Algebraic & Geometric Topology

Volume 26 (2026)

On embeddings of 4-manifolds in codimension 2

Abhijeet Ghanwat and Dishant M Pancholi

msp

Algebraic & Geometric Topology 26:4 (2026) 1293–1320
DOI: 10.2140/agt.2026.26.1293
Published: April 25, 2026

On embeddings of 4-manifolds in codimension 2

ABHIJEET GHANWAT AND DISHANT M PANCHOLI

We show that every closed orientable smooth 4-manifold admits a smooth embedding in a large class of closed 6-manifolds. In particular, we show that every smooth 4-manifold admits a smooth embedding in the complex projective 3-space. Our embedding technique also provides a new proof of embeddings of 4-manifolds in $\mathbb{R}^7$.

1 Introduction

A basic question in the field of geometric topology which concerns embeddings of manifolds can be stated as follows: given a pair of manifolds M and N, how many smooth embeddings of M exist in N?

A slightly simpler and related question is the question of finding which manifolds embed in a given manifold. Detailed investigations in this regard have led to the discovery of interesting invariants of manifolds. One of the earliest seminal results in this context is due to H. Whitney who showed that every closed manifold of dimension n admits an embedding in $\mathbb{R}^{2n}$. Subsequently, this result has been extensively generalized. Most notably, M. Hirsch [19] showed that every closed orientable odd-dimensional manifold M^{2n-1} admits a smooth embedding in $\mathbb{R}^{4n-3}$. This result, together with those by C. T. C. Wall [27] and V. Rokhlin [25], implies that every closed 3-manifold (orientable or otherwise) admits an embedding in $\mathbb{R}^5$.

For closed n-dimensional manifolds, combining the results of A. Haefliger [16], A. Haefliger and M. Hirsch [17], and W. Massey and F. Peterson [23], one knows that every such n-manifold embeds in $\mathbb{R}^{2n-1}$ when $n > 4$ and n is not a power of two. For 4-manifolds it was shown by M. Hirsch [20] and C. T. C. Wall (M. Hirsch mentions in [20] that C. T. C. Wall had independently proved this result) that every orientable PL 4-manifold admits a PL embedding in $\mathbb{R}^7$.

The purpose of this article is to show that there are smooth six-dimensional manifolds with relatively simple topology in which all closed-orientable smooth manifolds of dimension four embed. Ideally one would like to embed every closed smooth 4-dimensional manifold in $\mathbb{R}^6$. However, D. Ruberman [26] has shown that a closed smooth 4-manifold admits a smooth embedding in $\mathbb{R}^6$ if and only if it admits a spin structure and its signature is zero, a result which was also stated by S. Cappell and J. Shaneson in [7]. In particular, this implies that $\mathbb{C}P^2$ does not smoothly embed in $\mathbb{R}^6$.

Dishant M. Pancholi is thankful to the Simon's Foundation and ICTP, Trieste, Italy for the Simons Associateship, which allowed him to travel to ICTP, Trieste, Italy, where a part of the work related to this article was carried out. We are thankful to Prof. Yakov Eliashberg for constant encouragement and support. We are also thankful to Prof. S. Lakshmibala for suggestions regarding the presentation.
MSC2020: 57R40.

The simplest 6-manifold in which $\mathbb{C}P^2$ embeds is $\mathbb{C}P^3$. Furthermore, the question of embeddability of two important classes of closed orientable smooth 4-manifolds, namely, symplectic 4-manifolds and smooth algebraic surfaces have been extensively examined (see, for instance, [2; 9; 10]), and the question of their embeddability in $\mathbb{C}P^3$ is very important as any such embedding corresponds to a *Lefschetz pencil* of $\mathbb{C}P^3$ with given embedded submanifold as its generic fiber. We therefore investigate embeddings of 4-manifolds in $\mathbb{C}P^3$ and establish the following:

Theorem 1.1 *Every closed orientable smooth* 4*-manifold admits a smooth embedding in* $\mathbb{C}P^3$.

To the best of our knowledge, Theorem 1.1 above and Theorem 1.2, which establishes embedding of 4-manifolds in certain 6-manifolds of the type $N \times \mathbb{C}P^1$, are the only results demonstrating the existence of closed 6-manifolds in which all closed orientable smooth 4-manifolds embed.

The central idea for the proof of Theorem 1.1 is drawn from a well-known fact that given a projective embedding of a smooth algebraic surface, the standard Lefschetz pencil of the complex projective space generically induces a Lefschetz pencil structure on the surface. It was established by R. I. Baykur and O. Saeki [5; 6] that every closed orientable smooth 4-manifold admits a *simplified broken Lefschetz fibration* (SBLF), which can be regarded as a natural generalization of the Lefschetz pencil for an arbitrary smooth 4-manifold. This decomposition allows us to express any smooth 4-manifold as a singular fiber bundle over $\mathbb{C}P^1$ with a finite number of *Lefschetz singularities* and a unique *indefinite fold circle*. The advantage of this decomposition is that we can associate with any smooth 4-manifold certain data which comprise two constituents. These are an element of the *mapping class group* of a closed orientable surface of genus g expressed as a product of (positive) *Dehn twists*, corresponding to Lefschetz singularities, and a round handle attachment [4; 13] corresponding to the fold singularity.

Let us now briefly outline the argument establishing Theorem 1.1. We need Theorem 1.2 to prove Theorem 1.1. Hence, we begin by first stating and outlining the proof of Theorem 1.2.

Consider any closed orientable 4-manifold N which admits an embedding of a Hopf link which is *separable* in the sense of Definition 4.4. Roughly speaking, by a separable Hopf link in a manifold N, we mean that N admits a handle decomposition that satisfies the following property: the boundary of a 0-handle has a Hopf link, which is slice in the complement of the 0-handle. For any 4-manifold admitting separable Hopf link, we show:

Theorem 1.2 *Let M be an orientable closed smooth* 4*-manifold. Let N be a* 4*-manifold which admits a separable Hopf link. Then there exists an embedding* $\psi : M \to N \times \mathbb{C}P^1$.

Let us now outline the proof of Theorem 1.2. Given a closed orientable smooth 4-manifold M, consider the manifold M together with any given SBLF. We need to produce an embedding f of M in $N \times \mathbb{C}P^1$, where N is a 4-manifold admitting an embedding of separable Hopf link. The embedding will be produced such that the trivial fibration $\pi_2 : N \times \mathbb{C}P^1 \to \mathbb{C}P^1$ of $N \times \mathbb{C}P^1$ induces the given SBLF.[1]

[1] Strictly speaking one will only produce embeddings satisfying such properties up to isotopies and diffeomorphisms of source and target manifolds under consideration.

The three important steps for constructing the embedding f are the following: In the first step, using an appropriate generalization of techniques from [24], and a specific local embedding model for a given Lefschetz singularity, we provide an embedding of genus $g+1$ *Lefschetz subfibration* over a disc $\mathbb{D}^2$ in $N \times \mathbb{D}^2$, which is associated with the given SBLF. This embedding is such that the trivial product fibration $\pi_2 : N \times \mathbb{D}^2 \to \mathbb{D}^2$ induces the given *Lefschetz fibration*. This is the most important step in the proof and is detailed in Section 4. In fact, in Section 4 we show how to embed any Lefschetz fibration over a disc or $\mathbb{C}P^1$ in a trivial fibration over $\mathbb{C}P^1$ with fiber N.

Next, we use a local embedding model for fold singularities to produce an embedding of a submanifold $(\widetilde{M}, \partial\widetilde{M}) \subset M$ (having two disjoint boundary components) in $N \times I \times \mathbb{S}^1$. This embedding is constructed such that it agrees with the embedding in the first step near one of the boundary components of $\widetilde{M}$, and is a trivial fibration $\Sigma_g \times S^1$ near the other boundary component of $\widetilde{M}$. Here, Σ_g denotes a closed orientable surface of genus g. This provides us with a fiber-preserving embedding of $M \setminus \Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$. Finally, we extend the embedding of $M \setminus \Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$ using an embedding of $\Sigma_g \times \mathbb{D}^2$ in $N \times \mathbb{D}^2$ to obtain the embedding $f : M \hookrightarrow N \times \mathbb{C}P^1$. These two steps are discussed in Section 5. Embedding of M in $N \times \mathbb{C}P^1$ is the content of Theorem 1.2. Theorem 1.2 immediately implies Theorem 6.1 which establishes embeddings of smooth closed orientable 4-manifolds in $\mathbb{R}^7$.

Having outlined a proof of Theorem 1.2, let us now discuss how to establish embeddings of 4-manifolds in $\mathbb{C}P^3$ as claimed in Theorem 1.1. Given a smooth, orientable, closed 4-manifold, we first consider the manifold $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ together with a specific SBLF. Next, we notice that the *blow-up* of $\mathbb{C}P^3$ along $\mathbb{C}P^1$ is a fiber bundle over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^2$ with the property that the fiber bundle is trivial in the complement of the *exceptional divisor*.

We embed $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up of $\mathbb{C}P^3$ using this specific SBLF by observing that $\mathbb{C}P^2$ admits a separable Hopf link and hence a slight generalization of the argument necessary to establish Theorem 1.2 allows us to embed $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the *blow-up* of $\mathbb{C}P^3$. Further, we ensure that the embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up of $\mathbb{C}P^3$ is such that the fiber of the specific SBLF associated to $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ has certain specific intersection property with the exceptional divisor of the blow-up of $\mathbb{C}P^3$. This allows us to show that the embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up of $\mathbb{C}P^3$ is such that when we *blow-down* the blow-up of $\mathbb{C}P^3$, we get a manifold diffeomorphic to $\mathbb{C}P^3$ that has M as its embedded submanifold. The construction of the specific SBLF, blow-up and blow-down procedures, and the proof of Theorem 1.1 are discussed in the final section.

The mathematical preliminaries to carry out these steps are given in Sections 2 and 3. In particular, we discuss relevant aspects of *broken Lefschetz fibrations* in Section 2, and of mapping class groups in Section 3.

Finally, a few remarks on conventions used in this article. By a manifold we mean a smooth compact orientable manifold with or without boundary. We denote manifolds by capital letters M, N, etc. When we need to emphasize that we are working with a manifold with boundary, we use the notation $(M, \partial M)$ consisting of the pair M and the boundary ∂M of M. As usual, the notation Σ or Σ_g is used for denoting a closed orientable surface, with g indicating the genus.

2 Review of broken Lefschetz fibrations

Broken Lefschetz fibrations (BLF) were introduced by D. Auroux, S. K. Donaldson, and L. Katzarkov [1]. These are generalized Lefschetz fibrations. R. I. Baykur [3] established that every smooth orientable closed 4-manifold admits a broken Lefschetz fibration. The purpose of this section is to review a few definitions and results related to BLF. We refer to [3; 5] for a detailed discussion on BLF. Let us begin by recalling the definition of Lefschetz singularity.

Definition 2.1 (Lefschetz singularity) Let M be an oriented 4-manifold and Σ an oriented surface. Let $f: M \to \Sigma$ be a smooth map. A point $x \in M$ is said to be a Lefschetz singularity of the map f, provided there is an orientation preserving parameterization $\phi: U \subset M \to \mathbb{C}^2$, and an orientation preserving parameterization $\psi: V \subset \Sigma \to \mathbb{C}$ such that the following properties are satisfied:

(1) $x \in U$, and $\phi(x) = (0,0) \in \mathbb{C}^2$.

(2) $f(x) \in V$, and $\psi(f(x)) = 0 \in \mathbb{C}$.

(3) For the map $g: \mathbb{C}^2 \to \mathbb{C}$ given by $g(z_1, z_2) = z_1.z_2$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{C}^2 \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} \\ V & \xrightarrow{\psi} & \mathbb{C} \end{array}$$

Remark 2.2 (a) Observe that both M as well as Σ can have nonempty boundary, however, it follows from Definition 2.1 that the critical point c belongs to the interior $\mathring{M}$ of M, and $f(c) \in \mathring{\Sigma}$.

(b) Let $f: M \to S$ be a map with an isolated Lefschetz singularity at $c \in M$ such that $f(c) \in S$ is an isolated critical value. It is well known that generically the fiber over $f(c)$ is obtained by pinching a simple closed curve γ on a nearby smooth fiber Σ_g to a point. The curve γ is known as a *vanishing cycle*.

Next, we recall the definition of 1-*fold singularity*.

Definition 2.3 (1-fold singularity) Let M be an oriented 4-manifold, and let Σ be an oriented surface. Let $f: M \to \Sigma$ be a smooth map. A point $x \in M$ is said to be a 1-fold singularity of the map f, provided there is an orientation preserving parameterization $\phi: U \subset M \to \mathbb{R}^4$, and an orientation preserving parameterization $\psi: V \subset \Sigma \to \mathbb{R}^2$ such that the following properties are satisfied:

(1) $x \in U$, and $\phi(x) = (0,0,0,0) \in \mathbb{R}^4$.

(2) $f(x) \in V$, and $\psi(f(x)) = (0,0) \in \mathbb{R}^2$.

(3) For the map $h: \mathbb{R}^4 \to \mathbb{R}^2$ given by $h(t, x_1, x_2, x_3) = (t, -x_1^2 + x_2^2 + x_3^2)$, the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \mathbb{R}^4 \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle h} \\ V & \xrightarrow{\psi} & \mathbb{R}^2 \end{array}$$

Remark 2.4 (a) If a map $f : M \to \Sigma$ has a 1-fold singularity at x, then $x \in \mathring{M}$, and $f(x) \in \mathring{\Sigma}$.

(b) When the map h in the definition of 1-fold singularity is allowed to have the local model

$$(t, x_1, x_2, x_3) \to (t, \pm x_1^2 \pm x_2^2 \pm x_3^2),$$

the singularity is called a *fold singularity*. In this article, we will only need the local model around 1-fold singularity.

We are now in a position to recall the notion of a broken Lefschetz fibration (BLF).

Definition 2.5 (broken Lefschetz fibration) Let M a smooth oriented 4-manifold. By a broken Lefschetz fibration of M we mean a smooth surjective map $f : M \to \mathbb{C}P^1$ such that f has only 1-fold or Lefschetz singularities.

Remark 2.6 (a) Given a BLF $f : M \to \mathbb{C}P^1$, the inverse image $f^{-1}(y)$ for any regular value y is called a fiber of BLF.

(b) Generically, the image set of a component of 1-fold singularities on Σ is an immersed circle in $\mathring{\Sigma}$.

A BLF without 1-fold singularity is called a Lefschetz fibration. These singular fibrations are extremely useful in algebraic geometry [15] and symplectic geometry [10]. Let us now formally define a Lefschetz fibration.

Definition 2.7 (Lefschetz fibration) Let M be a smooth oriented 4-manifold. A smooth surjective map $f : M \to \Sigma$, where Σ is an oriented surface, having its singular points modeled only on Lefschetz singularities is called a Lefschetz fibration of M.

Remark 2.8 (a) Unlike a fiber bundle or Lefschetz fibration, the regular fibers of a BLF are typically not diffeomorphic. In fact, the 1-fold singularity in the definition of BLF corresponds to a round 1-handle attachment [4; 13]. Hence, if BLF has points having fold singularity, then the topology of the regular fiber changes as we cross the image of an immersed circle coming from a 1-fold singularity.

(b) The fibers of BLF need not be connected. However, it can be shown that every 4-manifold admits a BLF with connected fibers having genus at least 2. This follows from [3, Theorem 1.1].

Observe that a BLF provides us a decomposition of a smooth manifold into simple pieces. A more simplified form of this decomposition of a smooth 4-manifold is what we will need for this article. This simplification was introduced by R. I. Baykur [4], and the proof of this simplified decomposition was given by R. I. Baykur and O. Saeki [5; 6]. This decomposition is known as a simplified broken Lefschetz fibration. Let us recall the definition of this:

Definition 2.9 (simplified broken Lefschetz fibration (SBLF)) Let $f : M \to \mathbb{C}P^1$ be a BLF. We say that this BLF is a simplified broken Lefschetz fibration (SBLF) provided the function f satisfies the following additional properties:

(1) The set Z_f of all $x \in M$ admitting a 1-fold singularity model is connected.

(2) All fibers are connected.

(3) The map f is injective when restricted to Z_f as well as when restricted to the set, C_f, of Lefschetz singular points and the set C_f is contained in the connected component of $\mathbb{C}P^1 \setminus f(Z_f)$ which has regular fibers of higher genus.

Remark 2.10 (a) Throughout this article, we will assume without loss of generality that for any Lefschetz fibration (or BLF) $f : M \to \Sigma$ any regular fiber $f^{-1}(y)$ is connected, and f is injective when restricted to critical set C_f.

(b) Observe that the definition of SBLF implies that there exists a disc $\mathcal{D}$ contained in $\mathbb{C}P^1$ such that every $y \in \mathcal{D}$ is a regular value, and the genus of the fiber over y is minimum among all fibers of SBLF. We call this fiber the *lower genus fiber*.

(c) Topologically, the unique 1-fold singularity of SBLF corresponds to adding a 1-handle to a circle worth of lower genus fibers over $\partial\mathcal{D}$. This corresponds to an attachment of a round 1-handle to $f^{-1}(\mathcal{D})$ such that a generic fiber of SLBF over $\mathbb{C}P^1 \setminus \overline{\mathcal{D}}$ has genus one more than the fibers over $\mathcal{D}$.

In [5; 6], it was shown that every orientable smooth 4-manifold admits an SBLF.

Theorem 2.11 (R. I. Baykur, O. Saeki [5, Theorem 1]) *Given any generic map from a closed, connected, oriented, smooth* 4*-manifold* X *to* $\mathbb{C}P^1$*, there are explicit algorithms to modify it to an SBLF. In particular, every closed orientable smooth* 4*-manifold admits an SBLF. Furthermore, we can always construct an SBLF on* M *such that the genus of the lower genus figure is bigger than* 1.

We would like to point out that Theorem 2.11 is not stated as above in [5]. The statement regarding the lower bound on the genus of a lower genus fiber is not explicitly mentioned in [5, Theorem 1]. However, it follows from the application of [5, Theorem 1] followed by [5, Theorem 2]. For the sake of completeness, we discuss the proof of Theorem 2.11.

Proof To begin with, recall that by a *trisection* of a smooth orientable closed 4-manifold M one means a decomposition of M into three 4-dimensional handlebodies (thickening of a wedge of circles), meeting pairwise in 3-dimensional handlebodies, and all three 4-dimensional handlebodies intersect in a surface. A trisection corresponds to a Morse 2-function on M. If k' is the number of indefinite folds for the Morse 2-function associated to a given trisection and g' is the genus of the surface corresponding to the common intersections of three 4-dimensional handlebodies, one says that the 4-manifold has a (g', k')-trisection.

In order to produce an SBLF as stated in Theorem 2.11, we observe that given M, according to [5, Theorem 1], there exists an SBLF $f : M \to \mathbb{C}P^1$. Let g be the genus of the lower genus fiber of the SBLF. If $g > 1$, then we are through. In case, $g \leq 1$, we apply [5, Theorem 2] to produce a (g', k')-trisection from the given SBLF $f : M \to \mathbb{C}P^1$. According to [5, Theorem 2], we get a (g', k')-trisection with $g \geq 1$.

Next, we again apply the second part of [5, Theorem 2] to produce from this trisection a new SBLF. Observe that according to [5, Theorem 2], the new SBLF has a lower genus fiber having its genus $g' + 2$. Since $g' \geq 0$, the theorem follows. □

We would like to remark that the proof of the existence of SBLF with higher genus fiber also follows from the proof of Proposition 1.3 given in [3].

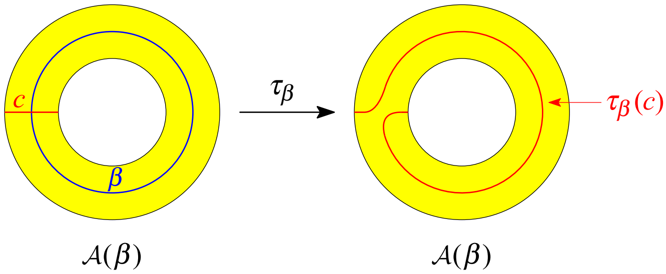

Figure 1: The figure is a pictorial description of the Dehn twist τ_β restricted to the neighborhood $\mathcal{A}(\beta) = \mathbb{S}^1 \times [0, 2\pi]$. The map τ_β is given by $\tau_\beta(\theta, t) = (\theta - t, t)$ when restricted to $\mathcal{A}(\beta)$. It sends the arc c — depicted as a red-colored arc in the picture on the left of the figure — to an arc isotopic to the arc $\tau_\beta(c)$ depicted in the picture on the right of the figure.

3 Mapping class groups of surfaces

In this section, we review some results related to mapping class groups of closed-orientable surfaces. Good references for the results discussed here are [12; 21]. Let us begin by recalling the definition of the mapping class group.

Definition 3.1 (mapping class group) Let Σ be a closed oriented surface. By the mapping class group of Σ, we mean the group of orientation preserving self diffeomorphisms of Σ up to isotopy.

We denote the mapping class group of a surface Σ by $\mathcal{MCG}(\Sigma)$. Next, let us discuss the notion of a *Dehn twist* along a simple closed curve embedded in a surface Σ. We refer to [12] for a more detailed discussion on Dehn twists.

Definition 3.2 (Dehn twist) Let Σ be an orientable surface. Let β be a simple closed curve embedded in the interior of Σ. By a Dehn twist along β, we mean a diffeomorphism which is identity outside an annulus neighborhood $\mathcal{A}(\beta)$ of β in Σ, and is given by τ_β on $\mathcal{A}(\beta)$ when restricted to $\mathcal{A}(\beta)$, where τ_β is the diffeomorphism of $\mathcal{A}(\beta)$ described in Figure 1.

M. Dehn [8] (see also [21]) established that the mapping class group of an orientable genus g surface Σ_g is generated by Dehn twists along simple closed curves embedded in Σ_g. W. Lickorish [22] further strengthened this result to show that the mapping class group of a closed orientable surface Σ_g is generated by Dehn twists along the curves a_i's, b_j's and c_k's as depicted in Figure 2. Following [24], we will refer to these curves as *Lickorish generators*.

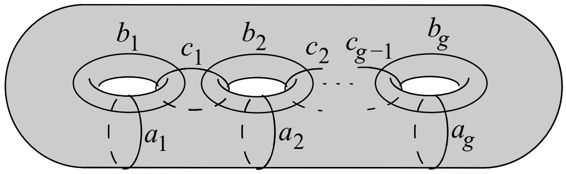

Figure 2: Dehn twists along curves a_i's, b_j's and c_k's generate the mapping class group of an orientable genus g surface.

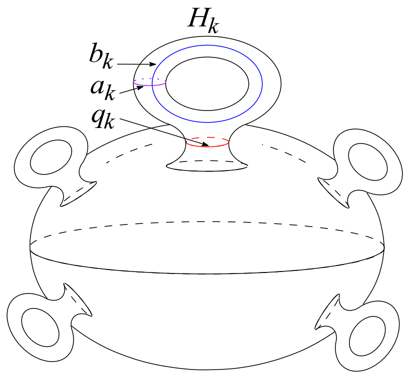

Figure 3: The figure shows a surface of genus g embedded in $\mathbb{R}^3$ as a boundary of a genus g handlebody S_g considered as a unit ball with g 1-handles attached to it.

We end this section with a proposition that is a consequence of [21, Lemma 3]. In order to state this proposition we need some terminology from [21].

Let us regard an orientable surface Σ_g of genus g as the boundary of a standard handlebody S_g. Here, a standard handlebody S_g consists of g 1-handles attached to the unit 3-ball in $\mathbb{R}^3$ as depicted in Figure 3.

Consider a typical handle H_k, as shown in Figure 3. Following [21], we say that a simple closed curve p *does not meet* the handle H_k provided it does not intersect the curve a_k depicted Figure 3.

Proposition 3.3 (Lickorish [21, Lemma 3]) *Let p be any simple closed curve on Σ_g. There exists a diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $\phi(p)$ does not meet any handle of Σ_g.*

4 Lefschetz fibration embedding

Recall from Remark 2.10 that a Lefschetz fibration (LF) of a closed oriented 4-manifold is a pair $(M, \pi : M \to \Sigma)$, where $\pi : M \to \Sigma$ is an LF and Σ is either a disc or $\mathbb{C}P^1$. Furthermore, we always assume that π is injective when restricted to the critical set. Given such an LF, in this section, we show that there exists an embedding of the LF into certain manifolds of type $N^4 \times \Sigma$ which is fiber preserving in the sense of Definition 4.10 provided the genus of the regular fiber is at least 2. This result (Theorem 4.11) can be regarded as the first step towards establishing Theorem 1.1.

4.1 Flexible embedding in standard position

Let us begin this subsection by reviewing the notion of *flexible embedding*.

Definition 4.1 (flexible embedding) Let M be an orientable closed smooth manifold. A smooth embedding $\phi : \Sigma_g \hookrightarrow M$ of a closed orientable surface Σ_g is said to be flexible provided for every $f \in \mathcal{MCG}(\Sigma_g)$ there exists a diffeomorphism ψ of M isotopic to the identity which maps $\phi(\Sigma_g)$ to itself and satisfies $\phi^{-1} \circ \psi \circ \phi = f$.

Next, we state a lemma regarding a flexible embedding of any surface of genus g into a 4-manifold N, which admits a separable Hopf link. In order to state this lemma, we need to introduce the following definitions:

Definition 4.2 (embedding in standard position) An embedding $\phi : \Sigma_g \hookrightarrow N$ of a surface Σ_g is said to be in a standard position provided the following properties are satisfied:

(1) Every simple closed curve γ on $\phi(\Sigma)$ is a boundary of a 2-disc $\mathbb{D}^2$ intersecting $\phi(\Sigma_g)$ only in γ.

(2) There exists a tubular neighborhood $\mathcal{N}(\mathbb{D})$ of the disc $\mathbb{D}^2$ having the boundary γ such that $\mathcal{N}(\mathbb{D})$ is the image of a coordinate chart $\phi_\gamma : \mathbb{C}^2 \to \mathcal{N}(\mathbb{D})$ satisfying the following:

- $\phi_\gamma^{-1}(\phi(\Sigma_g) \cap \mathcal{N}(\mathbb{D}))$ is $g^{-1}(1)$, where $g : \mathbb{C}^2 \to \mathbb{C}$ is the polynomial map $g(z_1, z_2) = z_1.z_2$.

Remark 4.3 The standard position embedding $\phi : \Sigma_g \to N$ in Definition 4.2 can be thought of as an embedding of Σ_g in N such that for every simple closed curve γ on $\phi(\Sigma_g)$, there exists a compact 4-ball B_γ^4 in N such that the intersection of $\phi(\Sigma_g)$ with B_γ^4 is a Hopf annulus in $\partial B\gamma$.

The equivalence between Definition 4.2 and Remark 4.3 plays an important role in the constructions of embeddings throughout this article. We now outline this equivalence. Let $\pi : D^4 \to D^2$ be a Lefschetz fibration of a 4-ball D^4 whose regular fiber is an annulus, and which has a single Lefschetz singularity at the origin $0 \in D^4$, with corresponding singular value $0 \in D^2$. Here, D^2 denotes the closed 2-disc of radius 2 in $\mathbb{R}^2$, centered at the origin. One can easily see the following:

(1) For each disc $D_r^2 \subset D^2$ of positive radius r, $\pi^{-1}(D_r^2)$ is diffeomorphic to a compact 4-ball and its boundary $S_r^3 = \partial(\pi^{-1}(D_r^2))$ is diffeomorphic to the 3-sphere.

(2) The Lefschetz fibration $\pi : \pi^{-1}(D_r^2) \to D_r^2$ induces an open book on $\partial\pi^{-1}(D_r^2)$ with a page an annulus $A_r = \pi^{-1}(re^{i0})$ and the monodromy a Dehn twist along the central curve of A_r.

(3) The annulus A_r is a Hopf annulus in $S_r^3 = \partial(\pi^{-1}(D_r^2))$.

(4) There are orientation preserving diffeomorphisms $\chi : \mathring{D}^4 \to \mathbb{C}^2$ and $\xi : \mathring{D}^2 \to \mathbb{C}^2$ such that $\xi(S^1) = S^1$ and the diagram

$$\begin{array}{ccc} \mathring{D}^4 & \xrightarrow{\chi} & \mathbb{C}^2 \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle g} \\ \mathring{D}^2 & \xrightarrow{\xi} & \mathbb{C} \end{array}$$

commutes, where $\mathring{D}^4$ and $\mathring{D}^2$ are the interiors of D^4 and D^2, respectively, and the map g is given by $g(z_1, z_2) = z_1.z_2$.

Hence, the 4-ball B_γ^4 in Remark 4.3 can be realized as the closure of the image of the map $\phi_\gamma \circ \chi : \pi^{-1}(D_1^2) \to \mathcal{N}(\mathbb{D})$.

Conversely, suppose we are given a 4-ball B_γ^4 in N such that the intersection of $\phi(\Sigma_g)$ with B_γ^4 is an Hopf annulus $\mathcal{H}$ in $\partial B\gamma$. We know that the 4-ball B_γ^4 admits a Lefschetz fibration $\pi' : B_\gamma^4 \to D_1^2$

with only one Lefschetz singularity at $p \in B^4_\gamma$ such that $\pi'(p) = 0 \in D^2_1$ and $\pi^{-1}(e^{i0}) = \mathcal{H}'$, where $\mathcal{H}'$ is a Hopf annulus in ∂B^4_γ. Since any two positive (negative) Hopf annuli in $S^3 = \partial B^4_\gamma$ are isotopic, we can assume $\pi'^{-1}(e^{i0}) = \mathcal{H}$. Now, we first identify B^4_γ with $\pi^{-1}(D^2_1) \subset D^4$ by a diffeomorphism $h : \pi^{-1}(D^2_1) \to B^4_\gamma$ such that the diagram

$$\begin{array}{ccc} N \supset B^4_\gamma & \xleftarrow{\;h\;} & \pi^{-1}(D^2_1) \subset D^4 \\ \big\downarrow \pi' & & \big\downarrow \pi \\ D^2_1 & \xleftarrow{\;\mathrm{Id}\;} & D^2_1 \subset D^2 \end{array}$$

commutes. Then the desired chart $\phi_\gamma : \mathbb{C}^2 \to \mathcal{N}(\mathbb{D})$ can be defined by appropriately extending the map $h \circ \chi^{-1} : \chi(\pi^{-1}(\mathring{D}^2_1)) \subset \mathbb{C}^2 \to N$. Under this chart, we have

$$\begin{aligned} \phi_\gamma^{-1}(\phi(\Sigma_g) \cap B^4_\gamma)) &= \chi\big(h^{-1}(\phi(\Sigma_g) \cap B^4_\gamma)\big) = \chi(h^{-1}(\mathcal{H})) \\ &= \chi\big(h^{-1}(\pi'^{-1}(e^{i0}))\big) = \chi(\pi^{-1}(e^{i0})) = g^{-1}(e^{i0}), \end{aligned}$$

as desired.

Definition 4.4 (separable Hopf link) We say that a link $l_1 \sqcup l_2$ in a 4-manifold N is a separable Hopf link provided the following properties are satisfied:

(1) There exist an embedding of a 4-ball $\mathbb{D}^4 = \mathbb{D}^2 \times \mathbb{D}^2$ in N such that $\partial\mathbb{D}^2 \times \{0\} \sqcup \{0\} \times \partial\mathbb{D}^2 = l_1 \sqcup l_2$.

(2) There exists two disjoint properly embedded discs $\mathcal{D}_1$ and $\mathcal{D}_2$ in $N \setminus (\mathbb{D}^2 \times \mathbb{D}^2)^\circ$ such that $\partial\mathcal{D}_1 = l_1$ and $\partial\mathcal{D}_2 = l_2$.

Lemma 4.5 *Let N be a 4-manifold which admits a separable Hopf link. Then there exists an embedding ϕ of any closed orientable surface Σ_g of genus g in N which satisfies the following*:

(1) *The embedding is flexible.*

(2) *The embedding is in a standard position.*

Before we establish this lemma, we would like to point out that the flexible embedding of Σ_g in N was first provided by S. Hirose and A. Yasuhara [18]. Our main observation is that we can achieve the additional property of the embedding being in a standard position, provided that we use Proposition 3.3 established by Lickorish [21] in conjunction with the techniques from [18].

Proof of Lemma 4.5 We want to construct an embedding of Σ_g in N which is both flexible and in a standard position. Let $l_1 \sqcup l_2$ be a separable Hopf link in N. It follows from the definition of separable link that there exists an embedded 4-ball $\mathbb{D}^4 = \mathbb{D}^2 \times \mathbb{D}^2$ in N such that $\partial\mathbb{D}^2 \times \{0\} \sqcup \{0\} \times \partial\mathbb{D}^2 = l_1 \sqcup l_2$, and there exists two disjoint properly embedded discs $\mathcal{D}_1$ and $\mathcal{D}_2$ in $N \setminus (\mathbb{D}^2 \times \mathbb{D}^2)^\circ$ such that $\partial\mathcal{D}_1 = l_1$ and $\partial\mathcal{D}_2 = l_2$. We regard the 4-ball $\mathbb{D}^4$ as the 4-ball $B^4(0,2)$ of radius 2 in $\mathbb{C}^2$ with its center at the origin. We will also regard $\mathbb{S}^3 \times [1,2]$ as the collar $B^4(0,2) \setminus B^4(0,1)$ contained in N.

Next, observe that the link $l_1 \times \{\frac{3}{2}\} \sqcup l_2 \times \{\frac{3}{2}\}$ bounds a Hopf band say $\mathcal{H}$ in $\mathbb{S}^3 \times \{\frac{3}{2}\}$. We embed a genus g surface Σ_g in $\mathbb{S}^3 \times \{\frac{3}{2}\} \subset \mathbb{S}^3 \times [1,2] \subset N$ as the boundary of standard genus g handle body H_g

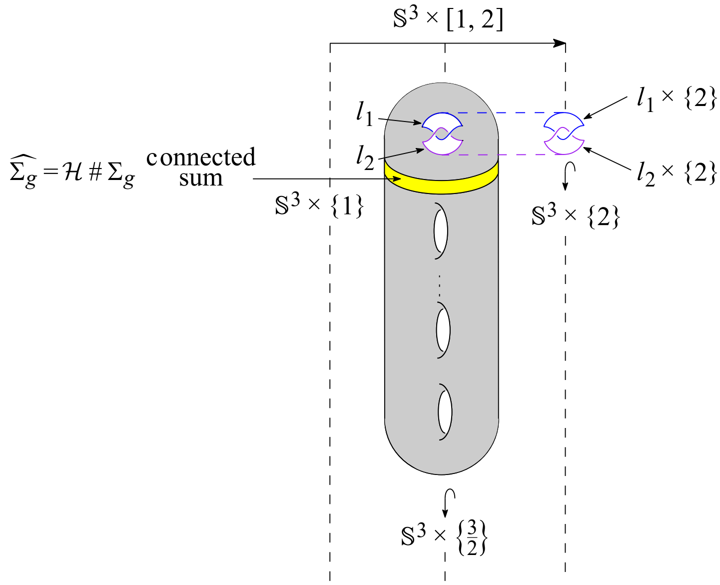

Figure 4: The figure depicts the embedding of the surface Σ_g which is flexible as well as in the standard position. The figure depicts the collar $\mathbb{S}^3 \times [1, 2] \subset N$ with dashed lines representing $\mathbb{S}^3$ at levels 1, 2 and $\frac{3}{2}$.

and disjoint form $\mathcal{H}$ as depicted in Figure 3. Then we take ambient connected sum of embedded Σ_g and $\mathcal{H}$ in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ to obtain a surface $\widehat{\Sigma_g}$ with two boundary components as shown in Figure 4. Thus by adding two cylinders $l_1 \sqcup l_2 \times \left[\frac{3}{2}, 2\right]$ and two disjoint discs $\mathcal{D}_1$, $\mathcal{D}_2$ to $\widehat{\Sigma_g}$, we obtain an embedding of a closed genus g surface. Let us denote this embedding — after smoothing the corners — by ϕ. For a pictorial description of the embedding ϕ, we refer the reader to Figure 4. We claim that the embedding $\phi : \Sigma_g \hookrightarrow N$ is both flexible and in standard position. Let us now establish this claim.

The claim that the embedding is flexible is already established in [18, Theorem 3.1]. Let us briefly review the argument. First of all, notice that every Lickorish generator γ of Σ_g embedded in N via ϕ has — up to an isotopy — a Hopf annulus neighborhood which is contained in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\} \subset N$. Next, recall that the mapping class group of Σ_g is generated by Dehn twists along Lickorish generators, and in $\mathbb{S}^3$ there exists a diffeomorphism isotopic to the identity which induces a Dehn twist on a given Hopf annulus fixing its boundary pointwise. In the proof of [18, Theorem 3.1] it is shown that this implies that there exists a diffeomorphism of N isotopic to the identity which induces a Dehn twist along a Lickorish generator of $\phi(\Sigma_g)$. The claim now follows by successive application of ambient isotopies of N inducing Dehn twists on Lickorish generators. See also [24, Lemma 15] for additional details.

Let us now show that the embedding is in a standard position. First of all notice that — by the very construction, any simple closed curve on $\phi(\Sigma_g)$ can be isotoped on the surface $\phi(\Sigma_g)$ such that it is contained in $\phi(\Sigma) \cap \mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$. We claim that any simple closed curve which does not meet handles[2]

[2]Recall that a simple closed curve p does not meet the handle H_k provided it does not intersect the curve a_k depicted in Figure 3.

of $\phi(\Sigma_g)$ satisfies both the properties necessary for an embedding to be in a standard position. This is because:

(1) All curves mentioned in the claim are unknots in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ hence they bound a disc in $\mathbb{S}^3 \times \left[1, \frac{3}{2}\right]$, that meets $\phi(\Sigma)$ only in the given curve.

(2) Any curve γ mentioned in the claim is isotopic to a simple closed curve C in $\phi(\Sigma_g) \cap \mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ via an isotopy of $\phi(\Sigma_g)$ such that C admits a neighborhood $\mathcal{N}(C)$ in $\phi(\Sigma_g)$ which is a Hopf band in $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$.

It follows from both the properties listed above that any simple closed curve C, which does not meet any handle, satisfies both the properties necessary for a surface to be in the standard position.

Now, according to Proposition 3.3, given any simple closed curve C, there exists a diffeomorphism of $\phi(\Sigma_g)$ which sends C to a curve which does not meet any handle. Since the embedding ϕ of Σ_g is flexible in N, given a simple closed curve C which meets some handles can be isotoped so that now it does not meet any handle. Hence, the claim that the embedding is also in a standard position follows. □

Remark 4.6 (1) Any simple closed curve C in $\phi(\Sigma_g)$ that does not meet handles of $\phi(\Sigma_g)$ can be isotoped to a curve C' in $\phi(\Sigma_g) \cap \mathbb{S}^3 \times \frac{3}{2}$ such that an annular neighborhood $\mathcal{A} = \mathcal{N}(C')$ of C' in $\phi(\Sigma_g)$ is a planar annulus in $\mathbb{S}^3 \times \frac{3}{2}$. Therefore, there exists an embedded 4-ball $B^4 = B^3 \times [0,1]$ in N such that $\mathcal{A} = B^3 \times \{0\} \cap \phi(\Sigma_g)$, $\mathcal{A}$ is a planar annulus on $\partial(B^3 \times \{0\})$.

(2) Since the embedded surface $\phi(\Sigma_g)$ is flexible, for given any simple closed curve γ, there exists an embedded 4-ball $B^4_\gamma = B^3_\gamma \times [0,1]$ in N such that $B^3_\gamma \times \{0\} \cap \phi(\Sigma_g) = \mathcal{A}_\gamma$ is a annular neighborhood of γ in $\phi(\Sigma_g)$ and $\mathcal{A}_\gamma$ is a planar annulus on $\partial(B^3_\gamma \times \{0\})$.

In what follows we will work with embeddings of surfaces in N constructed using the procedure described in the proof of Lemma 4.5. We will use the term *standard embedding* for any such embedding. More precisely, we have the following:

Definition 4.7 (standard embedding) Let N be a manifold admitting a separable Hopf link. An embedding ψ of a closed orientable surface Σ_g, which is isotopic to an embedding obtained by the procedure described in the proof of Lemma 4.5, will be called a standard embedding of Σ_g

We end this subsection by establishing an embedding result regarding the embeddings of mapping tori in $N \times \mathbb{S}^1$. Recall that given a manifold Σ, the mapping torus of Σ with monodromy g, where g is a diffeomorphism of Σ, is the quotient space $\Sigma \times [0,1]/\sim$, where $(x,0) \sim (g(x),1)$. Throughout this article we will consider mapping tori up to the ambient isotopy class of g in Σ. We will denote the mapping torus by $\mathcal{MT}(\Sigma, g)$. Notice that $\mathcal{MT}(\Sigma, g)$ is a fiber bundle over $\mathbb{S}^1$. Our next lemma establishes a fiber-preserving embedding of any mapping tours of Σ into $N \times \mathbb{S}^1$. More precisely:

Lemma 4.8 *Let N be a 4-manifold admitting a separable Hopf link and let $\phi : \Sigma_g \to N$ be a standard embedding of Σ_g. Let $d_\gamma : \Sigma_g \to \Sigma_g$ be a Dehn twist along a simple closed curve γ on Σ_g. Then there* *exists an embedding Ψ of $\mathcal{M}T(\Sigma_g, d_\gamma)$ in $N \times \mathbb{S}^1$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}T(\Sigma_g, d_\gamma) & \xrightarrow{\Psi} & N \times \mathbb{S}^1 \\ \downarrow \pi & & \downarrow \pi_2 \\ \mathbb{S}^1 & \xrightarrow{\text{Id}} & \mathbb{S}^1 \end{array}$$

Proof Let $\mathbb{D}^2$ be a disc in N such that it intersects $\phi(\Sigma_g)$ in only γ. Let $\mathcal{N}(\mathbb{D}^2)$ be a neighborhood in $\mathbb{D}^2$ such that $\mathcal{N}(\mathbb{D}^2)$ is the image of the coordinate chart $\phi_\gamma : \mathbb{C}^2 \to \mathcal{N}(\mathbb{D}^2)$ with $\phi_\gamma^{-1}(\phi(\Sigma_g) \cap \mathcal{N}(\mathbb{D}^2)) = g^{-1}(1)$, where the map $g : \mathbb{C}^2 \to \mathbb{C}$ is given by $g(z_1, z_2) = z_1.z_2$. Note that the monodromy of the Lefschetz fibration g over the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ is a Dehn twist along the central curve of the annulus $g^{-1}(1)$. Therefore, there is a flow $\rho_t : \mathbb{C}^2 \to \mathbb{C}^2,\ 0 \le t \le 1$, supported in $g^{-1}(\mathcal{N}(S^1))$ such that

(1) $\rho_t(g^{-1}(e^{i0})) = g^{-1}(e^{2\pi i t})$ and

(2) ρ_1 restricted to $g^{-1}(e^{i0})$ is a Dehn twist along the central curve of the annulus $g^{-1}(e^{i0})$,

where $\mathcal{N}(S^1)$ is a small annulus neighborhood of S^1 in $\mathbb{C}$. Using this flow, we can define a flow $\phi_\gamma \circ \rho_t \circ \phi_\gamma^{-1} : \mathcal{N}(\mathbb{D}^2) \to \mathcal{N}(\mathbb{D}^2),\ 0 \le t \le 1$. Since the flow ρ_t is supported in $g^{-1}(\mathcal{N}(S^1))$, the flow $\phi_\gamma \circ \rho_t \circ \phi_\gamma^{-1}$ on $\mathcal{N}(\mathbb{D}^2)$ can be extended to a flow $\xi_t : N \to N,\ 0 \le t \le 1$, by defining $\xi_t = \phi_\gamma \circ \rho_t \circ \phi_\gamma^{-1}$ on $\mathcal{N}(\mathbb{D}^2)$ and $\xi_t = \text{Id}$ in the complement of $\mathcal{N}(\mathbb{D}^2)$ in N. Now, the desired embedding $\Psi : \mathcal{M}T(\Sigma_g, d_\gamma) \to N \times \mathbb{S}^1$ is given by $\Psi(x, t) = (\xi_t \circ \phi(x), e^{2\pi i t})$. □

Definition 4.9 Let N be a manifold admitting a separable Hopf link and let $\phi : \Sigma_g \to N$ be a standard embedding of Σ_g. Let $d_\gamma : \Sigma_g \to \Sigma_g$ be a Dehn twist along a simple closed curve γ on Σ_g. Then, the embedding $\Psi_\gamma : \mathcal{M}T(\Sigma_g, d_\gamma) \to N \times \mathbb{S}^1$ constructed in the proof of the above lemma will be called the standard embedding of $\mathcal{M}T(\Sigma_g, d_\gamma)$ in $N \times \mathbb{S}^1$ with respect to the standard embedding ϕ and the Dehn twist d_γ.

Before we proceed, we would like to point out that Lemma 4.8 was implicitly established in [24].

4.2 The existence of Lefschetz fibration embedding

We are now in a position to state and prove our main result regarding *Lefschetz fibration embeddings*. As usual, we denote the map $N \times \mathbb{C}P^1$ to $\mathbb{C}P^1$ corresponding to the projection on the second factor by π_2.

Definition 4.10 (Lefschetz fibration embedding) Let $(M, \pi : M \to \Sigma)$ be a Lefschetz fibration, where Σ is a 2-disc or $\mathbb{C}P^1$. An embedding $f : M \to N \times \mathbb{C}P^1$ of a manifold M into a manifold $N \times \mathbb{C}P^1$ is said to be a *Lefschetz fibration embedding* provided $\pi_2 \circ f = i \circ \pi$, where i is an inclusion of $\mathbb{D}^2$ in $\mathbb{C}P^1$ when $\partial M \ne \varnothing$, otherwise it is the identity.

Theorem 4.11 *Let M be an orientable smooth 4-manifold. Let N be a 4-manifold which admits a separable Hopf link. Let $\pi : M \to \Sigma$, where Σ is either $\mathbb{C}P^1$ or a 2-disc $\mathbb{D}^2$ embedded in $\mathbb{C}P^1$, be a Lefschetz fibration (LF) of M having genus g fibers with $g \ge 2$. If the map π is injective when restricted to the set of critical points of π, then there exists a Lefschetz fibration embedding of (M, π) in $(N \times \mathbb{C}P^1, \pi_2)$.*

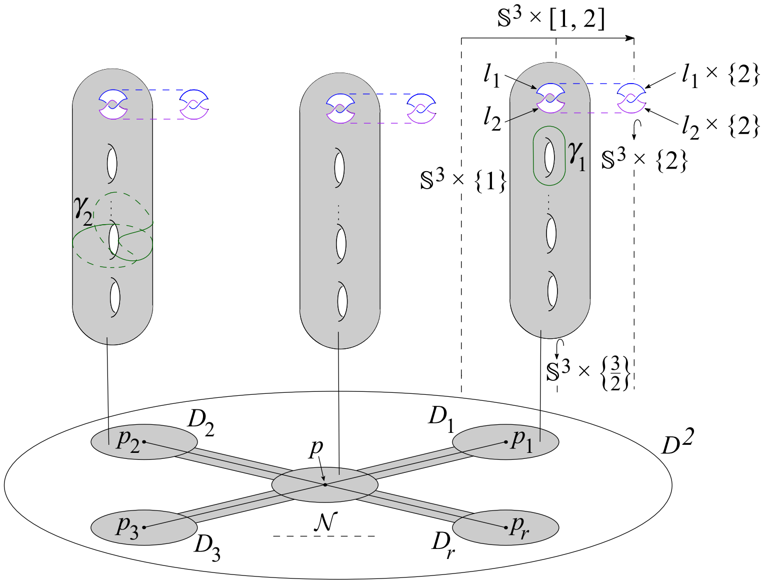

Figure 5: The figure depicts part of a Lefschetz fibration (M, π) over a disc embedded as a Lefschetz fibration in the (Lefschetz) fibration $\pi_2 : N \times D^2 \to D^2$. The embedding is such that the generic fiber of (M, π) is a flexible embedding in the standard position in N. The curves on the surface depict the vanishing cycles γ_i's.

Proof Let $c_1, c_2, \dots, c_k$ be k critical points of the Lefschetz fibration (M, π). Since the Lefschetz fibration π is injective when restricted to the set of critical points, points $\pi(c_1) = p_1, \pi(c_2) = p_2, \dots,$ and $\pi(c_k) = p_k$ are distinct points on Σ. Let γ_i be the vanishing cycle corresponding to the critical point c_i on a generic fiber Σ_g of the LF.

Let U_i be an open ball in M around c_i such that on U_i we have coordinates (z_1, z_2) such that π in this coordinates is given by $(z_1, z_2) \to z_1.z_2$. Let $\widetilde{D}_i = \pi(U_i) \subset \Sigma$. Let D_i be an open disc containing p_i with $\overline{D_i} \subset \widetilde{D}_i$.

Let $\widehat{\Sigma}$ be Σ in case $\Sigma = \mathbb{D}^2$ or $\widehat{\Sigma} = \mathbb{C}P^1 \setminus D$ where D is a small open disc in $\mathbb{C}P^1$ lying in the complement of the set $\{p_1, \dots, p_k\}$. We will first produce an embedding $\widehat{f}$ of the fibration π restricted to $\pi^{-1}(\widehat{\Sigma})$. Denote by $\widehat{M}$ the manifold $\pi^{-1}(\widehat{\Sigma})$. We have that $\widehat{M} = M$ when $\Sigma = \mathbb{D}^2$.

Embedding of $\widehat{M}$ Consider an embedding ϕ of the fiber Σ_g in N which is a standard embedding. Recall that the existence of such an embedding is the content of Lemma 4.5.

Using the flexibility of the embedding ϕ, we first produce $\widehat{f}$ restricted to $\widehat{M} \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i)$ in the manifold $N \times \left(\mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i\right)$ such that the following diagram commutes:

$$\begin{array}{ccc} M \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i) & \xrightarrow{\widehat{f}} & N \times \left(\mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i\right) \\ \downarrow \pi & & \downarrow \pi_2 \\ \mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i & \xrightarrow{\text{Id}} & \mathbb{C}P^1 \setminus \bigsqcup_{i=1}^k D_i \end{array} \tag{1}$$

Since the embedding of Σ_g in N is standard, by Lemma 4.8, there exists a standard embedding $\Psi_i : \mathcal{M}T(\Sigma_g, d_{\gamma_i}) \to N \times \mathbb{S}^1$ for each $1 \le i \le k$ with respect to the standard embedding ϕ and the Dehn twist d_{γ_i}. Note that for each i, the embedding Ψ_i is such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}T(\Sigma_g, d_{\gamma_i}) & \xrightarrow{\Psi_i} & N \times \mathbb{S}^1 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{S}^1 & \xrightarrow{\text{Id}} & \mathbb{S}^1 \end{array} \tag{2}$$

Next, considering $\partial D_i \subset \mathbb{C}P^1 = \mathbb{S}^1$ for each i, the embeddings Ψ_i's together give an embedding $\Psi : \bigsqcup_{i=1}^k \pi^{-1}(\partial D_i) \to \bigsqcup_{i=1}^k N \times \partial D_i$ such that $\pi_2 \circ \Psi = \pi$. Now for each i take an arc α_i connecting a point on ∂D_i to a fixed regular value p for the map π in $\widehat{\Sigma}$ as depicted in Figure 5. We can assume that $\widehat{\Sigma}$ is a regular neighborhood of the set $\bigsqcup D_i \cup \bigcup_i \alpha_i$. The flexibility of the embedding ϕ now implies that the embedding $\hat{f}$ restricted $\widehat{M} \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i)$ exists such that when restricted to $\partial \widehat{\Sigma}$ this embedding is an embedding of $\mathcal{M}T(\Sigma_g, F)$, where $F = \prod_{i=1}^k \tau_{\gamma_i}$ when $\widehat{\Sigma} = \Sigma$ and $F = \text{id}$ when $\widehat{\Sigma} = \mathbb{C}P^1 \setminus D$.

Our next step is to show how to extend this embedding to produce a Lefschetz fibration embedding $\hat{f}$ of $\widehat{M}$ in $N \times \mathbb{C}P^1$. For this the property that the embedding ϕ of Σ_g is also in the standard position is required.

Since the embedding ϕ is in a standard position — by the definition of an embedding in a standard position given in Definition 4.2 — there exists an embedding of $\phi_{\gamma_i} : \mathbb{C}^2 \hookrightarrow N$ which satisfies the second property listed in Definition 4.2.

Next, for each critical point c_i, we claim that the diagram

$$\begin{array}{ccccccc} U_i \subset M & \xrightarrow{\phi_i} & \mathbb{C}^2 & \xrightarrow{i} & \mathbb{C}^2 \times \mathbb{C} & \xrightarrow{f_{c_i}} & N \times \mathbb{C}P^1 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \pi_2} \\ \widetilde{D}_i & \xrightarrow{\psi_i} & \mathbb{C} & \xrightarrow{\text{Id}} & \mathbb{C} & \xrightarrow{\psi_i^{-1}} & \widetilde{D}_i \end{array} \tag{3}$$

commutes, where the definitions of the maps appearing in the diagram are as follows:

(1) $\phi_i : U_i \subset M \to \mathbb{C}^2$ and $\psi_i : \widetilde{D}_i \subset \mathbb{C}P^1 \to \mathbb{C}$ are orientation preserving parameterizations around critical point c_i of π and $\pi(c_i)$, respectively, such that left square commutes in the diagram above.

(2) $i : \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ are defined as $i(z_1, z_2) = (z_1, z_2, 0)$ and $g(z_1, z_2) = z_1.z_2$.

(3) $f_{c_i} : \mathbb{C}^2 \times \mathbb{C} \to N \times \mathbb{C}P^1$ and $P : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$ are defined as

$$f_{c_i}(z_1, z_2, z_3) = (\phi_{\gamma_i}(z_1, z_2), \quad \psi_i^{-1}(z_1.z_2 + z_3)), \quad P(z_1, z_2, z_3) = z_1.z_2 + z_3.$$

The commutativity of the middle square follows directly from the definitions of the maps g, i, and P. Also, the commutativity of the last square is clear by the definition of the map f_{c_i}. Next, we see that the commutative diagram (3) allows us to extend the embedding $\hat{f}$ to the embedding $\hat{f}_{c_i}$ of $\big(\widehat{M} \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i)\big) \cup U_i$. This is possible because $\hat{f}$ and $f_{c_i} \circ i \circ \phi_i$ agree on the overlapping region of the domain. Hence, $\hat{f}$ and $f_{c_i} \circ i \circ \phi_i$ together define a map $\hat{f}_{c_i}$.

Let us now notice that this allows us to extend the embedding $\widehat{f_{c_i}}$ to an embedding $\widehat{f_{c_i}}$ of the space $W_{c_i} = \widehat{M} \setminus \big(\bigcup_{l=1}^{i-1} \pi^{-1}(D_l) \cup \bigcup_{l=i+1}^{k} \pi^{-1}(D_l)\big)$ in $N \times \mathbb{C}P^1$ such that the following diagram commutes:

$$\begin{array}{ccc} W_{c_i} & \xrightarrow{\widehat{f_{c_i}}} & \widehat{f_{c_i}}(W_{c_i}) \subset N \times \mathbb{C}P^1 \\ \big\downarrow \pi & & \big\downarrow \pi_2 \\ \pi(W_{c_i}) \subset \mathbb{C}P^1 & \xrightarrow{\mathrm{Id}} & \pi_2(\widehat{f_{c_i}}(W_{c_i})) = \pi(W_{c_i}) \end{array} \tag{4}$$

Observe that by construction the embeddings $\widehat{f_{c_i}}$ and $\widehat{f_{c_j}}$ agree on $W_{c_i} \cap W_{c_j}$. Since $\widehat{M} = \bigcup_{i=1}^{k} W_{c_i}$ we get an embedding $\widehat{f}$ of $\widehat{M}$ with the required properties.

Embedding of M When $\Sigma = \mathbb{D}^2$ there is nothing to prove as in this case $\widehat{M} = M$. In the case when $\Sigma = \mathbb{C}P^1$, we recall that the embedding $\widehat{f}$ is constructed so that $\widehat{f}$ restricted to $\partial\widehat{M}$ is an embedding of $\Sigma_g \times \mathbb{S}^1$ in $N \times \mathbb{S}^1 = N \times \partial\widehat{\Sigma}$. When we regard the boundary $\partial N \times D$ as $N \times \mathbb{S}^1$, we get an embedding of $\mathcal{MT}(\Sigma_g, \mathrm{id})$ in $\mathcal{MT}(N, \mathrm{id}) = N \times \mathbb{S}^1$ via the embedding $\widehat{f}$. Hence, we get an embedding of a closed manifold $\widetilde{M}$ obtained by identifying $\partial\widehat{M}$ with $\partial\Sigma_g \times D$ along the common boundary via a diffeomorphism of $\Sigma_g \times \mathbb{S}^1$. Since the genus g of Σ_g is at least 2, it follows from the triviality of the group $\pi_1(\mathrm{Diff}_0(\Sigma_g))$ — the identity connected component of the group of diffeomorphisms of Σ_g — proved in [11, Theorem 1] that $\widetilde{M} = M$. Hence, we have the required embedding of M in $N \times \mathbb{C}P^1$. □

5 Embeddings of orientable 4-manifolds via SBLF

The purpose of this section is to establish Theorem 1.2. Recall that Theorem 1.2 claims that every closed orientable smooth 4-manifold admits an embedding in a manifold of type $N \times \mathbb{C}P^1$, where N is a 4-manifold admitting separable Hopf link. As mentioned in the introduction while outlining the proof, we will use the SBLF decomposition of a closed orientable smooth 4-manifold for constructing embeddings. We first need the following:

Definition 5.1 (1-fold simple singular fibration) Let $(M, \partial M)$ be an oriented smooth 4-manifold with boundary and let $f : M \to [-1, 1] \times \mathbb{S}^1$ be a smooth surjective map which satisfies the following:

(1) There exists a unique embedded circle Z_f in M of 1-fold singularities for f such that $f(Z_f)$ is an embedded circle in $[-1, 1] \times \mathbb{S}^1$ which is ambiently isotopic to the circle $\{0\} \times \mathbb{S}^1$.

(2) For every $x \in M \setminus Z_f$, $f(x)$ is a regular point for the map f.

(3) $\partial M = f^{-1}(\{-1\} \times \mathbb{S}^1 \sqcup \{1\} \times \mathbb{S}^1)$.

Then, we say that $f : M \to [-1, 1] \times \mathbb{S}^1$ is a 1*-fold simple singular fibration*.

Remark 5.2 (a) Since $f : M \to [-1, 1] \times \mathbb{S}^1$ has a unique embedded singular locus Z_f which projects to a circle C isotopic to $\{0\} \times \mathbb{S}^1$, the inverse image of any regular value is a closed surface Σ whose genus is either g or $g + 1$ for some $g \in \mathbb{N} \cup \{0\}$. We call a fiber with genus g as a lower genus fiber.

(b) Observe that as we cross the $f(Z_f)$, a round 1-handle is added to a manifold diffeomorphic to $\Sigma_g \times A$, where A is an annulus.

(c) We will always use the convention that fibers over $\{-1\} \times \mathbb{S}^1$ have lower genus.

Lemma 5.3 *Let $(M, \partial M)$ be an orientable smooth 4-manifold with boundary and $f : M \to [-1,1] \times \mathbb{S}^1$ be a 1-fold simple singular fibration. Let N be a 4-manifold which admits a separable Hopf link. Then, there exists an embedding $\psi : M \to N \times [-1,1] \times \mathbb{S}^1$ such that following properties are satisfied:*

(1) *The following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \times [-1,1] \times \mathbb{S}^1 \\ \downarrow f & & \downarrow \pi_2 \\ [-1,1] \times \mathbb{S}^1 & \xrightarrow{\mathrm{Id}} & [-1,1] \times \mathbb{S}^1 \end{array} \tag{5}$$

(2) *Given a standard embedding ϕ of a surface of genus $g+1$ in N, we can ensure that ψ restricted to any higher genus fiber sends the fiber to a surface in N which is isotopic to the given embedding ϕ.*

Proof Let us define $M_0 = f^{-1}(\{-1\} \times \mathbb{S}^1)$ and $M_1 = f^{-1}(\{1\} \times \mathbb{S}^1)$. We know that $\partial M = M_0 \sqcup M_1$. Observe that M_1 is a mapping torus over $\mathbb{S}^1$ with fiber Σ_{g+1}. Recall that any mapping torus over $\mathbb{S}^1$ is determined by its monodromy — an element of $\mathcal{MCG}(\Sigma_g)$. Let ϕ be the monodromy for the fiber bundle M_1 over $\mathbb{S}^1$. Further, since $f : (M, \partial M) \to [-1,1] \times \mathbb{S}^1$ is a 1-fold simple singular fibration, we have the following: there exists a homologically nontrivial curve c in Σ_{g+1} which is mapped to itself by ϕ [5, p. 10895], and the boundary component M_0 is obtained from M_1 by the following procedure:

First cut Σ_{g+1} along c, and attach to the resulting surface a pair of discs — say D_1 and D_2. Now form the mapping torus of the resulting surface Σ_g with monodromy the map ϕ restricted to Σ_g.

This also implies that we can obtain $(M, \partial M)$ by suitably adding a round 1-handle to $\Sigma_g \times \mathbb{S}^1$ along a pair of points in Σ_g times $\mathbb{S}^1$ such that each disc $D_i \times \mathbb{S}^1$ contains a circle of the round attaching sphere.

Now, let $i : \Sigma_{g+1} \subset N$ be a standard embedding of Σ_{g+1} in N. Since the embedding is standard, we know every simple closed curve γ on Σ_{g+1} bounds a disc D in N such that the intersection of this disc with N is γ. Furthermore, recall that any simple closed curve in a standard embedding of Σ_{g+1} can be assumed to be disjoint from the separable Hopf link, and the pair of disjoint discs that the link bounds. This implies that there exist a 4-ball B^4 containing the disc D such that $\Sigma_{g+1} \cap B^4$ is an annulus A and ∂A is a pair of unlinked unknots in ∂B^4 (see Remark 4.6). We call this link $L = L_1 \sqcup L_2$.

Since the embedding is standard, from Lemma 4.8 it follows that there exist a fiber preserving embedding of M_1 in $N \times \{1\} \times \mathbb{S}^1$. Since ϕ sends c to itself $\phi(c) = \pm c$. Since the curve c bounds disc in Σ_g, without loss of generality we can assume that $\phi(c) = c$.

We know that the embedding of a surface Σ_g obtained by cutting Σ_{g+1} along a curve $\hat{c}$ obtained by pushing c slightly away from itself in a small tubular neighborhood of c agrees with Σ_{g+1} everywhere except in a ball B^4 satisfying the property that $B^4 \cap \Sigma_{g+1}$ is fixed annulus having boundary a pair of unknot. Since the ball B^4 is disjoint from the separable Hopf link and the pair of disjoint discs that the

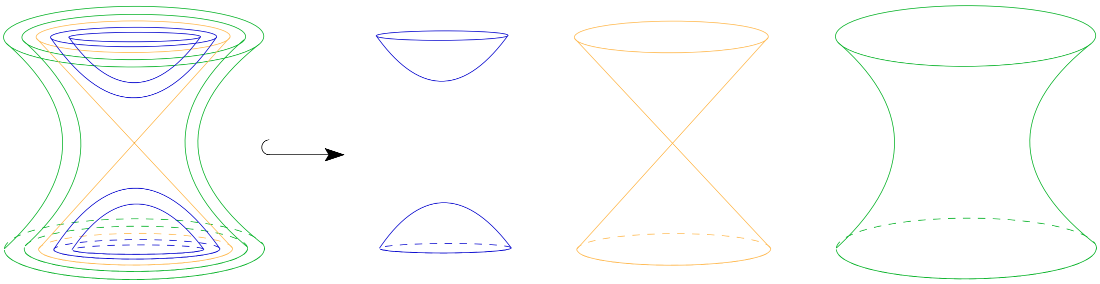

Figure 6: Simple Lefschetz fibration embedding.

link bounds, we get that the embedding of Σ_g given by cutting Σ_{g+1} is also standard. Hence, applying Lemma 4.8, we get an embedding of M_0 in $N \times \{-1\} \times \mathbb{S}^1$ which is also fiber preserving.

Observe that by very construction the embedding of $\partial M = M_0 \sqcup M_1$ can be extended to an embedding $\widehat{\psi}$ of $(M, \partial M) \setminus \mathcal{N}$ in $N \setminus B^4 \times [-1,1] \times \mathbb{S}^1$, where $\mathcal{N}$ is a neighborhood of 1-fold singularity. Furthermore, we can assume that the following diagram commutes:

$$\begin{array}{ccc} M \setminus \mathcal{N} & \xrightarrow{\widehat{\psi}} & N \setminus B^4 \times [-1,1] \times \mathbb{S}^1 \\ \big\downarrow f & & \big\downarrow \pi_2 \\ [-1,1] \times \mathbb{S}^1 & \xrightarrow{\mathrm{Id}} & [-1,1] \times \mathbb{S}^1 \end{array} \tag{6}$$

Hence, in order to establish the lemma, we need to extend the embedding constructed so far in the region $\mathcal{N}$. We can assume that $\mathcal{N}$ is a tubular neighborhood of the 1-fold critical locus, and hence can be identified with $B^3 \times \mathbb{S}^1$.

Let (x, y, z, θ) be coordinates on a tubular neighborhood $\mathcal{N} = B^3 \times \mathbb{S}^1$ of the singular locus Z_f of f such that f sends (x, y, z, θ) to $(-x^2 + y^2 + z^2, \theta)$. Let us embed $B^3 \times \mathbb{S}^1$ in $B^4(0,1) \times [-1,1] \times \mathbb{S}^1$. The embedding $\widehat{\psi_1} : B^3 \times \mathbb{S}^1 \to B^4(0,1) \times [-1,1] \times \mathbb{S}^1$ is defined as

$$\widehat{\psi_1}(x, y, z, \theta) = (x, y, z, 0, -x^2 + y^2 + z^2, \theta).$$

For a pictorial description of the embedding $\widehat{\psi_1}$, see Figure 6. We can see $\widehat{\psi_1}$ is defined such that following diagram commutes:

$$\begin{array}{ccc} B^3 \times \mathbb{S}^1 \subset M & \xrightarrow{\widehat{\psi_1}} & B^4(0,1) \times [-1,1] \times \mathbb{S}^1 \subset N \times [-1,1] \times \mathbb{S}^1 \\ \big\downarrow f & & \big\downarrow \pi_2 \\ [-1,1] \times \mathbb{S}^1 & \xrightarrow{\mathrm{Id}} & [-1,1] \times \mathbb{S}^1 \end{array} \tag{7}$$

Observe that the embedding $\widehat{\psi}_1$ has the property that for each (t,θ), the intersection of $\widehat{\psi}_1(f^{-1}(t,\theta))$ with $\partial B^4 \times \{(t,\theta)\}$ is a pair of unlinked unknots. Hence perturbing this embedding if necessary, it is possible to ensure that this pair is the pair $L = L_1 \sqcup L_2$ for each (t,θ).

Observe that for any $t < 0$ the embedding of $\Sigma_g \cap B^4 \times \{(t,\theta)\}$ produced by $\widehat{\psi}$ and the embedding of pair of disc bounding the unlink $L_1 \sqcup L_2$ produced by $\widehat{\psi}_1$ differ only up to bounding discs of each unknot L_i. Hence up to an isotopy, both embeddings agree. Similarly, for $t > 0$ embeddings $\widehat{\psi}$ and $\widehat{\psi}_1$ differ only up to annuli that the unlink $L = L_1 \sqcup L_2$ bound. Hence we can isotope further to ensure that for $t > 0$ they also agree. This implies that by perturbing the embedding $\widehat{\psi}$ we can assume that both embeddings agree near the boundary to produce an embedding ψ of M in $N \times [-1,1] \times \mathbb{S}^1$.

Clearly, ψ is the required embedding. This shows that we can produce an embedding of $(M, \partial M)$ in N satisfying the property (2). Since there always exists a standard embedding of Σ_{g+1}, the lemma follows. □

Let us now establish Theorem 1.2.

Proof of Theorem 1.2 Let M be a closed-oriented 4-manifold. By Theorem 2.11 there exists a smooth map $f : M \to \mathbb{C}P^1$ which defines SBLF such that the lower genus fiber Σ_g of f has genus bigger than 1.

Therefore, we can write $\mathbb{C}P^1 = D_1 \cup \mathcal{A} \cup D_2$, where D_1 is an embedded disc in $\mathbb{C}P^1$ containing all Lefschetz critical values of f, $\mathcal{A} = [-1,1] \times \mathbb{S}^1$ is an embedded annulus in $\mathbb{C}P^1$ with $\{0\} \times \mathbb{S}^1$ as the embedded image of 1-fold singularities of f, and D_2 is an embedded disc in $\mathbb{C}P^1$ containing no critical values of f such that $\partial D_1 = \{1\} \times \mathbb{S}^1$ and $\partial D_2 = \{-1\} \times \mathbb{S}^1$.

Since lower genus fiber has genus at least 2, we have a decomposition of M, $M = X_1 \sqcup X_2 \sqcup \Sigma_g \times D_2$ due to [11, Theorem 1] which satisfy the following properties:

(1) $f|_{X_1} : X_1 = f^{-1}(D_1) \to D_1$ is a Lefschetz fibration.

(2) $f|_{X_2} : X_1 = f^{-1}(\mathcal{A}) \to [-1,1] \times \mathbb{S}^1$ is a 1-fold simple singular fibration.

(3) $\Sigma_g \times D_2 = f^{-1}(D_2)$.

(4) Identifications along the boundaries of adjacent regions are always done by the identity map.

It follows from Theorem 4.11, and Lemma 5.3, that each piece of M embeds in $N \times \mathbb{C}P^1$. Also, it is clear from the second property listed in the statement of Lemma 5.3 that embeddings of each piece can be arranged such that in the overlapping region they agree. This clearly implies that we have an embedding of M in $N \times \mathbb{C}P^1$ as claimed. □

Remark 5.4 (a) The embedding $\psi : M \to N \times \mathbb{C}P^1$ produced in Theorem 1.2 satisfies $\psi \circ \pi_2 = f$, where $f : M \to \mathbb{C}P^1$ is an SBLF associated to M and $\pi_2 : N \times \mathbb{C}P^1 \to \mathbb{C}P^1$ is the projection onto the second factor of $N \times \mathbb{C}P^1$. In this case, the embedding ψ is called an *SBLF embedding*.

(b) In general, given a fiber bundle $\pi : X^6 \to \mathbb{C}P^1$ and an SBLF $f : M^4 \to \mathbb{C}P^1$, an embedding $\Psi : M^4 \to X^6$ is called an SBLF embedding if $\pi \circ \Psi = f$.

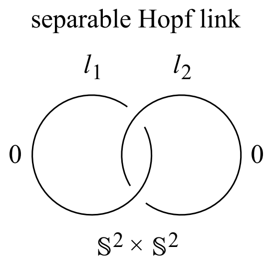

Figure 7: This figure depicts the Kirby diagram of $\mathbb{S}^2 \times \mathbb{S}^2$. Observe that attaching circles of 2-handles form a Hopf link in the boundary of the unique 0-handle, and they bound disjoint discs corresponding to attaching discs in $\mathbb{S}^2 \times \mathbb{S}^2$.

6 Embeddings in $\mathbb{R}^7$

In this section, we give a new proof of the fact that every closed smooth orientable 4-manifold admits a smooth embedding in $\mathbb{R}^7$.

Theorem 6.1 *Every closed orientable 4-manifold admits a smooth embedding in $\mathbb{R}^7$.*

Proof Consider the 4-manifold $\mathbb{S}^2 \times \mathbb{S}^2$. We observe that $\mathbb{S}^2 \times \mathbb{S}^2$ admits a separable Hopf link. This is because $\mathbb{S}^2 \times \mathbb{S}^2$ admits a handle decomposition consisting of a unique 0-handle H_0 on which a pair of two 2-handles are attached such that the attaching circles form a Hopf link in ∂H_0. For a pictorial description of this handle decomposition, we refer to Figure 7, where we have presented a Kirby diagram of $\mathbb{S}^2 \times \mathbb{S}^2$. This clearly implies that the Hopf link consisting of the pair of attaching circles is a separable Hopf link. Thus by Theorem 1.2, every 4-manifold embeds in $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{C}P^1 = \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. Now as $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ embeds in $\mathbb{R}^7$, we get the required embedding of M in $\mathbb{R}^7$. □

7 Embeddings in $\mathbb{C}P^3$

Let us now establish Theorem 1.1. As mentioned in the introduction, the first step of the proof involves the construction of a specific SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. We then use this SBLF to produce an embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the blow-up $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ of $\mathbb{C}P^3$ along $\mathbb{C}P^1$. Furthermore, we show that this embedding can be constructed such that when we blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we get an embedding of M in $\mathbb{C}P^3$. We begin by reviewing notions related to blow-up and blow-down.

7.1 Generalized Lefschetz pencil

Definition 7.1 (generalized Lefschetz pencil) Let M be an oriented smooth 4-manifold. A *generalized Lefschetz pencil* associated to M is a map $\pi : M \setminus B \to \mathbb{C}P^1$ such that the following properties are satisfied:

(1) B is finite.

(2) $\pi : M \setminus B \to \mathbb{C}P^1$ is a Lefschetz fibration.

(3) For every point $b \in B$ there is a parameterization — *not necessarily preserving orientations* — $\phi : U \subset M \to \mathbb{C}^2$ that satisfies the following:

(a) $b \in U$ and $\phi(b) = 0 \in \mathbb{C}^2$.

(b) For the map $g : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1$ given by $g(z_1, z_2) = [z_1 : z_2]$, the following diagram commutes:

$$\begin{array}{ccc} U \setminus \{b\} & \xrightarrow{\ \phi\ } & \mathbb{C}^2 \setminus \{0\} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle g} \\ \mathbb{C}P^1 & \xrightarrow{\ \mathrm{Id}\ } & \mathbb{C}P^1 \end{array} \tag{8}$$

In this case, we call B the base locus of a generalized Lefschetz pencil associated with M.

Remark 7.2 (a) We would like to emphasize that the notion of generalized Lefschetz pencil defined above is weaker than the notion of Lefschetz pencil. Generally one demands that M and $\mathbb{C}P^1$ are oriented and the parameterization $\phi : U \subset M \to \mathbb{C}^2$ is orientation preserving in Definition 7.1.

(b) If a fibration $\pi : M \setminus B \to \mathbb{C}P^1$ is a simplified broken Lefschetz fibration, then we say that the map π is a *generalized simplified broken Lefschetz pencil* (generalized SBLP in short) of M.

(c) If the fibration $\pi : M \setminus B \to \mathbb{C}P^1$ is a simplified broken Lefschetz fibration and the parameterization $\phi : U \subset M \to \mathbb{C}^2$ is orientation preserving, then the map π is called a *simplified broken Lefschetz pencil* (SBLP).

7.2 Topological blow-up and blow-down of 4-manifolds

We begin by recalling a few standard facts from [14] about the tautological line bundle over $\mathbb{C}P^1$ and the bundle (complex) dual to this bundle.

Consider the tautological line bundle $\tau_{\mathbb{C}P^1}$ over $\mathbb{C}P^1$, and the bundle $\tau^*_{\mathbb{C}P^1}$ dual to the bundle $\tau_{\mathbb{C}P^1}$. Let $\mathcal{Z}_{\ll}$ denote the zero section of the bundle $\tau_{\mathbb{C}P^1}$, and $\mathcal{Z}_{\tau^*}$ denote the zero section of the bundle $\tau^*_{\mathbb{C}P^1}$.

We know that $\tau_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\tau}$ and $\tau^*_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\tau^*}$ are diffeomorphic to $\mathbb{R}^4 \setminus \{0\}$ by diffeomorphisms coming from the restrictions of the projection of the second factor for the corresponding bundles. We fix this identification of the complement of zero sections with $\mathbb{R}^4 \setminus \{0\}$ for both of these bundles.

Definition 7.3 (topological blow-up) Let M a smooth 4-manifolds. Let p be a point in M. Let U be a neighborhood of p diffeomorphic to $\mathbb{R}^4$ via a diffeomorphism which sends p to $0 \in \mathbb{R}^4$. The manifold $\widehat{M}$ obtained by removing p from U and identifying $U \setminus \{p\}$ with either $\tau^*_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\ll^*}$ or with $\tau_{\mathbb{C}P^1} \setminus \mathcal{Z}_{\ll}$ is called a topological blow-up of M along p.

Remark 7.4 (a) The operation of topological blow-up of a manifold along a point corresponds to its connected sum with $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$. While performing a topological blow-up, if we use the tautological line bundle $\tau_{\mathbb{C}P^1}$, then we get $M \# \overline{\mathbb{C}P^2}$. On the other hand, if we use the dual bundle to $\tau_{\mathbb{C}P^1}$, then we get $M \# \mathbb{C}P^2$.

(b) Topological blow-up of M along p produces a manifold $\widehat{M}$ admitting an embedded $\mathbb{C}P^1$ with self intersection number ± 1. Recall that the usual blow-up always produces an embedded $\mathbb{C}P^1$ with self intersection -1.

(c) Throughout this discussion, an embedded $\mathbb{C}P^1$ in a 4-manifold M with self intersection number ± 1 will be called an *exceptional sphere* in M.

Definition 7.5 (topological blow-down) Let $\widehat{M}$ be a smooth 4-manifold admitting an embedded $\mathbb{C}P^1$ whose normal bundle is isomorphic to $\tau_{\mathbb{C}P^1}$ or $\tau^*_{\mathbb{C}P^1}$. That is the embedded $\mathbb{C}P^1$ is an exceptional sphere in $\widehat{M}$. In this case, we can carry out the process exactly opposite of the one described in the definition of blow-up, where we remove a tubular neighborhood of $\mathbb{C}P^1$ and replace it with a 4-ball. The resulting manifold M that we obtain as a result of this process is called a topological blow-down of $\widehat{M}$.

Remark 7.6 (1) Observe that given a manifold M admitting an embedding of $\mathbb{C}P^1$ with its self intersection number ± 1, we can perform topological blow-down operation.

(2) Suppose we are given a manifold $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Let E_1 and E_{-1} be two embedded $\mathbb{C}P^1$'s corresponding to zero sections of $\tau^*_{\mathbb{C}P^1}$ and $\tau_{\mathbb{C}P^1}$, respectively. Suppose $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ is an SBLF such that the intersection number of each fiber with E_1 is 1, and the intersection number of each fiber with E_{-1} is -1. Then the two operations of blow-downs corresponding to removal of E_1 and E_{-1} produces a generalized SBLP on M. This is because the SBLF restricted to a tubular neighborhood of E_1 is isomorphic $\tau^*_{\mathbb{C}P^1}$, while a tubular neighborhood of E_{-1} is isomorphic to $\tau_{\mathbb{C}P^1}$.

7.3 Construction of SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

The purpose of this subsection is to establish an SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which satisfies the property that the intersection of each fiber with two exceptional spheres E_1 and E_{-1} corresponding to zero sections is $+1$ and -1, respectively.

Lemma 7.7 *Consider a closed oriented smooth 4-manifold $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Then, there exists an SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ which satisfies the following:*

(1) *The lower genus fiber has its genus bigger than 1.*

(2) *The fibration agrees with the standard fibration in a tubular neighborhood of both exceptional spheres E_1 and E_{-1}.*

In particular, blowing down the SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ produces a generalized SBLP on M.

In [6, Theorem 6.5], R. I. Baykur and O. Saeki established the existence of a simplified broken Lefschetz pencil for any near symplectic manifold admitting connected singular locus for near symplectic structure. It is easy to see that following the proof of [6, Theorem 6.5] — essentially verbatim — provides a proof of Lemma 7.7.

Proof To begin with, notice that there exists an embedded surface Σ in $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which satisfies the following properties:

- The self intersection of Σ is 0.
- $\Sigma \cap E_1 = +1$ and $\Sigma \cap E_{-1} = -1$.
- Σ is connected and the genus of Σ is bigger than three.

Observe that since the self-intersection number of E_1 is $+1$ and the self-intersection number of E_{-1} is -1, it is easy to construct a disconnected surface consisting of disjoint union of two spheres. By making connected sums of these two spheres with an embedded surface bounding a 3-dimensional handlebody and embedded in B^4, it is easy to construct such a surface.

Consider the map $\pi : \Sigma \times \mathbb{D}^2 \to \mathbb{D}^2$, corresponding to the projection on the second factor, and regard $\mathbb{D}^2$ as embedded in $\mathbb{C}P^1$ as a southern hemisphere. This allows us to regard π as a map from a tubular neighborhood $\mathcal{N}(\Sigma)$ of Σ to southern hemisphere. Construct a map $g : \mathcal{N}(\Sigma) \cup \mathcal{N}(E_1) \cup \mathcal{N}(E_{-1}) \to \mathbb{C}P^1$ which satisfies the following:

(1) The map g when restricted to $\mathcal{N}(E_1)$ and $\mathcal{N}(E_{-1})$ is the surjection on $\mathbb{C}P^1$ coming from the bundle projections $\pi_{E_1} : \mathcal{N}(E_1) \to E_1$ and $\pi_{E_{-1}} : \mathcal{N}(E_{-1}) \to E_{-1}$.

(2) The map g agrees with π when restricted to $\mathcal{N}(\Sigma)$.

Next, extend the map g to a generic smooth map $\hat{f} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$. According to [6, Remark 4.5], this map can be modified to produce an SBLF $\hat{f} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ such that all the modifications performed while obtaining the SBLF from g are performed away from the region where g is defined.

Next, we convert the SBLF $\hat{f} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ to an SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ whose lower genus fiber is bigger than 2 by applying a technique similar to the one which provides a proof of Theorem 2.11 or a proof of [3, Proposition 1.3]. The SBLF $f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ can be ensured to satisfy the required properties because every fiber of f is homologous to the original fiber Σ and hence the intersection of fibers of f has same property that Σ had. □

Let us end this section with a convention: from now on the SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ described in the statement of Lemma 7.7 will be denoted by $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$.

7.4 Blow-up and blow-down of $\mathbb{C}P^3$ along $\mathbb{C}P^1$

Let us begin this subsection by making a convention. By a standard $\mathbb{C}P^1$ in $\overline{\mathbb{C}P^2}$, we mean a $\mathbb{C}P^1$ embedded in $\overline{\mathbb{C}P^2}$ with its normal bundle isomorphic to the dual of the tautological line bundle over $\mathbb{C}P^1$. On the other hand, by a standard $\mathbb{C}P^1$ in $\mathbb{C}P^n$, we mean $\{[z_0, z_1, \dots, z_n] \mid z_i = 0 \text{ for all } i \geq 2\}$, where $[z_0, z_1, \dots, z_n]$ denotes the homogeneous coordinates of $\mathbb{C}P^n$.

Consider $\mathbb{C}P^3$ and a standard $\mathbb{C}P^1$ embedded in it. Fix a local trivialization $\mathbb{D}^2 \times \mathbb{C}^2$ of the normal bundle $\mathcal{N}(\mathbb{C}P^1)$ of $\mathbb{C}P^1$ in $\mathbb{C}P^3$. Now consider $\mathbb{D}^2 \times \mathbb{C}P^1 \times \mathbb{C}^2$ and a subset V of $\mathbb{D}^2 \times \mathbb{C}P^1 \times \mathbb{C}^2$ given by

$$V = \{(w, l, z_1, z_2,) \mid \|z_1^2\| + \|z_2^2\| \leq 1 \text{ and } (z_1, z_2) \in l\},$$

where a point l in $\mathbb{C}P^1$ is identified with the complex linear subspace corresponding to that point.

Now, observe that the complement of $\mathbb{D}^2 \times \mathbb{C}P^1 \times \{(0,0)\}$ in V can be identified with the complement of $\mathbb{D}^2 \times \{(0,0)\}$ in $\mathbb{D}^2 \times \mathbb{C}^2$.

Choose two local trivializations $U_1 \times \mathbb{C}^2$ and $U_2 \times \mathbb{C}^2$ over open sets U_1 and U_2 such that U_1 and U_2 cover $\mathbb{C}P^1$. By the (topological) blow-up of $\mathbb{C}P^3$ along $\mathbb{C}P^1$ we mean the operation of removing $U_i \times \{(0,0)\}$ from $U_i \times \mathbb{C}^2$, for each i, and replacing it with the interior of V as discussed in the previous paragraph.

Remark 7.8 (1) An *exceptional divisor* of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ is the union of $\mathbb{D}^2 \times \mathbb{C}P^1 \times \{(0,0)\}$ over a finite collection V_s of trivializations of the bundle $\mathcal{N}(\mathbb{C}P^1)$. Again notice that the triviality of the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^3$ implies that the exceptional divisor is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

The notion of blow-up discussed above is a particular case of blow-up of a manifold along a submanifold. We refer [15, pp. 196 and 602] for a detailed discussion on blow-ups.

By a blow-down of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ we will mean the process exactly opposite to the process of blow-up. More precisely, let $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ be obtained by blowing up a $\mathbb{C}P^1$. Let E be the exceptional divisor obtained as a result of the blow-up. By blow-down of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we mean removal of a tubular neighborhood of E and replacing it by a tubular neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^3$.

We say that $\mathbb{C}P^3$ is obtained from $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ by blowing down along E. Since E is diffeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$, we sometimes do not distinguish between E and $\mathbb{C}P^1 \times \mathbb{C}P^1$ and say that $\mathbb{C}P^3$ is obtained from $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ by blowing down along $\mathbb{C}P^1 \times \mathbb{C}P^1$.

We end this subsection with the following:

Lemma 7.9 *Let $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ be a closed oriented smooth manifold. Let $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ be an SBLF on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ as in the statement of Lemma 7.7. If there exists an SBLF embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that each fiber of SBLF intersects the standard $\mathbb{C}P^1$ of the fiber $\mathbb{C}P^2$ of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ in two distinct but fixed algebraically canceling points, then there exists an embedding of M in $\mathbb{C}P^3$ such that the standard pencil of $\mathbb{C}P^3$ induces the generalized SBLP of M corresponding to the SBLF of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.*

Proof Let E_1 and E_{-1} be two exceptional divisors of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Recall the exceptional divisor of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ consists of the union of two local exceptional divisors of the type $U_i \times W$, where $W \subset \mathbb{C}P^1 \times \mathbb{C}^2$ consists of $\{(l, z_1, z_2) \mid (z_1, z_2) \in l\}$. Since by hypothesis the fiber of π_{spl} intersects the standard $\mathbb{C}P^1$ inside $\mathbb{C}P^2$ in a pair of fixed points, we can assume that the tubular neighborhoods of exceptional divisors $E_{\pm 1}$ are contained in $U_i \times W$, and since the embedding is fiber preserving it consists of $\{p_{\pm}\} \times W \subset U_1 \times W$.

Furthermore, by the definition of the blow-up, the fibration on $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ restricted to $U_1 \times W$ can be assumed to be given by $(u, l, z_1, z_2) \to l$. This clearly implies that when we blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ we get M embedded in $\mathbb{C}P^3$ with standard pencil of $\mathbb{C}P^3$ inducing the generalized SBLP on M associated to SBLF $\pi_{\mathrm{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$. □

7.5 Embeddings in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$

In this subsection, we establish SBLF embedding of the special SBLF $\pi_{\text{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Proposition 7.10 *Let M be a closed-oriented smooth 4-manifold. Consider $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and let $\pi_{\text{spl}} : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ be a special SBLF with the lower genus fiber having genus bigger than 1. There exists an SBLF embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that each fiber of SBLF intersects the standard $\mathbb{C}P^1$ in the fiber $\mathbb{C}P^2$ in a pair of canceling intersection points.*

Proof We will follow the line of argument we used to establish Theorem 1.2. Let us denote by π the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ obtained via blow-up of the standard pencil of $\mathbb{C}P^3$. We first consider neighborhoods of exceptional divisors E_1 and E_{-1} of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and embed them in a tubular neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ of $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ such that the embedding is fiber preserving. In order to produce this embedding recall that a tubular neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ is the union of two open sets $U_i \times W$, $i = 1, 2$.

Now consider a pair of points p_+, p_- in U_1, and consider spheres $\{p_\pm\} \times \mathbb{C}P^1$ embedded in $U_1 \times W$. Since tubular neighborhood of $E_{\pm 1}$ is isomorphic to tubular neighborhood of any sphere in $U_1 \times W$ of the form $\{p\} \times \mathbb{C}P^1$, where p is a point in U_1, we get that there exists an embedding of small neighborhoods of $E_{\pm 1}$ in a neighborhood of the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ such that π_{spl} restricted to this neighborhood agrees with restriction of π on the embedded neighborhoods.

Observe that the intersection of the embedded neighborhoods of $E_{\pm 1}$ with a fiber of the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ is a pair of discs satisfying the property that the intersection of this pair of discs with the boundary of a small tubular neighborhood of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ is a Hopf link. Furthermore, observe that since the embedding of the neighborhood of E_{-1} as a tubular neighborhood of $\{p_+\} \times \mathbb{C}P^1$ is orientation reversing, and the embedding of neighborhood of E_{+1} as a tubular neighborhood $\{p_-\} \times \mathbb{C}P^1$ is orientation preserving. This implies that if we establish that

(1) $\mathbb{C}P^2$ admits a separable Hopf link,

(2) there exists an embedding of any surface of genus g in $\mathbb{C}P^2$ which is standard embedding,

(3) the embedded surface Σ_g intersects the standard $\mathbb{C}P^1$ contained in $\mathbb{C}P^2$ in a pair of algebraically canceling points, and $\Sigma_g \cap \partial\mathcal{N}(\mathbb{C}P^1)$ is a Hopf link in $\partial\mathcal{N}(\mathbb{C}P^1)$, where $\mathcal{N}(\mathbb{C}P^1)$ is a fixed open tubular neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^2$,

then the triviality of the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ in the complement of the exceptional divisor implies that an argument similar to the one which established Theorem 1.2 produces the required SBLF embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Hence, the task at hand is to establish an embedding of a surface satisfying the three properties listed above. To this end, we observe that it is relatively easy to verify property (1) and get an embedding satisfying property (2). In fact, in section 4 we have already shown how to achieve these for various 4-manifolds. Hence our main focus will be on proving property (3), however, for the sake of completeness,

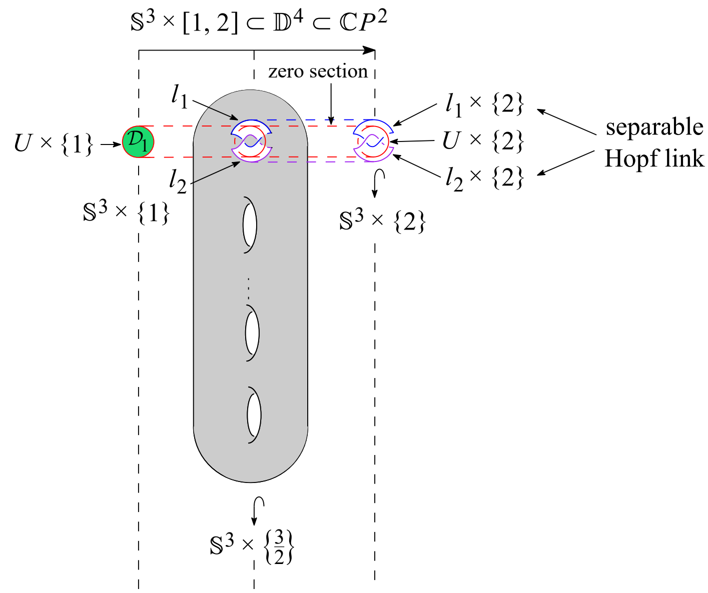

Figure 8: This figure depicts an embedded surface in $\mathbb{C}P^2$ which is flexible and in a standard position. The diagram focus on a collar $\mathbb{S}^3 \times [1,2]$ of a 4-ball $\mathbb{D}^4$ regarded as the unique zero handle H_0 of $\mathbb{C}P^2$. The circle U is the attaching circle of the unique 2-handle H_2. $U \times [1,2]$ with the core disc attached at $U \times \{2\}$ and the green disc at $U \times \{1\}$ forms the standard $\mathbb{C}P^1$ embedded in $\mathbb{C}P^2$.

we will again describe how $\mathbb{C}P^2$ satisfies property (1). After showing this we will discuss how a modification of embeddings of Σ_g discussed in section 4 produces a required embedding satisfying the remaining two properties.

To begin with, consider a handle decomposition of $\mathbb{C}P^2$ with the 0-handle H_0 corresponding to $B^4(0,2)$ — the 4-ball of radius 2 in $\mathbb{C}^2$ with its center at the origin — to which a 2-handle H_2 is attached along an unknot U with framing $+1$. Finally a 4-handle H_4 is attached to the 4-manifold, which is the union of the 0-handle $B^4(0,2)$ and the 2-handle H_2. Regarding H_0 as a ball, let $\mathbb{S}^3 \times [1,2]$ be a collar of ∂H_0. Let $U \times \{2\}$ be the attaching circle of H_2. Observe that any Hopf link consisting of a parallel copy of the attaching circle — say $l_1 \times \{2\}$ and a circle $l_2 \times \{2\}$ which links both the attaching circle and l_1 once as depicted in Figure 8 constitute a Hopf link that is separable. This is because $l_1 \times \{2\}$ bounds a parallel copy of the core of the 2-handle, and $l_2 \times \{2\}$ bounds a disc in the unique 4-handle.

Next, consider cylinders $l_i \times \left[\frac{3}{2}, 2\right]$, $i = 1,2$. They intersect $\mathbb{S}^3 \times \left\{\frac{3}{2}\right\}$ in $l_i \times \left\{\frac{3}{2}\right\}$. Observe that there exists a surface Σ_g with two boundary components whose boundary is the Hopf link $l_1 \times \left\{\frac{3}{2}\right\} \sqcup l_2 \times \left\{\frac{3}{2}\right\}$; see Figure 8. It follows from an argument similar to the one used in establishing Lemma 4.5 that the embedding is both flexible and in a standard position.

Regarding the standard $\mathbb{C}P^1$ as the union of the core of 2-handle H_2 with a disc $\mathbb{D}$ that $U \times \{2\}$ bounds, we see that the embedded Σ_g intersects $\mathbb{C}P^1$ in a pair of points. This pair has to be algebraically canceling as we can push the disc $\mathbb{D}$ down to produce an isotopy of $\mathbb{C}P^1$ that sends the $\mathbb{C}P^1$ to a new $\mathbb{C}P^1$ which consists of union of core of H_2, $U \times [1,2]$, and a disc $\mathbb{D}$ that $U \times \{1\}$ bounds. The disc that $U \times \{1\}$ bounds is denoted by a green disc in Figure 8. Notice that the isotoped $\mathbb{C}P^1$ is disjoint from Σ_g implying that the algebraic intersection of Σ_g with the standard $\mathbb{C}P^1$ is zero. □

Now we have established all the results necessary to establish Theorem 1.1.

7.6 Proof of Theorem 1.1

We need to prove that every smooth orientable closed 4-manifold admits an embedding in $\mathbb{C}P^3$.

Proof of Theorem 1.1 Let M be the given closed orientable 4-manifold. Consider the manifold $\widehat{M} = M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ thought of as a blow-up of M done at two distinct points p_1 and p_2. Recall that $\widehat{M}$ admits a pair of exceptional divisors — say E_1 and E_{-1} such that $E_1 \cap E_1 = 1$ while $E_{-1} \cap E_{-1} = -1$.

Next, apply Lemma 7.7 to produce the special SBLF $\pi_{\mathrm{spl}} : \widehat{M} \to \mathbb{C}P^1$ on $\mathbb{C}P^1$. Recall that this SBLF satisfies the following:

(1) The lower genus fiber has its genus bigger than 1.

(2) The fibration agrees with the standard fibration in a tubular neighborhood of both exceptional spheres E_1 and E_{-1}.

Now, by Proposition 7.10 there exists SBLF embedding of $\widehat{M}$ in $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

Also, notice that the intersection property of the embedded fiber of SBLF with standard $\mathbb{C}P^1$ contained in $\mathbb{C}P^2$ stated in Proposition 7.10 implies that the embedding is such that each fiber of the SBLF associated to $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ intersects the standard $\mathbb{C}P^1$ of a fiber $\mathbb{C}P^2$ of the fibration $\pi : \mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ in a pair of algebraically canceling points.

Finally, blow-down $\mathcal{B}_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along its exceptional divisor. Observe that Lemma 7.9 implies that blow-down produces an embedding of M in $\mathbb{C}P^3$ such that the standard Lefschetz pencil of $\mathbb{C}P^3$ induces an SBLP on M. □

References

[1] **D Auroux**, **S K Donaldson**, **L Katzarkov**, *Singular Lefschetz pencils*, Geom. Topol. 9 (2005) 1043–1114 MR

[2] **W P Barth**, **K Hulek**, **C A M Peters**, **A Van de Ven**, *Compact complex surfaces*, 2nd edition, Ergebnisse der Math. (3) 4, Springer (2004) MR

[3] **R I Baykur**, *Existence of broken Lefschetz fibrations*, Int. Math. Res. Not. 2008 (2008) art. id. rnn101 MR

[4] **R I Baykur**, *Topology of broken Lefschetz fibrations and near-symplectic four-manifolds*, Pacific J. Math. 240:2 (2009) 201–230 MR

[5] **R I Baykur**, **O Saeki**, *Simplified broken Lefschetz fibrations and trisections of 4-manifolds*, Proc. Natl. Acad. Sci. USA 115:43 (2018) 10894–10900 MR

[6] **R I Baykur**, **O Saeki**, *Simplifying indefinite fibrations on 4-manifolds*, Trans. Amer. Math. Soc. 376:5 (2023) 3011–3062 MR

[7] **S E Cappell**, **J L Shaneson**, *Embeddings and immersions of four-dimensional manifolds in $\mathbb{R}^6$*, from "Geometric topology" (Athens, 1977) (J C Cantrell, editor), Academic, New York (1979) 301–303 MR

[8] **M Dehn**, *Die Gruppe der Abbildungsklassen: das arithmetische Feld auf Flächen*, Acta Math. 69:1 (1938) 135–206 MR

[9] **S K Donaldson**, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. 44:4 (1996) 666–705 MR

[10] **S K Donaldson**, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. 53:2 (1999) 205–236 MR

[11] **C J Earle**, **J Eells**, *The diffeomorphism group of a compact Riemann surface*, Bull. Amer. Math. Soc. 73 (1967) 557–559 MR

[12] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR

[13] **D T Gay**, **R Kirby**, *Constructing Lefschetz-type fibrations on four-manifolds*, Geom. Topol. 11 (2007) 2075–2115 MR

[14] **R E Gompf**, **A I Stipsicz**, 4*-manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, Amer. Math. Soc., Providence, RI (1999) MR

[15] **P Griffiths**, **J Harris**, *Principles of algebraic geometry*, Wiley, New York (1994) MR

[16] **A Haefliger**, *Plongements différentiables dans le domaine stable*, Comment. Math. Helv. 37 (1962/63) 155–176 MR

[17] **A Haefliger**, **M W Hirsch**, *On the existence and classification of differentiable embeddings*, Topology 2 (1963) 129–135 MR

[18] **S Hirose**, **A Yasuhara**, *Surfaces in* 4*-manifolds and their mapping class groups*, Topology 47:1 (2008) 41–50 MR

[19] **M W Hirsch**, *On imbedding differentiable manifolds in euclidean space*, Ann. of Math. (2) 73 (1961) 566–571 MR

[20] **M W Hirsch**, *On embedding* 4*-manifolds in* R^7, Proc. Cambridge Philos. Soc. 61 (1965) 657–658 MR

[21] **W B R Lickorish**, *A representation of orientable combinatorial* 3*-manifolds*, Ann. of Math. (2) 76 (1962) 531–540 MR

[22] **W B R Lickorish**, *A finite set of generators for the homeotopy group of a* 2*-manifold*, Proc. Cambridge Philos. Soc. 60 (1964) 769–778 MR

[23] **W S Massey**, **F P Peterson**, *On the dual Stiefel–Whitney classes of a manifold*, Bol. Soc. Mat. Mexicana (2) 8 (1963) 1–13 MR

[24] **D M Pancholi**, **S Pandit**, **K Saha**, *Embeddings of* 3*-manifolds via open books*, J. Ramanujan Math. Soc. 36:3 (2021) 243–250 MR

[25] **V A Rohlin**, *The embedding of non-orientable three-manifolds into five-dimensional Euclidean space*, Dokl. Akad. Nauk SSSR 160 (1965) 549–551 MR In Russian

[26] **D Ruberman**, *Imbedding four-manifolds and slicing links*, Math. Proc. Cambridge Philos. Soc. 91:1 (1982) 107–110 MR

[27] **C T C Wall**, *All* 3*-manifolds imbed in* 5*-space*, Bull. Amer. Math. Soc. 71 (1965) 564–567 MR

ABHIJEET GHANWAT ghanwata16@gmail.com
Department of Mathematics, University of Georgia, Athens, GA, United States

DISHANT M PANCHOLI dishant@imsc.res.in
Institute for Mathematical Sciences, Chennai, India

Received: June 13, 2021 Revised: April 14, 2025

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2026 is US $795/year for the electronic version, and $1170/year (+$80, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 26 Issue 4 (pages 1229–1596) 2026