I]$ and $[D_2] = [[0, \infty)]$. We will write R in place of $[0, \infty)$, to simplify notation. In this case, $D_1 = \omega_1 I$ and $D_2 = T^m R$, where $\omega_1 \in \{A, B, T\}^*$ and $m \geq 0$. We have

$$[f_1, \omega_1 I] = [f_1\omega_1, I] \quad \text{and} \quad [f_2, T^m R] = [f_2 T^m, R].$$

Define $g : [0, \infty) \to [0, \infty)$ as

$$g(x) = \begin{cases} f_1\omega_1(x) & \text{if } x \in [0, 1), \\ f_2 T^{m-1}(x) & \text{if } x \in [1, \infty). \end{cases}$$

The pseudovertex $\{[g, R]\}$ expands to

$$\begin{aligned} \{[g, R]\} &= \{[g, I], [g, TR]\} \\ &= \{[f_1\omega_1, I], [f_2 T^{m-1}, TR]\} \\ &= \{[f_1, D_1], [f_2, T^m R]\} \\ &= \{[f_1, D_1], [f_2, D_2]\}. \end{aligned}$$

It follows that $\{[f_1, D_1], [f_2, D_2]\}$ is also a contracting vertex relative to $\mathcal{E}_i'$.

The assumption that $\mathcal{E}(b)$ is always finite is used in the proof of Theorem 9.9 in three ways:

(1) to prove that Γ acts cocompactly on the complexes $\Delta_n^{\mathcal{E}}$;

(2) to prove that the cell stabilisers have type F_∞, and

(3) to define a certain constant C_0.

We have already established (1) and (2) by other means: indeed, cell stabilisers are virtually finitely generated free abelian groups, and therefore have type F_∞, and the cocompactness of the actions on the complexes $\Delta_n^{\mathcal{E}_i}$ and $\Delta_n^{\mathcal{E}_i'}$ was proved as part of Proposition 9.5. The constant C_0 is the largest height (i.e., cardinality) of a contracting pseudovertex. Clearly we have an independent bound of $C_0 = 3$ when $S \in \{S_2, S_2'\}$, or $C_0 = 5$ when $S \in \{S_3, S_3'\}$. (Refer to the definitions of $\mathcal{E}_i$ and $\mathcal{E}_i'$ in Example 6.22.) □

Remark 9.11 We will offer a sketch of the argument here. This sketch is intended to remove some of the dependence on Theorem 9.9.

We check the hypotheses of Brown's finiteness criterion (Theorem 9.1). First, we note that $\Delta^{\mathcal{E}_i}$ and $\Delta^{\mathcal{E}_i'}$ are contractible by Theorem 9.2. It is clear that the relevant actions are cellular. Properties (3)(a) and (b) are settled in Proposition 9.5. Cell stabilisers are virtually finitely generated free abelian (and therefore of type F_∞) by Proposition 4.13.

This leaves only (3)(d) to check; i.e., we must show, for each $n \in \mathbb{N}$, that $\Delta_k^{\mathcal{E}_i}$ and $\Delta_k^{\mathcal{E}_i'}$ are $(n-1)$-connected for sufficiently large k. The proof of the latter follows a now-standard strategy: we show that the descending link of a vertex becomes highly connected as the height of the vertex increases. A few basics of this strategy are summarised in Subsection 7.2 of [6], although the methods of argument go back to [3; 1].

We consider $\Delta^{\mathcal{E}_2}$; the other cases are similar. Let v be a vertex of height k in $\Delta^{\mathcal{E}_2}$. The *descending link* of v is its link in $\Delta_k^{\mathcal{E}_2}$. Our analysis of the descending link uses the nerve theorem (as it appears in [2]; the nerve theorem is also Theorem 2.10 in [6]). Let

$$v = \{b_1, \dots, b_k\}.$$

We cover the descending link of v by a number of subcomplexes, called *partitioned downward links*, which are each determined by a partition of v, and which we now define.

Let $\mathcal{P}$ be a partition of v. The *partitioned downward star* $\mathrm{st}_{\downarrow}(v_{\mathcal{P}})$ (Definition 7.7 from [6]), is the subcomplex of $\Delta_k^{\mathcal{E}_2}$ consisting of the vertex v and all simplices resulting from $\mathcal{E}_2$-contractions that are supported within members of $\mathcal{P}$. For instance, if

$$\mathcal{P} = \{\{b_1, b_2\}, \{b_3, \dots, b_k\}\},$$

then a contraction supported on the subset $\{b_1, b_2\}$, or on the subset $\{b_3, b_4, b_7\}$ (if $k \geq 7$) (or indeed a combination of such contractions), results in a simplex of $\mathrm{st}_{\downarrow}(v_{\mathcal{P}})$, but a contraction supported on $\{b_2, b_3\}$ would not. We then define the partitioned downward link $\mathrm{lk}_{\downarrow}(v_{\mathcal{P}})$ as the link of v in $\mathrm{st}_{\downarrow}(v_{\mathcal{P}})$.

For each contracting pseudovertex $w \subseteq v$, we let

$$\mathcal{P}_w = \{v - w, w\}.$$

(We note that, in the current context, "contracting pseudovertex" is the same as "pseudovertex with two or three members", by the description of $\mathcal{E}_2$ from Example 6.22.) The collection

$$\mathcal{C} = \{\mathrm{lk}_{\downarrow}(v_{\mathcal{P}_w}) \mid w \text{ is a contracting pseudovertex}\}$$

is a cover of $\mathrm{lk}_{\downarrow}(v)$. We apply the nerve theorem to $\mathcal{C}$. The intersection of two members of $\mathcal{C}$ is another partitioned downward link,

$$\mathrm{lk}_{\downarrow}(v_{\mathcal{P}_{w'}}) \cap \mathrm{lk}_{\downarrow}(v_{\mathcal{P}_{w''}}) = \mathrm{lk}_{\downarrow}(v_{\mathcal{P}_{w'} \wedge \mathcal{P}_{w''}}),$$

where $\mathcal{P}_{w'} \wedge \mathcal{P}_{w''}$ is the coarsest common refinement of $\mathcal{P}_{w'}$ and $\mathcal{P}_{w''}$. The generalisation to finite intersections is straightforward.

For a partition $\mathcal{P} = \{P_1, \dots, P_\ell\}$ of v, there is a natural join structure (see Corollary 7.9 from [6]),

$$\mathrm{lk}_{\downarrow}(v_{\mathcal{P}}) \cong \mathop{\ast}_{j=1}^{\ell} \mathrm{lk}_{\downarrow}(P_j),$$

where the latter descending links depend only on the types of the pseudovertices P_j. (In the current case, the type is entirely determined by the cardinality.) Recall that, if X_1 and X_2 are n_1-connected and n_2-connected complexes (respectively), then the join $X_1 * X_2$ is $(n_1{+}n_2{+}2)$-connected. It follows that $\mathrm{lk}_{\downarrow}(v_{\mathcal{P}})$ is at least as connected as the most highly connected factor $\mathrm{lk}_{\downarrow}(P_j)$.

Finally, we note that a pseudovertex of height two or more has a nonempty descending link (since every such pseudovertex contains a contracting pseudovertex). This gives us the base case of an induction; the above considerations allow us to prove inductively that pseudovertices of increasing height have increasing connectivity. The actual induction is done in the proof of Theorem 8.2 from [6]; we omit further details. We will consider a similar induction in more detail in the next subsection.

9.5 The F_∞ property for the remaining groups

Theorem 9.12 *The groups $F(S)$, where $S \in \{S_2, S_3, S_2', S_3'\}$, and $T(S)$, where $S \in \{S_2, S_3\}$, have type F_∞.*

Proof We consider the group $F(S_2)$. The proofs that the other groups have type F_∞ differ in minor details.

We turn to an analysis of the descending link; all of the other ingredients of the proof can be assembled exactly as in Remark 9.11. Let

$$v = \{b_1, b_2, \dots, b_k\}$$

be either a vertex of $\Delta^{\mathcal{E}_2}$, or a pseudovertex. We assume that the b_i are linearly ordered, in the following sense: Each $b_i = [f_i, D_i]$, for appropriate f_i and $D_i \in \mathcal{D}^+_{\mathrm{gen}}$, where $f_i : D_i \to [0,1)$ is a locally S_2-embedding that is, moreover, continuous and increasing. We assume that $f_1(D_1), f_2(D_2), \dots, f_k(D_k)$ are arranged from left to right. With this assumption, each $\mathcal{E}_2$-contraction must be performed on two or three consecutive b_i. For a subset $K \subseteq \{1, \dots, k\}$, we define

$$\mathcal{P}_K = \{\{b_j \mid j \in K\}, \{b_j \mid j \notin K\}\}.$$

We then define

$$\mathcal{C} = \big\{\mathrm{lk}_\downarrow(v_{\mathcal{P}_K}) \mid K \in \{\{1,2\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{3,4,5\}\}\big\}.$$

(If $k < 5$, then the possible subsets K are restricted accordingly.)

We claim that $\mathcal{C}$ is a cover of $\mathrm{lk}_\downarrow(v)$. Indeed, let σ be a simplex in $\mathrm{lk}_\downarrow(v)$. Thus, there is an increasing sequence

$$v_0 < v_1 < v_2 < \cdots < v_{\ell-1} < v_\ell = v,$$

where each v_α is obtained by $\mathcal{E}_2$-expansion from v_0, and

$$\sigma = v_0 < v_1 < \cdots < v_{\ell-1}.$$

If $v_0 = \{b_0', b_1', \dots, b_q'\}$, where the members are linearly ordered, then there is a leftmost b_β' that is expanded when we pass from v_0 to v. In expanding at b_β', we replace b_β' with either two or three pairs from $\mathcal{B}$. The latter will occur consecutively in v. Thus, the result of expanding at b_β' will contribute either $\{b_\alpha, b_{\alpha+1}, b_{\alpha+2}\}$ or $\{b_\alpha, b_{\alpha+1}\}$ to v, for some α. All other expansions from v_0 to v will contribute a disjoint subset of b_i's to v. It follows easily from this that σ is contained in at least one member of the cover $\mathcal{C}$.

For instance, if the expansion at b_β' contributes b_1 and b_2 to v, any other expansion from v_0 must contribute some subset of $\{b_3, \dots, b_k\}$. Thus, in this case, $\sigma \subseteq \mathrm{lk}_\downarrow(v_{\mathcal{P}_{\{1,2\}}})$. If b_β' contributes b_2 and b_3, then $\sigma \subseteq \mathrm{lk}_\downarrow(v_{\mathcal{P}_{\{2,3\}}})$. If b_β' contributes b_m and b_{m+1}, for some $m \geq 3$, then $\sigma \subseteq \mathrm{lk}_\downarrow(v_{\mathcal{P}_{\{1,2\}}})$ (since, indeed, all expansions contribute some subset of $\{b_3, \dots, b_k\}$ under this hypothesis).

Now we establish the connectivity of the descending link, as a function of the height k. We note first that $\mathrm{lk}_{\downarrow}(v)$ is nonempty provided that $k \geq 2$. It follows from this that each $\mathrm{lk}_{\downarrow}(v_{P_K})$ is connected when $k \geq 7$, since each is a join of two nonempty complexes. Now, if $k \geq 7$, then $\mathrm{lk}_{\downarrow}(v)$ is connected, since it is covered by a collection $\mathcal{C}$ of nonempty subcomplexes, which have a nonempty intersection. (A contraction at $\{b_6, b_7\}$ lies in all of the partitioned descending links simultaneously.)

In general, $\mathrm{lk}_{\downarrow}(v)$ is n-connected provided that $k \geq 5n+7$. We have proved this already for $n=-1$ and $n=0$. Assume that the result is true for n. We consider a vertex v of height k at least $5n+12$. Each $\mathrm{lk}_{\downarrow}(v_{P_K})$ is $(n+1)$-connected, since each is a join of two complexes: one nonempty and one isomorphic to the descending link of a vertex of height at least $5n+7$, and therefore n-connected by induction. Moreover, any subcollection of $\mathcal{C}$ containing two of more members intersects in a subcomplex that is at least n-connected. (Any such intersection is a join, and one of the factors of the join is the descending link on $\{b_\gamma, \dots, b_k\}$, where $\gamma \leq 6$.)

By the nerve theorem [2], $\mathrm{lk}_{\downarrow}(v)$ is $(n+1)$-connected if t-fold intersections of the cover are $(n-t+2)$-connected and the nerve of the cover is $(n+1)$-connected. Since the nerve is easily seen to be a four-dimensional simplex, and t-fold intersections have the required connectivity (by the previous paragraph), $\mathrm{lk}_{\downarrow}(v)$ is $(n+1)$-connected, completing the induction.

By well-established principles (as summarised in Proposition 7.6 from [6], for instance), the connectivity of the subcomplex $\Delta_k^{\mathcal{E}_i}$ tends to infinity as k increases, completing the proof. □

10 The case of the Lodha–Moore group

Recall that $F(S_2')$ is the group of homeomorphisms of $[0,\infty)$ that is locally determined by the inverse semigroup generated by the set

$$\{A, B, C_2, T\},$$

where all of these are as defined in Definition 2.6. In this section, we will prove the following theorem:

Theorem 10.1 *The Lodha–Moore group G is isomorphic to an ascending HNN extension of $F(S_2')$ in which the stable letter is the translation $t \mapsto t+1$.*

In particular, G has type F_∞.

Throughout this section, we let $F(S_2')$ act on the entire real line, by simply defining each element of $F(S_2')$ to be the identity on $(-\infty, 0]$.

Definition 10.2 [8; 9] The *Lodha–Moore group G* is the group of homeomorphisms of the real line generated by three transformations, denoted by a, b, and c, and defined as

$$a(t) = t+1, \qquad b(t) = \begin{cases} t & \text{if } t \leq 0, \\ \frac{t}{1-t} & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 3 - \frac{1}{t} & \text{if } \frac{1}{2} \leq t \leq 1, \\ t+1 & \text{if } 1 \leq t, \end{cases} \qquad c(t) = \begin{cases} t & \text{if } t \leq 0, \\ \frac{2t}{t+1} & \text{if } 0 \leq t \leq 1, \\ t & \text{if } 1 \leq t. \end{cases}$$

Remark 10.3 We are using the notation a, b, and c for the generators of the Lodha–Moore group G, following the practice in [8; 9]. Of course, this notation contradicts our own practice of using lowercase "x" to denote the inverse of the partial transformation X (for $X \in \{A, B, C_2\}$). We will therefore continue to use the uppercase letters to denote our inverse semigroup generators, but will use "X^{-1}" to denote the inverse of X. For the remainder of the paper, a, b, and c refer to the generators in Definition 10.2.

Let us note that

$$b = A^{-1} \cup TB^{-1} \cup T^2T^{-1} \quad \text{and} \quad c = C \cup TT^{-1},$$

where union is interpreted in an obvious sense: b agrees with A^{-1} on $[0, 1/2)$ (the domain of A^{-1}), with TB^{-1} on $[1/2, 1)$ (the domain of TB^{-1}), etc. (The trivial action on negative numbers is implied in both of these definitions of b and c).

It follows directly that $\langle b, c\rangle \leq F(S_2')$.

Lemma 10.4 *Let $\mathcal{P}$ be a partition of $[0,\infty)$ into generating domains (Definition 3.1). There is an element $g \in \langle b, c\rangle$ such that either*

(1) *the leftmost member of $g(\mathcal{P})$ is $[0, 1)$, or*

(2) *the leftmost member of $g(\mathcal{P})$ is $[0,\infty)$.*

In either case, it can be arranged that the restriction $g_{|P} : P \to g(P)$ is in S_2', for each $P \in \mathcal{P}$, and that $g(\mathcal{P})$ is also a partition of $[0,\infty)$ into generating domains.

Proof It is rather clear from Definition 3.1 that a partition of $[0,\infty)$ by generating domains has members of two types:

(i) a (necessarily unique) member of the form $[n,\infty)$, for some integer $n \geq 0$, and

(ii) a collection of generating domains of the form $T^\alpha\omega \cdot [0, 1)$, where $\omega \in \{A, B\}^*$ (i.e., ω is a positive, possibly empty, word in the alphabet $\{A, B\}$), and $0 \leq \alpha < n$. Each domain of this form is contained in $[\alpha, \alpha + 1)$.

If $n = 0$, then $\mathcal{P} = \{[0,\infty)\}$, and we can simply let $g = \mathrm{id}_{\mathbb{R}}$. It is clear that g satisfies all of the required properties. (This uses the fact that the restriction of id to each generating domain is a member of S_2', which follows from Remark 2.2.)

Now suppose that $n > 0$. We prove the lemma by induction on the number m of generating domains of $\mathcal{P}$ that are contained in $[0, 1)$. If $m = 1$, then we can let $g = \mathrm{id}_{\mathbb{R}}$. Now suppose that $m > 1$. There are two types of generating domains of $\mathcal{P}$ that are contained in $[0, 1)$: those of the form $A\omega \cdot [0, 1)$, and those of the form $B\omega \cdot [0, 1)$. Both types must be present, since those of the first form are contained in $[0, 1/2)$, while those of the second form are contained in $[1/2, 1)$. We apply the transformation b from Definition 10.2. Using the description of b from Remark 10.3, we find that each domain $A\omega \cdot [0, 1)$ is carried to $\omega \cdot [0, 1)$, and each domain $B\omega \cdot [0, 1)$ is sent to $T\omega \cdot [0, 1)$. The element b acts on every generating domain of $\mathcal{P}$ in $[1,\infty)$ by the translation T. It follows that b carries the set $\mathcal{P}$ of generating domains to another set of generating domains, $b(\mathcal{P})$. We note that the restriction of b to each member of $\mathcal{P}$

is a member of the inverse semigroup S_2', and that the number m is reduced in the process (since each domain of the form $B\omega\cdot[0,1)$ is carried outside of $[0,1)$). We can then apply the inductive hypothesis to the partition $b(\mathcal{P})$ to produce an element $g\in\langle b,c\rangle$ such that $gb(\mathcal{P})$ has the required form, while the restriction of gb to each member of $\mathcal{P}$ is a member of S_2'. This completes the induction. $\square$

Remark 10.5 The element g produced in the proof of Lemma 10.4 is always a nonnegative power of b.

Proposition 10.6 *The group $F(S_2')$ is a subgroup of the Lodha–Moore group G.*

Proof Let $f\in F(S_2')$. There is a partition $\mathcal{P}_1$ of $[0,\infty)$ into finitely many generating domains such that $f_{|P}\in S_2'$ for each $P\in\mathcal{P}_1$, and such that $f(\mathcal{P}):=\mathcal{P}_2$ is also a partition of $[0,\infty)$ into generating domains. (This follows from Definition 2.8 and Remark 3.4.)

We will prove that $f\in G$ by induction on $|\mathcal{P}_1|$. If $|\mathcal{P}_1|=1$, then $f=\mathrm{id}_{\mathbb{R}}$, and $f\in G$. Let $|\mathcal{P}_1|=m$. Lemma 10.4 allows us to find $g_1,g_2\in G$ such that, for $i=1,2$,

(1) $g_i(\mathcal{P}_i)$ is a partition of $[0,\infty)$ into generating domains;

(2) $g_{i|P}\in S_2'$, for each $P\in\mathcal{P}_i$;

(3) the leftmost member of $g_i(\mathcal{P}_i)$ is $[0,1)$.

It follows from this that the element $g_2fg_1^{-1}$ carries the generating domain $[0,1)$ to the generating domain $[0,1)$ by a member of S_2'. By the characterisation of $\mathbb{S}(I,I)$ (Theorem 4.3), the restriction of $g_2fg_1^{-1}$ to $[0,1)$ is C^k, for some $k\in\mathbb{Z}$. It follows that $c^{-k}g_2fg_1^{-1}$ is equal to the identity on $[0,1)$. (Here we are using "c" to refer to the generator of G, as in Definition 10.2.) We note that $c^{-k}g_2fg_1^{-1}\in F(S_2')$ by construction, and that the domain and range of $c^{-k}g_2fg_1^{-1}$ are both partitioned into m pieces, each of which is a generating domain, and such that $c^{-k}g_2fg_1^{-1}$ matches these pieces by members of S_2'.

It follows from this that $a^{-1}c^{-k}g_2fg_1^{-1}a$ is a member of $F(S_2')$ that is similarly defined on $m-1$ pieces. It follows by induction that $a^{-1}c^{-k}g_2fg_1^{-1}a=g_3$, for some $g_3\in G$. Solving the latter equation for f, we find that $f\in G$, completing the induction. $\square$

Proof of Theorem 10.1 Since $\langle b,c\rangle\le F(S_2')$ by Remark 10.3 and $F(S_2')\le G$ by Proposition 10.6, $\langle a,F(S_2')\rangle=G$.

The map $g\mapsto aga^{-1}$ determines an injective endomorphism of $F(S_2')$. There is an induced homomorphism from the resulting HNN extension $F(S_2')_{*a}$ to G. Clearly this map is surjective by the previous paragraph. Injectivity follows from the fact that each nontrivial normal form $a^{-k}fa^{\ell}$ $(k,\ell\ge 0)$ maps to a nontrivial element of G. This proves that G is the required HNN extension of $F(S_2')$.

The fact that G has type F_∞ now follows from the fact that $F(S_2')$ has type F_∞ by Theorem 9.12 and from the fact that the ascending HNN extension of a type F_∞ group also has type F_∞. $\square$

Acknowledgements

I would like to thank Andy Moawad for suggesting corrections to an earlier version of this manuscript. Matt Zaremsky also suggested corrections, and pointed me to his article [10] with Yash Lodha, in which the authors proved that a certain group of piecewise projective homeomorphisms of the circle has type F_∞.

Their results anticipate the proof offered here that $T(S_2)$ has type F_∞. The question of whether a certain "V-like" Lodha–Moore group has type F_∞ is a conjecture in [10], which our proof that $V(S_2)$ has type F_∞ appears to resolve. The referee notes that, since the Lodha–Moore group is an ascending HNN extension of $F(S_2')$, the latter group must also be nonamenable, and have no free subgroups. Theorem 9.12 proves that $F(S_2')$ also has type F_∞. I do not know whether $F(S_2')$ is a *new* group with the indicated properties, however.

I thank the referee for suggesting numerous corrections and clarifications to an earlier draft of this paper.

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Received: March 5, 2023 Revised: June 14, 2024

Algebraic & Geometric Topology **26**:4 (2026) 1451–1464
DOI: 10.2140/agt.2026.26.1451
Published: April 25, 2026

Skew-rack cocycle invariants of closed 3-manifolds

TAKEFUMI NOSAKA

We establish a new approach to obtain 3-manifold invariants by means of Dehn surgery. In this approach, we introduce skew-racks with good involution and property FR, and define cocycle invariants as 3-manifold invariants.

1 Introduction

Every closed 3-manifold M with orientation can be obtained from a framed link in the 3-sphere S^3 by means of Dehn surgery. Since there is a one-to-one correspondence between closed 3-manifolds and framed links in the S^3 modulo either the Kirby moves [14] or Fenn–Rourke moves [7], any framed link invariant, which is invariant with respect to the moves, is a 3-manifold invariant. For example, in quantum topology, frameworks based on the Chern–Simons theory have produced many 3-manifold invariants, including the concepts of modular categories (see [16; 17]). In contrast, when examined from more classical viewpoints such as algebraic topology, the fundamental groups $\pi_1(M)$ of 3-manifolds contain useful information and are strong invariants. Further, as in the Dijkgraaf–Witten model [6], starting from a finite group G, we can define a certain weight of the set $\mathrm{Hom}(\pi_1(M), G)$ in terms of the group cohomology of G. However, apart from the quantum invariants and fundamental groups, there are relatively few procedures that yield 3-manifold invariants via Dehn surgery.

In this paper, we establish a new approach from Dehn surgery to yield 3-manifold invariants in a classical situation. In our approach, we focus on a class of skew-racks (see Section 3), which is an algebraic system, and a modification of quandles and biracks. As in quandle theory [4; 5; 9; 15], starting from skew-racks, we can define a set of colorings of framed links and weights of a set, where the weights are evaluated by birack 2-cocycles and are called *a cocycle invariant*, as a framed link invariant (see Section 3 for details). The aim of our study is to explore skew-racks such that the cocycle invariant is stable under the Fenn–Rourke moves. To this end, we define the property FR of skew-racks (Definition 4.1) and demonstrate (Theorem 4.2 and Proposition 6.2) that, in some situations, the associated cocycle invariant gives rise to a 3-manifold invariant. In Section 4, we establish several examples of skew-racks with property FR; for instance, from a group G and an involutive automorphism $\kappa : G \to G$, we can define a skew-rack with property FR (Examples 2.2 and 5.4).

Using the examples of skew-racks, we compute a set of colorings and several cocycle invariants—for example, we determine the invariants of the Brieskorn 3-manifolds as integral homology 3-spheres (Example 5.6). Following the computations, we present a comparison with the Dijkgraaf–Witten invariant

MSC2020: 16T25, 19J25, 20J06, 57K31.

and pose several problems (Problems 6.8 and 7.3). Finally, we attempt to make an application from the skew-racks above; in Section 7, we suggest several elementary approaches to find 3-manifolds, which are not the results of surgery of any knot in S^3. However, we were ultimately unable to find any examples of their application.

Conventional notation Every 3-manifold is understood to be connected, smooth, oriented, and closed.

2 Symmetric skew-racks and birack cocycle invariants

We introduce skew-racks as a special class of biracks (see [5; 7] for the definition of biracks). We define *a skew-rack* as a triple of a set X, a binary operation $\lhd : X \times X \to X$, and a bijection $\kappa : X \to X$ satisfying the following three axioms:

(SR1) For any $a, b \in X$, the equality $\kappa(a \lhd b) = \kappa(a) \lhd \kappa(b)$ holds.

(SR2) For any $b \in X$, the map $X \to X$ that sends x to $x \lhd b$ is a bijection.

(SR3) For any $a, b, c \in X$, the distributive law $(a \lhd b) \lhd c = (a \lhd \kappa(c)) \lhd (b \lhd c)$ holds.

As a special case, if $\kappa = \mathrm{id}_X$, the definition of skew-racks coincides with that of racks. We often denote the inverse map $\bullet \lhd b$ of the bijection as $\bullet \lhd^{-1} b$. Further, as a slight generalization of symmetric quandles in [12; 13], we define *a symmetric skew-rack* as a pair of a skew-rack $(X, \lhd, \kappa)$ and an involution $\rho : X \to X$ satisfying the following:

(SS1) For any $a, b \in X$, the equalities $(a \lhd b) \lhd \rho(b) = a$ and $\rho(a) \lhd \kappa(b) = \rho(a \lhd b)$ hold.

(SS2) The involutivity $\rho \circ \rho = \kappa \circ \kappa = \mathrm{id}_X$ and the commutativity $\rho \circ \kappa = \kappa \circ \rho$ hold.

Such a ρ is called *a good involution* (as in [12]). If $\kappa = \mathrm{id}_X$ and the equality $a \lhd a = a$ holds for any $a \in X$, the definition of symmetric biracks is the same as the original definition of symmetric quandles [12]. A few examples of symmetric skew-racks are as follows.

Example 2.1 Let X be a group G and let $\kappa : G \to G$ be an involutive automorphism. Define $x \lhd y$ by $\kappa(y^{-1})xy$, and $\rho(x)$ by x^{-1}. These maps then define a symmetric skew-rack structure on X.

Example 2.2 Let K be a group and let $f : K \to K$ be an involutive automorphism. Consider the direct products $X = K \times K$ and $\kappa = f \times f$. Define $(x, a) \lhd (y, b)$ by $(f(x)y^{-1}by, f(a))$ and $\rho(x, a) = (f(x), f(a)^{-1})$. Then, these $X, \lhd, \kappa, \rho$ define a symmetric skew-rack structure on $K \times K$. As discussed later (Sections 5–6), this skew-rack plays a key role in this paper.

Finally, we conclude this section by defining a bijection $\mathrm{Tw} : X \to X$ as follows:

Proposition 2.3 *Let $(X, \lhd, \kappa)$ be a skew-rack satisfying $\kappa^2 = \mathrm{id}_X$, as in* (SS2). *Define the map $\mathrm{Tw} : X \to X$ by setting $\mathrm{Tw}(x) = \kappa(x) \lhd^{-1} \kappa(x)$. Then, the map is bijective, where the inverse is the map $X \to X$ that sends x to $\kappa(x) \lhd x$.*

Proof When we let y be $\mathrm{Tw}(\kappa(x) \lhd x)$, we may show $y = x$. Note that $x \lhd \kappa(x) = y \lhd (x \lhd \kappa(x))$, which is equal to

$$\big((y \lhd^{-1} x) \lhd x\big) \lhd (x \lhd \kappa(x)) = \big((y \lhd^{-1} x) \lhd x\big) \lhd \kappa(x) = y \lhd \kappa(x).$$

Thus, by (SS2), we have $y = x$. Similarly, we can easily verify $\kappa(\mathrm{Tw}(x)) \lhd \mathrm{Tw}(x) = x$. □

3 Preliminaries: colorings and birack cocycle invariants

Our definition of X-colorings here is a slight modification of the classical X-colorings of quandles or biracks [4; 5; 8]. Let D be a framed link diagram D, and let $(X, \lhd, \kappa, \rho)$ be a symmetric skew-rack. Choose orientations o for each component of D, and denote by D^o the diagram with the orientations. In this paper, a *semiarc of* D means a path from a crossing to the next crossing along the diagram. Then, *an* X*-coloring* is a map $\mathcal{C} : \{\text{semiarc of } D\} \to X$ such that, for every crossing τ of D, the semiarcs around τ satisfy $\mathcal{C}(\gamma_\tau) = \kappa(\mathcal{C}(\beta_\tau))$ and $\mathcal{C}(\delta_\tau) = \mathcal{C}(\alpha_\tau) \lhd \mathcal{C}(\beta_\tau)$, where α_τ, β_τ, γ_τ, and δ_τ are the semiarcs shown in Figure 1. We denote by $\mathrm{Col}_X(D^o)$ the set of X-colorings of D^o. Then, as a basic fact in quandle theory (see [5; 8]), if two diagrams D^o and $(D')^{o'}$ are related by a Reidemeister move of type II, type III, or a doubled type I, there exists a canonical bijection $\mathcal{B}_{D^o,(D')^{o'}} : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X((D')^{o'})$. Moreover, thanks to the above axioms (SS1) and (SS2), if $D^{o'}$ is the same diagram D with opposite orientation, the correspondence $a \mapsto \rho(a)$ on the color of each semiarc on the opposite component defines a bijection $\mathcal{B}_{D^o,D^{o'}} : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X(D^{o'})$. In particular, the set $\mathrm{Col}_X(D^o)$ up to bijections does not depend on the choice of orientations of D. Accordingly, we sometimes use the expression $\mathrm{Col}_X(D)$ instead of $\mathrm{Col}_X(D^o)$. Finally, we should emphasize that the map $\mathrm{Tw}^{\pm 1}$ in Proposition 2.3 corresponds to an addition of a (∓ 1)-framing in an arc, as in the Reidemeister move of type I.

Next, we observe cocycle invariants of a symmetric skew-rack X. According to [4; 5; 9], a map $\phi : X^2 \to A$ for some abelian group A is called *a birack* 2*-cocycle* if

$$\text{(1)} \qquad \phi(a,b) + \phi(a \lhd b, c) = \phi(a, \kappa(c)) + \phi(a \lhd \kappa(c), b \lhd c), \qquad \phi(b,c) = \phi(\kappa(b), \kappa(c))$$

hold for any $a, b, c \in X$. Then, we define *the weight* (*of* τ), $\Phi(\tau)$, with respect to a crossing τ on D to be $\varepsilon_\tau \phi(\mathcal{C}(\alpha_\tau), \mathcal{C}(\beta_\tau)) \in A$, where ε_τ is the sign τ (as in Figure 1). We further define $\Phi_D(\mathcal{C}) \in A$ to be the sum $\sum_\tau \Phi(\tau)$, where τ runs over every crossing on D. Then, as is known [3; 5], if two diagrams D and D' are related by a Reidemeister move of type II, type III, or a doubled type I move, $\Phi_{D'} \circ \mathcal{B}_{D^o,(D')^{o'}} = \Phi_D$ holds as a map $\mathrm{Col}_X(D^o) \to A$. In other words, the map $\Phi_D : \mathrm{Col}_X(D^o) \to A$ up to bijections is an invariant of framed links with orientations. As in [4; 5], we call the map Φ *the* (*birack*) *cocycle invariant*.

Figure 1: Positive and negative crossings with eight labeled semiarcs.

Next, as an analogy of symmetric cocycle invariants in [12; 13], we discuss symmetric birack cocycles. We define a birack 2-cocycle $\phi : X^2 \to A$ to be *symmetric* if

$$\phi(a,b) = -\phi(a \lhd b, \rho(b)) = -\phi(\rho(a), \kappa(b)) \in A,$$

for any $a, b \in X$. Then, similarly to the discussion in [13, Theorem 6.3], we can easily confirm that the weight $\Phi(\tau)$ does not depend on the choice of orientations of D; neither does the map $\Phi_D : \mathrm{Col}_X(D^o) \to A$ up to bijections. In conclusion, the cocycle invariant $\Phi_D : \mathrm{Col}_X(D^o) \to A$ up to bijections is an invariant of framed links.

Finally, we briefly review surgery on links and Fenn–Rourke moves [7]. Let us regard a framed link diagram as the surgery on the framed link in the 3-sphere. Conventionally, every closed 3-manifold M can be expressed as the result of S^3 of surgery on a framed link. Furthermore, two framed links in S^3 have orientation-preserving homeomorphic results of surgery if and only if the framed links are related by a finite sequence of Fenn–Rourke moves and isotopies [7], where the Fenn–Rourke move is an operation between the framed links shown in Figures 3 and 4 in Section 4. Throughout this paper, for a framed link diagram D of a link L, we denote by M_D the result of surgery of S^3 on L.

4 Topological invariants from skew-racks with property FR

Our objective is to explore appropriate skew-racks that yield birack cocycle invariants that are invariant with respect to the Fenn–Rourke moves. In this section, we define skew-racks with property FR and the colorings of closed 3-manifolds.

For $\varepsilon \in \{\pm 1\}$ and $a_1, \dots, a_n \in X$, let us consider the bijection

$$A_{a_1,\dots,a_n} : X \to X, \quad x \mapsto \big(\cdots((x \lhd a_1) \lhd a_2) \lhd \cdots\big) \lhd a_n,$$

and define the subsets

$$\begin{aligned} &\mathrm{Ann}^{+1}(A_{a_1,\dots,a_n}) := \{x \in X \mid \kappa^{n+1}(x) = A_{a_1,\dots,a_n}(x) \lhd \kappa^{n+1}(x)\}, \\ (2)\qquad &\mathrm{Ann}^{-1}(A_{a_1,\dots,a_n}) := \{x \in X \mid \kappa^{n+1}(x) \lhd \kappa(A_{a_1,\dots,a_n}(x)) = A_{a_1,\dots,a_n}(x)\}. \end{aligned}$$

For the case $n = 0$, we define $\mathrm{Ann}^{\pm 1}(X)$ to be the subset $\{x \in X \mid x \lhd \kappa(x) = \kappa(x)\}$. Schematically speaking, as in Figure 2, the set $\mathrm{Ann}(X)^{\pm 1}$ is the set of X-colorings of the unknot of (± 1)-framing.

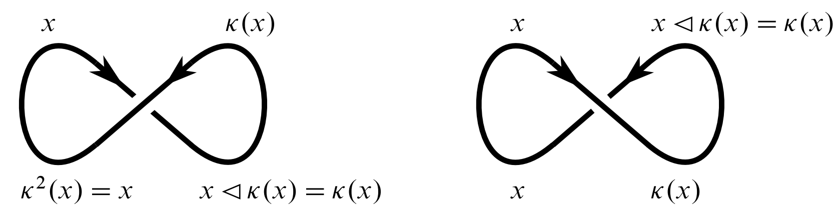

Figure 2: The coloring conditions of unknots of (± 1)-framing.

Definition 4.1 A symmetric skew-rack $(X, \lhd, \kappa, \rho)$ is said to have *property FR* if it satisfies the following:

(FR2) The subset $\mathrm{Ann}(X)$ is not empty, and is bijective to the set $\mathrm{Ann}^{\varepsilon}(A_{a_1,\dots,a_n})$ for arbitrary $n \in \mathbb{Z}$, $a_1, \dots, a_n \in X$ and $\varepsilon \in \{\pm 1\}$.

(FR2) For any $a_1, \dots, a_n \in X$ and $x \in \mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$, $y \in \mathrm{Ann}^{-1}(A_{a_1,\dots,a_n})$, the equalities

$$\kappa^{n+i}(a_i) = A_{a_1,\dots,a_n}(\kappa^{i+1}(a_i) \lhd x), \tag{3}$$

$$\kappa^{n+i}(a_i) \lhd \kappa^{n+1}(y) = A_{a_1 \lhd \kappa(y), a_2 \lhd \kappa^2(y), \dots, a_n \lhd \kappa^n(y)}(\kappa^{i+1}(a_i)) \tag{4}$$

hold, where $i \leq n$ is arbitrary.

Let us analyze the set of colorings of skew-racks with property FR.

Theorem 4.2 *Let $(X, \lhd, \kappa, \rho)$ be a symmetric skew-rack with property FR. Suppose that two framed link diagrams D and D' are related by a Fenn–Rourke move (as in Figures 3 and 4) and take orientations on D and D'.*

Then, for any coloring $\mathcal{C} \in \mathrm{Col}_X(D)$, there is uniquely another $\mathcal{C}' \in \mathrm{Col}_X(D')$ such that $\mathcal{C}(\alpha_i) = \mathcal{C}'(\alpha_i')$ and $\mathcal{C}(\beta_i) = \mathcal{C}'(\beta_i')$ for any $i \leq n$. Furthermore, the map

$$\mathcal{B} : \mathrm{Col}_X(D) \to \mathrm{Col}_X(D') \times \mathrm{Ann}(X), \quad \mathcal{C} \mapsto (\mathcal{C}', \mathcal{C}(\gamma)), \tag{5}$$

is bijective. Specifically, if X is of finite order, the rational number $|\mathrm{Col}_X(D)|/|\mathrm{Ann}(X)|^{\#D} \in \mathbb{Q}$ gives rise to a topological invariant of closed 3-manifolds.

Proof Take arcs γ, δ, α_i's, and β_i's as in Figures 3 and 4. By the properties of good involutions, the coloring conditions are independent of the choices of orientations of D. Thus, we fix the orientations of D and D' as shown in Figures 3 and 4. Given an X-coloring $\mathcal{C} \in \mathrm{Col}_X(D^o)$, define $a_i := \mathcal{C}(\alpha_i)$, $b_i := \mathcal{C}(\beta_i)$. We now show that the map $\mathcal{C}' : \{\text{semiarc of } D'\} \to X$ defined by $\mathcal{C}'(\alpha_i') = a_i$ and $\mathcal{C}'(\beta_i') = b_i$ gives rise to a unique X-coloring.

First, suppose that $\pm$ is positive and $x = \mathcal{C}(\gamma)$. The coloring condition on the arc δ is

$$\mathrm{Tw}(\kappa^{n+1}(x)) = \kappa^{n+1}(x) \lhd^{-1} \kappa^{n+1}(x) = \mathcal{C}(\delta) = A_{a_1,\dots,a_n}(x) \lhd \kappa^{n+1}(x);$$

hence, $x \in \mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$. Notice from (3) that

$$b_i = \kappa(a_i) \lhd \kappa^i(x) = \kappa^i(\kappa^{i+1}(a_i) \lhd x) = \kappa^i\bigl(A^{-1}_{a_1,\dots,a_n}(\kappa^{n+i}(a_i))\bigr) = A^{-1}_{\kappa^i(a_1),\dots,\kappa^i(a_n)}(\kappa^n(a_i)). \tag{6}$$

Meanwhile, the coloring condition on the arc δ_i in the right-hand side of Figure 3 is

$$A_{\kappa^i(a_1),\dots,\kappa^i(a_n)}(b_i) = \mathcal{C}(\delta_i) = \kappa^n(a_i),$$

which is equivalent to (6) exactly.

On the other hand, in the negative case of $\pm$ and $y = \mathcal{C}(\gamma)$, the condition on γ is equivalent to $y \in \mathrm{Ann}^{-1}(A_{a_1,\dots,a_n})$ by the definition of (2); notice that the coloring rule in D in Figure 3 implies $\kappa(b_i) = a_i \lhd \kappa^i(y)$ by the condition on the arc β_i. The coloring condition from β_i' to α_i' is equivalent

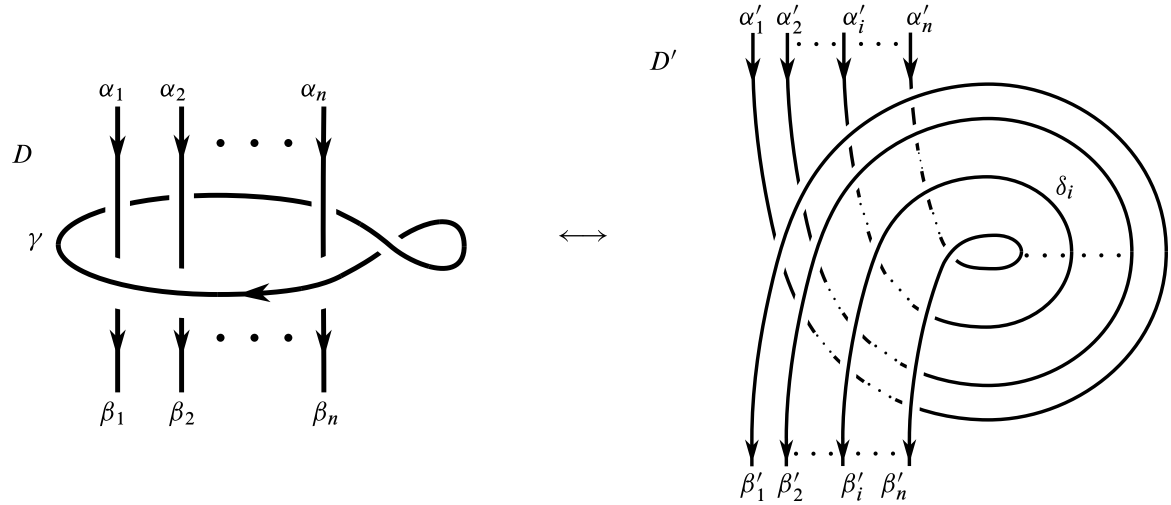

Figure 3: The positive Fenn–Rourke move and labeled semiarcs.

to $\kappa^n(b_i) = A_{\kappa^i(b_1),\dots,\kappa^i(b_n)}(a_i)$, that is, $\kappa^{n+i}(b_i) = A_{b_1,\dots,b_n}(\kappa^i(a_i))$, which directly follows from (4) since $\kappa(b_i) = a_i \lhd \kappa^i(y)$ by the condition on a_i.

Conversely, given an X-coloring $\mathcal{C}'$ of $(D')^{o'}$ and $x \in \mathrm{Ann}^{\pm 1}(A_{a_1,\dots,a_n}) \neq \varnothing$, we similarly can define an X-coloring $\mathcal{C}$ of D that sends α_i to $\mathcal{C}'(\alpha'_i)$, β_i to $\mathcal{C}'(\beta'_i)$, and γ to x.

In summary, by construction, the correspondence $\mathcal{C} \mapsto (\mathcal{C}', \mathcal{C}(\gamma))$ gives the required bijection $\mathcal{B}$. $\square$

Before moving on to the next section, we briefly discuss triviality of the invariants up to link homotopy. For this, consider the permutation group $\mathrm{Bij}(X)$ of a skew-rack X, and define a subgroup generated by

$$\{(\kappa(\bullet) \lhd a) \mid a \in X\} \cup \{(\bullet \lhd^{\epsilon_1} a_1) \lhd^{\epsilon_2} a_2 \mid a_i \in X,\ \epsilon_i \in \{\pm 1\}\}. \tag{7}$$

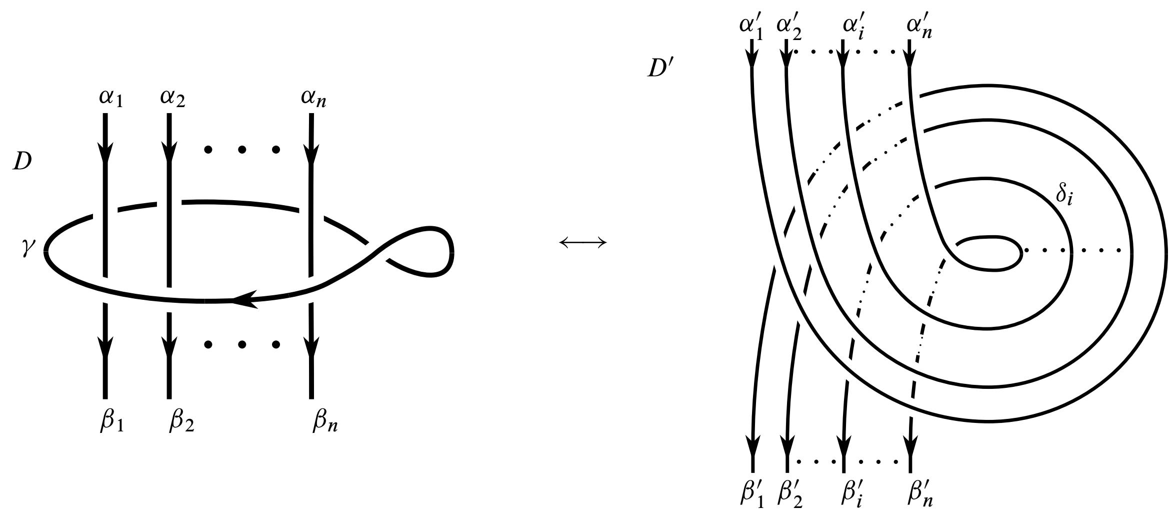

Figure 4: The negative Fenn–Rourke move and labeled semiarcs.

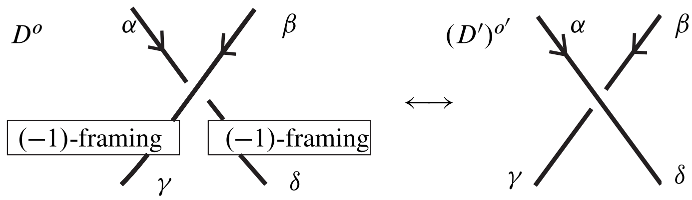

Figure 5: Diagrams D and D', where all semiarcs lie within a link component.

The subgroup, denoted by $\mathrm{Inn}_\kappa^{\mathrm{even}}(X)$, canonically has the right action on X. We say a skew-rack $(X, \lhd, \kappa)$ with property FR is *f-link homotopic* if $x \lhd^\varepsilon \kappa(x) = x \lhd^\varepsilon (x \cdot g)$ holds for any $x \in X$, $g \in \mathrm{Inn}_\kappa^{\mathrm{even}}(X)$, $\varepsilon \in \{\pm 1\}$.

Proposition 4.3 *Suppose a symmetric skew-rack $(X, \lhd, \kappa, \rho)$ with property FR is f-link homotopic. Then, if two framed link diagrams D and D' are transformed by the operation in Figure 5, there is a bijection $\mathcal{B}_f : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X((D')^{o'})$.*

Proof For a coloring $\mathcal{C} \in \mathrm{Col}_X(D^o)$, take $a \in X$ such that $\mathcal{C}(\alpha) = \kappa(a)$. Since α and β lie on the same link-component, there is $g \in \mathrm{Inn}_\kappa^{\mathrm{even}}(X)$ such that $\mathcal{C}(\beta) = a \cdot g$ from definition (7). Since $\mathrm{Tw}^{-1}(x) = x \lhd x$ by definition, the rule of colorings implies

$$\mathcal{C}(\gamma) = \mathrm{Tw}^{-1}(\kappa(a \cdot g)) = (a \cdot g) \lhd \kappa(a \cdot g), \quad \mathcal{C}(\delta) = \mathrm{Tw}^{-1}(\kappa(a)) \lhd (a \cdot g) = (a \lhd \kappa(a)) \lhd (a \cdot g).$$

Since X is f-link homotopic, $\mathcal{C}(\gamma) = (a \cdot g) \lhd^{-1} a$ and $\mathcal{C}(\delta) = a$. Thus, we can define another coloring $\mathcal{B}_f(\mathcal{C})$ of D' by $\mathcal{B}_f(\mathcal{C})(\alpha) = a$ and $\mathcal{B}_f(\mathcal{C})(\beta) = a \cdot g$. Since $\mathcal{C}(\gamma) = (a \cdot g) \lhd^{-1} a = \mathcal{B}_f(\mathcal{C})(\gamma)$ and $\mathcal{C}(\delta) = a = \mathcal{B}_f(\mathcal{C})(\delta)$ by definitions, the map $\mathcal{B}_f : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X((D')^{o'})$ is bijective, as required. □

Many 3-manifolds can be expressed as the results from S^3 of surgery along various framed knots, so to obtain nontrivial colorings, we consider skew-racks, which are not f-link homotopic.

5 Examples of skew-racks with property FR from groups

Here we provide examples of skew-racks with property FR. Throughout this section, we fix a group G, an automorphism $\kappa : G \to G$ satisfying $\kappa \circ \kappa = \mathrm{id}_G$, and a map $\delta : G \to G$ satisfying $\kappa \circ \delta = \delta \circ \kappa$. Consider the binary operation $\lhd : G \times G \to G$ defined by $x \lhd y = \kappa(x)\delta(y)$. Then, the twisting map Tw in Proposition 2.3 is given by $\mathrm{Tw}(g) = g\delta(g)^{-1}$.

Lemma 5.1 *These operations $(\lhd, \kappa)$ with $x \lhd y = \kappa(x)\delta(y)$ define a skew-rack of $X = G$ if and only if, for any $x, y \in G$,*

$$\delta(x)\delta(y) = \delta(y)\delta(x\delta(y)) \in G. \tag{8}$$

Let $\rho : G \to G$ be a good involution. Further, assume that the image $\mathrm{Im}(\delta) \subset G$ is a subgroup of G, and that the cardinality of the preimage $\delta^{-1}(d)$ is constant for any $d \in \mathrm{Im}(\delta)$. Then, the symmetric skew-rack on $X = G$ has property FR.

In addition, if the subgroup $\mathrm{Im}(\delta)$ is commutative, the skew-rack is f-link homotopic.

Proof Since the former part is shown by direct computation, we show only the remaining claims here. We now analyze the set $\mathrm{Ann}^{\varepsilon}(A_{a_1,\dots,a_n})$ in (2). First, suppose $\varepsilon = +1$. Then, the condition $\kappa^{n+1}(x) = A_{a_1,\dots,a_n}(x) \lhd \kappa^{n+1}(x)$ is equivalent to

$$\delta(\kappa^n(a_1))\delta(\kappa^{n-1}(a_2))\cdots\delta(\kappa(a_n))\delta(\kappa^{n+1}(x)) = 1. \tag{9}$$

Since $\mathrm{Im}(\delta)$ is a subgroup of G by assumption, the set $\mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$ is nonempty. Moreover, by the second assumption, the cardinality of $\mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$ does not depend on the choice of $a_1,\dots,a_n$, that is, X satisfies (FR1). As for (FR2), the equality (3) is shown by

$$A_{a_1,\dots,a_n}(\kappa^{i-1}(a_i)\lhd x) = \kappa^{n+i}(a_i)\delta(\kappa^{n+i}(x))\delta(\kappa^{n+i+1}(a_1))\delta(\kappa^{n+i}(a_2))\cdots\delta(\kappa^{i}(a_n)) = \kappa^{n+i+1}(a_i).$$

Next, we will show (4) in the case $\varepsilon = -1$. We can easily check $\kappa^{n+1}(x)\lhd\kappa\big(A_{a_1,\dots,a_n}(x)\big) = A_{a_1,\dots,a_n}(x)$ in (2) is equivalent to (9) exactly. Thus, similarly, the cardinality of $\mathrm{Ann}^{-1}(A_{a_1,\dots,a_n}) \neq \varnothing$ does not depend on the choice of $a_1,\dots,a_n$. In addition, for $x \in \mathrm{Ann}^{-1}(A_{a_1,\dots,a_n})$, the equality (4) is shown by

$$\begin{aligned}
&A_{a_1\lhd\kappa(x),a_2\lhd\kappa^2(x),\dots,a_n\lhd\kappa^n(x)}(\kappa^{i+1}(a_i))\\
&\qquad= \kappa^{i+n+1}(a_i)\delta\big(\kappa^{n-1}(a_1)\delta(\kappa^n(x))\big)\delta\big(\kappa^{n-2}(a_2)\delta(\kappa^n(x))\big)\cdots\delta\big(a_n\delta(\kappa^n(x))\big)\\
&\qquad= \kappa^{i+n+1}(a_i)\delta(\kappa^n(x))^{-1}\delta(\kappa^{n-1}(a_1))\delta(\kappa^n(a_2))\cdots\delta(a_n)\delta(\kappa^n(x)) = \kappa^{n+i}(a_i)\lhd\kappa^{n+1}(x).
\end{aligned}$$

Here, the second and third equalities are obtained from (8) and (9), respectively. Hence, X has property FR, as required.

Finally, we show the last statement. From the definition of the subgroup $\mathrm{Inn}_{\kappa}^{\mathrm{even}}(X)$, any $g \in \mathrm{Inn}_{\kappa}^{\mathrm{even}}(X)$ and $a \in G$ admit uniquely $b_1,\dots,b_n \in \mathrm{Im}(\delta)$ such that $a\cdot g = a\delta(b_1)\cdots\delta(b_n) \in G$. Since $\mathrm{Im}(\delta)$ is commutative, (8) means $\delta(a) = \delta(a\delta(b))$. Thus,

$$(z\lhd^{\varepsilon}\kappa(a))\lhd^{-\varepsilon}(a\cdot g) = z\delta(a)^{\varepsilon}\delta(a\delta(b_1)\cdots\delta(b_n))^{-\varepsilon} = z.$$

Therefore, the skew-rack is f-link homotopic by Proposition 4.3. □

We should point out that (8) comes with a few conditions. For example, if $|G| > 1$, the map δ is not surjective. In fact, if δ is surjective, then (8) with $x = 1$ is equivalent to $z^{-1}\delta(1)z = \delta(z)$ for any $z \in G$, which means that $\mathrm{Im}(\delta)$ is a conjugacy class, and contradicts the surjectivity. However, we provide some examples that satisfy the conditions in Lemma 5.1.

Example 5.2 First, we observe the case where δ is a group homomorphism. Then, we can easily check that (8) is equivalent to that of $\delta\circ\delta = 0$ and the image $\mathrm{Im}(\delta)$ is abelian. If so, the cardinality of $\delta^{-1}(k)$ is constant; thus, if X admits a good involution, the symmetric skew-rack has property FR, and is f-link homotopic by Lemma 5.1.

To avoid f-link homotopic skew-racks, we should focus on δ, which is not a homomorphism.

Example 5.3 (twisted conjugacy classes) Suppose we have a group automorphism $f : G \to G$, and define $\delta(x) = f(x^{-1})x$. Then, (8) is equivalent to $\mathrm{Im}(\delta\circ\delta) = \{1_G\}$. In general, we can easily check that, for any $g \in \mathrm{Im}(\delta)$, the preimage $\delta^{-1}(g)$ is bijective to the fixed-point subgroup $\{h \in G \mid f(h) = h\}$;

see, e.g., [2]. Thus, to apply Lemma 5.1, the remaining point is to analyze the situation such that the image $\mathrm{Im}(\delta)$ is a subgroup.

The image $\mathrm{Im}(\delta)$ is sometimes called as *a twisted conjugacy class* or *Reidemeister conjugacy class*. Prior works [2; 10] have investigated various conditions requiring that $\mathrm{Im}(\delta)$ be a subgroup and $\mathrm{Im}(\delta\circ\delta)=\{1_G\}$. However, many of the examples in those works satisfy that $\mathrm{Im}(\delta)$ is commutative Thus, it is difficult to find examples of pairs (G, f) satisfying that the resulting skew-racks are not f-link homotopic.

Example 5.4 Take a group K with a normal subgroup $N \trianglelefteq K$ and an involutive automorphism $f : K \to K$ satisfying $f(N) \subset N$. Let G be $K \times N$ and κ be $f \times f$. Define $\delta(x, y)$ as $(x^{-1}yx, 1)$, where $x, y \in K$. Next, we check the conditions in Lemma 5.1. Checking (8) is obvious: since $N = \{b^{-1}ab \mid a \in N,\ b \in K\}$, the image of δ is $N \times 1$ as a subgroup of G. Moreover, for any $(k, 1) \in K \times 1$, the preimage $\delta^{-1}(k, 1)$ is equal to $\{(y^{-1}ky, y) \in G \mid y \in K\}$, which is bijective to K. In conclusion, the symmetric skew-rack on G has property FR, by Lemma 5.1. For example, if $N = K$, the skew-rack on G is exactly equal to that in Example 2.2; here, we should remark $\mathrm{Tw}(x, a) = (a^{-1}x, a)$.

Finally, we compute a few colorings using the above skew-racks with property FR.

Example 5.5 For natural numbers $n, m \in \mathbb{N}$, we first compute colorings of the lens space $L(nm-1, n)$. Let $X = G$ be a skew-rack with good involution, which satisfies the conditions in Lemma 5.1. Let D be the Hopf link with framing (n, m). Then, M_D is known to be $L(nm-1, n)$. We fix two semiarcs α, β in each link-component on D. Then, from the definition of colorings, a coloring $\mathcal{C} \in \mathrm{Col}_X(D)$ satisfies

$$\kappa(\mathcal{C}(\alpha) \lhd \mathcal{C}(\beta)) = \mathrm{Tw}^n(\mathcal{C}(\alpha)), \quad \kappa(\mathcal{C}(\beta) \lhd \mathcal{C}(\alpha)) = \mathrm{Tw}^m(\mathcal{C}(\beta)). \tag{10}$$

Conversely, every $a, b \in X$ satisfying $\kappa(a \lhd b) = \mathrm{Tw}^n(a)$ and $\kappa(b \lhd a) = \mathrm{Tw}^m(b)$ yield a coloring of D. Since $\mathrm{Tw}^n(a) = a\delta(a)^{-n}$, (10) is equivalent to conditions $\mathcal{C}(\beta) = \delta(\mathcal{C}(\alpha))^m$ and $\delta(\mathcal{C}(\alpha))^{nm-1} = 1$. Hence, $\mathrm{Col}_X(D)$ is bijective to

$$\{(a, b) \in G^2 \mid \delta(a)^{nm-1} = 1,\ \delta(b) = \delta(a)^n\} \overset{1:1}{\longleftrightarrow} \{a \in G \mid \delta(a)^{nm-1} = 1\} \times \delta^{-1}(0), \tag{11}$$

where we use a bijection $\delta^{-1}(0) \leftrightarrow \{a \in G \mid \delta(a)^n = c\}$ for any $c \in G$. Therefore, the set $\mathrm{Col}_X(D)$ depends only on nm; it cannot classify the lens spaces of the forms $L(nm-1, n)$. In contrast, we later compute various cocycle invariants that can distinguish among different lens spaces (see Example 6.7).

Example 5.6 Next, we observe that the sets of colorings of integral homology 3-spheres seem to be strong invariants, where we consider the skew-rack on $X = K \times K$ in Example 2.2. Let $D_n^{\pm}$ be the $(2, n)$-torus knot with framing ± 1. Then, the resulting 3-manifold $M_{D_n^{\pm}}$ is the Brieskorn 3-manifold of the form $\Sigma(2, n, 2n \mp 1)$, as an integral homology 3-sphere. For a concrete group K, it is fairly easy to determine the set $\mathrm{Col}_X(D_q^{\pm})$ with the help of a computer program. A list of several computations of $|\mathrm{Col}_X(D_q^{\pm})|$ is provided in Table 1.

As seen in this example, it is reasonable to focus only on nonabelian groups K. In fact, if K is abelian, $(x, a) \lhd (y, b) = (x, a)$; hence, the coloring conditions are trivial; thus, considering the linking matrix of D, we can easily find a one-to-one correspondence $\mathrm{Col}_X(D) \simeq \mathrm{Hom}(H_1(M; \mathbb{Z}), K) \times K^{\sharp D}$.

p	$\lvert\mathrm{Col}_X(D_3^+)\rvert$	$\lvert\mathrm{Col}_X(D_3^-)\rvert$	$\lvert\mathrm{Col}_X(D_5^+)\rvert$	$\lvert\mathrm{Col}_X(D_5^-)\rvert$	$\lvert\mathrm{Col}_X(D_7^+)\rvert$	$\lvert\mathrm{Col}_X(D_7^-)\rvert$
3	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$
5	$121\lvert K\rvert$	$\lvert K\rvert$	$121\lvert K\rvert$	$\lvert K\rvert$	$25\lvert K\rvert$	$\lvert K\rvert$
7	$\lvert K\rvert$	$337\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$49\lvert K\rvert$
11	$2641\lvert K\rvert$	$\lvert K\rvert$	$2641\lvert K\rvert$	$2641\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$
13	$\lvert K\rvert$	$6553\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$

Table 1: Cardinality of $\mathrm{Col}_X(D_n^{\pm 1})$ for various p, n. Here, $K = \mathrm{SL}_2(\mathbb{F}_p)$ of order $p^3 - p$.

6 Cocycle invariants of 3-manifolds

As discussed in [3; 5; 15], there are various procedures to concretely find symmetric birack 2-cocycles. However, the condition that 2-cocycles must have invariance with respect to Fenn–Rourke moves seems strong. Nevertheless, we now investigate 2-cocycle invariants to obtain 3-manifold invariants. Throughout this section, we assume a symmetric skew-rack X with property FR, and a map ϕ from X^2 to an abelian group A.

We first introduce the property FR of birack 2-cocycles as follows.

Definition 6.1 Recall the bijection $\mathcal{B}$ in Theorem 4.2, and denote by 0_A the constant map to A whose image is zero. A symmetric birack 2-cocycle $\phi : X^2 \to A$ satisfies *property FR*, if $\Phi_D = (\Phi_{D'} \times 0_A) \circ \mathcal{B}$ holds for any diagrams D and D' in Figures 3 and 4. Here, Φ_D is the cocycle invariant explained in Section 3.

Such a ϕ is said to be *f-link homotopic* if X is f-link homotopic and, for any $a \in X$ and $g \in \mathrm{Inn}_\kappa^{\mathrm{even}}(X)$,

$$\phi((a\cdot g) \lhd \kappa(a), a\cdot g) + \phi(a, a \lhd \kappa(a)) = \phi(\kappa(a), a\cdot g) + \phi((a\cdot g) \lhd \kappa(a), a). \tag{12}$$

We will see (Proposition 6.2) that symmetric birack 2-cocycles with property FR yield topological invariants of closed 3-manifolds. Take two maps $F : Y \to A$ and $G : Z \to A$, where Y and Z are some sets. We call F *an FR-stabilization* of G if there is a bijection $B : Z \to Y \times \mathrm{Ann}(X)$ such that $g \circ B^{-1} = f \times 0_A$. More generally, F and G are *FR-equivalent* if F and G are related by a finite sequence of FR-(dis-)stabilizations. Then, the following proposition is fairly obvious by definitions.

Proposition 6.2 *Let ϕ be a symmetric birack 2-cocycle with property FR. Then, the correspondence $D \mapsto \Phi_D$ up to FR-equivalent relations is an invariant of closed 3-manifolds.*

Moreover, if X and ϕ are f-link homotopic, and if D and D' are related by the operation in Figure 5, then $\Phi_D = \Phi_{D'} \circ B_f$, where B_f is the bijection $\mathrm{Col}_X(D) \to \mathrm{Col}_X(D')$ in the proof of Proposition 4.3.

To conclude, to obtain 3-manifold invariants, it is important to find symmetric birack 2-cocycles with concrete expressions. In this context, we discuss Lemmas 6.3 and 6.4 below. Let $\widetilde{X}$ be $X \times A$. Define $\widetilde{\lhd} : \widetilde{X} \times \widetilde{X} \to \widetilde{X}$ by

$$(x,a) \,\widetilde{\lhd}\, (y,b) = (x \lhd y, a + \phi(x,y)), \quad (x, y \in X,\ a, b \in A),$$

and $\tilde{\kappa} : \widetilde{X} \to \widetilde{X}$ by $\tilde{\kappa}(x,a) = (\kappa(x), -a)$.

Lemma 6.3 (see [3, Section 3]) *These maps $\widetilde{\lhd}$, $\tilde{\kappa}$, $\tilde{\delta}$ define a skew-rack on $\widetilde{X} = X \times A$ if and only if ϕ is a birack 2-cocycle.*

Proof Describe the distribution law as in (SR3) as

$$\big((x,a)\,\widetilde{\lhd}\,(y,b)\big)\,\widetilde{\lhd}\,(z,c) = \big((x \lhd y) \lhd z, a + \phi(x,y) + \phi(x \lhd y, z)\big),$$
$$\big((x,a)\,\widetilde{\lhd}\,\tilde{\kappa}(z,c)\big)\,\widetilde{\lhd}\,\big((y,b)\,\widetilde{\lhd}\,(z,c)\big) = \big((x \lhd y) \lhd z, a + \phi(x,\kappa(z)) + \phi(x \lhd \kappa(z), y \lhd z)\big).$$

Furthermore, (SR1) implies $\phi(\kappa(x),\kappa(y)) = \phi(x,y)$. Hence, the desired claim follows directly from the definition (1) of birack 2-cocycle. □

Lemma 6.4 *Let ϕ be a birack 2-cocycle satisfying $\phi(a,b) = -\phi(\rho(a),\kappa(b))$. Then, the map $\overline{\phi} : X^2 \to A$ that sends (a,b) to $\phi(a,b) - \phi(a \lhd b, \rho(b))$ is a symmetric birack 2-cocycle.*

Proof It is easy to confirm $\overline{\phi}(a,b) + \overline{\phi}(a \lhd b, \rho(b)) = 0$. Thus, all that remains is to check the cocycle condition (1) of $\overline{\phi}$. For this, we may show

$$\phi(a \lhd b, \rho(b)) + \phi((a \lhd b) \lhd c, \rho(c)) = \phi\big(a \lhd \kappa(c), \rho(\kappa(c))\big) + \phi((a \lhd b) \lhd c, \rho(b \lhd c)). \tag{13}$$

Replace $(a \lhd b) \lhd c$, $\rho(b \lhd c)$, and $\rho(\kappa(c))$ with a, b, and c, respectively. Then, we can easily check that the replacement of (13) coincides with (1). □

Using these lemmas, we provide examples from several skew-racks in Example 5.4. Let $N \trianglelefteq K$ be groups and $f : K \to K$ be an involutive automorphism satisfying $f(N) \subset N$. Further, take a normalized group 2-cocycle $\theta : K \times K \to A$, where θ satisfies

$$\theta(x,y) - \theta(x,yz) + \theta(xy,z) - \theta(y,z) = 0, \quad \theta(1_K, x) = \theta(x, 1_K) = 0 \in A,$$

for any $x, y, z \in K$. Then, the product of $\widetilde{K} = K \times A$ has a group structure with operation $((x,a),(y,b)) \mapsto (xy, a+b+\theta(x,y))$ as a central extension of K. As is known in group cohomology, every central extension over K with fiber A can be expressed by the product for some θ. Then, from Example 5.4, we can define the symmetric skew-racks on $G = K \times N$ and $\widetilde{G} = \widetilde{K} \times \widetilde{N}$, which have property FR. Moreover, by the definition of $\lhd$ on $\widetilde{G}$, we obtain

$$\begin{aligned}
&\big((x,a),(y,b)\big)\\
&\quad \lhd \big((z,c),(w,d)\big) = \big(\tilde{\kappa}(x,a)(z^{-1}, -c - \theta(z,z^{-1}))(w,d)(z,c),\ (y,b)\big)\\
&\quad = \big(\big(f(x)z^{-1}wz, f(a) + d + \theta(f(x),z^{-1}) + \theta(f(x)z^{-1}, wz) + \theta(w,z) - \theta(z,z^{-1})\big), (y,b)\big) \in \widetilde{G}.
\end{aligned}$$

Inspired by Lemma 6.4, we obtain the following procedure for producing birack 2-cocycles:

Theorem 6.5 *Let $\lambda : N \to A$ be a group 1-cocycle. Then, the map*

$$\phi_{\lambda,\theta} : G^2 = (K \times N) \times (K \times N) \to A,$$
$$(x,y,z,w) \mapsto \lambda(y)\big(\theta(f(x),z^{-1}) + \theta(f(x)z^{-1}, wz) + \theta(z,w) - \theta(z,z^{-1})\big) \in A,$$

is a birack 2-cocycle of the skew-rack $G = K \times N$ in Example 2.2. If

$$\lambda(x)\theta(a,b) = \lambda(f(x))\theta(f(a), f(b))$$

hold for any $a, b \in K$, $x \in N$, the condition in Lemma 6.4 is true. Specifically, the cocycle $\overline{\phi_{\lambda,\theta}}$ mentioned in Lemma 6.4 is a symmetric birack 2-cocycle.

In general, it may seem difficult to find group 2-cocycles θ such that the associated map $\phi_{\lambda,\theta}$ has property FR. However, when K is a cyclic group, we give such examples of birack cocycles with property FR. More precisely, by a direction computation, we can show the following.

Proposition 6.6 *Let $p \in \mathbb{Z}$ be an odd prime. Let $K = N = \mathbb{Z}/p$, and take $\varepsilon \in \{\pm 1\}$ such that $f(x) = \varepsilon x$, Define group cocycles λ and θ by setting*

$$\lambda(x) = x, \quad \theta(x,y) = \frac{(x+\varepsilon y)^p - x^p - (\varepsilon y)^p}{p} = \sum_{j:1 \le j < p} j^{-1} x^j (\varepsilon y)^{p-j},$$

respectively, where $x, y \in \mathbb{Z}/p$. Then, $\overline{\phi_{\lambda,\theta}}(x,y,z,w) = 2y\theta(x,w)$, and the symmetric birack 2-cocycle $\overline{\phi_{\lambda,\theta}}$ has property FR and is f-link homotopic.

Example 6.7 Let D be the Hopf link with framings (n,m), as in Example 5.5. Recall that M_D is the lens space $L(nm-1,m)$. By (11), if $nm-1$ is divisible by p and $K = \mathbb{Z}/p$, then $\mathrm{Col}_X(D)$ is bijective to $(\mathbb{Z}/p)^2$. In addition, we can easily show that the cocycle invariant $\Phi_D : (\mathbb{Z}/p)^2 \to \mathbb{Z}/p$ is equal to the correspondence $(x,y) \mapsto -mx^2$, where we use the 2-cocycle $\overline{\phi_{\lambda,\theta}}$ in Proposition 6.6. For example, the invariant can distinguish between the lens spaces $L(11,1)$ and $L(11,3)$, which are not homotopy equivalent.

More generally, consider the lens space $L(p,q)$ and a framed diagram $D_{p,q}$ such that $M_{D_{p,q}} = L(p,q)$. Then, with the help of a computer program, if $p, q < 100$, it is fairly easy to check that the cocycle invariant $\Phi_{D_{p,q}} : (\mathbb{Z}/p)^{1+\#D_{p,q}} \to \mathbb{Z}/p$ is FR-equivalent to the map $\mathbb{Z}/p \to \mathbb{Z}/p; x \mapsto -qx^2$.

From this example, it is natural to pose the problem below, together with a relation to the Dijkgraaf–Witten invariant [6, §6]. We first briefly review the invariant. Fix a closed 3-manifold M with fundamental homology 3-class $[M] \in H_3(M;\mathbb{Z}) \cong \mathbb{Z}$. Let K be a group of finite order, and $\psi : K^3 \to A$ be a group 3-cocycle. Denote by BK the classifying space of K or the Eilenberg–Mac Lane space of type $(K,1)$, and $c_M : M \to B\pi_1(M)$ be a classifying map. Then, any group homomorphism $f : \pi_1(M) \to K$ induces a continuous map $f_* : B\pi_1(M) \to BK$. Since the (co)homology of BK equals that of K, we can define the pullback $(f_* \circ c_M)^*(\psi)$ as a 3-cocycle of M. Then, *the Dijkgraaf–Witten invariant* is defined as the map

$$\mathrm{DW}_\psi(M) : \mathrm{Hom}(\pi_1(M), K) \to A, \quad f \mapsto \langle (f_* \circ c_M)^*(\psi), [M] \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Kronecker map.

Problem 6.8 As in Example 2.2, let X be the symmetric skew-rack on $K \times K$. Let $\lambda : K \to A$ and $\theta : K^2 \to A$ be group cocycles, and ψ be the cup product $\lambda \smile \theta$ as a group 3-cocycle. Let D be a framed link diagram.

Then, is there a bijection $\mathcal{B} : \mathrm{Col}_X(D) \simeq \mathrm{Hom}(\pi_1(M), K) \times \mathrm{Ann}(X)^{\sharp D}$? Further, find a condition such that the birack 2-cocycle $\overline{\phi_{\lambda,\theta}}$ in Theorem 6.5 has property FR, and FR-equivalence between the cocycle invariant $\Phi : \mathrm{Col}_X(D) \to A$ and the Dijkgraaf–Witten invariant $\mathrm{DW}_\psi(M_D)$.

If this problem is correctly solved, we consequently obtain a diagrammatic computation of the Dijkgraaf–Witten invariant via the cocycle invariants and Dehn surgery.

7 Criteria for 3-manifolds that are not the result of surgery of any knot

As an application of the cocycle invariant, we provide two criteria to detect 3-manifolds that are not the result of surgery of any knot in S^3 (see [1, Section 7.1; 11] for the details of such 3-manifolds and other criteria). As in Example 5.4, we fix groups $N \trianglelefteq K$, and $X = K \times N$ with $f = \mathrm{id}_K$; recall that X is a skew-rack by $(x,a) \lhd (y,b) = (xy^{-1}by, a)$, and has property FR.

Proposition 7.1 *Suppose $|K| < \infty$ and that a framed link diagram D and a knot diagram of framing zero are related by a sequence of Fenn–Rourke moves and isotopy. Then, the invariant $|\mathrm{Col}_X(D)|/|K|^{\sharp D} \in \mathbb{Q}$ in Theorem 4.2 is larger than or equal to $|N|$.*

Proof We may suppose that D is a knot diagram of framing zero. For the proof, it is sufficient to construct $|K \times N|$ colorings on D. As in Figure 6, take semiarcs α_i and β_i in D, and denote by $\varepsilon_i \in \{\pm 1\}$ the sign of the crossing between α_i and β_i. For $(g,h) \in K \times N$, we define $\mathcal{C}_{g,h}(\alpha_i)$ to be $(h^{\sum_{j=1}^{i-1} \varepsilon_j} g, h) \in X = K \times N$. Since every β_i lies on the same link component, $\mathcal{C}_{g,h}(\beta_i) = (h^{n_i} g, h)$ for some $n_i \in \mathbb{Z}$. Hence, we can easily check that $\mathcal{C}_{g,h}$ defines an X-coloring as required. □

As a special case, let $K = N = \mathbb{Z}/2$. For $k_1, k_2, k_3 \in \mathbb{Z}/2$, we define a map $\phi_{k_1,k_2,k_3} : X \times X \to \mathbb{Z}/2$ by setting

$$\phi_{k_1,k_2,k_3}((x,a),(y,b)) = k_1 a + k_2 b + k_3 ab.$$

Then, by direct computation, it is not hard to show the following:

Proposition 7.2 *The map ϕ_{k_1,k_2,k_3} is a symmetric birack 2-cocycle with property FR, and is f-link homotopic. Furthermore, if a framed link diagram D is FR-equivalent to a knot diagram of framing zero, then the symmetric birack 2-cocycle invariant is trivial.*

Unfortunately, we could not find any new examples of framed link diagrams that are not FR-equivalent to any knot diagram of framing zero. We end this paper by presenting problems to be investigated in future work.

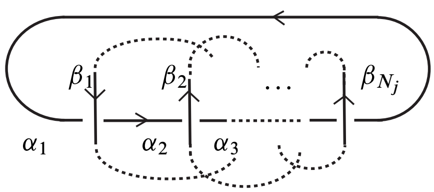

Figure 6: Semiarcs α_i and β_i in the knot diagram D.

Problem 7.3 As applications of the propositions above, find 3-manifolds that are not the surgery of any knot of framing zero. Establish stronger criteria than the propositions above, which are applicable to many framed link diagrams.

Acknowledgements

The author is grateful to Nozomu Sekino, Kimihiko Motegi, and Motoo Tange for their insightful discussions on Dehn surgery. He sincerely thanks the referee for careful readings of the paper.

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Received: May 20, 2023 Revised: August 16, 2024

Algebraic & Geometric Topology **26**:4 (2026) 1465–1505
DOI: 10.2140/agt.2026.26.1465
Published: April 25, 2026

The rational abelianization of the Chillingworth subgroup of the mapping class group of a surface

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The Chillingworth subgroup of the mapping class group of a compact oriented surface of genus g with one boundary component is defined as the subgroup whose elements preserve nonsingular vector fields on the surface up to homotopy. In this work, we determine the rational abelianization of the Chillingworth subgroup as a full mapping class group module. The abelianization is given by the first Johnson homomorphism and the Casson–Morita homomorphism for the Chillingworth subgroup. Additionally, we compute the order of the Euler class of a certain central extension related to the Chillingworth subgroup and determine the kernel of the Casson–Morita homomorphism for the Chillingworth subgroup.

1 Introduction

Throughout this paper, we assume that all surfaces are compact, connected, and oriented. Let $\Sigma_{g,1}$ (resp. $\Sigma_{g,*}$, Σ_g) denote a surface of genus g with one boundary component (resp. with a fixed base point, or with no boundary and no fixed point). The *mapping class group*, denoted by $\mathcal{M}_{g,1}$ (resp. $\mathcal{M}_{g,*}$, $\mathcal{M}_g$), is defined as the group of isotopy classes of orientation-preserving self-diffeomorphisms of the surface that fix the boundary or the base point pointwise. For oriented surface bundles, the structure group is the orientation-preserving diffeomorphism group of the surface. Except for a finite number of cases where the genus is small, this diffeomorphism group is homotopy equivalent to the mapping class group, which is discrete. As a result, their classifying spaces are homotopy equivalent, and therefore, the group cohomology of the mapping class group is equivalent to the characteristic classes of surface bundles. The mapping class group naturally acts on various structures of the surface. For example, it acts on the first integral homology group of the surface $H = H_1(\Sigma_{g,1};\mathbb{Z})$, preserving the intersection form. Consequently, the mapping class group acts on H via the integral symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$, called the symplectic representation. Through this, several important modules of the mapping class group can be

MSC2020: 20F38.

described by using representations of the integral symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$, or the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$ over $\mathbb{Q}$.

1A Chillingworth subgroups and related background

Chillingworth [7; 8] studied the action of the mapping class group on the set of homotopy classes of nonsingular vector fields, focusing on winding numbers. This action is described by a crossed homomorphism, known as the *Chillingworth homomorphism*, which maps to the first integral cohomology group of the surface. The *Chillingworth subgroup* is defined as the subgroup of the mapping class group consisting of elements that preserve vector fields on the surface up to homotopy (see Proposition 3.3 for alternative definitions). Chillingworth subgroups of $\mathcal{M}_{g,1}$, $\mathcal{M}_{g,*}$, and $\mathcal{M}_g$ are denoted by $\mathrm{Ch}_{g,1}$, $\mathrm{Ch}_{g,*}$, and Ch_g, respectively. Here, $\mathrm{Ch}_{g,*}$ and Ch_g are defined via certain natural homomorphisms $\mathcal{M}_{g,1}\to\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,*}\to\mathcal{M}_g$ between mapping class groups, where $\mathcal{M}_{g,1}\to\mathcal{M}_{g,*}$ is obtained by collapsing the boundary to a point, and $\mathcal{M}_{g,*}\to\mathcal{M}_g$ is obtained by forgetting the base point.

Johnson [17] discussed the kernel of the Chillingworth class on the Torelli group, where the Chillingworth class is defined as the Poincaré dual of the Chillingworth homomorphism. Trapp [41] introduced a $(2g+1)$-dimensional linear representation of the mapping class group $\mathcal{M}_{g,1}$, referred to as Trapp's representation. In that work, he used this representation to study the action of the mapping class group on the first homology group of the unit tangent bundle of the surface and characterized the Chillingworth subgroup as the kernel of this linear representation. Furthermore, the Chillingworth subgroup has been studied in other contexts. Childers [6] studied its relationship with the subgroup generated by the simply intersecting pair (SIP) maps. Blanchet, Palmer and Shaukat [5] mentioned it in the context of the action of the mapping class group on the Heisenberg group of the surface, which is defined as a certain quotient of the surface braid group or a certain central extension of the first integral homology group of the surface by the infinite cyclic group. However, the structure of the Chillingworth subgroup has not been well studied.

Before we get into the main topic of this paper, we will introduce some background information. The Chillingworth subgroup is an intermediate-sized group between two significant subgroups in the context of the mapping class group: the *Torelli group* $\mathcal{I}_{g,1}$ which is defined as the kernel of the action of the mapping class group on the first homology group of the surface and the *Johnson kernel* $\mathcal{K}_{g,1}$ which is defined as the subgroup generated by Dehn twists along separating simple closed curves on the surface. Specifically, $\mathcal{K}_{g,1}\subset\mathrm{Ch}_{g,1}\subset\mathcal{I}_{g,1}$.

The structure of the rational abelianization of the Torelli group as a mapping class group module was determined by Johnson [19] using the *Johnson homomorphism* $\tau_{g,1}(1):\mathcal{I}_{g,1}\to\bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$, which he introduced and is now known as the first Johnson homomorphism. The target space is the third exterior power of the first homology group of the surface with the mapping class group acting naturally on it as the symplectic group. Since the Torelli group is a normal subgroup of the mapping class group, the mapping class group acts on it by conjugation. Under these actions, the first Johnson homomorphism is equivariant with respect to the action of the mapping class group. The structure of the rational abelianization of the Johnson kernel as a mapping class group module was determined by

Dimca–Hain–Papadima [9], Morita–Sakasai–Suzuki [34] (in the case of closed surfaces without a base point), and Faes–Massuyeau [12, Theorem 3.2] (in the case of surfaces with one boundary component). The structure of the mapping class group module in this case is more complex but can be described as an extension of representations of the symplectic group (see Section 6).

The proof of Theorem A, which determines the rational abelianization of the Chillingworth subgroup, primarily involves analyzing the long exact sequence (inflation-restriction exact sequence)

$$H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q}) \to H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \to H_1(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_1(U;\mathbb{Q}) \to 0$$

induced by the first Johnson homomorphism $\tau_{g,1}(1):\mathrm{Ch}_{g,1} \to U$ restricted to the Chillingworth subgroup, where U is the image $\tau_{g,1}(1)(\mathrm{Ch}_{g,1}) \subset \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$.

We determine the rational abelianization of the Chillingworth subgroup using a result analogous to that of Hain [15] and the rational abelianization of the Johnson kernel by Faes and Massuyeau [12]. The former (Theorem B) corresponds to analyzing the leftmost map in the long exact sequence, while the latter corresponds to analyzing the third module from the left. Specifically, these are described by the (first) Johnson homomorphism $\tau_{g,1}(1)$ and the *Casson–Morita homomorphism* d. The first Johnson homomorphism is particularly important in the context of the Torelli group, as mentioned above, (see Section 2A); for example, the Johnson homomorphism for the Torelli group induces the rational abelianization of the Torelli group. The Casson–Morita homomorphism is closely related to the Casson invariant for homology 3-spheres (see Section 5) and provides one of the $\mathcal{M}_{g,1}$-invariant parts of the rational abelianization of the Johnson kernel $\mathcal{K}_{g,1}$.

In relation to the Casson–Morita homomorphism d, its properties on the Chillingworth subgroup, which are used in Theorems A and D, include its invariance under the action of the mapping class group and the determination of its image. These fundamental properties are summarized in Theorem C. Furthermore, although not directly relevant to Theorems A and D, Theorem C also includes an explicit description of the kernel of d for the Chillingworth subgroup, as part of the fundamental properties of d.

Before presenting the theorems, we introduce some notation: $[-]_{\mathrm{Sp}}$ represents the linear representations of the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$ corresponding to Young diagrams. For details, see Section 4.

The rational abelianization of the Chillingworth subgroup of the mapping class group is as follows.

Theorem A *For $g \geq 6$, the rational abelianizations of the Chillingworth subgroups of the mapping class groups of the surfaces are induced by the Johnson homomorphisms and the Casson–Morita homomorphism*

$$\begin{aligned} d \oplus \tau_{g,1}(1) &: \mathrm{Ch}_{g,1} \to (\mathbb{Z} \oplus U) \otimes \mathbb{Q} \cong [0]_{\mathrm{Sp}} \oplus [1^3]_{\mathrm{Sp}}, \\ \tau_{g,*}(1) &: \mathrm{Ch}_{g,*} \to U \otimes \mathbb{Q} \cong [1^3]_{\mathrm{Sp}}, \\ \tau_g(1) &: \mathrm{Ch}_g \to \overline{U} \otimes \mathbb{Q} \cong [1^3]_{\mathrm{Sp}}, \end{aligned}$$

where U and $\overline{U}$ are images of the Chillingworth subgroups under the first Johnson homomorphisms. Specifically, their targets and the first rational homology groups of the Chillingworth subgroups are isomorphic as mapping class group modules.

In particular, the actions of the mapping class group on these abelianizations of the Chillingworth subgroups factor through the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$*, and they decompose into irreducible representations of the rational symplectic group.*

Theorem B *The image (resp. kernel) of the homomorphisms between the second rational homology (resp. cohomology) induced by the first Johnson homomorphism*

$$\tau_{g,1}(1)=\tau_{g,1}(1)|_{\mathrm{Ch}_{g,1}}:\mathrm{Ch}_{g,1}\to U\subset \textstyle\bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z})$$

for the Chillingworth subgroup for the genus-g surface with one boundary is decomposed as mapping class group modules as

$$\mathrm{Im}\big((\tau_{g,1}(1))_*:H_2(\mathrm{Ch}_{g,1};\mathbb{Q})\to H_2(U;\mathbb{Q})\big)=\begin{cases}[2^21^2]_{\mathrm{Sp}}\oplus[1^4]_{\mathrm{Sp}}\oplus[1^6]_{\mathrm{Sp}} & (g\geq 6),\\ [2^21^2]_{\mathrm{Sp}}\oplus[1^4]_{\mathrm{Sp}} & (g=5),\\ [2^21^2]_{\mathrm{Sp}} & (g=4),\\ \{0\} & (g=3),\end{cases}$$

and

$$\mathrm{Ker}\big((\tau_{g,1}(1))^*:H^2(U;\mathbb{Q})\to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})\big)=\begin{cases}[0]_{\mathrm{Sp}}\oplus[2^2]_{\mathrm{Sp}}\oplus[1^2]_{\mathrm{Sp}} & (g\geq 4),\\ [0]_{\mathrm{Sp}}\oplus[2^2]_{\mathrm{Sp}} & (g=3).\end{cases}$$

The same holds for the Chillingworth subgroup in the case of a fixed base point $\mathrm{Ch}_{g,*}$.

Theorem B is used to prove Theorem A.

We examine the fundamental properties of the Casson–Morita homomorphism d for the Chillingworth subgroup, focusing on explicitly determining its kernel, which is crucial to Theorem A.

Theorem C *The Casson–Morita homomorphism* $d=d|_{\mathrm{Ch}_{g,1}}:\mathrm{Ch}_{g,1}\to\mathbb{Z}$ *satisfies these properties:*

(1) *The Casson–Morita homomorphism d is an $\mathcal{M}_{g,1}$-invariant homomorphism on the Chillingworth subgroup.*

(2) *The image* $\mathrm{Im}(d:\mathrm{Ch}_{g,1}\to\mathbb{Z})$ *of the Casson–Morita homomorphism for the Chillingworth subgroup corresponds to* $8\mathbb{Z}$.

(3) *The kernel* $\mathrm{Ker}(d:\mathrm{Ch}_{g,1}\to\mathbb{Z})$ *of the Casson–Morita homomorphism for the Chillingworth subgroup is given by the subgroup* $\langle T_{\gamma_1'}\rangle$ *generated by Dehn twists along the boundary of a genus-one subsurface with one boundary of the surface as shown in Figure 1, left, the normal subgroup* $\langle\langle B_0\rangle\rangle\lhd\mathcal{M}_{g,1}$ *(recall that* $\langle\langle\,\bullet\,\rangle\rangle$ *denotes the normal closure) generated by a certain element* $B_0:=T_{\gamma_2'}T_{\gamma_3'}{}^{-1}$ *called the homological genus-zero bounding pair map as shown in Figure 1, right, and the commutator subgroup* $[\mathcal{K}_{g,1},\mathcal{M}_{g,1}]$ *of the Johnson kernel and the full mapping class group as follows:*

$$\mathrm{Ker}(d:\mathrm{Ch}_{g,1}\to\mathbb{Z})=\langle\langle B_0\rangle\rangle\langle T_{\gamma_1'}\rangle[\mathcal{K}_{g,1},\mathcal{M}_{g,1}].$$

Additionally, we compute the order of the Euler class of the natural central extension

$$0\to\mathbb{Z}\to\mathrm{Ch}_{g,1}\to\mathrm{Ch}_{g,*}\to 1$$

related to the natural homomorphism $\mathcal{M}_{g,1}\to\mathcal{M}_{g,*}$. This is obtained by examining d on the Chillingworth subgroup.

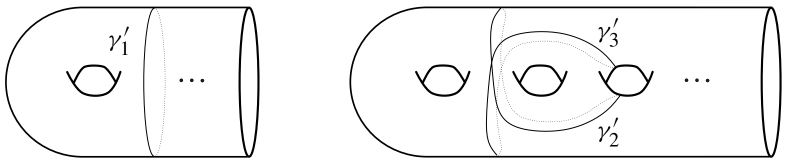


Figure 1: Left: the boundary curve γ_1' of a genus-one subsurface with one boundary of the surface defining the Dehn twist $T_{\gamma_1'}$. Right: Simple closed curves γ_2', γ_3' defining a homological genus-zero bounding pair map $B_0 := T_{\gamma_2'} T_{\gamma_3'}{}^{-1}$.

Theorem D *For $g \geq 6$, the order of the Euler class of the natural central extension*

$$0 \to \mathbb{Z} \to \mathrm{Ch}_{g,1} \to \mathrm{Ch}_{g,*} \to 1$$

equals $\frac{1}{2}g(g-1)$ in $H^2(\mathrm{Ch}_{g,*};\mathbb{Z})$, and the abelianization of the Chillingworth subgroup $(\mathrm{Ch}_{g,*})^{ab} \cong H_1(\mathrm{Ch}_{g,*};\mathbb{Z})$ for the surface with a base point has a $\frac{1}{2}g(g-1)$-torsion element.

2 Preliminaries

Let $\Sigma_{g,1}$ denote a connected, compact, oriented, genus-g surface with one boundary. We choose a base point on the boundary of the surface $\Sigma_{g,1}$ and let $\{\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g\}$ be a free generating set of the fundamental group $\pi_1(\Sigma_{g,1})$ of the surface as shown in Figure 2.

Given two elements γ_1, γ_2 in the fundamental group of the surface $\pi = \pi_1(\Sigma_{g,1})$, their product $\gamma_1\gamma_2$ indicates that we traverse γ_1 first, then γ_2. The commutator $[\gamma_1,\gamma_2]$ is defined by $\gamma_1\gamma_2\gamma_1{}^{-1}\gamma_2{}^{-1}$.

Let $H = H_1(\Sigma_{g,1};\mathbb{Z})$ be the first integral homology group of the surface and $\cdot : H \otimes H \to \mathbb{Z}$ be the intersection form of the first homology of the surface. We choose a symplectic basis $\{a_1,\ldots,a_g,b_1,\ldots,b_g\}$ of H as shown in Figure 3.

These elements are obtained through the Hurewicz homomorphism $\alpha_i \mapsto a_i$, $\beta_i \mapsto b_i$. The first integral cohomology group of the surface $H^* = H^1(\Sigma_{g,1};\mathbb{Z})$ is naturally isomorphic to the first homology group H of the surface as $\mathrm{Sp}(2g,\mathbb{Z})$-modules by the Poincaré duality: $a_i \leftrightarrow b_i^*$, $b_i \leftrightarrow -a_i^*$. Using this, henceforth H and H^* will be freely identified. Let $\mathcal{M}_{g,1}$ be the mapping class group of the surface, which is defined as the isotopy classes of orientation-preserving self-diffeomorphisms of the surface that

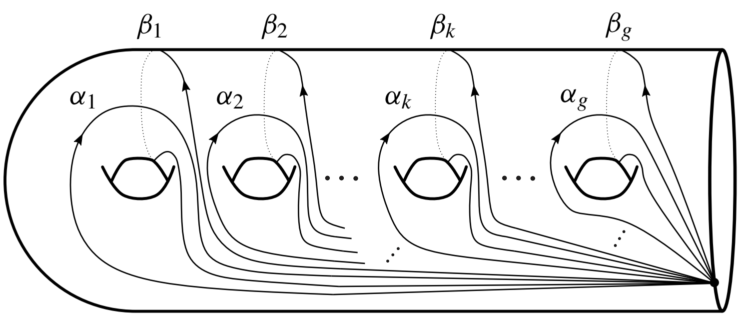


Figure 2: A generating system of the fundamental group of the surface $\pi_1(\Sigma_{g,1})$.

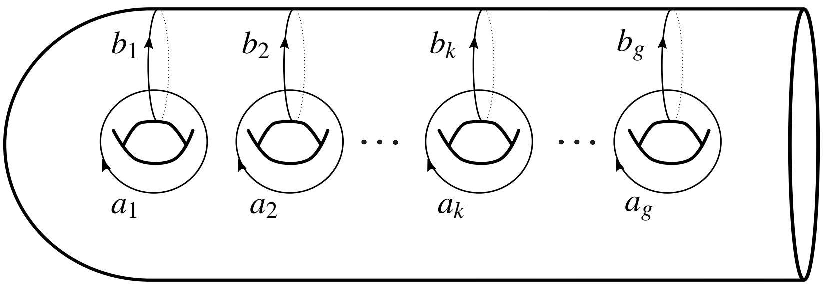


Figure 3: A symplectic basis of H.

are pointwise identities on the boundary of the surface. That is,

$$\mathcal{M}_{g,1} := \mathrm{Diff}^{(+)}(\Sigma_{g,1}, \partial\Sigma_{g,1})/(\text{isotopies fixing the boundary pointwise}).$$

A diffeomorphism that is the identity on the boundary is automatically orientation-preserving. The product $\varphi\psi$ in the mapping class group $\mathcal{M}_{g,1}$ indicates that we apply ψ first, then φ. For a simple closed curve $C \subset \mathrm{Int}(\Sigma_{g,1})$, let T_C be the (right-hand) Dehn twist along C.

2A Mapping class groups, fundamental groups, and Johnson homomorphisms

The action of the mapping class group on the fundamental group of the surface yields the Dehn–Nielsen representation $r : \mathcal{M}_{g,1} \to \mathrm{Aut}(\pi)$, which is known to be faithful. The mapping class group also acts naturally on the first integral homology group of the surface $H = H_1(\Sigma_{g,1}; \mathbb{Z})$ and this action preserves the intersection form of the surface. Hence, the mapping class acts on H as the integral symplectic group $\mathrm{Sp}(H, \cdot) \cong \mathrm{Sp}(2g, \mathbb{Z})$ and this action $\rho : \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbb{Z})$ is called the *symplectic representation*. It is known that the representation ρ is surjective classically, and we summarize in the short exact sequence

$$1 \to \mathcal{I}_{g,1} \to \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1,$$

where the kernel $\mathcal{I}_{g,1} := \mathrm{Ker}(\rho : \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbb{Z}))$ of the symplectic representation is called the *Torelli group* of the mapping class group.

The Johnson homomorphism, initially defined by Johnson, provides an abelian quotient of the Torelli group and is equivariant under the action of the mapping class group (see Johnson [17; 18]). It has been developed by Morita and formalized as a graded Lie algebra homomorphism using the free Lie algebra generated by H (see Morita [25; 29; 33]).

The mapping class group acts naturally on the nilpotent quotient of the fundamental group of the surface, denoted by $N_i := \pi/\Gamma_i$, where $\{\Gamma_i\}_{i\geq 1}$ is the lower central series of π, defined inductively by $\Gamma_1 := \pi$ and $\Gamma_{i+1} := [\Gamma_i, \pi]$.

Definition 2.1 These actions on $\{N_i\}_{i\geq 1}$ define a filtration of the mapping class group, denoted by $\mathcal{M}_{g,1}[i] := \mathrm{Ker}(\mathcal{M}_{g,1} \to \mathrm{Aut}(N_i))$, called the Johnson filtration.

Definition 2.2 The subgroup

$$\mathcal{K}_{g,1} := \langle \text{Dehn twists along bounding simple closed curves } (\textit{BSCC map}) \rangle$$

is called the Johnson kernel.

Proposition 2.3 *We have*

$$\mathcal{M}_{g,1}[1] = \mathcal{M}_{g,1}, \quad \mathcal{M}_{g,1}[2] = \mathcal{I}_{g,1} = \mathrm{Ker}(\rho), \quad \mathcal{M}_{g,1}[3] = \mathcal{K}_{g,1}.$$

The last was shown by Johnson [20].

For $\varphi \in \mathcal{M}_{g,1}[i+1]$ and $\gamma \in \pi$, we have $\varphi(\gamma)\gamma^{-1} \in \Gamma_{i+1}$ by definition. Therefore, this defines a homomorphism $\mathcal{M}_{g,1}[i+1] \to \mathrm{Hom}(H, \Gamma_{i+1}/\Gamma_{i+2})$. The associated graded abelian group $\{\Gamma_i/\Gamma_{i+1}\}_{i\geq 1}$ of $\{\Gamma_i\}_{i\geq 1}$ admits a Lie algebra structure over $\mathbb{Z}$ via commutators on π. It is well known that the associated graded Lie algebra $\{\Gamma_i/\Gamma_{i+1}\}_{i\geq 1}$ is isomorphic to $\mathcal{L}_{g,1} = \{\mathcal{L}_{g,1}[i]\}_{i\geq 1}$, which is the free Lie algebra generated by H over $\mathbb{Z}$, as a graded Lie algebra over $\mathbb{Z}$. For example, see [22]. Combining this with Poincaré duality, the homomorphism $\tau_{g,1}(i) : \mathcal{M}_{g,1}[i+1] \to H \otimes \mathcal{L}_{g,1}[i+1]$ is defined. By definition, we have $\mathrm{Ker}(\tau_{g,1}(i)) = \mathcal{M}_{g,1}[i+2]$.

Morita refined the target space using the structure of Lie algebra. Let

$$\mathfrak{h}_{g,1} = \{\mathfrak{h}_{g,1}(i)\}_{i\geq 1} := \bigl\{\mathrm{Ker}(H \otimes \mathcal{L}_{g,1}[i+1] \xrightarrow{\text{bracket}} \mathcal{L}_{g,1}[i+2])\bigr\}_{i\geq 1}$$

be the kernel of the bracket, which is a graded Lie subalgebra of $\mathrm{Hom}(H, \mathcal{L}_{g,1}) \cong \{H \otimes \mathcal{L}_{g,1}[i]\}_{i\geq 1}$.

Theorem 2.4 (Morita [28; 29]) *The image $\mathrm{Im}(\tau_{g,1}(i))$ lies in $\mathfrak{h}_{g,1}(i)$, and $\{\mathrm{Im}(\tau_{g,1}(i))\}_{i\geq 1}$ is a graded Lie subalgebra of $\mathfrak{h}_{g,1}$.*

Definition 2.5 (Morita) The homomorphism $\tau_{g,1}(i) : \mathcal{M}_{g,1}[i+1] \to \mathfrak{h}_{g,1}(i)$, known as the i-th Johnson homomorphism, is an $\mathcal{M}_{g,1}$-equivariant graded Lie algebra homomorphism

$$\{\tau_{g,1}(i)\}_{i\geq 1} : \{\mathcal{M}_{g,1}[i+1]/\mathcal{M}_{g,1}[i+2]\}_{i\geq 1} \to \{\mathfrak{h}_{g,1}(i)\}_{i\geq 1},$$

which is also called the Johnson homomorphism.

Remark 2.6 Originally, Johnson defined it as $\tau_{g,1}(1) : \mathcal{I}_{g,1} \to \bigwedge^3 H$, where

$$\textstyle\bigwedge^3 H = \{x \wedge y \wedge z := x \otimes (y \wedge z) + y \otimes (z \wedge x) + z \otimes (x \wedge y) \mid x, y, z \in H\} \subset H \otimes \bigwedge^2 H \cong H \otimes \mathcal{L}_{g,1}[2],$$

and showed in [17] its surjectivity.

The above argument gives the Johnson filtrations and the Johnson homomorphisms for the mapping class group $\mathcal{M}_{g,*}$ of the surface with a base point and the mapping class group $\mathcal{M}_g$ of the closed surface without a base point.

Definition 2.7 The homomorphism $\tau_{g,*}(i) : \mathcal{M}_{g,*}[i+1] \to \mathfrak{h}_{g,*}(i)$ is called the i-th Johnson homomorphism for $\mathcal{M}_{g,*}[i+1]$, where the target space $\mathfrak{h}_{g,*}(i)$ is defined as

$$\mathfrak{h}_{g,*} = \{\mathfrak{h}_{g,*}(i)\}_{i\geq 1} := \bigl\{\mathrm{Ker}(H \otimes \mathcal{L}_g[i+1] \xrightarrow{\text{bracket}} \mathcal{L}_g[i+2])\bigr\}_{i\geq 1},$$

where $\mathcal{L}_g := \mathcal{L}_{g,1}/\bigl(\omega_0 := \sum_{i=1}^{g}[a_i, b_i]\bigr)$. Similarly, the homomorphism $\tau_g(i) : \mathcal{M}_g[i+1] \to \mathfrak{h}_g(i)$ is called the i-th Johnson homomorphism for $\mathcal{M}_g[i+1]$, where the target space $\mathfrak{h}_g(i)$ is defined as

$$\mathfrak{h}_g := \mathfrak{h}_{g,*}/\mathcal{L}_g.$$

Remark 2.8 By a result of Labute [22], the Lie algebra $\mathcal{L}_g$ is isomorphic to $\{\Gamma_i\pi_1(\Sigma_g)/\,\Gamma_{i+1}\pi_1(\Sigma_g)\}_{i\geq 1}$, where $\Gamma_i\pi_1(\Sigma)$ is the i-th term of the lower central series of $\pi_1(\Sigma_g)$.

Remark 2.9 Originally, Johnson defined the first Johnson homomorphisms of these cases as $\tau_{g,*}(1) : \mathcal{I}_{g,*} \to \bigwedge^3 H$ and

$$\tau_g(1) : \mathcal{I}_g \to \textstyle\bigwedge^3 H/H := \bigwedge^3 H/\operatorname{Im}\Bigl(u : H \hookrightarrow \bigwedge^3 H,\ u(x) = \displaystyle\sum_{i=1}^{g} a_i \wedge b_i \wedge x\Bigr).$$

In this paper, for the sake of convenience, calculations using the first Johnson homomorphism are primarily performed using the original notation.

These Johnson homomorphisms commute with natural homomorphisms $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$ induced by collapsing the boundary and $\mathcal{M}_{g,*} \to \mathcal{M}_g$ induced by forgetting the base point. There exists the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{M}_{g,1}[i+2] \arrow[r] \arrow[d] & \mathcal{M}_{g,1}[i+1] \arrow[r, "\tau_{g,1}(i)"] \arrow[d] & \mathfrak{h}_{g,1}(i) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{M}_{g,*}[i+2] \arrow[r] \arrow[d] & \mathcal{M}_{g,*}[i+1] \arrow[r, "\tau_{g,*}(i)"] \arrow[d] & \mathfrak{h}_{g,*}(i) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{M}_{g}[i+2] \arrow[r] & \mathcal{M}_{g}[i+1] \arrow[r, "\tau_{g}(i)"] & \mathfrak{h}_{g}(i) \arrow[r] & 1
\end{tikzcd}$$

that commutes with the action of the mapping class group.

The short exact sequences

$$0 \to \mathbb{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to 1,$$
$$0 \to \mathbb{Z} \to \mathcal{I}_{g,1} \to \mathcal{I}_{g,*} \to 1,$$
$$0 \to \mathbb{Z} \to \mathcal{K}_{g,1} \to \mathcal{K}_{g,*} \to 1$$

are induced by natural homomorphisms $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$ and $\mathcal{M}_{g,*} \to \mathcal{M}_g$, where the second homomorphism from the left for each exact sequence is defined by $1 \mapsto T_\zeta$ and ζ is the boundary parallel loop of $\Sigma_{g,1}$. We also have the short exact sequences

$$1 \to \pi_1(\Sigma_g) \to \mathcal{M}_{g,*} \to \mathcal{M}_g \to 1,$$
$$1 \to \pi_1(\Sigma_g) \to \mathcal{I}_{g,*} \to \mathcal{I}_g \to 1,$$
$$1 \to [\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \to \mathcal{K}_{g,*} \to \mathcal{K}_g \to 1.$$

The second homomorphism from the left for each exact sequence is called the push map defined by dragging the base point of the fundamental group along the element of the fundamental group. More generally, Asada and Kaneko showed in [1] that $\pi_1(\Sigma_g) \cap \mathcal{M}_{g,*}[i+1] = \Gamma_i \pi_1(\Sigma_g)$ and

$$1 \to \Gamma_i \pi_1(\Sigma_g) \to \mathcal{M}_{g,*}[i+1] \to \mathcal{M}_g[i+1] \to 1,$$

where $\Gamma_i G$ is the i-th term of the lower central series of G.

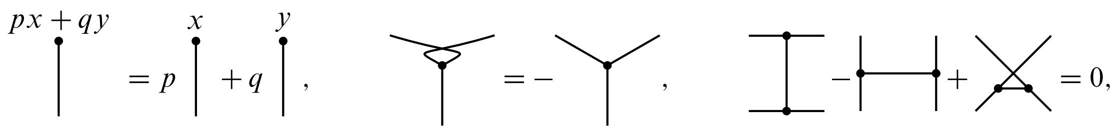
2B Tree diagrams and infinitesimal Dehn–Nielsen representations

The *infinitesimal Dehn–Nielsen representation*, introduced by Massuyeau [23], is an infinitesimal version of the Dehn–Nielsen representation $r : \mathcal{M}_{g,1} \to \mathrm{Aut}(\pi)$. It is described using an action on a certain complete Lie algebra defined by π, rather than the action on π itself.

The target space of the infinitesimal Dehn–Nielsen representation is represented by H-labeled trees (see [12; 23]) called *tree diagrams*.

Definition 2.10 A tree diagram is a finite, connected, unitrivalent graph whose trivalent vertices have cyclic order, and univalent vertices are colored by an element of H. The trivalent vertices of a tree diagram are called *nodes*, univalent vertices are called *leaves*, and the number of nodes in a tree diagram is called its *degree*.

Definition 2.11 [12; 23] Define $\mathcal{T}_d(H)$ as the free abelian group generated by degree-d tree diagrams modulo the relations



multilinearity AS relation (antisymmetry) IHX relation (Jacobi identity)

where $x, y \in H$ and $p, q \in \mathbb{Z}$. We define $\mathcal{T}(H) := \bigoplus_{d=1}^{\infty} \mathcal{T}_d(H)$, and $\widehat{\mathcal{T}(H)}$ as the degree completion of $\mathcal{T}(H)$. Similarly, we can define $\mathcal{T}(H_{\mathbb{Q}})$ over $\mathbb{Q}$ by taking the tensor product with $\mathbb{Q}$, giving us $\mathcal{T}(H_{\mathbb{Q}}) = \mathcal{T}(H) \otimes \mathbb{Q}$, and similarly for its completion $\widehat{\mathcal{T}(H_{\mathbb{Q}})}$, where the subscript $\mathbb{Q}$ means taking the tensor product $- \otimes \mathbb{Q}$.

Additionally, $\mathcal{T}(H)$ forms a graded Lie algebra over $\mathbb{Z}$, with the bracket $[\bullet, \bullet]_{\mathcal{T}}$ defined as

$$[P, Q]_{\mathcal{T}} := \sum_{\substack{v \in \mathrm{leaves}(P) \\ w \in \mathrm{leaves}(Q)}} (\mathrm{col}(P_v) \cdot \mathrm{col}(Q_w))(\text{graph obtained by gluing } P \text{ and } Q \text{ at } v \text{ and } w),$$

where leaves(P) is the set of leaves of P, col(P_v) is the color of the univalent vertex v, and P_v is the rooted tree obtained by viewing P as a rooted tree with root at vertex v. This bracket on $\mathcal{T}(H)$ is uniquely extended to the continuous bracket $[\bullet, \bullet]_{\widehat{\mathcal{T}}}$ on $\widehat{\mathcal{T}(H)}$. Then $(\widehat{\mathcal{T}(H)}, [\bullet, \bullet]_{\widehat{\mathcal{T}}})$ forms a complete graded Lie algebra over $\mathbb{Z}$. We can define similarly $(\widehat{\mathcal{T}(H_{\mathbb{Q}})}, [\bullet, \bullet]_{\widehat{\mathcal{T}}})$ over $\mathbb{Q}$.

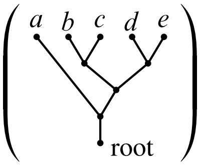
The direct sum $\mathfrak{h}_{g,1} = \bigoplus_{i=1}^{\infty} \mathfrak{h}_{g,1}(i)$ of the target spaces of the Johnson homomorphisms forms a Lie subalgebra of $\bigoplus_{i=1}^{\infty} H \otimes \mathcal{L}_{g,1}[i+1]$, and there exists a Lie algebra homomorphism $\eta : \mathcal{T}(H) \to \mathfrak{h}_{g,1}$.

Definition 2.12 The Lie algebra homomorphism $\eta : \mathcal{T}(H) \to \mathfrak{h}_{g,1}$ is defined by

$$\eta(P) := \sum_{v \in \mathrm{leaves}(P)} \mathrm{col}(P_v) \otimes \mathrm{brack}(P_v),$$

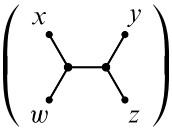
where $\mathrm{brack}(P_v)$ is the bracketification map defined by taking the iterated bracket as

$$\mathrm{brack}\left(\quad \right) = \big[a, [[b, c], [d, e]]\big].$$



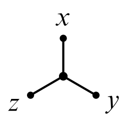
For example, in the case of $d = 2$,

$$\eta\left(\quad \right) = x \otimes [[y, z], w] + y \otimes [z, [w, x]] + z \otimes [[w, x], y] + w \otimes [x, [y, z]].$$



Especially, in the case of $d = 1$, we have the correspondence

$$\begin{matrix} x \wedge y \wedge z \\ \in \bigwedge^3 H \end{matrix} \longleftrightarrow \begin{matrix} x \otimes [y, z] + y \otimes [z, x] + z \otimes [x, y] \\ \in H \otimes \mathcal{L}_{g,1}[2] \end{matrix} \longleftrightarrow \quad \in \mathcal{T}_1(H).$$



The homomorphism η is not an isomorphism over $\mathbb{Z}$ (see [12]). However, if we take the tensor product with $\mathbb{Q}$, we have $\eta_{\mathbb{Q}} := \eta \otimes \mathrm{id}_{\mathbb{Q}}$, which is an isomorphism of Lie algebras over $\mathbb{Q}$ (see [14, Theorem 2]).

Definition 2.13 (Massuyeau [23]) Let $\widehat{\mathcal{L}_{g,1\mathbb{Q}}}$ be the degree completion of $\mathcal{L}_{g,1\mathbb{Q}} = \bigoplus_{i=1}^{\infty} \mathcal{L}_{g,1\mathbb{Q}}[i]$. The *symplectic expansion* (*logansion*), introduced by Massuyeau, is a map $\theta : \pi \to \widehat{\mathcal{L}_{g,1\mathbb{Q}}}$ that satisfies the following conditions:

(1) $\theta : \pi \to (\widehat{\mathcal{L}_{g,1\mathbb{Q}}}, \star)$ is a group homomorphism, where $\star$ is the product defined by the *Baker–Campbell–Hausdorff series* (*BCH product*) with respect to the bracket $[\bullet, \bullet]$ of $\mathcal{L}_{g,1\mathbb{Q}}$.

(2) $\theta(x) = [x] + (\text{degree} \geq 2)$, where $[x] \in H$ is the image of $x \in \pi$ under the Hurewicz homomorphism.

(3) $\theta(\zeta) = -\omega := -\sum_{i=1}^{g} [a_i, b_i]$, where $\zeta \in \pi$ is the boundary parallel loop of $\Sigma_{g,1}$.

Theorem 2.14 (Massuyeau [23, Lemma 2.16]) *There does exist a symplectic expansion* $\theta : \pi \to \widehat{\mathcal{L}_{g,1\mathbb{Q}}}$.

Before introducing the definition of the infinitesimal Dehn–Nielsen representation, we will introduce several maps. Let us fix a symplectic expansion $\theta : \pi \to \widehat{\mathcal{L}_{g,1\mathbb{Q}}}$.

First, we introduce the map $\varrho^\theta : \mathcal{I}_{g,1} \to \mathrm{IAut}_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}})$, where $\mathrm{IAut}_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}})$ is the automorphism group of $\widehat{\mathcal{L}_{g,1\mathbb{Q}}}$ whose elements induce the identity map on

$$\mathrm{gr}\mathcal{L}_{g,1\mathbb{Q}} = \bigoplus_{i=1}^{\infty} (\mathcal{L}_{g,1\mathbb{Q}}[i] / \mathcal{L}_{g,1\mathbb{Q}}[i+1])$$

and preserve the element $\omega = \sum_{i=1}^{g}[a_i, b_i] \in \mathcal{L}_{g,1}[2]$. The value $\varrho^\theta(f)$ for $f \in \mathcal{M}_{g,1}$ is induced by $x \mapsto f_*\theta(x)$.

Next, define $\log : \mathrm{IAut}_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}}) \to (\mathrm{Der}^+_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}}), \star)$, where the target space $\mathrm{Der}^+_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}})$ is the space of derivations of $\widehat{\mathcal{L}_{g,1\mathbb{Q}}}$ that strictly increase degrees and are trivial on ω. The target space has a natural Lie algebra structure. If we define the group structure by the BCH product induced from this bracket, then the map log defined by the series induces a group isomorphism (the inverse map is exp defined by the series).

Finally, the map $\mathrm{Der}^+_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}}) \to \widehat{\mathfrak{h}_{g,1}}$ is a canonical isomorphism induced by Poincaré duality. Specifically, it is the map induced by $D \mapsto \sum_{i=1}^{g}(b_i \otimes D(a_i) - a_i \otimes D(b_i))$.

Combining the above three maps with the map $\widehat{\eta_{\mathbb{Q}}^{-1}} : \widehat{\mathfrak{h}_{g,1}} \to \widehat{\mathcal{T}(H_{\mathbb{Q}})}$, we obtain the infinitesimal Dehn–Nielsen representation.

Definition 2.15 The infinitesimal Dehn–Nielsen representation $r^\theta : \mathcal{I}_{g,1} \to (\widehat{\mathcal{T}(H_{\mathbb{Q}})}, \star)$ is defined as the composition of the homomorphisms

$$\begin{tikzcd}
 & \mathrm{IAut}_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}}) \arrow[r, "\log"] & (\mathrm{Der}^+_\omega(\widehat{\mathcal{L}_{g,1\mathbb{Q}}}), \star) \arrow[r] & (\widehat{\mathfrak{h}_{g,1}}, \star) \arrow[dr] & \\
\mathcal{I}_{g,1} \arrow[ur, "\varrho^\theta"] \arrow[rrrr, dashed, "r^\theta"] & & & & (\widehat{\mathcal{T}(H_{\mathbb{Q}})}, \star)
\end{tikzcd}$$

We also define its degree-d part by composing with the projection $r_d^\theta : \mathcal{I}_{g,1} \xrightarrow{r^\theta} \widehat{\mathcal{T}(H_{\mathbb{Q}})} \to \mathcal{T}_d(H_{\mathbb{Q}})$; in particular, $\eta_{\mathbb{Q}} \circ r_i^\theta|_{\mathcal{M}_{g,1}[i+1]} : \mathcal{M}_{g,1}[i+1] \to \mathfrak{h}_{g,1\mathbb{Q}}(i)$ is nothing but the i-th Johnson homomorphism $\tau_{g,1}(i) : \mathcal{M}_{g,1}[i+1] \to \mathfrak{h}_{g,1}(i)$. Hence, the infinitesimal Dehn–Nielsen representation $r^\theta : \mathcal{I}_{g,1} \to \widehat{\mathcal{T}(H_{\mathbb{Q}})}$ on the $(i+1)$-st depth of the Johnson filtration $\mathcal{M}_{g,1}[i+1]$ is trivial up to degree-$(i-1)$-st part.

3 The action on the sets of homotopy classes of vector fields and the Chillingworth subgroups

Let X be a nonsingular vector field on the surface $\Sigma_{g,1}$ and $\Xi(\Sigma_{g,1})$ be the set of homotopy classes of nonsingular vector fields on the surface. A homotopy class of nonsingular vector fields on a surface induces a trivialization of the unit tangent bundle $\mathrm{UT}\Sigma_{g,1} \xrightarrow{\cong} \Sigma_{g,1} \times S^1$ of the surface up to homotopy.

Let γ be an oriented regular closed curve on the surface. The winding number of γ with respect to X denoted by $\omega_X(\gamma)$ is defined by the number of times its tangent transversely intersects with the section of the unit tangent bundle $\mathrm{UT}\Sigma_{g,1} \to \Sigma_{g,1}$ induced by X. Alternatively, we can compute the winding number by counting the points where the velocity vector is tangent to the vector field X, with the sign as shown in Figure 4.

The winding number function ω_X can be regarded as an element of $H^1(\mathrm{UT}\Sigma_{g,1}; \mathbb{Z})$. This element is characterized by the preimage of $1 \in H^1(S^1; \mathbb{Z})$ under the map $H^1(\mathrm{UT}\Sigma_{g,1}; \mathbb{Z}) \to H^1(S^1; \mathbb{Z})$. Conversely, for an arbitrary element $\omega \in H^1(\mathrm{UT}\Sigma_{g,1}; \mathbb{Z})$ which satisfies the condition, there exists a nonsingular vector field $X \in \Xi(\Sigma_{g,1})$ such that $\omega = \omega_X \in H^1(\mathrm{UT}\Sigma_{g,1}; \mathbb{Z})$; one can construct such an X

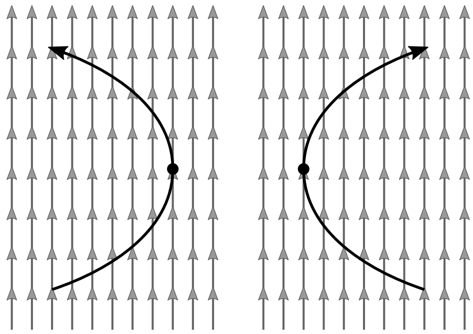


Figure 4: Signs of points where the velocity vector is tangent to the vector field.

by considering $\Sigma_{g,1}$ as a disk with $2g$ attached 1-handles and specifying the vector field on each 1-handle. This correspondence $\Xi(\Sigma_{g,1}) \leftrightarrow \{\text{preimage of } 1\} \subset H^1(\mathrm{UT}\Sigma_{g,1};\mathbb{Z})$ is one-to-one.

The action of the mapping class group $\mathcal{M}_{g,1}$ of the surface on $\Xi(\Sigma_{g,1})$ is described by the $H^1(\Sigma_{g,1};\mathbb{Z})$-affine space structure and *the Chillingworth homomorphism*, which is defined using the winding number function ω_X.

Let us fix a nonsingular vector field $X \in \Xi(\Sigma_{g,1})$. We recall the short exact sequence of the first cohomology

$$0 \to H^1(\Sigma_{g,1};\mathbb{Z}) \to H^1(\mathrm{UT}\Sigma_{g,1};\mathbb{Z}) \to H^1(S^1;\mathbb{Z}) \to 0,$$

which is equivariant under the action of the mapping class group.

Definition 3.1 For a nonsingular vector field X, the Chillingworth homomorphism $e_X : \mathcal{M}_{g,1} \to H^1(\Sigma_{g,1};\mathbb{Z})$ is defined by the equality $e_X(f)([\gamma]) := \omega_X(f \circ \gamma) - \omega_X(\gamma)$.

The Chillingworth *homomorphism* is not a homomorphism but a crossed homomorphism. It satisfies $e_X(fg) = e_X(f) + (f^{-1})^* e_X(g)$. The kernel of the Chillingworth homomorphism $\mathrm{Ker}(e_X) := {e_X}^{-1}(0)$ is a subgroup of the mapping class group that preserve the chosen vector field X up to homotopy. In particular, the Chillingworth homomorphism e_X depends on the choice of a vector field X.

The construction of a crossed homomorphism on the mapping class group using the unit tangent bundle and the winding number was also proposed by Mikio Furuta. For details, see Morita [32]. Earle independently introduced an essentially identical crossed homomorphism using a different approach. For details, see [10].

Let us consider the restriction of the Chillingworth homomorphism to the Torelli subgroup. The restricted Chillingworth homomorphism $e_X|_{\mathcal{I}_{g,1}}$ is a homomorphism in the usual sense. Moreover, the restricted Chillingworth homomorphism does not depend on the choice of a nonsingular vector field on the surface. This is because, due to the short exact sequence, the difference $\omega_X - \omega_{X'}$ for different nonsingular vector fields X and X' can be expressed by an element $h \in H^1(\Sigma_{g,1};\mathbb{Z})$. From this, we have $e_X(f) - e_{X'}(f) = (f^{-1})^* h - h$, whose right-hand side is always zero on the Torelli group $\mathcal{I}_{g,1}$.

Definition 3.2 The Chillingworth subgroup $\mathrm{Ch}_{g,1}$ is defined by the kernel $\mathrm{Ker}(e_X|_{\mathcal{I}_{g,1}})$ of the restricted Chillingworth homomorphism, and the Chillingworth subgroup of the surface with the base point $\mathrm{Ch}_{g,*}$ is defined similarly. We define the Chillingworth subgroup of the closed surface without a base point Ch_g as the image of the Chillingworth subgroup under the natural homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to \mathcal{M}_g$.

Morita proved that $H^1(\mathcal{M}_{g,1}; H^*) \cong H^1(\mathcal{M}_{g,1}; H)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$ in [26, Proposition 6.4] and that the crossed homomorphism (twisted 1-cocycle) e_X is a generator of $H^1(\mathcal{M}_{g,1}; H^*)$ in [32, Proposition 4.1]. Hence, the Chillingworth subgroup is characterized as below.

Proposition 3.3 (see [5; 7; 8; 41]) *The Chillingworth subgroup* $\mathrm{Ch}_{g,1}$ *has the following characterizations*:

(1) *the subgroup of the mapping class group whose elements preserve all nonsingular vector fields up to homotopy*;

(2) *the kernel* $\mathrm{Ker}(\mathcal{M}_{g,1} \curvearrowright \Xi(\Sigma_{g,1}))$ *of the action on the set of homotopy classes of nonsingular vector fields on the surface*;

(3) *the intersection of the kernel of a nontrivial crossed homomorphism with values in H or H^* and the Torelli group* $\mathcal{I}_{g,1}$;

(4) *the kernel* $\mathrm{Ker}(\mathcal{M}_{g,1} \curvearrowright H^1(\mathrm{UT}\Sigma_{g,1}; \mathbb{Z}))$ *of the action on the first cohomology of the unit tangent bundle of the surface*;

(5) *the kernel* $\mathrm{Ker}(\mathcal{M}_{g,1} \curvearrowright H_1(\mathrm{UT}\Sigma_{g,1}; \mathbb{Z}))$ *of the action on the first homology of the unit tangent bundle of the surface*;

(6) *the kernel* $\mathrm{Ker}(\mathcal{M}_{g,1} \curvearrowright \mathcal{H})$ *of the action on the Heisenberg group of the surface, where $\mathcal{H}$ is the Heisenberg group of the surface defined by $\mathcal{H} = \mathbb{Z} \times H$ as a set with the product defined by* $(n, x)(m, y) = (n + m + x \cdot y, x + y)$;

(7) *the kernel* $\mathrm{Ker}(\Phi_X : \mathcal{M}_{g,1} \to \mathrm{GL}(2g + 1, \mathbb{Z}))$ *of Trapp's representation, which is defined by* $\Phi_X(f) = \left[\begin{smallmatrix} 1 & e_X(f) \\ 0 & \rho(f) \end{smallmatrix}\right]$.

First, there exists the following relationship among the Chillingworth subgroup, the Torelli group, and the Johnson kernel.

Lemma 3.4 *For $g \geq 3$, we have $\mathcal{K}_{g,1} \subsetneq \mathrm{Ch}_{g,1} \subsetneq \mathcal{I}_{g,1}$, and for $g = 2$, we have $\mathcal{K}_{2,1} = \mathrm{Ch}_{2,1} \subsetneq \mathcal{I}_{2,1}$. Similarly, for $g \geq 3$, we have $\mathcal{K}_{g,*} \subsetneq \mathrm{Ch}_{g,*} \subsetneq \mathcal{I}_{g,*}$ and $\mathcal{K}_g \subsetneq \mathrm{Ch}_g \subsetneq \mathcal{I}_g$, and for $g = 2$, we have $\mathcal{K}_{2,*} = \mathrm{Ch}_{2,*} \subsetneq \mathcal{I}_{g,*}$ and $\mathcal{K}_2 = \mathrm{Ch}_2 = \mathcal{I}_2$.*

Furthermore, Ch_g is a finite-index normal subgroup of $\mathcal{I}_g$.

Lemma 3.5 *For $g \geq 3$, Ch_g is a normal subgroup of index $(g-1)^{2g}$ in $\mathcal{I}_g$, and the quotient $\mathcal{I}_g/\mathrm{Ch}_g$ is isomorphic to $(\mathbb{Z}/(g-1)\mathbb{Z})^{2g}$.*

Additionally, the relationships between the Chillingworth subgroups in each case can be summarized by the following short exact sequences.

Proposition 3.6 *There exist two short exact sequences*

$$0 \to \mathbb{Z} \to \mathrm{Ch}_{g,1} \to \mathrm{Ch}_{g,*} \to 1, \quad 1 \to [\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \to \mathrm{Ch}_{g,*} \to \mathrm{Ch}_g \to 1.$$

Lemmas 3.4–3.5 and Proposition 3.6 can be seen from the relationship between the Chillingworth subgroup and the Johnson homomorphism, which will be explained in the next subsection.

3A The first Johnson homomorphisms and the Chillingworth subgroups

Johnson [17] introduced the element $t_f \in H$ as the Poincaré dual of the value of the Chillingworth homomorphism. It is characterized by the property that $x \cdot t_f = e_X(f)(x)$ for all $x \in H_1(\Sigma_{g,1}; \mathbb{Z})$. Here, $t_\bullet$ is called the *Chillingworth class*. Johnson proved that the Chillingworth class factors through the first Johnson homomorphism.

Lemma 3.7 (Johnson [17, Theorem 2]) *The diagram*

$$\begin{tikzcd} \mathcal{I}_{g,1} \arrow[r, "\tau_{g,1}(1)"] \arrow[dr, "t"'] & \bigwedge^3 H \arrow[d, "2C_3"] \\ & H \end{tikzcd}$$

is $\mathcal{M}_{g,1}$-equivariant and commutative. Here, the $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant homomorphism $C_3 : \bigwedge^3 H \to H$ is defined by $x \wedge y \wedge z \mapsto (x \cdot y)z + (y \cdot z)x + (z \cdot x)y$ and called the ***contraction****.*

Moreover, Morita [30] constructed an extension of the Johnson homomorphism as a crossed homomorphism to the mapping class group $\mathcal{M}_{g,1} \to \frac{1}{2} \bigwedge^3 H$. By composing this extension with $2C_3$, one can also obtain a crossed homomorphism on $\mathcal{M}_{g,1}$.

Here, we introduce certain elements of a Torelli group that will appear in subsequent discussions.

Definition 3.8 For two disjoint nonseparating simple closed curves γ_1 and γ_2 on the surface $\Sigma_{g,1}$, when there exists a subsurface with genus h with the boundary components equal to $\gamma_1 \cup \gamma_2$, we call the map $\mathrm{BP}(\gamma_1, \gamma_2) := T_{\gamma_1} {T_{\gamma_2}}^{-1}$ the *genus-h bounding pair map* (*BP map*), which is an element of the Torelli group.

To prove Lemmas 3.4–3.5 and Proposition 3.6, we introduce some calculation formulas.

Proposition 3.9 (Johnson [17, Lemma 4A]) *Let $\{x_i, y_i\}_{i=1,\dots,h}$ be a symplectic basis of the first homology group of the subsurface defining a genus-h BP map $\mathrm{BP}(\gamma_1, \gamma_2)$. Then, we have*

$$\tau_{g,1}(1)(\mathrm{BP}(\gamma_1, \gamma_2)) = \sum_{i=1}^{h} x_i \wedge y_i \wedge [\gamma_1],$$

where γ_1 and γ_2 are endowed with the orientation induced from the subsurface.

Furthermore, using $C_3(x_i \wedge y_i \wedge [\gamma_1]) = [\gamma_1]$, we obtain the following:

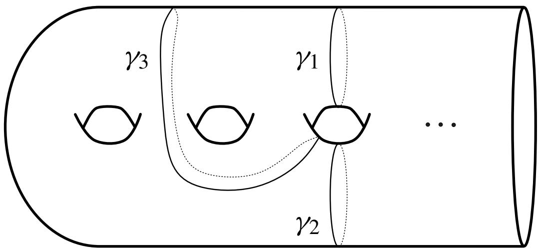


Figure 5: Some simple closed curves on the surface defining some BP maps.

Lemma 3.10 *Let* $\mathrm{BP}(\gamma_1, \gamma_2)$ *be a genus-h BP map. We have*

$$t_{\mathrm{BP}(\gamma_1,\gamma_2)} = 2C_3 \circ \tau_{g,1}(1)(\mathrm{BP}(\gamma_1, \gamma_2)) = 2h[\gamma_1].$$

Now, we prove Lemma 3.4.

Proof For $g = 2$, the contraction $C_3 : \bigwedge^3 H \to H$ is an isomorphism, and its kernel is trivial. Hence, the conditions $\tau_{2,1}(1)(f) = 0$ and $t_f = 0$ are equivalent, which implies that $\mathrm{Ch}_{2,1} = \mathcal{K}_{2,1}$. It follows that $\mathrm{Ch}_{2,*} = \mathcal{K}_{2,*}$ and $\mathrm{Ch}_2 = \mathcal{K}_2$. Moreover, in the case of genus-two closed surfaces without a base point, the target space of the first Johnson homomorphism $\bigwedge^3 H/H$ is trivial. Therefore $\mathcal{K}_2 = \mathcal{I}_2$; in particular, $\mathcal{K}_2 = \mathrm{Ch}_2 = \mathcal{I}_2$. Next, consider the case of $g \geq 3$ with one boundary component. Consider a genus-one BP map. This element is contained in $\mathcal{I}_{g,1}$ but not contained in $\mathrm{Ch}_{g,1}$, as its value under $t = 2C_3 \circ \tau_{g,1}(1)$ is nontrivial, as shown in Lemma 3.10. Therefore, we have $\mathrm{Ch}_{g,1} \subsetneq \mathcal{I}_{g,1}$. For $g \geq 3$, let us consider the element $\mathrm{BP}(\gamma_1, \gamma_2)\,\mathrm{BP}(\gamma_1, \gamma_3)^{-2}$ as in Figure 5. This element is contained in $\mathrm{Ch}_{g,1}$ but not in $\mathcal{K}_{g,1}$. Specifically, $2C_3 \circ \tau_{g,1}(1)(\mathrm{BP}(\gamma_1, \gamma_2)\,\mathrm{BP}(\gamma_1, \gamma_3)^{-2}) = 2C_3(a_1 \wedge b_1 \wedge b_3 - a_2 \wedge b_2 \wedge b_3) = 0$, indicating this element is contained in the Chillingworth subgroup $\mathrm{Ch}_{g,1}$. However, $\tau_{g,1}(1)(\mathrm{BP}(\gamma_1, \gamma_2)\,\mathrm{BP}(\gamma_1, \gamma_3)^{-2}) = a_1 \wedge b_1 \wedge b_3 - a_2 \wedge b_2 \wedge b_3$ is nonzero, implying this element is not contained in the Johnson kernel $\mathcal{K}_{g,1}$. Therefore, we have $\mathcal{K}_{g,1} \subsetneq \mathrm{Ch}_{g,1}$. The same argument can be applied for $\mathrm{Ch}_{g,*}$ for $g \geq 2$, and Ch_g for $g \geq 3$ cases. □

Before Lemma 3.5, we discuss Proposition 3.6. Johnson [17] discusses the kernel of the homomorphism $\mathcal{I}_{g,1} \to \mathcal{I}_g$. We obtain the following:

Lemma 3.11 (Johnson [17]) *By composing the push map* $\pi_1(\Sigma_g) \hookrightarrow \mathcal{I}_{g,*}$ *with the first Johnson homomorphism* $\tau_{g,*}(1) : \mathcal{I}_{g,*} \to \bigwedge^3 H$, *we obtain*

$$\tau_{g,*}(1)(\gamma) = -\sum_{i=1}^{g} a_i \wedge b_i \wedge [\gamma],$$

for $\gamma \in \pi_1(\Sigma_g)$. *In particular,* $t_\gamma = -2(g-1)[\gamma]$.

This shows that $t \mod (2g-2) : \mathcal{I}_g \to H \otimes (\mathbb{Z}/(2g-2)\mathbb{Z})$ is well defined.

Using this lemma, we will now proceed to prove Proposition 3.6.

Proof Since
$$[\pi_1(\Sigma_g), \pi_1(\Sigma_g)] = \mathcal{K}_{g,*} \cap \pi_1(\Sigma_g) \subset \mathrm{Ch}_{g,*} \cap \pi_1(\Sigma_g),$$
we only need to prove that
$$[\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \supset \mathrm{Ch}_{g,*} \cap \pi_1(\Sigma_g).$$
Let γ be an element of $\pi_1(\Sigma_g) \cap \mathrm{Ch}_*$. Since γ is contained in $\mathrm{Ch}_{g,*}$, we have $t_\gamma = 0$. From Lemma 3.11, we have $-2(g-1)[\gamma] = 0$ in $H = H_1(\Sigma_{g,*}; \mathbb{Z}) \cong H_1(\Sigma_g; \mathbb{Z}) \cong \pi_1(\Sigma_g)/[\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$. Since H is a free abelian group, $[\gamma] = 0$ in $\pi_1(\Sigma_g)/[\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$. Therefore $\gamma \in [\pi_1(\Sigma_g), \pi_1(\Sigma_g)]$. □

We denote the kernel of the contraction $\mathrm{Ker}(C_3) \subset \bigwedge^3 H$ as U. Note that U is a rank-$\left(\binom{2g}{3} - 2g\right)$ free abelian group and a $\mathrm{Sp}(2g, \mathbb{Z})$-submodule of $\bigwedge^3 H$. We denote the image of U under the natural homomorphism $U \hookrightarrow \bigwedge^3 H \to \bigwedge^3 H/H$ as $\overline{U}$. By definition, these coincide with the images of the Chillingworth subgroups under the Johnson homomorphisms: $\tau_{g,1}(1)(\mathrm{Ch}_{g,1}) = \tau_{g,*}(1)(\mathrm{Ch}_{g,*}) = U$ and $\tau_g(1)(\mathrm{Ch}_g) = \overline{U}$.

Finally, we prove Lemma 3.5, which follows from Lemma 3.12. Before stating the lemma, we define the map $v : H \oplus U \to \bigwedge^3 H$ as
$$v : H \oplus U \to \textstyle\bigwedge^3 H, \quad (x, Y) \mapsto \left(\sum_{i=1}^{g} a_i \wedge b_i \wedge x\right) + Y.$$

Lemma 3.12 *For $g \geq 3$, the quotient*
$$\mathcal{I}_g/\mathrm{Ch}_g \cong \left(\textstyle\bigwedge^3 H/H\right)/\overline{U} = \mathrm{Coker}\left(v : H \oplus U \to \textstyle\bigwedge^3 H\right)$$
is isomorphic to $(\mathbb{Z}/(g-1)\mathbb{Z})^{2g}$.

Proof Let us take a basis of U as

(i) $a_i \wedge a_j \wedge a_k$, $b_i \wedge b_j \wedge b_k$ for distinct i, j, k,

(ii) $a_i \wedge a_j \wedge b_k$, $a_i \wedge b_j \wedge b_k$ for distinct i, j, k,

(iii) $a_1 \wedge a_2 \wedge b_2 - a_1 \wedge a_i \wedge b_i$ for $i \geq 3$, $a_j \wedge a_1 \wedge b_1 - a_j \wedge a_i \wedge b_i$ for $i \geq 3$, $j \geq 2$, $i \neq j$, $b_1 \wedge a_2 \wedge b_2 - a_1 \wedge b_i \wedge b_i$ for $i \geq 3$, and $b_j \wedge a_1 \wedge b_1 - b_j \wedge a_i \wedge b_i$ for $i \geq 3$, $j \geq 2$, $i \neq j$,

and take a basis of $\bigwedge^3 H$ as (i), (ii), (iii), and

(iv) $a_i \wedge a_j \wedge b_j$ for $i \neq j$, $b_i \wedge a_j \wedge b_j$ for $i \neq j$.

The representation matrix of $v : H \oplus U \to \bigwedge^3 H$ with respect to the above basis is
$$(I_{\binom{g}{3}})^{\oplus 2} \oplus (I_{g\binom{g-1}{2}})^{\oplus 2} \oplus \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & & & & 0 \\ 1 & & -1 & & & \vdots \\ 1 & & & -1 & & \\ \vdots & & & & \ddots & 0 \\ 1 & 0 & \cdots & & 0 & -1 \end{bmatrix}^{\oplus 2g},$$

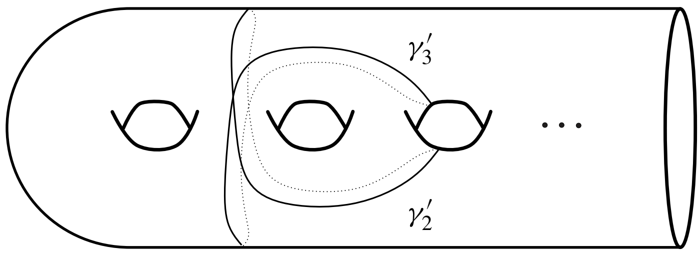


Figure 6: Some simple closed curves on the surface defining the element B_0.

where I_n is the identity matrix of size $n \times n$ and the rightmost matrix is of $(g-1)\times(g-1)$ size. We compute the invariant factor of it as

$$(I_{\binom{g}{3}})^{\oplus 2} \oplus (I_{g\binom{g-1}{2}})^{\oplus 2} \oplus \begin{bmatrix} g-1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}^{\oplus 2g}.$$

Hence, the quotient $\mathcal{I}_g/\mathrm{Ch}_g$ is isomorphic to $(\mathbb{Z}/(g-1)\mathbb{Z})^{2g}$. □

Lemma 3.12 is restated as follows.

Proposition 3.13 *By Lemma 3.11, the map* $t \mod (2g-2) : \mathcal{I}_g \to H \otimes \mathbb{Z}/(2g-2)\mathbb{Z}$ *is a well-defined homomorphism, with its kernel being* Ch_g, *and its image being* $2H \otimes \mathbb{Z}/(2g-2)\mathbb{Z} \cong H \otimes \mathbb{Z}/(g-1)\mathbb{Z}$.

As stated previously, the composition $U \to \bigwedge^3 H \to \bigwedge^3 H/H$ is not an isomorphism. However, if we take the tensor product with $\mathbb{Q}$, then the composition

$$U \otimes \mathbb{Q} \to \left(\bigwedge^3 H\right) \otimes \mathbb{Q} \to \left(\bigwedge^3 H/H\right) \otimes \mathbb{Q}$$

becomes an isomorphism as $\mathrm{Sp}(2g,\mathbb{Q})$-modules. We use the notation $\bigwedge^3 H_{\mathbb{Q}} := \left(\bigwedge^3 H\right) \otimes \mathbb{Q} = \bigwedge^3 (H_{\mathbb{Q}})$, $U_{\mathbb{Q}} := U \otimes \mathbb{Q}$ and so forth.

Proposition 3.14 *For $g \geq 4$, Chillingworth subgroups* $\mathrm{Ch}_{g,1}$, $\mathrm{Ch}_{g,*}$ *and* Ch_g *are normally generated by one element and the Johnson kernel in the full mapping class group.*

Proof We consider the exact sequence

$$1 \to \mathcal{K}_{g,1} \to \mathrm{Ch}_{g,1} \to U \to 1$$

induced by the Johnson homomorphism for the Chillingworth subgroup. The Chillingworth subgroup $\mathrm{Ch}_{g,1}$ is generated by $\mathcal{K}_{g,1}$ together with lifts of elements of U under the surjective homomorphism $\tau_{g,1}(1) : \mathrm{Ch}_{g,1} \to U$. Let us take the conjugacy class of a certain element $B_0 := \mathrm{BP}(\gamma_2', \gamma_3') := T_{\gamma_2'} T_{\gamma_3'}^{-1}$ as shown in Figure 6 (which we call a *homological genus-zero* (*or one minus one*) *bounding pair map*). The image of this conjugacy class under the Johnson homomorphism is surjective onto U. Equivalently, U is generated by $\tau_{g,1}(1)(B_0) = a_1 \wedge b_1 \wedge b_3 - a_2 \wedge b_2 \wedge b_3$ as an $\mathrm{Sp}(2g,\mathbb{Z})$-module. To show this, it suffices to construct all the basis elements of U given in the proof of Lemma 3.12(i), (ii), and (iii) by applying appropriate elements of $\mathrm{Sp}(2g,\mathbb{Z})$ to $a_1\wedge b_1\wedge b_3-a_2\wedge b_2\wedge b_3$. By suitably permuting the indices and applying the matrices determined by $a_i\mapsto b_i\mapsto -a_i$, we can construct all the elements of type (iii). Next, if we subtract the result of applying

$$\begin{cases} a_1\mapsto a_1+b_1-b_4, \\ a_4\mapsto a_4+b_4-b_1, \\ a_i\mapsto a_i \quad (i\neq 1,4), \\ b_i\mapsto b_i \end{cases}$$

to the original element $a_1\wedge b_1\wedge b_3-a_2\wedge b_2\wedge b_3$, we obtain $b_4\wedge b_1\wedge b_3$. From this element, we can similarly construct elements of types (i) and (ii) using the same argument. Therefore, $\mathrm{Ch}_{g,1}$ is normally generated by B_0 and the Johnson kernel. For $\mathrm{Ch}_{g,*}$ and Ch_g, we obtain similar results via the natural surjective homomorphisms $\mathrm{Ch}_{g,1}\to\mathrm{Ch}_{g,*}$ and $\mathrm{Ch}_{g,*}\to\mathrm{Ch}_g$. □

4 Proof of Theorem B

By the general theory of representation, a finite-dimensional polynomial representation of the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$ corresponds bijectively to those of $\mathrm{Sp}(2g,\mathbb{C})$ and the Lie algebra $\mathfrak{sp}(2g,\mathbb{C})$. These representations are parametrized by Young diagrams. We use a notation in conformity to Fulton–Harris [13].

We denote the one-dimensional trivial representation $\mathbb{Q}$ by $[0]_{\mathrm{Sp}}$, and the natural representation $H_{\mathbb{Q}}$ by $[1]_{\mathrm{Sp}}$. For a Young diagram corresponding to $n_1\geq n_2\geq\cdots\geq n_l\geq 1$, $l\leq g$, we define $[n_1n_2\cdots n_l]_{\mathrm{Sp}}$ as below:

- Let $m_1\geq m_2\geq\cdots\geq m_k$ be the transpose of $n_1\geq n_2\geq\cdots\geq n_l\geq l$.
- The vector
$$(a_1\wedge a_2\wedge\cdots\wedge a_{m_1})\otimes(a_1\wedge a_2\wedge\cdots\wedge a_{m_2})\otimes\cdots\otimes(a_1\wedge a_2\wedge\cdots\wedge a_{m_k})$$
within $\left(\bigwedge^{m_1}H_{\mathbb{Q}}\right)\otimes\left(\bigwedge^{m_2}H_{\mathbb{Q}}\right)\otimes\cdots\otimes\left(\bigwedge^{m_k}H_{\mathbb{Q}}\right)$ generates an irreducible subrepresentation.
- This irreducible representation is denoted by $[n_1n_2\cdots n_l]_{\mathrm{Sp}}$, and the vector
$$(a_1\wedge a_2\wedge\cdots\wedge a_{m_1})\otimes(a_1\wedge a_2\wedge\cdots\wedge a_{m_2})\otimes\cdots\otimes(a_1\wedge a_2\wedge\cdots\wedge a_{m_k})$$
is called the *highest weight vector* of $[n_1n_2\cdots n_l]_{\mathrm{Sp}}$.

We abbreviate $[2211]_{\mathrm{Sp}}$, $[111111]_{\mathrm{Sp}}$ and so forth as $[2^21^2]_{\mathrm{Sp}}$, $[1^6]_{\mathrm{Sp}}$ and so forth.

These representations are naturally isomorphic to their dual representation. For example, $H_{\mathbb{Q}}^*$ and its dual $H_{\mathbb{Q}}$ are isomorphic as representations of $\mathrm{Sp}(2g,\mathbb{Q})$ via the Poincaré duality, and are denoted by $[1]_{\mathrm{Sp}}$. We have $\bigwedge^3 H_{\mathbb{Q}}=[1^3]_{\mathrm{Sp}}\oplus[1]_{\mathrm{Sp}}$ and $\bigwedge^3 H_{\mathbb{Q}}/H_{\mathbb{Q}}\cong U_{\mathbb{Q}}=[1^3]_{\mathrm{Sp}}$.

The following proposition follows from the irreducibility of $U_{\mathbb{Q}}=[1^3]_{\mathrm{Sp}}$.

Proposition 4.1 *For $g \geq 3$,*

$$(\tau_{g,1}(1))_* : H_1(\mathrm{Ch}_{g,1}; \mathbb{Q}) \to H_1(U; \mathbb{Q})$$

is surjective and

$$(\tau_{g,1}(1))^* : H^1(U; \mathbb{Q}) \to H^1(\mathrm{Ch}_{g,1}; \mathbb{Q})$$

is injective. The same holds for the $\mathrm{Ch}_{g,*}$ and Ch_g cases.

Hain studied the homomorphism $(\tau_g(1))^* : H^2\big(\bigwedge^3 H/H; \mathbb{Q}\big) \to H^2(\mathcal{I}_g; \mathbb{Q})$ between the second rational cohomology induced by the Johnson homomorphism and determined the kernel of this map as $\mathrm{Sp}(2g, \mathbb{Q})$-modules using representation theory.

Lemma 4.2 (Hain [15, Lemma 10.2]) *For $g \geq 3$,*

$$\begin{aligned} H^2\big(\textstyle\bigwedge^3 H/H; \mathbb{Q}\big) &\cong H_2\big(\textstyle\bigwedge^3 H/H; \mathbb{Q}\big) \cong H^2(U; \mathbb{Q}) \cong H_2(U; \mathbb{Q}) \cong \textstyle\bigwedge^2 U_{\mathbb{Q}} \\ &= \begin{cases} [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}} & (g \geq 6), \\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} & (g = 5), \\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} & (g = 4), \\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} & (g = 3) \end{cases} \end{aligned}$$

holds as $\mathrm{Sp}(2g, \mathbb{Q})$-modules.

Theorem 4.3 (Hain [15]) *For $g \geq 3$,*

$$\mathrm{Ker}\big((\tau_g(1))^* : H^2\big(\textstyle\bigwedge^3 H/H; \mathbb{Q}\big) \to H^2(\mathcal{I}_g; \mathbb{Q})\big) = [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}}$$

holds as $\mathrm{Sp}(2g, \mathbb{Q})$-modules.

Moreover, the dual of the preceding theorem implies that the image of the homomorphism

$$(\tau_g(1))_* : H_2(\mathcal{I}_g; \mathbb{Q}) \to H_2\big(\textstyle\bigwedge^3 H/H; \mathbb{Q}\big)$$

between the second rational homology induced by the Johnson homomorphism is decomposed as $\mathrm{Sp}(2g, \mathbb{Q})$-modules as follows:

Theorem 4.4 (Hain [15]) *For $g \geq 3$,*

$$\mathrm{Im}\big((\tau_g(1))_* : H_2(\mathcal{I}_g; \mathbb{Q}) \to H_2\big(\textstyle\bigwedge^3 H/H; \mathbb{Q}\big)\big) = \begin{cases} [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}} & (g \geq 6), \\ [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} & (g = 5), \\ [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} & (g = 4), \\ \{0\} & (g = 3) \end{cases}$$

holds as $\mathrm{Sp}(2g, \mathbb{Q})$-modules.

For $g \geq 3$, the homomorphism $\mathrm{Ch}_{g,1} \hookrightarrow \mathcal{I}_{g,1} \to \mathcal{I}_{g,*} \to \mathcal{I}_g$ induces the $\mathcal{M}_{g,1}$-equivariant commutative diagram

$$\begin{tikzcd}
\bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, hook] \arrow[ddd, bend right=60, "\mathrm{id}_{\bigwedge^2 U_{\mathbb{Q}}}"'] & H^2\big(\bigwedge^3 H/H;\mathbb{Q}\big) \arrow[r, "(\tau_g(1))^*"] \arrow[d] & H^2(\mathcal{I}_g;\mathbb{Q}) \arrow[d] \\
\bigwedge^2 H_{\mathbb{Q}} \oplus (H_{\mathbb{Q}} \otimes U_{\mathbb{Q}}) \oplus \bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, equal] & H^2\big(\bigwedge^3 H;\mathbb{Q}\big) \arrow[r, "(\tau_{g,*}(1))^*"] \arrow[d] & H^2(\mathcal{I}_{g,*};\mathbb{Q}) \arrow[d] \\
\bigwedge^2 H_{\mathbb{Q}} \oplus (H_{\mathbb{Q}} \otimes U_{\mathbb{Q}}) \oplus \bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\cong"] \arrow[d, two heads] & H^2\big(\bigwedge^3 H;\mathbb{Q}\big) \arrow[r, "(\tau_{g,1}(1))^*"] \arrow[d] & H^2(\mathcal{I}_{g,1};\mathbb{Q}) \arrow[d] \\
\bigwedge^2 U_{\mathbb{Q}} \arrow[r, "\cong"] & H^2(U;\mathbb{Q}) \arrow[r, "(\tau_{g,1}(1))^*"] & H^2(\mathrm{Ch}_{g,1};\mathbb{Q})
\end{tikzcd}$$

By Theorem 4.3 and this commutative diagram, the kernel

$$\mathrm{Ker}\big((\tau_{g,1}(1))^* : H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})\big)$$

contains $[0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}}$, and taking the dual of this, we obtain

$$\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big) \subset \begin{cases} [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}} & (g \geq 6), \\ [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} & (g = 5), \\ [1^2]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} & (g = 4), \\ \{0\} & (g = 3). \end{cases}$$

In fact, the summand $[1^2]_{\mathrm{Sp}}$ is not contained in the image

$$\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big).$$

In this subsection, we show that any other summands except for $[0]_{\mathrm{Sp}}$, $[2^2]_{\mathrm{Sp}}$, and $[1^2]_{\mathrm{Sp}}$ are contained in the image $\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big)$.

Now, we introduce some $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphisms to detect specific irreducible component, and *abelian cycles* (see [39; 40]).

Let V be a representation of $\mathrm{Sp}(2g,\mathbb{Q})$.

(1) **The contraction** For $k \geq 2$, $C_k : \bigwedge^k H_{\mathbb{Q}} \to \bigwedge^{k-2} H_{\mathbb{Q}}$ is defined by

$$x_1 \wedge \cdots \wedge x_k \mapsto \sum_{i<j} (-1)^{i+j+1} (x_i \cdot x_j) x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_k.$$

Also, the kernel of the contraction $\mathrm{Ker}(C_k)$ corresponds to an irreducible representation denoted by $[1^k]_{\mathrm{Sp}}$.

(2) **The canonical inclusion** $i_V^k : \bigwedge^k V \hookrightarrow \bigotimes^k V$ is defined by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sign}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

(3) **The multiplication** $\phi_V^{m,n} : (\bigwedge^m V) \otimes (\bigwedge^n V) \to \bigwedge^{m+n} V$ is defined by

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n}.$$

(4) **The Jacobi identity map** $j_V : \bigwedge^3 V \to V \otimes \bigwedge^2 V$ is defined by

$$v_1 \wedge v_2 \wedge v_3 \mapsto v_1 \otimes (v_2 \wedge v_3) + v_2 \otimes (v_3 \wedge v_1) + v_3 \otimes (v_1 \wedge v_2).$$

Next, we introduce *abelian cycles* which give concrete elements of the image

$$\operatorname{Im}((\tau_{g_1}(1))_* : H_2(\mathrm{Ch}_{g,1}; \mathbb{Z}) \to H_2(U; \mathbb{Z}))$$

of the homomorphism between the second rational homology induced by the first Johnson homomorphism.

Definition 4.5 [39, Subsection 4.3; 40, page 103, Step 2] Let G be a group and $c : \mathbb{Z}^2 \to G$ be a homomorphism. The image of the fundamental class $1 \in H_2(\mathbb{Z}^2; \mathbb{Z})$ under the induced homomorphism $c_* : H_2(\mathbb{Z}^2; \mathbb{Z}) \to H_2(G; \mathbb{Z})$ is called the *abelian cycle*.

Let $\{e_1, e_2\}$ denote the standard basis of $\mathbb{Z}^2$. Recall that for a finitely generated free abelian group A, the second homology group $H_2(A; \mathbb{Z})$ is naturally isomorphic to the second exterior power $\bigwedge^2 A$.

Proposition 4.6 [37, Lemma 2.1; 39, Lemma 4.5] *Let A be a finitely generated free abelian group and $c : \mathbb{Z}^2 \to A$ be a homomorphism. Then the abelian cycle with respect to c coincides with*

$$c(e_1) \wedge c(e_2) \in \bigwedge^2 A \cong H_2(A; \mathbb{Z}).$$

If we apply this to $\mathbb{Z}^2 \xrightarrow{c} \mathrm{Ch}_{g,1} \xrightarrow{\tau_{g,1}(1)} U$ where $c(e_i) = f_i \in \mathrm{Ch}_{g,1}$ for $i = 1, 2$, we obtain the following.

Proposition 4.7 *Let f_1 and f_2 be mutually commutative elements in $\mathrm{Ch}_{g,1}$. Then the element*

$$\tau_{g,1}(1)(f_1) \wedge \tau_{g,1}(1)(f_2) \in \bigwedge^2 U_{\mathbb{Q}} \cong H_2(U; \mathbb{Q})$$

belongs to the image

$$\operatorname{Im}((\tau_{g_1}(1))_* : H_2(\mathrm{Ch}_{g,1}; \mathbb{Q}) \to H_2(U; \mathbb{Q})).$$

Additionally, we introduce elements of $\mathrm{Sp}(2g, \mathbb{Z}) \subset \mathrm{Sp}(2g, \mathbb{Q})$ that will appear several times. Let I denote the identity matrix and for distinct $1 \le i, j \le g$, we define the matrix $A_{i,j}$ by the transformation

$$A_{i,j} := \begin{cases} a_i \mapsto a_i + b_i - b_j, \\ a_j \mapsto a_j + b_j - b_i, \\ a_k \mapsto a_k \quad (k \neq i, j), \\ b_k \mapsto b_k. \end{cases}$$

Proposition 4.8 *For $g \ge 4$, the summand $[2^2 1^2]_{\mathrm{Sp}}$ is contained in the image*

$$\operatorname{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1}; \mathbb{Q}) \to H_2(U; \mathbb{Q})\big).$$

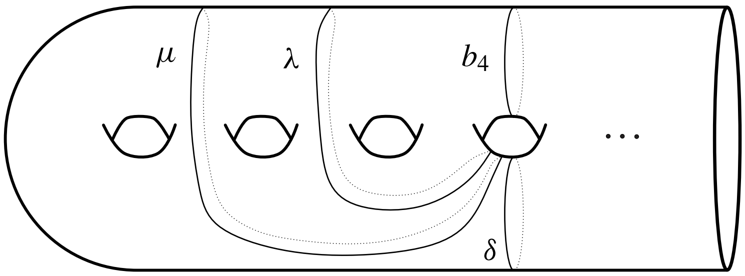


Figure 7: Some simple closed curves on the surface defining an abelian cycle which detects the summand $[2^2 1^2]_{\mathrm{Sp}}$.

Proof We take some simple closed curves on the surface as in Figure 7 and we define a homomorphism $\mathbb{Z}^2 \to \mathrm{Ch}_{g,1}$ by

$$\begin{aligned} e_1 &\mapsto \mathrm{BP}(b_4, \delta)\mathrm{BP}(b_4, \mu)^{-1}\mathrm{BP}(b_4, \lambda)^{-1} = T_{b_4}{}^{-1} T_\delta{}^{-1} T_\mu T_\lambda, \\ e_2 &\mapsto \mathrm{BP}(b_4, \mu)\mathrm{BP}(b_4, \lambda)^{-2} = T_{b_4}{}^{-1} T_\mu{}^{-1} T_\lambda{}^2. \end{aligned}$$

We confirm that these two elements are contained in $\mathrm{Ch}_{g,1}$:

$$\begin{aligned} C_3 \circ \tau_{g,1}(1)(\mathrm{BP}(b_4, \delta)\,\mathrm{BP}(b_4, \mu)\,\mathrm{BP}(b_4, \lambda)) &= C_3(a_1 \wedge b_1 \wedge b_4 - a_3 \wedge b_3 \wedge b_4) = 0, \\ C_3 \circ \tau_{g,1}(1)(\mathrm{BP}(b_4, \mu)\mathrm{BP}(b_4, \lambda)^{-2}) &= C_3(a_2 \wedge b_2 \wedge b_4 - a_3 \wedge b_3 \wedge b_4) = 0. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \zeta_1 &:= (a_1 \wedge b_1 \wedge b_4 - a_3 \wedge b_3 \wedge b_4) \wedge (a_2 \wedge b_2 \wedge b_4 - a_3 \wedge b_3 \wedge b_4) \\ &= \begin{pmatrix} (a_1 \wedge b_1 \wedge b_4) \wedge (a_2 \wedge b_2 \wedge b_4) + (a_2 \wedge b_2 \wedge b_4) \wedge (a_3 \wedge b_3 \wedge b_4) \\ \hfill + (a_3 \wedge b_3 \wedge b_4) \wedge (a_1 \wedge b_1 \wedge b_4) \end{pmatrix} \in \bigwedge^2 U_{\mathbb{Q}} \end{aligned}$$

as an element of $\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1}; \mathbb{Q}) \to H_2(U; \mathbb{Q})\big)$.

To prove Proposition 4.8, it is enough to show that ζ_1 is nontrivial on the summand $[2^2 1^2]_{\mathrm{Sp}}$. We detect the nontriviality by using an $\mathrm{Sp}(2g, \mathbb{Q})$-equivariant homomorphism

$$G_1 : \bigwedge^2 U_{\mathbb{Q}} \to \big(\bigwedge^4 H_{\mathbb{Q}}\big) \otimes \big(\bigwedge^2 H_{\mathbb{Q}}\big) \supset [2^2 1^2]_{\mathrm{Sp}}$$

as the composition of the maps

$$\begin{aligned} \bigwedge^2 U_{\mathbb{Q}} &\hookrightarrow \bigwedge^2\big(\bigwedge^3 H_{\mathbb{Q}}\big), \\ i^2_{\bigwedge^3 H_{\mathbb{Q}}} : \bigwedge^2\big(\bigwedge^3 H_{\mathbb{Q}}\big) &\to \bigotimes^2\big(\bigwedge^3 H_{\mathbb{Q}}\big), \\ \mathrm{id}_{\bigwedge^3 H_{\mathbb{Q}}} \otimes j_{H_{\mathbb{Q}}} : \big(\bigwedge^3 H_{\mathbb{Q}}\big) \otimes \big(\bigwedge^3 H_{\mathbb{Q}}\big) &\to \big(\bigwedge^3 H_{\mathbb{Q}}\big) \otimes H_{\mathbb{Q}} \otimes \big(\bigwedge^2 H_{\mathbb{Q}}\big), \\ \phi^{3,1}_{H_{\mathbb{Q}}} \otimes \mathrm{id}_{\bigwedge^2 H_{\mathbb{Q}}} : \big(\bigwedge^3 H_{\mathbb{Q}}\big) \otimes H_{\mathbb{Q}} \otimes \big(\bigwedge^2 H_{\mathbb{Q}}\big) &\to \big(\bigwedge^4 H_{\mathbb{Q}}\big) \otimes \big(\bigwedge^2 H_{\mathbb{Q}}\big). \end{aligned}$$

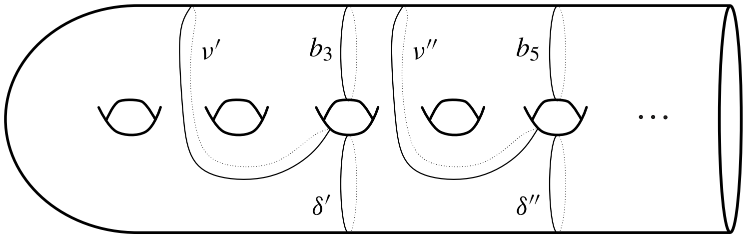


Figure 8: Some simple closed curves on the surface defining an abelian cycle which detects the summand $[1^4]_{\mathrm{Sp}}$.

Using this homomorphism and appropriate elements of $\mathrm{Sp}(2g,\mathbb{Q})$, we compute

$$\begin{aligned}
&\begin{pmatrix}(a_1\wedge b_1\wedge b_4)\wedge(a_2\wedge b_2\wedge b_4)+(a_2\wedge b_2\wedge b_4)\wedge(a_3\wedge b_3\wedge b_4)\\ \qquad +(a_3\wedge b_3\wedge b_4)\wedge(a_1\wedge b_1\wedge b_4)\end{pmatrix}\\
&\quad\xmapsto{I-A_{2,3}}\begin{pmatrix}-2(a_1\wedge b_1\wedge b_4)\wedge(b_2\wedge b_3\wedge b_4)+(a_2\wedge b_2\wedge b_4)\wedge(b_2\wedge b_3\wedge b_4)\\ \qquad +(a_3\wedge b_3\wedge b_4)\wedge(b_2\wedge b_3\wedge b_4)\end{pmatrix}\\
&\quad\xmapsto{I-A_{1,2}} 3(b_1\wedge b_2\wedge b_4)\wedge(b_2\wedge b_3\wedge b_4)\\
&\quad\xmapsto{i^2_{\wedge^3 H_{\mathbb{Q}}}} 3(b_1\wedge b_2\wedge b_4)\otimes(b_2\wedge b_3\wedge b_4)-3(b_2\wedge b_3\wedge b_4)\otimes(b_1\wedge b_2\wedge b_4)\\
&\quad\xmapsto{\mathrm{id}_{\wedge^3 H_{\mathbb{Q}}}\otimes j_{H_{\mathbb{Q}}}}\begin{pmatrix}3(b_1\wedge b_2\wedge b_4)\otimes(b_2\otimes(b_3\wedge b_4)+b_3\otimes(b_4\wedge b_2)+b_4\otimes(b_2\wedge b_3))\\ -3(b_2\wedge b_3\wedge b_4)\otimes(b_1\otimes(b_2\wedge b_4)+b_2\otimes(b_4\wedge b_1)+b_4\otimes(b_1\wedge b_2))\end{pmatrix}\\
&\quad\xmapsto{\phi^{3,1}_{H_{\mathbb{Q}}}\otimes\mathrm{id}_{\wedge^2 H_{\mathbb{Q}}}} -6(b_4\wedge b_2\wedge b_1\wedge b_3)\otimes(b_4\wedge b_2)\\
&\quad\xmapsto[b_4\mapsto b_1,\,b_1\mapsto b_3,\,b_3\mapsto b_4]{a_4\mapsto a_1,\,a_1\mapsto a_3,\,a_3\mapsto a_4,} -6(b_1\wedge b_2\wedge b_3\wedge b_4)\otimes(b_1\wedge b_2)\\
&\quad\xmapsto{b_i\mapsto a_i,\,a_i\mapsto -b_i\ (i=1,2,3,4)} -6(a_1\wedge a_2\wedge a_3\wedge a_4)\otimes(a_1\wedge a_2).
\end{aligned}$$

This vector $-6(a_1\wedge a_2\wedge a_3\wedge a_4)\otimes(a_1\wedge a_2)$ is a highest weight vector of $\left(\bigwedge^4 H_{\mathbb{Q}}\right)\otimes\left(\bigwedge^2 H_{\mathbb{Q}}\right)$. Hence $G_1(\zeta_1)$ is nontrivial on the summand $[2^2 1^2]_{\mathrm{Sp}}$, and $[2^2 1^2]_{\mathrm{Sp}}$ is contained in the image

$$\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q})\to H_2(U;\mathbb{Q})\big). \qquad \square$$

Proposition 4.9 *For $g\geq 5$, the summand $[1^4]_{\mathrm{Sp}}$ is contained in the image*

$$\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q})\to H_2(U;\mathbb{Q})\big).$$

Proof We take some simple closed curves on the surface as in Figure 8 and we define a homomorphism $\mathbb{Z}^2\to\mathrm{Ch}_{g,1}$ and an abelian cycle as

$$e_1\mapsto \mathrm{BP}(b_3,\delta')\,\mathrm{BP}(b_3,\nu')^{-2}=T_{b_3}{}^{-1}T_{\delta'}{}^{-1}T_{\nu'}{}^{2},\quad e_2\mapsto \mathrm{BP}(b_5,\delta'')\,\mathrm{BP}(b_5,\nu'')^{-4}=T_{b_5}{}^{-3}T_{\delta''}{}^{-1}T_{\nu''}{}^{4}.$$

Similarly, we obtain

$$\zeta_2 := (a_1 \wedge b_1 \wedge b_3 - a_2 \wedge b_2 \wedge b_3) \wedge (a_1 \wedge b_1 \wedge b_5 + a_2 \wedge b_2 \wedge b_5 + a_3 \wedge b_3 \wedge b_5 - 3 + a_4 \wedge b_4 \wedge b_5)$$

as an element of the image $\operatorname{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big)$.

To prove Proposition 4.9, it is enough to show that ζ_2 is nontrivial on the summand $[1^4]_{\mathrm{Sp}}$, and we detect the nontriviality by using an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism $G_2 : \bigwedge^2 U_{\mathbb{Q}} \to \bigwedge^4 H_{\mathbb{Q}} \supset [1^4]_{\mathrm{Sp}}$ as the composition of the maps

$$\begin{aligned} \textstyle\bigwedge^2 U_{\mathbb{Q}} &\hookrightarrow \textstyle\bigwedge^2\big(\bigwedge^3 H_{\mathbb{Q}}\big), \\ i^2_{\bigwedge^3 H_{\mathbb{Q}}} : \textstyle\bigwedge^2\big(\bigwedge^3 H_{\mathbb{Q}}\big) &\to \textstyle\bigotimes^2\big(\bigwedge^3 H_{\mathbb{Q}}\big), \\ \phi_{H^{3,3}_{\mathbb{Q}}} : \big(\textstyle\bigwedge^3 H_{\mathbb{Q}}\big) \otimes \big(\textstyle\bigwedge^3 H_{\mathbb{Q}}\big) &\to \textstyle\bigwedge^6 H_{\mathbb{Q}}, \\ C_6 : \textstyle\bigwedge^6 H_{\mathbb{Q}} &\to \textstyle\bigwedge^4 H_{\mathbb{Q}}. \end{aligned}$$

Similarly, we compute

$$\begin{aligned}
\zeta_2 &= \begin{pmatrix} (a_1 \wedge b_1 \wedge b_3) \wedge (a_1 \wedge b_1 \wedge b_5) + (a_1 \wedge b_1 \wedge b_3) \wedge (a_2 \wedge b_2 \wedge b_5) \\ + (a_1 \wedge b_1 \wedge b_3) \wedge (a_3 \wedge b_3 \wedge b_5) \\ - 3(a_1 \wedge b_1 \wedge b_3) \wedge (a_4 \wedge b_4 \wedge b_5) - (a_2 \wedge b_2 \wedge b_3) \wedge (a_1 \wedge b_1 \wedge b_5) \\ - (a_2 \wedge b_2 \wedge b_3) \wedge (a_2 \wedge b_2 \wedge b_5) \\ - (a_2 \wedge b_2 \wedge b_3) \wedge (a_3 \wedge b_3 \wedge b_5) + 3(a_2 \wedge b_2 \wedge b_3) \wedge (a_4 \wedge b_4 \wedge b_5) \end{pmatrix} \\
&\xmapsto{i^2_{\bigwedge^3 H_{\mathbb{Q}}}} \begin{pmatrix} (a_1 \wedge b_1 \wedge b_3) \otimes (a_1 \wedge b_1 \wedge b_5) - (a_1 \wedge b_1 \wedge b_5) \otimes (a_1 \wedge b_1 \wedge b_3) \\ + (a_1 \wedge b_1 \wedge b_3) \otimes (a_2 \wedge b_2 \wedge b_5) - (a_2 \wedge b_2 \wedge b_5) \otimes (a_1 \wedge b_1 \wedge b_3) \\ + (a_1 \wedge b_1 \wedge b_3) \otimes (a_3 \wedge b_3 \wedge b_5) - (a_3 \wedge b_3 \wedge b_5) \otimes (a_1 \wedge b_1 \wedge b_3) \\ - 3(a_1 \wedge b_1 \wedge b_3) \otimes (a_4 \wedge b_4 \wedge b_5) + 3(a_4 \wedge b_4 \wedge b_5) \otimes (a_1 \wedge b_1 \wedge b_3) \\ - (a_2 \wedge b_2 \wedge b_3) \otimes (a_1 \wedge b_1 \wedge b_5) + (a_1 \wedge b_1 \wedge b_5) \otimes (a_2 \wedge b_2 \wedge b_3) \\ - (a_2 \wedge b_2 \wedge b_3) \otimes (a_2 \wedge b_2 \wedge b_5) + (a_2 \wedge b_2 \wedge b_5) \otimes (a_2 \wedge b_2 \wedge b_3) \\ - (a_2 \wedge b_2 \wedge b_3) \otimes (a_3 \wedge b_3 \wedge b_5) + (a_3 \wedge b_3 \wedge b_3) \otimes (a_2 \wedge b_2 \wedge b_3) \\ + 3(a_2 \wedge b_2 \wedge b_3) \otimes (a_4 \wedge b_4 \wedge b_5) - 3(a_4 \wedge b_4 \wedge b_5) \otimes (a_2 \wedge b_2 \wedge b_3) \end{pmatrix} \\
&\xmapsto{\phi_{H^{3,3}_{\mathbb{Q}}}} 6a_2 \wedge b_2 \wedge a_4 \wedge b_4 \wedge b_3 \wedge b_5 - 6a_1 \wedge b_1 \wedge a_4 \wedge b_4 \wedge b_3 \wedge b_5 \\
&\xmapsto{C_6} 6a_2 \wedge b_2 \wedge b_3 \wedge b_5 - 6a_1 \wedge b_1 \wedge b_3 \wedge b_5 \ (\neq 0) \\
&\xmapsto{C_4} 6b_3 \wedge b_5 - 6b_3 \wedge b_5 = 0.
\end{aligned}$$

Since this abelian cycle is nontrivial on the kernel $\operatorname{Ker}(C_4) = [1^4]_{\mathrm{Sp}}$, it follows that $G_2(\zeta_2)$ is nontrivial on the summand $[1^4]_{\mathrm{Sp}}$, and $[1^4]_{\mathrm{Sp}}$ is contained in the image $\operatorname{Im}\big((\tau_{g,1}(1))* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big)$. $\square$

Proposition 4.10 *For $g \geq 6$, the summand $[1^6]_{\mathrm{Sp}}$ is contained in the image*

$$\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big).$$

Proof For $g \geq 6$, the same abelian cycle as in Proposition 4.9 is also nontrivial on the summand $[1^6]_{\mathrm{Sp}}$, and we check this by using an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism $G_3 : \bigwedge^2 U_{\mathbb{Q}} \to \bigwedge^6 H_{\mathbb{Q}} \supset [1^6]_{\mathrm{Sp}}$ as the composition of the maps

$$\begin{aligned} \textstyle\bigwedge^2 U_{\mathbb{Q}} &\hookrightarrow \textstyle\bigwedge^2\big(\bigwedge^3 H_{\mathbb{Q}}\big), \\ i^2_{\bigwedge^3 H_{\mathbb{Q}}} : \textstyle\bigwedge^2\big(\bigwedge^3 H_{\mathbb{Q}}\big) &\to \textstyle\bigotimes^2\big(\bigwedge^3 H_{\mathbb{Q}}\big), \\ \phi_{H^{3,3}_{\mathbb{Q}}} : \big(\textstyle\bigwedge^3 H_{\mathbb{Q}}\big) \otimes \big(\textstyle\bigwedge^3 H_{\mathbb{Q}}\big) &\to \textstyle\bigwedge^6 H_{\mathbb{Q}}. \end{aligned}$$

The result is

$$\begin{aligned} \zeta_3 := \zeta_2 &= (a_1\wedge b_1\wedge b_3 - a_2\wedge b_2\wedge b_3)\wedge(a_1\wedge b_1\wedge b_5 + a_2\wedge b_2\wedge b_5 + a_3\wedge b_3\wedge b_5 - 3a_4\wedge b_4\wedge b_5) \\ &= \big((a_1\wedge b_1\wedge b_3)\wedge(a_1\wedge b_1\wedge b_5) + (a_1\wedge b_1\wedge b_3)\wedge(a_2\wedge b_2\wedge b_5) + (a_1\wedge b_1\wedge b_3)\wedge(a_3\wedge b_3\wedge b_5) \\ &\quad -3(a_1\wedge b_1\wedge b_3)\wedge(a_4\wedge b_4\wedge b_5) - (a_2\wedge b_2\wedge b_3)\wedge(a_1\wedge b_1\wedge b_5) - (a_2\wedge b_2\wedge b_3)\wedge(a_2\wedge b_2\wedge b_5) \\ &\qquad - (a_2\wedge b_2\wedge b_3)\wedge(a_3\wedge b_3\wedge b_5) + 3(a_2\wedge b_2\wedge b_3)\wedge(a_4\wedge b_4\wedge b_5)\big) \\ &\xmapsto{I-A_{4,6}} 3(a_2\wedge b_2\wedge b_3)\wedge(b_6\wedge b_4\wedge b_5) - 3(a_1\wedge b_1\wedge b_3)\wedge(b_6\wedge b_4\wedge b_5) \\ &\xmapsto{I-A_{1,2}} 6(b_1\wedge b_2\wedge b_3)\wedge(b_4\wedge b_5\wedge b_6) \\ &\xmapsto{i^2_{\bigwedge^3 H_{\mathbb{Q}}}} 6(b_1\wedge b_2\wedge b_3)\otimes(b_4\wedge b_5\wedge b_6) - 6(b_4\wedge b_5\wedge b_6)\wedge(b_1\wedge b_2\wedge b_3) \\ &\xmapsto{\phi_{H^{3,3}_{\mathbb{Q}}}} 12 b_1\wedge b_2\wedge b_3\wedge b_4\wedge b_5\wedge b_6 \\ &\xmapsto{b_i\mapsto a_i,\ a_i\mapsto -b_i\ (i=1,2,3,4,5,6)} 12 a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_5\wedge a_6. \end{aligned}$$

This vector $12a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_5\wedge a_6$ is a highest weight vector of $\bigwedge^6 H_{\mathbb{Q}}$. Hence $G_3(\zeta_3)$ is nontrivial on the summand $[1^6]_{\mathrm{Sp}}$, and $[1^6]_{\mathrm{Sp}}$ is contained in the image $\mathrm{Im}((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q}))$. □

Propositions 4.8–4.10 and Theorem 4.3 together imply that the summands $[0]_{\mathrm{Sp}}$ and $[2^2]_{\mathrm{Sp}}$ are contained in the kernel

$$\mathrm{Ker}\big((\tau_{g,1}(1))^* : H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})\big),$$

whereas the summands $[2^2 1^2]_{\mathrm{Sp}}$, $[1^4]_{\mathrm{Sp}}$ and $[1^6]_{\mathrm{Sp}}$ are not contained in it. Next, for $g \geq 4$, we prove that the summand $[1^2]_{\mathrm{Sp}}$ is not contained in the image

$$\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big)$$

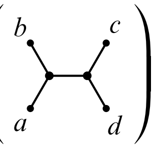
and is contained in the kernel

$$\mathrm{Ker}\big((\tau_{g,1}(1))^* : H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})\big).$$

Proposition 4.11 *The diagram*

$$\begin{tikzcd}
H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \arrow[r, "(-)^{ab}{}_*"'] \arrow[d, "(\tau_{g,1}(1))_*"] & H_2(\mathrm{Ch}^{ab}_{g,1};\mathbb{Q}) \cong \bigwedge^2 H_1(\mathrm{Ch}_{g,1};\mathbb{Q}) \arrow[r, "\text{bracket}"] \arrow[d, two heads, "\bigwedge^2(\tau_{g,1}(1))_*"] & (\Gamma_2(\mathrm{Ch}_{g,1})/\,\Gamma_3(\mathrm{Ch}_{g,1}))\otimes\mathbb{Q} \arrow[r] \arrow[d] & 0 \\
H_2(U;\mathbb{Q}) \arrow[r, "\cong"'] \arrow[drr, dashed, bend right, "s"'] & \bigwedge^2 H_1(U;\mathbb{Q}) \arrow[r, "\text{bracket}"] \arrow[d, "\bigwedge^2 \eta_{\mathbb{Q}}^{-1}"] & (\mathcal{K}_{g,1}/\mathcal{M}_{g,1}[4])\otimes\mathbb{Q} \cong \mathrm{Im}(\tau_{g,1}(2))\otimes\mathbb{Q} \arrow[d, "\eta_{\mathbb{Q}}^{-1}"] & \\
& \bigwedge^2 \mathcal{T}_1(H_{\mathbb{Q}}) \arrow[r, "{[\bullet,\bullet]_{\mathcal{T}}}"] & \mathcal{T}_2(H_{\mathbb{Q}}) \arrow[d, "q"] & \\
& & \bigwedge^2 H_{\mathbb{Q}} &
\end{tikzcd}$$

is $\mathcal{M}_{g,1}$*-equivariant and commutative, where the first row is exact, and the* $\mathrm{Sp}(2g,\mathbb{Q})$*-equivariant homomorphism* $q:\mathcal{T}_2(H_{\mathbb{Q}})\to\bigwedge^2 H_{\mathbb{Q}}$ *(see [31]) is defined by*

$q\Bigg($ 

$\Bigg)$ $$:= 4(a\cdot b)(c\wedge d)+4(c\cdot d)(a\wedge b) + 2(d\cdot a)(b\wedge c)+2(b\cdot c)(d\wedge a)+2(a\cdot c)(b\wedge d)+2(d\cdot b)(c\wedge a).$$

Proof For the exactness of the first row of the diagram, see [16, page 24, diagram 1.11]. The commutativity in the upper left follows from the naturality of homology with respect to group homomorphisms, while the commutativity in the lower right follows from $\eta_{\mathbb{Q}}^{-1}:\mathfrak{h}_{g,1\mathbb{Q}}\to\mathcal{T}(H_{\mathbb{Q}})$ being a homomorphism of Lie algebras. Regarding the commutativity in the upper right, it arises from the brackets induced by commutators within the mapping class group, and the vertical natural homeomorphisms. Recall that

$$\Gamma_2 Ch_{g,1}\subset\Gamma_2\mathcal{I}_{g,1}=\Gamma_2\mathcal{M}_{g,1}[2]\subset\mathcal{M}_{g,1}[3]=\mathcal{K}_{g,1}\quad\text{and}\quad\Gamma_3 Ch_{g,1}\subset\Gamma_3\mathcal{I}_{g,1}=\Gamma_3\mathcal{M}_{g,1}[2]\subset\mathcal{M}_{g,1}[4],$$

due to the fact $\{\mathcal{M}_{g,1}[i+1]/\mathcal{M}_{g,1}[i+2]\}_{i\geq 1}=\{\mathrm{Im}(\tau_{g,1}(i))\}_{i\geq 1}$ forms a graded Lie algebra under the commutator:

$$\begin{tikzcd}
\bigwedge^2(\mathrm{Ch}_{g,1}\,/\,\Gamma_2(\mathrm{Ch}_{g,1})) \arrow[r, "\text{bracket}"] \arrow[d] & \Gamma_2(\mathrm{Ch}_{g,1})/\,\Gamma_3(\mathrm{Ch}_{g,1}) \arrow[d] \\
\bigwedge^2(\mathrm{Ch}_{g,1}\,/\mathcal{M}_{g,1}[3]) \arrow[r, "\text{bracket}"] & \mathcal{M}_{g,1}[3]/\mathcal{M}_{g,1}[4]
\end{tikzcd}$$ □

Proposition 4.12 *For* $g\geq 4$, *the summand* $[1^2]_{\mathrm{Sp}}\subset H_2(U;\mathbb{Q})\cong\bigwedge^2 U_{\mathbb{Q}}$ *does not appear in the image*

$$\mathrm{Im}\big((\tau_{g,1}(1))_*:H_2(\mathrm{Ch}_{g,1};\mathbb{Q})\to H_2(U;\mathbb{Q})\big),$$

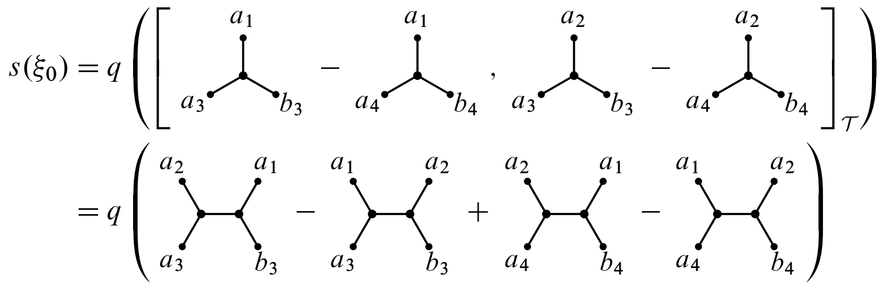
and the summand appears in the kernel

$$\mathrm{Ker}\big((\tau_{g,1}(1))^*:H^2(U;\mathbb{Q})\to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})\big).$$

Proof If $[1^2]_{\mathrm{Sp}} \subset H_2(U;\mathbb{Q}) \cong \bigwedge^2 U_{\mathbb{Q}}$ appears in the image of $(\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})$, then the $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism $s\colon H_2(U;\mathbb{Q}) \to \bigwedge^2 H_{\mathbb{Q}}$ has to be trivial on $[1^2]_{\mathrm{Sp}}$ because of the commutativity of the diagram and the exactness of the first row. Let

$$\xi_0 = (a_1 \wedge a_3 \wedge b_3 - a_1 \wedge a_4 \wedge b_4) \wedge (a_2 \wedge a_3 \wedge b_3 - a_2 \wedge a_4 \wedge b_4)$$

be an element of $\bigwedge^2 U_{\mathbb{Q}} \cong H_2(U;\mathbb{Q})$. We compute the value of ξ_0 under s as



$$\begin{aligned}
&= \big(2(b_3\cdot a_3)(a_2\wedge a_1) - 2(b_3\cdot a_3)(a_1\wedge a_2) + 2(b_4\cdot a_4)(a_2\wedge a_1) - 2(b_4\cdot a_4)(a_1\wedge a_2)\big)\\
&= 2a_1\wedge a_2 + 2a_1\wedge a_2 + 2a_1\wedge a_2 + 2a_1\wedge a_2\\
&= 8a_1\wedge a_2.
\end{aligned}$$

The vector $8a_1 \wedge a_2$ is a highest weight vector of $\bigwedge^2 H_{\mathbb{Q}}$. Hence the $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism $s : H_2(U;\mathbb{Q}) \to \bigwedge^2 H_{\mathbb{Q}}$ is nontrivial on the summand $[1^2]_{\mathrm{Sp}}$, which leads to a contradiction. Therefore, the summand $[1^2]_{\mathrm{Sp}} \subset H_2(U;\mathbb{Q})$ never appears in the image of $(\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})$ and it does appear in the kernel of $(\tau_{g,1}(1))^* : H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})$ for $g \geq 4$. □

From the above considerations, we conclude:

Theorem 4.13 (Theorem B) *For $g \geq 3$, we have*

$$\mathrm{Im}\big((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q})\big) = \begin{cases} [2^21^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} \oplus [1^6]_{\mathrm{Sp}} & (g \geq 6),\\ [2^21^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}} & (g=5),\\ [2^21^2]_{\mathrm{Sp}} & (g=4),\\ \{0\} & (g=3),\end{cases}$$

and

$$\mathrm{Ker}\big((\tau_{g,1}(1))^* : H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})\big) = \begin{cases} [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} & (g \geq 4),\\ [0]_{\mathrm{Sp}} \oplus [2^2]_{\mathrm{Sp}} & (g=3)\end{cases}$$

as $\mathrm{Sp}(2g,\mathbb{Q})$-modules, and the same holds for the $\mathrm{Ch}_{g,*}$ case.

Moreover, for a 2-cocycle v representing an element of the kernel

$$\mathrm{Ker}\big((\tau_{g,1}(1))^* : H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})\big) \cong (\tau_{g,1}(2)\otimes\mathbb{Q})^*,$$

we explicitly construct a coboundary of $(\tau_{g,1}(1))^*(v)$.

Proposition 4.14 *Let* $[v] \in \operatorname{Ker}(H^2(U,\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q})) \cong (\operatorname{Im}(\tau_{g,1}(2)) \otimes \mathbb{Q})^* \cong \mathcal{T}_2(H_{\mathbb{Q}})^*$, *where* v *is a representative* 2*-cocycle. Then the* 1*-cochain* $v \circ 2r_2^\theta$ *cobounds* $(\tau_{g,1}(1))^*(v)$.

Proof We identify $\operatorname{Ker}(H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q}))$ with the image of the map

$$[\bullet,\bullet]^* : \mathcal{T}_2(H_{\mathbb{Q}})^* \cong (\operatorname{Im}(\tau_{g,1}(2)) \otimes \mathbb{Q})^* \hookrightarrow H^2(U;\mathbb{Q})$$

dual of the Lie bracket. For $[v] \in \operatorname{Ker}(H^2(U;\mathbb{Q}) \to H^2(\mathrm{Ch}_{g,1};\mathbb{Q}))$, the 2-cocycle

$$(\tau_{g,1}(1))^*(v) = v \circ [\tau_{g,1}, \tau_{g,1}] = v \circ [r_1^\theta, r_1^\theta]_{\mathcal{T}}$$

is cobounded by $v \circ r_2^\theta$. In fact, from the second-degree part of the BCH series, we compute

$$\begin{aligned}
\delta(v \circ 2r_2^\theta)(f,g) &= v \circ 2(r_2^\theta(fg) - r_2^\theta(f) - r_2^\theta(g)) \\
&= v \circ 2\big(\big(\tfrac{1}{2}[r_1^\theta(f), r_1^\theta(g)]_{\mathcal{T}} + r_2^\theta(f) + r_2^\theta(g)\big) - r_2^\theta(f) - r_2^\theta(g)\big) \\
&= v \circ 2\big(\tfrac{1}{2}[r_1^\theta(f), r_1^\theta(g)]_{\mathcal{T}}\big) \\
&= v \circ [r_1^\theta(f), r_1^\theta(g)]_{\mathcal{T}}.
\end{aligned}$$

□

5 The Casson–Morita homomorphism $d : \mathrm{Ch}_{g,1} \to \mathbb{Z}$ for the Chillingworth subgroup

Morita [25] introduced a certain *map* $d : \mathcal{M}_{g,1} \to \mathbb{Z}$ related to the Casson invariant.

Definition 5.1 A Heegaard embedding $\Sigma_{g,1} \to \Sigma_g \to S^3$ is a map such that cutting along it decomposes into two genus-g handlebodies V_g^+ and V_g^-.

Definition 5.2 For an element $\varphi \in \mathcal{I}_{g,1}$, the integral homology 3-sphere M_φ is defined as the 3-manifold obtained by regluing V_g^- to V_g^+ along φ.

The Casson invariant λ : {integral homology 3-spheres} $\to \mathbb{Z}$ is one of the fundamental invariants of integral homology 3-spheres. In particular, we can consider the Casson invariant for M_φ. He found that the Casson invariant can be interpreted as a secondary invariant associated with the characteristic classes of the surface and studied this mapping in detail. Morita showed in [28, Proposition 2.1] that the map $\varphi \mapsto \lambda(M_\varphi) =: \lambda^*(\varphi)$ (the map λ^* depends on the choice of a Heegaard embedding and there is no canonical choice) is a homomorphism on the Johnson kernel $\mathcal{K}_{g,1}$. He defined a homomorphism $d = d|_{\mathcal{K}_{g,1}} : \mathcal{K}_{g,1} \to \mathbb{Z}$ related to λ^*. He called this homomorphism the core of the Casson invariant, and we call it the *Casson–Morita homomorphism*.

To define the Casson–Morita homomorphism, we introduce some 2-cocycles of the full mapping class group $\mathcal{M}_{g,1}$.

Definition 5.3 Let $\tau : \mathcal{M}_{g,1} \times \mathcal{M}_{g,1} \to \mathbb{Z}$ be the *Meyer cocycle* characterized by the signature of the 4-manifold defined by the surface Σ_g bundle over a pair of pants $\Sigma_{0,3}$ with corresponding monodromies (see [24]). Next, let $k : \mathcal{M}_{g,1} \to H^{(*)}$ be a crossed homomorphism representing a generator of $H^1(\mathcal{M}_{g,1}; H^{(*)}) \cong \mathbb{Z}$, for example the Chillingworth homomorphism $k = e_X$. We define the 2-cocycle $c : \mathcal{M}_{g,1} \times \mathcal{M}_{g,1} \to \mathbb{Z}$ by $c(\varphi,\psi) := k(\varphi^{-1}) \cdot k(\psi)$ called the *intersection cocycle*.

These 2-cocycles are related by $[-3\tau] = e_1 = [c] \in H^2(\mathcal{M}_{g,1};\mathbb{Z})$, where e_1 is the first *Mumford–Morita–Miller class* (see [24; 25; 27]). Therefore, there exists a map $d : \mathcal{M}_{g,1} \to \mathbb{Z}$ such that the coboundary δd coincides with $c + 3\tau$ as 2-cocycles. Moreover, for $g \geq 3$, $H^1(\mathcal{M}_{g,1};\mathbb{Z}) = 0$ holds (Mumford [35], Birman [3] and Powell [36] showed this for the closed case. For the general case, see a Korkmaz's survey [21]). Hence, such a map $d : \mathcal{M}_{g,1} \to \mathbb{Z}$ is uniquely determined. Therefore, we will always assume $g \geq 3$ from now on.

We have the following by definition.

Proposition 5.4 *For $f, g \in \mathcal{M}_{g,1}$, we have*

$$d(fg) = d(f) + d(g) - k(f^{-1}) \cdot k(g) - 3\tau(f, g).$$

Definition 5.5 We define the homomorphism $d = d|_{\mathrm{Ch}_{g,1}} : \mathrm{Ch}_{g,1} \to \mathbb{Z}$ given above as the Casson–Morita homomorphism on $\mathrm{Ch}_{g,1}$.

By this equality, $d = d|_{\mathrm{Ch}_{g,1}} : \mathrm{Ch}_{g,1} \to \mathbb{Z}$ is a homomorphism on the Chillingworth subgroup because the Meyer cocycle τ is realized as a normalized 2-cocycle on $\mathrm{Sp}(2g,\mathbb{Z})$, hence vanishes on the Torelli group $\mathcal{I}_{g,1}$ (see [24]), and the crossed homomorphism k is trivial on the Chillingworth subgroup $\mathrm{Ch}_{g,1}$.

Remark 5.6 The Casson–Morita *map* $d : \mathcal{M}_{g,1} \to \mathbb{Z}$ depends on the choice of a crossed homomorphism $k : \mathcal{M}_{g,1} \to H^{(*)}$.

Proposition 5.7 *The Casson–Morita homomorphism $d : \mathrm{Ch}_{g,1} \to \mathbb{Z}$ does not depend on the choice of a crossed homomorphism $k : \mathcal{M}_{g,1} \to H^*$.*

Proof Let $k_0, k_1 : \mathcal{M}_{g,1} \to H^*$ be two crossed homomorphisms representing a generator of the group $H^1(\mathcal{M}_{g,1}; H^*) \cong \mathbb{Z}$. For $i = 0, 1$, we denote the intersection cocycles determined by k_i as c_i and the Casson–Morita homomorphisms as d_i. First, when $k_1 = -k_0$, since $c_0 = c_1$, we have $d_0 = d_1$ on $\mathcal{M}_{g,1}$. Therefore, it suffices to consider the case where k_1 is cohomologous to k_0 in $H^1(\mathcal{M}_{g,1}; H^*)$. In this case, we can write $k_1(f) - k_0(f) = (f^{-1})^*h - h$ for some element $h \in H^*$. Then, $d_1(f) - d_0(f) = (k_1(f^{-1}) - k_0(f)) \cdot h$, and the right side is always 0 on the Chillingworth subgroup $\mathrm{Ch}_{g,1}$. Indeed, the calculation proceeds by direct computation as follows:

$$\begin{aligned}
\delta(d_1 - d_0)(f, g) &= (c_1 - c_0)(f, g) \\
&= k_1(f^{-1}) \cdot k_1(g) - k_0(f^{-1}) \cdot k_0(g) \\
&= (k_0(f^{-1}) + f^*h - h) \cdot (k_0(g) + (g^{-1})^*h - h) - k_0(f^{-1}) \cdot k_0(g) \\
&= k_0(f^{-1}) \cdot ((g^{-1})^*h - h) + (f^*h - h) \cdot k_0(g) + (f^*h - h) \cdot ((g^{-1})^*h - h) \\
&= (g^*k_0(f^{-1}) - k_0(f^{-1})) \cdot h - ((f^{-1})^*k_0(g) - k_0(g)) \cdot h \\
&\qquad\qquad + (g^*f^*h - f^*h - g^*h) \cdot h \\
&= (k_0((fg)^{-1}) - k_0(f^{-1}) - k_0(g^{-1})) \cdot h - (k_0(fg) - k_0(f) - k_0(g)) \cdot h \\
&\qquad\qquad + ((fg)^*h - f^*h - g^*h) \cdot h \\
&= \delta\big(\bullet \mapsto (k_0(\bullet^{-1}) + (\bullet)^*h - h - k_0(\bullet)) \cdot h\big)(f, g) \\
&= \delta\big(\bullet \mapsto (k_1(\bullet^{-1}) - k_0(\bullet)) \cdot h\big)(f, g).
\end{aligned}$$

Since $H^1(\mathcal{M}_{g,1};\mathbb{Z})=0$, the coboundary is unique, leading to

$$d_1(f)-d_0(f)=(k_1(f^{-1})-k_0(f))\cdot h,$$

and hence d_1 and d_0 coincide on $\mathrm{Ch}_{g,1}$. $\square$

Morita [25] gave some properties and formulas of the Casson–Morita map.

Proposition 5.8 (Morita [25, page 320, Theorem 5.3]) (1) *The Casson–Morita homomorphism $d:\mathcal{M}_{g,1}\to\mathbb{Z}$ defined by a crossed homomorphism $k:\mathcal{M}_{g,1}\to H$ is stable with respect to the genus of the surface if k is stable with respect to the genus. Specifically, for the homomorphism $\mathcal{M}_{g,1}\to\mathcal{M}_{g+1,1}$ induced by gluing a genus-1 surface with 2 boundary components, denoted by $\Sigma_{1,2}$, to the boundary of $\Sigma_{g,1}$ to obtain $\Sigma_{g+1,1}$, the diagram*

$$\begin{tikzcd}
\mathcal{M}_{g,1} \arrow[d] \arrow[dr, "d"] & \\
\mathcal{M}_{g+1,1} \arrow[r, "d"] & \mathbb{Z}
\end{tikzcd}$$

commutes if the diagram

$$\begin{tikzcd}
\mathcal{M}_{g,1} \arrow[r, "k"] \arrow[d] & H_1(\Sigma_{g,1};\mathbb{Z}) \arrow[d] \\
\mathcal{M}_{g+1,1} \arrow[r, "k"] & H_1(\Sigma_{g+1,1};\mathbb{Z})
\end{tikzcd}$$

commutes.

(2) *Let T_γ be a genus-h BSCC map, meaning that γ bounds a genus-h subsurface of the surface. Then the value under d is $4h(h-1)$. In particular, the Casson–Morita homomorphism $d:\mathcal{K}_{g,1}\to\mathbb{Z}$ on the Johnson kernel is $\mathcal{M}_{g,1}$-invariant, and its image coincides with $8\mathbb{Z}$.*

Proposition 5.9 (part of Theorem C) *The Casson–Morita map $d:\mathrm{Ch}_{g,1}\to\mathbb{Z}$ on the Chillingworth subgroup is also an $\mathcal{M}_{g,1}$-invariant homomorphism.*

Proof Consider $h\in\mathrm{Ch}_{g,1}$ and $f\in\mathcal{M}_{g,1}$. Then, we have

$$\begin{aligned}
d(fhf^{-1}) &= d(fh)+d(f^{-1})-k((fh)^{-1})\cdot k(f^{-1})-3\tau(fh,f^{-1})\\
&= (d(f)+d(h)-k(f^{-1})\cdot k(h)-3\tau(f,h))\\
&\qquad\qquad -d(f)-(k(h^{-1})+h^*k(f^{-1}))\cdot k(f^{-1})-3\tau(f,f^{-1})\\
&= d(f)+d(h)-d(f)-3\tau(f,\mathrm{id}_{\Sigma_{g,1}})-k(f^{-1})\cdot k(f^{-1})-3\tau(f,f^{-1})\\
&= d(h),
\end{aligned}$$

where we used following properties of the Meyer cocycle $\tau:\mathcal{M}_{g,1}\times\mathcal{M}_{g,1}\to\mathbb{Z}$ (see [24]).

(1) *The Meyer cocycle factors through $\mathrm{Sp}(2g,\mathbb{Z})$, which means that for any $h_1,h_2\in\mathcal{I}_{g,1}$, we have $\tau(fh_1,gh_2)=\tau(f,g)$.*

(2) $\tau(f,f^{-1})=0$.

(3) $\tau(f,\mathrm{id}_{\Sigma_{g,1}})=0$. $\square$

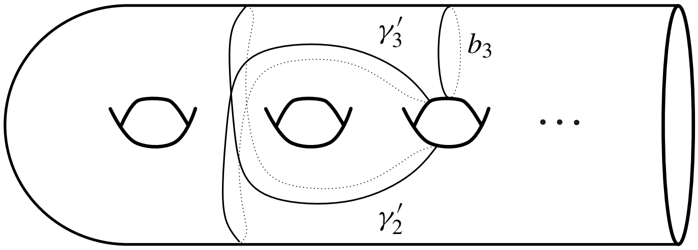


Figure 9: Simple closed curves γ_2', γ_3' defining a homological genus-zero bounding pair map $B_0 := T_{\gamma_2'}T_{\gamma_3'}{}^{-1}$ and b_3.

Proposition 5.10 (part of Theorem C) *The image of the Casson–Morita homomorphism* $d : \mathrm{Ch}_{g,1} \to \mathbb{Z}$ *on the Chillingworth subgroup is also* $8\mathbb{Z}$.

Proof Let k denote the Chillingworth class t. By

$$d(\text{genus-}h \text{ BSCC map}) = 4h(h-1),$$

the image of the Casson–Morita homomorphism restricted to the Johnson kernel $\mathcal{K}_{g,1}$ is $8\mathbb{Z}$. Let us consider the element $B_0 := \mathrm{BP}(\gamma_2', \gamma_3') := T_{\gamma_2'}T_{\gamma_3'}{}^{-1}$ in Figure 9. We have $d(B_0) = 0$. Indeed, we consider $k(\mathrm{BP}(\gamma_2', b_3)) - k(\mathrm{BP}(\gamma_3', b_3)) = 2[\gamma_2'] - 2[\gamma_3'] = 0$ and using $k(fg) = k(f) + f_*k(g)$, we have $k(T_{\gamma_2'}) = k(T_{\gamma_3'})$, and note that $\rho(T_{\gamma_2'}) = \rho(T_{\gamma_3'}) \in \mathrm{Sp}(2g, \mathbb{Z})$. Moreover, considering the braid relations

$$T_{a_3}T_{\gamma_i'}T_{a_3} = T_{\gamma_i'}T_{a_3}T_{\gamma_i'}$$

and using $k(T_{\gamma_2'}) = k(T_{\gamma_3'})$ and $\rho(T_{\gamma_2'}) = \rho(T_{\gamma_3'})$, we have $d(T_{\gamma_2'}) = d(T_{\gamma_3'}) = -d(T_{\gamma_3'}^{-1})$ and consequently, $d(B_0) = 0$. Since $\mathrm{Ch}_{g,1}$ is normally generated by the Johnson kernel $\mathcal{K}_{g,1}$ and B_0, the image of the map $d : \mathrm{Ch}_{g,1} \to \mathbb{Z}$ coincides with $8\mathbb{Z}$. □

We also determine the kernel of the Casson–Morita homomorphism for the Chillingworth subgroup. Before discussing it, we present Faes's result on the Johnson kernel, which provides the motivation for our study.

Theorem 5.11 (Faes [11, Remark 2.15]) *For* $g \geq 2$, *the kernel of the Casson–Morita homomorphism restricted to the Johnson kernel is given by*

$$\mathrm{Ker}(d|_{\mathcal{K}_{g,1}} : \mathcal{K}_{g,1} \to \mathbb{Z}) = \langle T_{\gamma_1'} \rangle [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}],$$

where $T_{\gamma_1'}$ *is the Dehn twist along* γ_1' *also called a genus-one BSCC map as shown in Figure 10,* $\langle T_{\gamma_1'} \rangle$ *is the subgroup generated by* $T_{\gamma_1'}$, *and* $[\mathcal{K}_{g,1}, \mathcal{M}_{g,1}]$ *is the commutator subgroup of the Johnson kernel and the full mapping class group.*

Proof sketch This theorem is essentially based on the result $H^1(\mathcal{K}_{g,1}; \mathbb{Z})^{\mathcal{M}_{g,1}} \cong \mathbb{Z} \oplus \mathbb{Z}$ by Morita [28], where the superscript means the $\mathcal{M}_{g,1}$-invariants. Morita introduced an $\mathcal{M}_{g,1}$-invariant homomorphism

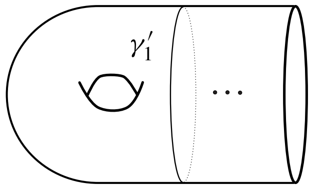


Figure 10: A simple closed curve γ_1' on the surface.

$d' : \mathcal{K}_{g,1} \to \mathbb{Z}$, distinct from d, and showed that $H^1(\mathcal{K}_{g,1};\mathbb{Q})^{\mathcal{M}_{g,1}}$ is generated by d and d'. Faes [11] took linear combinations $\frac{d}{8}$ and $\frac{4d'-5d}{12}$, and proved that these two elements give an isomorphism

$$\overline{\left(\frac{d}{8}, \frac{4d'-5d}{12}\right)} : \mathcal{K}_{g,1}/[\mathcal{K}_{g,1}, \mathcal{M}_{g,1}] \cong H_1(\mathcal{K}, \mathbb{Z})_{\mathcal{M}_{g,1}} \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z},$$

and we have $\left(\frac{d}{8}, \frac{4d'-5d}{12}\right)(T_{\gamma_1'}) = (0,1)$. In particular, the intersection of these kernels

$$\operatorname{Ker}\left(\left(\frac{d}{8}, \frac{4d'-5d}{12}\right) : \mathcal{K}_{g,1} \to \mathbb{Z} \oplus \mathbb{Z}\right) = \operatorname{Ker}(d|_{\mathcal{K}_{g,1}}) \cap \operatorname{Ker}(d'|_{\mathcal{K}_{g,1}})$$

coincides with the commutator subgroup $[\mathcal{K}_{g,1}, \mathcal{M}_{g,1}]$ of the Johnson kernel and the full mapping class group. For any elements $\varphi \in \operatorname{Ker}(d|_{\mathcal{K}_{g,1}})$, the element

$$T_{\gamma_1'}^{\left(-\frac{4d'(\varphi)-5d(\varphi)}{12}\right)}\varphi = T_{\gamma_1'}^{\left(-\frac{d'(\varphi)}{3}\right)}\varphi$$

is contained in

$$\operatorname{Ker}(d|_{\mathcal{K}_{g,1}}) \cap \operatorname{Ker}(d'|_{\mathcal{K}_{g,1}}) = [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}].$$

Therefore, we have

$$\operatorname{Ker}(d|_{\mathcal{K}_{g,1}}) = \langle T_{\gamma_1'} \rangle [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}]. \qquad \square$$

Theorem 5.12 (Theorem C) *For $g \geq 4$, the kernel of the Casson–Morita homomorphism on the Chillingworth subgroup is given by*

$$\operatorname{Ker}(d : \mathrm{Ch}_{g,1} \to \mathbb{Z}) = \langle\langle B_0 \rangle\rangle \langle T_{\gamma_1'} \rangle [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}],$$

where the element $B_0 := T_{\gamma_2'} T_{\gamma_3'}{}^{-1}$ is a homological genus-zero BP map as shown in Figure 11, and $\langle\langle B_0 \rangle\rangle$ is the normal subgroup of $\mathcal{M}_{g,1}$ generated by B_0, and where $T_{\gamma_1'}$ is the Dehn twist along the simple closed curve γ_1' in Figure 10.

Proof The element B_0 satisfies $d(B_0) = 0$ and U is generated by

$$\tau_{g,1}(1)(B_0) = a_1 \wedge b_1 \wedge b_3 - a_2 \wedge b_2 \wedge b_3$$

as an $\mathrm{Sp}(2g,\mathbb{Z})$-module (see the proof of Proposition 3.14). Therefore, for any $\varphi \in \operatorname{Ker}(d : \mathrm{Ch}_{g,1} \to \mathbb{Z})$, there exists an $\psi \in \langle\langle B_0 \rangle\rangle$ such that $\tau_{g,1}(1)(\psi) = \tau_{g,1}(1)(\varphi)$, which implies that $\psi^{-1}\varphi \in \operatorname{Ker}(d|_{\mathcal{K}_{g,1}})$. Hence, we have

$$\operatorname{Ker}(d : \mathrm{Ch}_{g,1} \to \mathbb{Z}) = \langle\langle B_0 \rangle\rangle \langle T_{\gamma_1'} \rangle [\mathcal{K}_{g,1}, \mathcal{M}_{g,1}]. \qquad \square$$

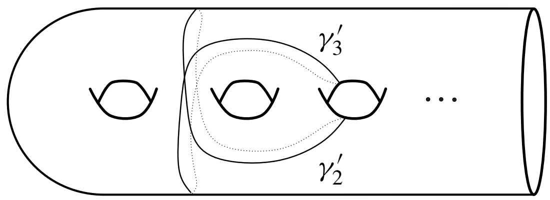


Figure 11: Some simple closed curves on the surface defining a homological genus-zero BP map.

6 Proof of Theorem A

For the Torelli group, the rational abelianization is obtained from the first Johnson homomorphism as a mapping class group module. More precisely, Johnson [19] showed that the abelianization of the Torelli group is isomorphic to the direct sum of the target space of the Johnson homomorphism and additional 2-torsion parts, arising from the Birman–Craggs homomorphism. The latter is closely related to spin structures and the Rokhlin invariant (see [4]). For the Chillingworth subgroup, the first Johnson homomorphism provides one abelian quotient. In addition, the Casson–Morita homomorphism is a homomorphism on the Chillingworth subgroup that is nontrivial on the kernel of the first Johnson homomorphism. This allows us to combine both homomorphisms to obtain a better lower bound for the rational abelianization $H_1(\mathrm{Ch}_{g,1};\mathbb{Q}) \cong (\mathrm{Ch}_{g,1})^{ab} \otimes \mathbb{Q}$ of the Chillingworth subgroup:

$$d \oplus \tau_{g,1}(1) : \mathrm{Ch}_{g,1} \to (8\mathbb{Z} \oplus U) \otimes \mathbb{Q}.$$

To determine the rational abelianization of the Chillingworth subgroup, we consider the inflation-restriction exact sequence of the rational homology for the short exact sequence

$$1 \to \mathcal{K}_{g,1} \to \mathrm{Ch}_{g,1} \to U \to 0$$

induced by the first Johnson homomorphism for the Chillingworth subgroup as follows:

$$H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q}) \cong \textstyle\bigwedge^2 U_{\mathbb{Q}} \to H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \to H_1(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_1(U;\mathbb{Q}) \cong U_{\mathbb{Q}} \to 0.$$

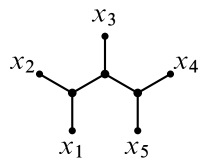
This exact sequence is equivariant under the natural action of the mapping class group. Having already determined the $\mathcal{M}_{g,1}$-module structures of the image

$$\mathrm{Im}((\tau_{g,1}(1))_* : H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to H_2(U;\mathbb{Q}))$$

and $U_{\mathbb{Q}}$, we only have to determine the $\mathcal{M}_{g,1}$-module structure of $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U$, where the subscript U means the U-coinvariant of the first rational homology group $H_1(\mathcal{K}_{g,1};\mathbb{Q})$ of the Johnson kernel. To study the structure of $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U$, we use the rational abelianization of the Johnson kernel $\mathcal{K}_{g,1}$ by Faes and Massuyeau [12]. The rational abelianization of the Johnson kernel was originally computed for the case of the closed surface $\mathcal{K}_g$ by Dimca–Hain–Papadima [9] and Morita–Sakasai–Suzuki [34].

Here, we introduce a certain homomorphism Tr_3, which is needed to describe the rational abelianization of the Johnson kernel.

Definition 6.1 (Morita [29, Section 6]) The $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism $\mathrm{Tr}_3 : \mathcal{T}_3(H_{\mathbb{Q}}) \to \mathrm{S}^3 H_{\mathbb{Q}}$ is called Morita's trace map and is defined by

$\mathrm{Tr}_3\Bigg($  $\Bigg) := 2(x_5\cdot x_1)x_2x_3x_4 + 2(x_1\cdot x_4)x_5x_3x_2 + 2(x_4\cdot x_2)x_1x_3x_5 + 2(x_2\cdot x_5)x_2x_3x_1,$

where $\mathrm{S}^3 H_{\mathbb{Q}}$ is the third symmetric power of $H_{\mathbb{Q}}$.

Theorem 6.2 (Faes–Massuyeau [12, Theorem 3.2]) *For $g \geq 6$, the rational abelianization of the Johnson kernel $H_1(\mathcal{K}_{g,1};\mathbb{Q})$ as an $\mathcal{M}_{g,1}$-module is given by the Casson–Morita homomorphism d and the truncations of the infinitesimal Dehn–Nielsen representation (r_2^θ, r_3^θ) as*

$$d \oplus (r_2^\theta, r_3^\theta) : \mathcal{K}_{g,1} \to \mathbb{Q} \oplus (\mathcal{T}_2(H_{\mathbb{Q}}) \times \mathrm{Ker}(\mathrm{Tr}_3)) \subset \mathbb{Q} \oplus (\mathcal{T}_2(H_{\mathbb{Q}}) \times \mathcal{T}_3(H_{\mathbb{Q}})).$$

The homomorphism $d \oplus (r_2^\theta, r_3^\theta)$ induces the isomorphism $H_1(\mathcal{K}_{g,1};\mathbb{Q}) \cong \mathbb{Q} \oplus (\mathcal{T}_2(H_{\mathbb{Q}}) \times \mathrm{Ker}(\mathrm{Tr}_3))$ as $\mathcal{M}_{g,1}$-modules.

Remark 6.3 The action of the mapping class group $\mathcal{M}_{g,1}$ on $\mathbb{Q} \oplus (\mathcal{T}_2(H_{\mathbb{Q}}) \times \mathrm{Ker}(\mathrm{Tr}_3))$ does not factor through the integral symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ because of the influence of the bracket.

Next, to study the action of $U \cong \mathrm{Ch}_{g,1}/\mathcal{K}_{g,1}$ on the rational abelianization of the Johnson kernel $H_1(\mathcal{K}_{g,1};\mathbb{Q})$, we summarize the behavior by conjugation.

Lemma 6.4 *For $f \in \mathcal{I}_{g,1}$ and $h \in \mathcal{K}_{g,1}$, we have*

$$d \oplus (r_2^\theta, r_3^\theta)(fhf^{-1}) = d \oplus (r_2^\theta, r_3^\theta)(h) + \big(0, (0, [r_1^\theta(f), r_2^\theta(h)]_{\mathcal{T}})\big).$$

Proof Since the Casson–Morita homomorphism $d : \mathcal{K}_{g,1} \to \mathbb{Z}$ is $\mathcal{M}_{g,1}$-invariant, it suffices to consider the conjugation action on (r_2^θ, r_3^θ). For $f \in \mathcal{I}_{g,1}$ and $h \in \mathcal{K}_{g,1}$, we compute the part of $r^\theta(fhf^{-1})$ up to the third degree, that is, modulo the terms of degree four and higher, $\widehat{\mathcal{T}_{\geq 4}} = \widehat{\bigoplus_{i\geq 4}\mathcal{T}_i(H_{\mathbb{Q}})} \subset \widehat{\mathcal{T}(H_{\mathbb{Q}})}$, as

$$\begin{aligned} r^\theta(fhf^{-1}) &= r^\theta(f) \star r^\theta(hf^{-1}) \\ &\equiv \left(\begin{aligned} &r^\theta(f) + r^\theta(hf^{-1}) + \tfrac{1}{2}[r^\theta(f), r^\theta(hf^{-1})]_{\widehat{\mathcal{T}}} \\ &+ \tfrac{1}{12}\left(\begin{aligned} &[r^\theta(f), [r^\theta(f), r^\theta(hf^{-1})]_{\widehat{\mathcal{T}}}]_{\widehat{\mathcal{T}}} \\ &\qquad - [r^\theta(hf^{-1}), [r^\theta(f), r^\theta(hf^{-1})]_{\widehat{\mathcal{T}}}]_{\widehat{\mathcal{T}}} \end{aligned}\right) \end{aligned}\right) \pmod{\widehat{\mathcal{T}_{\geq 4}}}. \end{aligned}$$

Using $r^\theta = r_1^\theta + r_2^\theta + \cdots$, we expand the brackets up to terms of degree 3, we have

$$\begin{aligned} &r^\theta(fhf^{-1}) \\ &\quad \equiv \left(\begin{aligned} &r^\theta(f) + r^\theta(hf^{-1}) \\ &+ \tfrac{1}{2}\Big([r_1^\theta(f), r_1^\theta(hf^{-1})]_{\mathcal{T}} + [r_1^\theta(f), r_2^\theta(hf^{-1})]_{\mathcal{T}} + [r_2^\theta(f), r_1^\theta(hf^{-1})]_{\mathcal{T}}\Big) \\ &+ \tfrac{1}{12}\Big([r_1^\theta(f), [r_1^\theta(f), r_1^\theta(hf^{-1})]_{\mathcal{T}}]_{\mathcal{T}} - [r_1^\theta(hf^{-1}), [r_1^\theta(f), r_1^\theta(hf^{-1})]_{\mathcal{T}}]_{\mathcal{T}}\Big) \end{aligned}\right) \pmod{\widehat{\mathcal{T}_{\geq 4}}}. \end{aligned}$$

By separating the BCH product by each degree, we note that

$$r_1^\theta(hf^{-1}) = r_1^\theta(h) + r_1^\theta(f^{-1}) = r_1^\theta(h) - r_1^\theta(f)$$

and

$$r_2^\theta(hf^{-1}) = r_2^\theta(h) + r_2^\theta(f^{-1}) + \tfrac{1}{2}[r_1^\theta(h), r_1^\theta(f^{-1})] = r_2^\theta(h) - r_2^\theta(f) - \tfrac{1}{2}[r_1^\theta(h), r_1^\theta(f)]_{\mathcal{T}}$$

are obtained. Using this,

$$r^\theta(fhf^{-1}) \equiv \begin{pmatrix} r^\theta(f) + r^\theta(hf^{-1}) + \frac{1}{2}([r_1^\theta(f), r_1^\theta(h) - r_1^\theta(f)]_{\mathcal{T}} \\ \qquad + [r_1^\theta(f), r_2^\theta(h) - r_2^\theta(f) - \frac{1}{2}[r_1^\theta(h), r_1^\theta(f)]_{\mathcal{T}}]_{\mathcal{T}} \\ \qquad\qquad + [r_2^\theta(f), r_1^\theta(h) - r_1^\theta(f)]_{\mathcal{T}}) \\ + \frac{1}{12}\begin{pmatrix} [r_1^\theta(f), [r_1^\theta(f), r_1^\theta(h) - r_1^\theta(f)]_{\mathcal{T}}]_{\mathcal{T}} \\ \qquad - [r_1^\theta(h) - r_1^\theta(f), [r_1^\theta(f), r_1^\theta(h) - r_1^\theta(f)]_{\mathcal{T}}]_{\mathcal{T}} \end{pmatrix} \end{pmatrix} \pmod{\widehat{\mathcal{T}_{\geq 4}}}.$$

Since h is an element of the Johnson kernel $\mathcal{K}_{g,1}$ and r_1^θ is the first Johnson homomorphism, we have $r_1^\theta(h) = 0$, which implies

$$\begin{aligned} r^\theta(fhf^{-1}) &\equiv \begin{pmatrix} r^\theta(f) + r^\theta(hf^{-1}) + \frac{1}{2}([r_1^\theta(f), 0 - r_1^\theta(f)]_{\mathcal{T}} \\ \qquad + [r_1^\theta(f), r_2^\theta(h) - r_2^\theta(f) - \frac{1}{2}[0, r_1^\theta(f)]_{\mathcal{T}}]_{\mathcal{T}} + [r_2^\theta(f), 0 - r_1^\theta(f)]_{\mathcal{T}}) \\ + \frac{1}{12}\Big([r_1^\theta(f), [r_1^\theta(f), 0 - r_1^\theta(f)]_{\mathcal{T}}]_{\mathcal{T}} - [0 - r_1^\theta(f), [r_1^\theta(f), 0 - r_1^\theta(f)]_{\mathcal{T}}]_{\mathcal{T}}\Big) \end{pmatrix} \pmod{\widehat{\mathcal{T}_{\geq 4}}} \\ &= r^\theta(f) + r^\theta(hf^{-1}) + \tfrac{1}{2}\big([r_1^\theta(f), r_2^\theta(h) - r_2^\theta(f)]_{\mathcal{T}} - [r_2^\theta(f), r_1^\theta(f)]_{\mathcal{T}}\big) \pmod{\widehat{\mathcal{T}_{\geq 4}}} \\ &= r^\theta(f) + r^\theta(hf^{-1}) + \tfrac{1}{2}[r_1^\theta(f), r_2^\theta(h)]_{\mathcal{T}} \pmod{\widehat{\mathcal{T}_{\geq 4}}}. \end{aligned}$$

Similarly, by expanding the BCH product and collecting nontrivial terms up to the third degree, we have

$$\begin{aligned} r^\theta(fhf^{-1}) &\equiv \begin{pmatrix} r^\theta(f) + r^\theta(h) - r^\theta(f) + \frac{1}{2}[r_1^\theta(f), r_2^\theta(h)]_{\mathcal{T}} \\ \qquad\qquad + \frac{1}{2}[r^\theta(h), -r^\theta(f)]_{\widehat{\mathcal{T}}} \\ + \frac{1}{12}\Big([r^\theta(h), [r^\theta(h), -r^\theta(f)]_{\widehat{\mathcal{T}}}]_{\widehat{\mathcal{T}}} - [-r^\theta(f), [r^\theta(h), -r^\theta(f)]_{\widehat{\mathcal{T}}}]_{\widehat{\mathcal{T}}}\Big) \end{pmatrix} \pmod{\widehat{\mathcal{T}_{\geq 4}}} \\ &\equiv r^\theta(f) + r^\theta(h) - r^\theta(f) + \tfrac{1}{2}[r_1^\theta(f), r_2^\theta(h)]_{\mathcal{T}} - \tfrac{1}{2}[r_2^\theta(h), r_1^\theta(f)]_{\mathcal{T}} \pmod{\widehat{\mathcal{T}_{\geq 4}}} \\ &= r^\theta(h) + [r_1^\theta(f), r_2^\theta(h)]_{\mathcal{T}} \pmod{\widehat{\mathcal{T}_{\geq 4}}}. \end{aligned}$$

Therefore, for $f \in \mathcal{I}_{g,1}$ and $h \in \mathcal{K}_{g,1}$, we have

$$(r_2^\theta, r_3^\theta)(fhf^{-1}) = (r_2^\theta, r_3^\theta)(h) + \big(0, [r_1^\theta(f), r_2^\theta(h)]_{\mathcal{T}}\big). \qquad \square$$

Proposition 6.5 *For $g \geq 3$, the $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant bracket map*

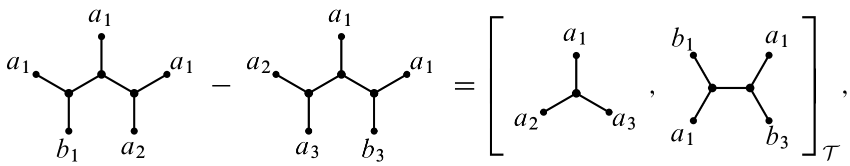
$$[\bullet,\bullet]_{\mathcal{T}} : (r_1^\theta(\mathrm{Ch}_{g,1}) \otimes \mathbb{Q}) \otimes (r_2^\theta(\mathcal{K}_{g,1}) \otimes \mathbb{Q}) \to \mathrm{Ker}(\mathrm{Tr}_3) \xrightarrow[\cong]{\eta^{\mathbb{Q}}} \mathrm{Im}(\tau_{g,1}(3)) \otimes \mathbb{Q}$$

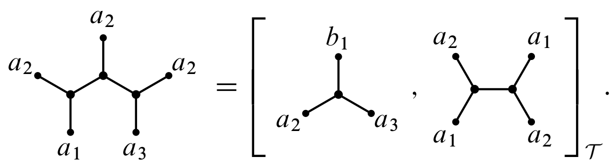
is surjective.

Proof The target space

$$\mathrm{Ker}(\mathrm{Tr}_3) \cong \mathrm{Im}(\tau_{g,1}(3)) \otimes \mathbb{Q}$$

(regarding the inclusion $\eta(\mathrm{Ker}(\mathrm{Tr}_3)) \supset \mathrm{Im}(\tau_{g,1}(3))$, see Morita [29, Theorem 6.1]; for the coincidence, see [33]) is isomorphic to $[31^2]_{\mathrm{Sp}} \oplus [21]_{\mathrm{Sp}}$ as representations of the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$, which is shown by Asada and Nakamura [2, Theorem A-(iii)]. Therefore, it is sufficient to show that the bracket map is nontrivial on both the summands $[31^2]_{\mathrm{Sp}}$ and $[21]_{\mathrm{Sp}}$. Let us consider two elements $\xi_1, \xi_2 \in [r_1^\theta(\mathrm{Ch}_{g,1}) \otimes \mathbb{Q}, r_2^\theta(\mathcal{K}_{g,1}) \otimes \mathbb{Q}]_{\mathcal{T}} \subset \mathrm{Ker}(\mathrm{Tr}_3) \subset r_3^\theta(\mathrm{Ch}_{g,1}) \otimes \mathbb{Q} \subset \mathcal{T}_3(H_{\mathbb{Q}})$ defined as

$\xi_1 :=$ 

$\xi_2 := -2$ 

We also define an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant homomorphism to detect these summands as

$$\mathrm{Ker}(\mathrm{Tr}_3) \hookrightarrow \mathcal{T}_3(H_{\mathbb{Q}}) \hookrightarrow H_{\mathbb{Q}} \otimes \mathcal{L}_{g,1}(4)_{\mathbb{Q}} \hookrightarrow \bigotimes^5 H_{\mathbb{Q}}$$

and

$$\bigotimes^5 H_{\mathbb{Q}} \xrightarrow{x_1\otimes x_2\otimes x_3\otimes x_4\otimes x_5 \mapsto (x_1\cdot x_2)(x_3\wedge x_4)\otimes x_5} \left(\bigwedge^2 H_{\mathbb{Q}}\right) \otimes H_{\mathbb{Q}}.$$

The value of ξ_1 under the above homomorphism is $9(a_1 \wedge a_2) \otimes a_1$ which hits the highest weight vector of the summand $[21]_{\mathrm{Sp}}$, and the value of ξ_2 under the above homomorphism is 0, but ξ_2 itself is nontrivial. Hence, ξ_2 purely lies in the summand $[31^2]_{\mathrm{Sp}}$. Therefore, the bracket map is surjective over $\mathbb{Q}$. (We calculated with a Mathematica program based on the method described in [38, p. 22, 7. Appendix].) □

We get the following directly from Lemma 6.4, Proposition 6.5 and the fact, shown by Morita [25, Proposition 1.2] and Hain [15], that $\mathrm{Im}(\tau_{g,1}(2)) \otimes \mathbb{Q} = \mathfrak{h}_{g,1\mathbb{Q}}(2) \cong \mathcal{T}_2(H_{\mathbb{Q}}) \cong [2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [0]_{\mathrm{Sp}}$.

Proposition 6.6 *The U-coinvariant $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U$ of the first rational homology of the Johnson kernel are isomorphic to $\mathbb{Q} \oplus \mathcal{T}_2(H_{\mathbb{Q}})$ via the homomorphism $(d, r_2^\theta = \tau_{g,1}(2))$ as $\mathcal{M}_{g,1}$-modules. In particular, the action of the mapping class group on $H_1(\mathcal{K}_{g,1};\mathbb{Q})_U$ factors through the rational symplectic group $\mathrm{Sp}(2g,\mathbb{Q})$, and decomposes as $[0]_{\mathrm{Sp}} \oplus ([2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [0]_{\mathrm{Sp}})$.*

We now proceed to Theorem A.

Theorem 6.7 (part of Theorem A) *For $g \geq 6$, the first rational (co)homology of the Chillingworth subgroup $\mathrm{Ch}_{g,1}$ for the genus-g surface with one boundary is induced by the Casson–Morita homomorphism and* *the first Johnson homomorphism for the Chillingworth subgroup* $d \oplus \tau_{g,1}(1) : \mathrm{Ch}_{g,1} \to 8\mathbb{Z} \oplus U$, *and satisfies*

$$(\mathrm{Ch}_{g,1})^{ab} \otimes \mathbb{Q} \cong H_1(\mathrm{Ch}_{g,1};\mathbb{Q}) \cong [1^3]_{\mathrm{Sp}} \oplus [0]_{\mathrm{Sp}},$$
$$((\mathrm{Ch}_{g,1})^{ab} \otimes \mathbb{Q})^* \cong H^1(\mathrm{Ch}_{g,1};\mathbb{Q}) \cong [1^3]_{\mathrm{Sp}} \oplus [0]_{\mathrm{Sp}}$$

as $\mathcal{M}_{g,1}$*-modules.*

Proof Now, we handle the inflation-restriction exact sequence of the rational homology for the short exact sequence

$$1 \to \mathcal{K}_{g,1} \to \mathrm{Ch}_{g,1} \to U \to 0$$

to determine the first rational homology group $H_1(\mathrm{Ch}_{g,1};\mathbb{Q})$ of the Chillingworth subgroup. For $g \geq 6$, we have determined the image

$$\mathrm{Im}\big(H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \to \textstyle\bigwedge^2 U_{\mathbb{Q}}\big) \cong [1^6]_{\mathrm{Sp}} \oplus [2^2 1^2]_{\mathrm{Sp}} \oplus [1^4]_{\mathrm{Sp}}$$

of the homomorphism between the second rational homology induced by the first Johnson homomorphism for the Chillingworth subgroup, the U-coinvariant

$$H_1(\mathcal{K}_{g,1};\mathbb{Q})_U \cong [0]_{\mathrm{Sp}} \oplus ([2^2]_{\mathrm{Sp}} \oplus [1^2]_{\mathrm{Sp}} \oplus [0]_{\mathrm{Sp}})$$

of the first rational homology of the Johnson kernel and $U_{\mathbb{Q}} \cong [1^3]_{\mathrm{Sp}}$. By adding the information obtained from the above to the long exact sequence, we obtain

$$\begin{tikzcd}
H_2(\mathrm{Ch}_{g,1};\mathbb{Q}) \arrow[r, "(\tau_{g,1(1)})^*"] & \underset{\substack{([1^6]_{\mathrm{Sp}}\oplus[2^2 1^2]_{\mathrm{Sp}}\oplus[1^4]_{\mathrm{Sp}}) \\ \oplus([2^2]_{\mathrm{Sp}}\oplus[1^2]_{\mathrm{Sp}}\oplus[0]_{\mathrm{Sp}})}}{\bigwedge^2 U_{\mathbb{Q}}} \arrow[r] & \underset{\substack{([2^2]_{\mathrm{Sp}}\oplus[1^2]_{\mathrm{Sp}}\oplus[0]_{\mathrm{Sp}}) \\ \oplus[0]_{\mathrm{Sp}}}}{H_1(\mathcal{K}_{g,1};\mathbb{Q})_U} \arrow[dll, rounded corners] \\
\underset{\substack{\text{“}[0]_{\mathrm{Sp}} \\ \oplus[1^3]_{\mathrm{Sp}}\text{”}}}{H_1(\mathrm{Ch}_{g,1};\mathbb{Q})} \arrow[r] & \underset{[1^3]_{\mathrm{Sp}}}{U_{\mathbb{Q}}} \arrow[r] & 0.
\end{tikzcd}$$

The preceding argument alone does not determine whether $H_1(\mathrm{Ch}_{g,1};\mathbb{Q})$ decomposes into a direct sum of two summands, $[0]_{\mathrm{Sp}}$ and $[1^3]_{\mathrm{Sp}}$, as $\mathcal{M}_{g,1}$-modules. However, $\dim_{\mathbb{Q}} H_1(\mathrm{Ch}_{g,1};\mathbb{Q}) = \dim_{\mathbb{Q}}([0]_{\mathrm{Sp}} \oplus [1^3]_{\mathrm{Sp}})$. Combining this with the lower bound of the rational abelianization of the Chillingworth subgroup already obtained, $d \oplus \tau_{g,1}(1) : \mathrm{Ch}_{g,1} \to \mathbb{Q} \oplus U_{\mathbb{Q}} \cong [0]_{\mathrm{Sp}} \oplus [1^3]$ gives a rational abelianization of the Chillingworth subgroup. Therefore, this long exact sequence splits at the $H_1(\mathrm{Ch}_{g,1};\mathbb{Q})$ as $\mathcal{M}_{g,1}$-modules. □

Corollary 6.8 *For* $g \geq 6$, *the rank of the abelianization of the Chillingworth subgroup* $\mathrm{Ch}_{g,1}$ *for the surface* $\Sigma_{g,1}$ *is* $\frac{1}{3}(2g-1)(2g^2-2g-3)$.

Next, we consider the case of the Chillingworth subgroup $\mathrm{Ch}_{g,*}$ with a fixed base point.

Theorem 6.9 (part of Theorem A) *For $g \geq 6$, the first rational (co)homology of the Chillingworth subgroup $\mathrm{Ch}_{g,*}$ for the genus-g surface with a base point is induced by the first Johnson homomorphism $\tau_{g,*}(1) : \mathrm{Ch}_{g,*} \to U$, and satisfies*

$$(\mathrm{Ch}_{g,*})^{ab} \otimes \mathbb{Q} \cong H_1(\mathrm{Ch}_{g,*}; \mathbb{Q}) \cong [1^3]_{\mathrm{Sp}}, \quad ((\mathrm{Ch}_{g,*})^{ab} \otimes \mathbb{Q})^* \cong H^1(\mathrm{Ch}_{g,*}; \mathbb{Q}) \cong [1^3]_{\mathrm{Sp}}$$

as $\mathcal{M}_{g,*}$-modules.

Proof We consider the central extension

$$0 \to \mathbb{Z} \to \mathrm{Ch}_{g,1} \to \mathrm{Ch}_{g,*} \to 1$$

induced by the natural homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$ and the inflation-restriction exact sequence for it

$$\cdots \to H_1(\mathbb{Z}; \mathbb{Q}) \to H_1(\mathrm{Ch}_{g,1}; \mathbb{Q}) \to H_1(\mathrm{Ch}_{g,*}; \mathbb{Q}) \to 0.$$

The Casson–Morita homomorphism induces the map $\mathbb{Z} \hookrightarrow \mathrm{Ch}_{g,1} \xrightarrow{d} \mathbb{Z}$, $1 \mapsto 4g(g-1)$ which is nontrivial by the formula by Morita. Therefore, the homomorphism $H_1(\mathbb{Z}; \mathbb{Q}) \to H_1(\mathrm{Ch}_{g,1}; \mathbb{Q})$ is nontrivial and the image

$$\mathrm{Im}(H_1(\mathbb{Z}; \mathbb{Q}) \to H_1(\mathrm{Ch}_{g,1}; \mathbb{Q}))$$

coincides with the summand $[0]_{\mathrm{Sp}}$. We have the exact sequence

$$\cdots \to \underset{[0]_{\mathrm{Sp}}}{H_1(\mathbb{Z}; \mathbb{Q})} \to \underset{[0]_{\mathrm{Sp}} \oplus [1^3]_{\mathrm{Sp}}}{H_1(\mathrm{Ch}_{g,1}; \mathbb{Q})} \to \underset{[1^3]_{\mathrm{Sp}}}{H_1(\mathrm{Ch}_{g,*}; \mathbb{Q})} \to 0.$$

In particular, the rational abelianization of the Chillingworth subgroup $\mathrm{Ch}_{g,*}$ for the surface with a base point $\mathrm{Ch}_{g,1}$ is induced by the first Johnson homomorphism for the Chillingworth subgroup alone. □

Corollary 6.10 *For $g \geq 6$, the rank of the abelianization of the Chillingworth subgroup $\mathrm{Ch}_{g,*}$ for the surface $\Sigma_{g,*}$ is $\frac{2}{3} g(2g+1)(g-2)$.*

Finally, we consider the case of the Chillingworth subgroup Ch_g without a fixed base point and boundary components.

Theorem 6.11 (part of Theorem A) *For $g \geq 6$, the first rational (co)homology of the Chillingworth subgroup Ch_g for the genus-g closed surface Ch_g is induced by the first Johnson homomorphism $\tau_g(1) : \mathrm{Ch}_g \to \overline{U}$, and satisfies*

$$(\mathrm{Ch}_g)^{ab} \otimes \mathbb{Q} \cong H_1(\mathrm{Ch}_g; \mathbb{Q}) \cong [1^3]_{\mathrm{Sp}}, \quad ((\mathrm{Ch}_g)^{ab} \otimes \mathbb{Q})^* \cong H^1(\mathrm{Ch}_g; \mathbb{Q}) \cong [1^3]_{\mathrm{Sp}}$$

as $\mathcal{M}_g$-modules.

Proof Let us consider the short exact sequence

$$1 \to [\pi_1(\Sigma_g), \pi_1(\Sigma_g)] \to \mathrm{Ch}_{g,*} \to \mathrm{Ch}_g \to 1$$

induced by the natural homomorphism $\mathcal{M}_{g,*} \to \mathcal{M}_g$ and the long exact sequence for it

$$\cdots \to H_1(\mathrm{Ch}_{g,*};\mathbb{Q}) \to H_1(\mathrm{Ch}_g;\mathbb{Q}) \to 0.$$

Since the rational abelianization of the Chillingworth subgroup $H_1(\mathrm{Ch}_{g,*};\mathbb{Q})$ is irreducible as an $\mathcal{M}_g$-module and there exists the first Johnson homomorphism $\tau_g(1) : \mathrm{Ch}_g \to \overline{U} \subset \bigwedge^3 H/H$, the natural homomorphism $H_1(\mathrm{Ch}_{g,*};\mathbb{Q}) \to H_1(\mathrm{Ch}_g;\mathbb{Q})$ is an isomorphism. □

Corollary 6.12 *For $g \geq 6$, the rank of the abelianization of the Chillingworth subgroup Ch_g for the surface Σ_g is also $\frac{2}{3}g(2g+1)(g-2)$.*

In the last of this section, we also mention the Euler class of the central extension

$$0 \to \mathbb{Z} \to \mathrm{Ch}_{g,1} \to \mathrm{Ch}_{g,*} \to 1$$

induced by the natural homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$.

Theorem 6.13 (Theorem D) *The Euler class $e \in H^2(\mathrm{Ch}_{g,*};\mathbb{Z})$ of the central extension*

$$0 \to \mathbb{Z} \to \mathrm{Ch}_{g,1} \to \mathrm{Ch}_{g,*} \to 1$$

is a $\frac{g(g-1)}{2}$-torsion element.

Proof We consider the inflation-restriction exact sequence of the integral cohomology

$$\cdots \to H^1(\mathrm{Ch}_{g,1};\mathbb{Z}) \to H^1(\mathbb{Z};\mathbb{Z}) \to H^2(\mathrm{Ch}_{g,*};\mathbb{Z}) \to \cdots$$

for the central extension. The value of the genus-g BSCC map under the $d \oplus \tau_{g,1}(1)$ is $(4g(g-1), 0) \in 8\mathbb{Z} \oplus U$. Therefore, the image

$$\mathrm{Im}(H^1(\mathrm{Ch}_{g,1};\mathbb{Z}) \cong 8\mathbb{Z} \oplus U \to H^1(\mathbb{Z};\mathbb{Z}))$$

of the natural homomorphism is generated by the homomorphism defined by $1 \mapsto \frac{4g(g-1)}{8} = \frac{g(g-1)}{2}$, and the cokernel is isomorphic to the cyclic group of order $\frac{g(g-1)}{2}$, and the Euler class $e \in H^2(\mathrm{Ch}_{g,*};\mathbb{Z})$ is a $\frac{g(g-1)}{2}$-torsion element in the second integral cohomology group $H^2(\mathrm{Ch}_{g,*};\mathbb{Z})$. □

Applying the universal coefficient theorem to the preceding, we obtain the following corollary.

Corollary 6.14 *For $g \geq 6$, the abelianization $H_1(\mathrm{Ch}_{g,*};\mathbb{Z}) \cong (\mathrm{Ch}_{g,*})^{ab}$ of the Chillingworth subgroup $\mathrm{Ch}_{g,*}$ for the genus-g surface with a base point has $\frac{g(g-1)}{2}$-torsion elements.*

Acknowledgements

The author would like to thank Takuya Sakasai and Quentin Faes for their valuable discussions and constructive feedback. The author also appreciates Nariya Kawazumi for his useful comments. The author is also grateful to the anonymous referee for a careful review and suggestions, which have helped to improve the clarity and quality of this paper.

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Received: October 18, 2023 Revised: September 5, 2024

msp
Algebraic & Geometric Topology **26**:4 (2026) 1507–1528
DOI: 10.2140/agt.2026.26.1507
Published: April 25, 2026

Acylindrical hyperbolicity for Artin groups with a visual splitting

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We establish a criterion that implies the acylindrical hyperbolicity of many Artin groups admitting a visual splitting. This gives a variety of new examples of acylindrically hyperbolic Artin groups, including many Artin groups of FC-type.

Our approach relies on understanding when parabolic subgroups are weakly malnormal in a given Artin group. We formulate a conjecture for when this happens, and prove it for several classes of Artin groups, including all spherical-type, all two-dimensional, and all even FC-type Artin groups. In addition, we establish some connections between several conjectures about Artin groups, related to questions of acylindrical hyperbolicity, weak malnormality of parabolic subgroups, and intersections of parabolic subgroups.

1 Introduction

Background and motivation Artin groups are generalizations of braid groups, with many connections to Coxeter groups. Artin groups remain largely mysterious in general, both from an algebraic and geometric viewpoint, although significant progress has been made in studying specific classes: spherical-type, FC-type, and 2-dimensional Artin groups, etc. (see Section 2 for the definition of each class). For Artin groups outside of these classes, it remains unknown in general whether they have solvable word problem, contain torsion elements, or have nontrivial centers. Geometrically, while no Artin groups other than free groups are hyperbolic (as they otherwise contain $\mathbb{Z}^2$-subgroups), it is expected that they are all CAT(0), although this is still open even for braid groups. Some classes of Artin groups have been shown to satisfy other notions of nonpositive curvature; see, for instance, [21; 23; 25; 26]

Acylindrical hyperbolicity is a notion encapsulating the idea of a group "having hyperbolic directions", a very weak form of hyperbolic behaviour. Many groups of geometric interest are known to be acylindrically hyperbolic, and despite its generality, this notion is strong enough to have important consequences for the structure of the group. (We refer the reader to [37; 38] for a discussion of the consequences.)

In the case of Artin groups, there is a clear conjectural picture of when they are expected to be acylindrically hyperbolic:

Conjecture (acylindrical hyperbolicity conjecture) *Let A_Γ be an irreducible Artin group. Then the central quotient $A_G/Z(A_\Gamma)$ is acylindrically hyperbolic.*

This conjecture essentially states that Artin groups are expected to be acylindrically hyperbolic unless they "clearly cannot be". Indeed, reducible Artin groups cannot be acylidrically hyperbolic as they split as direct products of infinite groups. Spherical-type Artin groups cannot be acylindrically hyperbolic as

MSC2020: primary 20F65; secondary 20F36, 20F67.

they have an infinite cyclic centre (but the acylindrical hyperbolicity of their central quotient was proved by Calvez and Wiest [8]). Note that Artin groups of nonspherical type are conjectured to have a trivial centre, so in that case the conjecture states that an Artin group of nonspherical type is acylindrically hyperbolic if and only if it is irreducible. This conjecture has been proved for several families of Artin groups already; see Section 2 for details.

Beside being interesting in its own right, the question of acylindrical hyperbolicity for Artin groups has applications to some well-known open problems for these groups. A first possible application is to the centre of these groups, as Artin groups of infinite type are conjectured to have a trivial centre. Since it is known that acylindrically hyperbolic groups have a finite centre, showing this property is a possible first step towards proving the triviality of the centre. Another possible application comes from the isomorphism problem, which asks which labelled graphs produce isomorphic Artin groups. Very little is currently known about the isomorphism problem for Artin groups; see, for instance, [14; 17; 31; 39; 41]. For instance, it is not even known whether being a spherical-type Artin group, or being an irreducible Artin group, is invariant under isomorphism. Since acylindrically hyperbolic groups have finite centres and do not split as direct products of infinite factors, a positive answer to the acylindrical hyperbolicity conjecture would imply that both aforementioned properties are indeed invariant under isomorphism.

A particular family of Artin groups all of whose elements are expected to be acylindrically hyperbolic is the family of Artin groups whose presentation is not a complete graph. Such Artin groups have the useful feature of decomposing as amalgamated products of standard parabolic subgroups (over a standard parabolic subgroup). Such splittings, which we will refer to as *visual splittings* as they can be read directly from the presentation graph, have been used to derive properties of the Artin group from the properties of the corresponding parabolic subgroups; see, for instance, [11; 12; 18; 35]. For groups splitting as amalgamated products, and more generally for groups acting on trees, there is a useful acylindrical hyperbolicity criterion due to Minsayan–Osin [34]. In a nutshell (see Theorem 3.1 for the precise statement), a nonvirtually cyclic group G splitting as an amalgamated product $G = A *_C B$ is acylindrically hyperbolic as soon as the edge group C is *weakly malnormal* in G, i.e., as soon as C intersects one of its conjugates along a finite subgroup. In the case of Artin groups admitting a visual splitting, this amounts to understanding when standard parabolic subgroups are weakly malnormal in the ambient group. We introduce the following conjecture, which provides a complete description of when this is expected to happen:

Conjecture (weak malnormality conjecture) *A proper standard parabolic subgroup of A_Γ is weakly malnormal if and only if it does not contain a standard parabolic subgroup that is a direct factor of A_Γ.*

In particular, if A_Γ is irreducible, then every proper standard parabolic subgroup is weakly malnormal.

In trying to apply the criterion of Minsayan–Osin, we are thus led to study the intersections of standard parabolic subgroups. Such intersections have been heavily studied in recent years in connection with other problems about Artin groups. In particular, the following conjecture is particularly relevant:

Conjecture (intersection conjecture) *The intersection of any two parabolic subgroups in A_Γ is again a parabolic subgroup.*

This conjecture has been proved for a few families of Artin groups but remains open in general. For some families of Artin groups, certain weaker versions have been established; see Section 2 for more details.

In this paper, we study the connections between these three conjectures, and show the acylindrical hyperbolicity of new classes of Artin groups.

Statement of results We now state the main results of this article. Our main theorem is a criterion for showing the acylindrical hyperbolicity of an Artin group admitting a visual splitting, under a mild assumption on the amalgamating subgroup:

Theorem A *Let A_Γ be an irreducible Artin group that splits visually as an amalgamated product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. If the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ, then A_Ω is weakly malnormal in A_Γ. In particular, A_Γ is acylindrically hyperbolic.*

As a consequence of this result, we also obtain the following result showing the connection between the three conjectures at the centre of this article:

Theorem B *Suppose A_Γ is irreducible and Γ is not a clique.*

- *If A_Γ satisfies the intersection conjecture, then it also satisfies the weak malnormality conjecture.*
- *If A_Γ satisfies the weak malnormality conjecture, then it also satisfies the acylindrical hyperbolicity conjecture.*

Theorem A can be used to show the acylindrical hyperbolicity of many new classes of Artin groups. For instance, using the existing results about intersections of parabolic subgroups in Artin groups of FC type [2], we obtain the following:

Corollary C *Even Artin groups of FC type satisfy the acylindrical hyperbolicity conjecture. In addition, any Artin group of FC type that visually splits over a spherical-type parabolic subgroup satisfies the acylindrical hyperbolicity conjecture.*

After this article was first released, Kato–Oguni announced a proof of the acylindrical hyperbolicity conjecture for the class of free-of-infinity Artin groups [28], which contains the above classes.

Note that in Theorem A, the condition on the edge group is strictly weaker than requiring that the whole Artin group A_Γ satisfies the intersection conjecture. In Section 3.3, we give examples of how to check this property in some cases, by using a framework of Godelle–Paris [19] to construct suitable CAT(0) cube complexes associated to Artin groups. As an application, we derive the acylindrical hyperbolicity of some new Artin groups whose underlying graph is a cone (Corollary 3.19), and for which the intersection conjecture is currently unknown.

In some special cases, the main hypothesis of Theorem A may be verified simply by checking that the edge group is weakly malnormal in one of the vertex groups. We thus provide a list of Artin groups for which we know the weak malnormality conjecture, as this allows us to quickly verify this condition, and also allows us to construct new examples of acylindrically hyperbolic Artin groups (see Corollary 4.7):

Theorem D *The weak malnormality conjecture holds for the following classes of groups*:

- *Artin groups satisfying the hypothesis of Theorem A* (*for instance*, *even Artin groups of FC type*),
- *Artin groups of spherical type*,
- *two-dimensional Artin groups.*

Organisation of the paper In Section 2, we recall the terminology and some standard results about Artin groups and their parabolic subgroups. In Section 3, we prove Theorem A and Corollary C by studying in detail the action of these Artin groups on their Bass–Serre trees. We also use CAT(0) cube complexes introduced by Godelle–Paris to prove the acylindrical hyperbolicity of additional classes of Artin groups. In Section 4, we prove Theorems B and D by studying the geometry of the orbits of parabolic subgroups in a suitable complex (depending on the case: Bass–Serre tree of a splitting, or Deligne complex of the group).

2 Preliminaries on Artin groups

Definition 2.1 Let Γ be a graph with vertices labelled by the set S and any edge between s and t labelled by an integer $m_{st} \in \{2, 3, \dots\}$. Define the Artin group by the presentation

$$A_\Gamma = \langle S \mid \underbrace{stst\dots}_{m_{st}\text{ terms}} = \underbrace{tsts\dots}_{m_{st}\text{ terms}} \quad \text{for all edges in } \Gamma\rangle.$$

Note that in this definition, two vertices s and t which are not joined by an edge in Γ have the free relation and in this case we define $m_{st} = \infty$. The graph Γ is often called the presentation graph. In some literature, a different defining graph is used called the Dynkin diagram, wherein edges with $m_{st} = 2$ are omitted, while edges with $m_{st} = \infty$ are included.

For every Artin group, there is an associated Coxeter group, which is obtained by adding to the Artin presentation the relation $s^2 = 1$ for all generators s. An Artin group is called *spherical-type* or *finite-type* if the corresponding Coxeter group is finite. Such Artin groups have well-understood algebraic and geometric properties in comparison to their infinite-type cousins.

The *dimension* of an Artin group A_Γ is the maximal size of a subgraph $\Omega \subset \Gamma$ such that A_Ω is spherical-type. So for example, a 2-dimensional Artin group is one where the presentation graph Γ is not discrete and for which the only spherical-type parabolic subgroups are either cyclic (1-generator) or dihedral (2-generator) Artin groups.

Definition 2.2 An Artin group A_Γ is said to be

- of *spherical type* if the corresponding Coxeter group W_Γ is finite,
- of *FC type* if for every induced complete subgraph $\Gamma' \subset \Gamma$, the corresponding Coxeter group $W_{\Gamma'}$ is finite,

- *two-dimensional* if Γ is not discrete and for every triangle of Γ with vertices a, b, c, we have
$$\frac{1}{m_{ab}}+\frac{1}{m_{bc}}+\frac{1}{m_{ac}}\le 1$$
(this is equivalent to requiring that A_Γ has cohomological dimension 2, and also coincides with the notion of dimension of an Artin group introduced above),
- of *even type* if all labels of Γ are even.

2.1 Parabolic subgroups

For an induced subgraph Ω of Γ, the Artin group A_Ω embeds as a subgroup of A_Γ [29]. We call such a subgroup a *standard parabolic subgroup*. A *parabolic subgroup* is a conjugate of some standard parabolic subgroup. Parabolic subgroups play a central role in our understanding of Artin groups. For example, many conjectures about Artin groups can be reduced to the case of parabolic subgroups corresponding to cliques in Γ, that is, if A_Ω satisfies the conjecture for every clique Ω in Γ, then the conjecture also holds for A_Γ [12; 18].

We say that an Artin group A_Γ is *reducible* if there exist two induced disjoint subgraphs Γ_1, Γ_2 of Γ such that $V(\Gamma)=V(\Gamma_1)\cup V(\Gamma_2)$, and every vertex of Γ_1 is connected to every vertex of Γ_2 by an edge labelled 2. In that case, we have that A_Γ decomposes as the direct product $A_{\Gamma_1}\times A_{\Gamma_2}$, and the parabolic subgroups A_{Γ_1}, A_{Γ_2} are called *direct factors* of A_Γ. If no such subgraphs Γ_1, Γ_2 exist, the Artin group A_Γ is called *irreducible*.

The behaviour of parabolic subgroups will be key to the discussion which follows. As noted above, we do not know in general if intersections of parabolic subgroups are parabolic. A useful fact proven by Blufstein and Paris [5], is that if $P\subseteq P'$ are two parabolic subgroups of A_Γ, then P is also a parabolic subgroup of P'. The following lemma will be often used in this article.

Lemma 2.3 *Let A_Γ be an Artin group. Then for every sequence of parabolic subgroups $H_0\subsetneq\cdots\subsetneq H_n$, we have $n\le|V(\Gamma)|$. In particular, there is an upper bound on the length of chains of parabolic subgroups.*

Proof Say H_i is a conjugate of A_{Ω_i}. By Blufstein–Paris [5], for each i, H_{i-1} is a (proper) parabolic subgroup of H_i, so Ω_{i-1} must be a proper subgraph of Ω_i, that is, $\Omega_0\subsetneq\cdots\subsetneq\Omega_n\subseteq\Gamma$. □

A geometric construction that has played a primary role in the study of Artin groups is the Deligne complex. For an infinite-type Artin group A_Γ, let $\mathcal{P}_\Gamma$ denote the poset consisting of cosets $aA_T\subset A_\Gamma$ such that A_T is a spherical-type parabolic subgroup. Partially order $\mathcal{P}_\Gamma$ by inclusion. The *Deligne complex* D_Γ is the cell complex whose vertices are the elements of $\mathcal{P}_\Gamma$ and whose cells are cubes spanned by intervals $[aA_T,aA_{T'}]$ for pairs $aA_T\subseteq aA_{T'}$. The Artin group acts on the Deligne complex by left multiplication, and the vertex corresponding to the coset aA_T is stabilised by the parabolic subgroup $(A_T)^a:=aA_Ta^{-1}$.

There are two well-known metrics on D_Γ. One is the standard cubical metric; this metric is CAT(0) if and only if A_Γ is FC-type. The other is a piecewise Euclidean metric, called the Moussong metric, in which the metric on a cube $[aA_T,aA_{T'}]$ depends on the shape of the Coxeter cell for $W_{T'}$. It is conjectured that the Moussong metric is CAT(0) for all infinite-type Artin groups. This has been shown to hold for all 2-dimensional Artin groups [11], some 3-dimensional Artin groups [10], and a class known as locally reducible Artin groups [9].

2.2 Visual splittings

Acylindrically hyperbolic groups do not have infinite direct factors. Likewise a group that factors as a direct product clearly has proper subgroups which are not weakly malnormal. Thus, we will focus on irreducible Artin groups in this paper. We will also focus on Artin groups that can be decomposed as amalgamated products.

Definition 2.4 (visual splitting of an Artin group) A *visual splitting* of an Artin group A_Γ is a splitting as an amalgamated free product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ where Γ_1 and Γ_2 are proper, induced subgraphs of Γ and $\Omega = \Gamma_1 \cap \Gamma_2$. This happens precisely when $\Gamma = \Gamma_1 \cup \Gamma_2$. Note that under such assumptions on Γ_1 and Γ_2, the associated splitting is nontrivial, i.e., $A_\Omega \neq A_{\Gamma_1}, A_{\Gamma_2}$.

Note that such a splitting exists if and only if Γ is not a clique. Indeed, assume that the two vertices $s, t \in V(\Gamma)$ are not connected by an edge in Γ. Then one checks that A_Γ splits as the amalgamated product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ with $\Gamma_1 = \Gamma - \{s\}$, $\Gamma_2 = \Gamma - \{t\}$, and $\Omega = \Gamma - \{s, t\}$.

When an Artin group $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ admits a visual splitting, we can sometimes derive properties of A_Γ using properties of A_{Γ_1}, A_{Γ_2}, and A_Ω. This can, for instance, be done to study Artin groups of FC type, as such groups can always be decomposed as a sequence of nested amalgamated free products, where the final splitting has spherical-type edge groups.

2.3 Existing results for the main conjectures

In this section we will review known results for the main conjectures, and prove some elementary implications that we will use later.

Intersection conjecture In 1983, van der Lek showed that the intersection of standard parabolic subgroups $A_{\Gamma_1} \cap A_{\Gamma_2}$ is always parabolic [29]. More recently, the more general intersection conjecture, namely the property that the intersection of any two parabolics is again parabolic, has been proven to hold for the following classes of Artin groups:

- spherical-type Artin groups [15],
- right-angled Artin groups and other graphs of groups [3],
- large type (i.e., all labels satisfy $m_{s,t} \geq 3$) Artin groups [16],
- (2,2)-free two-dimensional Artin group, i.e., Γ does not have two consecutive edges labelled by 2 and the cohomological dimension of A_Γ is 2 [4],
- Euclidean type of the form $\widetilde{A}_n$ and $\widetilde{C}_n$ [22],
- even FC type Artin groups [2].

While this conjecture remains open in general, there are several other classes of Artin groups for which some weaker version of the intersection property is known to hold. For example, in [36] it is shown that in FC-type Artin groups, intersections of two spherical-type parabolics are parabolic, and this was further generalized by Möller, Paris, and Varghese [35] to include the case where just one of the two parabolics is spherical-type.

The proof in [36] that intersections of spherical-type parabolics are parabolic for FC-type Artin groups uses the fact that the cubical metric on the Deligne complex D_Γ is CAT(0) for these groups. This argument can be generalized to other Artin groups acting on CAT(0) spaces where the intersection property is known for the stabilisers of vertices. We include the proof here for completeness.

Proposition 2.5 *Let A_Γ be an Artin group acting on a polyhedral complex X with a piecewise Euclidean* CAT(0) *metric, where each cell stabiliser is a parabolic subgroup of A_Γ. Assume that the action is without inversions, that is, the stabiliser of each cell pointwise fixes the entire cell. Let $\mathcal{P}$ be the collection of parabolic subgroups that appear as stabilisers of the cells of X. Suppose that the stabiliser of every vertex of X satisfies the intersection conjecture. Then the intersection of any two elements of $\mathcal{P}$ is again a parabolic subgroup of A_Γ.*

Proof Let P and P' be two parabolics in $\mathcal{P}$. There exist cells σ and σ' in X such that P, P' are the stabilisers of σ, σ', respectively. Let x, x' be two points in the interior of σ, σ', respectively. Since $P \cap P'$ fixes both σ and σ', it fixes x and x', and hence the unique CAT(0) geodesic γ between them. Moreover, because the action is without inversions, it fixes the subcomplex consisting of the union of all the cells that contain a point of γ in their interior. In particular, it fixes some edge path ρ that contains all vertices of σ and σ'. Thus, $P \cap P'$ is equal to the pointwise stabiliser of ρ. The result now follows from the following:

Claim *Let $v_0, \dots, v_k$ be a combinatorial path in X, with stabilisers $P_0, \dots, P_k$, respectively. Then the intersection $\bigcap_{0 \le i \le k} P_i$ is a parabolic subgroup of A_Γ.*

Let us prove this claim by induction on $k \ge 1$. For $k = 1$, the intersection $P_0 \cap P_1$ is the stabiliser of a single edge. By hypothesis, the stabiliser of the edge is a parabolic subgroup in $\mathcal{P}$.

Now suppose that we have proved the result for some $k \ge 1$. Consider a combinatorial path $v_0, \dots, v_{k+1}$ with stabilisers $P_0, \dots, P_{k+1}$. By the induction hypothesis, we have that $\bigcap_{0 \le i \le k} P_i$ is a parabolic subgroup of A_Γ. Note that this is a subgroup of P_k, and hence a parabolic subgroup of P_k by [5]. We also know that $P_k \cap P_{k+1}$ is the stabiliser of the last edge of this path, and so $P_k \cap P_{k+1}$ is a parabolic subgroup of P_k.

We can thus write

$$\bigcap_{0 \le i \le k+1} P_i = \Big(\bigcap_{0 \le i \le k} P_i \Big) \cap (P_k \cap P_{k+1}).$$

Since the intersection conjecture holds in P_k by assumption, it follows that $\bigcap_{0 \le i \le k+1} P_i$ is a parabolic subgroup of P_k, and hence of A_Γ. □

In particular, for Artin groups for which the Moussong metric on Deligne complex is known to be CAT(0), the lemma above applies to show that intersections of spherical-type parabolics are parabolic. As noted above, this holds for 2-dimensional Artin groups, some 3-dimensional Artin groups, and locally reducible Artin groups.

Remark 2.6 The above proposition can be generalised to actions on complexes that satisfy other forms of nonpositive curvature, as in [4; 16]. We leave it to the reader to check that the key geometric feature necessary for the proof to carry over is the following property: if an element $g \in A_\Gamma$ fixes two vertices v, v' of X, then it also fixes some combinatorial path of X from v to v'. Such a weak form of convexity is satisfied by many forms of nonpositive curvature.

We can also obtain more examples of groups that satisfy the intersection conjecture by taking products of groups where the intersection conjecture is known.

Lemma 2.7 *Suppose that A_Γ is a reducible Artin group with direct factors $A_\Gamma = A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$. If the intersection conjecture holds for all parabolic subgroups in each direct factor A_{Γ_i} then the intersection conjecture holds for A_Γ.*

Proof Given two parabolics P, Q of A_Γ, one can decompose them as direct products

$$P = P_1 \times \cdots \times P_k, \quad Q = Q_1 \times \cdots \times Q_k$$

with each P_i, Q_i a parabolic subgroup of A_{Γ_i}. We thus have

$$P \cap Q = (P_1 \cap Q_1) \times \cdots \times (P_k \cap Q_k)$$

and the result follows from the fact that each A_{Γ_i} satisfies the intersection conjecture. □

We also add the following examples, which will be used later in this article.

Lemma 2.8 *Suppose that A_Γ is an Artin group with 3 or fewer generators. Then A_Γ satisfies the intersection conjecture.*

Proof The result is clear if A_Γ has one generator. If it has two generators, it is either a free group (in particular, a right-angled Artin group) or a dihedral Artin group (in particular, a spherical-type Artin group), and so the result follows from [3] and [15], respectively. Let us assume that A_Γ has three generators. If A_Γ is spherical-type (i.e., in the case of a triangle Artin group $(2,3,n)$ for $3 \leq n \leq 5$ or $(2,2,n)$ for $n \geq 2$), this holds by [15], so assume it is infinite type. If A_Γ is a free group or the triangle Artin group $(2,2,\infty)$, then it is a right-angled Artin group, and the result follows from [3]. Otherwise, A_Γ is two-dimensional and $(2,2)$-free, and the result follows from [4]. □

Acylindrical hyperbolicity In this paragraph, we review previously known results about acylindrically hyperbolic Artin groups. For Artin groups of spherical type, the acylindrical hyperbolicity conjecture was proved by Calvez–Wiest [8], following earlier result for braid groups [6; 33]. Thus, the acylindrical hyperbolicity conjecture reduces to the case of Artin groups of infinite type. Such groups are conjectured to have a trivial centre, so the conjecture asks whether these groups are acylindrically hyperbolic. Currently, the acylindrical hyperbolicity conjecture is known for several classes of Artin groups (we refer the reader to these articles for the definition of some of these classes), which we list below:

- spherical-type Artin groups, by Calvez–Wiest [8],
- right-angled Artin groups, by Osin [37],
- two-dimensional Artin groups, by Vaskou [40], following earlier work for XXL-type Artin groups (i.e., all labels satisfy $m_{s,t} \geq 5$) by Haettel [21], for XL-type Artin groups (i.e., all labels satisfy $m_{s,t} \geq 4$) by Martin–Przytycki [30], and for some two-dimensional Artin groups admitting a specific CAT(0) model by Kato–Oguni [27],
- Euclidean-type Artin groups, by Calvez [7],
- Artin groups whose graph is not a join, by Charney–Morris-Wright [12], following previous work by Chatterji–Martin [13],
- some relatively extra-large type Artin groups, by Goldman [20],
- some locally reducible Artin groups, by Mastrocola [32].

In this article, we add to this list the class of even Artin groups of FC type, among other new examples. Note that after this article was first released, Kato–Oguni announced a proof for the class of free-of-infinity Artin groups [28], which contains in particular the class of (nonspherical) even Artin groups of FC type.

3 Artin groups with visual splittings and acylindrical hyperbolicity

The goal of this section is to obtain new criteria for proving acylindrical hyperbolicity and apply them to get new examples of acylindrically hyperbolic Artin groups. We will focus primarily on the case of Artin groups with a visual splitting. In this case, there is a clear connection between malnormality and acylindricity given by the following theorem of Minasyan and Osin [34].

Theorem 3.1 (see [34]) *Suppose G splits as an amalgamated product of groups $G = A *_C B$ with $A \neq C \neq B$. If C is weakly malnormal in G, then G is either virtually cyclic or acylindrically hyperbolic.*

Thus a key to proving acylindricity for A_Γ is understanding when parabolic subgroups are weakly malnormal.

3.1 The main acylindrical hyperbolicity criterion

The main result of this subsection is the following.

Theorem 3.2 *Let A_Γ be an irreducible Artin group that splits visually as an amalgamated product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. If the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ, then A_Ω is weakly malnormal in A_Γ. In particular, A_Γ is acylindrically hyperbolic.*

In order to prove this result, we need the following characterisation of normal parabolic subgroups of Artin groups. We start with the irreducible case:

Lemma 3.3 *Let A_Γ be an irreducible Artin group. Then the only normal standard parabolic subgroups of A_Γ are A_Γ and the trivial subgroup.*

Proof Let $\varnothing \subsetneq \Gamma' \subsetneq \Gamma$ be a strict subgraph of Γ. Let Γ_D, Γ'_D be the Dynkin diagrams corresponding to the presentation graph Γ, Γ', respectively (i.e., no edge if $m_{s,t} = 2$, and edges with label ∞ for pairs s, t that are not connected in Γ). Since A_Γ is irreducible, Γ_D is connected and Γ'_D is a strict induced subgraph of Γ_D. Thus, there exists an edge of Γ_D connecting a vertex $s \in \Gamma'_D$ and $t \notin \Gamma'_D$. It follows from van der Lek that

$$A_{\Gamma'} \cap A_{\{s,t\}} = \langle s \rangle.$$

We can also compute that

$$t A_{\Gamma'} t^{-1} \cap A_{\{s,t\}} = t(A_{\Gamma'} \cap A_{\{s,t\}})t^{-1} = t\langle s \rangle t^{-1}.$$

Thus in order to show that $tA_{\Gamma'}t^{-1} \neq A_{\Gamma'}$ it is sufficient to show that t does not normalise $\langle s \rangle$. Suppose by contradiction that $t\langle s \rangle t^{-1} = \langle s \rangle$. Then $\langle s \rangle$ is normal in $A_{\{s,t\}}$ and the corresponding quotient is either trivial or infinite cyclic, depending on the parity of m_{st}. Thus, $A_{\{s,t\}}$ is either cyclic or $\mathbb{Z}$-by-$\mathbb{Z}$, and in particular virtually abelian. But since $m_{st} \geq 3$ as s, t are joined by an edge in the Dynkin diagram, we have that $A_{\{s,t\}}$ contains nonabelian free subgroups, a contradiction. Thus we know that t does not normalise $\langle s \rangle$ and $A_{\Gamma'}$ cannot be a normal subgroup. □

Corollary 3.4 *Let A_Γ be an Artin group. Then a standard parabolic subgroup is normal if and only if it is a product of direct factors of A_Γ.*

Proof Decompose A_Γ as a product of irreducible Artin groups $A_\Gamma = A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$. If $A_{\Gamma'}$ is a product of groups of the form $A_{\Gamma'} = A_{\Gamma_{i_1}} \times \cdots \times A_{\Gamma_{i_k}}$ then $A_{\Gamma'}$ is a direct factor of A_Γ and so must be a normal subgroup.

Conversely, if $A_{\Gamma'}$ is normal in A_Γ then for all i, $A_{\Gamma'} \cap A_{\Gamma_i}$ is normal in A_{Γ_i}. By Lemma 3.3, we see that $A_{\Gamma'} \cap A_{\Gamma_i}$ must be equal to either A_{Γ_i} or to the trivial group. This implies that $A_{\Gamma'}$ is a direct product of factors of A_Γ. □

Proof of Theorem 3.2 Let $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ be a visual splitting and let T be the Bass–Serre tree of this splitting.

Claim 1 *Let e be an edge of T with vertices v and w. The trivial subgroup is the only parabolic subgroup of $\mathrm{Stab}(e)$ that is normal in both $\mathrm{Stab}(v)$ and $\mathrm{Stab}(w)$.*

Up to conjugation, it is enough to show that the trivial subgroup is the only parabolic subgroup of A_Ω that is normal in both A_{Γ_1} and A_{Γ_2}. Suppose to the contrary that H is normal in both A_{Γ_1} and A_{Γ_2}. Since every element of A_Γ is a product of elements of A_{Γ_1} and A_{Γ_2}, H is also normal in A_Γ. By assumption, A_Γ is irreducible, so applying Corollary 3.4 we conclude that H must be trivial. This proves the claim.

Claim 2 *Let γ be a geodesic segment of T, and let $\mathrm{Stab}_*(\gamma)$ be the pointwise stabiliser of γ. If $\mathrm{Stab}_*(\gamma)$ is nontrivial, then we can extend γ to a geodesic segment $\gamma' \supsetneq \gamma$ such that $\mathrm{Stab}_*(\gamma') \subsetneq \mathrm{Stab}_*(\gamma)$.*

Let $v_1, \dots, v_n$ be the vertices of γ. Up to the action of A_Γ, we can assume that $\mathrm{Stab}_*(v_1) = A_{\Gamma_1}$ and $\mathrm{Stab}_*(v_2) = A_{\Gamma_2}$. Suppose for every geodesic segment γ' of T extending γ, we have $\mathrm{Stab}_*(\gamma') = \mathrm{Stab}_*(\gamma)$. We first show that for every $g \in A_\Gamma$ such that the translate $g\gamma$ has as its closest-point projection on γ the single point $\{v_n\}$, we also have $\mathrm{Stab}_*(g\gamma) = \mathrm{Stab}_*(\gamma)$ (and in particular, such an element $g \in A_\Gamma$ normalises $\mathrm{Stab}_*(\gamma)$). For such an element g, the geodesic from v_1 to any point of $g\gamma$ contains γ. Thus, the minimal subtree T_g of T containing γ and $g\gamma$ can be written as the union of at most two geodesic segments γ_1, γ_2 extending γ, and thus

$$\mathrm{Stab}_*(\gamma) \cap \mathrm{Stab}_*(g\gamma) = \mathrm{Stab}_*(T_g) = \mathrm{Stab}_*(\gamma_1) \cap \mathrm{Stab}_*(\gamma_2) = \mathrm{Stab}_*(\gamma).$$

Thus, $\mathrm{Stab}_*(\gamma) \subset \mathrm{Stab}_*(g\gamma) = g\mathrm{Stab}_*(\gamma)g^{-1}$. Since $\mathrm{Stab}_*(\gamma)$ is a parabolic subgroup by assumption on the splitting, it follows from Lemma 2.3 that this inclusion is actually an equality, for otherwise the sequence $(g^n \mathrm{Stab}_*(\gamma) g^{-n})_{n \geq 0}$ would form an unbounded strict chain of parabolic subgroups.

Let T' be the subtree of T consisting of all the points of T whose closest-point projection on γ is v_n. Since the action of A_Γ on T is cocompact and T' is an unbounded subtree of T, we can pick an element $g \in A_\Gamma$ such that $g\gamma \subset T'$ and $d(\gamma, g\gamma) > |\gamma|$ (where as usual d denotes the path metric of T and $|\gamma|$ the combinatorial length of γ). These conditions imply that for every $h_1 \in \mathrm{Stab}_*(gv_1) = gA_{\Gamma_1}g^{-1}$ and every $h_2 \in \mathrm{Stab}_*(gv_2) = gA_{\Gamma_2}g^{-1}$, we also have $h_1 g\gamma \subset T'$ and $h_2 g\gamma \subset T'$. Thus, we get that g as well as every element of the form $h_1 g$ or $h_2 g$ normalises $\mathrm{Stab}_*(\gamma)$, for $h_i \in gA_{\Gamma_i}g^{-1}$. Thus, both $gA_{\Gamma_1}g^{-1}$ and $gA_{\Gamma_2}g^{-1}$ normalise $\mathrm{Stab}_*(\gamma)$. Since A_Γ is generated by A_{Γ_1} and A_{Γ_2}, hence by $gA_{\Gamma_1}g^{-1}$ and $gA_{\Gamma_2}g^{-1}$, it follows that $\mathrm{Stab}_*(\gamma)$ is normal in A_Γ. Since A_Γ is irreducible, it follows from Lemma 3.3 that $\mathrm{Stab}_*(\gamma)$ is trivial, which proves the claim.

By Claim 2, we can construct a sequence of geodesic segments $\gamma_0 \subsetneq \gamma_1 \subsetneq \cdots$, such that the sequence of stabilisers $\mathrm{Stab}_*(\gamma_i)$ strictly decreases as long as they are not trivial. By assumption on the splitting, we have that each $\mathrm{Stab}_*(\gamma_i)$ is a parabolic subgroup of A_Γ. It now follows from Lemma 2.3 that $\mathrm{Stab}_*(\gamma_i)$ becomes trivial after finitely many steps.

We thus have a finite length geodesic segment γ with trivial point stabiliser. Let $e_1, \dots, e_n$ be the edges of γ. Since T is a tree, we have $\mathrm{Stab}_*(\gamma) = \mathrm{Stab}(e_1) \cap \mathrm{Stab}(e_n)$, and by assumption $\mathrm{Stab}_*(\gamma) = \{1\}$. Since edge stabilisers are conjugates of A_Ω by construction, it follows that A_Ω is weakly malnormal in A_Γ, hence A_Γ is acylindrically hyperbolic by Theorem 3.1. □

3.2 Applications to some classes of Artin groups

Recall that A_Γ is even FC-type if all edge labels in Γ are even, and all cliques in Γ generate spherical-type parabolics.

Theorem 3.5 *Even FC-type Artin groups satisfy the acylindrical hyperbolicity conjecture.*

Proof Assume that A_Γ is irreducible. If A_Γ is spherical-type, this follows from Calvez–Wiest [8]. If A_Γ is not spherical-type, then it admits a visual splitting. Since even FC-type Artin groups satisfy the intersection property by [2], the intersection of any two conjugates of the edge group for this splitting must be parabolic and so by Theorem 3.2, A_Γ is acylindrically hyperbolic. $\square$

For general FC-type Artin groups, we cannot directly apply Theorem 3.2 as we do not know that they satisfy the intersection conjecture. We therefore ask the following:

Question 3.6 Do FC-type Artin groups satisfy the intersection conjecture?

We can nonetheless prove acylindrical hyperbolicity under more restrictive conditions.

Theorem 3.7 *Let A_Γ be an irreducible Artin group such that the Deligne complex D_Γ admits a piecewise Euclidean* CAT(0) *metric. If A_Γ splits visually over a spherical-type parabolic, then A_Γ is acylindrically hyperbolic.*

Proof Stabilisers of simplices in the Deligne complex are precisely the spherical-type parabolic subgroups. Since spherical-type Artin groups satisfy the intersection conjecture [15], it follows from Proposition 2.5 that the intersection of any two spherical-type parabolic subgroups of A_Γ is again a parabolic subgroup. In particular, the splitting satisfies the hypothesis of Theorem 3.2. $\square$

In particular, Theorem 3.7 applies to FC-type Artin groups, locally reducible Artin groups, and certain Artin groups of dimension 3 (namely those for which all the irreducible three-dimensional parabolic subgroups are isomorphic to the braid group B_4), whenever they split visually over a spherical-type parabolic.

3.3 A weaker version of the intersection property

In Theorem 3.2, the condition on the intersections of conjugates of the edge group A_Ω is a priori weaker than requiring that A_Γ satisfies the intersection conjecture. In this section, we show how to obtain this weaker condition in cases where the intersection conjecture may not be known for A_Γ. We start with the following observation.

Theorem 3.8 *Let A_Γ be an irreducible Artin group with a visual splitting $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. If both A_{Γ_1} and A_{Γ_2} satisfy the intersection conjecture, then the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ. In particular, A_Γ is acylindrically hyperbolic.*

Proof Since A_{Γ_1} and A_{Γ_2} both satisfy the intersection conjecture, the fact that the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ is a direct consequence of Proposition 2.5 applied to the action of A_Γ on the (CAT(0)) Bass–Serre tree of the splitting $A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. $\square$

In the rest of this section, we show how one may understand the intersection of two conjugates of A_Ω even when the vertex groups are not known to satisfy the intersection conjecture. As with the original proofs of the intersection property for some families of Artin groups (see Section 2.3), the idea is to realise the conjugates of A_Ω as stabilisers in a suitable CAT(0) complex and apply Proposition 2.5. We recall a relevant framework of Godelle–Paris to construct such complexes [19].

Definition 3.9 A *complete cover* $\mathcal{U}$ of Γ is a collection of induced subgraphs of Γ that contains every edge of Γ and is stable under taking induced subgraphs (including the empty graph). Given an element $\Gamma' \in \mathcal{U}$, the corresponding standard parabolic subgroup $A_{\Gamma'}$ is called a *$\mathcal{U}$-standard parabolic subgroup*, and a conjugate of $A_{\Gamma'}$ is called a *$\mathcal{U}$-parabolic subgroup*.

Given a complete cover $\mathcal{U}$, one defines the corresponding *Godelle–Paris cube complex $X_{\mathcal{U}}$* as follows: vertices of $X_{\mathcal{U}}$ correspond to cosets of $\mathcal{U}$-standard parabolic subgroups, and cubes correspond to the intervals (for the inclusion) between gA_{Γ_1} and gA_{Γ_2}, whenever $g \in A_\Gamma$ and $\Gamma_1 \subset \Gamma_2$ are in $\mathcal{U}$.

Note that A_Γ acts on $X_{\mathcal{U}}$ by left multiplication on left cosets. This action is cocompact and without inversion.

For instance, if A_Γ is FC-type and $\mathcal{U}$ consists of all cliques of Γ, we recover the cubical Deligne complex of Charney–Davis [11]. More generally, if $\mathcal{U}$ consists of all the cliques of Γ, one recovers the Godelle–Paris clique complex [19].

There is a complete characterisation of when the standard cubical metric on this complex is CAT(0). We need the following definition:

Definition 3.10 Let $\mathcal{U}$ be a complete cover of Γ. We define a simplicial complex $L_{\mathcal{U}}$ as follows: the vertices of $L_{\mathcal{U}}$ are the vertices of Γ, and a set of vertices $v_0, \dots, v_k$ of $L_{\mathcal{U}}$ span a k-simplex of $L_{\mathcal{U}}$ if and only if the induced subgraph of Γ spanned by $v_0, \dots, v_k$ belongs to $\mathcal{U}$.

Note that $L_{\mathcal{U}}$ is isomorphic to the link of any vertex of $X_{\mathcal{U}}$ corresponding to the empty subgraph of Γ.

Theorem 3.11 [19, Theorem 4.2] *Let $\mathcal{U}$ be a complete cover of Γ. Then $X_{\mathcal{U}}$ is a* CAT(0) *cube complex if and only if $L_{\mathcal{U}}$ is a flag simplicial complex.*

Remark 3.12 In [19], the above theorem is stated with the additional condition that the $\mathcal{U}$-standard parabolic subgroups satisfy the $K(\pi, 1)$-conjecture. However, the reader can follow the proof and check that this assumption is not needed in order to prove that $X_{\mathcal{U}}$ is CAT(0). (Godelle–Paris need this assumption to prove a subsequent theorem showing that an Artin group satisfies the $K(\pi, 1)$-conjecture if all its free-of-infinity standard parabolic subgroups satisfy that conjecture.)

Corollary 3.13 *Let A_Γ be an Artin group. Let $\mathcal{U}$ be a complete cover of Γ. Assume that*

- *$L_{\mathcal{U}}$ is a flag simplicial complex,*
- *for every Γ' in $\mathcal{U}$, the standard parabolic $A_{\Gamma'}$ satisfies the intersection conjecture.*

Then the intersection of any two $\mathcal{U}$-parabolic subgroups is again a parabolic subgroup of A_Γ.

Proof By Theorem 3.11, we have that $X_{\mathcal{U}}$ is a CAT(0) cube complex. By construction, the action is without inversion and the stabilisers of cubes are precisely the $\mathcal{U}$-parabolic subgroups of A_Γ. Thus, the result follows from Proposition 2.5. □

In particular, to show that the intersection of two conjugates of a standard parabolic A_Ω is again a parabolic subgroup, it is enough to include Ω in a suitable complete cover of Γ. We now give an example of a geometric condition on Ω that guarantees that such a cover exists.

Definition 3.14 We say that an induced subgraph Ω is 2-*convex* in Γ if every geodesic path of Γ of length 2 with endpoints in Ω is contained in Ω.

Lemma 3.15 *Let Γ be a simplicial graph, and let Ω be a 2-convex subgraph of Γ. Let $\mathcal{U}$ be the complete cover consisting of all the cliques of Γ and all the induced subgraphs of Ω. Then the simplicial complex $L_{\mathcal{U}}$ is flag.*

Proof Let $T \subset V(\Gamma)$ be a set of vertices that are pairwise connected by edges in $L_{\mathcal{U}}$. To show that T spans a simplex of $L_{\mathcal{U}}$, it is enough to show that either $T \subset \Omega$ or the vertices of T span a clique of Γ.

We can thus assume that T is not contained in Ω, and let us show that T spans a clique of Γ. We decompose T as a disjoint union $T = T_1 \cup T_2$, where $T_1 := T \cap V(\Omega)$ and $T_2 := T - T_1$. Note that by construction of $\mathcal{U}$, two vertices of Γ are adjacent in $L_{\mathcal{U}}$ if and only if they are both contained in Ω or they are connected by an edge of Γ. By assumption, any two vertices of T are adjacent in $L_{\mathcal{U}}$, thus, for every $t \in T_2$ and $t' \in T - \{t\}$, we have that t and t' are connected by an edge in Γ.

It remains to show that any two vertices $t \neq t' \in T_1$ are connected by an edge of Γ. Suppose by contradiction that there exists a pair $t, t' \in T_1$ that is not connected by an edge of Γ. Since T_2 is not empty, we can pick an element $s \in T_2$ and the previous argument shows that t, s, t' forms a path in Γ. Since t, t' are not adjacent in Γ, this path is geodesic. By 2-convexity of Ω, we get that $s \in T_1$, a contradiction. Thus, the vertices of T_1 span a simplex of Γ, and it now follows that T spans a clique of Γ, and hence spans a simplex in $L_{\mathcal{U}}$. □

The following is now a direct consequence of Corollary 3.13:

Corollary 3.16 *Let A_Γ be an Artin group, and let Ω be a 2-convex subgraph of Γ. Assume that A_Ω as well as every clique standard parabolic subgroup of A_Γ satisfy the intersection conjecture. Then the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ.*

Proposition 3.17 *Let A_Γ be an irreducible Artin group that visually splits over a standard parabolic subgroup A_Ω. Assume that*

- *Ω is 2-convex in Γ,*
- *A_Ω and all clique parabolic subgroups of A_Γ satisfy the intersection conjecture.*

Then A_Γ is acylindrically hyperbolic.

Proof It follows from Corollary 3.16 that the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ. The result thus follows from Theorem 3.2. □

Application We can use this result to obtain new examples of acylindrically hyperbolic Artin groups not covered by the recent results of Charney–Morris-Wright [12].

Definition 3.18 Let C_n denote the graph that is a cycle on n vertices. The *wheel* W_n is the graph obtained from C_n by adding a new vertex (the apex) and connecting it to every vertex of C_n.

Corollary 3.19 *Let A_Γ be an irreducible Artin group whose underlying graph is a wheel W_n with $n \geq 6$. Then A_Γ is acylindrically hyperbolic.*

Proof Since $n \geq 6$, A_{W_n} visually splits over a parabolic subgroup A_Ω where Ω is a geodesic of length 2 containing the apex, that is 2-convex in W_n. Since all 3-generated Artin groups satisfy the intersection conjecture by Lemma 2.8, it follows that A_Ω and all clique parabolic subgroups of A_Γ satisfy the intersection conjecture. The result now follows from Proposition 3.17. $\square$

4 The weak malnormality conjecture

Next we consider the weak malnormality conjecture. The following reduction lemma shows that it is enough to deal with irreducible Artin groups:

Lemma 4.1 *Let $A_{\Gamma_1}, \dots, A_{\Gamma_k}$ be irreducible Artin groups that satisfy the weak malnormality conjecture. Then the direct product $A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$ also satisfies the weak malnormality conjecture.*

Proof Let $A_{\Gamma'}$ be a standard parabolic subgroup of the direct product $A_\Gamma := A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$. It is clear that if $A_{\Gamma'}$ contains one of the (normal) direct factors A_{Γ_i}, then it is not weakly malnormal. Thus, let us assume that $A_{\Gamma'}$ contains no direct factor. Note that $A_{\Gamma'}$ decomposes as the direct product $A_{\Gamma'} = A_{\Gamma'_1} \times \cdots \times A_{\Gamma'_k}$ where for each i, $\Gamma'_i := \Gamma' \cap \Gamma_i$. Since $A_{\Gamma'}$ does not contain any of the A_{Γ_i}, each $A_{\Gamma'_i}$ is a proper parabolic subgroup of A_{Γ_i}, hence weakly malnormal in A_{Γ_i} since by assumption, A_{Γ_i} satisfies the weak malnormality conjecture. Thus, for each i we can pick $g_i \in A_{\Gamma_i}$ such that $A_{\Gamma'_i} \cap A_{\Gamma'_i}^{g_i}$ is finite. Now set $g := g_1 \cdots g_k$. We have

$$A_{\Gamma'} \cap A_{\Gamma'}^{g} = (A_{\Gamma'_1} \cap A_{\Gamma'_1}^{g_1}) \times \cdots \times (A_{\Gamma'_k} \cap A_{\Gamma'_k}^{g_k}),$$

which is finite. Thus, $A_{\Gamma'}$ is weakly malnormal in A_Γ. $\square$

4.1 Connections between the three main conjectures

Proposition 4.2 *Let A_Γ be an irreducible Artin group that visually splits as an amalgamated product over a standard parabolic subgroup A_Ω. If A_Ω is weakly malnormal in A_Γ, then every proper parabolic subgroup of A_Γ is weakly malnormal. Thus A_Γ satisfies the weak malnormality conjecture.*

Proof Let T be the Bass–Serre of the splitting $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. Let P be a standard parabolic subgroup of A_Γ. Note that we have a splitting $P = P_1 *_{P_\Omega} P_2$, where $P_i := P \cap A_{\Gamma_i}$ and $P_\Omega = P \cap A_\Omega$. The Bass–Serre tree T' of that induced splitting embeds isometrically in T.

Since P is a proper parabolic subgroup, we have $T' \neq T$, hence we can pick a vertex v of T', and an edge $e = [v, w]$ that is not in T'. Since $\mathrm{Stab}(e)$ is conjugated to A_Ω, it is weakly malnormal in A_Γ. Choose $h \in A_\Gamma$ such that $\mathrm{Stab}(e) \cap \mathrm{Stab}(he)$ is finite. Let γ be the geodesic path in T with initial edge e and final edge he. In particular, the pointwise stabiliser of γ is finite.

Claim *There exist elements $g_1, g_2 \in A_\Gamma$ such that the trees $g_1 T'$ and $g_2 T'$ are disjoint, and the unique geodesic between them contains γ.*

If either (or both) endpoints of γ are translates of w, we can extend γ by a single edge at that endpoint to obtain a geodesic path $\gamma' \supseteq \gamma$ both of whose endpoints are translates of v. Say the initial edge of γ' is

$g_1 e$ and the final edge is $g_2 e$. Since e is not contained in T', $g_i e$ is not contained in $g_i T'$ for $i = 1, 2$. Thus γ' intersects $g_i T'$ in a single point. It now follows from standard arguments on the geometry of trees that $g_1 T'$ and $g_2 T'$ are disjoint, and γ' is the unique geodesic between them, which proves the claim.

Since P stabilises the tree T', the conjugate P^{g_i} stabilises the tree $g_i T'$, and it follows that the intersection $P^{g_1} \cap P^{g_2}$ stabilises the unique geodesic between these disjoint trees. Thus it fixes pointwise the path γ'. Since $\gamma' \supseteq \gamma$, it has finite stabiliser and we conclude that $P^{g_1} \cap P^{g_2}$ is finite. Hence P is weakly malnormal in A_Γ. □

Combining the results above, we now conclude:

Corollary 4.3 *Suppose A_Γ is irreducible and Γ is not a clique.*

- *If A_Γ satisfies the intersection conjecture, then it also satisfies the weak malnormality conjecture.*
- *If A_Γ satisfies the weak malnormality conjecture, then it also satisfies the acylindrical hyperbolicity conjecture.*

Proof If Γ is not a clique, A_Γ admits a visual splitting over some standard parabolic A_Ω. The first bullet point is the direct application of Theorem 3.2 and Proposition 4.2. For the second bullet point, since A_Γ is irreducible and satisfies the weak malnormality conjecture, A_Ω is weakly malnormal, so the result follows from Theorem 3.1. □

One might wonder if the converse of these implications also hold. It is possible to obtain a partial converse to the implication in the second bullet point, assuming that A_Γ acts acylindrically on some hyperbolic space such that the geometry of the action is "compatible" with the parabolic subgroups, in the following sense:

Lemma 4.4 *Let A_Γ be an irreducible Artin group, and assume that A_Γ is acylindrically hyperbolic, with a cobounded acylindrical action on a hyperbolic graph X. Suppose that for every proper parabolic subgroup $A_{\Gamma'}$, the following holds: Let $X_{\Gamma'} \subset X$ denote the $A_{\Gamma'}$-orbit of some chosen point of X. Then $X_{\Gamma'}$ is quasiconvex in X and its limit set $\Lambda X_{\Gamma'}$ is a strict subset of the Gromov boundary ∂X.*

Then A_Γ satisfies the weak malnormality conjecture.

Proof The set of limit points of loxodromic elements of A_Γ is dense in ∂X (see, for instance, Theorem 2.6 in [24]), so since $\Lambda X_{\Gamma'}$ is a proper closed subset of ∂X, we can pick a loxodromic element $g \in A_\Gamma$ such that $\Lambda g \cap \Lambda X_{\Gamma'} = \varnothing$. By hyperbolicity of X and quasiconvexity of $X_{\Gamma'}$, there exist constants ℓ and D (that depend only on the space X and the quasiconvexity constants of $X_{\Gamma'}$) such that if two translates of $X_{\Gamma'}$ are at distance at least ℓ, then the diameter of the closest projection of one on the other is bounded above by D. By acylindricity of the action, we can pick a constant L such that if $x, y \in X$ are at distance at least L, there are only finitely many elements $h \in A_\Gamma$ such that $d(x, hx) \le D$ and $d(y, hx) \le D$.

Using North-South dynamics of the action, we can now pick a large power $n \ge 0$ such that $X_{\Gamma'}$ and $g^n X_{\Gamma'}$ are disjoint, the diameter of the closest projection on each other is bounded above by D, and such that their distance is greater than L. Let $x \in X_{\Gamma'}$ and $y \in g^n X_{\Gamma'}$ be a pair of points that realises the distance between these two translates. We get in particular that an element $h \in A_{\Gamma'} \cap A_{\Gamma'}^{g^n}$ sends the pair x, y to another pair realising the distance between these two translates. Thus, for every $h \in A_{\Gamma'} \cap A_{\Gamma'}^{g^n}$, we have $d(x, hx) \leq D$ and $d(y, hx) \leq D$. Since $d(x, y) \geq L$ by construction, the acylindricity implies that the set of such h is finite. Thus, $A_{\Gamma'} \cap A_{\Gamma'}^{g^n}$ is finite, and $A_{\Gamma'}$ is weakly malnormal. □

Thus, we ask the following question:

Question 4.5 Let A_Γ be an irreducible Artin group, and assume that A_Γ is acylindrically hyperbolic, with a cobounded acylindrical action on a hyperbolic graph X. Let $A_{\Gamma'}$ be a proper parabolic subgroup of A_Γ, and let $X_{\Gamma'} \subset X$ denote the $A_{\Gamma'}$-orbit of some chosen point of X. Do we have that $X_{\Gamma'}$ is quasiconvex in X, with limit set $\Lambda X_{\Gamma'} \neq \partial X$?

4.2 Artin groups satisfying the weak malnormality conjecture

In this section we will show that the weak malnormality conjecture holds for several classes of Artin groups, which allows us to prove that new classes of Artin groups are acylindrically hyperbolic.

Proposition 4.6 *The weak malnormality conjecture holds for the following classes of groups*:

- *Artin groups satisfying the hypothesis of Theorem 3.2* (*for instance*, *even Artin group of FC type*),
- *Artin groups of spherical type*,
- *two-dimensional Artin groups.*

Using the criterion of Minasyan–Osin [34], this implies the acylindrical hyperbolicity of many groups admitting a visual splitting:

Corollary 4.7 *Let A_Γ be an Artin group with a visual splitting $A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ and assume that A_Ω does not contain a direct factor of A_{Γ_1}* (*which holds in particular if A_{Γ_1} is irreducible*). *Suppose that A_{Γ_1} is one of the following*:

- *an Artin group satisfying the hypothesis of Theorem 3.2* (*for instance*, *an even Artin group of FC type*),
- *an Artin group of spherical type*,
- *a two-dimensional Artin group.*

Then A_Γ is acylindrically hyperbolic.

Proof By Proposition 4.6, the hypotheses of the corollary imply that A_Ω is weakly malnormal in A_{Γ_1} and hence also in A_Γ, so the result follows from Theorem 3.1. □

The proof of Proposition 4.6 will occupy the remainder of this section.

Spherical-type Artin groups Although the intersection conjecture is known to hold for spherical-type Artin groups, we cannot apply Corollary 4.3 since the defining graph Γ is always a clique. Nevertheless, we can prove:

Lemma 4.8 *Artin groups of spherical type satisfy the weak malnormality conjecture.*

Proof By Lemma 4.1, it is enough to deal with the irreducible spherical case. Suppose that A_Γ is irreducible and of spherical type, and let $A_{\Gamma'}$ be a proper parabolic subgroup.

If A_Γ is not cyclic or of dihedral type, then by Theorem 3 of [1], A_Γ contains a subgroup isomorphic to $A_{\Gamma'} * \mathbb{Z}$, and where the free factor $A_{\Gamma'}$ is the proper parabolic subgroup under study. A standard argument from actions on trees shows that $A_{\Gamma'}$ is weakly malnormal in $A_{\Gamma'} * \mathbb{Z}$, hence it is weakly malnormal in A_Γ.

If A_Γ is cyclic, there is nothing to prove. Suppose that A_Γ is dihedral, with standard generators s, t, and let us show that $A_{\Gamma'} = \langle s\rangle$ is weakly malnormal. We know from Lemma 3.3 that there exists $g \in A_\Gamma$ such that $\langle s\rangle^g \neq \langle s\rangle$. Let us show by contradiction that $\langle s\rangle^g \cap \langle s\rangle = \{1\}$, which will prove weak malnormality. Let $x \in \langle s\rangle^g \cap \langle s\rangle$ be a nontrivial element, and let $n, m \geq 1$ be such that $x = s^n = gs^m g^{-1}$. By applying the homomorphism $A_\Gamma \to \mathbb{Z}$ sending both generators to 1, we see that $n = m$. Thus, g lies in the centralizer $C(s^n) = C(s)$, the latter equality following, for instance, from Lemma 7 of [14]. It follows that $\langle s\rangle^g = \langle s\rangle$, a contradiction. □

Even Artin groups of FC-type

Lemma 4.9 *Let A_Γ be an Artin group satisfying the hypotheses of Theorem 3.2. Then A_Γ satisfies the weak malnormality conjecture.*

Proof The edge group A_Ω is weakly malnormal in A_Γ by Theorem 3.2, so this is now a direct consequence of Proposition 4.2. □

In particular, we get the following:

Corollary 4.10 *Even FC-type Artin groups satisfy the weak malnormality conjecture.*

Proof By Lemma 4.1, it is enough to assume that A_Γ is irreducible and by Lemma 4.8 we may assume that it is not of spherical-type, that is, Γ is not a clique. The result now follows from Lemma 4.9. □

Note that the previous corollary is also a direct consequence of Corollary 4.3.

Two-dimensional Artin groups An Artin group A_Γ is two-dimensional if Γ has at least one edge (i.e., A_Γ is not a free group) and any three vertices in Γ generate an infinite-type parabolic subgroup. Recall that the intersection conjecture has not yet been proved for two-dimensional Artin groups, with currently the largest subclass for which it has been proved being the class of two-dimensional Artin groups whose presentation graph does not contain two adjacent edges with label 2 [4].

In this section we introduce another strategy for proving the weak malnormality conjecture using an action of A_Γ on the Deligne complex, which we can apply to two-dimensional Artin groups.

Proposition 4.11 *Let A_Γ be an Artin group such that*

- *D_Γ is* CAT(0) *with respect to either the cubical metric or the Moussong metric*,
- *there exists a vertex v of D_Γ with unbounded link and such that Stab(v) is weakly malnormal in A_Γ.*

Then A_Γ satisfies the weak malnormality conjecture.

Corollary 4.12 *Two-dimensional Artin groups satisfy the weak malnormality conjecture.*

Proof First suppose A_Γ contains an edge e labelled $k > 2$. Then it cannot be reducible, since e together with any vertex in the opposite direct factor would generate a spherical-type subgroup of rank 3. The Moussong metric on the Deligne complex D_Γ is CAT(0), by Charney–Davis [11]. By Lemma 5.7 of Vaskou [40], there exists vertices a, b in Γ connected by an edge labelled > 2 such that the subgroup $A_{a,b}$ is weakly malnormal in A_Γ. Viewing $A_{a,b}$ as a vertex in D_Γ, it has unbounded link by Proposition E of [40]. Thus, we can apply Proposition 4.11 to conclude that every proper parabolic subgroup in A_Γ is weakly malnormal.

If all edges of Γ are labelled 2 then A_Γ is a RAAG, hence even FC-type, so the result follows from Theorem 3.5. □

Proof of Proposition 4.11 For a proper parabolic subgroup $A_{\Gamma'}$, the Deligne complex $D_{\Gamma'}$ embeds equivariantly as a strict convex subcomplex of the CAT(0) space D_Γ that is stabilised by $A_{\Gamma'}$. (This is easily verified for the cubical metric. For the Moussong metric, see Lemma 5.1 of [9].) We want to construct a translate $gD_{\Gamma'}$ such that the following is satisfied:

- There is a unique geodesic realising the distance between $D_{\Gamma'}$ and $gD_{\Gamma'}$.
- The pointwise stabiliser of that geodesic is finite.

This will imply that $A_{\Gamma'} \cap gA_{\Gamma'}g^{-1}$ is trivial, hence $A_{\Gamma'}$ is weakly malnormal.

Since checking that $A_{\Gamma'}$ is weakly malnormal is equivalent to checking that any of its conjugates is weakly malnormal, we will consider instead a translate $kD_{\Gamma'}$ for some $k \in A_\Gamma$, such that the vertex v from the proposition's statement is not contained in $kD_{\Gamma'}$. We first observe that the projection of $kD_{\Gamma'}$ onto the link, $\mathrm{lk}(v)$, has diameter at most π. To see this, let x, y be two points in $kD_{\Gamma'}$ and let α_x, α_y be the geodesics connecting x to v and y to v. If the angle between α_x and α_y was $\geq \pi$, a standard argument of CAT(0) geometry would imply that the concatenation of α_x and α_y is a geodesic from x to y. Since $kD_{\Gamma'}$ is convex in D_Γ, this geodesic must lie entirely in $kD_{\Gamma'}$. This contradicts our assumption that $v \notin kD_{\Gamma'}$.

Since $\mathrm{Stab}(v)$ is weakly malnormal, there exists a translate w of v such that $\mathrm{Stab}(v) \cap \mathrm{Stab}(w)$ is finite, and hence the geodesic $\gamma = [v, w]$ connecting them has finite pointwise stabiliser. Since the link of v is unbounded and $\mathrm{Stab}(v)$ acts cocompactly on it, we can pick an element $h \in \mathrm{Stab}(v)$ such that the distance in $\mathrm{lk}(v)$ between the projection of $kD_{\Gamma'}$ and $h\gamma$ is at least π. The stabiliser of $h\gamma$ is conjugate to that of γ, hence it is also finite. Next, since the link of hw is also unbounded, we can pick an element $g \in \mathrm{Stab}(hw)$ such that the distance in $\mathrm{lk}(hw)$ between $h\gamma$ and $gh\gamma$ is at least π. And finally, since the distance in $\mathrm{lk}(v)$ between the projection of $kD_{\Gamma'}$ and $h\gamma$ is at least π the same holds for the distance in $\mathrm{lk}(gv)$ the between the projection of $gkD_{\Gamma'}$ and $gh\gamma$. (See Figure 1.)

It follows that for any points $x \in kD_{\Gamma'}$ and $y \in gkD_{\Gamma'}$, the concatenation of the geodesics $[x, v]$, $[v, hw]$, $[hw, gv]$, $[gv, y]$ is the (unique) geodesic from x to y. In particular, taking x to be the nearest point projection of v on $kD_{\Gamma'}$ and y to be the nearest point projection of gv on $gkD_{\Gamma'}$, we obtain a

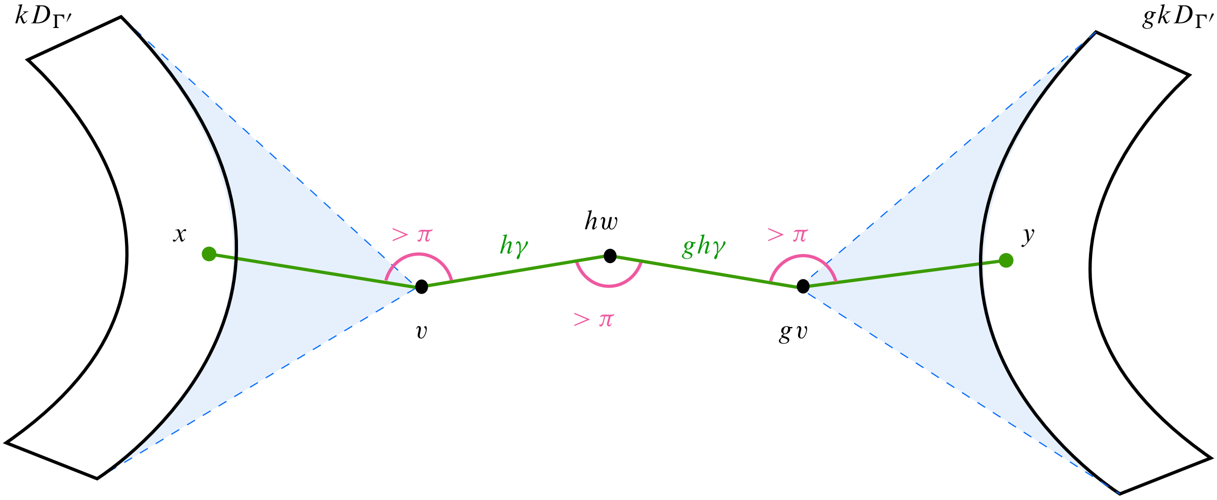

Figure 1: A geodesic (green) between $kD_{\Gamma'}$ and $gkD_{\Gamma'}$, obtained by concatenating several geodesic segments making an angle greater than π at their intersection point. The angles at v between any two points of $kD_{\Gamma'}$ are smaller than π ("visual cone" in light blue).

unique length-minimizing path between $kD_{\Gamma'}$ and $gkD_{\Gamma'}$. Since $kA_{\Gamma'}k^{-1} \cap gkA_{\Gamma'}(gk)^{-1}$ preserves both of these subcomplexes, it must fix this path. In particular, it lies in the pointwise stabilser of $h\gamma$. We conclude that this intersection is finite and hence $A_{\Gamma'}$ is weakly malnormal. □

Acknowledgements

Part of this work was conducted during the 2023 ICMS workshop "Polyhedral Products: a Path Between Homotopy Theory and Geometric Group Theory", as well as during the 2023 AIM workshop "Geometry and topology of Artin groups". The authors warmly thank the organisers, as well as the personnel of these institutes, for providing a supportive and mathematically rich environment. The authors also thank the anonymous referee for helpful remarks and suggestions.

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Received: May 27, 2024 Revised: March 10, 2025

Algebraic & Geometric Topology 26:4 (2026) 1529–1548
DOI: 10.2140/agt.2026.26.1529
Published: April 25, 2026

Equivariant preimage theory for G-maps

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Let X and Y be closed G-manifolds and $B \subset Y$ a closed invariant nonempty subset where G is a finite group. For any G-map $f : X \to Y$ and for every subgroup $H \le G$, we introduce a Nielsen type number $N(f^H, B^H)$ which is a lower bound for the number of connected components of WH-orbits of $(f^H)^{-1}(B^H)$. This theory generalizes existing Nielsen type numbers for various G and B with an application to the Nielsen Borsuk–Ulam theory for the minimal number of coincidences of $f(x) = f\tau(x)$ where $f : X \to Y$ and τ a free involution on X.

1 Introduction

It is well known that the Brouwer fixed point theorem is equivalent to the Borsuk–Ulam theorem. The former says that for any (continuous) self-map $f : X \to X$, the fixed point set $\operatorname{Fix} f = \{x \in X \mid f(x) = x\}$ is nonempty when $X = D^n$ is the closed n-disk. The latter is equivalent to the following: for any $\mathbb{Z}_2$-equivariant map $\varphi : S^n \to \mathbb{R}^n$, the preimage $\varphi^{-1}(\{0\})$ is nonempty, where the $\mathbb{Z}_2$-actions on S^n and on $\mathbb{R}^n$ are the usual antipodal actions. Lefschetz generalized Brouwer's result to coincidences of two maps $f, g : X \to Y$ between closed orientable manifolds of the same dimension. If we let $F : X \to Y \times Y$ be given by $F(x) = (f(x), g(x))$ and $\Delta = \{(y, y) \mid y \in Y\}$, then the coincidence set $C(f, g) = \{x \in X \mid f(x) = g(x)\} = F^{-1}(\Delta)$. Thus the coincidence theorem of Lefschetz asserts that $F^{-1}(\Delta) \neq \varnothing$ if the Lefschetz coincidence number $L(f, g)$ is nonzero. Subsequently, many authors study the general problem of determining whether $\Phi^{-1}(B) \neq \varnothing$ for a mapping $\Phi : W \to Z$ and $B \subset Z$ a closed subspace (e.g., Dobreńko [4], Frolkina [7], Gonçalves and Wong [14], Ha and Lee [15], and Liu and Zhao [16]). In other words, fixed point and coincidence point problems, as well as Borsuk–Ulam type theorems can be formulated as a preimage problem $\varphi^{-1}(B)$ for $\varphi : X \to Y \supset B$, where the latter is under the presence of a group action.

While the above-mentioned problems study whether the preimage set $\varphi^{-1}(B)$ is nonempty or not, the results do not give any information about the *size* of $\varphi^{-1}(B)$. Nielsen (fixed point or coincidence point) theory gives a geometric count of the number of connected components in $\varphi^{-1}(B)$. Such a theory was developed by Dobreńko and Kucharski [5], who introduced a Nielsen type number for the number of preimages of a map $f : X \to Y \supset B$. Under appropriate dimension conditions, a minimality theorem was established. From the (co)homological aspect, the algebraic size of $\varphi^{-1}(B)$ has been studied by Gonçalves and Wong [14] using local coefficients, and various cohomological index theories by Conner and Floyd [1], Fadell and Husseini [6], and Yang [23], among others. There is a vast literature on

MSC2020: primary 55M20; secondary 57S99.

Borsuk–Ulam and Bourgin–Yang type results and their applications to nonlinear analysis (see, e.g., Mawhin and Willem [18]).

The main objective of this paper is to give a geometric approach, a la Nielsen, similar to the nonequivariant setting of [5] and of [7], and to introduce a Nielsen type number that yields a lower bound for the number of connected components of $\varphi^{-1}(B)$ for a G-equivariant map $\varphi : X \to Y \supset B$ between G-spaces. Here B is a closed G-invariant subset and G is a finite group. We develop this theory using both the universal coverings and the Hopf coverings, generalizing the previous works on nonequivariant settings in [5; 7; 15; 16]. The algebraic approach here establishes an equivariant Reidemeister number which is an upper bound for the equivariant Nielsen preimage number for $\varphi^{-1}(B)$.

This equivariant preimage theory also generalizes existing works. If $X = Y$ and $B = \Delta_X$ is the diagonal with $f = 1 \times h$, $h : X \to X$, then we recover the equivariant Nielsen fixed point theory of [22]. Similarly if $B = \{a\}$ is a point in Y^G, we recover the equivariant Nielsen root theory of [20]. On the other hand, when G is trivial, our setting reduces to the nonequivariant Nielsen and Reidemeister settings of [5] and of [15; 16].

If $G = \mathbb{Z}_2 = \langle \tau \rangle$ is generated by a free involution τ on X, our equivariant Nielsen equivalence coincides with the Nielsen coincidence equivalence for Borsuk–Ulam coincidences studied by Cotrim, de Melo and Vendrúscolo [2; 3; 19]. In this setting of a free involution τ on X, our work provides a Reidemeister number for the Nielsen Borsuk–Ulam theory in [2; 3], and this should facilitate computation in future work in this direction. It is easy to see that one can generalize the Borsuk–Ulam coincidences to the study of the set of points $x \in X$ such that the orbit $\{x, \tau(x), \dots, \tau^{k-1}(x)\}$ is mapped to the same value under f, i.e., $f(x) = f(\tau(x)) = \cdots = f(\tau^{k-1}(x))$ for a free $\mathbb{Z}_k = \langle \tau \rangle$ action on X and a map $f : X \to Z$.

This paper is organized as follows. In Section 2, we introduce the concept of G-preimage classes utilizing the framework of universal covering. In Section 3, we define G-Nielsen preimage classes using the geometric essentiality for such classes. Furthermore, we provide an interpretation of the G-Nielsen preimage classes as the nonempty G-preimage classes established in the previous section. In Section 4, we present an algebraic approach to derive an upper bound for the number of essential G-Nielsen preimage classes via the universal cover. When using a Hopf cover, we also obtain in Section 5 a sharper upper bound. Finally, in Section 6, we conclude the paper by showcasing the practical application of our invariants in the context of the Nielsen Borsuk–Ulam theory of [2; 3; 19].

Throughout this paper, X and Y are connected, locally pathwise-connected and semilocally simply connected spaces. For any group G, a G-space X is assumed to have an effective G-action, i.e., if $g \cdot x = x$ for all $x \in X$ then $g = 1_G$.

2 G-preimage classes

Let X and Y be G-spaces where G is a finite group, $\varnothing \neq B \subset Y$ be a G-invariant closed subset of Y, and $f : X \to Y$ be a G-map. In this section, we use liftings to the universal cover to define the G-preimage classes.

In [7], a preimage problem is denoted by $f : X \to Y \supset B$. Our setting will be referred as a G-preimage problem and it will be denoted by $f : X \to_G Y \supset B$.

If $x_0 \in f^{-1}(B)$ then the orbit of x_0, $G \cdot x_0 = \{g \cdot x_0 \mid g \in G\}$, is also contained in $f^{-1}(B)$, that is, $f^{-1}(B)$ is a G-invariant subspace of X. In what follows, the set $f^{-1}(B)$ will be partitioned into the so-called *G-preimage classes*.

Let $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$ be the universal coverings of X and Y, respectively. It is well known from the nonequivariant preimage theory (see, e.g., [15] or [16]) that

$$f^{-1}(B) = \bigcup_{\tilde{f},\widetilde{B}} \eta_X(\tilde{f}^{-1}(\widetilde{B})),$$

where $\tilde{f}$ ranges over all liftings of f with respect to universal coverings $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$, and $\widetilde{B}$ ranges over all path components of $\eta_Y^{-1}(B)$. Each subset $\eta_X(\tilde{f}^{-1}(\widetilde{B}))$ of $f^{-1}(B)$ is referred to as a *preimage class*. Since we assume that f is an equivariant map and that B is a G-invariant subset, it follows that the group G acts on the set of preimage classes. More precisely, for each $g \in G$, the set

$$g \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$$

is itself a preimage class. To see this, consider a lifting $\widetilde{\psi}_g : \widetilde{Y} \to \widetilde{Y}$ of the homeomorphism $\psi_g : Y \to Y$ given by multiplication by g, i.e., $y \mapsto g \cdot y$. It is straightforward to verify that $\widetilde{\psi}_g(\widetilde{B})$ is also a path component of $\eta_Y^{-1}(B)$. Furthermore, one can check that

$$g \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B})) = \eta_X(\tilde{f}_0(\widetilde{B}_0)),$$

where

$$\tilde{f}_0 = \widetilde{\psi}_g \circ \tilde{f} \circ \tilde{\tau}_{g^{-1}},$$

with $\widetilde{\psi}_g : \widetilde{Y} \to \widetilde{Y}$ and $\tilde{\tau}_{g^{-1}} : \widetilde{X} \to \widetilde{X}$ being liftings of the maps $Y \to Y$, $y \mapsto g \cdot y$, and $X \to X$, $x \mapsto g^{-1} \cdot x$, respectively. Also, we define $\widetilde{B}_0 = \widetilde{\psi}_g(\widetilde{B})$. Since $\tilde{f}_0$ is a lifting of f, it follows that $g \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$ is indeed a preimage class of f.

As an immediate consequence of the above observation, we obtain the following result.

Lemma 2.1 *The preimage set*

$$f^{-1}(B) = \bigcup_{\tilde{f},\widetilde{B}} G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B})),$$

where $\tilde{f}$ ranges over all liftings of f with respect to universal coverings $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$, and $\widetilde{B}$ ranges over all path components of $\eta_Y^{-1}(B)$.

Following [16], a pair $(\tilde{f}, \widetilde{B})$ as in Lemma 2.1 is called a *lifting data pair* for preimage of f at B.

Lemma 2.2 *For any two lifting data pairs $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ of f at B, either*

$$G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) = G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) \quad \text{or} \quad G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = \varnothing.$$

Proof Suppose $x_0 \in G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$. Then, $x_0 = g_1 \cdot \eta_X(a) = g_2 \cdot \eta_X(b)$, for some $a \in \tilde{f}_1^{-1}(\widetilde{B}_1)$, $b \in \tilde{f}_2^{-1}(\widetilde{B}_2)$, and $g_1, g_2 \in G$. Therefore, $\eta_X(a) = g \cdot \eta_X(b)$, where $g = g_1^{-1} g_2$.

Let $\tau_g : X \to X$ be the homeomorphism given by $x \mapsto g \cdot x$ and let $\tilde{\tau}_g : \widetilde{X} \to \widetilde{X}$ be a lifting of τ_g. Then

$$\eta_X(\tilde{\tau}_g(b)) = g \cdot \eta_X(b) = \eta_X(a).$$

Hence, there exists $\alpha \in \mathrm{Cov}(\eta_X)$ such that $a = \alpha(\tilde{\tau}_g(b))$.

Let $\psi_{g^{-1}} : Y \to Y$ be the homeomorphism given by $y \mapsto g^{-1} \cdot y$, and let $\widetilde{\psi}_{g^{-1}} : \widetilde{Y} \to \widetilde{Y}$ be a lifting of $\psi_{g^{-1}}$. Then

$$\begin{aligned}
\eta_Y(\widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g(b)) &= \psi_{g^{-1}}\big(\eta_Y(\tilde{f}_1(\alpha(\tilde{\tau}_g(b))))\big) \\
&= g^{-1} \cdot f(\eta_X(\alpha(\tilde{\tau}_g(b)))) \\
&= g^{-1} \cdot f(\eta_X(\tilde{\tau}_g(b))) \\
&= g^{-1} \cdot f(g \cdot \eta_X(b)) \\
&= f(\eta_X(b)) \\
&= \eta_Y(\tilde{f}_2(b)).
\end{aligned}$$

Hence, there exists $\beta \in \mathrm{Cov}(\eta_Y)$ such that

$$\beta(\widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g(b)) = \tilde{f}_2(b). \tag{2-1}$$

Since $\beta \circ \widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g$ and $\tilde{f}_2$ are liftings of f that coincide at a point, they are the same, that is,

$$\beta \circ \widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g = \tilde{f}_2. \tag{2-2}$$

Thus, $\tilde{\delta} = \beta \circ \widetilde{\psi}_{g^{-1}}$ is a lifting of $\psi_{g^{-1}}$, $\tilde{\sigma} = \alpha \circ \tilde{\tau}_g$ is a lifting of τ_g, and

$$\tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma} = \tilde{f}_2. \tag{2-3}$$

Moreover, since $\tilde{f}_1(a) \in \widetilde{B}_1$ and

$$\tilde{\delta}(\tilde{f}_1(a)) = \beta \circ \widetilde{\psi}_{g^{-1}}(\tilde{f}_1(a)) = \beta \circ \widetilde{\psi}_{g^{-1}}(\tilde{f}_1(\alpha(\tilde{\tau}_g(b)))) = \tilde{f}_2(b) \in \widetilde{B}_2,$$

we have

$$\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2. \tag{2-4}$$

Thus,

$$\begin{aligned}
\eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) &= \eta_X((\tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma})^{-1}(\widetilde{B}_2)) \\
&= \eta_X(\tilde{\sigma}^{-1}(\tilde{f}_1^{-1}(\tilde{\delta}^{-1}(\widetilde{B}_2)))) \\
&= \eta_X(\tilde{\sigma}^{-1}(\tilde{f}_1^{-1}(\widetilde{B}_1))) \subset g \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)).
\end{aligned}$$

Therefore, $G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1))$.

Similarly, by using that $\tilde{f}_1 = \tilde{\delta}^{-1} \circ \tilde{f}_2 \circ \tilde{\sigma}^{-1}$, we conclude that

$$G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)).$$

Therefore, $G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1))$. □

Definition 2.3 Lemma 2.1 asserts that the preimage $f^{-1}(B)$ is a disjoint union of subsets of the form $G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. Each one of such subsets is called a *G-preimage class* of f at B.

Following the observation made before Lemma 2.1, each G-preimage class is a union of ordinary nonequivariant preimage classes.

The next result follows immediately from the proof of the Lemma 2.2.

Corollary 2.4 *Two lifting data pairs, $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$, define the same G-preimage class of f at B if and only if*

$$\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1},$$

where $\tilde{\delta} : \widetilde{Y} \to \widetilde{Y}$ is a lifting of the homeomorphism $\psi_g : Y \to Y$, $\psi_g(y) = g \cdot y$, and $\tilde{\sigma} : \widetilde{X} \to \widetilde{X}$ is a lifting of the homeomorphism $\tau_g : X \to X$, $\tau_g(x) = g \cdot x$, for some $g \in G$, and $\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$.

3 G-Nielsen preimage classes

We now introduce an equivariant analog of the Nielsen equivalence of [5] (see also [7]).

Definition 3.1 Two points $x_0, x_1 \in f^{-1}(B)$ are said to be G-Nielsen equivalent, denoted by $x_0 \sim_G x_1$, if

(i) $x_0 = g \cdot x_1$ for some $g \in G$ or

(ii) there exists a path γ in X from x_0 to $g \cdot x_1$ and a path β in B from $f(x_0)$ to $f(g \cdot x_1)$ such that $f \circ \gamma \sim \beta$ relative to the endpoints, for some $g \in G$.

The above relation splits $f^{-1}(B)$ into equivalence classes, the so-called *G-Nielsen preimage classes* of f at B.

In nonequivariant preimage theory, two points $x_0, x_1 \in f^{-1}(B)$ are said to be Nielsen related with respect to the subset B, $x_0 \sim x_1$, if there exists a path γ in X from x_0 to x_1 and a path β in B from $f(x_0)$ to $f(x_1)$ such that $f \circ \gamma \sim \beta$ relative to the end points (see [5, Definition 1.2]). Consequently, we have that $x_0, x_1 \in f^{-1}(B)$ are G-Nielsen equivalent if and only if x_0 and $g \cdot x_1$ are Nielsen equivalent (in the sense of standard preimage theory), for some $g \in G$. Analogously to [21, Theorem 2.1, page 32], one can show that two points $x_0, x_1 \in f^{-1}(B)$ are Nielsen equivalent with respect to the subset B if and only if there is a lifting data pair $(\tilde{f}, \widetilde{B})$ such that $x_0, x_1 \in \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. In other words, the set of Nielsen preimage classes coincides with the set of nonempty preimage classes.

Furthermore, the same relationship holds for G-Nielsen preimage classes and G-preimage classes: every G-Nielsen preimage class is a nonempty G-preimage class, as we now show.

Proposition 3.2 *Let $x_0, x_1 \in f^{-1}(B)$. Then $x_0 \sim_G x_1$ if and only if $x_0, x_1 \in G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$ for some lifting data pair $(\tilde{f}, \widetilde{B})$.*

Proof Let $x_0, x_1 \in f^{-1}(B)$ be related G-Nielsen preimage points with respect to the subset B. As we commented, it means that x_0 is Nielsen related to $g \cdot x_1$ (in the standard sense of preimage theory), for some $g \in G$. In turn, this is equivalent to the existence of a lifting data pair $(\tilde{f}, \widetilde{B})$ such that $x_0, g \cdot x_1 \in \eta_X(\tilde{f}^{-1}(\widetilde{B}))$, as we pointed out above. Therefore, $x_0, x_1 \in G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$.

On the other hand, let $(\tilde{f}, \widetilde{B})$ be a lifting data pair such that $x_0, x_1 \in G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. Suppose that $x_0 = g_0 \cdot z_0$ and $x_1 = g_1 \cdot z_1$, where $g_0, g_1 \in G$ and $z_0, z_1 \in \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. Since every nonempty preimage class is a Nielsen preimage class, there exists a path γ from z_0 to z_1 such that $f \circ \gamma$ is homotopic, relative to the endpoints, to a path β in B. Now, consider the path $g_0\gamma$, defined by $t \mapsto g_0 \cdot \gamma(t)$. This is a path from $x_0 = g_0 \cdot z_0$ to $g_0 \cdot z_1$ such that $f \circ (g_0\gamma) = g_0(f \circ \gamma)$ is homotopic to $g_0\beta$ relative to the endpoints. Since B is G-invariant, the path $g_0\beta$ also lies in B, implying that $x_0 \sim_G z_1$. Consequently, we obtain $x_0 \sim_G g_1 \cdot z_1 = x_1$, as desired. □

3.1 Topological essentiality of a G-Nielsen preimage class

Definition 3.3 Let $\{f_t : X \to Y\}$ be a G-homotopy of $f_0 = f$. A preimage point $x_0 \in f^{-1}(B)$ of f at B is $\{f_t\}_G$-related to a preimage point $x_1 \in f_1^{-1}(B)$ of f_1 at B, denoted by $x_0\{f_t\}_G\, x_1$, if x_0 is $\{f_t\}$-related to $g \cdot x_1$, for some $g \in G$. This means that there exist paths γ in X from x_0 to $g \cdot x_1$ and β in B from $f_0(x_0)$ to $f_1(g \cdot x_1)$ such that $\{f_t(\gamma(t))\} \sim \beta$ relative to the endpoints.

Similar to the nonequivariant case, the $\{f_t\}_G$ relation above induces a one-to-one correspondence between the G-preimage classes of $f = f_0$ and the G-preimage classes of f_1, as it is stated below. The proof is straightforward.

Lemma 3.4 *Let $\{f_t : X \to Y\}$ be a G-homotopy of f, and let $x_0 \in f^{-1}(B)$ and $x_1 \in f_1^{-1}(B)$ be such that $x_0\{f_t\}_G\, x_1$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the G-preimage classes of f and f_1, respectively, such that $x_i \in \mathcal{A}_i$, $i = 0, 1$. Then $x_0' \in \mathcal{A}_0$ if and only if $x_0'\{f_t\}_G x_1$, and $x_1' \in \mathcal{A}_1$ if and only if $x_0\{f_t\}_G x_1'$.*

In other words, the relation $x_0\{f_t\}_G x_1$ induces a correspondence from $\mathcal{A}_0$ to $\mathcal{A}_1$ under $\{f_t\}$, which is denoted by $\mathcal{A}_0\{f_t\}_G\mathcal{A}_1$.

Definition 3.5 A G-Nielsen preimage class of f at B is *essential* if given any G-homotopy $\{f_t : X \to Y\}$ from f it is $\{f_t\}_G$-related to a G-Nielsen preimage class of f_1 at B. Otherwise, it is called *inessential*. The *G-Nielsen preimage number* of f at B is defined as the number of essential G-preimage classes; it is denoted by $N_G(f; B)$. If X is a compact space, then $0 \le N_G(f; B) < \infty$.

This Nielsen number $N_G(f, B)$ has the usual properties that it is a G-homotopy invariant and is a lower bound for the number of connected components of the preimages. We have the following.

Proposition 3.6 *Given a G-preimage problem $f : X \to_G Y \supset B$,*

(1) *if $f : X \to Y$ is G-homotopic to h then $N_G(h, B) = N_G(f, B)$,*

(2) $N_G(f, B) \le \pi_0(f^{-1}(B))$.

4 Classes of lifting data pairs

Let $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$ be the universal covering of X and Y, respectively.

Since X is a G-space, each $g \in G$ can be associated to the homeomorphism of X, $\tau_g : X \to X$, given by $\tau_g(x) = g \cdot x$. The same for the G-space Y, where we will denote by $\psi_g : Y \to Y$ the homeomorphism given by $\psi_g(y) = g \cdot y$.

We will consider the following groups:

$$\pi_X = \{\tilde{\alpha} \in \mathrm{Homeo}(\widetilde{X}) \mid \eta_X \circ \tilde{\alpha} = \eta_X\} = \mathrm{Cov}(\eta_X),$$
$$\hat{\pi}_X = \{\tilde{\tau}_g \in \mathrm{Homeo}(\widetilde{X}) \mid \eta_X \circ \tilde{\tau}_g = \tau_g \circ \eta_X \text{ for some } g \in G\},$$

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{\tau}_g"] \arrow[d, "\eta_X"'] & \widetilde{X} \arrow[d, "\eta_X"] \\ X \arrow[r, "\tau_g"] & X \end{tikzcd}$$

Analogously:

$$\pi_Y = \{\tilde{\gamma} \in \mathrm{Homeo}(\widetilde{Y}) \mid \eta_Y \circ \tilde{\gamma} = \eta_Y\} = \mathrm{Cov}(\eta_Y),$$
$$\hat{\pi}_Y = \{\widetilde{\psi}_g \in \mathrm{Homeo}(\widetilde{Y}) \mid \eta_Y \circ \widetilde{\psi}_g = \psi_g \circ \eta_Y \text{ for some } g \in G\},$$

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\widetilde{\psi}_g"] \arrow[d, "\eta_Y"'] & \widetilde{Y} \arrow[d, "\eta_Y"] \\ Y \arrow[r, "\psi_g"] & Y \end{tikzcd}$$

Note that $\hat{\pi}_X$ and $\hat{\pi}_Y$ are extensions of π_X and π_Y, respectively. The elements in π_X and π_Y are the ones in $\hat{\pi}_X$ and $\hat{\pi}_Y$, respectively, that cover the identity $\mathrm{Id} = \tau_e$, where $e \in G$ is the identity.

Remark 4.1 In general, the short exact sequence $1 \to \pi_X \to \hat{\pi}_X \to G \to 1$ does not split so G does not act on π_X unless π_X is abelian. Moreover, every $g \in G$ induces a homeomorphism $\theta_g : X \to X$, which in turn induces an isomorphism $\pi_1(X, x_0) \to \pi_1(X, gx_0)$ but gx_0 need not be the same as x_0.

Let

$$\Gamma = \{(\tilde{\delta}, \tilde{\sigma}) \in \hat{\pi}_Y \times \hat{\pi}_X \mid \eta_Y \circ \tilde{\delta} = \psi_g \circ \eta_Y \text{ and } \eta_X \circ \tilde{\sigma} = \tau_g \circ \eta_X \text{ for some } g \in G\}.$$

Definition 4.2 Two lifting data pairs $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ are said to be equivalent if there is $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$ such that $\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1}$ and $\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$.

Lemma 4.3 *Let $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ be two lifting data pairs.*

(1) *If the two pairs are equivalent, then $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) = G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$.*

(2) *If the two pairs are not equivalent, then $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = \varnothing$.*

Proof (1) Since $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ are equivalent,

$$\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1} \quad \text{and} \quad \tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$$

for some $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$, which means that $\eta_Y \circ \tilde{\delta} = \psi_g \circ \eta_Y$ and $\eta_X \circ \tilde{\sigma} = \tau_g \circ \eta_X$ for some $g \in G$. Thus,

$$\tilde{f}_2^{-1}(\widetilde{B}_2) = (\tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1})^{-1}(\tilde{\delta}(\widetilde{B}_1)) = \tilde{\sigma}(\tilde{f}_1^{-1}(\widetilde{B}_1)).$$

Therefore,

$$\eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = \eta_X(\tilde{\sigma}(\tilde{f}_1^{-1}(\widetilde{B}_1))) = \tau_g(\eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1))) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)).$$

By Lemma 2.2, $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) = G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$.

(2) Suppose on the contrary that $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$ contains a point x_0. Then, by following the proof of Lemma 2.2, there is $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$ such that $\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$ and $\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1}$. □

Next, we define the so-called G-Reidemeister preimage number $R_G(f, B)$ of f at B. Such number is an upper bound for the number $N_G(f; B)$ — the G-Nielsen preimage number of f at B — defined previously. There are two possible approaches: either by using universal covering or by using Hopf covering. First, we use the universal covering to define $R_G(f, B)$.

4.1 G-Reidemeister preimage number via universal covering

Once and for all, let us fix a lifting data pair $(\tilde{f}, \widetilde{B})$ of f at B. Note that given an arbitrary data pair $(\tilde{f}_1, \widetilde{B}_1)$,

$$\tilde{f}_1^{-1}(\widetilde{B}_1) = (\alpha \circ \tilde{f})^{-1}(\widetilde{B})$$

for some $\alpha \in \operatorname{Cov}(\eta_Y)$. Thus,

$$f^{-1}(B) = \bigcup_{\alpha \in \operatorname{Cov}(\eta_Y)} G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B}))$$

and, from what was shown before, given $\alpha, \beta \in \operatorname{Cov}(\eta_Y)$,

$$G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B})) = G \cdot \eta_X((\beta \circ \tilde{f})^{-1}(\widetilde{B}))$$

if and only if there is $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$ such that $\tilde{\delta}(\widetilde{B}) = \widetilde{B}$ and $\beta \circ \tilde{f} = \tilde{\delta} \circ (\alpha \circ \tilde{f}) \circ \tilde{\sigma}^{-1}$.

If $\tilde{\alpha} \in \pi_X$ then $\tilde{f} \circ \tilde{\alpha}$ is a lifting of f. Therefore, there exists a unique element $\tilde{f}_\pi(\tilde{\alpha}) \in \pi_Y$ such that

$$\tilde{f} \circ \tilde{\alpha} = \tilde{f}_\pi(\tilde{\alpha}) \circ \tilde{f} \tag{4-1}$$

and, consequently, $\tilde{f}_\pi : \pi_X \to \pi_Y$ is a group homomorphism.

Similarly, given $\tilde{\alpha} \in \hat{\pi}_X$, there exists a unique element $\Phi(\tilde{\alpha}) \in \hat{\pi}_Y$ such that

$$\tilde{f} \circ \tilde{\alpha} = \Phi(\tilde{\alpha}) \circ \tilde{f}. \tag{4-2}$$

Therefore, $\Phi : \hat{\pi}_X \to \hat{\pi}_Y$ is a group homomorphism.

Let $\Gamma_{\widetilde{B}}$ be the subgroup of Γ given by

$$\Gamma_{\widetilde{B}} = \{(\tilde{\delta}, \tilde{\sigma}) \in \Gamma \mid \tilde{\delta}(\widetilde{B}) = \widetilde{B}\}.$$

Then $\Gamma_{\widetilde{B}}$ acts on π_Y via: given $\alpha \in \pi_Y$ and $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}$,

$$(\tilde{\delta}, \tilde{\sigma}) \cdot \alpha = \tilde{\delta}\, \alpha\, \Phi(\tilde{\sigma})^{-1} \in \pi_Y.$$

Such action splits π_Y into disjoint orbit sets: given $\alpha \in \pi_Y$, the orbit of α is the set

$$\{\tilde{\delta}\, \alpha\, \Phi(\tilde{\sigma})^{-1} \mid (\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}\}.$$

The orbit set, $\pi_Y / \Gamma_{\widetilde{B}}$, will be denoted by $\mathcal{R}_G[f, B]$, its cardinality will be denoted by $R_G(f, B)$, and we call $R_G(f, B)$ the *G-Reidemeister preimage number* of f at B.

Theorem 4.4 *Let $\alpha, \beta \in \mathrm{Cov}(\eta_Y)$. Then*

$$G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B})) = G \cdot \eta_X((\beta \circ \tilde{f})^{-1}(\widetilde{B}))$$

if and only if

$$\beta = (\tilde{\delta}, \tilde{\sigma}) \cdot \alpha = \tilde{\delta}\, \alpha\, \Phi(\tilde{\sigma})^{-1} \quad \textit{for some } (\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}.$$

Proof Let $\alpha, \beta \in \mathrm{Cov}(\eta_Y)$. From Lemma 4.3,

$$G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B})) = G \cdot \eta_X((\beta \circ \tilde{f})^{-1}(\widetilde{B}))$$

if and only if $(\alpha \circ \tilde{f}, \widetilde{B})$ and $(\beta \circ \tilde{f}, \widetilde{B})$ are equivalent lifting data pairs, which means that

$$\beta \circ \tilde{f} = \tilde{\delta} \circ (\alpha \circ \tilde{f}) \circ \tilde{\sigma}^{-1},$$

for some $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}$. By the definition of the group homomorphism $\Phi : \hat{\pi}_X \to \hat{\pi}_Y$,

$$\tilde{f} \circ \tilde{\sigma}^{-1} = \Phi(\tilde{\sigma})^{-1} \circ \tilde{f}.$$

Therefore,

$$\beta \circ \tilde{f} = \tilde{\delta} \circ \alpha \circ \Phi(\tilde{\sigma})^{-1} \circ \tilde{f},$$

and so

$$\beta = \tilde{\delta} \circ \alpha \circ \Phi(\tilde{\sigma})^{-1}. \qquad \square$$

Corollary 4.5 *The number of G-preimage classes of f at B is the G-Reidemeister preimage number of f at B. Therefore, $N_G(f, B) \le R_G(f, B)$.*

Remark 4.6 Theorem 4.4 reduces to the classical (nonequivariant) Reidemeister action in [15] or [16]. The G-action induces an action on $\pi_Y \equiv \pi_1(Y)$ by the group $\Gamma_{\widetilde{B}}$ and thus a G-(Reidemeister or Nielsen) preimage class is a finite union of nonequivariant (Reidemeister or Nielsen) preimage classes.

5 G-Reidemeister preimage number via Hopf covering

In [20], a Nielsen root theory for G-maps via an equivariant analog of the approach of Brooks using Hopf lifts was developed. In this section, we will develop the analogous construction to the case of an equivariant preimage problem, generalizing that of [7] in the nonequivariant case.

The map $f : X \to Y$ induces a homomorphism $f_\sharp : \pi_1(X) \to \pi_1(Y)$ on fundamental groups, and there exists a covering $\eta : \widehat{Y} \to Y$ such that $\eta_\sharp \pi_1(\widehat{Y}) = f_\sharp \pi_1(X)$, so we can lift f through η to $\hat{f} : X \to \widehat{Y}$, that is, $f = \eta \circ \hat{f}$. The map $\hat{f}$ is called a Hopf lifting of f, and η a Hopf covering for f.

Frolkina [7] proved the following (nonequivariant setting):

Theorem 5.1 [7, Theorem 2] *Let $(\widehat{Y}, p)$ and $\hat{f}$ be a Hopf covering and a Hopf lift for $f : X \to Y \supset B$. Let $\{f_t\} : X \to Y$ be a homotopy from $f_0 = f$ to f_1 and $\{\hat{f}_t\} : X \to \widehat{Y}$ its lift such that $\hat{f}_0 = \hat{f}$. Then:*

1. *Two preimage points $x_0, x_1 \in f^{-1}(B)$ are Nielsen equivalent if and only if the points $\hat{f}(x_0)$ and $\hat{f}(x_1)$ lie in the same path component of the set $p^{-1}(B)$.*
2. *Nielsen classes of $f : X \to Y \supset B$ are precisely nonempty sets of the form $\hat{f}^{-1}(C)$, where C is a path component of the set $p^{-1}(B)$.*
3. *A point $x_0 \in f_0^{-1}(B)$ is $\{f_t\}$-related to a point $x_1 \in f_1^{-1}(B)$ if and only if the points $\hat{f}_0(x_0)$, $\hat{f}_1(x_1)$ are contained in the same path component of the set $p^{-1}(B)$.*
4. *A preimage class $A_0 \subset f_0^{-1}(B)$ is $\{f_t\}$-related to a class $A_1 \subset f_1^{-1}(B)$ if and only if the sets $\hat{f}_0(A_0)$ and $\hat{f}_1(A_1)$ are contained in one path component of the set $p^{-1}(B)$.*
5. *A preimage class $A_0 \subset f_0^{-1}(B)$ is $\{f_t\}$-related to a class $A_1 \subset f_1^{-1}(B)$ if and only if A_0 and A_1 are 0- and 1-sections of some preimage class of $F : X \times I \to Y \supset B$, where $F(x,t) = f_t(x)$.*

In [20], it was shown that in the setting of $f : X \to Y$ being a G-map, $(\widehat{Y}, \eta)$ a Hopf covering of f and $\hat{f}$ a Hopf lifting of f, there is an action of G on $\widehat{Y}$ under which $\hat{f} : X \to \widehat{Y}$ and $\eta : \widehat{Y} \to Y$ are G-maps, among other properties, as we recall below.

Denote by $\mathfrak{D}(\eta) = \{\delta \in \mathrm{Homeo}(\widehat{Y}) \mid \eta\delta = \eta\}$ the group of deck transformations of η. Let

$$\Gamma_G(\widehat{Y}) = \{\hat{g} \in \mathrm{Homeo}(\widehat{Y}) \mid \eta\hat{g} = g\eta \text{ for some } g \in G\},$$

where g can be regarded as a homeomorphism of Y (previously, we denoted such homeomorphism by ψ_g). Now, there is a short exact sequence

$$1 \to \mathfrak{D}(\eta) \xhookrightarrow{i} \Gamma_G(\widehat{Y}) \xrightarrow{p} G \to 1$$

where i is the inclusion and $p(\hat{g}) = g$ is the projection (the projection is well defined because the action of G on Y is supposed to be effective).

The map f induces a group homomorphism $\varphi : G \to \Gamma_G(\widehat{Y})$ as follows. Pick a point $x_0 \in X$, and let $\hat{f}(x_0) \in \widehat{Y}$. There is a unique lift $\varphi(g)$ of g such that

$$\varphi(g)\hat{f}(x_0) = \hat{f}(gx_0).$$

Now, $\hat{f}g, \varphi(g)\hat{f} : X \to \widehat{Y}$ are both liftings of the same map $fg = gf : X \to Y$, and they agree at x_0. Therefore, $\hat{f}g = \varphi(g)\hat{f}$. The map φ defined under such construction depends on f, η and $\hat{f}$.

Lemma 5.2 [20, Lemma 3.6] *The map $\varphi : G \to \Gamma_G(\widehat{Y})$ is a group homomorphism and is a section to p, i.e., $p \circ \varphi = 1_G$. In particular, $\Gamma_G(\widehat{Y}) = \mathfrak{D}(\eta) \rtimes G$.*

Remark 5.3 When η is a regular cover, we have $\mathfrak{D}(\eta) = \pi_1(Y)/f_\sharp(\pi_1(X)$.

Lemma 5.4 [20, Lemma 3.7] *The maps $\hat{f} : X \to \widehat{Y}$ and $\eta : \widehat{Y} \to Y$ are equivariant maps.*

Theorem 5.5 [20, Theorem 3.8] *If $f' : X \to Y$ is G-homotopic to f, then they induce the same action on the Hopf covering space $\widehat{Y}$ of Y.*

Consider the restriction of the G-action on $\widehat{Y}$ given by φ on the set $\eta^{-1}(B)$. Since η is equivariant and B is G-invariant, $\eta^{-1}(B)$ becomes a G-set.

We now prove an equivariant analog of Theorem 5.1.

Theorem 5.6 *Let $(\widehat{Y}, \eta)$ and $\hat{f}$ be a Hopf covering and a Hopf lift for $f : X \to_G Y \supset B$. Let $\{f_t\} : X \to Y$ be a G-homotopy from $f_0 = f$ to f_1 and $\{\hat{f}_t\} : X \to \widehat{Y}$ its lift such that $\hat{f}_0 = \hat{f}$. Then:*

(1) *Two preimage points $x_0, x_1 \in f^{-1}(B)$ are G-Nielsen equivalent if and only if the points $\hat{f}(x_0)$, $g \cdot \hat{f}(x_1)$ lie in the same path component of the set $\eta^{-1}(B)$, for some $g \in G$.*

(2) *The G-Nielsen preimage classes of $f : X \to_G Y \supset B$ are precisely the nonempty sets of the form $G \cdot \hat{f}^{-1}(C)$, where C is a path component of the set $\eta^{-1}(B)$; and a class $G \cdot \hat{f}^{-1}(C)$ is essential if and only if $\hat{f}_1^{-1}(C) \neq \varnothing$ for any G-homotopy $\{\hat{f}_t\}$ beginning at $\hat{f}_0 = \hat{f}$.*

(3) *A point $x_0 \in f_0^{-1}(B)$ is $\{f_t\}_G$-related to a point $x_1 \in f_1^{-1}(B)$ if and only if $\hat{f}_0(x_0)$ and $g \cdot \hat{f}_1(x_1)$ are contained in the same path component of the set $\eta^{-1}(B)$, for some $g \in G$.*

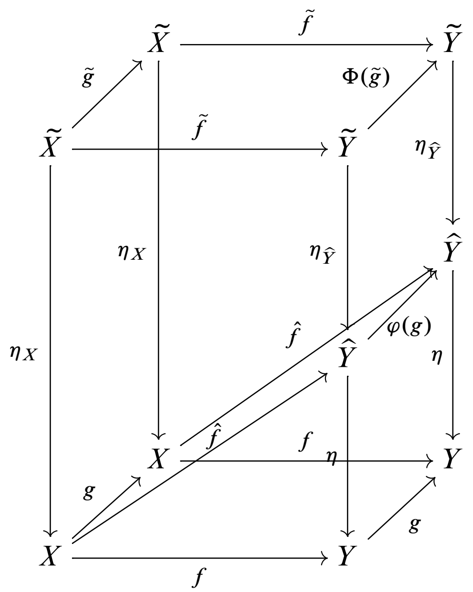
(4) *A G-preimage class $\mathcal{A}_0 \subset f_0^{-1}(B)$ is $\{f_t\}_G$-related to a class $\mathcal{A}_1 \subset f_1^{-1}(B)$ if and only if the sets $\hat{f}_0(\mathcal{A}_0)$ and $g \cdot \hat{f}_1(\mathcal{A}_1)$ are contained in one path component of the set $\eta^{-1}(B)$.*

(5) *A G-preimage class $\mathcal{A}_0 \subset f_0^{-1}(B)$ is $\{f_t\}_G$-related to a G-preimage class $\mathcal{A}_1 \subset f_1^{-1}(B)$ if and only if $\mathcal{A}_0$ and $\mathcal{A}_1$ are the 0- and 1-sections of some G-preimage class of $F : X \times I \to_G Y \supset B$, where $F(x,t) = f_t(x)$, and the action of G on $X \times I$ is given by $g \cdot (x,t) = (g \cdot x, t)$.*

Proof Similar to [7, Theorem 2], one can note that (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). So, it is sufficient to prove (3).

By definition, a point $x_0 \in f_0^{-1}(B)$ is $\{f_t\}_G$-related to a point $x_1 \in f_1^{-1}(B)$ if and only if x_0 is $\{f_t\}$-related to $g \cdot x_1$, for some $g \in G$, which is, by Theorem 5.1, equivalent to $\hat{f}(x_0)$ and $\hat{f}(g \cdot x_1)$ lying in the same path component of $\eta^{-1}(B)$. Since $\hat{f}(g \cdot x_1) = g \cdot \hat{f}(x_1)$, the result follows. □

Consider the commutative diagram



where $g : X \to X$ denotes the homeomorphism $x \mapsto g \cdot x$, $g : Y \to Y$ denotes the homeomorphism $y \mapsto g \cdot y$, and $\hat{f} : X \to \widehat{Y}$ is a Hopf lifting of f and $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ is a lifting of $\hat{f}$ with respect to the universal coverings $\eta_X : \widetilde{X} \to X$ and $\eta_Y = \eta \circ \eta_{\widehat{Y}} : \widetilde{Y} \to Y$, where $\eta_{\widehat{Y}} : \widetilde{Y} \to \widehat{Y}$ is a universal covering of $\widehat{Y}$. Let $\widehat{B}$ be a path component of $\eta^{-1}(B)$ and $\widetilde{B}$ a path component of $\eta_{\widehat{Y}}^{-1}(\widehat{B})$, so $\widetilde{B}$ is a path component of $\eta_Y^{-1}(B)$. Then:

(1) The G-preimage classes are of the form $\{G \cdot (\hat{\alpha} \hat{f})^{-1}(\widehat{B})\}$ where $\widehat{B}$ is a path component of $\eta^{-1}(B)$ and $\hat{\alpha} \in \mathscr{D}(\eta)$. Indeed, from Theorem 5.6(2), the G-Nielsen preimage classes of $f : X \to_G Y \supset B$ are precisely the nonempty sets of the form $G \cdot \hat{f}^{-1}(C)$, where C is a path component of $\eta^{-1}(B)$; fix base points $b_0 \in B$, $\hat{b}_0 \in \widehat{B}$ and $c_0 \in C$ such that $\hat{b}_0, c_0 \in \eta^{-1}(b_0)$. Let $\hat{\alpha} \in \mathscr{D}(\eta)$ be such that $\hat{\alpha}(\hat{b}_0) = c_0$. Thus, $\hat{\alpha}(\widehat{B}) = C$. Therefore, $(\hat{\alpha}\, \hat{f})^{-1}(\widehat{B}) = \hat{f}^{-1}(\hat{\alpha}^{-1}(\widehat{B})) = \hat{f}^{-1}(C)$. For a general G-preimage class (eventually an empty one), we have the following.

(2) Suppose $\tilde{\alpha} \in \pi_Y$ covers $\hat{\alpha}$ and $\widetilde{B}$ covers $\widehat{B}$. Then

$$G \cdot (\hat{\alpha} \hat{f})^{-1}(\widehat{B}) = G \cdot \eta_X (\tilde{\alpha} \tilde{f})^{-1}(\widetilde{B}).$$

In fact, let $x = \eta_X(\tilde{x})$, with $\tilde{x} \in (\tilde{\alpha} \tilde{f})^{-1}(\widetilde{B})$ be arbitrary. Then

$$\hat{\alpha}\, \hat{f}(x) = \hat{\alpha}\, \hat{f}(\eta_X(\tilde{x})) = \hat{\alpha}\, \eta_{\widehat{Y}}\, \tilde{f}(\tilde{x}) = \eta_{\widehat{Y}}\, \tilde{\alpha}\, \tilde{f}(\tilde{x}) \in \widehat{B} \quad \text{since } \tilde{\alpha} \tilde{f}(\tilde{x}) \in \widetilde{B}.$$

Therefore,

$$G \cdot \eta_X (\tilde{\alpha} \tilde{f})^{-1}(\widetilde{B}) \subset G \cdot (\hat{\alpha} \hat{f})^{-1}(\widehat{B}).$$

On the other hand, let $x \in (\hat{\alpha}\, \hat{f})^{-1}(\widehat{B})$ be arbitrary. And let $\tilde{\alpha} \in \pi_Y$ be an arbitrary element such that $\eta_{\widehat{Y}}\, \tilde{\alpha} = \hat{\alpha}\, \eta_{\widehat{Y}}$, i.e., $\tilde{\alpha}$ covers $\hat{\alpha}$. Let $\tilde{x} \in \widetilde{X}$ be such that $\eta_X(\tilde{x}) = x$. Therefore,

$$\eta_{\widehat{Y}}(\tilde{\alpha} \tilde{f}(\tilde{x})) = \hat{\alpha} \eta_{\widehat{Y}} \tilde{f}(\tilde{x}) = \hat{\alpha} \hat{f} \eta_X(\tilde{x}) = \hat{\alpha} \hat{f}(x) \in \widehat{B},$$

so $\tilde{\alpha}\tilde{f}(\tilde{x}) \in \eta_{\widehat{Y}}^{-1}(\widehat{B})$. Let C be the path component of $\eta_{\widehat{Y}}^{-1}(\widehat{B})$ such that $\tilde{\alpha}\tilde{f}(\tilde{x}) \in C$. Then $x \in \eta_X((\tilde{\alpha}\tilde{f})^{-1}(C))$. Let $\tilde{b} \in \widetilde{B}$ be such that $\eta_{\widehat{Y}}(\tilde{b}) = \hat{\alpha}\hat{f}(x)$ and let $\beta \in \mathscr{D}(\eta_{\widehat{Y}})$ be the unique element such that $\beta(\tilde{\alpha}\tilde{f}(\tilde{x})) = \tilde{b}$. Then $\beta\tilde{\alpha} \in \pi_Y$, $\beta\tilde{\alpha}$ covers $\hat{\alpha}$, and $x \in G \cdot \eta_X((\beta\tilde{\alpha})\tilde{f})^{-1}(\widetilde{B})$.

Therefore,

$$G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B}) = \bigcup_{\tilde{\alpha} \text{ covers } \hat{\alpha},\, \tilde{\alpha}(\widetilde{B}) = \widehat{B}} G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}).$$

With the above equality established, we conclude the following:

(a) If $G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B}) = \varnothing$ then $G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}) = \varnothing$, for any $\tilde{\alpha} \in \pi_Y$ that covers $\hat{\alpha}$ and $\tilde{\alpha}(\widetilde{B}) = \widehat{B}$.

(b) If $G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$ then

$$G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}) = G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B})$$

because, in this case, both $G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B})$ and $G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B})$ are G-Nielsen preimage classes and $G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}) \subset G \cdot (\hat{\alpha}\hat{f})^{-1}(\widehat{B})$.

(c) If $G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$ for some $\tilde{\alpha} \in \pi_Y$ that covers $\hat{\alpha}$, then $G \cdot \eta_X(\tilde{\beta}\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$ for any $\tilde{\beta} \in \pi_Y$ that covers $\hat{\alpha}$. Indeed, suppose $x = \eta_X(\tilde{x})$, with $\tilde{x} \in (\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B})$. Let $\tilde{\beta} \in \pi_Y$ be any element that covers $\hat{\alpha}$. Then $\eta_{\widehat{Y}}(\tilde{\beta}\tilde{f}(\tilde{x})) = \eta_{\widehat{Y}}(\tilde{\alpha}\tilde{f}(\tilde{x})) \in \widehat{B}$, i.e., $\tilde{\beta}\tilde{f}(\tilde{x}) \in \eta_{\widehat{Y}}^{-1}(\widehat{B})$, but not necessarily $\tilde{\beta}\tilde{f}(\tilde{x}) \in \widetilde{B}$. Anyway, let $b_0 = \tilde{\beta}\tilde{f}(\tilde{x})$ and $b_1 = \tilde{\alpha}\tilde{f}(\tilde{x}) \in \widetilde{B}$. Let $\gamma : I \to \widetilde{Y}$ be a path with $\gamma(0) = b_0$ and $\gamma(1) = b_1$. Then $\eta_{\widehat{Y}} \circ \gamma$ is a loop in $\widehat{Y}$ with base point $\hat{b} = \hat{\alpha}\hat{f}(x)$. Therefore, $\eta(\eta_{\widehat{Y}}\gamma)$ is a loop in Y with base point $\eta(\hat{b}) = f(x)$. Since $\eta_\sharp(\pi_1(Y)) = f_\sharp(\pi_1(X))$, there is a loop $\rho : I \to X$ with base point x such that $f\rho \sim \eta(\eta_{\widehat{Y}}\gamma)$ rel $\{0, 1\}$. Let $\tilde{\rho} : I \to \widetilde{X}$ be a lifting of ρ such that $\tilde{\rho}(0) = \tilde{x}$. Now,

$$\eta\eta_{\widehat{Y}}(\tilde{\beta}\tilde{f}\tilde{\rho}) = \eta\eta_{\widehat{Y}}\tilde{f}\tilde{\rho} = f\eta_X\tilde{\rho} = f\rho \sim \eta\eta_{\widehat{Y}}\gamma.$$

Since $\tilde{\beta}\tilde{f}\tilde{\rho}(0) = \tilde{\beta}\tilde{f}(\tilde{x}) = \gamma(0)$, it follows from [17, Lemma 3.3, page 152] that

$$\tilde{\beta}\tilde{f}\tilde{\rho}(1) = \gamma(1) = b_1 \in \widetilde{B}.$$

Therefore, $(\tilde{\beta}\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$.

(d) From the above items (a)–(c), it follows that if $\tilde{\alpha} \in \pi_Y$ covers $\hat{\alpha}$ and $\widetilde{B}$ covers $\widehat{B}$ then

$$G \cdot (\hat{\alpha}\hat{f})^{-1}(\widehat{B}) = G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}).$$

(3) Moreover, $G \cdot (\hat{\alpha}\hat{f})^{-1}(\widehat{B}) = G \cdot (\hat{\beta}\hat{f})^{-1}(\widehat{B})$ if and only if

$$\hat{\beta} = \hat{\delta}\hat{\alpha}\varphi(g)^{-1} \tag{5-1}$$

for some $\hat{\delta} \in \Gamma_G(\widehat{Y})$, $\hat{\delta}(\widehat{B}) = \widehat{B}$ and $g \in G$ such that $\hat{\delta}$ covers g.

Let $\Gamma_h = \{\hat{\delta} = (\hat{\delta}', g) \in \Gamma_G(\hat{y}) \mid \hat{\delta} \text{ covers } g \text{ and } \hat{\delta}(\widehat{B}) = \widehat{B}\}$.

Definition 5.7 Equation (5-1) defines a (Hopf–Reidemeister) action of Γ_h on $\mathfrak{D}(\eta)$. We define the equivariant G-Hopf–Reidemeister number $R_{G,h}(f, B)$ to be the cardinality of the set of orbits of the action $\hat{\alpha} \mapsto \hat{\delta}\hat{\alpha}\varphi(g)^{-1}$ given by (5-1). Furthermore, by Theorem 5.6, $N_G(f, B) \le R_{G,h}(f, B)$.

From the previous section, we have defined $R_G(f, B)$. Next, we will relate these two Reidemeister numbers by showing that $R_{G,h}(f, B) \le R_G(f, B)$.

Note that for any homeomorphism $\zeta : \widetilde{Y} \to \widetilde{Y}$ that belongs to $\hat{\pi}_Y$, i.e.,

$$(\eta \circ \eta_{\widehat{Y}}) \circ \zeta = g \cdot (\eta \circ \eta_{\widehat{Y}}) \quad \text{for some element } g \in G,$$

there is a unique homeomorphism $\hat{\zeta} : \widehat{Y} \to \widehat{Y}$ that ζ covers it (also, $\hat{\zeta}$ belongs to $\Gamma_G(\widehat{Y})$). In fact, let $\tilde{y}_0 \in \widetilde{Y}$ be a base point and let $\hat{y}_0 = \eta_{\widehat{Y}}(\tilde{y}_0)$. Then

$$\eta\, \eta_{\widehat{Y}}\, \zeta(\tilde{y}_0) = g \cdot \eta\, \eta_{\widehat{Y}}(\tilde{y}_0) = g \cdot \eta(\hat{y}_0) = \eta(g \cdot \hat{y}_0).$$

Therefore, there is a unique element $\delta \in \mathfrak{D}(\eta)$ such that

$$\delta\,(\eta_{\widehat{Y}}\, \zeta(\tilde{y}_0)) = g \cdot \hat{y}_0.$$

Let $\hat{\zeta} : \widehat{Y} \to \widehat{Y}$ be given by $\hat{\zeta}(y) = \delta^{-1}(g \cdot y)$. Then:

(a) $\hat{\zeta}$ belongs to $\Gamma_G(\widehat{Y})$:

$$\eta\, \hat{\zeta}(y) = \eta\, \delta^{-1}(g \cdot y) = \eta(g \cdot y) = g \cdot \eta(y).$$

(b) ζ covers $\hat{\zeta}$: Let $\beta : \widetilde{Y} \to \widetilde{Y}$ be the unique lifting of $\hat{\zeta}$ such that

$$\beta(\tilde{y}_0) = \zeta(\tilde{y}_0).$$

It easy to see that β belongs to $\hat{\pi}_Y$ and, consequently, $\beta = \zeta$.

Now, we let $\Lambda([\alpha]) = \langle \hat{\alpha} \rangle$ where $[\,\cdot\,]$ denotes the classes using the universal cover and $\langle\,\cdot\,\rangle$ denotes the classes using Hopf coverings. To see that this is well defined, let $[\alpha] = [\beta]$. Thus, $\beta = \tilde{\delta}\alpha\Phi(\tilde{\sigma})^{-1}$. The corresponding map $\hat{\beta}$ is given by $\hat{\delta}\hat{\alpha}\varphi(g)^{-1}$. This can be verified using the commutative diagram above. Also, note that $\tilde{\delta}(\widetilde{B}) = \widetilde{B}$. Since $\eta_{\widehat{Y}}\tilde{\delta} = \hat{\delta}\eta_{\widehat{Y}}$, it follows that $\hat{\delta}(\widehat{B}) = \widehat{B}$ where $\widehat{B} = \eta_{\widehat{Y}}(\widetilde{B})$. We have just shown that $\Lambda : [\,\cdot\,] \to \langle\,\cdot\,\rangle$ is surjective.

Now we have:

Proposition 5.8 $$N_G(f, B) \le R_{G,h}(f, B) \le R_G(f, B).$$

Remark 5.9 It should be pointed out that both $R_G(f, B)$ and $R_{G,h}(f, B)$ are well defined and independent of the lifts $\tilde{f}$ and $\hat{f}$ or the Hopf covering $\widehat{Y}$. Moreover, when $B = \{a\}$ is a singleton where $a \in Y^G$, our equivariant Nielsen preimage theory reduces to that of [20]. The equivariant Reidemeister root number defined in [20] using Hopf liftings coincides with $R_{G,h}(f, B)$ for $B = \{a\}$.

Given a G-space Z, we say that Z is *G-connected* if for any subgroup H, Z^H is connected. If G is a finite group and $f : X \to Y$ is a G-map between two G-connected spaces then for each subgroup $H \leq G$, $f^H : X^H \to Y^H$ is a WH-map between WH-spaces X^H and Y^H, where f^H is the restriction of f on the fixed point set X^H and $WH = NH/H$ is the Weyl group of H in G where NH is the normalizer of H in G. Thus, for each $H \leq G$, the previous sections will yield the invariants $N_{WH}(f^H, B^H)$, $R_{WH}(f^H, B^H)$ and $R_{WH,h}(f^H, B^H)$.

6 Application to Nielsen Borsuk–Ulam theory

In recent years, a theory called the "Nielsen Borsuk–Ulam theory" has been developed (see [2; 3; 19]). This theory is not only to consider the question of the existence of Borsuk–Ulam type coincidences but also to study the minimum number of such coincidences using methods inspired by Nielsen theory for fixed points and coincidences. In what follows, we will show that this theory is a special case of the "equivariant preimage theory for G-maps".

6.1 Borsuk–Ulam property (BUP)

In the literature, several authors have been studying the so-called Borsuk–Ulam property (see, for example, [8; 9; 10; 11; 13]).

The classical Borsuk–Ulam theorem states that for every continuous function $f : S^n \to \mathbb{R}^n$, there exists a point $z \in S^n$ such that $f(z) = f(-z)$, where $-z$ is the antipode of z on the sphere S^n. This result leads to a more general question: given a free involution τ on a space X, does every continuous function $f : X \to Y$ have the property that $f(x) = f(\tau(x))$ for some $x \in X$? If the answer is affirmative, it is said that the triple (X, τ, Y) has the Borsuk–Ulam property or, briefly, BUP. More generally, one can replace the sphere S^n with a topological space X equipped with a free $\mathbb{Z}_p$-action, where p is prime, and Euclidean space $\mathbb{R}^n$ with a topological space Y. In this setup, one possible question is: given $f : X \to Y$ a continuous function, does there exist $x \in X$ such that $f(x) = f(g \cdot x) = f(g^2 \cdot x) = \cdots = f(g^{p-1} \cdot x)$, where $\mathbb{Z}_p = \langle g \rangle$? Another way to pose this question is as follows: Let $\tau : X \to X$ be defined by $\tau(x) = g \cdot x$. Thus, the $\mathbb{Z}_p$-action on X is determined by the homeomorphism τ, and vice versa. One can ask if, given a continuous function $f : X \to Y$, the set of coincidences among the multiple maps $f, f \circ \tau, \ldots, f \circ \tau^{p-1}$,

$$\mathrm{Coin}(f, f \circ \tau, f \circ \tau^2, \ldots, f \circ \tau^{p-1}) = \{x \in X \mid f(x) = f(\tau(x)) = \cdots = f(\tau^{p-1}(x))\},$$

is nonempty. When the answer is positive for every continuous function from X to Y, we will say that the triple (X, τ, Y) has the Borsuk–Ulam property (briefly, BUP). Also, a homotopy class $\beta \in [X, Y]$ is said to have the BUP with respect to τ when for every continuous function $f : X \to Y$ representing β, there exists a point $x \in X$ such that $f(x) = f(\tau(x)) = \cdots = f(\tau^{p-1}(x))$.

The problem of determining the BUP for a triple (X, τ, Y) can be translated into an equivariant context as follows: let X be a topological space equipped with a free $\mathbb{Z}_p$-action, and let Y be a topological space.

As before, let $\tau : X \to X$ be defined by $\tau^p(x) = x$. Consider the $\mathbb{Z}_p$-action on Y^p determined by the homeomorphism $\tau' : Y^p \to Y^p$ given by

$$\tau'(y_1, y_2, \dots, y_p) = (y_2, y_3, \dots, y_p, y_1).$$

Let $\Delta_p(Y) = \{(y_1, \dots, y_p) \in Y^p \mid y_1 = y_2 = \cdots = y_p\}$ be the diagonal in Y^p. Note that this construction produces two types of isotropy subgroups: if $y \in Y^p \setminus \Delta_p(Y)$, then the isotropy subgroup of $\mathbb{Z}_p$ at y is Id; if $y \in \Delta_p(Y)$, then the isotropy subgroup of $\mathbb{Z}_p$ at y is $\mathbb{Z}_p$.

There is a bijection between the set of continuous functions from X to Y and the set of equivariant maps from X to Y^p: given a continuous function $f : X \to Y$, define $\varphi_f : X \to Y^p$ as

$$\varphi_f(x) = (f(x), f(\tau(x)), \dots, f(\tau^{p-1}(x))).$$

Thus, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, "\varphi_f"'] & X \arrow[d, "\varphi_f"] \\ Y^p \arrow[r, "\tau'"] & Y^p \end{tikzcd}$$

In other words, φ_f is equivariant. On the other hand, if $g : X \to Y^p$, $g(x) = (g_1(x), g_2(x), \dots, g_p(x))$, is an equivariant map, then $\varphi_{g_1} = g$.

Note that, under this bijection, given a continuous function $f : X \to Y$, there exists $x \in X$ such that $f(x) = f(\tau(x)) = \cdots = f(\tau^{p-1}(x))$ if and only if $\varphi_f^{-1}(\Delta_p(Y)) \neq \varnothing$.

Furthermore, $\{f_t\}$ is a homotopy between f_0 and f_1 if and only if $\{\varphi_{f_t}\}$ is a $\mathbb{Z}_p$-homotopy between φ_{f_0} and φ_{f_1} where $\varphi_{f_t}(x) = (f_t(x), f_t(\tau x), \dots, f_t(\tau^{p-1} x))$. Thus two continuous functions $f, f' : X \to Y$ are homotopic if and only if φ_f and $\varphi_{f'}$ are $\mathbb{Z}_p$-homotopic. Also,

$$\mathrm{Coin}(f, f \circ \tau, \dots, f \circ \tau^{p-1}) = \varphi_f^{-1}(\Delta_p(Y)).$$

Therefore, the study of the set of Borsuk–Ulam type coincidences, $\mathrm{Coin}(f, f \circ \tau, \dots, f \circ \tau^{p-1})$, is equivalent to the study of the $\mathbb{Z}_p$-preimage problem for φ_f at $\Delta_p(Y)$.

Because of the above observation, we will use the following nomenclature and notation: given X a $\mathbb{Z}_p$-space and $f : X \to Y$ a continuous map, a $\mathbb{Z}_p$-Nielsen preimage class for $\varphi_f : X \to_{\mathbb{Z}_p} Y^p \supset \Delta_p(Y)$ will be called a Borsuk–Ulam class of f (compare with [19]), $N_{\mathbb{Z}_p}(\varphi_f; \Delta_p(Y))$ will be denoted by $N_{\mathrm{BU}}(f)$, $R_{\mathbb{Z}_p}(\varphi_f, \Delta_p(Y))$ by $R_{\mathrm{BU}}(f)$ and $R_{\mathbb{Z}_p,h}(\varphi_f, \Delta_p(Y))$ by $R_{\mathrm{BU},h}(f)$.

6.2 Borsuk–Ulam coincidences as a $\mathbb{Z}_p$-preimage problem

As before, let X and Y be connected, locally pathwise-connected and semilocally simply connected spaces. Suppose X is a free $\mathbb{Z}_p$-space and let $\tau : X \to X$ be the homeomorphism given by $\tau(x) = g \cdot x$, where $\mathbb{Z}_p = \langle g \rangle$. Given a continuous function $f : X \to Y$, a point $x \in X$ such that $f(x) = f(\tau^i(x))$, $i = 1, \dots, p-1$, will be referred as a Borsuk–Ulam type coincidence for (f, τ).

On the cartesian product Y^p, consider the $\mathbb{Z}_p$-action determined by the homeomorphism $\tau' : Y^p \to Y^p$ given by

$$\tau'(y_1, y_2, \dots, y_p) = (y_2, y_3, \dots, y_p, y_1)$$

and let $B = \Delta_p(Y) = \{(y_1, \dots, y_p) \in Y^p \mid y_1 = y_2 = \cdots = y_p\}$ be the thin diagonal in Y^p. As we pointed above, B is invariant with respect to τ'. Also, the map $\varphi_f : X \to Y^p$ given by

$$\varphi_f(x) = (f(x), f(\tau(x)), \dots, f(\tau^{p-1}(x)))$$

is a $\mathbb{Z}_p$-equivariant map.

Let $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$ be universal coverings of X and Y, respectively. Then

$$\eta = \eta_Y \times \cdots \times \eta_Y : \widetilde{Y}^p \to Y^p$$

is a universal covering of Y^p.

Proposition 6.1 *Let $\widetilde{B}$ be the path component of $\eta^{-1}(\Delta_p(Y))$ that contains the thin diagonal $\Delta_p(\widetilde{Y})$. Then $\widetilde{B} = \Delta_p(\widetilde{Y})$.*

Proof Let $\widetilde{B}$ be the path component of $\eta^{-1}(\Delta_p(Y))$ that contains the thin diagonal $\Delta_p(\widetilde{Y})$. Let $(\tilde{y}_1, \dots, \tilde{y}_p) \in \widetilde{B}$ be an arbitrary point and consider $\lambda : I \to \widetilde{B}$ a continuous path from $(\tilde{y}, \dots, \tilde{y})$ to $(\tilde{y}_1, \dots, \tilde{y}_p)$. Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_p(t))$. Since $\widetilde{B} \subset \eta^{-1}(\Delta_p(Y))$,

$$\eta(\lambda(t)) = (\eta_Y(\lambda_1(t)), \dots, \eta_Y(\lambda_p(t))) \in \Delta_p(Y) \quad \text{for all } t \in I,$$

that is,

$$\eta_Y(\lambda_1(t)) = \eta_Y(\lambda_2(t)) \quad \text{for all } t \in I.$$

Since $\lambda_1(0) = \cdots = \lambda_p(0) = \tilde{y}$, it follows that $\lambda_1(1) = \cdots = \lambda_p(1)$, that is, $\tilde{y}_1 = \cdots = \tilde{y}_p$.

Hence, $\widetilde{B} \subset \Delta_p(\widetilde{Y})$. Therefore, $\widetilde{B} = \Delta_p(\widetilde{Y})$. □

In the special case of $p = 2$, one can show that

$$\hat{\pi}_{Y\times Y} = \pi_{Y\times Y} \rtimes \mathbb{Z}_2.$$

Indeed, consider the covering $\tilde{\tau}' : \widetilde{Y} \times \widetilde{Y} \to \widetilde{Y} \times \widetilde{Y}$ of τ' given by

$$\tilde{\tau}'(\tilde{y}_1, \tilde{y}_2) = (\tilde{y}_2, \tilde{y}_1).$$

Then the short exact sequence

$$1 \longrightarrow \pi_{Y\times Y} \hookrightarrow \hat{\pi}_{Y\times Y} \xrightarrow{\text{proj}} \mathbb{Z}_2 \longrightarrow 1$$

splits

$$\begin{tikzcd} 1 \arrow[r] & \pi_{Y\times Y} \arrow[r] & \hat{\pi}_{Y\times Y} \arrow[r, "\text{proj}"] & \mathbb{Z}_2 \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd}$$

where $s : \mathbb{Z}_2 \to \hat{\pi}_{Y\times Y}$ is the homomorphism such that $s(\bar{1}) = \tilde{\tau}'$.

Therefore, $\hat{\pi}_{Y\times Y} = \pi_{Y\times Y} \rtimes \mathbb{Z}_2$.

Remark 6.2 The equivariant Nielsen theory developed in this paper can also be applied to the Nielsen Borsuk–Ulam setting even when the $\mathbb{Z}_p$-action is not free. Furthermore, one can develop a Nielsen Borsuk–Ulam type theory for an arbitrary finite group G and arbitrary G-invariant subspace B. For instance, G can be taken to be the symmetric group and B to be the *fat* diagonal. Such a Borsuk–Ulam problem has already been studied in [12]. The applications to these various Borsuk–Ulam type settings will be further developed in a forthcoming work.

6.3 Maps to a topological group

In [9] and [19], the authors considered self-maps of the torus. We now give a different proof of some of their results.

Proposition 6.3 *Let $f : T^2 \to T^2$ be a continuous function and $\tau : T^2 \to T^2$ a free involution on the 2-torus T^2. If τ is an orientation preserving map then all Borsuk–Ulam classes of f (with respect to τ) are inessential. Consequently, $N_{\mathrm{BU}}(f) = 0$, which gives the existence of $f' \sim f$ such that $\mathrm{Coin}(f', f' \circ \tau) = 0$; that means, $\beta = [f]$ does not have the BUP.*

Proof Let F be an essential Borsuk–Ulam class of f. Thus, F is a finite disjoint union of ordinary coincidence classes of f and $f \circ \tau$. From the classical coincidence theory, the ordinary coincidence classes of f and $f \circ \tau$ have the same coincidence index. Since τ is orientation-preserving, it follows from [3, Definition 2.5] that the BU-index of F has the same sign as that of an ordinary coincidence class. This implies that either $N_{\mathrm{BU}}(f) = 0$ when $\mathrm{ind}(f; F) = 0$, or equivalently the Lefschetz coincidence number $L(f, f \circ \tau) = 0$, or all BU-classes are essential, or equivalently when $\mathrm{ind}(f; F) \neq 0$. Denote by A_f and A_τ the matrices associated to the map f and to the map τ, respectively. Then $L(f, f \circ \tau) = 0$ if and only if $\det(A_f - A_f A_\tau) = \det A_f \cdot \det(I - A_\tau)$ vanishes. Since τ is orientation-preserving, it follows from [9] that τ is equivalent to a map that lifts to the map $\tau_1 : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (x + 1/2, y)$. Thus, we conclude that $\det(I - A_\tau) = 0$. □

Corollary 6.4 *Let $\tau : T^2 \to T^2$ be a free involution that preserves orientation. If $\beta \in [T^2, T^2]$ is a homotopy class then β does not have the Borsuk–Ulam property with respect to τ.*

Proof Let $f : T^2 \to T^2$ be an arbitrary self-map on the 2-torus. Therefore, from Proposition 6.3, there is $f' \sim f$ such that f' has no Borsuk–Ulam coincidences. Hence, for any homotopy class $\beta \in [T^2, T^2]$, β does not have BUP with respect to τ. □

The above result was already proved in [9, Theorem 1] and in [19, Theorem 4.1] using different techniques.

We end this paper with the following slight generalization of the setting of self-maps of the torus. Let X be a closed connected manifold with a free involution τ and $f : X \to K$ a map where K is a compact connected topological group. The inversion $\mu : K \to K$ given by $\mu(k) = k^{-1}$ is an involution on K. Define $\varphi_f : X \to K$ by

$$\varphi_f(x) = f(\tau(x)) \cdot [f(x)]^{-1} = f(\tau(x)) \cdot \mu(f(x)).$$

Then

$$\begin{aligned}\varphi_f(\tau(x)) = f(\tau^2(x))\cdot\mu(f(\tau(x)) = f(x)\cdot[f(\tau(x))]^{-1} \\ = [f(\tau(x))\cdot[f(x)]^{-1}]^{-1} = \mu(\varphi_f(x)).\end{aligned} \tag{6-1}$$

It follows that φ_f is a $\mathbb{Z}_2$-equivariant map where $\mathbb{Z}_2 \cong \langle\tau\rangle \cong \langle\mu\rangle$. Moreover, f is homotopic to f' if and only if φ_f is $\mathbb{Z}_2$-homotopic to $\varphi_{f'}$. Now,

$$C(f, f\tau) = \{x \in X \mid f(x) = f\tau(x)\} = \varphi_f^{-1}(e),$$

where $e \in K$ is the unit element of the group K. Thus, we are in the equivariant root problem as in [20].

Acknowledgements

This work is supported by FAPESP of Brazil grant 2020/10874-1.

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Received: June 1, 2024 Revised: April 22, 2025

Algebraic & Geometric Topology 26:4 (2026) 1549–1564
DOI: 10.2140/agt.2026.26.1549
Published: April 25, 2026

Homotopy commutativity in quasitoric manifolds

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We prove that the loop space of a quasitoric manifold is homotopy commutative if and only if the underlying polytope is a product of 3-simplices $(\Delta^3)^n$ and the characteristic matrix is equivalent to a matrix of certain type. Quasitoric manifolds over $(\Delta^3)^n$ include generalized Bott manifolds, and we also construct an infinite family of homotopy nonequivalent generalized Bott manifolds over $(\Delta^3)^n$, only half of them have homotopy commutative loop spaces. In particular, for each $n \geq 2$, there are infinitely many homotopy types of $6n$-dimensional quasitoric manifolds having homotopy (non)commutative loop spaces.

1 Introduction

Quasitoric manifolds were introduced by Davis and Januszkiewicz [8] as a topological counterpart of smooth projective toric varieties. By definition, a quasitoric manifold is a closed manifold of dimension $2n$ equipped with a locally standard action of T^n such that the orbit space M/T^n is isomorphic to an n-dimensional simple polytope as a manifold with corners. Recall that every toric variety is constructed from a fan, a combinatorial object. There is a similar combinatorial construction of quasitoric manifolds, each of which is equivalent (in a precise sense defined in Section 2) to that associated to a simple polytope P and a certain characteristic matrix over P. Here, we remark that our equivalences of quasitoric manifolds are weaker than those in [8] as they respect a fixed isomorphism $M/T^n \cong P$ while ours do not.

It is well known that properties of a toric variety are described in terms of the corresponding fan, which exhibits a fascinating connection between algebraic geometry and combinatorics. Then it may be possible to describe topological properties of a quasitoric manifold in terms of the underlying simple polytope and the characteristic matrix, which also exhibits a fascinating connection between topology and combinatorics. There are examples of such descriptions for quasitoric manifolds, cohomology and Chern classes as in [8].

The understanding of a given space goes often through the study of its loop space. A first question is then whether or not it is commutative, up to homotopy. In this paper, we study the homotopy commutativity of the loop space of a quasitoric manifold. See [1; 2; 9; 10; 14; 15; 16; 19; 25] for other results on the loop spaces of quasitoric manifolds and related spaces. Complex projective spaces are special quasitoric manifolds, and the homotopy commutativity of their loop spaces were determined by Ganea [13]. The first result completely determines whether or not the loop space of any quasitoric manifold is homotopy commutative in terms of the underlying simple polytope and the characteristic matrix. Let Δ^n and E_n denote the n-simplex and the n-dimensional identity matrix.

MSC2020: primary 57S12; secondary 55P35, 55Q15.

Theorem 1.1 *The loop space of a quasitoric manifold over a simple polytope P is homotopy commutative if and only if $P = (\Delta^3)^n$ and the characteristic matrix is equivalent to*

$$\begin{pmatrix} E_3 & a_{11} & & a_{12} & & a_{13} & & & a_{1n} \\ & a_{21} & E_3 & a_{22} & & a_{23} & & & a_{2n} \\ & a_{31} & & a_{32} & E_3 & a_{33} & & & a_{3n} \\ & \vdots & & \vdots & & \vdots & \ddots & & \vdots \\ & a_{n1} & & a_{n2} & & a_{n3} & & E_3 & a_{nn} \end{pmatrix} \tag{1-1}$$

for $a_{ij} \in \mathbb{Z}^3$ such that

$$a_{ii} = {}^t(1,1,1) \quad \text{and} \quad (1,1,1)a_{ij} \equiv 0 \mod 2 \quad (i \neq j), \tag{1-2}$$

where the facets of $(\Delta^3)^n$ are ordered as in Section 4.

Remarks on Theorem 1.1 are in order. First, equivalences of characteristic matrices will be defined in Section 2. Second, the loop spaces of quasitoric manifolds over a common simple polytope have the same homotopy type. Then Theorem 1.1 may indicate that there are quasitoric manifolds whose loop spaces are homotopy equivalent but not H-equivalent, which is verified by Theorem 1.2 below. Third, we can further consider the higher homotopy commutativity of the loop space of a quasitoric manifold if it is homotopy commutative. Actually, by looking at the cohomology of a quasitoric manifold, we can find a nontrivial quadruple higher Whitehead product if its loop space is homotopy commutative. Then if the loop space of a quasitoric manifold is homotopy commutative, it is not a C_4-space in the sense of Williams [24], so it is not very highly homotopy commutative. Fourth, every characteristic matrix over $(\Delta^3)^n$ is equivalent to the matrix (1-1) satisfying the first condition of (1-2) (Lemma 4.1). Then the second condition of (1-2) guarantees that the loop space of a quasitoric manifold over $(\Delta^3)^n$ is homotopy commutative. On the other hand, $\mathbb{C}P^n$ is a quasitoric manifold over Δ^n, and in particular, a characteristic matrix of $\mathbb{C}P^3$ is

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{1-3}$$

Then Theorem 1.1 recovers Ganea's result [13] that the loop space of $\mathbb{C}P^n$ is homotopy commutative if and only if $n = 3$, where $\mathbb{C}P^n$ is a quasitoric manifold over Δ^n. Thus Theorem 1.1 can be thought of as an extension of Ganea's result. See [20; 21] for other extensions of Ganea's result.

As mentioned above, every characteristic matrix over Δ^3 is equivalent to (1-3), so every quasitoric manifold over Δ^3 is equivalent to $\mathbb{C}P^3$ (see [8, Example 1.18]). However, in general, it is quite hard to describe all characteristic matrices over a given simple polytope, and this is the case for $(\Delta^3)^n$ with $n \geq 2$ as in [7]. Then one cannot immediately see how many nonequivalent quasitoric manifolds over $(\Delta^3)^n$ for $n \geq 2$ there are, whose loop spaces are (not) homotopy commutative. For each $n \geq 2$, we construct an infinite family of homotopy nonequivalent quasitoric manifolds over $(\Delta^3)^n$, only half of them have homotopy commutative loop spaces. Let B be as in (1-3), and let

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $k > 0$ and $n \geq 2$, we define a $3n \times 4n$ matrix

$$B(k,n) = \begin{pmatrix} B & kN & & & \\ & B & kN & & \\ & & \ddots & \ddots & \\ & & & B & kN \\ & & & & B \end{pmatrix}.$$

The matrix $B(k,n)$ is a characteristic matrix over $(\Delta^3)^n$, so we get the corresponding quasitoric manifold $M(k,n)$ over $(\Delta^3)^n$. Note that $M(k,n)$ is defined only for $n \geq 2$ and a positive integer k. Observe that Theorem 1.1 shows that the loop space of $M(k,n)$ is homotopy commutative if and only if k is even. The second result concerns the homotopy types of $M(k,n)$, which implies that for each $n \geq 2$, there are infinitely many homotopy types of $6n$-dimensional quasitoric manifolds having homotopy (non)commutative loop spaces.

Theorem 1.2 *The quasitoric manifolds $M(k,n)$ and $M(l,n)$ are not homotopy equivalent for $k \neq l$.*

Remarks on Theorem 1.2 are in order. First, the quasitoric manifold $M(k,n)$ is a so-called generalized Bott manifold, that is, $M(k,n)$ is obtained by iterated "nice" $\mathbb{C}P^3$-bundles starting from a point. Second, we will actually prove that $M(k,n)$ satisfies cohomological rigidity; $M(k,n)$ and $M(l,n)$ are equivalent if and only if their integral cohomology are isomorphic. Third, if M_i is a quasitoric manifold over P_i for $i = 1, \dots, n$, then $M_1 \times \cdots \times M_n$ is a quasitoric manifold over $P_1 \times \cdots \times P_n$. Hence one can construct a quasitoric manifold over $(\Delta^3)^n$ from quasitoric manifolds over $(\Delta^3)^m$ for $m < n$. However, $M(k,n)$ is not equivalent to a product of nontrivial quasitoric manifolds (Proposition 4.4), so it is an "atomic" quasitoric manifold over $(\Delta^3)^n$.

The paper is organized as follows. In Section 2, we recall the basics of quasitoric manifolds, and show a loop space decomposition of quasitoric manifolds. Then by using this decomposition, we deduce that the loop space of a quasitoric manifold is homotopy commutative only if the underlying simple polytope is a product of Δ^3. In Section 3, we extend the method of Barrat, James and Stein [4] for computing Whitehead products. In Section 4, we prove Theorem 1.1 by applying the results in Sections 2 and 3. We also prove Theorem 1.2 by a direct cohomology computation.

2 Loop space decomposition

In this section, we recall the basic properties of quasitoric manifolds, and show a loop space decomposition of a quasitoric manifold. Then by applying this decomposition, we prove that the loop space of a quasitoric manifold is not homotopy commutative unless its underlying simple polytope is $(\Delta^3)^n$.

First, we define characteristic matrices over a simple polytope and equivalences among them. Let P be an n-dimensional convex polytope. A codimension one face of P will be called a facet. We say that P is simple if exactly n facets of P meet at each vertex. For example, simplices are simple polytopes, and a product of simple polytopes is a simple polytope. Suppose that P is simple and has m facets $F_1, \dots, F_m$. A characteristic matrix over P is an integer matrix $(a_1 \cdots a_m)$ for $a_1, \dots, a_m \in \mathbb{Z}^n$ such that $\det(a_{i_1} \cdots a_{i_n}) = \pm 1$ whenever $F_{i_1} \cap \cdots \cap F_{i_n} \neq \varnothing$ for $i_1 < \cdots < i_n$. Since an automorphism of P as a combinatorial polytope permutes facets, it acts on characteristic matrices over P by column permutation. We define that characteristic matrices A and B over P are equivalent if

$$A = \alpha \cdot (QBD) \tag{2-1}$$

for $Q \in \mathrm{GL}_n(\mathbb{Z})$, a diagonal matrix D with diagonal entries ± 1 and an automorphism α of P.

Next, we recall the construction of a quasitoric manifold using a moment-angle complex. Let K be a simplicial complex with vertex set $[m] = \{1, 2, \dots, m\}$, where an ordering of vertices is given. The moment-angle complex for K is defined by

$$Z_K = \bigcup_{\sigma \in K} Z(\sigma),$$

where $Z(\sigma) = X_1 \times \cdots \times X_m$ such that $X_i = D^2$ for $i \in \sigma$ and $X_i = S^1$ for $i \notin \sigma$. Note that the m-dimensional torus T^m acts naturally on Z_K. We will use the following obvious property of a moment-angle complex. For $\varnothing \neq I \subset [m]$, let

$$K_I = \{\sigma \in K \mid \sigma \subset I\}.$$

Lemma 2.1 *For $\varnothing \neq I \subset [m]$, Z_{K_I} is a retract of Z_K.*

Proof We can identify Z_{K_I} with the subspace

$$\{(x_1, \dots, x_m) \in Z_K \mid x_i \text{ is the basepoint for } i \in I\}$$

of Z_K. □

Let P be an n-dimensional simple polytope with m facets. Let $K(P)$ denote the boundary of the dual simplicial polytope of P. Then $K(P)$ is an $(n-1)$-dimensional simplicial sphere with m vertices. Let A be a characteristic matrix over P. Then the kernel of the linear map $A : \mathbb{Z}^m \to \mathbb{Z}^n$ defines a split subtorus $T(A)$ of dimension $m-n$ which acts freely on $Z_{K(P)}$. Let

$$M(A) = Z_{K(P)}/T(A).$$

By [8], we have:

Proposition 2.2 *The orbit space $M(A)$ is a quasitoric manifold over P such that every quasitoric manifold over P is equivalent to $M(A)$ for some characteristic matrix A over P.*

Let M and N be quasitoric manifolds of dimension $2n$. A map $f : M \to N$ is weakly equivariant if there is an automorphism $\theta : T^n \to T^n$ such that

$$f(tx) = \theta(t) f(x)$$

for $t \in T^n$ and $x \in M$. We say that M and N are equivalent if there is a weakly equivariant homeomorphism between them. Note that if M and N are equivalent, their underlying simple polytopes are isomorphic. Then equivalent quasitoric manifolds are essentially the same. As remarked in Section 1, our equivalences of quasitoric manifolds are weaker than those in [8] as Davis and Januszkiewicz demand equivalences to preserve an extra structure, a fixed isomorphism between M/T^n and a simple polytope. By [8], we also have:

Proposition 2.3 *The quasitoric manifolds $M(A_1)$ and $M(A_2)$ over P are equivalent if and only if the characteristic matrices A_1 and A_2 are equivalent as in* (2-1).

Now we prove a loop decomposition of a quasitoric manifold.

Proposition 2.4 *Let M be a quasitoric manifold over an n-dimensional simple polytope P with m facets. Then there is a homotopy equivalence*

$$\Omega M \simeq T^{m-n} \times \Omega Z_{K(P)}.$$

Proof By Proposition 2.2, there is a homotopy fibration $Z_{K(P)} \to M \to BT^{m-n}$, so we get an H-fibration

$$\Omega Z_{K(P)} \to \Omega M \to T^{m-n}.$$

By [6, Theorem 3.4.7], $Z_{K(P)}$ is 2-connected, so $\Omega Z_{K(P)}$ is simply connected. Then the map $\Omega M \to T^{m-n}$ has a section, implying the above H-fibration splits. □

We record an obvious fact about homotopy commutativity.

Lemma 2.5 *Let X, Y be H-groups, and let $f : X \to Y$ be an H-map. If X is not homotopy commutative and f has a left homotopy inverse, then Y is not homotopy commutative.*

Proof Let $g : Y \to X$ be a left homotopy inverse of f. If Y is homotopy commutative, then by definition, the Samelson product $\langle f, f\rangle$ is trivial, implying a contradiction

$$0 \neq \langle 1_X, 1_X\rangle = g \circ f \circ \langle 1_X, 1_X\rangle = g \circ \langle f, f\rangle = 0. \qquad \square$$

Now we consider conditions on the underlying polytope of a quasitoric manifold M that guarantee ΩM is not homotopy commutative. Let K be a simplicial complex. We say that a nonempty subset I of the vertex set of K is a minimal nonface of K if I is not a simplex of K and all proper subsets of I are simplices of K. Equivalently, $K_I = \partial\Delta^{|I|-1}$.

Lemma 2.6 *Let P be a simple polytope. If $K(P)$ has a minimal nonface of cardinality* 2, 3 *or* ≥ 5, *then the loop space of a quasitoric manifold over P is not homotopy commutative.*

Proof By Proposition 2.4 and Lemma 2.5, it suffices to show $\Omega Z_{K(P)}$ is not homotopy commutative. Let I be a minimal nonface of $K(P)$ of cardinality k. Then $Z_{K(P)_I} = Z_{\partial\Delta^{k-1}} = S^{2k-1}$, so by Lemma 2.1, S^{2k-1} is a retract of $Z_{K(P)}$. By [3], the Whitehead product $[1_{S^{2k-1}}, 1_{S^{2k-1}}]$ is nontrivial for $k = 3$ and $k \geq 5$, so by the adjointness of Whitehead products and Samelson products [22], ΩS^{2k-1} is not homotopy commutative for $k = 3$ and $k \geq 5$. Thus by Proposition 2.4 and Lemma 2.5, $\Omega Z_{K(P)}$ is not homotopy commutative either.

Now we suppose $k = 2$. Let M be a quasitoric manifold over P. Since S^3 is a retract of $Z_{K(P)}$, $H^3(Z_{K(P)}; \mathbb{Q})$ has a basis $\{u_1, \ldots, u_l\}$ for some $l \geq 1$. By Proposition 2.2, there is a homotopy fibration

$$Z_{K(P)} \to M \to BT^{m-n},$$

where m is the number of facets of P and $n = \dim P$. By [6, Theorem 3.4.7], $Z_{K(P)}$ is 2-connected, so in the Serre spectral sequence of the above homotopy fibration, each u_i is transgressive. Moreover, by [8, Proposition 3.10], $H^*(M; \mathbb{Q})$ is generated by elements of degree two, so the transgression images of $u_1, \ldots, u_l$ are linearly independent. Then we get

$$H^*(M; \mathbb{Q}) = \mathbb{Q}[t_1, \ldots, t_{m-n}]/(q_1, \ldots, q_l), \quad |t_i| = 2,$$

for $* \leq 5$, where q_i is the transgression image of u_i. This readily implies that the minimal Sullivan model for M is given by

$$(\mathbb{Q}[t_1, \ldots, t_{m-n}] \otimes \Lambda(x_1, \ldots, x_l), d), \quad dt_i = 0,\ dx_i = q_i,$$

in dimension ≤ 4. Since $|q_i| = 4$, q_i is a quadratic polynomial in $t_1, \ldots, t_{m-n}$. Thus by Proposition 13.16 of [11], M has nontrivial Whitehead product, implying ΩM is not homotopy commutative. □

Lemma 2.7 *Let P be a simple polytope. If $K(P)$ has intersecting distinct minimal nonfaces, then the loop space of a quasitoric manifold over P is not homotopy commutative.*

Proof Let I_1, I_2 be minimal nonfaces of $K(P)$ with $I_1 \neq I_2$ and $I_1 \cap I_2 \neq \varnothing$. Let $|I_1 \cap I_2| = j > 0$ and $|I_k| = i_k + j$ for $k = 1, 2$. Then for $k = 1, 2$,

$$Z_{K(P)_{I_k}} = S^{2(i_k+j)-1}.$$

Let $\iota_k : Z_{K(P)_{I_k}} \to Z_{K(P)_{I_1 \cup I_2}}$ be the inclusion, and let v_k be a generator of $H^{2(i_k+j)-1}(Z_{K(P)_{I_k}}) \cong \mathbb{Z}$. Then by Lemma 2.1, there is $u_k \in H^{2(i_k+j)-1}(Z_{K(P)_{I_1 \cup I_2}})$ satisfying $\iota_k^*(u_k) = v_k$ for $k = 1, 2$. Now we assume that the Whitehead product $[\iota_1, \iota_2]$ is trivial. Then there is a homotopy commutative diagram

$$\begin{tikzcd} Z_{K(P)_{I_1}} \vee Z_{K(P)_{I_2}} \arrow[r, "\iota_1+\iota_2"] \arrow[d] & Z_{K(P)_{I_1 \cup I_2}} \arrow[d, equal] \\ Z_{K(P)_{I_1}} \times Z_{K(P)_{I_2}} \arrow[r, "\mu"] & Z_{K(P)_{I_1 \cup I_2}} \end{tikzcd}$$

Hence $\mu^*(u_1) = v_1 \times 1$ and $\mu^*(u_2) = 1 \times v_2$, so

$$\mu^*(u_1u_2) = \mu^*(u_1)\mu^*(u_2) = v_1 \times v_2 \neq 0.$$

Thus we get $u_1u_2 \neq 0$. On the other hand, since $K(P)_{I_1 \cup I_2}$ has at least two minimal nonfaces, it is not a full simplex, implying $\dim Z_{K(P)_{I_1 \cup I_2}} \leq 2(i_1 + i_2 + j) - 1$. Then

$$|u_1u_2| = 2(i_1 + i_2 + j) - 2 + 2j > 2(i_1 + i_2 + j) - 1 \geq \dim Z_{K(P)_{I_1 \cup I_2}}$$

as $j > 0$, so we get $u_1u_2 = 0$, a contradiction. Thus the Whitehead product $[\iota_1, \iota_2]$ is nontrivial, so $\Omega Z_{K(P)_{I_1 \cup I_2}}$ is not homotopy commutative. Therefore by Lemma 2.5, $\Omega Z_{K(P)}$ is not homotopy commutative too. □

Now we are ready to prove:

Proposition 2.8 *Let M be a quasitoric manifold over a simple polytope P. If the loop space of M is homotopy commutative, then $P = (\Delta^3)^n$.*

Proof Suppose ΩM is homotopy commutative. Then by Lemmas 2.6 and 2.7, minimal nonfaces of $K(P)$ are of cardinality 4 and pairwise disjoint. Then

$$K(P) = \underbrace{\partial\Delta^3 \star \cdots \star \partial\Delta^3}_{n} \star \Delta^l$$

for some $l \geq -1$, where $\Delta^{-1} = \{\varnothing\}$. Since $K(P)$ is a simplicial sphere, we have $l = -1$, so

$$K(P) = \underbrace{\partial\Delta^3 \star \cdots \star \partial\Delta^3}_{n}.$$

Thus we obtain $P = (\Delta^3)^n$, as stated. □

We further consider a condition equivalent to the loop space of a quasitoric manifold over $(\Delta^3)^n$ being homotopy commutative. As in the proof of Proposition 2.8, if $P = (\Delta^3)^n$, then $K(P)$ is the join of n copies of $\partial\Delta^3$, implying

$$Z_{K(P)} = (S^7)^n.$$

Let M be a quasitoric manifold over $(\Delta^3)^n$. Then by Proposition 2.4, there is a homotopy equivalence

$$\Omega M \simeq (S^1)^n \times (\Omega S^7)^n,$$

which is not necessarily an H-equivalence. For $i = 1, \ldots, n$, let $a_i : S^1 \to \Omega M$ and $b_i : S^6 \to \Omega M$ be the composite maps

$$S^1 \xrightarrow{g_i} (S^1)^n \to \Omega M \quad \text{and} \quad S^6 \xrightarrow{E} \Omega S^7 \xrightarrow{g_i} (\Omega S^7)^n \to \Omega M,$$

where g_i and E denote the i-th inclusion and the suspension map, respectively.

Lemma 2.9 *Let M be a quasitoric manifold over $(\Delta^3)^n$. The loop space of M is homotopy commutative if and only if the Samelson products $\langle a_i, b_j \rangle$ for $i, j = 1, \ldots, n$ are trivial.*

Proof For $i = 1, \dots, n$, let $\bar{b}_i : \Omega S^7 \to \Omega M$ denote the composite of the i-th inclusion $\Omega S^7 \to (\Omega S^7)^n$ and the natural map $(\Omega S^7)^n \to \Omega M$. By [18, Proposition 1], ΩM is homotopy commutative if and only if the Samelson products $\langle a_i, a_j \rangle$, $\langle a_i, \bar{b}_j \rangle$, $\langle \bar{b}_i, \bar{b}_j \rangle$ for $i, j = 1, \dots, n$ are trivial. Clearly, $\langle a_i, a_j \rangle$ are trivial. By [13, Lemma 2.1], $\langle a_i, \bar{b}_j \rangle = 0$ if and only if $\langle a_i, b_j \rangle = 0$, and $\langle \bar{b}_i, \bar{b}_j \rangle = 0$ if and only if $\langle b_i, b_j \rangle = 0$. Note that each $b_i : S^6 \to \Omega M$ lifts to a map $\tilde{b}_i : S^6 \to (\Omega S^7)^n$. Then the Samelson products $\langle b_i, b_j \rangle$ in ΩM lift to the Samelson products $\langle \tilde{b}_i, \tilde{b}_j \rangle$ in $(\Omega S^7)^n$. Hence since $(\Omega S^7)^n$ is homotopy commutative, $\langle \tilde{b}_i, \tilde{b}_j \rangle$ are trivial, implying so are $\langle b_i, b_j \rangle$. □

3 Computation of Whitehead products

In this section, we extend the method of Barrat, James, and Stein [4] computing Whitehead products. The coefficients of cohomology will be the integers $\mathbb{Z}$.

Let X be a simply connected finite complex satisfying a homotopy fibration

$$(S^{2d-1})^n \xrightarrow{\phi} X \xrightarrow{\pi} (\mathbb{C}P^\infty)^n \tag{3-1}$$

for $d \geq 3$ such that

$$H^*(X) = \mathbb{Z}[t_1, \dots, t_n]/(q_1, \dots, q_n), \quad |t_i| = 2,\ |q_i| = 2d,$$

where for $i = 1, \dots, n$, t_i corresponds to the fundamental class of the i-th $\mathbb{C}P^\infty$ in $(\mathbb{C}P^\infty)^n$ and $q_i \in \mathbb{Z}[t_1, \dots, t_n]$ is the transgression image of a generator of $H^{2d-1}(S^{2d-1})$ for the i-th S^{2d-1} in $(S^{2d-1})^n$.

Lemma 3.1 *The sequence $q_1, \dots, q_n$ in $\mathbb{Z}[t_1, \dots, t_n]$ is regular.*

Proof For any field $\mathbb{F}$, $\mathbb{F}[t_1, \dots, t_n]$ is Cohen–Macaulay, and the Krull dimension of $H^*(X) \otimes \mathbb{F}$ is zero as X is a finite complex. Then the sequence $q_1, \dots, q_n$ is regular in $\mathbb{F}[t_1, \dots, t_n]$ for any field $\mathbb{F}$, so the sequence $q_1, \dots, q_n$ is regular in $\mathbb{Z}[t_1, \dots, t_n]$ too, as stated. □

We consider the cofiber Y of the map $\phi : (S^{2d-1})^n \to X$. For $i = 1, \dots, n$, let $\beta_i : S^{2d-1} \to X$ be the composite

$$S^{2d-1} \xrightarrow{i\text{-th incl}} (S^{2d-1})^n \xrightarrow{\phi} X.$$

Then by degree reasons,

$$Y_{4d-2} = X_{4d-2} \cup_{\beta_1} e^{2d} \cup_{\beta_2} \cdots \cup_{\beta_n} e^{2d}, \tag{3-2}$$

where Y_k denotes the k-skeleton of Y.

Lemma 3.2 *For $* \leq 2d+3$,*

$$H^*(Y) = \mathbb{Z}[t_1, \dots, t_n]/(t_i q_j \mid i, j = 1, \dots, n).$$

Proof Let $u_i \in H^{2d-1}((S^{2d-1})^n)$ denote the generator corresponding to the i-th S^{2d-1} in $(S^{2d-1})^n$. By Lemma 3.1, the elements $q_1,\dots,q_n$ of $\mathbb{Z}[t_1,\dots,t_n]$ are linearly independent, so we may assume

$$\tau(u_i)=q_i$$

for $i=1,\dots,n$, where τ denotes the transgression in the Serre spectral sequence for the homotopy fibration (3-1), implying

$$\delta(u_i)=\pi^*(q_i)$$

for the connecting map $\delta : H^{*-1}((S^{2d-1})^n)\to H^*(X,(S^{2d-1})^n)$ of the long exact sequence for the pair $(X,(S^{2d-1})^n)$ and the map

$$\pi^* : H^*((\mathbb{C}P^\infty)^n)\to H^*(X,(S^{2d-1})^n). \tag{3-3}$$

By degree reasons, the kernel of the composite

$$H^{2d}((\mathbb{C}P^\infty)^n)\xrightarrow{\pi^*} H^{2d}(X,(S^{2d-1})^n)\to H^{2d}(X)$$

is generated by $q_1,\dots,q_n$. Then the map (3-3) for $*=2d$ is an isomorphism. Thus since $\widetilde{H}^*(Y)\cong H^*(X,(S^{2d-1})^n)$, it follows from (3-2) that the map (3-3) is an isomorphism for $1\le *\le 2d$. On the other hand, the $(2d+2)$-dimensional part of the ideal $(q_1,\dots,q_n)$ in $\mathbb{Z}[t_1,\dots,t_n]$ is generated by t_iq_j for $i,j=1,\dots,n$. Then by (3-2), the proof is finished. □

Let $\bar{\pi} : Y\to(\mathbb{C}P^\infty)^n$ denote an extension of the map $\pi : X\to(\mathbb{C}P^\infty)^n$. Then by [12, Theorem 1.1], the homotopy fiber of $\bar{\pi} : Y\to(\mathbb{C}P^\infty)^n$ has the homotopy type of the join $(S^1)^n\star(S^{2d-1})^n$. We consider the cofiber $\overline{Y}$ of the fiber inclusion of $\bar{\pi}$. Let $g_i : A\to A^n$ denote the i-th inclusion for $i=1,\dots,n$, and let γ_{ij} denote the composite

$$S^{2d+1}=S^1\star S^{2d-1}\xrightarrow{g_i\star g_j}(S^1)^n\star(S^{2d-1})^n\xrightarrow{\text{incl}} Y.$$

Then the map

$$\bigvee_{i,j=1}^n g_i\star g_j : \bigvee_{i,j=1}^n S^{2d+1}\to(S^1)^n\star(S^{2d-1})^n$$

is an inclusion of the $(2d+1)$-skeleton and has a left homotopy inverse, implying

$$\overline{Y}_{2d+2}=Y_{2d+2}\cup_{\gamma_{11}}e^{2d+2}\cup_{\gamma_{12}}\cdots\cup_{\gamma_{nn}}e^{2d+2}. \tag{3-4}$$

Lemma 3.3 *For* $*\le 2d+2$,

$$H^*(\overline{Y})=\mathbb{Z}[t_1,\dots,t_n].$$

Proof Since $(S^1)^n\star(S^{2d-1})^n$ is homotopy equivalent to a wedge of spheres, all $(2d+3)$-cells of $\overline{Y}$ are attached to Y_{2d+2}. Then by Lemma 3.2 and (3-4), the $(2d+3)$-cells of $\overline{Y}$ do not kill any cohomology class of $\overline{Y}_{2d+2}$, implying $H^*(\overline{Y})=H^*(\overline{Y}_{2d+2})$ for $*\le 2d+2$. Now by Lemma 3.1, t_iq_j for $i,j=1,\dots,n$ are linearly independent in $\mathbb{Z}[t_1,\dots,t_n]$. Then by arguing as in the proof of Lemma 3.2, the statement is proved. □

Lemma 3.4 *The homotopy group $\pi_{2d+1}(Y)$ is a free abelian group generated by γ_{ij} for $i, j = 1, \dots, n$.*

Proof The statement follows from the homotopy exact sequence of the homotopy fibration

$$(S^1)^n \star (S^{2d-1})^n \to Y \to (\mathbb{C}P^\infty)^n,$$

where the $(2d+1)$-skeleton of $(S^1)^n \star (S^{2d-1})^n$ is described as above. □

For $i = 1, \dots, n$, let $\alpha_i : S^2 \to X$ be a map whose Hurewicz image is the dual of t_i, and let $\overline{\beta}_i : (D^{2d}, S^{2d-1}) \to (Y, X)$ denote the obvious extension of $\beta_i : S^{2d-1} \to X$. Then

$$\delta(\overline{\beta}_i) = \beta_i$$

for the connecting homomorphism $\delta : \pi_*(Y, X) \to \pi_{*-1}(X)$. We consider the relative Whitehead product $[\alpha_i, \overline{\beta}_j] \in \pi_{2d+1}(Y, X)$. See [5] for the definition. By [5, (3.5)],

$$\delta([\alpha_i, \overline{\beta}_j]) = -[\alpha_i, \beta_j].$$

Let $\overline{\eta} : (D^{2d+1}, S^{2d}) \to (D^{2d}, S^{2d-1})$ be the obvious extension of the Hopf map $\eta : S^{2d} \to S^{2d-1}$. By [17, Theorem (1.4)] (see [23, (5.8)]), we can compute $\pi_{2d+1}(Y, X)$ as follows.

For a commutative ring R, let $R\{a_1, \dots, a_k\}$ denote the free R-module with a basis $\{a_1, \dots, a_k\}$.

Lemma 3.5 $\pi_{2d+1}(Y, X) = \mathbb{Z}\{[\alpha_i, \overline{\beta}_j] \mid i, j = 1, \dots, n\} \oplus \mathbb{Z}_2\{\overline{\beta}_i \circ \overline{\eta} \mid i = 1, \dots, n\}$.

Let $\beta = \beta_1 \vee \dots \vee \beta_n : (S^{2d-1})^{\vee n} \to X$ and $\overline{\beta} = \overline{\beta}_1 \vee \dots \vee \overline{\beta}_n : ((D^{2d})^{\vee n}, (S^{2d-1})^{\vee n}) \to (Y, X)$. Let $\iota : (A, *) \to (A, B)$ and $\rho : A \to A/B$ denote the inclusion and the pinch map, respectively. There is a commutative diagram

$$\begin{tikzcd}
 & \pi_{2d+1}((D^{2d})^{\vee n}, (S^{2d-1})^{\vee n}) \arrow[r, "\delta", "\cong"'] \arrow[d, "\overline{\beta}_*"] & \pi_{2d}((S^{2d-1})^{\vee n}) \arrow[d, "\beta_*"] \\
\pi_{2d+1}(Y) \arrow[r, "\iota_*"] \arrow[d, "\rho_*"] & \pi_{2d+1}(Y, X) \arrow[r, "\delta"] \arrow[d, "\rho_*"] & \pi_{2d}(X) \\
\pi_{2d+1}(Y/X) \arrow[r, equal] & \pi_{2d+1}(Y/X) &
\end{tikzcd} \tag{3-5}$$

in which the middle row is exact. By the homotopy exact sequence for the homotopy fibration (3-1), we can see the map β_* is an isomorphism. Then there is $\epsilon_{ij} \in \pi_{2d+1}((D^{2d})^{\vee n}, (S^{2d-1})^{\vee n})$ such that

$$\beta_* \circ \delta(\epsilon_{ij}) = -[\alpha_i, \beta_j], \tag{3-6}$$

implying

$$\delta([\alpha_i, \overline{\beta}_j] - \overline{\beta}_*(\epsilon_{ij})) = -[\alpha_i, \beta_j] - \beta_* \circ \delta(\epsilon_{ij}) = 0.$$

Hence by Lemma 3.4,

$$[\alpha_i, \overline{\beta}_j] - \overline{\beta}_*(\epsilon_{ij}) = \iota_*(\zeta_{ij}) \tag{3-7}$$

such that ζ_{ij} is a linear combination of $\gamma_{kl} \in \pi_{2d+1}(Y)$ for $k, l = 1, \dots, n$.

Lemma 3.6 *The Whitehead product $[\alpha_i, \beta_j]$ vanishes if and only if $\rho_*(\zeta_{ij}) = 0$, where $\rho : Y \to Y/X$ denotes the pinch map.*

Proof Since β_* in (3-5) is an isomorphism, it follows from (3-6) that $[\alpha_i, \beta_j] = 0$ if and only if $\epsilon_{ij} = 0$. By (3-2),

$$(Y/X)_{4d-2} = \underbrace{S^{2d} \vee \cdots \vee S^{2d}}_{n},$$

so $\rho_* \circ \overline{\beta}_*$ in (3-5) is an isomorphism. Then $\epsilon_{ij} = 0$ if and only if $\rho_* \circ \overline{\beta}_*(\epsilon_{ij}) = 0$. On the other hand, $\rho_*([\alpha_i, \overline{\beta}_j]) = 0$ as $\rho_*(\alpha_i) = 0$, so by (3-7), we get

$$\rho_* \circ \overline{\beta}_*(\epsilon_{ij}) + \rho_*(\zeta_{ij}) = \rho_*([\alpha_i, \overline{\beta}_j]) = 0.$$

Thus $\rho_* \circ \overline{\beta}_*(\epsilon_{ij}) = 0$ if and only if $\rho_*(\zeta_{ij}) = 0$. □

Now we are ready to prove:

Proposition 3.7 *The Whitehead products $[\alpha_i, \beta_j]$ are trivial for $i, j = 1, \dots, n$ if and only if $\mathrm{Sq}^2 q_k = 0$ in $\mathbb{Z}_2[t_1, \dots, t_n]$ for all k.*

Proof By the homotopy fibration (3-1), we can see that $\pi_{2d+1}(X)$ is a finite group, so the map ι_* in (3-5) is injective by Lemma 3.4. In particular, $\mathrm{Im}\,\iota_*$ is a free abelian group. On the other hand, by Lemma 3.5 and (3-7), the subgroup A of $\pi_{2d+1}(Y, X)$ generated by $\iota_*(\zeta_{ij})$ for $i, j = 1, \dots, n$ is a maximal free abelian subgroup of $\pi_{2d+1}(Y, X)$. Then since $A \subset \mathrm{Im}\,\iota_*$, we obtain $A = \mathrm{Im}\,\iota_*$, implying that $\rho_*(\zeta_{ij}) = 0$ for $i, j = 1, \dots, n$ if and only if $\rho_*(\gamma_{ij}) = 0$ for $i, j = 1, \dots, n$.

By Lemma 3.2, the $(2d)$-cells in (3-2) correspond to $q_1, \dots, q_n$, and by (3-2) and (3-4),

$$(\overline{Y}/X)_{2d+2} = (\underbrace{S^{2d} \vee \cdots \vee S^{2d}}_{n}) \cup_{\rho_*(\gamma_{11})} e^{2d+2} \cup_{\rho_*(\gamma_{12})} \cdots \cup_{\rho_*(\gamma_{nn})} e^{2d+2}$$

such that the $(2d+2)$-cells may be considered to be corresponding to $t_i q_j$ for $i, j = 1, \dots, n$. Then as the generator of $\pi_{2d+1}(S^{2d}) \cong \mathbb{Z}_2$ is detected by Sq^2, we get that $\rho_*(\gamma_{ij}) = 0$ if and only if $\mathrm{Sq}^2 q_k$ does not include the terms $t_i q_j$ in $H^*(\overline{Y}/X; \mathbb{Z}_2)$ for $k = 1, \dots, n$. Note that in $H^*(\overline{Y}; \mathbb{Z}_2)$, $\mathrm{Sq}^2 q_k$ must belong to the ideal $(q_1, \dots, q_n)$ and every degree $2d+2$ element of $(q_1, \dots, q_n)$ is a linear combination of $t_i q_j$ for $i, j = 1, \dots, n$. Then since the natural map $H^*(\overline{Y}/X; \mathbb{Z}_2) \to H^*(\overline{Y}; \mathbb{Z}_2)$ is injective for $2d \le * \le 2d+2$, the above condition on $\mathrm{Sq}^2 q_k$ in $H^*(\overline{Y}/X; \mathbb{Z}_2)$ is equivalent to that $\mathrm{Sq}^2 q_k = 0$ in $H^*(\overline{Y}; \mathbb{Z}_2) = \mathbb{Z}_2[t_1, \dots, t_n]$ $(* \le 2d+2)$ for $k = 1, \dots, n$. □

4 Quasitoric manifolds over $(\Delta^3)^n$

In this section, we prove Theorems 1.1 and 1.2. We fix an ordering of the facets of $(\Delta^3)^n$ as in [7] to consider characteristic matrices over $(\Delta^3)^n$. Let F_1, F_2, F_3, F_4 be the facets of Δ^3, where any choice of ordering will do by symmetry. Then facets of $(\Delta^3)^n$ are

$$F_{ij} = (\Delta^3)^{i-1} \times F_j \times (\Delta^3)^{n-i}$$

for $i = 1, \dots, n$ and $j = 1, 2, 3, 4$. We fix an ordering of facets as

$$F_{11}, F_{12}, F_{13}, F_{14}, F_{21}, F_{22}, F_{23}, F_{24}, \dots, F_{n1}, F_{n2}, F_{n3}, F_{n4},$$

where this ordering is used in Theorems 1.1 and 1.2.

Lemma 4.1 *Every characteristic matrix over $(\Delta^3)^n$ is equivalent to a matrix*

$$\begin{pmatrix} E_3 & a_{11} & & a_{12} & & a_{13} & & a_{1n} \\ & a_{21} & E_3 & a_{22} & & a_{23} & & a_{2n} \\ & a_{31} & & a_{32} & E_3 & a_{33} & & a_{3n} \\ & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ & a_{n1} & & a_{n2} & & a_{n3} & E_3 & a_{nn} \end{pmatrix}$$

for $a_{ij} \in \mathbb{Z}^3$ such that $a_{ii} = {}^t(1, 1, 1)$ for $i = 1, \dots, n$.

Proof Let $B = (b_1^1\ b_2^1\ b_3^1\ b_4^1\ b_1^2\ b_2^2\ b_3^2\ b_4^2\ \cdots\ b_1^n\ b_2^n\ b_3^n\ b_4^n)$ be a characteristic matrix over $(\Delta^3)^n$, where $b_j^i \in \mathbb{Z}^{3n}$. Since the facets of $(\Delta^3)^n$ except for $F_{14}, F_{24}, \dots, F_{n4}$ meet at a vertex, the matrix $Q = (b_1^1\ b_2^1\ b_3^1\ b_1^2\ b_2^2\ b_3^2\ \cdots\ b_1^n\ b_2^n\ b_3^n)$ is invertible, so B is equivalent to

$$Q^{-1}B = \begin{pmatrix} E_3 & c_{11} & & c_{12} & & c_{13} & & c_{1n} \\ & c_{21} & E_3 & c_{22} & & c_{23} & & c_{2n} \\ & c_{31} & & c_{32} & E_3 & c_{33} & & c_{3n} \\ & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ & c_{n1} & & a_{n2} & & c_{n3} & E_3 & c_{nn} \end{pmatrix}$$

for $c_{ij} \in \mathbb{Z}^3$. Since the facets of $(\Delta^3)^n$ except for $F_{14}, \dots, F_{i-1,4}, F_{ij}, F_{i+1,4}, \dots, F_{n4}$ meet at a vertex for $i = 1, \dots, n$ and $j = 1, 2, 3$,

$$\det\begin{pmatrix} E_{3(i-1)} & & \\ & C_{ij} & \\ & & E_{3(n-i)} \end{pmatrix} = \pm 1$$

for $i = 1, \dots, n$ and $j = 1, 2, 3$, where C_{ij} is the $3{\times}4$ matrix $(E_3\ \ c_{ii})$ with j-th column removed. Then we get $c_{ii} = {}^t(\pm 1, \pm 1, \pm 1)$. Since multiplying columns and rows of a characteristic matrix by -1 yields an equivalent characteristic matrix, we obtain that $Q^{-1}B$ is equivalent to the matrix in the statement. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 By Propositions 2.2 and 2.8, we only need to consider a quasitoric manifold $M(A)$ over $(\Delta^3)^n$ such that A is a characteristic matrix over $(\Delta^3)^n$ in Lemma 4.1. By [8, Theorem 4.14],

$$H^*(M(A)) = \mathbb{Z}[t_{ij} \mid i = 1, \dots, n,\ j = 1, 2, 3, 4]/I + J, \quad |t_{ij}| = 2,$$

where $I = (t_{i1}t_{i2}t_{i3}t_{i4} \mid i = 1, \dots, n)$ and

$$J = \left(t_{ij} + \sum_{k=1}^{n} a_{ik}^{j} t_{k4} \;\middle|\; i = 1, \dots, n,\ j = 1, 2, 3 \right),$$

where $a_{ik} = {}^t(a_{ik}^1, a_{ik}^2, a_{ik}^3)$. So we get

$$H^*(M(A)) = \mathbb{Z}[t_1, \dots, t_n]/(q_1, \dots, q_n), \quad |t_i| = 2, \tag{4-1}$$

such that

$$q_i = t_i \prod_{j=1}^{3} \biggl(\sum_{k=1}^{n} a_{ik}^j t_k \biggr),$$

where we put $t_i = t_{i4}$. Now

$$\begin{aligned}\mathrm{Sq}^2 q_i &= \biggl(t_i + \sum_{j=1}^{3} \sum_{k=1}^{n} a_{ik}^j t_k \biggr) q_i = \biggl((1 + a_{ii}^1 + a_{ii}^2 + a_{ii}^3) t_i + \sum_{k \neq i} (a_{ik}^1 + a_{ik}^2 + a_{ik}^3) t_k \biggr) q_i \\ &= \sum_{k \neq i} (a_{ik}^1 + a_{ik}^2 + a_{ik}^3) t_k q_i\end{aligned}$$

because $a_{ii} = {}^t(1,1,1)$. Thus by Lemma 3.1 and Proposition 3.7, the Whitehead products $[\alpha_i, \beta_j]$ are trivial for $i, j = 1, \dots, n$ if and only if $(1,1,1)a_{ij} = a_{ij}^1 + a_{ij}^2 + a_{ij}^3 \equiv 0 \bmod 2$ for all $i \neq j$. On the other hand, by the adjointness of Whitehead products and Samelson products [22], the Whitehead product $[\alpha_i, \beta_j]$ is trivial if and only if the Samelson product $\langle a_i, b_j \rangle$ is trivial, where a_i and b_j are as in Section 3. Therefore by Lemma 2.9, the proof is finished. $\square$

Hereafter, let k be a positive integer. For $n \geq 1$, we define a graded algebra

$$H(k,n) = \mathbb{Z}[t_1, \dots, t_n]/(t_1^4 + k t_1^3 t_2, \dots, t_{n-1}^4 + k t_{n-1}^3 t_n, t_n^4), \quad |t_i| = 2.$$

We need the following properties of $H(k,n)$.

Lemma 4.2 *If $x \in H(k,n)$ satisfies $|x| = 2$ and $x^4 = 0$, then $x = a t_n$ for some $a \in \mathbb{Z}$.*

Proof Since $|x| = 2$, we may put $x = a_1 t_1 + \dots + a_n t_n$ for integers $a_1, \dots, a_n \in \mathbb{Z}$. Note that the set $\{t_{i_1} t_{i_2} t_{i_3} t_{i_4} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq n,\ i_1 < i_4\}$ is a basis of the degree-8 part of $H(k,n)$. We express x^4 as a linear combination of this basis. Then x^4 includes the term $6a_i^2 a_j^2 t_i^2 t_j^2$ for $i \neq j$, implying $a_i a_j = 0$ for $i \neq j$. This readily implies $x = a_i t_i$ for some $1 \leq i \leq n$. On the other hand, $t_i^4 = 0$ in $H(k,n)$ if and only if $i = n$. $\square$

Lemma 4.3 *If connected graded rings A, B have nontrivial elements of degree two, then $H(k,n)$ is not isomorphic to $A \otimes B$.*

Proof We prove the statement by induction on n. For $n = 1$, the statement holds because the degree-two part of $H(k,1)$ is isomorphic to $\mathbb{Z}$. We assume the $n = m$ case, and prove the $n = m+1$ case. Suppose that there is an isomorphism $f : H(k, m+1) \to A \otimes B$. We may put $f(t_{m+1}) = a + b$ for $a \in A$ and $b \in B$, where $|a| = |b| = 2$. Then

$$0 = f(t_{m+1}^4) = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4.$$

Since $H(k,m+1) \cong A \otimes B$, A and B are isomorphic to polynomial rings in degrees < 8, implying $a = 0$ or $b = 0$. We may assume $b = 0$. Then f induces an isomorphism

$$\bar{f} : H(k,m+1)/(t_{m+1}) \to (A/(a)) \otimes B.$$

Since $H(k,m+1)/(t_{m+1}) \cong H(k,m)$, it follows from the assumption that $A/(a) = \mathbb{Z}$. So $t = f^{-1}(\bar{f}(t_m))$ and t_{m+1} are linearly independent in $H(k,m+1)$, and $t^4 = t_{m+1}^4 = 0$. This is a contradiction by Lemma 4.2, and therefore $H(k,m+1)$ is not isomorphic to $A \otimes B$. □

For the rest of the paper, we set $n \geq 2$. Let $M(k,n)$ denote the quasitoric manifold in Theorem 1.2. Then by (4-1),

$$H^*(M(k,n)) = H(k,n). \tag{4-2}$$

We remark that $M(k,n)$ is a generalized Bott manifold such that $M(k,n+1)$ is the projectivization of a complex vector bundle $E \oplus \underline{\mathbb{C}}^3 \to M(k,n)$, where E is the complex line bundle with total Chern class $c(E) = 1 + kt_1$ and $\underline{\mathbb{C}}$ denotes the trivial bundle. We show atomicity of $M(k,n)$ with respect to products of quasitoric manifolds.

Proposition 4.4 *The quasitoric manifold $M(k,n)$ is not homotopy equivalent to a product of two nontrivial quasitoric manifolds.*

Proof Let $M(k,n) \simeq M \times N$ for nontrivial quasitoric manifolds M, N. Then by the Künneth formula,

$$H(k,n) \cong H^*(M) \otimes H^*(N).$$

Thus the statement follows from Lemma 4.3. □

Now we start to prove Theorem 1.2. The following lemma is immediate from Lemma 4.2.

Lemma 4.5 *Every graded algebra isomorphism $f : H(k,n) \xrightarrow{\cong} H(l,n)$ satisfies*

$$f(t_n) = \pm t_n.$$

For $j = 1, \dots, n$, we define an ideal of $H(k,n)$ by

$$I_j(k,n) = (t_{n-j+1}, t_{n-j+2}, \dots, t_n).$$

Then we get a sequence

$$I_1(k,n) \subset I_2(k,n) \subset \cdots \subset I_n(k,n).$$

Lemma 4.6 *Every graded algebra isomorphism $f : H(k,n) \xrightarrow{\cong} H(l,n)$ satisfies*

$$f(I_j(k,n)) = I_j(l,n)$$

for $j = 1, \dots, n$.

Proof We show $f(I_j(k,n)) = I_j(l,n)$ by induction on j. For $j = 1$, $f(I_1(k,n)) = I_1(l,n)$ by Lemma 4.5. Assume that the statement holds for $j = 1, \dots, p$. Then the map f induces an isomorphism

$$\bar{f} : H(k,n)/I_p(k,n) \xrightarrow{\cong} H(l,n)/I_p(l,n).$$

On the other hand, there is a natural isomorphism

$$H(m,n)/I_p(m,n) \cong H(m,n-p)$$

for any positive integer m such that $I_1(m,n-p)$ in $H(m,n-p)$ lifts to $I_{p+1}(m,n)$ in $H(m,n)$. By the induction hypothesis, $\bar{f}(I_1(k,n-p)) \subset I_1(k,n-p)$ through the above natural isomorphism. Thus we get $f(I_{p+1}(k,n)) = I_{p+1}(l,n)$. □

Proposition 4.7 *$H(k,n) \cong H(l,n)$ if and only if $k = l$.*

Proof The if part is trivial, and we consider the only if part. First, we consider the $n = 2$ case. Suppose there is an isomorphism $f : H(k,2) \xrightarrow{\cong} H(l,2)$. By Lemma 4.6,

$$f(t_1) = \epsilon_1(t_1 + ct_2) \quad \text{and} \quad f(t_2) = \epsilon_2 t_2$$

for $\epsilon_1, \epsilon_2 = \pm 1$ and an integer c, so we get

$$0 = f(t_1^4 + kt_1^3t_2) = (4c - l + k\epsilon_1\epsilon_2)t_1^3t_2 + (6c^2 + 3k\epsilon_1\epsilon_2 c)t_1^2t_2^2 + (4c^3 + 3k\epsilon_1\epsilon_2 c^2)t_1t_2^3.$$

Then since $t_1^3t_2$, $t_1^2t_2^2$, $t_1t_2^3$ are linearly independent in $H(l,n)$, we obtain

$$6c^2 + 3k\epsilon_1\epsilon_2 c = 0, \quad 4c^3 + 3k\epsilon_1\epsilon_2 c^2 = 0, \quad 4c - l + k\epsilon_1\epsilon_2 = 0.$$

By the first two equations, we get $c = 0$, so by the third equation, we obtain $k = l$, as desired.

Next, we consider the $n > 2$ case. Let $H(m)$ denote the subalgebra of $H(m,n)$ generated by t_{n-1} and t_n. Then there is a canonical isomorphism

$$H(m) \cong H(m,2).$$

Suppose there is an isomorphism $f : H(k,n) \xrightarrow{\cong} H(l,n)$. Then by Lemma 4.6, the map f restricts to an isomorphism

$$H(k) \xrightarrow{\cong} H(l).$$

Thus by the $n = 2$ case, we get $k = l$. □

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2 The first statement follows from Theorem 1.1, and the second statement follows from (4-2) and Proposition 4.7. □

Acknowledgements

Kishimoto and Tsutaya were partially supported by JSPS KAKENHI grants JP22K03284 and JP22K03317, respectively. Tong was partially supported by JST SPRING grant JPMJSP2110. The authors are grateful to Jérôme Scherer and anonymous referees for useful comments.

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Received: August 15, 2024 Revised: March 30, 2025

Algebraic & Geometric Topology **26**:4 (2026) 1565–1584
DOI: 10.2140/agt.2026.26.1565
Published: April 25, 2026

Tautological rings of fibrations

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We study the analogue of tautological rings of fibre bundles in the context of fibrations with Poincaré fibre, i.e., the ring obtained by fibre integrating powers of the fibrewise Euler class. We discuss how to compute the Euler ring with tools from rational homotopy theory and completely determine the tautological ring for even spheres, complex projective spaces and some products of odd spheres.

1 Introduction

Let $\pi : E \to B$ be a fibration with fibre X an oriented Poincaré duality space of formal dimension d (see [35, Chapter 1]), which if X is a simply connected finite CW complex just means that one can choose a fundamental class $[X] \in H_d(X;\mathbb{Z})$ that induces the Poincaré duality isomorphism for all local coefficient systems. It is called *oriented* if the corresponding local coefficient system $\mathcal{H}^d(X;\mathbb{Z})$ is trivial and we choose an isomorphism $\mathcal{H}^d(X;\mathbb{Z}) \cong \mathbb{Z}$. In [19, Section 3] the authors construct for such fibrations a *fibrewise Euler class* $e^{\mathrm{fw}}(\pi) \in H^d(E;\mathbb{Z})$ which extends the construction for smooth fibre bundles, i.e., if X is a closed, oriented manifold and $\pi : E \to B$ is an oriented fibre bundle, then the fibrewise Euler class agrees with the Euler class of the vertical tangent bundle $T_\pi E \to E$ which, if π is a smooth submersion, is defined as the kernel of the differential of the projection map $T_\pi E := \ker(D\pi : TE \to TB) \subset TE$.

Using the fibrewise Euler class we can extend the construction of *tautological classes* of smooth fibre bundles [14; 16; 23; 27] to fibrations. Recall that given a smooth, oriented fibre bundle $\pi : E \to B$ with fibre M a closed manifold and a class $c \in H^{|c|}(\mathrm{B\,SO}(d);\mathbb{Q})$, the fibre integral

$$\kappa_c(\pi) := \int_\pi c(T_\pi E) \in H^{|c|-d}(B;\mathbb{Q})$$

is a characteristic class of the bundle and the subring $R^*(\pi) \subset H^*(B;\mathbb{Q})$ generated by all tautological classes is called the *tautological ring*.

As we can define fibre integration more generally for oriented fibrations with Poincaré fibre (see [6, Section 8] and Section 3), we obtain analogous characteristic classes of oriented fibrations $\pi : E \to B$ with Poincaré fibre by setting

$$\kappa_i(\pi) := \pi_!(e^{\mathrm{fw}}(\pi)^{i+1}) \in H^{i\cdot d}(B;\mathbb{Z}),$$

where $\pi_! : H^*(E;\mathbb{Z}) \to H^{*-d}(B;\mathbb{Z})$ is another notation for the fibre integration map. In particular, for the universal oriented X-fibration that classifies oriented X-fibrations [30]

$$X \hookrightarrow E \xrightarrow{\pi} \mathrm{BhAut}^+(X), \tag{1-1}$$

MSC2020: 55R40.

where the base $\mathrm{BhAut}^+(X)$ is the classifying space of the monoid of orientation-preserving homotopy self-equivalences $\mathrm{hAut}^+(X)$, we can study the subring generated by the tautological classes.

Note that while tautological classes of fibrations can be defined integrally, we only study the tautological ring rationally due to the limitations of the techniques we use, and from now on we always use rational coefficients which we omit from the notation.

Definition 1.1 Let X be an oriented Poincaré duality space of formal dimension d. The *Euler ring* $E^*(X)$ is the subring of $H^*(\mathrm{BhAut}^+(X))$ generated by all tautological classes

$$\kappa_i := \pi_!(e^{\mathrm{fw}}(\pi)^{i+1}) \in H^{i\cdot d}(\mathrm{BhAut}^+(X)), \tag{1-2}$$

where $e^{\mathrm{fw}}(\pi) \in H^d(E)$ is the fibrewise Euler class of the universal oriented X-fibration in (1-1).

Unlike the smooth tautological ring, the computation of the Euler ring $E^*(X)$ is a purely homotopy-theoretic problem. The main content of this paper is to work out how one can use the models from rational homotopy theory for fibrations to compute Euler rings. Specifically, we determine representatives of the fibre integration maps and fibrewise Euler classes in terms of the algebraic models from rational homotopy theory. The computation of the Euler ring turns out to be particularly tractable for rationally elliptic spaces and we determine the Euler ring for two classes of examples.

Theorem A *The Euler ring of complex projective space is* $E^*(\mathbb{C}P^n) \cong \mathbb{Q}[\kappa_1, \dots, \kappa_{n-1}, \kappa_{n+1}]$.

Theorem B *Let X be either rationally equivalent to $S^{2k+1} \times \cdots \times S^{2k+1}$ for $k \geq 1$ or a finite CW complex rationally equivalent to a product of two odd-dimensional simply connected spheres of different dimension. Then $E^*(X) = \mathbb{Q}$.*

Some further computations of Euler rings can be found in the author's thesis [25, Section 4.2]. One can also extend the definition of the Euler ring for fibrations with extra structure to obtain, for example, better homotopy-theoretic approximations to the smooth tautological ring [2]. Finally, these techniques can be used to infer properties about smooth tautological rings [26].

2 Rational homotopy theory of fibrations

The classifying space $\mathrm{BhAut}^+(X)$ is rarely simply connected even if X is, and therefore we cannot immediately apply the results from rational homotopy theory. Instead, for a simply connected Poincaré duality space X we study the universal 1-connected fibration

$$X \hookrightarrow E \xrightarrow{\pi} \mathrm{BhAut}_0(X), \tag{2-1}$$

over the classifying space of the path component of the identity $\mathrm{hAut}_0(X) \subset \mathrm{hAut}^+(X)$ (or equivalently the induced fibration over the universal covering of $\mathrm{BhAut}^+(X)$). Many people have studied rational models of (2-1) (see [21; 32; 33]) and fibrations in general (see [11; 18]), and we will use these algebraic models to compute the image of the Euler ring

$$E_0^*(X) \subset H^*(\mathrm{BhAut}_0(X)) \tag{2-2}$$

induced by the natural map $\mathrm{BhAut}_0(X) \to \mathrm{BhAut}^+(X)$. We discuss in Section 5 how one can in some cases upgrade the computation of $E_0(X)$ to a computation of the full Euler ring $E^*(X)$.

2.1 Rational models of fibrations

In the following, we use standard terminology from rational homotopy theory as discussed in [11]. Due to different conventions in the literature, we use cohomological grading conventions for cdgas and differential graded modules and homological grading conventions for dg Lie algebras.

Throughout this paper, we denote a cdga model of a map of connected spaces $\pi : E \to B$, typically a fibration with simply connected fibre, by $\pi^* : \mathrm{R} \to \mathrm{S}$ and we also assume, unless stated otherwise, that cdga models are connected (i.e., $\mathrm{R}^0 = \mathrm{S}^0 = \mathbb{Q}$). The following cdga models for fibrations enjoy good homotopical properties.

Definition 2.1 [11, Section 14] A *relative Sullivan algebra* is a cdga $(\mathrm{R} \otimes \Lambda V, D)$ so that $\mathrm{id}_{\mathrm{R}} \otimes 1 : \mathrm{R} \to \mathrm{R} \otimes \Lambda V$ is a map of cdgas, and $V = \bigoplus_{p \geq 1} V^p$ is a graded vector space with an exhaustive filtration $V(0) \subset V(1) \subset \cdots$ of graded subspaces so that $D|_{1 \otimes V(0)} : V(0) \to \mathrm{R}$ and $D|_{1 \otimes V(k)} : V(k) \to \mathrm{R} \otimes \Lambda V(k-1)$. A *relative Sullivan model* of a map of cdgas $\pi^* : \mathrm{R} \to \mathrm{S}$ is a relative Sullivan algebra $\mathrm{S} = (\mathrm{R} \otimes \Lambda V, D)$ together with a quasi-isomorphism $\mathrm{S}' \xrightarrow{\simeq} \mathrm{S}$ of R-algebras.

By [11, Proposition 14.3] any map $\pi^* : \mathrm{R} \to \mathrm{S}$ of connected cdgas admits a relative Sullivan model if $H(\pi^*) : H^1(\mathrm{R}) \to H^1(\mathrm{S})$ is injective. Since this condition is always satisfied if π^* is the model of a fibration $\pi : E \to B$ with connected fibre, we can always find relative Sullivan models in this case. Moreover, if either the base or fibre has finite type and $\pi_1(B)$ acts trivially on the cohomology of the fibre, then $\mathrm{S} \otimes_{\mathrm{R}} \mathbb{Q}$ is a Sullivan model of the fibre by [18, Theorem 20.3].

There is a convenient way to obtain relative Sullivan algebras via differential graded Lie algebras. We follow [2] in our convention (which in turn is based on [33]), in defining Chevalley–Eilenberg complex of a dg Lie algebra L as the dg coalgebra

$$\mathscr{C}^{\mathrm{CE}}(L) = (\Lambda^c sL, D = d_1 + d_2),$$

where $\Lambda^c sL$ denotes the cofree, conilpotent, cocommutative coalgebra on the suspension $(sL)_* = L_{*-1}$ and the differentials are determined by their corestrictions

$$d_1(sl) = -s d_L(l), \quad d_2(sl_1 \wedge sl_2) = (-1)^{|l_1|} s[l_1, l_2].$$

If a dg Lie algebra acts on a cdga A through derivations, then the Chevalley–Eilenberg cochain complex is

$$\mathscr{C}_{\mathrm{CE}}(L; A) := \big(\mathrm{Hom}(\mathscr{C}^{\mathrm{CE}}(L), A), \partial + t\big),$$

where

$$\partial(f) = d_A \circ f - (-1)^{|f|} f \circ D,$$

$$t(f)(sl_1 \wedge \cdots \wedge sl_n) = \sum_{i=1}^{n} (-1)^{|sl_i|(|f|+|sl_1|+\cdots+|sl_{i-1}|)} l_i \cdot f(sl_1 \wedge \cdots \wedge \widehat{sl_i} \wedge \cdots \wedge sl_n), \tag{2-3}$$

and $\mathscr{C}_{\mathrm{CE}}(L;A)$ is a cdga via the convolution product. An element $f \in \mathscr{C}^{\mathrm{CE}}(L;A)$ is an *n-cochain* if $f(sl_1 \wedge \cdots \wedge sl_k) = 0$ unless $k = n$ and we identify 0-cochains with A. Moreover, if A is a Sullivan algebra then $\mathscr{C}_{\mathrm{CE}}(L,A)$ is a relative Sullivan algebra over the Chevalley–Eilenberg cochain complex $\mathscr{C}_{\mathrm{CE}}(L) := \mathscr{C}_{\mathrm{CE}}(L;\mathbb{Q})$ via the map on Chevalley–Eilenberg cochain complexes induced by the unit $\eta : \mathbb{Q} \to A$.

With these definitions in place, we can describe a cdga model of (2-1). Let X be a simply connected space of finite type and with minimal Sullivan model $(\Lambda V, d)$. Consider the dg Lie algebra $(\mathrm{Der}^+(\Lambda V), [d,-])$ of positive-degree derivations, where a derivation $\theta \in \mathrm{Der}(\Lambda V)$ has degree n if it *lowers*[1] the degree by n (additionally, $\theta \in \mathrm{Der}^+(\Lambda V)_1$ only if $[d,\theta] = 0$). It was first proved by Sullivan [32] that $\mathrm{Der}^+(\Lambda V)$ is a dg Lie model of $\mathrm{BhAut}_0(X)$, and one can further describe a model of (2-1) as follows.

Theorem 2.2 *Let X be a 1-connected space of finite type with minimal Sullivan model $(\Lambda V, d)$ and unit $\eta : \mathbb{Q} \to \Lambda V$. Then*

$$\mathscr{C}^*_{\mathrm{CE}}(\mathrm{Der}^+(\Lambda V);\mathbb{Q}) \xrightarrow{\eta_*} \mathscr{C}^*_{\mathrm{CE}}(\mathrm{Der}^+(\Lambda V);\Lambda V) \tag{2-4}$$

is a relative Sullivan model of the universal oriented 1-connected fibration (2-1).

Remark 2.3 (i) In a previous version of this article we proved this result via a comparison of (2-4) with Tanré's model [33] of the universal 1-connected fibration (see also [25]). It has been pointed out to the author by Andrey Lazarev that instead one can derive this result more directly from [21]. Another proof is due to Alexander Berglund and is based on rational models for the bar construction in terms of Chevalley–Eilenberg complexes [1]. The statement of Theorem 2.2 can be found as a special case of [2, Proposition 3.6] and we refer to these papers for a proof.

(ii) Theorem 2.2 is particularly useful for rationally elliptic spaces because the dg Lie algebra $\mathrm{Der}^+(\Lambda V)$ is finite dimensional in contrast to Tanré's model which is only of finite type. However, if X is rationally hyperbolic then $\mathrm{Der}^+(\Lambda V)$ is not even of finite type and it seems more feasible to study other models of (2-1) instead, for example, based on a dg Lie algebra model of the fibre (see, for example, [3; 31]).

(iii) The theory of minimal models in its most general form [18, Theorem 20.3] applies more generally to give a model for (2-1) for any connected space X of finite type and minimal Sullivan model $(\Lambda V, d)$. It seems plausible to the author that the algebraic tools we develop in subsequent sections generalise to study the Euler ring $E^*_0(X)$ of nonsimply connected Poincaré duality spaces X. However, minimal Sullivan models of nonsimply connected spaces are considerably more complicated and the dg Lie algebra of derivations is intractable for computations, so that we have not worked in this generality.

Example 2.4 Let $X = S^{2n}$ with Sullivan model

$$A_n = (\Lambda(x,y), |x| = 2n, |y| = 4n-1, d = x^2 \cdot \partial/\partial y).$$

[1] This slightly confusing convention is due to using homological grading conventions for dg Lie algebras and cohomological grading conventions for cdgas and can be avoided if one sticks to just one.

Then $\mathrm{Der}^+(A_n)$ is 3-dimensional with basis $\eta_{2n-1} := x \cdot \partial/\partial y$, $\eta_{2n} := \partial/\partial x$ and $\eta_{4n-1} := \partial/\partial y$ and differential $[d, \eta_{2n}] = -2\eta_{2n-1}$. Hence, the inclusion of the abelian Lie algebra with trivial differential $\mathfrak{g} := \mathbb{Q}\{\eta_{4n-1}\} \hookrightarrow \mathrm{Der}^+(A_n)$ is a quasi-isomorphism of dg Lie algebras and we get a cdga quasi-isomorphism $\mathscr{C}^*_{\mathrm{CE}}(\mathrm{Der}^+(A_n); \mathbb{Q}) \to \mathscr{C}^*_{\mathrm{CE}}(\mathfrak{g}; \mathbb{Q}) = (\Lambda z_{4n}, d = 0)$ as well as with coefficients in A_n. Thus, the cdga model of the universal 1-connected S^{2n}-fibration in Theorem 2.2 is equivalent to

$$(\Lambda(z_{4n}), d = 0) \to (\Lambda(z_{4n}) \otimes \Lambda(x, y), D(x) = 0, D(y) = x^2 + z_{4n}),$$

where z_{4n} is the 1-cochain dual to $s\eta_{4n-1}$.

3 Fibre integration in rational homotopy theory

Before we discuss rational models for fibre integration, we recall the definition suitable for oriented fibrations as a special case of the following construction: Let $\pi : E \to B$ be a fibration with fibre X and $H^*(X) = 0$ for $* > d$ and $H^d(X)$ nontrivial. Given a $\pi_1(B)$-module homomorphism $\phi : \mathscr{H}^d(X) \to \mathbb{Q}$, we can define *$\phi$-integration* as the composition

$$\phi_! : H^*(E) \twoheadrightarrow E_\infty^{*-d,d} \subset E_2^{*-d,d} = H^{*-d}(B; \mathscr{H}^d(X)) \xrightarrow{H(\phi)} H^{*-d}(B), \tag{3-1}$$

where we project $H^*(E)$ onto the d-th row of the E_∞-page of the Serre spectral sequence, which is possible since $H^*(X) = 0$ for $* > d$, and $E_\infty^{*-d,d} \subset E_2^{*-d,d}$ as there are no differentials into this row.

Note that because the cohomological Serre spectral sequence is compatible with cup product, there is a push-pull identity

$$\phi_!(\pi^*(x) \smile y) = x \smile \phi_!(y) \tag{3-2}$$

for $x \in H^*(B)$ and $y \in H^*(E)$, and thus $\phi_!$ is a $H^*(B)$-module map.

Definition 3.1 Let $\pi : E \to B$ be an orientable fibration with Poincaré fibre X of formal dimension d and let $\varepsilon_X : H^d(X) \to \mathbb{Q}$ be an orientation of X. Then

$$\pi_! := (\varepsilon_X)_! : H^*(E) \to H^{*-d}(B)$$

is called *fibre integration* of $\pi : E \to B$.

3.1 Chain level fibre integration

In the main result of this section we show that there exists a chain level representative of fibre integration. More precisely, given a relative Sullivan model $\mathrm{S} = (\mathrm{R} \otimes \Lambda V, D)$ of an oriented fibration $\pi : E \to B$ with Poincaré fibre X, there exists a R-module homomorphism $\Pi : \mathrm{S} \to \mathrm{R}$ of degree $-d$ which is unique up to homotopy and induces fibre integration on cohomology. This is most conveniently expressed in terms of differential Ext groups.

Let R be a connected cdga and M, N a differential graded R-modules. Then R-module homomorphisms $\mathrm{Hom}_{\mathrm{R}}(M, N)$ is a R-module with differential $D(f) := d_N f - (-1)^{|f|} f d_M$. Recall from [11, Section 7] that M is a *semifree* R*-module* if there is an exhaustive filtration $M(0) \subset M(1) \subset \cdots \subset M$ so that $M(0)$ and $M(k)/M(k-1)$ are free R-modules for all $k \geq 1$. A *semifree resolution* of a R-module M is a semifree R-module M' with a quasi-isomorphism $M' \xrightarrow{\simeq} M$ and we denote by $M \otimes^{\mathbb{L}}_{\mathrm{R}} N$ the derived tensor product of two R-modules given by $M' \otimes_R N$ for some semifree resolution M. Following [12], the differential Ext groups for R-modules M, N are defined as

$$\mathrm{Ext}_{\mathrm{R}}(M, N) := H(\mathrm{Hom}_{\mathrm{R}}(M', N)) \tag{3-3}$$

for a semifree resolution $M' \xrightarrow{\simeq} M$.

Proposition 3.2 *Let $\pi : E \to B$ be a fibration with connected base and total space and 1-connected fibre X. Assume that $\pi_1(B)$ acts trivially on $H^*(X)$ and that $H^*(X)$ is of finite type and nontrivial in degree d and vanishes for $* > d$. Let $\pi^* : \mathrm{R} \to \mathrm{S}$ be a cdga model of π. Then the augmentation induces an isomorphism*

$$\mathrm{Ext}^{-d}_{\mathrm{R}}(\mathrm{S}, \mathrm{R}) \xrightarrow{\cong} \mathrm{Ext}^{-d}_{\mathbb{Q}}(\mathrm{S} \otimes^{\mathbb{L}}_{\mathrm{R}} \mathbb{Q}, \mathbb{Q}) \cong \mathrm{Hom}(H^d(X), \mathbb{Q}), \tag{3-4}$$

and given $\phi \in \mathrm{Hom}(H^d(X), \mathbb{Q})$ the chain level representative $[\Phi] \in \mathrm{Ext}^{-d}_{\mathrm{R}}(\mathrm{S}, \mathrm{R})$ induces ϕ-integration.

Remark 3.3 Proposition 3.2 generalises [13, Theorem A] where they identify fibre integration as elements in differential Ext groups for fibrations over Poincaré duality spaces and for pullbacks from such fibrations. Moreover, fibre integration can be identified rationally as a map of parametrised suspension spectra $\pi_! : \Sigma^{\infty}_B B_+ \to \Sigma^{\infty-d}_B E_+$, and by [12, Theorem 1.1] the set of homotopy classes of such maps is given by differential Ext groups which is consistent with our result.

Proof Let $\mathrm{S}' \xrightarrow{\simeq} \mathrm{S}$ be a relative Sullivan model. Then $\mathrm{Ext}^*(\mathrm{S}, \mathrm{R}) = H^*(\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R}))$ since S' is a semifree resolution by [11, Lemma 14.1]. We consider the exhaustive filtration of $\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R})$ given by $F^p = \mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R}^{\geq p})$. According to [5, Theorem 9.3] the corresponding spectral sequence is conditionally convergent to the completion

$$\begin{aligned}\varprojlim_p \mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R})/\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R}^{\geq p}) &\cong \varprojlim_p \mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R}/\mathrm{R}^{\geq p}) \\ &\cong \mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \varprojlim_p \mathrm{R}/\mathrm{R}^{\geq p}) \\ &\cong \mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R}),\end{aligned}$$

where we have used that S' is a projective R-module for the first isomorphism. The E_1-page is $H(\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathbb{Q})) \otimes B^p$. By the assumption on the fibration it follows from [18] that the Sullivan fibre $\mathrm{S}' \otimes_{\mathrm{R}} \mathbb{Q} = (\Lambda V, d)$ is a cdga model of X. Hence, we have that the E_1-page can be simplified as $E_1^{p,q} = H^q((\Lambda V)^\vee) \otimes \mathrm{R}^p \cong \mathrm{Hom}^q(H^*(X), \mathbb{Q}) \otimes \mathrm{R}^p$. Since $\pi_1(B)$ acts trivially, the differential on the E_1-page is given by $\mathrm{id} \otimes d_{\mathrm{R}} : E_1^{p,q} \to E_1^{p+1,q}$ (if R was simply connected this follows for degree reasons but in general it follows from the fact the cdga model of the universal 1-connected X-fibration in Theorem 2.2 has a simply connected base and by assumption the relative Sullivan model is obtained by base change along a map $\mathscr{C}_{\mathrm{CE}}(\mathrm{Der}^+(\Lambda V)) \to \mathrm{R}$). Hence the E_2-page is $E_2^{p,q} = H^p(\mathrm{R}) \otimes H^q((\Lambda V)^\vee)$.

In particular, the spectral sequence vanishes for $q < -d$. Since the gradings are such that the differentials are $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$, there are only finitely many nontrivial differentials. This implies that the derived E_∞-page is zero and so by [5, Theorem 7.1] the spectral sequence converges strongly,

$$E_2^{p,q} = H^p(\mathrm{R}) \otimes H^q((\Lambda V)^\vee) \cong H^p(B) \otimes \mathrm{Hom}^q(H^*(X), \mathbb{Q}) \Rightarrow H(\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R})),$$

and we can recover $H(\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R}))$ from the entries of the E_∞-page. The only contribution with total degree $-d$ comes from $E_\infty^{0,-d} \cong E_2^{0,-d} \cong \mathrm{Hom}(H^d(X), \mathbb{Q})$ which proves the first part of the statement.

It remains to show that for $\phi = H^d(\Phi \otimes_{\mathrm{R}} \mathbb{Q}) : H^d(\Lambda V, d) \to \mathbb{Q}$ the induced ϕ-integration map coincides with $H(\Phi) : H^*(\mathrm{S}') \to H^*(\mathrm{R})$. First, we note that $\mathrm{S}' = (\mathrm{R} \otimes \Lambda V, D)$ has a filtration $F^p = \mathrm{R}^{\geq p} \otimes \Lambda V$ and that the corresponding spectral sequence converges as $E_2^{p,q} = H^p(\mathrm{R}) \otimes H^q(\Lambda V) \Rightarrow H^{p+q}(\mathrm{S}')$. In fact, R also has an analogous filtration $G^p = \mathrm{R}^{\geq p}$ with only nontrivial differential on the E_1-page. Then Φ induces a map between these two filtrations and the map on E_2-pages is precisely ϕ-integration defined using this spectral sequence. As we have defined ϕ-integration using the Serre spectral sequence, it remains to show that this spectral sequence is isomorphic to the Serre spectral sequence. Grivel has shown in [17] that the above filtration gives rise to the Serre spectral sequence if the base is simply connected, and this has been generalised by Halperin [18] if $\pi_1(B)$ acts nilpotently on the cohomology of the fibre. More precisely, it follows from the proof of [17, Theorem 6.4], respectively [18, Section 20], that the comparison map of $\mathrm{R} \to \mathrm{S}'$ with $A_{PL}(B) \to A_{PL}(E)$ is compatible with Dress' construction of the Serre spectral sequence [10] and induces an isomorphism on the E_2-pages. □

We can use Proposition 3.2 to build a representative of fibre integration for an oriented fibration $\pi : E \to B$ and oriented Poincaré fibre (X, ε_X): Consider a relative Sullivan model $\pi^* : \mathrm{R} \to (\mathrm{R} \otimes \Lambda V, D)$. Pick a chain level representative ε of the orientation $\varepsilon_X \in H^{-d}(\mathrm{Hom}(\Lambda V, \mathbb{Q}))$ of X. By Proposition 3.2 there is a cycle $\Pi \in \mathrm{Hom}^{-d}(\mathrm{S}', \mathrm{R})$ unique up to chain homotopy that satisfies

$$\Pi(1 \otimes \chi) = \varepsilon_X(\chi) \in \mathrm{R}^0 = \mathbb{Q} \tag{3-5}$$

for all $\chi \in (\Lambda V)^d$ and that induces fibre integration on cohomology

$$\pi_! : H^*(E; \mathbb{Q}) \cong H^*(\mathrm{S}') \xrightarrow{H(\Pi)} H^{*-d}(\mathrm{R}) \cong H^{*-d}(B; \mathbb{Q}).$$

We demonstrate this technique in the following example.

Example 3.4 Recall the relative Sullivan model of the universal 1-connected fibration for an even-dimensional sphere $X = S^{2n}$ as discussed in Example 2.4. We choose as orientation $\varepsilon_X : A_n \to \mathbb{Q}$ the homomorphism determined by $\varepsilon_X(x) = 1$. For degree reasons $\Pi(yx^k) = 0$ and since Π has to be a chain map we have $0 = \Pi(D(yx^k)) = \Pi(x^{k+2} + z_{4n}x^k)$. This determines a $\Lambda(z_{4n})$-module map $\Pi : (\Lambda(z_{4n}, x, y), D) \to (\Lambda(z_{4n}), d = 0)$ by

$$\Pi(yx^k) = 0 \qquad \text{and} \qquad \Pi(x^n) = \begin{cases} 0, & n = 2k, \\ (-1)^k z_{4n}^k, & n = 2k+1, \end{cases}$$

which is a chain map by construction and induces fibre integration on cohomology as it satisfies (3-5).

4 The fibrewise Euler class

The definition of the fibrewise Euler class in [19, Definition 3.1.1] uses constructions in the category of parametrised spectra. And while there has been a lot of progress to adapt the tools from rational homotopy theory to the context of parametrised stable homotopy theory [7; 8], there is a simpler way to define the fibrewise Euler class with rational coefficients so that we can avoid discussing the category of parametrised spectra and their rational models altogether.

We begin by describing a special case of the definition of the fibrewise Euler class. Consider an oriented fibration $\pi : E \to B$ with fibre X so that both X and B are oriented Poincaré duality spaces of dimensions d and b, respectively. Then the total space is also an oriented Poincaré duality space [15], and we can define the Umkehr map of the fibrewise diagonal $\Delta : E \to E \times_B E$ by

$$\Delta_! : H^*(E) \xrightarrow[\cong]{D_E} H_{b+d-*}(E) \xrightarrow{\Delta_*} H_{b+d-*}(E \times_B E) \xrightarrow[\cong]{D^{-1}_{E\times_B E}} H^{*+d}(E \times_B E), \tag{4-1}$$

where D_E and $D_{E\times_B E}$ denote the Poincaré duality isomorphisms. It is shown in [19, Section 3] that the fibrewise Euler class of $\pi : E \to B$ agrees with

$$e^{\mathrm{fw}}(\pi) = \Delta^*(\Delta_!(1)) \in H^d(E). \tag{4-2}$$

This is sufficient to define the fibrewise Euler class with rational coefficients, because for any space X the rational homology groups are isomorphic to rationalised stable framed bordism $H_*(X;\mathbb{Q}) \cong \Omega_*^{\mathrm{sfr}}(X)\otimes\mathbb{Q}$, and so we can determine a rational cohomology class by defining its evaluation on framed bordism classes. Hence, denoting by E the total space of the universal 1-connected fibration (2-1), given a stably framed bordism class $[f : M^d \to E, \xi] \in \Omega_d^{\mathrm{sfr}}(E) \otimes \mathbb{Q}$, we can consider the pullback of the X-fibration $p_1 : E \times_{\mathrm{BhAut}_0(X)} E \to E$ along f. The pullback $\pi : f^*(E \times_{\mathrm{BhAut}_0(X)} E) \to M$ has a section s via the diagonal, and we can associate to it an Euler class e_f via (4-2). Then the fibrewise Euler class $e^{\mathrm{fw}}(\pi) \in H^d(E;\mathbb{Q})$ agrees with the class defined by the pairing $\langle e^{\mathrm{fw}}(\pi), [f : M \to E, \xi]\rangle = \langle s^*(e_f), [M]\rangle$ since the construction in [19] coincides with (4-2) if the base is a Poincaré duality space.

4.1 A cocycle representative via rational homotopy theory

The idea to obtain cocycle representatives of the fibrewise Euler class is simply to construct a chain level representative of the Umkehr map $\Delta_! : H^*(E) \to H^{*+d}(E \times_B E)$ in terms of the algebraic models. In the following, we denote by $\mathbb{Q}[n]$ the graded vector space with $\mathbb{Q}$ in degree n and for an R-module M define $M[n] := M \otimes \mathbb{Q}[n]$.

Proposition 4.1 *Let* $\mathrm{R} \to \mathrm{S}$ *be a relative Sullivan model of an oriented fibration* $\pi : E \to B$ *with connected base and total space and simply connected Poincaré fibre* X *of formal dimension* d*. Let* ε_X *be the orientation and* $\Pi \in \mathrm{Hom}_{\mathrm{R}}^{-d}(\mathrm{S},\mathrm{R})$ *be a cocycle representative of fibre integration. Then the map*

$$\overline{\Pi} : \mathrm{S}[-d] \to \mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{R}), \quad e \mapsto (e' \mapsto (-1)^{d+d\cdot|e|}\Pi(e\cdot e')),$$

is a quasi-isomorphism of R*-modules.*

Proof It is a simple check that $\overline{\Pi}$ defines a R-module homomorphism. By assumption, $\mathrm{S} = (\mathrm{R} \otimes \Lambda V, D)$ is a relative Sullivan algebra and thus has a filtration which induces the Serre spectral sequence as discussed in the proof of Proposition 3.2. In the same proof we have described a filtration of $\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}, \mathrm{R})$ which strongly converges because there is a horizontal vanishing line. The map $\overline{\Pi}$ is compatible with the two filtrations and induces a map of the associated spectral sequences. The induced map on the E_2-page is given by

$$E_2^{p,q} = H^p(\mathrm{R}) \otimes H^q(\Lambda V, d) \xrightarrow{\mathrm{Id} \otimes \bar{\varepsilon}_X} H^p(\mathrm{R}) \otimes \mathrm{Hom}(H^{q-d}(\Lambda V), \mathbb{Q}),$$

where $\bar{\varepsilon}_X : H^q(\Lambda V) \to \mathrm{Hom}(H^{d-q}(\Lambda V), \mathbb{Q})$ is the adjoint of $H^q(\Lambda V) \otimes H^{d-q}(\Lambda V) \xrightarrow{\cup} H^d(\Lambda V) \xrightarrow{\varepsilon_X} \mathbb{Q}$. Since $(H^*(X; \mathbb{Q}), \varepsilon_X)$ is an oriented Poincaré duality algebra, $\overline{\Pi}$ induces an isomorphism of E_2-pages. □

This enables us to define an algebraic Umkehr map as follows. Let $\mathrm{R} \to \mathrm{S}$ be a relative Sullivan model of $\pi : E \to B$ and $\Pi : \mathrm{S} \to \mathrm{R}$ a chain level representative of fibre integration. Then $\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}$ is a Sullivan model of $E \times_B E$, the multiplication $\mu : \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S} \to \mathrm{S}$ is a model of the fibrewise diagonal $\Delta : E \to E \times_B E$ and $\Pi \otimes \Pi : \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S} \to \mathrm{R} \otimes_{\mathrm{R}} \mathrm{R} = \mathrm{R}$ is a chain level representative of fibre integration for $E \times_B E$. Since S is R-semifree, we can find a lift of R-modules

$$\begin{tikzcd} \mathrm{S}'[-d] \arrow[r, dashed, "\Delta_!"] \arrow[d, "\overline{\Pi}"', "\simeq"] & \mathrm{S}' \otimes_{\mathrm{R}} \mathrm{S}'[-2d] \arrow[d, "\overline{\Pi \otimes \Pi}"', "\simeq"] \\ \mathrm{Hom}_{\mathrm{R}}(\mathrm{S}', \mathrm{R}) \arrow[r, "\mu^*"] & \mathrm{Hom}_{\mathrm{R}'}(\mathrm{S}' \otimes_{\mathrm{R}} \mathrm{S}', \mathrm{R}) \end{tikzcd} \tag{4-3}$$

which is unique up to homotopy and therefore obtain a well-defined class

$$[\mu(\Delta_!(1)] \in H^d(\mathrm{S}). \tag{4-4}$$

Proposition 4.2 *Let $\pi : E \to B$ be an oriented fibration with connected base and total space and simply connected Poincaré fibre X of formal dimension d so that $\pi_1(B)$ acts trivially on $H^*(X)$. If $\mathrm{R} \to \mathrm{S}$ is a relative Sullivan model and $\Delta_! : \mathrm{S} \to \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}$ an Umkehr map as in* (4-3)*, then $\mu(\Delta_!(1)) \in \mathrm{S}^d$ is a representative of the fibrewise Euler class $e^{\mathrm{fw}}(\pi) \in H^d(E)$.*

Proof We first observe that the definition of the class in (4-4) is natural with respect to pullbacks: For a map $f : B' \to B$ with cdga model $\phi : \mathrm{R} \to \mathrm{R}'$, a model of the pullback $\pi : f^*E \to B'$ is given by $\mathrm{R}' \otimes_{\mathrm{R}} \mathrm{S}$ and a model on the map of total spaces is given by sending $\Phi(s) = 1 \otimes s \in \mathrm{R}' \otimes_{\mathrm{R}} \mathrm{S}$ for $s \in \mathrm{S}$ by [18, Section 20.6]. It follows from Proposition 3.2 that a cocycle representative of fibre integration is given by $\mathrm{R}' \otimes_{\mathrm{R}} \Pi : \mathrm{R}' \otimes_{\mathrm{R}} \mathrm{S} \to \mathrm{R}' \otimes_{\mathrm{R}} \mathrm{R} \cong \mathrm{R}'$ and therefore $\mathrm{R}' \otimes_B \Delta_!$ is a model of the Umkehr map so that (4-4) in this case is given by $[1 \otimes \mu(\Delta_!(1))] = \Phi\big(\mu(\Delta_!(1))\big) \in \mathrm{R}' \otimes_{\mathrm{R}} \mathrm{S}$.

Hence, it suffices to prove that (4-4) coincides with the class defined in (4-2) for fibrations where the base space is a Poincaré duality space (or even just for closed, stably framed manifolds). So let us assume that the base space is a closed manifold B or more generally a Poincaré duality space. Then $B \to *$ is a fibration with Poincaré fibre and we can apply the results of the previous section to get a chain level representative of fibre integration map $\Pi_B \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{R}, \mathbb{Q})$ corresponding to evaluating a fundamental class.

Suppose $\mathrm{R} \to \mathrm{S}$ is a Sullivan model of the fibration and let $\Pi \in \mathrm{Hom}_{\mathrm{R}}(\mathrm{S}, \mathrm{R})$ be a model of fibre integration of π. Then $([B] \cap \text{-}) \circ \pi_! : H^{b+d}(E) \to H^b(B) \to \mathbb{Q}$ is an orientation of the Poincaré algebra $H^*(E; \mathbb{Q})$ itself and therefore $\Pi_B \circ \Pi \in \mathrm{Hom}_{\mathbb{Q}}(\mathrm{S}, \mathbb{Q})$ is a cocycle representative. Define $\Pi_E := \Pi_B \circ \Pi$ so that

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}, \mathrm{R}) \arrow[rr, bend right, "(\Pi_B)_*"'] & \mathrm{S}[-d] \arrow[l, "\simeq", "\overline{\Pi}"'] \arrow[r, "\simeq", "\overline{\Pi}_E"'] & \mathrm{Hom}_{\mathbb{Q}}(\mathrm{S}), \mathbb{Q})
\end{tikzcd}$$

is a commuting diagram, where $(\Pi_B)_* f = \Pi_B \circ f$ for $f \in \mathrm{Hom}_R(S, R)$. We can choose chain level representatives of fibre integration of $E \times_B E \to B$ and $E \times_B E \to *$ as $\Pi \otimes \Pi : \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S} \to \mathrm{R} \otimes_{\mathrm{R}} \mathrm{R} \cong \mathrm{R}$ and $\Pi_{E \times_B E} := \Pi_B \circ (\Pi \otimes \Pi)$ by the same arguments as above. We therefore have a diagram

$$\begin{tikzcd}
\mathrm{S}[-d] \arrow[r, dashed, "\Delta_!"] \arrow[d, "\overline{\Pi}"', "\simeq"] \arrow[dd, bend right=60, "\overline{\Pi}_E"', "\simeq"] & \mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}[-2d] \arrow[d, "\simeq"', "\overline{\Pi \otimes \Pi}"] \arrow[dd, bend left=60, "\simeq"', "\overline{\Pi}_{E \times_B E}"] \\
\mathrm{Hom}_{\mathrm{R}}(\mathrm{S}, \mathrm{R}) \arrow[r, "\Delta^*"] \arrow[d, "(\Pi_B)_*"'] & \mathrm{Hom}_{\mathrm{R}}(\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}, \mathrm{R}) \arrow[d, "(\Pi_B)_*"] \\
\mathrm{Hom}_{\mathbb{Q}}(\mathrm{S}, \mathbb{Q}) \arrow[r, "\Delta^*"] & \mathrm{Hom}_{\mathbb{Q}}(\mathrm{S} \otimes_{\mathrm{R}} \mathrm{S}, \mathbb{Q})
\end{tikzcd}$$

where the dashed maps denote the Umkehr map from (4-3). The upper square commutes up to homotopy by construction and therefore so does the outer square by commutativity of the lower square. This shows that $\Delta_!$ is a cochain level representative of the Gysin map and thus $[\mu(\Delta_!(1))] \in H^d(S)$ agrees with (4-2). $\square$

4.2 The fibrewise Euler class of Leray–Hirsch fibrations

In the case of fibrations $\pi : E \to B$ with oriented Poincaré fibre (X, ε_X) which are Leray–Hirsch, i.e., where the restriction map $H^*(E) \to H^*(X)$ is surjective, the definition of fibre integration and the fibrewise Euler class can be simplified significantly. Surjectivity of the restriction map implies that $H^*(E)$ is a free $H^*(B)$-module, and we may denote by $1, e_1, \dots, e_k \in H^*(E)$ a $H^*(B)$-basis of the cohomology of E that restricts to a basis $1, x_1 = i^*(e_1), \dots, x_k = i^*(e_k) \in H^*(X)$ of the fibre. If X is a Poincaré complex of formal dimension d, we can order the basis such that $|e_k| = d$ and all other $|e_i|$ have lower degree. Since fibre integration is a $H^*(B)$-module map. It suffices to determine $\pi_!$ on a basis and for degree reasons $\pi_!(e_i) = 0$ for $i < k$. If we set $\pi_!(e_k) = \varepsilon_X(x_k)$ this restricts to the orientation on the fibre hence determines fibre integration as

$$\pi_!\Big(\sum_{i=0}^{k} b_i \cdot e_i\Big) = \varepsilon_X(x_k) \cdot b_k \tag{4-5}$$

for $b_i \in H^*(B)$. Since the fibre is Poincaré, the (fibrewise) intersection pairing

$$\langle \text{-}, \text{-} \rangle : H^*(E) \otimes_{H^*(B)} H^*(E) \xrightarrow{\pi_!(\text{-} \cup \text{-})} H^*(B) \tag{4-6}$$

is nondegenerate. This enables us to mimic the construction of the Euler class as the dual of the fibrewise diagonal.

Proposition 4.3 [27] *Let $\pi : E \to B$ be an oriented fibration with Poincaré fibre which is Leray–Hirsch. Let $e_0, \dots, e_k \in H^*(E)$ be an $H^*(B)$-module basis and denote by $e_0^\#, \dots, e_k^\# \in H^*(E)$ the dual basis under the nondegenerate pairing* (4-6). *Then the fibrewise Euler class is*

$$e^{\mathrm{fw}}(\pi) = \sum_{i=0}^{k} (-1)^{|e_i|} e_i e_i^\# \in H^d(E;\mathbb{Q}). \tag{4-7}$$

Example 4.4 Let $X = S^{2n}$ be an even-dimensional sphere and recall the cdga model and fibre integration from the Examples 2.4 and 3.4. Then 1 and x are a $H^*(\mathrm{BhAut}_0(S^{2n});\mathbb{Q})$-basis of the cohomology of the total space that restricts to a basis of $H^*(S^{2n})$ on the fibre, i.e., the fibration is Leray–Hirsch. Note that the formula for fibre integration in (4-5) gives the same result as our construction of Π in Example 3.4. We can apply the above proposition to find a representative of the fibrewise Euler class. The dual basis with respect to the pairing induced by $\pi_! = H(\Pi)$ is $x^\# = 1$ and $1^\# = x$ (since $\pi_!(x \cdot 1^\#) = \pi_!(x^2) = 0$), and we find that the fibrewise Euler class is represented by $e^{\mathrm{fw}}(\pi) = 2x$.

4.3 Fibrations with positively rationally elliptic fibre

A simply connected space X is called *rationally elliptic* if both $\dim H(X;\mathbb{Q})$ and $\dim \pi(X) \otimes \mathbb{Q}$ are finite-dimensional vector spaces. If moreover $\chi(X) > 0$ then X is called *positively rationally elliptic*. Algebraic models for (positively) elliptic spaces are quite rigid. For example, one can show that any positively rationally elliptic space satisfies rational Poincaré duality. The main result in this section is a simple, closed formula for the fibrewise Euler class of a fibration with positively rationally elliptic fibre.

The minimal Sullivan model of a rationally elliptic space $\Lambda = (\Lambda V, d)$ is free on a finite-dimensional vector space V. Hence, the dg Lie algebra model $\mathrm{Der}^+(\Lambda)$ for $\mathrm{BhAut}_0(X)$ is finite dimensional which makes the study of fibrations with rationally elliptic fibres tractable via Theorem 2.2. Moreover, a famous conjecture due to Halperin states that any fibration with positively elliptic fibre (and trivial holonomy action) is Leray–Hirsch. This conjecture is known to be true for a large number of examples [22; 28; 34] and by [22] it is equivalent to $\pi_{2i-1}(\mathrm{BhAut}_0(X)) \otimes \mathbb{Q} = 0$ for all $i \in \mathbb{N}$. Since $\pi_{2i-1}(\mathrm{BhAut}_0(X)) \otimes \mathbb{Q}$ is isomorphic to $H_{2i}(\mathrm{Der}^+(\Lambda), [d,-])$, this condition can easily be checked in examples.

We recall a few results about positively rationally elliptic space from [11, Section 32]. A pure Sullivan algebra is a cdga $\Lambda = (\Lambda V, d)$ with $d(V^{\mathrm{even}}) = 0$ and $d(V^{\mathrm{odd}}) \subset \Lambda V^{\mathrm{even}}$, and we define $P = V^{\mathrm{odd}}$ and $Q = V^{\mathrm{even}}$. Observe that pure Sullivan algebras Λ are bigraded with additional lower grading given by $\Lambda_k = \Lambda Q \otimes \Lambda^k P$. By [11, Proposition 32.10] a minimal Sullivan model of a positively rationally elliptic space is isomorphic to a pure Sullivan algebra with $\dim P = \dim Q$ and so that $d|_P$ maps a basis of P to a regular sequence of the graded polynomial ring ΛQ. In this case, the cohomology is concentrated in lower grading 0, i.e., $H^*(\Lambda) = H_0^*(\Lambda)$.

This bigrading is inherited by the dg Lie algebra $\mathrm{Der}^+_{*,*}(\Lambda)$ where a derivation θ has bidegree (m,n) if θ lowers the internal degree by $|m|$ and $\theta(\Lambda Q \otimes \Lambda^k P) \subset \Lambda Q \otimes \Lambda^{k+n} P$. The following statement has been explained to the author by Alexander Berglund.

Lemma 4.5 *Let Λ be a pure Sullivan model for a positively rationally elliptic space that satisfies the Halperin conjecture. There exists an abelian dg Lie algebra with trivial differential $\mathfrak{a} \subset \mathrm{Der}^+_{*,-1}(\Lambda)$ that is quasi-isomorphic to $\mathrm{Der}^+(\Lambda)$.*

Proof If Λ is a pure Sullivan model for a positively rationally elliptic space, the projection map $\psi : \Lambda \to H(\Lambda)$ is a quasi-isomorphism. It induces a quasi-isomorphism of chain complexes $\mathrm{Der}^+(\Lambda) \to \mathrm{Der}^+_{\psi}(\Lambda, H(\Lambda))$ (analogous to [3, Lemma 3.5]). Since the $H(\Lambda) = H_0(\Lambda)$, we see that $H_{*,>0}(\mathrm{Der}^+(\Lambda))$ is contained in the kernel and thus is trivial. The Halperin conjecture implies that $H_*(\mathrm{Der}^+(\Lambda))$ is concentrated in odd degrees and therefore $H_*(\mathrm{Der}^+(\Lambda)) \cong H_{\mathrm{odd},-1}(\mathrm{Der}^+\,\Lambda)$ as claimed. Hence, any choice of representatives in $\mathrm{Der}^+_{*,-1}(\Lambda)$ of a basis of $H_*(\mathrm{Der}^+(\Lambda))$ spans an abelian dg Lie subalgebra with trivial differential that is quasi-isomorphic to $\mathrm{Der}^+(\Lambda)$. □

Halperin's conjecture implies that $\mathrm{BhAut}_0(X) \simeq_{\mathbb{Q}} \prod_{i=1}^k K(\mathbb{Q}, 2n_i)$ and so the cohomology of the total space E of (2-1) is a free module on $H^*(X;\mathbb{Q})$ over a positively and evenly graded polynomial ring. Moreover, it follows from Lemma 4.5 and Theorem 2.2 that as a ring $H^*(E)$ is a complete intersection over $H^*(\mathrm{BhAut}_0(X);\mathbb{Q})$ (see proof of Theorem 4.7). Here, by a *complete intersection* over a commutative ring R we mean a finite R-algebra S that is isomorphic to $\mathrm{R}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ for $f_1, \dots, f_n \in \mathrm{R}[x_1, \dots, x_n]$ (see [29]).

For a finite R-algebra S, we can define the trace of an endomorphism $\mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{S}) \cong \mathrm{S} \otimes_{\mathrm{R}} \mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{R})$ via the evaluation $\mathrm{S} \otimes_{\mathrm{R}} \mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{R}) \to \mathrm{R}$. In particular, we can associate to any $s \in \mathrm{S}$ the trace of the endomorphism $s \cdot - : \mathrm{S} \to \mathrm{S}$ and we obtain an element $\mathrm{Tr}_{\mathrm{S}/\mathrm{R}} \in \mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{R})$. We will use the following result about complete intersections.

Proposition 4.6 [29] *Let* R *be a commutative ring and $f_1, \dots, f_n \in \mathrm{R}[x_1, \dots, x_n]$ for a nonnegative integer n. Assume that $\mathrm{S} = \mathrm{R}[x_1 \dots, x_n]/(f_1, \dots, f_n)$ is a finite* R*-algebra. Then*

(i) S *is a projective* R*-module*;[2]

(ii) $\mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{R})$ *is a free of rank* 1 *as an* S*-module*;

(iii) *there is a generator λ of $\mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{R})$ as an* S*-module such that* $\mathrm{Tr}_{\mathrm{S}/\mathrm{R}} = \det(\partial f_i/\partial x_j) \cdot \lambda$.

We recognise (iii) as an analogue for complete intersections of the relation between fibre integration, the fibrewise Euler class and the Becker–Gottlieb transfer $\tau_\pi : \Sigma^\infty B_+ \to \Sigma^\infty E_+$ for fibrations with Poincaré fibres. The transfer map induces a map on cohomology $\mathrm{trf}^*_\pi : H^*(E) \to H^*(B)$ and if the fibre is a Poincaré complex one can show that

$$\mathrm{trf}^*_\pi(x) = \pi_!(e^{\mathrm{fw}}(\pi) \cdot x)$$

for all $x \in H^*(E)$. If we consider the universal 1-connected fibration (2-1) of a positively rationally elliptic space that satisfies the Halperin conjecture, its algebraic model $\mathrm{R} \to \mathrm{S}$ is equivalent to a complete intersection over a polynomial ring and we can identify the Becker–Gottlieb transfer and fibre integration with the trace $\mathrm{Tr}_{\mathrm{S}/\mathrm{R}}$ and λ, respectively, which therefore leads to an identification of the fibrewise Euler class.

[2] In our applications, R is a positively graded polynomial ring so that S is in fact free.

Theorem 4.7 *Let X be a simply connected, oriented Poincaré duality space that is positively rationally elliptic and satisfies the Halperin conjecture. Then a cdga model of* (2-1) *is given by a complete intersection* $\mathrm{S}=\mathrm{R}[x_1,\dots,x_n]/(f_1,\dots,f_n)$ *over a polynomial ring* R *and the fibrewise Euler class is given by*

$$e^{\mathrm{fw}}(\pi)=\det\left(\frac{\partial f_i}{\partial x_j}\right)\in \mathrm{S}. \tag{4-8}$$

Remark 4.8 The first part of the theorem improves a result by Kuribayashi [20, Theorem 1.1] where he showed that the cohomology ring of the total space of the universal 1-connected fibration (2-1) for positively rationally elliptic spaces that satisfy the Halperin conjecture is a complete intersection over the cohomology of the base.

Proof Let $\Lambda=(\mathbb{Q}[x_1,\dots,x_n]\otimes\Lambda(y_1,\dots,y_n),d)$ be a pure Sullivan model of X with differential $d(y_i)=\bar{f_i}\in\mathbb{Q}[x_1,\dots,x_n]$ where $\bar{f_1},\dots,\bar{f_n}$ is a regular sequence. The cohomology ring $H(\Lambda)$ is a Poincaré duality algebra and $\det(\partial\bar{f_i}/\partial x_j)\in H^*(X)$ is a generator in top degree [24; 28], so that the orientation is defined by $\varepsilon_X(\det(\partial\bar{f_i}/\partial x_j)):=\chi(X)$. Now let $\mathfrak{a}\subset\mathrm{Der}^+_{*,-1}(\Lambda)$ be an abelian dg Lie subalgebra quasi-isomorphic to $\mathrm{Der}^+(\Lambda)$ from Lemma 4.5. Then a model of the universal 1-connected fibration is given by

$$\mathrm{R}:=\mathscr{C}_{\mathrm{CE}}(\mathfrak{a})\to\mathscr{C}_{\mathrm{CE}}(\mathfrak{a};\Lambda)\cong(\mathrm{R}\otimes\Lambda(x_i,y_i),D)$$

by Theorem 2.2. And because $\mathfrak{a}\subset\mathrm{Der}^+_{*,-1}(\Lambda)$, we see that $D(x_i)=0$ and $D(y_i)=f_i\in\mathrm{R}[x_1,\dots,x_n]$ are polynomials that satisfy $\bar{f_i}=f_i\otimes_{\mathrm{R}}1\in\mathrm{R}[x_1,\dots,x_n]\otimes_{\mathrm{R}}\mathbb{Q}$. This implies $\mathrm{S}:=\mathrm{R}[x_1,\dots,x_n]/(f_1,\dots,f_n)$ is a complete intersection over R and the projection map $\mathscr{C}_{\mathrm{CE}}(\mathfrak{a};\Lambda)\to\mathrm{S}$ is a quasi-isomorphism.

One can show that the trace $\mathrm{Tr}_{\mathrm{S}/\mathrm{R}}$ and the transfer tr^*_π agree using the general theory in [9] or by directly checking that

$$\mathrm{Tr}_{\mathrm{S}/\mathrm{R}}(x)=\pi_!(e^{\mathrm{fw}}(\pi)\cdot x), \tag{4-9}$$

as in [27, Lemma 2.3] using that S is a finite free R-algebra. Now consider λ from Proposition 4.6. Then $\lambda(\det(\partial f_i/\partial x_j))=\mathrm{Tr}_{\mathrm{S}/\mathrm{R}}(1)=\chi(X)$ by (4-9) which agrees with $\pi_!(\det(\partial f_i/\partial x_j))=\varepsilon_X(\det(\partial\bar{f_i}/\partial x_j))$. Therefore, it follows from degree reasons that $\pi_!=\lambda$ and we obtain (4-8) from the identity

$$\det(\partial f_i/\partial x_j)\cdot\pi_!=\det(\partial f_i/\partial x_j)\cdot\lambda=\mathrm{Tr}_{E/B}=\mathrm{trf}^*_\pi=e^{\mathrm{fw}}(\pi)\cdot\pi_!$$

since $\mathrm{Hom}_{\mathrm{R}}(\mathrm{S},\mathrm{R})$ is a free S-module by (ii). □

5 Computations

Before we come to the computation of the Euler ring, we record a few general facts. First we observe that in some cases it is sufficient to compute the Euler ring of the universal 1-connected fibration $E_0(X)\subset H^*(\mathrm{BhAut}_0(X))$.

Lemma 5.1 *If $\pi_0(\mathrm{hAut}^+(X))$ is finite, then $i:\mathrm{BhAut}_0(X)\to\mathrm{BhAut}^+(X)$ induces an isomorphism $E^*(X)\cong E^*_0(X)$.*

Proof By naturality of the fibrewise Euler class, $i : \mathrm{BhAut}_0(X) \to \mathrm{BhAut}^+(X)$ induces a surjection of Euler rings $E^*(X) \twoheadrightarrow E_0^*(X)$. It follows from the spectral sequence associated to the fibre sequence

$$\mathrm{BhAut}_0(X) \to \mathrm{BhAut}^+(X) \to \mathrm{B}\pi_0(\mathrm{hAut}^+(X))$$

that $H^*(\mathrm{BhAut}(X);\mathbb{Q}) \cong H^*(\mathrm{BhAut}_0(X);\mathbb{Q})^{\pi_0(\mathrm{hAut}^+(X))}$ if $\pi_0(\mathrm{hAut}^+(X))$ is finite. In particular, i induces an injection on rational cohomology. $\square$

For a positively rationally elliptic space there is a simple criterion to check if it has finitely many homotopy automorphisms.

Proposition 5.2 *Let X be a simply connected Poincaré duality space that is positively rationally elliptic space. If the cohomology ring $H^*(X;\mathbb{Q})$ has finitely many orientation-preserving automorphisms (i.e., algebra automorphisms that preserve the orientation $\varepsilon_X \in H^d(X;\mathbb{Q})^\vee$), then $\pi_0(\mathrm{hAut}^+(X))$ is finite.*

Proof The statement follows if the kernel of $\mathscr{E}(X) \to \mathrm{Aut}(H^*(X;\mathbb{Q}), \varepsilon_X)$ is finite. Rationalisation induces a map $\mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}})$ which has finite kernel [32, Theorem 10.2], so that it suffices to prove that the subgroup of $\mathscr{E}(X_{\mathbb{Q}})$ of homotopy equivalences that induce the identity on cohomology is trivial. By [32, 10.3], homotopy automorphism of $X_{\mathbb{Q}}$ are the same as homotopy classes of automorphisms of a minimal Sullivan model.

For a positively rationally elliptic space we can choose $\Lambda = \bigl(\Lambda(x_i, y_i)_{i=1,\dots,n}, d = \sum_i f_i \partial y_i\bigr)$ as a model where $f_1, \dots, f_n \in \Lambda(x_i)_{i=1,\dots,n}$ is a regular sequence. Let $\phi : \Lambda \to \Lambda$ be an automorphism so that $H(\phi) = \mathrm{Id}$. Then $[x_i - \phi(x_i)] = 0$ and we can pick a coboundary $\xi_i \in \Lambda_{>0}$ satisfying $d\xi_i = x_i - \phi(x_i) \in (f_1, \dots, f_k) + \Lambda_{>0}$. We want to define a homotopy $H : \Lambda \to \Lambda \otimes \Lambda(t, dt)$ by setting $H(x_i) = \phi(x_i) + t(x_i - \phi(x_i)) - \xi_i dt$. Then $dH(x_i) = 0 = Hd(x_i)$ and $\varepsilon_1 \circ H(x_i) = x_i$ and $\varepsilon_0 \circ H(x_i) = \phi(x_i)$. It remains to define $H(y_i)$, i.e., we have to find a coboundary for $f_i(H(x_1), \dots, H(x_n)) \in \Lambda \otimes \Lambda(t, dt)$. Observe that $\Lambda \otimes \Lambda(t, dt)$ is a positively elliptic Sullivan algebra as well since $f_1, \dots, f_n, t$ is a regular sequence in $\mathbb{Q}[x_1, \dots, x_n, t]$, and $f_i(H(x_1), \dots, H(x_n)) - d\phi(y_i) \in (\Lambda \otimes \Lambda^+(t, dt)) \oplus (\Lambda \otimes \Lambda(t, dt))_{>0}$. Since both summands are acyclic, there is $\zeta_i \in \Lambda \otimes \Lambda(t, dt)$ so that $d\zeta_i = f_i(H(x_1), \dots, H(x_n)) - d\phi(y_i)$, and we can set $H(y_i) := \zeta_i + \phi(y_i)$. This shows ϕ is homotopic to a map which is the identity on $\Lambda(x_i)$.

Given an automorphism $\psi : \Lambda \to \Lambda$ with $\psi(x_i) = x_i$, then $d\psi(y_i) = \psi(f_i) = f_i = d(y_i)$ and hence $y_i - \psi(y_i)$ is a cocycle in $\Lambda_{>0}$. Let ξ_i be a coboundary $d\xi_i = y_i - \psi(y_i)$. Then $\psi \simeq_H \mathrm{Id}_\Lambda$ via $H(x_i) := x_i$ and $H(y_i) := \psi(y_i) + t(y_i - \psi(y_i)) - \xi_i dt$. $\square$

Corollary 5.3 *Let X be a simply connected Poincaré duality space with $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$ for some n and $|x|$ even. Then $\pi_0(\mathrm{hAut}^+(X))$ is finite.*

Proof For any choice of orientation of X, the group $\mathrm{Aut}(H^*(X;\mathbb{Q}), \varepsilon_X)$ is trivial if n is odd and $\mathbb{Z}/2$ if n is even. Hence, $\pi_0(\mathrm{hAut}^+(X))$ is finite by Proposition 5.2 $\square$

Finally, in some cases the Euler ring is finitely generated so that the computation of the Euler ring amounts to computing the ideal of relations among a generating set, and which simplifies some computations below.

Proposition 5.4 *Let X be a Poincaré duality space so that $H^*(X)$ is concentrated in even degrees and let $n = \dim H^*(X;\mathbb{Q})$. Then $E^*(X)$ is generated by $\kappa_1, \dots, \kappa_{n-2}, \kappa_n$.*

Proof This follows directly from the tools developed in [27]. Applying [27, Corollary 2.7] for $x = e^{\mathrm{fw}}(\pi)$, the corresponding monic polynomial $\rho_x(z) \in H^*(\mathrm{BhAut}^+(X);\mathbb{Q})[z]$ of degree n is given by

$$\rho_x(z) = \frac{(-1)^n}{n!} \sum_{\sigma \in \Sigma_{n+1}} \mathrm{sgn}(\sigma) \kappa_{l(\gamma_2)} \cdots \kappa_{l(\gamma_{q(\sigma)})} \cdot z^{l(\gamma_1)-1},$$

where $\sigma = \gamma_1 \cdots \gamma_{q(\sigma)}$ is the cycle decomposition of σ, $l(\gamma_i)$ denotes the length of γ_i and 1 is contained in the support of γ_1. Then [27, Corollary 2.7] implies that $\rho_x(e^{fw}(\pi)) = 0 \in H^*(E)$ and by fibre integrating $0 = e^{\mathrm{fw}}(\pi)^i \cdot \rho(e^{\mathrm{fw}}(\pi))$, one can decompose κ_{n+i} in terms of κ-classes of lower degree for $i \geq 0$; except when $i = 1$ as $\rho_x(z)$ has a constant term$-\kappa_n/n$ which cancels the leading term $\pi_!(e^{\mathrm{fw}}(\pi)^{n+1})$ in $\pi_!\big(e^{\mathrm{fw}}(\pi)\rho_x(e^{\mathrm{fw}}(\pi))\big)$. □

5.1 The Euler ring of even spheres

Proposition 5.5 *The Euler ring of an even-dimensional sphere is $E^*(S^{2n}) \cong \mathbb{Q}[\kappa_2]$ where $\kappa_2^k = 2^{k-1}\kappa_{2k}$ and all odd κ_{2i+1} vanish.*

Proof We have seen in Example 4.4 that $e^{\mathrm{fw}}(\pi) = 2x \in \Lambda(x, y, z_{4n})$ and in Example 3.4 that fibre integration is given by $\Pi(x^{2k}) = 0$ and $\Pi(x^{2k+1}) = (-1)^k z_{4n}^k$. Thus $\kappa_{2k} = 2^{1-k}(-2^3 z_{4n})^k = 2^{1-k}\kappa_2^k$. Since $\pi_0(\mathrm{hAut}^+(S^{2n}))$ is trivial, the 1-connected universal fibration is the universal fibration and the result follows. □

5.2 The Euler ring of complex projective space

A minimal model of $\mathbb{C}P^n$ is

$$P_n := (\Lambda(x, y), |x| = 2, |y| = 2n+1, d = x^{n+1}\partial/\partial y)$$

with orientation $\varepsilon_{\mathbb{C}P^n}(x^n) = 1$ induced by integral Poincaré duality, and we apply Theorem 2.2 to compute a model for the minimal 1-connected $\mathbb{C}P^n$-fibration.

Proposition 5.6 *A cdga model of the universal 1-connected $\mathbb{C}P^n$-fibration is given by*

$$\text{(5-1)} \quad B_n := (\mathbb{Q}[x_2, \dots, x_{n+1}], |x_i| = 2i, d = 0) \xrightarrow{\pi} E_n := \left(B_n[x]\Big/\left(x^{n+1} + \sum_{i=2}^{n+1} x_i \cdot x^{n+1-i}\right), |x| = 2, d = 0\right),$$

and the fibrewise Euler class in E_n is represented by

$$e^{\mathrm{fw}}(\pi) = (n+1) \cdot x^n + \sum_{i=2}^{n} (n+1-i) \cdot x_i \cdot x^{n-i} \in E_n. \tag{5-2}$$

Proof Note that $\mathrm{Der}^+(P_n)$ has a (vector space) basis given by $\theta_i := x^{n+1-i}\partial/\partial y$ for $i = 1,\dots,n+1$ of degree $2i-1$ and $\eta := \partial/\partial x$ of degree 2. The only nontrivial differential on the derivation Lie algebra is given by $[d,\eta] = -(n+1)\theta_1$. Since $\mathbb{C}P^n$ satisfies the Halperin conjecture, it follows from Lemma 4.5 that there exists a quasi-isomorphic abelian dg Lie subalgebra $\mathfrak{a}_n \subset \mathrm{Der}^+(P_n)$ with trivial differential, which in this case is easy to identify as $\mathfrak{a}_n := \mathbb{Q}\{\theta_2,\dots,\theta_{n+1}\}$. The statement follows directly from Theorem 4.7 for this choice of $\mathfrak{a}_n$. □

Theorem 5.7 *The Euler ring of complex projective space is* $E^*(\mathbb{C}P^n) \cong \mathbb{Q}[\kappa_1,\dots,\kappa_{n-1},\kappa_{n+1}]$.

Proof By Corollary 5.3, $\pi_0(\mathrm{hAut}^+(\mathbb{C}P^n)$ is finite (in fact one can easily see $\pi_0(\mathrm{hAut}(\mathbb{C}P^n)) \cong \mathbb{Z}/2$), and so $E^*(\mathbb{C}P^n) \cong E^*_0(\mathbb{C}P^n)$ by Lemma 5.1. The Euler ring is generated by $\kappa_1,\dots,\kappa_{n-1},\kappa_{n+1}$ by Proposition 5.4, so it remains to show they are algebraically independent which follows if $\det(\partial\kappa_i/\partial x_j)$ is nonzero.

It turns out that the polynomials representing the κ_i are quite complicated so that it is difficult to give a closed formula for the determinant of the Jacobian. We will resolve this issue by focussing on the terms containing x_{n+1} because it is the variable of the highest degree and it is not contained in $e^{\mathrm{fw}}(\pi)$ so that it only arises through fibre integrating x^k for $k > n$. It will be sufficient to consider elements modulo decomposables, i.e., for $x, y \in B_n$ then $x \sim y$ if $x - y \in (B_n^+)^2$. We will start with the following observation about fibre integration.

Observation 1 *If* $k = 2,\dots,n+1$ *then* $\pi_!(x^{n+k}) \sim -x_k \in B_n$.

Proof Rewriting x^{n+2} in terms of the module basis $\{1, x, \dots, x^n\}$, one can see that $\pi_!(x^{n+2}) = x_2$. Then $\pi_!(x^{n+k}) = \pi_!(x^{n+1}\cdot x^{k-1}) = -\sum_{i=2}^{n+1} x_i \cdot \pi_!(x^{n+k-i})$ and by induction over k, the only indecomposable contribution is for $i = k$. □

Observation 2 *For* $i = 1,\dots,n-1$ *the highest power of* x_{n+1} *in* $\kappa_i(x_2,\dots,x_{n+1})$ *is* $i-1$ *and the coefficient* $c_i \in \mathbb{Q}[x_2,\dots,x_n]$ *of* x_{n+1}^{i-1} *satisfies* $c_i \sim (-1)^i i(n+1)^i(n-i)\cdot x_{n+1-i}$.

Proof It follows from degree considerations that the highest power of x_{n+1} is $i-1$ and c_i equals $A\cdot x_{n+1-i} +$ decomposables. It remains to determine the coefficient A. When expanding $e^{\mathrm{fw}}(\pi)$ using (5-2), the only relevant contributions are

$$\begin{aligned}(n+1)^{i+1}x^{n(i+1)} - (i+1)(n+1-(n+1-i))x_{n+1-i}x^{n-(n+1-i)}\cdot(n+1)^i x^{ni}\\ = (n+1)^{i+1}(x^{n+1})^{i-1}\cdot x^{2n-i+1} - i(i+1)(n+1)^i x_{n+1-i}\cdot(x^{n+1})^{i-1}\cdot x^n.\end{aligned}$$

Now rewrite $x^{n+1} = \sum_{i=2}^{n+1} x_i x^{n+1-i}$ and collect all terms containing x_{n+1}^{i-1} and $x_{n+1}^{i-2}x_{n+1-i}$ (we can ignore the rest because it cannot contribute to A) to get

$$(n+1)^{i+1}(x_{n+1}^{i-1} + (i-1)x_{n+1}^{i-2}x_{n+1-i}\cdot x^i)\cdot x^{2n-i+1} - i(i+1)(n+1)^i x_{n+1-i}\cdot x_{n+1}^{i-1}\cdot x^n.$$

The statement follows by fibre integrating and discarding decomposables as in Observation 1 above. □

Observation 3 *The highest contribution of* x_{n+1} *in* κ_{n+1} *is* $(-1)^n(n+1)^{n+2}\cdot x_{n+1}^n$.

Proof The expression for $e^{\mathrm{fw}}(\pi)^{n+2}$ contains $(n+1)^{n+2}\cdot x^{n(n+2)} = (n+1)^{n+2}\cdot (x^{n+1})^n\cdot x^n$. This is the only summand that fibre integrates to a multiple of x_{n+1}^n, i.e., $\kappa_{n+1} = (-1)^n(n+1)^{n+2}\cdot x_{n+1}^n + \cdots$ where we can ignore all other terms. □

We can now analyse $\det(\partial\kappa_i/\partial x_j)$ which contains the summand

$$\frac{\partial\kappa_1}{\partial x_n}\cdot\frac{\partial\kappa_2}{\partial x_{n-1}}\cdots\frac{\partial\kappa_{n-1}}{\partial x_2}\cdot\frac{\partial\kappa_{n+1}}{\partial x_{n+1}}.$$

It follows from Observations 2 and 3 that the above expression contains $C\cdot x_{n+1}^N$, where C is a nonzero constant and $N=\frac{1}{2}n(n-1)$. This is the only possible way to get a monomial in $\det\bigl(\frac{\partial\kappa_i}{\partial x_j}\bigr)$ that contains only x_{n+1}. Hence, the determinant does not vanish and the generating set $\kappa_1,\dots,\kappa_{n-1},\kappa_{n+1}$ is algebraically independent. □

Remark 5.8 Theorem 5.7 has been studied in the smooth case for $n=2$ in [27] by studying the natural smooth 2-torus action on $\mathbb{C}P^2$ and that implies our result in this case as well. This has been extended by Dexter Chua to $n\le 4$, but for large n the algebra becomes intractable.

5.3 The Euler ring of products of odd spheres

The main result of this section is the computation of $E_0^*(X)$ for a simply connected Poincaré duality space X that is rationally equivalent to a product of odd spheres $\prod_{i=1}^{2n} S^{2k_i+1}$ for some $n,k_i>0$.

Theorem 5.9 *Let X be a simply connected Poincaré duality space that is rationally equivalent to a product of odd spheres $\prod_{i=1}^{2n} S^{2k_i+1}$ for some $n,k_i>0$. Then the fibrewise Euler class of the universal 1-connected X-fibration $E\to \mathrm{BhAut}_0(X)$ is trivial and hence $E_0^*(X)\cong\mathbb{Q}$.*

Proof The minimal model of X is given by an exterior algebra $A_X=(\Lambda(x_i)_{1\le i\le 2n}, d=0)$ and by Theorem 2.2 the model of the universal 1-connected fibration is given by

$$B_X := \mathscr{C}^*_{\mathrm{CE}}(\mathrm{Der}^+(A_X);\mathbb{Q}) \to E_X := \mathscr{C}^*_{\mathrm{CE}}(\mathrm{Der}^+(A_X);A_X)\cong (B_X\otimes A_X, D). \tag{5-3}$$

Observe that $\mathscr{C}^*_{\mathrm{CE}}(\mathrm{Der}^+(A_X);A_X)$ is a finitely generated B_X-module as the minimal Sullivan model is finite dimensional. Let $\varepsilon_X : A_X\to\mathbb{Q}$ denote a rational orientation, then $B_X\otimes\varepsilon_X : (B_X\otimes A_X, D)\to B_X$ is the only module homomorphism that restricts to ε_X on the fibre and thus is a representative of fibre integration by Proposition 3.2.

Moreover, $\mathrm{Hom}_{B_X}(E_X,B_X)$ is a minimal semifree module and therefore the quasi-isomorphism $\overline{\Pi}: E_X[-d]\to \mathrm{Hom}_{B_X}(E_X,B_X)$ from Proposition 4.1 is in fact an isomorphism by uniqueness of minimal free resolutions [11, Example 8, Chapter 6] (similarly for $\overline{\Pi\otimes\Pi}$). Hence, the algebraic Umkehr map is given by

$$\Delta_!: E_X[-|F|] \xrightarrow[\cong]{\overline{\Pi}} \mathrm{Hom}_{B_X}(E_X,B_X) \xrightarrow{\Delta^*} \mathrm{Hom}_{B_X}(E_X\otimes_{B_X}E_X, B_X) \xrightarrow[\cong]{(\overline{\Pi\otimes\Pi})^{-1}} E_X\otimes_{B_X} E_X[-2|F|].$$

The composition of $\overline{\Pi}$ with the vector space isomorphism $\mathrm{Hom}_{B_X}(E_X,B_X)\cong (A_X)^\vee\otimes B_X$ is given by $\bar{\varepsilon}_X\otimes\mathrm{Id}_{B_X}$ where $\bar{\varepsilon}_X : A_X\to (A_X)^\vee$ is the adjoint of $\varepsilon_X : A_X\otimes A_X\to\mathbb{Q}$. The same statement holds for

$\overline{\Pi \otimes \Pi}$ with the appropriate choice of orientation on $X \times X$ given by $\varepsilon_{X\times X} := \varepsilon_X \otimes \varepsilon_X : (A_X \otimes A_X)^{\otimes 2} \to \mathbb{Q}$. Note that $\Delta^* \overline{\Pi}(1)$ is contained in $(A_X \otimes A_X)^\vee \otimes 1$ so that $(\overline{\Pi \otimes \Pi})^{-1} \Delta^* \overline{\Pi}(1)$ is in $A_X \otimes A_X \otimes 1 \subset E_X \otimes_{B_X} E_X$. A direct computation shows that

$$\Delta_!(1) = \sum_{S_1 \sqcup S_2 = F} \pm x_{S_1} \otimes x_{S_2} \in E_X \otimes_{B_X} E_X$$

for some signs that can be worked out. Hence, the fibrewise Euler class is

$$e^{\mathrm{fw}}(\pi) = \Delta^* \circ \Delta_!(1) = \sum_{S_1 \sqcup S_2 = F} \pm x_F$$

and since $\Pi(e^{fw}(\pi)) = \chi(X) = 0$, the summands must cancel. $\square$

One can easily see that the group of homotopy self-equivalences $\pi_0(\mathrm{hAut}^+(X))$ is not finite in most cases, and so we cannot infer that the full Euler ring is trivial in general except in the two cases below.

Theorem 5.10 *Let X be either rationally equivalent to $(S^{2k+1})^{\times n}$ or a finite CW complex rationally equivalent to $S^{2k+1} \times S^{2l+1}$ for $1 < k < l$ and n even. Then $E^*(X) = \mathbb{Q}$.*

Proof We start with the proof of the second case. By [32, Theorem 10.3] the group $\pi_0(\mathrm{hAut}^+(X))$ is commensurable with an arithmetic subgroup of the homotopy classes automorphisms of A_X. If $X \simeq_{\mathbb{Q}} S^{2k+1} \times S^{2l+1}$ then the group of automorphisms of A_X modulo homotopy is $\mathbb{Q}^\times \times \mathbb{Q}^\times$ and the arithmetic subgroups of this linear algebraic group are finite and hence by commensurability so is $\pi_0(\mathrm{hAut}^+(X))$. Hence, $E^*(X) \cong E_0^*(X) = \mathbb{Q}$ by Lemma 5.1 and Theorem 5.9.

We need to introduce some notation in the first case. Let $\pi : E \to \mathrm{BhAut}^+(X)$ denote the universal orientated X-fibration and $\pi_0 : E_0 \to \mathrm{BhAut}_0(X)$ the universal 1-connected X-fibration. The cdga model of the total space E_0 is a free algebra on generators $x_1, \dots, x_n, y^1, \dots, y^n$ with differential $D(x_i) = y^i$ and therefore $H(E_0) = \mathbb{Q}$. It is well known that E is homotopy equivalent to the classifying space of pointed homotopy automorphisms $\mathrm{BhAut}_*^+(X)$ and there is a fibration sequence $E_0 \to E \xrightarrow{H} \mathrm{B}\pi_0(\mathrm{hAut}_*^+(X))$ which induces an isomorphism $H^*(E) \cong H^*(\mathrm{B}\pi_0(\mathrm{hAut}_*^+(X)))$ since the rational cohomology of E_0 is trivial. In particular, $e^{\mathrm{fw}}(\pi) = H^* e$ for some $e \in H^{n\cdot(2k+1)}\big(\mathrm{B}\pi_0(\mathrm{hAut}_*^+(X))\big)$. It follows from the commutative diagram

$$\begin{tikzcd} E \arrow[r, "H", "\simeq_{\mathbb{Q}}"'] \arrow[d, "\pi"] & \mathrm{B}\pi_0(\mathrm{hAut}_*^+(X)) \arrow[d, equal] \\ \mathrm{BhAut}^+(X) \arrow[r, "h"] & \mathrm{B}\pi_0(\mathrm{hAut}^+(X)) \end{tikzcd}$$

that the fibrewise Euler class $e^{\mathrm{fw}}(\pi) = H^* e = \pi^* h^* e$ is pulled back from the base, and therefore that the fibre integrals $\pi_!(\pi^*(h^*e)^k) = (h^*e)^k \cdot \pi_!(1) = 0$ all vanish. $\square$

Remark 5.11 There are more cases when the group of homotopy self-equivalences $\mathscr{E}(X)$ is finite for a space X that is rationally equivalent to a product of odd-dimensional spheres, and the second case in Theorem 5.10 is merely one of the simplest to establish. But we expect that the Euler ring vanishes in general even if $\mathscr{E}(X)$ is not finite using recent results in [4], where Berglund and Zeman have given a rational description of $\mathrm{BhAut}(X)$ via a fibre sequence

$$\mathrm{BhAut}_u(X) \to \mathrm{BhAut}(X) \to \mathrm{B}\Gamma(X),$$

where $\Gamma(X)$ is a certain arithmetic group and $\mathrm{BhAut}_u(X)$ is the classifying space of normal unipotent X-fibrations. They provide $\Gamma(X)$-equivariant models for $\mathrm{BhAut}_u(X)$ so that one can obtain an algebraic model for $\mathrm{BhAut}(X)$. We expect that one can extend the results in this paper using their results to study Euler rings in more generality, and in particular that the Euler ring $E^*(X)$ vanishes for X a product of odd spheres in general.

Acknowledgements

This paper is part of my PhD project and I would like to thank my supervisor Oscar Randal-Williams for many enlightening discussions and his help and patience. I would also like to thank Alexander Berglund for many useful discussions and for his help with the revision of this article, and Andrey Lazarev for pointing out modern references for models of the universal fibration. For the revision of this article I have been supported by the Knut and Alice Wallenberg foundation through grant 2019.0519.

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Received: October 11, 2024 Revised: April 9, 2025

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Algebraic & Geometric Topology **26**:4 (2026) 1585–1596
DOI: 10.2140/agt.2026.26.1585
Published: April 25, 2026

On the analog category of finite groups

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We show that the analog category of a finite group is essentially proportional to the size of its largest Sylow subgroup. We conclude that the universal upper bound given by the order of the group is very far from optimal.

1 Introduction

We continue the probabilistic reimagining of the foundations of topological robotics [7], begun simultaneously in [10] and [5], in which motion planning is conducted according to continuously varying probability measures on the relevant space of paths. The resulting "analog" invariants, which bound their classical counterparts — the Lusternik–Schnirelmann category and topological complexity — from below, display surprisingly subtle behavior.

For example, the analog category of an aspherical space with torsion-free fundamental group is equal to the cohomological dimension of that group [10, Theorem 1.1], a direct analogue of the Eilenberg–Ganea theorem. Thus, in this case, analog category equals category. On the other hand, in the case of a finite fundamental group, the classical category is always infinite, while in our setting we have the following; see Section 2 for the definition of $\mathrm{acat}(G)$.

Universal upper bound [10, Theorem 7.2] *If G is finite, then* $\mathrm{acat}(G)+1\le |G|$.[1]

We show here that this bound is a very bad one in three senses: the groups for which it is sharp are highly constrained; the difference of the two sides is arbitrarily large; and their ratio is arbitrarily small. Provisionally, let us call G *a-special* if $|G|=ap^s$ with p a prime, $(a,p)=1$, and $p^s>a$ (thus, a 1-special group is simply a p-group).

Theorem 1.1 *In what follows, G refers to a finite group.*

(1) *If G is not a-special for some $a\in\{1,2,3\}$, then the universal upper bound is strict for G.*

(2) *If G is 1- or 2-special, then the universal upper bound is sharp for G.*[2]

(3) *For any $N\ge 0$ and $\epsilon>0$, there exists a G such that* $\mathrm{acat}(G)+1\le |G|-N$ *and* $\mathrm{acat}(G)+1\le \epsilon|G|$.

[1] For stating our results, it is convenient to work with the "unreduced" category, which differs from the "reduced" convention by 1. To avoid confusion, we will simply write $\mathrm{acat}(G)$ for the latter and $\mathrm{acat}(G)+1$ for the former. Both conventions are common throughout the literature on Lusternik–Schnirelmann category and topological complexity.

[2] Unfortunately, we do not know whether the universal upper bound is sharp for 3-special groups.

MSC2020: 55M30.

In particular, the universal upper bound is sharp for p-groups and strict for almost all other groups. For groups of the latter type, we show that the analog category is roughly proportional to the size of the largest Sylow subgroup.

Theorem 1.2 *Let G be a finite group not of prime power order. If $P \leq G$ is a Sylow subgroup of maximal order, then*

$$\min\left\{2, \frac{|N(P)|}{|P|}\right\} \leq \frac{\operatorname{acat}(G)+1}{|P|} \leq 3,$$

where $N(P)$ denotes the normalizer of P in G.

In more prosaic terms, the lower bound is 1 when the largest Sylow subgroup of G is self-normalizing,[3] and otherwise it is 2; for example, the lower bound of 2 obtains for all nilpotent groups.

Prior to our work here, the quantity $\operatorname{acat}(G)$ was almost completely unknown apart from the universal upper bound and a calculation for cyclic groups of prime order [4]. Strictly speaking, this last calculation was of an a priori different invariant, the distributional category; we show here that the two coincide for finite groups, a special case of [10, Conjecture 1.2].

Theorem 1.3 *For any finite group G, the analog and distributional category of G coincide.*

In fact, the same argument may be used to show that the analog and distributional versions of the r-th sequential topological complexity TC_r coincide for every r.

1.1 Conventions

We write Δ^S for the simplex spanned by the set S. Thus, a point in Δ^S is a formal sum $\sum_{s\in S} t_s s$, where the *barycentric coordinates* t_s are nonnegative numbers summing to 1, all but finitely many of which vanish. In the case $S = \{1,\dots,n\}$, we make the abbreviation $\Delta^{\{1,\dots,n\}} = \Delta^{n-1}$; explicitly, this choice amounts to the slightly nonstandard notational convention

$$\Delta^{n-1} := \left\{(t_1,\dots,t_n) \in [0,1]^n : \sum_{i=1}^{n} t_i = 1\right\}.$$

We work in Steenrod's convenient category of topological spaces [12]; see [10, Appendix A] for a summary of relevant facts about these spaces. Topological spaces are implicitly convenient, as are limits — including products — and mapping spaces. Convenient colimits, when they exist, are the same as ordinary colimits. The adjective "compact" refers to the definition in terms of open covers. We write BG for the classifying space of the (discrete) group G, i.e., the geometric realization of its nerve. We write EG for the universal cover of BG and $X^{hG} = \operatorname{Map}^G(EG, X)$ for the space of homotopy fixed points of the G-space X, where Map^G denotes the space of G-equivariant maps.

[3] As shown in [8], the admission of a self-normalizing Sylow subgroup places strong constraints on a group.

2 The analog category of a group

The purpose of this section is to establish the following formula, which the reader may take as a definition of the analog category $\operatorname{acat}(G)$. We recall that Δ^G denotes the simplex spanned by the elements of G, and Δ^G_n is its n-skeleton.

Proposition 2.1 *For any group G, we have* $\operatorname{acat}(G) = \min\{n \mid (\Delta^G_n)^{hG} \neq \varnothing\}$.

We begin with a brief review of the invariants of [10], which are defined in terms of the set $\mathcal{P}(X)$ of probability measures with finite support on the topological space X. We view $\mathcal{P}(X)$ as a topological space with the quotient topology inherited from the various maps

$$X^n \times \Delta^{n-1} \to \mathcal{P}(X), \quad (x,t) \mapsto \sum_{i=1}^{n} t_i \delta_{x_i}.$$

We write $\mathcal{P}_n(X) \subseteq \mathcal{P}(X)$ for the subspace of measures with support of cardinality at most n.

Classically, spaces of probability measures are often topologized using the Lévy–Prokhorov metric, which metrizes the topology of weak convergence when the background space X is a separable metric space. We direct the reader to Section 5 below for some comparisons between the two approaches. The twin advantages of ours are its generality, as we do not even require the background space to be metrizable, and its excellent technical features, which are summarized in the following result.

Theorem 2.2 [10, Theorem 2.7] *The functor $\mathcal{P}$ is an endofunctor on the category of convenient spaces, which preserves homotopy, sifted colimits, quotient maps, and closed embeddings.*

Given a map $f : X \to Y$, we may consider the space of probability measures on X with fiberwise support over Y, namely

$$\mathcal{P}(f) = \left\{ \sum_{i=1}^{n} t_i \delta_{x_i} \in \mathcal{P}(X) : f(x_1) = f(x_2) = \cdots = f(x_n) \right\},$$

and we set $\mathcal{P}_n(f) = \mathcal{P}_n(X) \cap \mathcal{P}(f)$. Sending an element of $\mathcal{P}(f)$ to the point in whose fiber it is supported defines a continuous map to Y.

Definition 2.3 The *analog sectional category* of the map $f : X \to Y$ is the least n such that $\mathcal{P}_{n+1}(f) \to Y$ admits a section. The *analog category* of the space X, denoted by $\operatorname{acat}(X)$, is the analog sectional category of the evaluation map $(X, x_0)^{([0,1],\{0\})} \to X$, where $x_0 \in X$ is any basepoint.

As our interest here lies solely in the aspherical context, we permit ourselves the abusive abbreviation $\operatorname{acat}(G) = \operatorname{acat}(BG)$.

For the proof of the proposition stated above, we require the following standard fact.

Lemma 2.4 *For any G-space X, the homotopy fixed point space X^{hG} is canonically weakly equivalent to the space of sections of the canonical map $EG \times_G X \to BG$.*

Proof Consider the following commutative diagram of mapping spaces:

$$\begin{tikzcd}
\mathrm{Map}^G(EG, EG \times X) \arrow[r] \arrow[d] & \mathrm{Map}^G(EG, EG \times_G X) \arrow[r, equal, "\sim"] \arrow[d] & \mathrm{Map}(BG, EG \times_G X) \arrow[d] \\
\mathrm{Map}^G(EG, EG) \arrow[r] & \mathrm{Map}^G(EG, BG) \arrow[r, equal, "\sim"] & \mathrm{Map}(BG, BG)
\end{tikzcd}$$

The section space in question is the fiber of the right-hand vertical map over id_{BG}, while X^{hG} is the fiber of the left-hand vertical map over id_{EG}. The claim follows after noting that the vertical maps are fibrations and the left-hand square a homotopy pullback. $\square$

Proof of Proposition 2.1 It is well known that, for any group G, there is a commutative diagram of topological spaces

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & (BG, x_0)^{([0,1],\{0,1\})} \arrow[d] \\
EG \arrow[r] \arrow[d, "\pi"'] & (BG, x_0)^{([0,1],\{0\})} \arrow[d] \\
BG \arrow[r, equal] & BG
\end{tikzcd}$$

in which the horizontal arrows are homotopy equivalences and the vertical columns are (Hurewicz) fiber sequences. It follows from [10, Corollary 5.3] that $\mathrm{acat}(G)$ is the analog sectional category of π, which is a fiber bundle with structure group G; therefore, by [10, Corollary 5.8], we have

$$\mathcal{P}_{n+1}(\pi) \cong EG \times_G \mathcal{P}_{n+1}(G)$$

as spaces over BG, and it is easy to see that $\mathcal{P}_{n+1}(G) \cong \Delta_n^G$ as G-spaces, so the claim follows from Lemma 2.4. $\square$

Corollary 2.5 *If X is a contractible G-space with an equivariant map $X \to \Delta_n^G$, then* $\mathrm{acat}(G) \leq n$.

Proof Contractibility implies that $X^{hG} \neq \varnothing$, since an equivariant map $EG \to X$ may be constructed by elementary obstruction theory; alternatively, the map $EG \times_G X \to BG$ is a trivial fibration, since $EG \to \mathrm{pt}$ is a homotopy equivalence, and Lemma 2.4 applies. It follows that $(\Delta_n^G)^{hG}$ receives a map from a nonempty space, and hence is itself nonempty. The claim follows from Proposition 2.1. $\square$

Remark 2.6 Essentially the same argument shows that the r-th analog topological complexity of BG, as defined in [10], is equal to the least n such that the G^r-space $\Delta_n^{G^r/G}$ admits a homotopy fixed point.

3 Designer complexes

This section is concerned with the construction of certain contractible equivariant cell complexes, which, via Corollary 2.5 and obstruction theory, will be the key to proving our main results. The ideas here are mostly taken from the work of Assadi [1], following Oliver, Conner and Floyd, Smith, and others, but the specificity of our situation permits some simplification and hence a relatively self-contained account.

In what follows, the group G is always finite. As a matter of terminology, we say that a space is p-acyclic if its mod p reduced homology vanishes.

Definition 3.1 Let X be a G-complex. Given a prime p, we say that X is *Smith p-acyclic* if X^P is p-acyclic for every nontrivial p-subgroup P. We say that X is *Smith acyclic* if X is Smith p-acyclic for every prime p.

The relevance of this definition lies in its connection to obstruction theory.

Proposition 3.2 *Let X be a G-complex of dimension m.*

(1) *If X is Smith $(p$-$)$acyclic, then a $(p$-$)$acyclic G-complex may be obtained from X by attaching free cells of dimension at most $m+1$.*

(2) *If X is acyclic, then a contractible G-complex may be obtained by attaching free cells of dimension at most 3.*

Proof For the first claim, if X is Smith acyclic, then [1, Proposition I.1.6] guarantees that we may achieve acyclicity below degree m and $\mathbb{Z}[G]$-projectivity in degree m by attaching free cells of dimension at most m. Thus, by the Eilenberg swindle, we may achieve acyclicity by further cell attachments of dimensions m and $m+1$. The Smith p-acyclic case is similar, invoking [1, Lemma II.1.5] instead, and obtaining instead a mod p homology group of degree m projective over $\mathbb{F}_p[G]$.

For the second claim, note first that X is path connected by acyclicity. We first kill the fundamental group of X by attaching free 2-cells indexed by a set I. Calling the resulting complex Y, the long exact sequence for the pair (Y, X) shows that

$$\widetilde{H}_i(Y) \cong \begin{cases} \bigoplus_I \mathbb{Z}[G] & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a simply connected acyclic G-complex may be obtained by attaching free 3-cells, and any such complex is contractible by Whitehead's theorem. □

Our main construction will proceed inductively and one prime at a time. In order to state the main result, we require the following definition, which will form the basis for our induction.

Definition 3.3 Let G be a finite group and p a prime dividing $|G|$.

(1) A subgroup $H \leq G$ is called a *p-intersection* if it is an intersection of p-Sylow subgroups.

(2) The *(p-Sylow) depth* of the p-intersection H is the largest d for which there is a chain $H = H_d < H_{d-1} < \cdots < H_1 = P$ of proper inclusions with P a p-Sylow subgroup and each H_i a p-intersection.

(3) The *(p-Sylow) depth* of a p-subgroup $H \leq G$ is the maximal depth of a p-intersection containing H.

(4) The *(p-Sylow) depth* of G, denoted by $d_p(G)$, is the maximal depth of a p-intersection in G.

We adopt the convention that $d_p(G) = 0$ if and only if $(p, |G|) = 1$.

It is easy to see that the depth of H as a p-subgroup coincides with its depth as a p-intersection. It is also easy to see that the $d_p(G)$ is bounded above by the number of distinct p-Sylow subgroups of G, as well as by the exponent of p in $|G|$.

Lemma 3.4 *Let H be a p-intersection of depth d and K any p-subgroup containing H. The depth of K is at most d, with equality if and only if $H = K$.*

Proof We may assume that $H \neq K$. Supposing that K has depth $s \geq d$, we obtain the chain of inclusions

$$H < K \leq H_s < H_{s-1} < \cdots < H_1 = P,$$

implying that the depth of H is at least $s+1 > d$, a contradiction. □

Corollary 3.5 *Let H_1 and H_2 be p-intersections. If $H_1 < H_2$, then the depth of H_1 is greater than the depth of H_2.*

Corollary 3.6 *If K is a p-subgroup of depth d, then K is contained in a unique p-intersection of depth d.*

Proof Let $H_1 \neq H_2$ be p-intersections of depth d containing K. Then $H_1 \cap H_2$ is a p-intersection properly contained in H_1, and hence of strictly greater depth by Corollary 3.5. It follows that K has depth greater than d, a contradiction. □

We write $\mathcal{I}_d = \mathcal{I}_d(p)$ for the set of p-intersections (in G, implicitly) of depth d, regarded as a G-set under conjugation.

Convention 3.7 For the remainder of this section, we assume that G is a finite group not of prime power order.

We come now to the main construction (compare [1, Theorem II.1.4]).

Theorem 3.8 *Let p be a prime dividing $|G|$. For $0 \leq d \leq d_p(G)$, there are G-complexes $X_p(G)_d$ with the following properties:*

(1) *$X_p(G)_0$ is free of dimension 1.*

(2) *$X_p(G)_{d+1}$ is obtained from $X_p(G)_d$ by attaching cells of dimension at most $d+2$ with isotropy in $\mathcal{I}_{d+1}$.*

(3) *$X_p(G)_d^P$ is p-acyclic for every nontrivial p-subgroup P of depth at most d.*

Lemma 3.9 *Let $H \leq G$ be a subgroup and X an $N(H)$-space. For any $g \in G$, there is a canonical homeomorphism*

$$(G \times_{N(H)} X)^{gHg^{-1}} \cong X^H.$$

Proof Consider the standard decomposition $G \times_{N(H)} X = \bigsqcup_{[g_i] \in G/N(H)} g_i X$, and take g to be one of our coset representatives. Writing $ghg^{-1}g_i = g_j h'$, we have $ghg^{-1} \cdot g_i x = g_j(h' \cdot x)$. It follows that $g_i x$ is fixed by gHg^{-1} if and only if x is fixed by H and $[g] = [g_i]$, which is to say $g = g_i$. Thus, the desired homeomorphism is given (from right to left) by $x \mapsto gx$. □

Proof of Theorem 3.8 We proceed by simultaneous induction on d and $d_p(G)$. In the case $d = d_p(G) = 0$, we let $X_p(G)_0$ be any connected 1-dimensional free G-complex, e.g., a Cayley graph. Notice that the third condition is vacuous in this case. For $d = 0$ and general G, we choose a p-Sylow subgroup P and set

$$X_p(G)_0 = G \times_{N(P)} X_p(N(P)/P)_0.$$

For $d = 1$, let P be as above and consider $X_p(N(P)/P)_0$. Since P is p-Sylow, the group ring $\mathbb{F}_p[N(P)/P]$ is semisimple by Maschke's theorem. It follows that $\widetilde{H}_1\big(X_p(N(P)/P)_0; \mathbb{F}_p\big)$ is projective over $\mathbb{F}_p[N(P)/P]$; therefore, by Proposition 3.2 and the Eilenberg swindle, we may attach free cells of dimensions 1 and 2 to obtain a p-acyclic $N(P)/P$-complex $\overline{X}_p(N(P)/P)_0$. Finally, we define

$$X_p(G)_1 = G \times_{N(P)} \overline{X}_p(N(P)/P)_0.$$

The second property holds by construction, and the third follows from the observation that, by Lemma 3.9, the fixed set of any p-Sylow subgroup is homeomorphic to $\overline{X}_p(N(P)/P)_0$, which is p-acyclic by construction.

In the general case, choose a p-intersection $H \leq G$ of depth $d+1$ and consider the $N(H)/H$-space $X_p(G)_d^H$. We claim that this space is Smith p-acyclic; indeed, given a nontrivial p-subgroup $P \leq N(H)/H$, we have $(X_p(G)_d^H)^P = X_p(G)_d^{\widetilde{P}}$, where $H \leq \widetilde{P}$ is the subgroup of $N(H)$ corresponding to P, and the depth of $\widetilde{P}$ is at most d by Lemma 3.4, since P was assumed nontrivial, so the claim follows by induction. Therefore, by Proposition 3.2, we may achieve p-acyclicity after attaching free $N(H)/H$-cells of dimension at most $d+2$, and, indexing these cells by $i \in I$, we achieve the same result G-equivariantly for all conjugates of H at once via the construction

$$G \times_{N(H)} \left(\bigsqcup_{i \in I} N(H)/H \times D^{n_i} \right) \sqcup_{G \times \bigsqcup_{i \in I} N(H)/H \times S^{n_i - 1}} X_p(G)_d.$$

By Corollary 3.5, this construction does not alter the fixed set of any member of $\mathcal{I}_{d+1}$ not conjugate to H; therefore, we may define $X_p(G)_{d+1}$ to be the result of iterating the construction over $\mathcal{I}_{d+1}/G$.

The second condition holds by construction. To check the third, we note that, if P has depth less than d, then $X_p(G)_d^P = X_p(G)_{d-1}^P$ by construction, since P is contained in no member of $\mathcal{I}_d$ by definition, and the latter is p-acyclic by induction. On the other hand, if P has depth d, then P is contained in a *unique* $H \in \mathcal{I}_d$ by Corollary 3.6, so $X_p(G)_d^P = X_p(G)_d^H$, which was constructed to be p-acyclic. □

We write $X_p(G) = X_p(G)_{d_p(G)}$.

Corollary 3.10 *There is a G-complex $X(G)$ with the following properties*:

(1) *$X(G)$ is obtained from $\bigsqcup_{p \mid |G|} X_p(G)$ by attaching free cells of dimension at most $\max_p d_p(G) + 2$.*

(2) *$X(G)$ is contractible.*

Proof By construction, given $p \neq q$ dividing $|G|$, every p-subgroup of G acts without fixed points on $X_q(G)$, so the third condition of Theorem 3.8 implies that the disjoint union in question is Smith acyclic. Since the dimension of $X_p(G)$ is $d_p(G)+1$, and since $\max_p d_p(G)+2 \geq 3$, Proposition 3.2 shows that we may achieve first acyclicity, then contractibility after the indicated type of cell attachment. □

4 Proofs of the main results

Our strategy will be to exploit Corollary 2.5 by applying obstruction theory to the complex $X(G)$ constructed in the previous section. In order to proceed, we require information on the connectivity of fixed-point sets.

Proposition 4.1 *For any $H \leq G$ and any $n \in \mathbb{Z}$, there is a canonical $N(H)/H$-equivariant homeomorphism*

$$(\Delta_n^G)^H \cong \Delta_k^{G/H},$$

where $k = \lfloor (n+1)/|H| \rfloor - 1$.

Corollary 4.2 *For any $H \leq G$ and $n < |G|$, the connectivity of $(\Delta_n^G)^H$ is exactly $\lfloor (n+1)/|H| \rfloor - 2$.*

Proof of Proposition 4.1 Since H is finite, we may define a function $f : \Delta^{G/H} \to \Delta^G$ by the formula

$$f(t)_g = \frac{1}{|H|} t_{gH}.$$

As the restriction of a linear map, this function is continuous, and its image is H-fixed by inspection; thus, we may we view f as a map to $(\Delta^G)^H$. As such, it is $N(H)/H$-equivariant and injective by inspection, and we claim that it is also surjective. To see why, note that a point in Δ^G is fixed by H if and only if the barycentric coordinate of gh is independent of $h \in H$ for every $g \in G$. Thus, given $t \in (\Delta^G)^H$, setting $t_{gH} = |H| t_g$ defines an element of $f^{-1}(t)$. The homeomorphism $(\Delta^G)^H \cong \Delta^{G/H}$ follows, since both sides are compact and Hausdorff.

Now, a point of Δ^G lies in Δ_n^G if and only if at most $n+1$ of its barycentric coordinates are nonzero. We conclude that f identifies $(\Delta_n^G)^H$ with the subspace of $\Delta^{G/H}$ in which at most $(n+1)/|H|$ barycentric coordinates are nonzero, as desired. □

In what follows, we write $X_p^\wedge$ for the completion of the space X at the prime p; see [11], for example. We recall that, according to one of several results known collectively as the "generalized Sullivan conjecture", due to Carlsson [2, Theorem B(c)] and Dwyer, Miller and Neisendorfer [6], the natural map $(X^P)_p^\wedge \to (X_p^\wedge)^{hP}$ is a weak equivalence for any p-group P and finite-dimensional P-CW complex X.

Proposition 4.3 *For any p-subgroup $P \leq G$, we have* $\operatorname{acat}(G) \geq |P|-1$. *If P is not self-normalizing, then* $\operatorname{acat}(G) \geq 2|P|-1$.

Lemma 4.4 *Let $H \leq G$ be any subgroup. If H is not self-normalizing, then $(G/H)^{hN(H)/H} = \varnothing$.*

Proof Our assumptions imply that $N(H)/H$ is a nontrivial group acting without fixed point on the discrete space G/H. An easy exercise shows that such a space admits no equivariant map from any connected $N(H)/H$-space with nontrivial action, and hence no homotopy fixed point. □

Proof of Proposition 4.3 We begin with a few elementary observations regarding the following commutative diagram of canonical maps, to which we will appeal throughout the argument:

$$\begin{tikzcd}
\mathrm{Map}(\mathrm{pt}, \Delta_n^G)^P \arrow[r] \arrow[d, equal] & (\mathrm{Map}(\mathrm{pt}, \Delta_n^G)^P)^\wedge_p \arrow[r] \arrow[d, equal] & \big(\mathrm{Map}(EG, (\Delta_n^G)^\wedge_p)\big)^P \arrow[d] & \mathrm{Map}(EG, \Delta_n^G)^P \arrow[l] \arrow[d] \\
(\Delta_n^G)^P \arrow[r] & ((\Delta_n^G)^P)^\wedge_p \arrow[r] & ((\Delta_n^G)^\wedge_p)^{hP} & (\Delta_n^G)^{hP} \arrow[l]
\end{tikzcd}$$

First, since the canonical map $EP \to EG$ is a P-equivariant homotopy equivalence, the third and fourth vertical arrows are weak equivalences. Second, since $N(P)/P$ acts canonically on the P-fixed set of any G-space, the arrows in the top row are all equivariant maps between $N(P)/P$-spaces. Third, by the Sullivan conjecture, the second map in the bottom row is a weak equivalence, and hence in the top row as well.

For the first claim of the proposition, it suffices to show that $(\Delta_n^G)^{hP} = \varnothing$ for $n < |P| - 1$, since $(\Delta_n^G)^{hG}$ has a canonical map to this space. In this range, we have $(\Delta_n^G)^P = \varnothing$ by Proposition 4.1; in particular, this space is p-complete, so the first map in this row is also a weak equivalence (in fact an equality, but this is irrelevant). We conclude that the target of the rightmost map is empty, so its source must be so as well.

For the second claim, it suffices as before to show that $(\Delta_n^G)^{hN(P)} = \varnothing$ for $|P| - 1 \le n < 2|P| - 1$. In this range, Proposition 4.1 instead identifies $(\Delta_n^G)^P$ with the discrete $N(P)/P$-space G/P, which is also p-complete. As before, it follows that the first arrow in the bottom row of the diagram above is a weak equivalence, and hence in the top row as well. We also conclude from Lemma 4.4 that $(\Delta_n^G)^P$ has no homotopy fixed points for the action of $N(P)/P$. Since the homotopy fixed-points functor preserves weak equivalences, a diagram chase as in the previous paragraph yields the conclusion that

$$(\mathrm{Map}(EG, \Delta_n^G)^P)^{hN(P)/P} = \varnothing.$$

An easy exercise shows that this space is weakly equivalent to $(\Delta_n^G)^{hN(P)}$, and the claim follows. □

Proof of Theorem 1.2 The lower bound follows from Proposition 4.3. For the upper bound, it suffices by Corollary 2.5 to construct an equivariant map $X(G) \to \Delta_n^G$ for $n \ge 3q - 1$, where q is the largest prime power dividing $|G|$. To begin, since $X_p(G)_0$ is free of dimension 1 for each prime p, there is no obstruction to constructing a map to Δ_n^G provided $n \ge 1$, which certainly holds in our situation. Proceeding inductively, we may extend an equivariant map from $X_p(G)_d$ to $X_p(G)_{d+1}$ provided $(\Delta_n^G)^H$ is $(d+1)$-connected for every $H \in \mathcal{I}_{d+1}(p)$; we will consider this question in a moment. Finally, equivariant maps from the various $X_p(G)$ may be extended to $X(G)$ provided $n \ge \max_p d_p(G) + 2$, which certainly holds in our situation, since $d_p(G)$ is bounded above by the exponent of p in $|G|$.

Now, for $H \in \mathcal{I}_{d+1}(p)$, the connectivity of $(\Delta_n^G)^H$ is $\lfloor (n+1)/|H| \rfloor - 2$ by Corollary 4.2, and $|H| \le p^{s-d}$, where p^s is the largest power of p dividing $|G|$. Thus, it suffices to establish the inequality

$$1 + \frac{d}{3} \le \frac{q}{p^{s-d}}$$

for every prime p dividing $|G|$. Since $q \geq p^s$ by definition, the claim follows from the obvious inequality

$$1 + \frac{d}{3} \leq p^d. \qquad \square$$

Proof of Theorem 1.1 Writing $|G| = qr$ with q as above and appealing to Theorem 1.2, we obtain the inequality

$$\frac{\operatorname{acat}(G) + 1}{|G|} \leq \frac{3}{r}.$$

If $r \notin \{1, 2, 3\}$, then the right-hand side of this inequality is strictly less than 1, implying the first claim. The second claim is immediate from Proposition 4.3 and the universal upper bound. For the third claim, fix N and ϵ and choose distinct primes $p > q$ and a natural number s so that $N \leq (q-3)p^s$ and $\epsilon \leq 3/q$. In this case, the desired bounds hold for the cyclic group of order $p^s q$. $\square$

5 Analog vs. distributional

The goal of this section is to prove Theorem 1.3, claiming that the analog category of the finite group G coincides with its distributional category in the sense of [5; 9]. We begin by recalling the relevant definitions.[4]

Throughout, we will use the subscript LP to refer to the Lévy–Prokhorov metric; thus, we have $\mathcal{P}_n(X)_{\mathrm{LP}}$ for X metric, and we have $\mathcal{P}_n(f)_{\mathrm{LP}}$ for continuous f with metric source. The rule for turning a definition of an analog invariant into that of a distributional invariant is to add this subscript; thus, the distributional sectional category of $f : X \to Y$ with metric source is the least n for which $\mathcal{P}_{n+1}(f)_{\mathrm{LP}} \to Y$ admits a section, and this definition specializes to the definition of distributional category as in Definition 2.3.

Remark 5.1 An issue deserving of care is that the distributional category is not defined if X is not metrizable — for example, $X = BG$ with G infinite. One possible workaround is to appeal to the fact that any CW complex is metrizable up to homotopy equivalence [3]. Fortunately, since we confine our discussion here to finite groups, the issue does not arise.

Our main technical result comparing these notions is the following.

Proposition 5.2 *Let $f : X \to Y$ be a map with X metric and Y convenient. If f is proper, then the analog and distributional sectional category of f coincide.*

For the proof, we require the following.

Lemma 5.3 *For a metric space X, the identity function $\mathcal{P}_n(X) \to \mathcal{P}_n(X)_{\mathrm{LP}}$ is continuous for every finite $n \geq 0$. If X is compact, then each of these maps is a homeomorphism.*

Note that the first claim of Lemma 5.3 is simply the claim that the quotient topology on $\mathcal{P}(X)$ is finer than the Lévy–Prokhorov topology when both are defined, which can also be seen by considering an explicit basis for the latter [9, Section 8].

[4] For the sake of easier reading, we depart from the notation of [5].

Proof of Lemma 5.3 For the first claim, let X be a metric space, and consider the commutative diagram

$$\begin{tikzcd} \operatorname{colim}_K \mathcal{P}(K) \arrow[r] \arrow[d] & \operatorname{colim}_K \mathcal{P}(K)_{\mathrm{LP}} \arrow[d] \\ \mathcal{P}(X) \arrow[r] & \mathcal{P}(X)_{\mathrm{LP}} \end{tikzcd}$$

where K ranges over compact subspaces of X. The collection of such is filtered, and hence sifted, so the left-hand arrow is a homeomorphism by Theorem 2.2. Thus, in order to establish continuity of the bottom arrow, it suffices to establish continuity of the top arrow; in other words, we may assume that X itself is compact. From the definition of $\mathcal{P}(X)$, continuity is equivalent to continuity of each of the maps

$$X^n \times \Delta^{n-1} \to \mathcal{P}(X)_{\mathrm{LP}}, \quad (x, k) \mapsto \sum_{i=1}^{n} t_i \delta_{x_i},$$

which is to say sequential continuity, since the source is metric. Since X is compact, and hence separable, the topology on the target is the topology of weak convergence of measures, so sequential continuity follows from continuity of the composite

$$X^n \times \Delta^{n-1} \xrightarrow{f^n \times \iota} \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\langle -,- \rangle} \mathbb{R},$$

where ι is the inclusion and $f : X \to \mathbb{R}$ is an arbitrary continuous function.

For the second claim, if X is compact, then so is $\mathcal{P}_n(X)$. Since $\mathcal{P}_n(X)_{\mathrm{LP}}$, as a metric space, is Hausdorff, and since the map in question is a continuous bijection, the claim follows. □

Proof of Proposition 5.2 It follows from Lemma 5.3 that the top arrow in the commutative diagram

$$\begin{tikzcd} \mathcal{P}_n(f) \arrow[rr, "\mathrm{id}"] \arrow[dr] & & \mathcal{P}_n(f)_{\mathrm{LP}} \arrow[dl] \\ & Y & \end{tikzcd}$$

is continuous, so a section of the left-hand map determines a section of the right. Thus, the distributional sectional category bounds the analog from below. For the reverse inequality, we will show that any section σ of the right-hand map is continuous when considered as a map to $\mathcal{P}_n(f)$. By assumption, the space Y is the colimit of its compact subsets; therefore, since $\sigma|_K$ factors through $\mathcal{P}_n(f|_{f^{-1}(K)})_{\mathrm{LP}}$, we may assume without loss of generality that Y itself is compact. In this case, since f is proper, it follows that X is also compact, and Lemma 5.3 implies the claim. □

Lemma 5.4 *For any $n \geq 0$, the functor $\mathcal{P}_n(-)_{\mathrm{LP}}$ preserves homotopy, and hence homotopy equivalence.*

Proof Using the topological basis given in [5, Section 3.1], it is easy to check that, for any metric space Y, the assignment

$$\left(\sum_{i=1}^{m} t_i \delta_{x_i}, y \right) \mapsto \sum_{i=1}^{m} t_i \delta_{(x_i, y)}$$

defines a continuous map $\mathcal{P}_n(X)_{\mathrm{LP}} \times Y \to \mathcal{P}_n(X \times Y)_{\mathrm{LP}}$. The claim follows easily after taking $Y = [0, 1]$. □

Proof of Theorem 1.3 It suffices to show that the distributional category of BG is the distributional sectional category of the map $\pi : EG \to BG$; indeed, the corresponding analog statement is true, as shown in the course of proving Proposition 2.1, and π is proper, so Proposition 5.2 applies. Considering the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & (BG, x_0)^{([0,1],\{0,1\})} \arrow[d] \\
EG \arrow[r] \arrow[d, "\pi"'] & (BG, x_0)^{([0,1],\{0\})} \arrow[d] \\
BG \arrow[r, equal] & BG
\end{tikzcd}$$

from the proof of Proposition 2.1, the claim follows by noting that the construction $f \mapsto \mathcal{P}_n(f)_{\mathrm{LP}}$ preserves Hurewicz fibrations by [5, Proposition 5.1] and that the construction $X \mapsto \mathcal{P}(X)_{\mathrm{LP}}$ preserves homotopy equivalence by Lemma 5.4. $\square$

Remark 5.5 An alternative argument establishing that the distributional sectional category is a homotopy invariant of fibrations is given in [9, Proposition 5.3].

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Received: November 27, 2024 Revised: February 24, 2025

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 26 Issue 4 (pages 1229–1596) 2026