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Recollements and stratification

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We develop various aspects of the theory of recollements of ∞-categories, including a symmetric monoidal refinement of the theory. Our main result establishes a formula for the gluing functor of a recollement on the right-lax limit of a locally cocartesian fibration determined by a sieve-cosieve decomposition of the base. As an application, we prove a reconstruction theorem for sheaves in an ∞-topos stratified over a finite poset P in the sense of Barwick, Glasman, and Haine. Combining our theorem with methods of Ayala, Mazel-Gee, and Rozenblyum, we then prove a conjecture of Barwick, Glasman, and Haine that asserts an equivalence between the ∞-category of P-stratified ∞-topoi and that of toposic locally cocartesian fibrations over P^{op}.

1 Introduction

The theory of recollements plays an important and ubiquitous role throughout topology, algebraic geometry, and representation theory. It is a common axiomatization of, on the one hand, the adjunctions

$$\mathbf{Shv}(U) \;\substack{\xhookrightarrow{\;j_!\;} \\ \xleftarrow{\;j^*\;} \\ \xhookrightarrow[\;j_*\;]{}}\; \mathbf{Shv}(X) \;\substack{\xrightarrow{\;i^*\;} \\ \xhookleftarrow[\;i_*\;]{}}\; \mathbf{Shv}(Z)$$

associated to ∞-categories of sheaves of spaces on a topological space X decomposed by an open subspace $j : U \hookrightarrow X$ and its closed complement $i : Z = X \setminus U \hookrightarrow X$, and, on the other hand, the adjunctions

$$\mathbf{QCoh}_Z(X) \;\substack{\xhookrightarrow{\;j_!\;} \\ \xleftarrow{\;j^*\;} \\ \xhookrightarrow[\;j_*\;]{}}\; \mathbf{QCoh}(X) \;\substack{\xrightarrow{\;i^*\;} \\ \xhookleftarrow{\;i_*\;} \\ \xrightarrow[\;i^!\;]{}}\; \mathbf{QCoh}(U)$$

associated to stable ∞-categories of quasicoherent complexes on a qcqs scheme X with open subscheme $i : U \hookrightarrow X$, where $\mathbf{QCoh}_Z(X)$ denotes those quasicoherent complexes set-theoretically supported on $Z = X \setminus U$. The fully faithful left adjoint $j_!$ is the definitional embedding of $\mathbf{QCoh}_Z(X)$ in $\mathbf{QCoh}(X)$, whereas the fully faithful right adjoint j_* embeds $\mathbf{QCoh}_Z(X)$ as $\mathbf{QCoh}(X^{\wedge}_Z) \subset \mathbf{QCoh}(X)$, the full subcategory of quasicoherent complexes on X complete along Z; see [4].

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Recollements were introduced by Beilinson, Bernstein, and Deligne [7] in the context of derived categories of perverse sheaves and were later defined by Lurie in the ∞-categorical context in the course of his study of constructible sheaves on stratified spaces [14, §A]. The goal of this article is to continue the development of the general theory of recollements from [14, §A.8], which we recapitulate in Section 2 beginning with the basic Definition 2.1. Our first contribution is to establish a symmetric monoidal refinement of this theory:

1.1 Definition (Definition 2.20) Let $\mathcal{X}$ be a symmetric monoidal ∞-category that admits finite limits. Then a recollement

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}$$

is *symmetric monoidal* if the localization functors $j_* j^*$ and $i_* i^*$ are compatible with the symmetric monoidal structure, so that $\mathcal{U}$ and $\mathcal{Z}$ uniquely inherit symmetric monoidal structures from $\mathcal{X}$ such that the functors j^* and i^* uniquely refine to (strong) symmetric monoidal functors.

Recall that Lurie shows that given a recollement $(\mathcal{U}, \mathcal{Z})$ on $\mathcal{X}$, if we define the *gluing functor* of the recollement to be $\phi = i^* j_*$ then we may reconstruct $\mathcal{X}$ as the fiber product $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$, where $\mathrm{Ar}(\mathcal{Z}) := \mathrm{Fun}(\Delta^1, \mathcal{Z})$ is the ∞-category of arrows in $\mathcal{Z}$.[1] Now given a lax symmetric monoidal functor $\phi : \mathcal{U} \to \mathcal{Z}$ of symmetric monoidal ∞-categories, we may construct a certain *canonical* symmetric monoidal structure on $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ (Definition 2.25). We then have:

1.2 Theorem (Theorem 2.30) *Let $\mathcal{X}$ be a symmetric monoidal ∞-category decomposed by a symmetric monoidal recollement $(\mathcal{U}, \mathcal{Z})$. Then the natural equivalence $\mathcal{X} \xrightarrow{\simeq} \mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ refines to an equivalence of symmetric monoidal ∞-categories. In other words, the lax symmetric monoidal structure on the gluing functor reconstructs the symmetric monoidal structure on $\mathcal{X}$.*

1.3 Remark Although this result is a simple exercise in the theory of ∞-operads, it appears that our work was the first to give a proof, and indeed a construction of the canonical symmetric monoidal structure. The work of Ayala, Mazel-Gee, and Rozenblyum has since placed this sort of construction within the context of endowing right-lax limits with $\mathcal{O}$-monoidal structure [2, §4.4].

Our next contribution is motivated by the following problem from equivariant stable homotopy theory:

1.4 Problem *Let G be a finite group and $\mathcal{F}$ a G-family (i.e., a set of subgroups of G closed under taking subgroups and conjugation). Given a (genuine) G-spectrum $X \in \mathbf{Sp}^G$ that is $\mathcal{F}$-complete and a subgroup $H \leq G$ not in $\mathcal{F}$, give a formula for the H-geometric fixed points of X in terms of the K-geometric fixed points of X ranging over $K \in \mathcal{F}$.*

Recollement theory is relevant here because any G-family $\mathcal{F}$ defines a recollement on $\mathbf{Sp}^G$ whose open part is spanned by the $\mathcal{F}$-complete G-spectra (see [15; 20]). In fact, we may further recast this problem using the stratification theory of Ayala, Mazel-Gee, and Rozenblyum [1; 2]. In their work, they

[1] To be precise, Lurie doesn't quite formulate his result in this way. See Observation 2.9 and the discussion thereafter.

construct a certain locally cocartesian fibration $\mathbf{Sp}^G_{\phi\text{-locus}} \to P$, where P is the poset of conjugacy classes of subgroups of G and the fiber over $[H]$ is $\mathrm{Fun}(BW_G H, \mathbf{Sp})$ for $W_G H = N_G H/H$ the Weyl group, such that one has a canonical equivalence

$$\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), \mathbf{Sp}^G_{\phi\text{-locus}}) \simeq \mathbf{Sp}^G, \tag{1-1}$$

where $\mathrm{sd}(P)$ is the barycentric subdivision[2] of P regarded as a locally cocartesian fibration over P via the functor that takes the maximum, and the left-hand side denotes the full subcategory spanned by those functors $\mathrm{sd}(P) \to \mathbf{Sp}^G_{\phi\text{-locus}}$ over P preserving locally cocartesian edges. The idea is that this equivalence parametrizes a G-spectrum in terms of its geometric fixed points, and indeed given a G-spectrum X, under this equivalence X transports to a functor $\mathrm{sd}(P) \to \mathbf{Sp}^G$ that sends $[H]$ to $\Phi^H X$. Now by definition any G-family $\mathcal{F}$ defines a *sieve* (i.e., a downward closed subposet) in P, and the $\mathcal{F}$-recollement on $\mathbf{Sp}^G$ transports to a recollement on $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), \mathbf{Sp}^G_{\phi\text{-locus}})$ given by the pair

$$\big(\mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{F}}(\mathrm{sd}(\mathcal{F}), \mathbf{Sp}^G_{\phi\text{-locus}}|_{\mathcal{F}}), \mathrm{Fun}^{\mathrm{cocart}}_{/(P\setminus\mathcal{F})}(\mathrm{sd}(P\setminus\mathcal{F}), \mathbf{Sp}^G_{\phi\text{-locus}}|_{P\setminus\mathcal{F}})\big).$$

Establishing a pointwise formula for the gluing functor of this recollement would then yield a solution to Problem 1.4. In general, we prove:

1.5 Theorem (Theorem 3.26) *Let P be a poset and let P_0 be a sieve in P. Let $\mathrm{sd}(P)_0 \subset \mathrm{sd}(P)$ be the subposet on those strings that originate in P_0, and note that $\max|_{\mathrm{sd}(P)_0}$ remains a locally cocartesian fibration. Then for every locally cocartesian fibration $C \to P$, the restriction functor*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P)_0, C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C|_{P_0})$$

is a trivial fibration.

Theorem A (Theorem 3.32, Proposition 3.36, and Theorem 3.39) *Let P be a down-finite poset[3] and let $p : C \to P$ be a locally cocartesian fibration such that for every $p \in P$, the fiber C_p admits finite limits, and for every $p \le q$, the associated pushforward functor $C_p \to C_q$ preserves finite limits. Then for every sieve-cosieve decomposition $P_0, P_1 = P \setminus P_0$ of P, we obtain a recollement*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C|_{P_0}) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/P_1}(\mathrm{sd}(P_1), C|_{P_1}),$$

where j^*, i^* are given by restriction and their fully faithful right adjoints j_*, i_* are describable by the following pointwise formulas:

1. *For every $x \in P_1$, let $J_x \subset \mathrm{sd}(P)$ be the subposet on strings $[a_0 < \cdots < a_n < x]$, $n \ge 0$ with $a_i \in P_0$. Then for every $[f : \mathrm{sd}(P_0) \to C|_{P_0}]$ on the left-hand side, if we let $\bar{f}$ denote the unique extension of f over $\mathrm{sd}(P)_0$ given by Theorem 1.5, then $j_*(f)$ evaluates on $x \in P_1$ to $\lim(\bar{f}|_{J_x} : J_x \to C_x)$.*
2. *For every $[f : \mathrm{sd}(P_1) \to C|_{P_1}]$ on the right-hand side, $i_*(f)$ evaluates on $x \in P_0$ to the final object $* \in C_x$.*

[2] Recall that $\mathrm{sd}(P)$ is the poset whose objects are strings $[a_0 < \cdots < a_n]$ in P and whose morphisms are string inclusions.
[3] A poset P is *down-finite* if for every $p \in P$, the subposet $P^{\le p}$ is finite.

1.6 Remark In [20], we use Theorem A to answer Problem 1.4 in the form of [20, Theorem F].

In fact, we prove a more general theorem where we replace P and the sieve P_0 by any ∞-category S and functor $\pi : S \to \Delta^1$ determining a sieve-cosieve decomposition of S, at the possible cost of demanding more conditions on our locally cocartesian fibration $p : C \to S$.

1.7 Remark Conceptually, a locally cocartesian fibration $C \to P$ is the unstraightening of a *left-lax* diagram $P \to \mathbf{Cat}_\infty$, and the ∞-category $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ is then the *right-lax limit* of this left-lax diagram (see [2, §A]). Theorem A then amounts to an *existence theorem* for the (pointwise) right-lax Kan extension of $[C \to P]$ along a functor $\pi : P \to \Delta^1$, along with a *transitivity property* of right-lax Kan extensions with respect to the composite $P \to \Delta^1 \to *$.

Although Theorem A may appear innocuous, we can leverage it to great effect in inductive arguments that build up the right-lax limit of a locally cocartesian fibration from its strata. For example, we will use Theorem A to establish the theory of 1*-generated and extendable objects* in Section 4, which furnishes a proof of an assertion of Nikolaus and Scholze [17, Remark II.4.8] on decomposing the ∞-category of bounded-below C_{p^n}-spectra as an iterated pullback; for a precise statement, see Remark 4.19.

In this paper, our main application of Theorem A will be to prove a *reconstruction theorem* for sheaves on an ∞-topos stratified over a finite poset P that was conjectured in the work of Barwick, Glasman, and Haine [5, Remark 8.2.7]. We recall the definition of a P-stratified ∞-topos as Definition 5.5 and that of a toposic locally cocartesian fibration as Definition 5.11. The reader may want to bear in mind the example of a P-stratified ∞-topos given by $\mathbf{Shv}(X)$ for X a topological space equipped with a continuous map $\pi : X \to P$, where we endow P with the Alexandroff topology (so that its open sets are cosieves).

Theorem B (Theorems 5.13 and 5.22) *Let $\mathcal{X}$ be an ∞-topos equipped with a P-stratification $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ for a finite poset P. Then we may functorially associate to $(\mathcal{X}, \pi_*)$ a locally cocartesian fibration $\mathcal{G}(\mathcal{X}) \to P^{\mathrm{op}}$ such that we have a canonical equivalence*

$$\Theta_P : \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X})) \xrightarrow{\simeq} \mathcal{X}. \tag{1-2}$$

Moreover, Θ_P is the counit of an adjoint equivalence

$$\lim^{\mathrm{rlax}} : \mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}} \rightleftarrows \mathbf{StrTop}_{\infty,P} : \mathcal{G} \tag{1-3}$$

between the ∞-category of toposic locally cocartesian fibrations over P^{op} and the ∞-category of P-stratified ∞-topoi.

1.8 Remark We explain how to interpret Theorem B as a reconstruction theorem. Define the *p-th stratum* $\mathcal{X}_p$ to be $\mathbf{Shv}(\{p\}) \times_{\mathbf{Shv}(P),\pi_*} \mathcal{X}$, where the fiber product is formed in the ∞-category $\mathbf{Top}_\infty$ of ∞-topoi and geometric morphisms thereof. (For example, if $\mathcal{X} = \mathbf{Shv}(X)$ for a P-stratified space $\pi : X \to P$, then $\mathcal{X}_p \simeq \mathbf{Shv}(X_p)$.) Let

$$\Phi^p : \mathcal{X} \rightleftarrows \mathcal{X}_p : \rho_p$$

denote the associated geometric morphism adjunction. Then ρ_p is fully faithful, and we in fact define

$$\mathcal{G}(\mathcal{X}) := \{(x, p) \in \mathcal{X} \times P^{\mathrm{op}} : x \in \mathcal{X}_p\}$$

with respect to $\rho_p : \mathcal{X}_p \hookrightarrow \mathcal{X}$, so that $\mathcal{G}(\mathcal{X})_p \simeq \mathcal{X}_p$ (Construction 5.10). Now under the equivalence Θ_P, a sheaf (i.e., object) $x \in \mathcal{X}$ transports to a functor $f_x : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})$ whose value on $[p]$ is given by $\Phi^p(x)$ (Remark 5.16). The functor Θ_p then sends f_x to the limit of its projection into $\mathcal{X}$.

1.9 Remark The strategy of our proof of Theorem B is heavily inspired by the work of Ayala, Mazel-Gee, and Rozenblyum, who assert a similar statement in the setting of presentable stable ∞-categories [2, Theorem A]. Note however that our proof of the equivalence (1-2) (but not (1-3)) is independent of any explicit use of $(\infty, 2)$-category theory in the form of the fibrational mate correspondence for locally cocartesian fibrations (that is, [2, Lemmas A.3.6 and A.3.7]), which we recall in this paper as Theorem 5.21. Indeed, we instead use Theorem A as the basis for an inductive argument that establishes (1-2). Similarly, one can supply an alternative proof of the comparable part of [2, Theorem A] using the same strategy; as already mentioned, we implement this idea in context of equivariant stable homotopy theory in our proof of the equivalence (1-1) in the form of [20, Theorem F] (see the discussion below [20, Theorem 2.42]).

1.1 What's new in this paper

We briefly comment on the relation of this paper to [18], which we have since split up into this paper and [19; 20]. Sections 2, 3, and 4 of this paper are lightly revised versions of the corresponding sections of [18], whereas Section 5 on the application to stratified ∞-topoi is entirely new. Also, in the intervening time since we wrote [18], Ayala, Mazel-Gee, and Rozenblyum released their work on stratified noncommutative geometry [2]; this is an expansion of [1] and bears greatly on many of the topics treated in this paper. As such, we have added a few remarks throughout (in particular, Remark 3.44 and the new Section 3.2.4) explaining how our work relates to [2]. One of the main takeaways here is that one can leverage Theorem A to remove the presentability hypotheses in [2, Theorem A]. Finally, our application to the description of bounded-below C_{p^n}-spectra as given in [18] relied on some work that has now been moved into [19].

2 Recollements

In this section, we establish the basic theory of recollements, expanding upon [4; 14, §A.8]. After setting up the definitions and summarizing Lurie's results on recollements, we explain a symmetric monoidal refinement of the theory of recollements, connect the theory of stable symmetric monoidal recollements to that of smashing localizations, and record some useful projection formulas. We conclude by proving a few lemmas concerning families of recollements that we will need in [19; 20].

2.1 Definition [14, Definition A.8.1] Let $\mathcal{X}$ be an ∞-category that admits finite limits and let $\mathcal{U}, \mathcal{Z} \subset \mathcal{X}$ be full subcategories that are stable under equivalences. Then $(\mathcal{U}, \mathcal{Z})$ is a *recollement* of $\mathcal{X}$ if the inclusion functors $j_* : \mathcal{U} \subset \mathcal{X}$ and $i_* : \mathcal{Z} \subset \mathcal{X}$ admit left exact left adjoints j^* and i^* such that:

(1) $j^* i_*$ is equivalent to the constant functor at the terminal object $*$ of $\mathcal{U}$.

(2) j^* and i^* are *jointly conservative*, i.e., if $f : x \to y$ is a morphism in $\mathcal{X}$ such that $j^* f$ and $i^* f$ are equivalences, then f is an equivalence.

We will call $\mathcal{U}$ the *open* part of the recollement, $\mathcal{Z}$ the *closed* part of the recollement, and $i^* j_*$ the *gluing functor*.[4]

The main purpose of the theory of recollements is to codify the various "fracture square" decompositions that recur throughout algebra and topology. Abstractly, we have:

2.2 Proposition *Let $(\mathcal{U}, \mathcal{Z})$ be a recollement of $\mathcal{X}$ and let $\eta_j : \mathrm{id} \to j_* j^*$, $\eta_i : \mathrm{id} \to i_* i^*$ denote the unit transformations. Then we have a pullback square of functors*

$$\begin{array}{ccc} \mathrm{id} & \xrightarrow{\eta_i} & i_* i^* \\ {\scriptstyle \eta_j}\downarrow & & \downarrow{\scriptstyle i_* i^* \eta_j} \\ j_* j^* & \xrightarrow{\eta_i j_* j^*} & i_* i^* j_* j^* \end{array}$$

Proof By joint conservativity of the left-exact functors j^* and i^*, it suffices to check that we have a pullback square after applying j^* and i^*, which is clear. □

Next, we define morphisms of recollements.

2.3 Definition Suppose that $(\mathcal{U}_1, \mathcal{Z}_1)$ and $(\mathcal{U}_2, \mathcal{Z}_2)$ are recollements on $\mathcal{X}_1$ and $\mathcal{X}_2$. Then a functor $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a *morphism of recollements* if F sends j_1^*-equivalences to j_2^*-equivalences and i_1^*-equivalences to i_2^*-equivalences. Let **Recoll** denote the resulting ∞-category of recollements, and let $\mathbf{Recoll}^{\mathrm{lex}}$ be the full subcategory on those morphisms of recollements that are also left-exact.

2.4 Observation Suppose that $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism of recollements $(\mathcal{U}_1, \mathcal{Z}_1) \to (\mathcal{U}_2, \mathcal{Z}_2)$. Then we may define $F_U = j_2^* F j_{1*} : \mathcal{U}_1 \to \mathcal{U}_2$ and $F_Z = i_2^* F i_{1*}$ so that we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{U}_1 & \xleftarrow{j_1^*} & \mathcal{X}_1 & \xrightarrow{i_2^*} & \mathcal{Z}_1 \\ \downarrow{\scriptstyle F_U} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_Z} \\ \mathcal{U}_2 & \xleftarrow{j_2^*} & \mathcal{X}_2 & \xrightarrow{i_2^*} & \mathcal{Z}_2 \end{array}$$

such that F is left-exact if and only if F_U and F_Z are left-exact. Conversely, if we are given such a commutative diagram, then F is a morphism of recollements. Indeed, for any morphism $[f : x \to y] \in \mathcal{X}_1$ such that $j^*(f)$ (resp. $i^*(f)$) is an equivalence, $j^* F(f) \simeq F j^*(f)$ (resp. $i^* F(f) \simeq F i^*(f)$) is an equivalence. Moreover, since $F_U \simeq j^* F j_*$ and $F_Z \simeq i^* F i_*$, it follows that functors $\mathcal{U}_1 \to \mathcal{U}_2$ and $\mathcal{Z}_1 \to \mathcal{Z}_2$ induced by F as a morphism of recollements are then canonically equivalent to F_U and F_Z.

[4]Our convention on which subcategory is open and which is closed matches that for constructible sheaves, whereas other authors (e.g., [4]) use the opposite convention, which matches that for quasicoherent sheaves. Also note that in [14, Definition A.8.1], Lurie calls the open part C_1 and the closed part C_0.

2.5 Observation In the situation of Observation 2.4, by adjunction we get natural transformations $\nu : F j_{1*} \Rightarrow j_{2*} F_U$ and $\nu' : F i_{1*} \Rightarrow i_{2*} F_Z$. Note that if F preserves the terminal object, then ν' is an equivalence; indeed, for all $z \in \mathcal{Z}_1$ we then have

$$j_2{}^* F i_{1*}(z) \simeq F_U j_1{}^* i_{1*}(z) \simeq F_U(*) \simeq *,$$

so the unit map $F i_{1*}(z) \to i_{2*} i_2^* F i_{1*}(z) = i_{2*} F_Z(z)$ is an equivalence. In particular, if F is left exact, then ν' is an equivalence [14, Remark A.8.10]. On the other hand, ν is an equivalence if and only if

$$\nu'' : F_Z i_1{}^* j_{1*} \Rightarrow i_2^* j_{2*} F_U$$

is an equivalence — indeed, the "only if" direction is obvious, and for the "if" direction we may readily check that $j_2{}^* \nu$ and $i_2{}^* \nu$ are equivalences and then invoke the joint conservativity of $j_2{}^*$ and $i_2{}^*$.

2.6 Definition If ν'' in Observation 2.5 is an equivalence, then we call F a *strict* morphism of recollements. Let $\mathbf{Recoll}_{\mathrm{str}} \subset \mathbf{Recoll}$ and $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}} \subset \mathbf{Recoll}^{\mathrm{lex}}$ be the wide subcategories on the strict morphisms.

2.7 Remark If $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a strict left-exact morphism of recollements, then F is an equivalence if and only if F_U and F_Z are equivalences [14, Proposition A.8.14].

2.8 Definition Let $\pi : \mathcal{M} \to \Delta^1$ be a functor of ∞-categories with fibers $\mathcal{M}_0 = \mathcal{Z}$ and $\mathcal{M}_1 = \mathcal{U}$. Then π is a *left-exact correspondence* [14, Definition A.8.6] if

(1) π is a cartesian fibration, so determines a functor $\phi : \mathcal{U} \to \mathcal{Z}$;

(2) $\mathcal{U}$ and $\mathcal{Z}$ admit finite limits and ϕ is left-exact.

A *morphism of left-exact correspondences* is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ over Δ^1. In terms of the left-exact functors ϕ_1 and ϕ_2, this corresponds to a right-lax commutative diagram

$$\begin{array}{ccc} \mathcal{U}_1 & \xrightarrow{\phi_1} & \mathcal{Z}_1 \\ {\scriptstyle F_U}\downarrow & \swarrow & \downarrow{\scriptstyle F_Z} \\ \mathcal{U}_2 & \xrightarrow[\phi_2]{} & \mathcal{Z}_2 \end{array}$$

Let $\mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ denote the resulting ∞-category of left-exact correspondences as a full subcategory of $(\mathbf{Cat}_\infty)_{/\Delta^1}$, and let $\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ be the wide subcategory on those morphisms that preserve cartesian edges, so that the right-lax commutativity is actually strict. Note that under the straightening correspondence, $\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ is the full subcategory of $\mathrm{Ar}(\mathbf{Cat}_\infty)$ on left-exact functors $\phi : \mathcal{U} \to \mathcal{Z}$.

If F_U and F_Z are also left-exact, we say that the morphism F of left-exact correspondences is *left-exact*. We may then view (lax) commutative squares as residing inside the category $\mathbf{Cat}^{\mathrm{lex}}_\infty$ itself. Let $\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \subset \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ and $\mathrm{Ar}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \subset \mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty)$ denote the resulting wide subcategories.

2.9 Observation Let $\mathcal{M} \to \Delta^1$ be a left-exact correspondence and let $\mathcal{X} = \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$ be its ∞-category of sections. Let $\mathcal{U} \subset \mathcal{X}$ be the full subcategory on the cartesian sections and let $\mathcal{Z} \subset \mathcal{X}$ be the full subcategory on those sections σ such that $\sigma(1)$ is a terminal object of $\mathcal{U}$. Then $(\mathcal{U}, \mathcal{Z})$ is a recollement of $\mathcal{X}$ [14, Proposition A.8.7]. Moreover, the formation of sections

$$\mathcal{M} \rightsquigarrow \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$$

carries morphisms of left-exact correspondences to morphisms of recollements, and thereby defines a functor[5]

$$\lim{}^{\mathrm{rlax}} : \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Recoll},$$

which is an equivalence of ∞-categories by [14, Proposition A.8.8] (which shows full faithfulness) and [14, Proposition A.8.11] (which shows that if $(\mathcal{U}, \mathcal{Z})$ is a recollement of $\mathcal{X}$, then $\mathcal{X}$ is equivalent to the right-lax limit of $i^* j_* : \mathcal{U} \to \mathcal{Z}$). Furthermore, in view of the discussion in Observation 2.5, $\lim^{\mathrm{rlax}}$ restricts to equivalences of subcategories

$$\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}_{\mathrm{str}}, \quad \mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{lex}}, \quad \mathrm{Ar}(\mathbf{Cat}^{\mathrm{lex}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}.$$

We next explain how to identify the ∞-category of sections of a cartesian fibration classified by the functor $\phi : \mathcal{U} \to \mathcal{Z}$ with the pullback $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$. For an efficient proof, we will use the machinery of marked simplicial sets [12, §3]. Recall that $\Delta^{n\natural}$ denotes the n-simplex with its last edge $\{n-1, n\}$ marked, and likewise for the marked horn $\Lambda^{n\natural}_n$. Moreover, given a cartesian fibration $\pi : C \to B$, we let $C^\natural$ denote the marking of all π-cartesian edges, for which $(C^\natural, \pi)$ is fibrant in the cartesian model structure on $s\mathbf{Set}^+_{/B}$.

2.10 Construction Let $\pi : \mathcal{M} \to \Delta^1$ be a cartesian fibration. By the dual of [21, Lemma 2.23], we have a trivial fibration $\mathrm{Ar}^{\mathrm{cart}}(\mathcal{M}) \to \mathrm{Ar}(\Delta^1) \times_{\mathrm{ev}_1, \Delta^1, \pi} \mathcal{M}$, which restricts to a trivial fibration $\mathrm{ev}_1 : \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathcal{M}_1$. Let χ be a section of ev_1.

Because the map $i : \Lambda^{2\natural}_2 \to \Delta^{2\natural}$ is right marked anodyne, with the structure map $\sigma^0 : \Delta^2 \to \Delta^1$, $(\sigma^0)^{-1}(0) = \{0, 1\}$ and $(\sigma^0)^{-1}(1) = \{2\}$, we have a trivial fibration

$$i^* : \mathrm{Fun}_{/\Delta^1}(\Delta^{2\natural}, \mathcal{M}^\natural) \to \mathrm{Fun}_{/\Delta^1}(\Lambda^{2\natural}_2, \mathcal{M}^\natural) \cong \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \times_{\mathrm{ev}_1, \mathcal{M}_1, \mathrm{ev}_1} \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}).$$

Let κ be a section of i^*. The section χ yields a functor

$$f = (\mathrm{id}, \chi \circ \mathrm{ev}_1) : \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \times_{\mathcal{M}_1} \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}).$$

Let $g = \kappa \circ f$. Then the various maps fit into the commutative diagram

$$\begin{array}{ccccc} \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) & \xrightarrow{g} & \mathrm{Fun}_{/\Delta^1}(\Delta^{2\natural}, \mathcal{M}^\natural) & \xrightarrow{\mathrm{ev}_{01}} & \mathrm{Fun}(\Delta^1, \mathcal{M}_0) \\ \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_{12}} & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathcal{M}_1 & \xrightarrow{\chi} & \mathrm{Fun}^{\mathrm{cart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) & \xrightarrow{\mathrm{ev}_0} & \mathcal{M}_0 \end{array}$$

[5] We denote this by $\lim^{\mathrm{rlax}}$ in view of the interpretation of the sections of a cartesian fibration as defining the right-lax limit of the corresponding functor.

2.11 Lemma *The natural map* $\mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathrm{Ar}(\mathcal{M}_0) \times_{\mathrm{ev}_1, \mathcal{M}_0} \mathcal{M}_1$ *is an equivalence, so the outer square is a homotopy pullback square of* ∞*-categories.*

Proof Because the sections χ and κ are equivalences, the map g is an equivalence. Moreover, because the map $\Lambda^2_1 \to \Delta^2$ is inner anodyne, the rightmost square is a homotopy pullback square. □

2.12 Corollary *Suppose that* $(\mathcal{U}, \mathcal{Z})$ *is a recollement of* $\mathcal{X}$ *and consider the commutative*[6] *diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{i^*\eta_j} & \mathrm{Ar}(\mathcal{Z}) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathcal{U} & \xrightarrow{\phi = i^* j_*} & \mathcal{Z} \end{array}$$

where $\eta_j : \mathcal{X} \to \mathrm{Ar}(\mathcal{X})$ *is the functor that sends* x *to the unit map* $x \to j_* j^* x$*. Then the induced map*

$$\mathcal{X} \xrightarrow{\simeq} \mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$$

is an equivalence of ∞*-categories.*

Proof Combine Lemma 2.11 with the equivalence $\lim^{\mathrm{rlax}} : \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}$ of Observation 2.9. □

2.13 Remark In view of Corollary 2.12, given a recollement $(\mathcal{U}, \mathcal{Z})$ of $\mathcal{X}$ with gluing functor $\phi = i^* j_*$ we will often write objects $x \in \mathcal{X}$ as $[u, \alpha : z \to \phi(u)]$ or $[u, z, \alpha]$.

Given a left-exact functor $\phi : \mathcal{U} \to \mathcal{Z}$, we may also extract the resulting recollement from the *cocartesian* fibration classified by ϕ, even though it is difficult to encode the right-lax functoriality when working with cocartesian fibrations.

2.14 Observation Let S be an ∞-category and $C \to S$ a cocartesian fibration. Recall from [6; 21, Recollection 5.17] that the *dual cartesian fibration* $C^\vee \to S^{\mathrm{op}}$ is defined to have n-simplices[7]

$$\begin{array}{ccc} {}_\natural\mathrm{TwAr}((\Delta^n)^{\mathrm{op}}) & \longrightarrow & {}_\natural C \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow \\ ((\Delta^n)^{\mathrm{op}})^\sharp & \longrightarrow & S^\sharp \end{array}$$

where we mark the cocartesian edges in C and $\mathrm{TwAr}((\Delta^n)^{\mathrm{op}})$. In fact, because the functor $\mathrm{TwAr}'(-) : s\mathbf{Set}^+_{/S} \to s\mathbf{Set}^+_{/S}$ of [21, Proposition 5.18] preserves colimits, it follows that, for all simplicial sets A over S^{op},

$$\mathrm{Hom}_{/S^{\mathrm{op}}}(A, C^\vee) \cong \mathrm{Hom}_{/S}(\mathrm{TwAr}'(A^{\mathrm{op}}), {}_\natural C).$$

Consequently, we obtain an equivalence

$$\mathrm{Fun}_{/S^{\mathrm{op}}}(S^{\mathrm{op}}, C^\vee) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{TwAr}(S), C).$$

[6] We can obtain a commutative diagram of simplicial sets using standard techniques in quasicategory theory.

[7] Here, $\mathrm{TwAr}(-)$ is the *twisted arrow* ∞*-category*. We use the directionality convention of [3] instead of [14, §5.2.1], so twisted arrows are contravariant in the source and covariant in the target.

Now note that the barycentric subdivision $\mathrm{sd}(\Delta^1) = [0 \to 01 \leftarrow 1]$ is isomorphic to the twisted arrow category $\mathrm{TwAr}(\Delta^1)$. Therefore, for a cocartesian fibration $C \to \Delta^1$, we deduce that

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\mathrm{sd}(\Delta^1), C) \simeq \mathrm{Fun}_{/\Delta^1}(\Delta^1, C^\vee)$$

and hence by Lemma 2.11 we can decompose $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\mathrm{sd}(\Delta^1), C)$ as a pullback square $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ for a choice of pushforward functor $\phi : \mathcal{U} \to \mathcal{Z}$ (where $\mathcal{U} \simeq C_0$ and $\mathcal{Z} \simeq C_1$). This observation will be important for us when we discuss recollements on right-lax limits in the next section.

2.1 Stable recollements

2.15 Definition Let $\mathcal{X}$ be a stable ∞-category and let $(\mathcal{U}, \mathcal{Z})$ be a recollement of $\mathcal{X}$. Then this recollement is *stable* if $\mathcal{U}$ and $\mathcal{Z}$ are stable subcategories. Let $\mathbf{Recoll}^{\mathrm{stab}}$ (resp. $\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$) be the full subcategory of $\mathbf{Recoll}^{\mathrm{lex}}$ (resp. $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}$) whose objects are the stable recollements.

2.16 Definition If $\mathcal{M} \to \Delta^1$ is a left-exact correspondence, then $\mathcal{M}$ is *exact* if the functor $\phi : \mathcal{M}_1 \to \mathcal{M}_0$ is an exact functor of stable ∞-categories. Let $\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{stab}}_\infty)$ (resp. $\mathrm{Ar}(\mathbf{Cat}^{\mathrm{stab}}_\infty)$) be the full subcategory of $\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{lex}}_\infty)$ (resp. $\mathrm{Ar}(\mathbf{Cat}^{\mathrm{lex}}_\infty)$) on the exact correspondences.

2.17 Remark The functor $\lim^{\mathrm{rlax}}$ of Observation 2.9 restricts to equivalences

$$\mathrm{Ar}^{\mathrm{rlax}}(\mathbf{Cat}^{\mathrm{stab}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{stab}}, \quad \mathrm{Ar}(\mathbf{Cat}^{\mathrm{stab}}_\infty) \xrightarrow{\simeq} \mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}.$$

2.18 Observation Let $(\mathcal{U}, \mathcal{Z})$ be a stable recollement of $\mathcal{X}$. Then $j^* : \mathcal{X} \to \mathcal{U}$ admits a fully faithful left adjoint[8] $j_!$, i_* admits a right adjoint $i^!$, and we have norm maps $\mathrm{Nm} : j_! \to j_*$ and $\mathrm{Nm}' : i^! \to i^*$ that fit into fiber sequences

$$j_! \to j_* \to i_* i^* j_* \quad \text{and} \quad i^! \to i^* \to i^* j_* j^*,$$

where the other maps are induced by the unit transformations for $j^* \dashv j_*$ and $i^* \dashv i_*$. On objects $x = [u, z, \alpha] \in \mathcal{X}$, these amount to the fiber sequences

$$[u, 0, 0] \to [u, \phi u, \mathrm{id}] \to [0, \phi u, 0] \quad \text{and} \quad \mathrm{fib}(\alpha) \to z \xrightarrow{\alpha} \phi u.$$

Considering the various unit and counit transformations and the norm maps, we may extend the pullback square of Proposition 2.2 to a commutative diagram

$$\begin{array}{ccccc}
 & & i_* i^! & \xrightarrow{\simeq} & i_* i^! \\
 & & \downarrow & & \downarrow{\scriptstyle i_* \mathrm{Nm}'} \\
j_! j^* & \longrightarrow & \mathrm{id} & \longrightarrow & i_* i^* \\
\downarrow{\scriptstyle \simeq} & & \downarrow & & \downarrow \\
j_! j^* & \xrightarrow[\mathrm{Nm}\, j^*]{} & j_* j^* & \longrightarrow & i_* i^* j_* j^*
\end{array}$$

in which every row and column is a fiber sequence.

[8] For the existence of $j_!$, we only need that $\mathcal{Z}$ admits an initial object $\varnothing$ [14, Corollary A.8.13]. Then $j_!$ is defined by the formula $j_!(u) = [u, \varnothing \to \phi(u)]$.

2.19 Observation In the stable case, the datum of the closed part of a recollement determines the entire recollement. More precisely, if $\mathcal{Z} \subset \mathcal{X}$ is a stable reflective and coreflective subcategory of $\mathcal{X}$ and we define $\mathcal{U}$ to be the full subcategory on those objects $u \in \mathcal{X}$ such that $\mathrm{Map}_{\mathcal{X}}(z, u) \simeq *$ for all $z \in \mathcal{Z}$, then $(\mathcal{U}, \mathcal{Z})$ is a stable recollement of $\mathcal{X}$ [14, Proposition A.8.20], and conversely, if $(\mathcal{U}, \mathcal{Z})$ is a stable recollement of $\mathcal{X}$ then $j_* : \mathcal{U} \subset \mathcal{X}$ is defined as above from $\mathcal{Z}$. We may also identify $j_!(\mathcal{U})$ as given by those objects $u \in \mathcal{X}$ such that $\mathrm{Map}_{\mathcal{X}}(u, z) \simeq *$ for all $z \in \mathcal{Z}$.

Moreover, $F : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism of stable recollements $(\mathcal{U}_1, \mathcal{Z}_1) \to (\mathcal{U}_2, \mathcal{Z}_2)$ if and only if $F|_{\mathcal{Z}_1} \subset \mathcal{Z}_2$ and $F|_{j_!(\mathcal{U}_1)} \subset j_!(\mathcal{U}_2)$ (in particular, we then have $j_{2!} F_U \simeq F j_{1!}$). This is because $\mathcal{Z}$ coincides with the j^*-null objects and $j_!(\mathcal{U})$ with the i^*-null objects. Given this, F is then a strict morphism of stable recollements if and only if we also have that $F|_{j_*(\mathcal{U}_1)} \subset j_*(\mathcal{U}_2)$.

2.2 Symmetric monoidal recollements

We now extend the theory of recollements to the situation where $\mathcal{X}$ admits a symmetric monoidal structure $(\mathcal{X}, \otimes, \mathbb{1})$. In what follows, we will call an adjunction $F : C \rightleftarrows D : G$ between symmetric monoidal ∞-categories *symmetric monoidal* if F is (strong) symmetric monoidal.

2.20 Definition Let $\mathcal{X}$ be a symmetric monoidal ∞-category that admits finite limits. Then a recollement $(\mathcal{U}, \mathcal{Z})$ of $\mathcal{X}$ is *symmetric monoidal* if the localization functors $j_* j^*$ and $i_* i^*$ are compatible with the symmetric monoidal structure in the sense of [14, Definition 2.2.1.6], i.e., if $f : x \to x'$ is a j^*- or i^*-equivalence, then so is $f \otimes \mathrm{id} : x \otimes y \to x' \otimes y$ for any $y \in \mathcal{X}$.

A morphism $F : (\mathcal{U}, \mathcal{Z}) \to (\mathcal{U}', \mathcal{Z}')$ of recollements on $\mathcal{X}$ and $\mathcal{X}'$ is *symmetric monoidal* if the functor $F : \mathcal{X} \to \mathcal{X}'$ is symmetric monoidal. Let $\mathbf{Recoll}^{\otimes}$ denote the ∞-category of symmetric monoidal recollements and morphisms thereof.

2.21 Observation In the situation of Definition 2.20, by [14, Proposition 2.2.1.9] $\mathcal{U}$ and $\mathcal{Z}$ obtain symmetric monoidal structures such that the adjunctions $j^* \dashv j_*$ and $i^* \dashv i_*$ are symmetric monoidal. In particular, the gluing functor $i^* j_*$ is lax symmetric monoidal. Furthermore, if F is a morphism of symmetric monoidal recollements, then the induced functors F_U and F_Z of Observation 2.5 are also symmetric monoidal.

2.22 Remark Most of the results of this subsection will extend verbatim to an arbitrary reduced ∞-operad $\mathcal{O}^{\otimes}$. We leave the details to the reader.

We first show that given a lax symmetric monoidal functor $\phi : \mathcal{U} \to \mathcal{Z}$, the recollement $\lim^{\mathrm{rlax}} \phi$ is symmetric monoidal. We first recall the pointwise symmetric monoidal structure on a functor ∞-category.

2.23 Construction Let $p : C^{\otimes} \to \mathbf{Fin}_*$ be an ∞-operad, and let K be a simplicial set. We have the cotensor $p^K : (C^{\otimes})^K \to \mathbf{Fin}_*$ defined by

$$\mathrm{Hom}_{/\mathbf{Fin}_*}(A, (C^{\otimes})^K) \cong \mathrm{Hom}_{/\mathbf{Fin}_*}(A \times K, C^{\otimes}).$$

Then p^K is again an ∞-operad: this follows from the observation that for any $\mathfrak{O}$-anodyne morphism $A \to B$ of preoperads (with $\mathfrak{O}$ the defining categorical pattern for the model structure on preoperads), $A \times K \to$

$B \times K$ is again $\mathfrak{O}$-anodyne [14, Proposition B.1.9]. Moreover, if p is in addition a cocartesian fibration, then p^K is also a cocartesian fibration. The fiber of p^K over $\langle n \rangle$ is $\mathrm{Fun}(K, C^{\times n}) \simeq \prod_{i=1}^n \mathrm{Fun}(K, C)$, and for the unique active map $\langle n \rangle \to \langle 1 \rangle$, if $\phi : C^{\times n} \to C$ is a choice of pushforward functor encoded by p, then the postcomposition by ϕ functor $\phi_* : \mathrm{Fun}(K, C^{\times n}) \to \mathrm{Fun}(K, C)$ is a choice of pushforward functor encoded by p^K. In other words, p^K is the pointwise symmetric monoidal structure on $\mathrm{Fun}(K, C)$.

We will also need the following lemma.

2.24 Lemma *Let $C^\otimes$ be a symmetric monoidal ∞-category. Then the functor*

$$e_L : (C^\otimes)^{K \star L} \to (C^\otimes)^L$$

induced by $L \subset K \star L$ is a cocartesian fibration of ∞-operads.

Proof Because e_L is induced by the monomorphism $L \subset K \star L$, e_L is a fibration of ∞-operads. Using the inert-active factorization system on an ∞-operad, it then suffices to prove the following two properties of e_L:

(1) For every object $\langle n \rangle \in \mathbf{Fin}_*$, $(e_L)_{\langle n \rangle}$ is a cocartesian fibration.

(2) For every active edge $\alpha : \langle n \rangle \to \langle 1 \rangle$ and commutative square

$$\begin{array}{ccc} f = (f_1, \dots, f_n) & \longrightarrow & f' = \bigotimes_{i=1}^n f_i \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta'} \\ g = (g_1, \dots, g_n) & \longrightarrow & g' = \bigotimes_{i=1}^n g_i \end{array}$$

in $(C^\otimes)^{K \star L}$ with the horizontal edges as $p^{K \star L}$-cocartesian edges covering α, if θ is $(e_L)_{\langle n \rangle}$-cocartesian then θ' is $(e_L)_{\langle 1 \rangle}$-cocartesian.

For (1), by [21, Lemma 4.8] we have that $(e_L)_{\langle n \rangle} : \mathrm{Fun}(K \star L, C^{\times n}) \to \mathrm{Fun}(L, C^{\times n})$ is a cocartesian fibration. Moreover, $\theta : f \to g$ is a $(e_L)_{\langle n \rangle}$-cocartesian edge if and only if its image in $\mathrm{Fun}(K, C^{\times n})$ is an equivalence. This proves (2), since the n-fold tensor product of equivalences is always an equivalence. □

We are now ready to define the symmetric monoidal structure on a right-lax limit.

2.25 Definition Suppose $\phi^\otimes : \mathcal{U}^\otimes \to \mathcal{Z}^\otimes$ is a lax symmetric monoidal functor of symmetric monoidal ∞-categories (i.e., a map of ∞-operads). Consider the pullback square of ∞-operads

$$\begin{array}{ccc} (\mathcal{Z}^\otimes)^{\Delta^1} \times_{\mathcal{Z}^\otimes} \mathcal{U}^\otimes & \longrightarrow & (\mathcal{Z}^\otimes)^{\Delta^1} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathcal{U}^\otimes & \xrightarrow{\phi^\otimes} & \mathcal{Z}^\otimes \end{array}$$

By Lemma 2.24, ev_1 is a cocartesian fibration, so $(\mathcal{Z}^\otimes)^{\Delta^1} \times_{\mathcal{Z}^\otimes} \mathcal{U}^\otimes \to \mathcal{U}^\otimes \to \mathbf{Fin}_*$ is a cocartesian fibration and therefore a symmetric monoidal ∞-category. This defines the *canonical* symmetric monoidal structure on the right-lax limit of ϕ.

2.26 Remark In Definition 2.25, at the level of objects the tensor product on $\mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$ is defined in the following way: Suppose we are given two objects $x = [u, z, \alpha]$ and $x' = [u', z', \alpha']$. Then $x \otimes x' = [u \otimes u', z \otimes z', \gamma]$, where γ is given by the composite map

$$z \otimes z' \xrightarrow{\alpha \otimes \alpha'} \phi(u) \otimes \phi(u') \to \phi(u \otimes u')$$

using the lax symmetric monoidality of ϕ for the second map.

2.27 Proposition *If $\phi : \mathcal{U} \to \mathcal{Z}$ is a lax symmetric monoidal left-exact functor, then $\lim^{\mathrm{rlax}} \phi$ is a symmetric monoidal recollement with respect to the canonical symmetric monoidal structure on $\mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$.*

Proof We only need to observe that in Definition 2.25, the two evaluation maps $j^* : \mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U} \to \mathcal{U}$ and $i^* : \mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U} \to \mathrm{Ar}(\mathcal{Z}) \xrightarrow{\mathrm{ev}_0} \mathcal{Z}$ are symmetric monoidal. □

We next wish to show that given a symmetric monoidal recollement $(\mathcal{U}, \mathcal{Z})$ of $\mathcal{X}$, the symmetric monoidal structure on $\mathcal{X}$ is the canonical one of Definition 2.25. We first observe that the unit transformation of a symmetric monoidal adjunction is itself a lax symmetric monoidal functor.

2.28 Lemma *Let $C^{\otimes}$ and $D^{\otimes}$ be symmetric monoidal ∞-categories and let $F : C \rightleftarrows D : G$ be a symmetric monoidal adjunction. Then the unit transformation $\eta : C \to \mathrm{Ar}(C)$ lifts to a lax symmetric monoidal functor $\eta^{\otimes} : C^{\otimes} \to (C^{\otimes})^{\Delta^1}$ such that $\mathrm{ev}_1 \eta^{\otimes} \simeq G^{\otimes} F^{\otimes}$ and $\mathrm{ev}_0 \eta^{\otimes} \simeq \mathrm{id}$.*

Proof Let $\mathcal{M} \to \Delta^1$ be the bicartesian fibration classified by the adjunction. We may factor (or define) η as the composition

$$C \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \subset \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \simeq \mathrm{Ar}(C) \times_C D \to \mathrm{Ar}(C),$$

where we use Lemma 2.11 for the identification of the sections of $\mathcal{M}$. Let $\mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$ be equipped with its canonical symmetric monoidal structure. Because F is symmetric monoidal, the inclusion $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \subset \mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M})$ defines a symmetric monoidal structure on $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M})$ by restriction such that the equivalence $\mathrm{ev}_0 : \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^1}(\Delta^1, \mathcal{M}) \xrightarrow{\simeq} C$ is an equivalence of symmetric monoidal ∞-categories. Also, the projection $\mathrm{Fun}_{/\Delta^1}(\Delta^1, \mathcal{M}) \to \mathrm{Ar}(C)$ is lax symmetric monoidal by definition. We deduce that η lifts to a lax symmetric monoidal functor $\eta^{\otimes}$ with the indicated properties. □

2.29 Proposition *Let $(\mathcal{U}, \mathcal{Z})$ be a symmetric monoidal recollement of $\mathcal{X}$. Then the functor $\mathcal{X} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{X})$ realizing the pullback square of functors*

$$\begin{array}{ccc} \mathrm{id} & \longrightarrow & i_* i^* \\ \downarrow & & \downarrow \\ j_* j^* & \longrightarrow & i_* i^* j_* j^* \end{array}$$

lifts to a lax symmetric monoidal functor $\mathcal{X}^{\otimes} \to (\mathcal{X}^{\otimes})^{\Delta^1 \times \Delta^1}$. Consequently, if $A \in \mathcal{X}$ is an algebra object, then we have an equivalence of algebras

$$A \simeq (j_* j^*)(A) \times_{(i_* i^* j_* j^*)(A)} (i_* i^*)(A).$$

Proof By Lemma 2.28, the symmetric monoidal adjunction $j^* \dashv j_*$ yields a lax symmetric monoidal functor

$$(\eta_j)^{\otimes} : \mathcal{X}^{\otimes} \to (\mathcal{X}^{\otimes})^{\Delta^1}.$$

We also have the induced symmetric monoidal adjunction $\hat{i}^* : \mathrm{Ar}(\mathcal{X}) \rightleftarrows \mathrm{Ar}(\mathcal{Z}) : \hat{i}_*$ which yields a lax symmetric monoidal functor

$$(\eta_{\hat{i}})^{\otimes} : (\mathcal{X}^{\otimes})^{\Delta^1} \to (\mathcal{X}^{\otimes})^{\Delta^1 \times \Delta^1}.$$

The composite $(\eta_{\hat{i}})^{\otimes} \circ (\eta_j)^{\otimes}$ then defines the desired functor. □

2.30 Theorem *Suppose $(\mathcal{U}, \mathcal{Z})$ is a symmetric monoidal recollement of $\mathcal{X}$. Then the equivalence*

$$\mathcal{X} \xrightarrow{\simeq} \mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$$

of Corollary 2.12 refines to an equivalence of symmetric monoidal ∞-categories, where we equip $\mathrm{Ar}(\mathcal{Z}) \times_{\mathcal{Z}} \mathcal{U}$ with the canonical symmetric monoidal structure of Definition 2.25.

Proof By Lemmas 2.28 and 2.31, we have a commutative diagram of ∞-operads

$$\begin{array}{ccc} \mathcal{X}^{\otimes} & \xrightarrow{(i^*)^{\otimes}(\eta_j)^{\otimes}} & (\mathcal{Z}^{\otimes})^{\Delta^1} \\ {\scriptstyle (j^*)^{\otimes}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathcal{U}^{\otimes} & \xrightarrow{(i^*)^{\otimes}(j_*)^{\otimes}} & Z^{\otimes} \end{array}$$

such that the induced functor $\theta^{\otimes} : \mathcal{X}^{\otimes} \to (\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes}$ covers the map θ of Corollary 2.12. Since θ is an equivalence, to show that $\theta^{\otimes}$ is an equivalence it suffices to check that $\theta^{\otimes}$ is strongly symmetric monoidal. But this follows from the symmetric monoidality of the jointly conservative functors j^*, i^*. □

We include the following simple strictification result for completeness.

2.31 Lemma *Suppose we have a homotopy commutative square of ∞-operads*

$$\begin{array}{ccc} A^{\otimes} & \xrightarrow{F'} & B^{\otimes} \\ \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle G} \\ C^{\otimes} & \xrightarrow{F} & D^{\otimes} \end{array}$$

in the sense that there is the data of a homotopy $\theta : G \circ F' \xRightarrow{\simeq} F \circ G'$, over $\mathbf{Fin}_*$,

$$\begin{array}{ccc} A^{\otimes} \times \{0\} & \xrightarrow{F'} & B^{\otimes} \\ \downarrow & & \downarrow{\scriptstyle G} \\ A^{\otimes} \times \Delta^1 & \xrightarrow{\theta} & D^{\otimes} \\ \uparrow & & \uparrow{\scriptstyle F} \\ A^{\otimes} \times \{1\} & \xrightarrow{G'} & C^{\otimes} \end{array}$$

such that θ sends every edge $(a,0) \to (a,1)$ to an equivalence. Suppose also that G is a fibration of ∞-operads, i.e., a categorical fibration [14, 2.1.2.10]*. Then there exists a functor $F'' : A^{\otimes} \to B^{\otimes}$ homotopic to F' as a map of ∞-operads such that the square*

$$\begin{array}{ccc} A^{\otimes} & \xrightarrow{F''} & B^{\otimes} \\ \downarrow{\scriptstyle G'} & & \downarrow{\scriptstyle G} \\ C^{\otimes} & \xrightarrow{F} & D^{\otimes} \end{array}$$

strictly commutes.

Proof Given an ∞-operad $O^{\otimes}$, let $O^{\otimes,\natural}$ denote the marked simplicial set $(O^{\otimes}, \mathcal{E})$ where $\mathcal{E}$ is the collection of inert morphisms in $O^{\otimes}$ [14, 2.1.4.5]. Consider the lifting problem in marked simplicial sets

$$\begin{array}{ccc} A^{\otimes,\natural} \times \{0\} & \xrightarrow{F'} & B^{\otimes,\natural} \\ \downarrow & \overset{\bar{\theta}}{\nearrow} & \downarrow{\scriptstyle G} \\ A^{\otimes,\natural} \times (\Delta^1)^{\sharp} & \xrightarrow{\theta} & D^{\otimes,\natural} \end{array}$$

Because G is assumed to be a fibration of ∞-operads, G is a fibration in the model structure on ∞-preoperads [14, 2.1.4.6]. Hence, the dotted lift $\bar{\theta}$ exists. If we then let $F'' = \bar{\theta}|_{A^{\otimes} \times \{1\}}$, the claim follows. □

We next turn to morphisms of symmetric monoidal recollements.

2.32 Observation Suppose we have a commutative diagram of symmetric monoidal ∞-categories and lax symmetric monoidal functors

$$\begin{array}{ccc} \mathcal{U}^{\otimes} & \xrightarrow{\phi^{\otimes}} & \mathcal{Z}^{\otimes} \\ {\scriptstyle F_U{}^{\otimes}}\downarrow & & \downarrow{\scriptstyle F_Z{}^{\otimes}} \\ \mathcal{U}'^{\otimes} & \xrightarrow{\phi'^{\otimes}} & \mathcal{Z}'^{\otimes} \end{array}$$

Then by way of the commutative diagram

$$\begin{array}{ccccc} (\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes} & \longrightarrow & (\mathcal{Z}^{\otimes})^{\Delta^1} & \xrightarrow{F_Z^{\otimes}} & (\mathcal{Z}'^{\otimes})^{\Delta^1} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathcal{U}^{\otimes} & \xrightarrow{\phi^{\otimes}} & \mathcal{Z}^{\otimes} & \xrightarrow{F_Z^{\otimes}} & \mathcal{Z}'^{\otimes} \\ & {\scriptstyle F_U^{\otimes}}\searrow & & \nearrow{\scriptstyle \phi'^{\otimes}} & \\ & & \mathcal{U}'^{\otimes} & & \end{array}$$

we obtain a lax symmetric monoidal functor $F^{\otimes} : (\mathcal{Z}^{\otimes})^{\Delta^1} \times_{\mathcal{Z}^{\otimes}} \mathcal{U}^{\otimes} \to (\mathcal{Z}'^{\otimes})^{\Delta^1} \times_{\mathcal{Z}'^{\otimes}} \mathcal{U}'^{\otimes}$, which is symmetric monoidal if $F_U^{\otimes}$ and $F_Z^{\otimes}$ are symmetric monoidal.

Let $\mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_{\infty}^{\otimes,\mathrm{lax}}) \subset \mathrm{Ar}(\mathbf{Cat}_{\infty}^{\otimes,\mathrm{lax}})$ be the subcategory whose objects are left-exact lax symmetric monoidal functors and whose morphisms are through symmetric monoidal functors. Then by the above construction[9] we may lift the functor $\lim^{\mathrm{rlax}} : \mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \to \mathbf{Recoll}_{\mathrm{str}}$ to

$$(\lim^{\mathrm{rlax}})^{\otimes} : \mathrm{Ar}_{\mathrm{lex}}(\mathbf{Cat}_\infty^{\otimes,\mathrm{lax}}) \to \mathbf{Recoll}_{\mathrm{str}}^{\otimes}.$$

An elaboration of Theorem 2.30 shows that $(\lim^{\mathrm{rlax}})^{\otimes}$ is an equivalence; we leave the details to the reader.

One also has a lift of $\lim^{\mathrm{rlax}} : \mathrm{Ar}^{\mathrm{rlax}}_{\mathrm{lex}}(\mathbf{Cat}_\infty) \to \mathbf{Recoll}$ if one considers right-lax commutative squares of ∞-operads. Since the details in this case are more involved, we leave a precise formulation to the reader.

Our next goal is to establish certain *projection formulas* satisfied by a (stable) symmetric monoidal recollement. First, we note the following about the situation in which the symmetric monoidal ∞-category $\mathcal{X}$ is in addition closed.

2.33 Observation Let $\mathcal{X}$ be a closed symmetric monoidal ∞-category and let $F(-,-)$ denote its internal hom. If $(\mathcal{U},\mathcal{Z})$ is a symmetric monoidal recollement of $\mathcal{X}$, then we define

$$F_{\mathcal{U}}(u,u') = j^* F(j_* u, j_* u') \quad \text{and} \quad F_{\mathcal{Z}}(z,z') = i^* F(i_* z, i_* z')$$

to be internal homs for $\mathcal{U}$ and $\mathcal{Z}$, so that $\mathcal{U}$ and $\mathcal{Z}$ are closed symmetric monoidal. Indeed, since $j^* \dashv j_*$ is monoidal, we have

$$\begin{aligned}&\mathrm{Map}_{\mathcal{U}}(w, j^*F(j_*u, j_*v)) \simeq \mathrm{Map}_{\mathcal{X}}(j_*w, F(j_*u, j_*v)) \simeq \mathrm{Map}_{\mathcal{X}}(j_*w \otimes j_*v, j_*v),\\ &\mathrm{Map}_{\mathcal{U}}(j^*(j_*w \otimes j_*u), v) \simeq \mathrm{Map}_{\mathcal{U}}(w \otimes u, v),\end{aligned}$$

and similarly for $F_{\mathcal{Z}}(-,-)$. Moreover we have natural equivalences

$$F(x, j_*u) \simeq j_* F_{\mathcal{U}}(j^*x, u), \quad F(x, i_*z) \simeq i_* F_{\mathcal{Z}}(i^*x, z).$$

For example, we may check

$$\begin{aligned}\mathrm{Map}_{\mathcal{X}}(x, F(y, j_*u)) &\simeq \mathrm{Map}_{\mathcal{X}}(x \otimes y, j_*u) \simeq \mathrm{Map}_{\mathcal{U}}(j^*x \otimes j^*y, u)\\ &\simeq \mathrm{Map}_{\mathcal{U}}(j^*x, F_{\mathcal{U}}(j^*y, u)) \simeq \mathrm{Map}_{\mathcal{X}}(x, j_*F_{\mathcal{U}}(j^*y, u)).\end{aligned}$$

This implies that the unit maps

$$\begin{aligned}F(j_*u, j_*u') &\to j_*j^*F(j_*u, j_*u') = j_*F_{\mathcal{U}}(u,u'),\\ F(i_*z, i_*z') &\to i_*i^*F(i_*z, i_*z') = i_*F_{\mathcal{Z}}(z,z')\end{aligned}$$

are equivalences.

2.34 Proposition (projection formulas) *Let $(\mathcal{U},\mathcal{Z})$ be a stable*[10] *symmetric monoidal recollement of $\mathcal{X}$.*

(1) *The natural maps $\alpha : i_*(z) \otimes x \to i_*(z \otimes i^*x)$ and $\beta : j_!(u \otimes j^*x) \to j_!(u) \otimes x$ are equivalences.*

(2) *The fiber sequence $j_! j^* x \to x \to i_* i^* x$ is equivalent to*

$$j_!(1_U) \otimes x \to x \to i_*(1_Z) \otimes x.$$

[9] Technically, to make a rigorous construction we may work at the level of preoperads and then pass to the underlying ∞-categories.

[10] We do not require stability for the $i^* \dashv i_*$ projection formula. For the assertions that only involve $j_!$, we only need that $\mathcal{X}$ be pointed.

Now suppose also that $\mathcal{X}$ is closed symmetric monoidal.

(3) *We have natural equivalences $F(j_! u, x) \simeq j_* F_{\mathcal{U}}(u, j^* x)$ and $F(i_* z, x) \simeq i_* F_{\mathcal{Z}}(z, i^! x)$.*

(4) *The fiber sequence $i_* i^! x \to x \to j_* j^* x$ is equivalent to*

$$F(i_* 1_Z, x) \to x \to F(j_! 1_U, x).$$

(5) *We have natural equivalences $j^* F(x, y) \simeq F_{\mathcal{U}}(j^* x, j^* y)$ and $F_{\mathcal{Z}}(i^* x, i^! y) \simeq i^! F(x, y)$.*

Proof For (1), it's easily checked that $i^*\alpha$, $j^*\alpha$ and $i^*\beta$, $j^*\beta$ are equivalences, hence α and β are equivalences. Item (2) then follows as a corollary. For (3), we have sequences of equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{X}}(y, F(j_! u, x)) &\simeq \mathrm{Map}_{\mathcal{X}}(y \otimes j_! u, x) \simeq \mathrm{Map}_{\mathcal{X}}(j_!(j^* y \otimes u), x) \simeq \mathrm{Map}_{\mathcal{U}}(j^* y \otimes u, j^* x) \\ &\simeq \mathrm{Map}_{\mathcal{U}}(j^* y, F_{\mathcal{U}}(u, j^* x)) \simeq \mathrm{Map}_{\mathcal{X}}(y, j_* F_{\mathcal{U}}(u, j^* x)), \\ \mathrm{Map}_{\mathcal{X}}(y, F(i_* z, x)) &\simeq \mathrm{Map}_{\mathcal{X}}(y \otimes i_* z, x) \simeq \mathrm{Map}_{\mathcal{X}}(i_*(i^* y \otimes z), x) \simeq \mathrm{Map}_{\mathcal{Z}}(i^* y \otimes z, i^! x) \\ &\simeq \mathrm{Map}_{\mathcal{Z}}(i^* y, F_{\mathcal{Z}}(z, i^! x)) \simeq \mathrm{Map}_{\mathcal{Z}}(y, i_* F_{\mathcal{Z}}(z, i^! x)). \end{aligned}$$

If we let $u = 1_U$, then $F_{\mathcal{U}}(1_U, v) \simeq v$, hence $F(j_! 1_U, x) \simeq j_* F_{\mathcal{U}}(1_U, j^* x) \simeq j_* j^* x$. Item (4) then follows as a corollary. For (5), we have sequences of equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{U}}(u, j^* F(x, y)) &\simeq \mathrm{Map}_{\mathcal{X}}(j_! u, F(x, y)) \simeq \mathrm{Map}_{\mathcal{X}}(j_! u \otimes x, y) \simeq \mathrm{Map}_{\mathcal{X}}(j_!(u \otimes j^* x), y) \\ &\simeq \mathrm{Map}_{\mathcal{U}}(u \otimes j^* x, j^* y) \simeq \mathrm{Map}_{\mathcal{U}}(u, F_{\mathcal{U}}(j^* x, j^* y)), \\ \mathrm{Map}_{\mathcal{Z}}(z, F_{\mathcal{Z}}(i^* x, i^! y)) &\simeq \mathrm{Map}_{\mathcal{Z}}(z \otimes i^* x, i^! y) \simeq \mathrm{Map}_{\mathcal{X}}(i_*(z \otimes i^* x), y) \simeq \mathrm{Map}_{\mathcal{X}}(i_* z \otimes x, y) \\ &\simeq \mathrm{Map}_{\mathcal{X}}(i_* z, F(x, y)) \simeq \mathrm{Map}_{\mathcal{Z}}(z, i^! F(x, y)). \end{aligned}$$ □

From Proposition 2.34, we immediately deduce the fundamental decomposition formula for objects in a stable symmetric monoidal recollement.

2.35 Corollary (decomposition formula) *Suppose that $(\mathcal{U}, \mathcal{Z})$ is a stable symmetric monoidal recollement of a closed symmetric monoidal stable ∞-category $\mathcal{X}$. Then for all $x \in \mathcal{X}$, we have a commutative diagram*

$$\begin{array}{ccccc} x \otimes j_!(1_U) & \longrightarrow & x & \longrightarrow & x \otimes i_*(1_Z) \\ \downarrow{\simeq} & & \downarrow & & \downarrow \\ F(j_!(1_U), x) \otimes j_!(1_U) & \longrightarrow & F(j_!(1_U), x) & \longrightarrow & F(j_!(1_U), x) \otimes i_*(1_Z) \end{array}$$

in which the right-hand square is a pullback square.

For example, Corollary 2.35 abstracts the well-known fracture square decomposition of a G-spectrum with respect to a family of subgroups, and conversely can be used to deduce it (see [20, §2.2]).

Finally, we record the following relation between stable symmetric monoidal recollements and smashing localizations.

2.36 Observation Suppose $\mathcal{X}$ is a symmetric monoidal stable ∞-category and $\mathcal{Z} \subset \mathcal{X}$ is a reflective and coreflective subcategory that determines a stable recollement $(\mathcal{U}, \mathcal{Z})$ on $\mathcal{X}$. Then this recollement is symmetric monoidal if and only if $i_* i^*$ is compatible with the symmetric monoidal structure on $\mathcal{X}$ and the resulting projection formula for $i^* \dashv i_*$ holds, i.e., the natural map $i_* z \otimes x \to i_*(z \otimes i^* x)$ is an equivalence for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$. Indeed, the "only if" direction hold by Proposition 2.34, and for the "if" direction, we only need to show that for every $x \in \mathcal{X}$ such that $j^* x \simeq 0$, $j^*(x \otimes y) \simeq 0$ for every $y \in \mathcal{X}$. But $j^* x \simeq 0$ if and only if $x \simeq i_* i^* x$, and then

$$j^*(x \otimes y) \simeq j^*(i_* i^* x \otimes y) \simeq j^*(i_*(i^* x \otimes i^* y)) \simeq 0.$$

Suppose further that $\mathcal{X}$ and $\mathcal{Z}$ are presentable. In view of [15, Proposition 5.29], $\mathcal{Z}$ is a *smashing localization* of $\mathcal{X}$ in the sense that $\mathcal{Z} \simeq \mathbf{Mod}_{\mathcal{X}}(A)$ for $A = i_* i^* 1$ an idempotent E_∞-algebra in $\mathcal{X}$. We deduce that smashing localizations of $\mathcal{X}$ are in bijective correspondence with stable symmetric monoidal recollements of $\mathcal{X}$. Moreover, if $F : \mathcal{X} \to \mathcal{X}'$ is a morphism of symmetric monoidal recollements $(\mathcal{U}, \mathcal{Z}) \to (\mathcal{U}', \mathcal{Z}')$, then

$$F i_* i^* 1 \simeq i'_* i'^* F(1) \simeq i'_* i'^* 1,$$

so F preserves the defining idempotent E_∞-algebras.

2.3 Families of recollements

We conclude this section with a few extensions of recollement theory to the parametrized setting. Let S be an ∞-category, let $\mathcal{X}_\bullet : S \to \mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}$ be a functor, and let $\mathcal{X}, \mathcal{U}, \mathcal{Z} \to S$ be the cocartesian fibrations obtained via the Grothendieck construction. Then in view of Observation 2.5 and the strictness assumption, we have S-adjunctions [21, Definition 8.3][11]

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}.$$

In what follows, we use the following terminology from [21]:

(1) An *S-∞-category* is a cocartesian fibration $C \to S$.

(2) Given two S-∞-categories $C, D \to S$, the *∞-category of S-functors* $\mathrm{Fun}_S(C, D)$ is notation for $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(C, D)$.

We first show that the procedure of taking S-functor categories yields a recollement.

2.37 Lemma *For any S-∞-category K, $(\mathrm{Fun}_S(K, \mathcal{U}), \mathrm{Fun}_S(K, \mathcal{Z}))$ is a recollement of $\mathrm{Fun}_S(K, \mathcal{X})$.*

Proof By [21, Proposition 8.4], we have induced adjunctions given by postcomposition

$$\mathrm{Fun}_S(K, \mathcal{U}) \underset{\bar{j}_*}{\overset{\bar{j}^*}{\leftrightarrows}} \mathrm{Fun}_S(K, \mathcal{X}) \underset{\bar{i}_*}{\overset{\bar{i}^*}{\rightleftarrows}} \mathrm{Fun}_S(K, \mathcal{Z}),$$

[11] Recall given two cocartesian fibrations $C, D \to S$ that a relative adjunction $F : C \rightleftarrows D : G$ with respect to S in the sense of Lurie [14, Definition 7.3.2.2] is said to be an *S-adjunction* if F and G both preserve cocartesian edges.

where it is clear that $\bar{j}^* \bar{j}_* \simeq \mathrm{id}$ and $\bar{i}^* \bar{i}_* \simeq \mathrm{id}$, hence $\bar{j}_*$ and $\bar{i}_*$ are fully faithful. By [12, Proposition 5.4.7.11], the hypothesis that for all $f : s \to t$ the restriction functors $f^* : \mathcal{X}_t \to \mathcal{X}_s$ preserve finite limits ensures that $\mathrm{Fun}_S(K, \mathcal{X})$ admits finite limits (which are computed fiberwise), and similarly the induced restriction functors f_U^* and f_Z^* preserve finite limits, so $\mathrm{Fun}_S(K, \mathcal{U})$, $\mathrm{Fun}_S(K, \mathcal{Z})$ admit finite limits and $\bar{j}^*, \bar{i}^*$ preserve finite limits. Since $j^* i_* \simeq 0$ and the terminal object $0 \in \mathrm{Fun}_S(K, \mathcal{U})$ is given by $K \to S \xrightarrow{\underline{0}} \mathcal{U}$ for the cocartesian section $\underline{0} : S \to \mathcal{U}$ that selects the terminal object in each fiber, we get that $\bar{j}^* \bar{i}_* \simeq 0$. Finally, since a morphism f in $\mathrm{Fun}_S(K, \mathcal{X})$ is an equivalence if and only if $f(k)$ is an equivalence for all $k \in K$, we deduce that $\bar{j}^*$ and $\bar{i}^*$ are jointly conservative using the joint conservativity of j^* and i^*. □

2.38 Corollary *The forgetful functors* $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}} \to \mathbf{Cat}_\infty$ *and* $\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}} \to \mathbf{Cat}^{\mathrm{stab}}_\infty$ *create limits.*

Proof The first statement follows from Lemma 2.37 by taking $K = S$ and using that the ∞-category of cocartesian sections computes the limit of a diagram of ∞-categories [12, §3.3.3]. We note that the proof of Lemma 2.37 shows that the evaluation functors at any $s \in S$ are left-exact and strict morphisms of recollements, so the limit resides in $\mathbf{Recoll}^{\mathrm{lex}}_{\mathrm{str}}$. Finally, because limits in $\mathbf{Cat}^{\mathrm{stab}}_\infty$ are created in $\mathbf{Cat}_\infty$, the second statement follows. □

We can also use Lemma 2.37 to compute S-colimits in $\mathcal{X}$. For clarity, let us revert to the nonparametrized case $S = *$ for the next two results; the S-analogues will also hold by the same reasoning.

2.39 Lemma *Let* $(\mathcal{U}, \mathcal{Z})$ *be a recollement of* $\mathcal{X}$ *and suppose that* $\mathcal{U}$ *and* $\mathcal{Z}$ *admit* K*-indexed colimits. Then* $\mathcal{X}$ *admits* K*-indexed colimits.*

Proof With respect to the recollement of $\mathrm{Fun}(K, \mathcal{X})$ of Lemma 2.37, the constant diagram functor $\delta : \mathcal{X} \to \mathrm{Fun}(K, \mathcal{X})$ is obviously a morphism of recollements. Passing to left adjoints, we obtain a right-lax commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}(K, \mathcal{U}) & \xrightarrow{\bar{i}^* \bar{j}_*} & \mathrm{Fun}(K, \mathcal{Z}) \\ {\scriptstyle \mathrm{colim}} \downarrow & \swarrow & \downarrow {\scriptstyle \mathrm{colim}} \\ \mathcal{U} & \xrightarrow[i^* j_*]{} & \mathcal{Z} \end{array}$$

which induces a morphism of recollements $\mathrm{colim} : \mathrm{Fun}(K, \mathcal{X}) \to \mathcal{X}$. We claim that colim is left adjoint to δ. In fact, if $\mathcal{M}, \mathcal{M}^K \to \Delta^1$ are the cartesian fibrations classified by $i^* j_*$ and $\bar{i}^* \bar{j}_*$ respectively, then we have a map $\delta : \mathcal{M}^K \to \mathcal{M}$ of cartesian fibrations and by [14, Proposition 7.3.2.6] a relative left adjoint $\mathrm{colim} : \mathcal{M}^K \to \mathcal{M}$. The formation of sections sends relative adjunctions to adjunctions, which proves the claim. We deduce that $\mathcal{X}$ admits K-indexed colimits. □

2.40 Corollary *Suppose* $\mathcal{U}$ *and* $\mathcal{Z}$ *are presentable* ∞*-categories and* $\phi : \mathcal{U} \to \mathcal{Z}$ *is a left-exact accessible functor. Then* $\mathcal{X} = \lim^{\mathrm{rlax}} \phi$ *is a presentable* ∞*-category.*

Proof By Lemma 2.39, $\mathcal{X}$ admits all small colimits. By [12, Corollary 5.4.7.17], $\mathcal{X}$ is accessible. We conclude that $\mathcal{X}$ is presentable. □

Finally, we describe how recollements interact with an ambidextrous adjunction (e.g., the adjunction between restriction and induction for equivariant spectra).

2.41 Lemma *Let $(\mathcal{U},\mathcal{Z})$ and $(\mathcal{U}',\mathcal{Z}')$ be stable recollements on $\mathcal{X}$ and $\mathcal{X}'$ and let $f^*:\mathcal{X}\to\mathcal{X}'$ be an exact functor such that $f^*|_{i_*(\mathcal{Z})}\subset i_*(\mathcal{Z}')$ (so f^* is not necessarily a morphism of recollements, but we still may define ${f_U}^*:=j'^*f^*j_*$, ${f_Z}^*:=i'^*f^*i_*$, and have ${f_U}^*j^*\simeq j'^*{f_U}^*$).*

(1) *Suppose that $f^*|_{j_!(\mathcal{U})}\subset j'_!(\mathcal{U}')$ and f^* admits a right adjoint f_*. Then:*

(a) *The essential image of $f_*j'_*$ lies in $j_*(\mathcal{U})$, so $f^*\dashv f_*$ restricts to an adjunction*

$$ {f_U}^*:\mathcal{U}\rightleftarrows\mathcal{U}':f_{U*} $$

with $j_*f_{U*}\simeq f_*j'_*$.

(b) *The natural map $j^*f_*\to f_{U*}j'^*$ is an equivalence.*

(c) *The essential image of $f_*i'_*$ lies in $i_*(\mathcal{Z})$, so $f^*\dashv f_*$ restricts to an adjunction*

$$ {f_Z}^*:\mathcal{Z}\rightleftarrows\mathcal{Z}':f_{Z*} $$

with $i_*f_{Z*}\simeq f_*i'_*$.

(2) *Suppose that $f^*|_{j_*(\mathcal{U})}\subset j'_*(\mathcal{U}')$ and f^* admits a left adjoint $f_!$. Then:*

(a) *The essential image of $f_*j'_!$ lies in $j_!(\mathcal{U})$, so $f_!\dashv f^*$ restricts to an adjunction*

$$ f_{U!}:\mathcal{U}'\rightleftarrows\mathcal{U}:{f_U}^* $$

with $j_!f_{U!}\simeq f_!j'_!$.

(b) *The natural map $f_{U!}j^*\to j'^*f_!$ is an equivalence.*

(c) *The essential image of $f_!i'_*$ lies in $i_*(\mathcal{Z})$, so $f_!\dashv f^*$ restricts to an adjunction*

$$ f_{Z!}:\mathcal{Z}'\rightleftarrows\mathcal{Z}:{f_Z}^* $$

with $i_*f_{Z!}\simeq f_!i'_*$.

(d) *The natural map $i^*f_{Z!}\to f_{Z!}i'^*$ is an equivalence.*

(3) *Suppose that $f^*\in\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$, f^* admits left and right adjoints $f_!$ and f_*, and we have the ambidexterity equivalence $f_!\simeq f_*$. Then $f_*\in\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$ and we additionally have ambidexterity equivalences $f_{U!}\simeq f_{U*}$ and $f_{Z!}\simeq f_{Z*}$.*

Proof We first prove the assertions of (1). For (1)(a), for any $u'\in\mathcal{U}'$ because we have for all $z\in\mathcal{Z}$ that

$$ \mathrm{Map}_{\mathcal{X}}(i_*z,f_*j'_*u')\simeq\mathrm{Map}_{\mathcal{U}'}(j'^*f^*i_*z,u')\simeq\mathrm{Map}_{\mathcal{U}'}(f_U^*j'^*i_*z,u')\simeq *, $$

we get $f_*j'_*u'\in j_*(\mathcal{U})$. For (1)(b), the assertion holds because the map is adjoint to the equivalence $f^*j_!\to j'_!{f_U}^*$. For (1)(c), for any $z'\in\mathcal{Z}'$ we have

$$ j^*f_*i'_*z'\simeq f_{U*}j^*i'_*z'\simeq f_{U*}0\simeq 0, $$

hence $f_* i'_* z' \in i_*(\mathcal{Z})$. Next, the assertions of (2) hold by a dual argument; we note that the extra assertion (2)(d) holds because $f_!$ now commutes with $j_!$ instead of j_*. Finally, for (3) the functor $f_! \simeq f_*$ is in $\mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$ by combining (1)(a), (1)(c), and (2)(a). For the ambidexterity assertions, the equivalence $f_{Z!} \simeq f_{Z*}$ is clear because the embedding $i_* : \mathcal{Z} \subset \mathcal{X}$ is unambiguous, whereas for $f_{U!} \simeq f_{U*}$ we note that the sequence of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathcal{U}}(u, f_{U!}u') &\simeq \mathrm{Map}_{\mathcal{X}}(j_! u, f_! j'_! u') \simeq \mathrm{Map}_{\mathcal{X}}(j_! u, f_* j'_! u') \simeq \mathrm{Map}_{\mathcal{X}'}(f^* j_! u, j'_! u') \\ &\simeq \mathrm{Map}_{\mathcal{X}'}(j'_! f_U{}^* u, j'_! u') \simeq \mathrm{Map}_{\mathcal{U}'}(f_U{}^* u, u')\end{aligned}$$

demonstrates that $f_{U!}$ is right adjoint to $f_U{}^*$ and hence $f_{U!} \simeq f_{U*}$. □

2.42 Corollary *Let G be a finite group. Suppose that $\mathcal{X}_\bullet : \mathcal{O}_G^{\mathrm{op}} \to \mathbf{Recoll}^{\mathrm{stab}}_{\mathrm{str}}$ is a functor such that the underlying G-∞-category $\mathcal{X}$ is G-stable* [16, Definition 7.1]*. Then $\mathcal{U}$ and $\mathcal{Z}$ are G-stable and all of the functors appearing in the diagram of G-adjunctions*

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}$$

are G-exact.

Proof By Lemma 2.41, it only remains to check the Beck–Chevalley condition for $\mathcal{U}$ and $\mathcal{Z}$ to show the existence of finite G-products. But this follows from the same condition on $\mathcal{X}$, since the restriction and induction functors $(f_-)^*$, $(f_-)_*$ commute with the inclusion functors $(j_\bullet)_*$, $(j_\bullet)_!$, and $(i_\bullet)_*$. □

2.43 Definition In the situation of Corollary 2.42, we say that $(\mathcal{U}, \mathcal{Z})$ is a *G-stable G-recollement* of $\mathcal{X}$.

3 Recollements on lax limits of ∞-categories

Let S be an ∞-category throughout this section. Suppose $p : C \to S$ is a locally cocartesian fibration classified by a 2-functor into $\mathbf{Cat}_\infty$ [12, Definition 1.1.5.1; 13, §3], so for every 2-simplex $\Delta^2 \to S$, we have a lax commutative diagram of ∞-categories

$$\begin{array}{ccc} C_0 & \xrightarrow{\;F_{02}\;} & C_2 \\ & \searrow^{F_{01}} \;\Downarrow\; \nearrow_{F_{12}} & \\ & C_1 & \end{array}$$

and the higher-dimensional simplices of S supply coherence data. Then the 2-functoriality of f yields two notions of lax limit corresponding to choosing two possible orientations for morphisms — informally, the *left-lax* limit of f has objects given by tuples $(x_i \in C_i, \alpha_{ij} : F_{ij}(x_i) \to x_j)$, whereas the *right-lax* limit of f has objects given by tuples $(x_i \in C_i, \alpha_{ij} : x_j \to F_{ij}(x_i))$. To give rigorous meaning to these notions, we may circumvent giving a precise formulation of the lax universal property (for instance, as carried out in [8]) and instead *define* the left-lax limit to be the ∞-category of sections

$$\lim^{\mathrm{llax}} f = \lim^{\mathrm{llax}} C := \mathrm{Fun}_{/S}(S, C)$$

and the right-lax limit to be the ∞-category

$$\lim^{\mathrm{rlax}} f = \lim^{\mathrm{rlax}} C := \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C),$$

where $\mathrm{sd}(S)$ is the *barycentric subdivision* of S (Definition 3.22) that is locally cocartesian over S via the *max* functor (Construction 3.24), and we let $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(-,-)$ be the full subcategory on those functors over S that preserve *locally cocartesian* edges. Viewing f itself as a *left-lax diagram* in $\mathbf{Cat}_\infty$, we may thereby speak of left-lax and right-lax limits of left-lax diagrams of ∞-categories; dually, we may also speak of left-lax and right-lax limits of right-lax diagrams of ∞-categories encoded as locally cartesian fibrations. We refer to [1, §1; 2, §A] for a more detailed discussion.[12]

3.1 Definition Let $S' \subset S$ be a full subcategory. Then S' is a *sieve* if for every morphism $x \to y$ in S, if $y \in S'$, then $x \in S'$. Dually, S' is a *cosieve* if $(S')^{\mathrm{op}}$ is a sieve in S^{op}.

Given a sieve $S_0 \subset S$ and cosieve $S_1 \subset S$, we say that S_0 and S_1 form a *sieve-cosieve decomposition* of S if S_0 and S_1 are disjoint and any object $x \in S$ lies either in S_0 or S_1.

3.2 Remark Sieves and cosieves are necessarily stable under equivalences. Given a sieve-cosieve decomposition (S_0, S_1) of S, we may define a functor $\pi : S \to \Delta^1$ that sends each object $x \in S$ to the integer $i \in \{0,1\}$ such that $x \in S_i$. Conversely, any functor $\pi : S \to \Delta^1$ determines a sieve-cosieve decomposition of S by taking its fibers over 0 and 1.

Our main goal in this section is to describe how sieve-cosieve decompositions of S produces recollements on right-lax limits of locally cocartesian fibrations $p : C \to S$ (Theorem 3.39).

3.3 Remark As we saw in Observation 2.9, a recollement itself is an example of a right-lax limit over Δ^1. Given a working theory of (pointwise) right-lax Kan extensions, our results should follow from the usual transitivity property of Kan extensions applied to the factorization $S \xrightarrow{\pi} \Delta^1 \to *$. However, we are not aware of such a theory that also affords the explicit description of the gluing functor given in Theorem 3.32; indeed, Theorem 3.32 should precisely amount to a pointwise formula for the right-lax Kan extension along π. We refer the interested reader to the discussion in [11, §2.2] for more on this question.

3.1 Recollements on right-lax limits of strict diagrams

Before entering into our study of left-lax diagrams, let us consider the simpler case of strict diagrams $f : S \to \mathbf{Cat}_\infty$. For this case, right-lax limits are modeled by sections of the *cartesian* fibration that classifies f. Thus suppose that $p : C \to S$ is a cartesian fibration, $\pi : S \to \Delta^1$ is a functor, and let $p_0 : C_0 \to S_0$, $p_1 : C_1 \to S_1$ denote the pullbacks of p to the fibers S_0, S_1 of π. Given a section $F : S \to C$ of p, let $j^* F : S_1 \to C_1$ be its restriction over S_1 and let $i^* F : S_0 \to C_0$ be its restriction over S_0. We obtain functors

$$j^* : \mathrm{Fun}_{/S}(S, C) \to \mathrm{Fun}_{/S_1}(S_1, C_1), \quad i^* : \mathrm{Fun}_{/S}(S, C) \to \mathrm{Fun}_{/S_0}(S_0, C_0).$$

[12] We follow [1, §1] in referring to these two types of lax limits as "left" and "right", even though lax and oplax are more standard nomenclature. The terminology is consistent with the usage of left for cocartesian-type constructions and right for cartesian-type constructions (e.g., left and right fibrations).

We first explain when j^* and i^* admit right adjoints. Suppose $G : S_1 \to C_1$ is a section of p_1. For every $x \in S$, let

$$G_x : (S_1)_{x/} := S_1 \times_S S_{x/} \to S_1 \xrightarrow{G} C_1 \subset C$$

be the composite functor and consider the commutative diagram

$$\begin{array}{ccc} (S_1)_{x/} & \xrightarrow{G_x} & C \\ \downarrow & \overline{G_x} \nearrow & \downarrow p \\ (S_1)^{\triangleleft}_{x/} & \longrightarrow & S \end{array}$$

where the cone point is sent to x. By [12, Corollary 4.3.1.11], if for every $s \in S$, C_s admits $(S_1)_{x/}$-indexed limits, and for every $f : s \to t$, the pullback functor $f^* : C_t \to C_s$ preserves $(S_1)_{x/}$-indexed limits, then there exists a dotted lift $\overline{G_x}$ which is a p-limit of G_x. If this holds for all $x \in S$, then by the dual of [12, Lemma 4.3.2.13], the p-right Kan extension j_*G exists and is computed pointwise by these p-limits. Moreover, by [12, Proposition 4.3.2.17], the right adjoint j_* then exists and is computed objectwise by j_*G.

Now let $H : S_0 \to C_0$ be a section of p_0. The same results hold for computing i_*H. However, the slice ∞-categories $(S_0)_{x/}$ are empty when $x \in S_1$. Therefore, the hypotheses above amount to supposing that for all $s \in S$, C_s admits a terminal object, and for all $f : s \to t$, the pullback functor f^* preserves this terminal object.

Finally, let $\mathcal{K} = \{K_\alpha\}_{\alpha \in A}$ be a class of simplicial sets and suppose that for all $K \in \mathcal{K}$ and $s \in S$, the fiber C_s admits K-indexed limits, and for all $f : s \to t$, the pullback functor f^* preserves K-indexed limits. Then by [12, dual of Proposition 5.4.7.11 and Remark 5.4.7.13], $\mathrm{Fun}_{/S}(S, C)$ admits K-indexed limits such that the evaluation functors $\mathrm{ev}_s : \mathrm{Fun}_{/S}(S, C) \to C_s$ preserve K-indexed limits — in other words, the K-indexed limits in $\mathrm{Fun}_{/S}(S, C)$ are computed fiberwise.

3.4 Definition (standard existence assumptions, strict version) Let $p : C \to S$ be a cartesian fibration and let $\pi : S \to \Delta^1$ be a functor. We say that p satisfies the *standard recollement existence assumptions* with respect to π if:

(1) For all $s \in S$, C_s admits finite limits, and for all morphisms $f : s \to t$ in S, the pullback functor $f^* : C_t \to C_s$ preserves finite limits.

(2) For all $x \in S$, C_s admits $(S_1)_{x/}$-indexed limits, and for all morphisms $f : s \to t$ in S, the pullback functor $f^* : C_t \to C_s$ preserves $(S_1)_{x/}$-indexed limits.

Let us now suppose that we are in the situation of Definition 3.4.

3.5 Proposition *The adjunctions*

$$\mathrm{Fun}_{/S_1}(S_1, C_1) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Fun}_{/S}(S, C) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}_{/S_0}(S_0, C_0)$$

together exhibit $\mathrm{Fun}_{/S}(S, C)$ *as a recollement of* $\mathrm{Fun}_{/S_1}(S_1, C_1)$ *and* $\mathrm{Fun}_{/S_0}(S_0, C_0)$.

Proof Note the functors j^* and i^* are left exact by the fiberwise computation of limits in section ∞-categories. Because $(S_0)_{x/} = \varnothing$ for all $x \in S_1$, we get that $j^* i_*$ is the constant functor at the terminal object of $\mathrm{Fun}_{/S_1}(S_1, C_1)$. Finally, i^* and j^* are jointly conservative because equivalences are detected objectwise in $\mathrm{Fun}_{/S}(S, C)$. □

3.6 Remark If the fibers of p are moreover stable ∞-categories, then the left-exact pullback functors f^* are necessarily exact and the recollement of Proposition 3.5 is stable.

3.7 Example Let $C \simeq D \times S$ and p be the projection to S. Then the recollement of Proposition 3.5 simplifies to

$$\mathrm{Fun}(S_1, D) \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}(S, D) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}(S_0, D),$$

where $j : S_1 \to S$ and $i : S_0 \to S$ now denote the inclusions. Recollement theory then gives a calculational technique for computing the right Kan extension $\phi_* F$ of a functor $F : S \to D$ along $\phi : S \to T$. Namely, if we let $\phi_0 = \phi \circ i$, $\phi_1 = \phi \circ j$, $F_0 = F|_{S_0}$, and $F_1 = F|_{S_1}$, the pullback square Proposition 2.2 yields a pullback square

$$\begin{array}{ccc} \phi_* F & \longrightarrow & (\phi_0)_* F_0 \\ \downarrow & & \downarrow \\ (\phi_1)_* F_1 & \longrightarrow & (\phi_0)_*((j_* F_1)|_{S_0}) \end{array}$$

3.2 Recollements on right-lax limits of left-lax diagrams

We now seek to establish the analogue of Proposition 3.5 for right-lax limits of locally cocartesian fibrations. Although the ideas are straightforward, the categorical details turn out to be considerably more involved. We begin by proving some needed extensions to the theory of relative right Kan extensions initiated in [12, §4.1–3], which play a technical role in our construction of the recollement adjunctions. We then construct the barycentric subdivision $\mathrm{sd}(S)$ (Definition 3.22, but also see Observation 3.23), and extend the cocartesian pushforward of [21, Lemma 2.23] to the locally cocartesian situation (Theorems 3.20 and 3.26). Finally, given a sieve-cosieve decomposition of S and suitable hypotheses on the locally cocartesian fibration $p : C \to S$, we establish localizations in Theorem 3.32, Corollary 3.34, and Proposition 3.36, and show that these together constitute a recollement of the right-lax limit of p in Theorem 3.39.

3.2.1 Relative right Kan extension In [12, Proposition 4.3.1.10], Lurie gives a criterion for when a colimit diagram in a fiber of a locally cocartesian fibration is a relative colimit. In contrast, we will also need a separate understanding of when a *limit* diagram in a fiber is a relative limit. As indicated in Lemma 3.8, in this situation we can give an unconditional statement.

3.8 Lemma *Let S be an ∞-category and let $f : C \to S$ be a locally cocartesian fibration. Let $s \in S$ be an object and $\bar{p} : K^{\lhd} \to C_s$ a limit diagram that extends p. Then, viewed as a diagram in C, $\bar{p}$ is a*

f-limit diagram [12, 4.3.1.1], *i.e., the commutative square*

$$\begin{array}{ccc} C_{/\bar{p}} & \longrightarrow & C_{/p} \\ \downarrow & & \downarrow \\ S_{/f\bar{p}} & \longrightarrow & S_{/fp} \end{array}$$

is a homotopy pullback square.

Proof It suffices to show that

$$C_{/\bar{p}} \to C_{/p} \times_{S_{/fp}} S_{/f\bar{p}}$$

is a trivial Kan fibration. To this end, let $A \to B$ be a monomorphism of simplicial sets and consider the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & C_{/\bar{p}} \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & C_{/p} \times_{S_{/fp}} S_{/f\bar{p}} \end{array}$$

This transposes to the lifting problem

$$\begin{array}{ccc} A \star K^{\lhd} \cup_{A \star K} B \star K & \xrightarrow{\beta} & C \\ \downarrow & \overset{\gamma}{\nearrow} & \downarrow f \\ B \star K^{\lhd} & \xrightarrow[\alpha]{} & S \end{array}$$

Our approach will be to first pushforward to the fiber C_s using that f is a locally cocartesian fibration and then solve the lifting problem in C_s using that $\bar{p}$ is a limit diagram.

To begin, because $\bar{p}$ is a diagram in the fiber C_s, the map α factors as $B \star K^{\rhd} \to B \star \Delta^0 \xrightarrow{\alpha'} S$ with $\alpha'|_{\Delta^0} = \{s\}$. We may define a map $r : (B \star \Delta^0) \times \Delta^1 \to B \star \Delta^0$ such that $r_0 = \mathrm{id}$ and r_1 is constant at Δ^0 in the following way: let $\pi : B \star \Delta^0 \to \Delta^1$ be the structure map of the join which sends B to $\{0\}$ and Δ^0 to $\{1\}$, and let ρ be the composite $(B \star \Delta^0) \times \Delta^1 \xrightarrow{\pi \times \mathrm{id}} \Delta^1 \times \Delta^1 \xrightarrow{\max} \Delta^1$, so the fiber of ρ over $\{0\}$ is $B \times \{0\}$. Then, recalling that maps $L \to X \star Y$ of simplicial sets over Δ^1 are equivalently specified by pairs of maps $(f_0 : L_0 \to X,\ f_1 : L_1 \to Y)$, r is the map over Δ^1 with respect to ρ and π given by $B \subset B \star \Delta^0$ and the constant map to Δ^0. Now let

$$h^{\alpha} : (B \star K^{\lhd}) \times \Delta^1 \to (B \star \Delta^0) \times \Delta^1 \xrightarrow{r} B \star \Delta^0 \xrightarrow{\alpha'} S,$$

so $h^{\alpha}_0 = \alpha$ and h^{α}_1 is constant at $\{s\}$. Also denote by h^{α} the restrictions of h^{α} to $(B \star K) \times \Delta^1$, $(A \star K^{\lhd}) \times \Delta^1$, and $(A \star K) \times \Delta^1$.

Let $\mathfrak{P} = (M_S, T, \varnothing)$ be the categorical pattern on $s\mathbf{Set}^{+}_{/S}$ that yields the locally cocartesian model structure, so M_S consists of all the edges in S, T consists of all the degenerate 2-simplices in S, and the fibrant objects are the locally cocartesian fibrations. By the criterion of [14, Lemma B.1.10] applied to $K \to B \star K$ (with the degenerate edges marked) and $\{0\} \to (\Delta^1)^\sharp$, the inclusion map of marked simplicial sets

$$(B \star K) \times \{0\} \cup_{(K \times \{0\})} K \times (\Delta^1)^\sharp \to (B \star K) \times (\Delta^1)^\sharp$$

is $\mathfrak{P}$-anodyne, and likewise replacing $K \to B \star K$ with $K^{\lhd} \to A \star K^{\lhd}$ and $K \to A \star K$. Using left properness of the locally cocartesian model structure, we deduce that the morphism

$$\begin{array}{c} (A \star K^{\lhd} \cup_{A \star K} B \star K) \times \{0\} \cup_{K^{\lhd} \times \{0\}} K^{\lhd} \times (\Delta^1)^\sharp \\ \downarrow \\ (A \star K^{\lhd} \cup_{A \star K} B \star K) \times (\Delta^1)^\sharp \end{array}$$

is $\mathfrak{P}$-anodyne. Consider the commutative square

$$\begin{array}{ccc} (A \star K^{\lhd} \cup_{A \star K} B \star K) \times \{0\} \cup_{K^{\lhd} \times \{0\}} K^{\lhd} \times (\Delta^1)^\sharp & \longrightarrow & {}_\natural C \\ \downarrow & \overset{h^\beta}{\nearrow} & \downarrow f \\ (A \star K^{\lhd} \cup_{A \star K} B \star K) \times (\Delta^1)^\sharp & \xrightarrow[h^\alpha]{} & S^\sharp \end{array}$$

where ${}_\natural C$ denotes the marking on C given by the f-locally cocartesian edges and the top horizontal map restricted to the first factor is β and to the second factor $K^{\lhd} \times (\Delta^1)^\sharp$ is the constant homotopy $K^{\lhd} \times \Delta^1 \xrightarrow{\mathrm{pr}} K^{\lhd} \xrightarrow{\bar{p}} C$. Then the dotted lift h^β exists, and the image of h^β_1 is contained in the fiber C_s.

Now consider the commutative triangle

$$\begin{array}{ccc} A \star K^{\lhd} \cup_{A \star K} B \star K & \xrightarrow{h^\beta_1} & C_s \\ \downarrow & \overset{\gamma_1}{\nearrow} & \\ B \star K^{\lhd} & & \end{array}$$

Because $\bar{p} : K^{\lhd} \to C_s$ is a limit diagram, the map $(C_s)_{/\bar{p}} \to (C_s)_{/p}$ is a trivial Kan fibration. Therefore, the dotted lift γ_1 exists.

Next, define a map

$$\theta = (\theta', \theta'') : (B \times \Delta^1) \star K^{\lhd} \to (B \star K^{\lhd}) \times \Delta^1$$

by its factors

$$\theta' : (B \times \Delta^1) \star K^{\lhd} \xrightarrow{\mathrm{pr} \star \mathrm{id}} B \star K^{\lhd},$$
$$\theta'' : (B \times \Delta^1) \star K^{\lhd} \xrightarrow{\mathrm{pr} \star \mathrm{id}} \Delta^1 \star K^{\lhd} \to \Delta^1 \star \Delta^0 \cong \Delta^2 \xrightarrow{\sigma^1} \Delta^1.$$

Here $\sigma^1 : \Delta^2 \to \Delta^1$ is the standard degeneracy map, so $\sigma^1(0) = 0$, $\sigma^1(1) = 1$, and $\sigma^1(2) = 1$. Also denote by θ the restriction to $(A \times \Delta^1) \star K^{\lhd}$, etc. Let

$$X = (A \times \Delta^1) \star K^{\lhd} \cup_{(A \times \Delta^1) \star K} (B \times \Delta^1) \star K \cup_{(A \times \{1\}) \star K^{\lhd} \cup_{(A \times \{1\}) \star K} (B \times \{1\}) \star K} B \star K^{\lhd}$$

and consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{(h^{\beta}\circ\theta)\cup\gamma_1} & C \\ {\scriptstyle\lambda}\big\downarrow & \overset{h^{\gamma}}{\nearrow} & \big\downarrow{\scriptstyle f} \\ (B\times\Delta^1)\star K^{\lhd} & \xrightarrow[h^{\alpha}\circ\theta]{} & S \end{array}$$

(where for commutativity, we use that $\theta_1 : (B\times\{1\})\star K^{\lhd} \to (B\star K^{\lhd})\times\{1\}$ is an isomorphism). By the dual of [12, Lemma 2.1.2.4] applied to $A\to B$ and the right anodyne map $\{1\}\to\Delta^1$, the map

$$\lambda' : A\times\Delta^1 \cup_{A\times\{1\}} B\times\{1\} \to B\times\Delta^1$$

is right anodyne. Then by [12, Lemma 2.1.2.3] applied to λ' and the map $K\to K^{\lhd}$, λ is inner anodyne. Thus the dotted lift h^{γ} exists. Finally, let $\gamma = h^{\gamma}_0$ and observe that γ is a solution to the original lifting problem of interest. □

We briefly digress to complete the theory of Kan extensions by constructing relative Kan extensions along general functors (see Lurie's remark at the beginning of [12, §4.3.3]). Recall the relative join construction $-\star_{-}-$ of [21, Definition 4.1] along with its bifibration property [21, Lemma 4.8].

3.9 Definition Consider the commutative diagram of ∞-categories

$$\begin{array}{ccc} X & \xrightarrow{F} & C \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & \xrightarrow{\alpha} & S \end{array}$$

where $p : C\to S$ is a categorical fibration. Suppose given the data of a functor $G : Y\to C$ over S and a homotopy $h : X\times\Delta^1\to C$ over S with $h_0 = G\circ\phi$ and $h_1 = F$. Let $\pi : Y\star_Y X\to Y$ be the structure map and let $\overline{G} : Y\star_Y X \xrightarrow{\pi} Y \xrightarrow{G} C$. Since $\mathrm{Fun}(Y\star_Y X, C)\to\mathrm{Fun}(Y,C)\times\mathrm{Fun}(X,C)$ is a bifibration, we may select an edge $\overline{G}\to\overline{F}$ that is cocartesian over $h : G\circ\phi\to F$ in $\mathrm{Fun}(X,C)$ with degenerate image id_G in $\mathrm{Fun}(Y,C)$. Then we say that G is a *p-right Kan extension of F* along ϕ (exhibited via h) if the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & C \\ {\scriptstyle\iota_X}\big\downarrow & \overset{\overline{F}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ Y\star_Y X & \xrightarrow{\alpha\circ\pi} & S \end{array}$$

exhibits $\overline{F}$ as a p-right Kan extension of F in the sense of [12, Definition 4.3.2.2].

3.10 Remark In the initial setup of Definition 3.9, given $\overline{F} : Y\star_Y X\to C$ a map over S extending $F : X\to C$, let $G = \overline{F}|_Y : Y\to C$ and let $h : X\times\Delta^1 \xrightarrow{h'} Y\star_Y X \xrightarrow{\overline{F}} C$ with h' specified by the pair (ϕ, id_Y) (see the definition [21, Definition 4.1] of $-\star_Y-$ as $j_* : s\mathbf{Set}_{/Y\times\partial\Delta^1}\to s\mathbf{Set}_{/Y\times\Delta^1}$ for the inclusion $j : Y\times\partial\Delta^1\to Y\times\Delta^1$). Then $\overline{F}$ is a p-right Kan extension in the sense of [12, Definition 4.3.2.2] if and only if G is a p-right Kan extension along ϕ in the sense of Definition 3.9. Moreover, we have an equivalence of ∞-categories $X \times_{Y \star_Y X} (Y \star_Y X)_{y/} \simeq X \times_Y Y_{y/}$ implemented by pulling back the functors $\iota_Y : Y \subset Y \star_Y X$ and $\pi : Y \star_Y X \to Y$ and the respective induced functors on the slice categories via $X \subset Y \star_Y X$. Because of this, Lurie's existence and uniqueness theorem [12, Proposition 4.3.2.15] for p-right Kan extensions applies to show that the p-right Kan extension G of F along ϕ exists if and only if for every $y \in Y$, the diagram $X \times_Y Y_{y/} \to X \xrightarrow{F} C$ extends to a p-limit diagram (which then computes the value of G on y). Moreover, there is then a contractible space of choices for G.

3.11 Remark The situation of Definition 3.9 globalizes in the following manner. Suppose every functor $F : X \to C$ admits a p-right Kan extension to $\overline{F} : Y \star_Y X \to C$. By [12, Proposition 4.3.2.17], the restriction functor $(\iota_X)^* : \mathrm{Fun}_{/S}(Y \star_Y X, C) \to \mathrm{Fun}_{/S}(X, C)$ then admits a right adjoint $(\iota_X)_*$ which is computed on objects as $F \mapsto \overline{F}$. We also have a relative adjunction [14, Definition 7.3.2.2]

$$\iota_Y : Y \rightleftarrows Y \star_Y X : \pi$$

over Y (hence over S) where ι_Y is left adjoint to π. From this, we obtain an adjunction

$$\pi^* : \mathrm{Fun}_{/S}(Y, C) \rightleftarrows \mathrm{Fun}_{/S}(Y \star_Y X, C) : (\iota_Y)^*,$$

where π^* is left adjoint to $(\iota_Y)^*$. Composing these two adjunctions, we obtain the adjunction

$$\phi^* : \mathrm{Fun}_{/S}(Y, C) \rightleftarrows \mathrm{Fun}_{/S}(X, C) : \phi_*,$$

where ϕ_* is given on objects by sending F to its p-right Kan extension along ϕ.

3.12 Corollary *Suppose we have a commutative diagram of ∞-categories*

$$\begin{array}{ccc} X & \xrightarrow{F} & C \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\alpha} & S \end{array}$$

where p is a locally cocartesian fibration and ϕ is a cartesian fibration. Suppose that for every $y \in Y$, the limit of $F|_{X_y} : X_y \to C_{\alpha(y)}$ exists. Then the p-right Kan extension $G : Y \to C$ of F along ϕ exists and $G(y) \simeq \varprojlim F|_{X_y}$. If G exists for all F, then we have an adjunction

$$\phi^* : \mathrm{Fun}_{/S}(Y, C) \rightleftarrows \mathrm{Fun}_{/S}(X, C) : \phi_*,$$

where $\phi_*(F) \simeq G$.

Proof We need to show that for every $y \in Y$, the p-limit of $F^y : X \times_Y Y_{y/} \to X \xrightarrow{F} C$ exists. By Lemma 3.8, the p-limit of $F|_{X_y}$ exists and is computed as the limit of $F|_{X_y}$ viewed as a diagram in $C_{\alpha(y)}$. Because ϕ is a cartesian fibration, we have a retraction $r : X \times_Y Y_{y/} \to X_y$ to the inclusion $i : X_y \to X \times_Y Y_{y/}$ such that r is right adjoint to i (on objects, r is given by the formula $r(x, y \xrightarrow{e} \phi(x)) = e^*(x)$, where $e^* : X_{\phi(x)} \to X_y$ is the pullback functor encoded by the lifting property of the cartesian fibration ϕ).

As a left adjoint, i is right cofinal.[13] However, since $r \circ i = \mathrm{id}$, we moreover have that r is right cofinal by the right cancellative property of right cofinal maps [12, Proposition 4.1.1.3(2)]. Hence, by [12, Proposition 4.3.1.7] applied to r and a p-limit diagram $(X_y)^{\triangleleft} \to C$, the p-limit of F^y exists and is computed as the limit of $F|_{X_y}$ in $C_{\alpha(y)}$. The claim now follows from Remark 3.10. □

3.2.2 Barycentric subdivision and locally cocartesian pushforward Our main goal in this subsection is to first define the barycentric subdivision $\mathrm{sd}(S)$ (Definition 3.22) consisting of conservative functors $\sigma : [n] \to S$ (i.e., *strings* in S) along with its *maximum* functor $\max_S : \mathrm{sd}(S) \to S$, $[\sigma : [n] \to S] \mapsto \sigma(n)$, which is a locally cocartesian fibration (Lemma 3.25). This allows us to define the right-lax limit of a locally cocartesian fibration $p : C \to S$ as

$$\lim^{\mathrm{rlax}} C := \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C).$$

We will then show that for any sieve $S_0 \subset S$, if we let $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$ denote the full subcategory of strings that originate in S_0, then the inclusion $\mathrm{sd}(S_0) \hookrightarrow \mathrm{sd}(S)_0$ is a locally cocartesian equivalence over S,[14] or equivalently, for any locally cocartesian fibration $p : C \to S$, the restriction functor

$$\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S)_0, C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C|_{S_0})$$

is a trivial fibration (Theorem 3.26(2)). A choice of inverse then amounts to a choice of *locally cocartesian pushforward*. This will be the formal half of extending an object in $\lim^{\mathrm{rlax}} C|_{S_0}$ to one in $\lim^{\mathrm{rlax}} C$ itself, which we take up in the next subsection.

To set the stage for our work, we first introduce a few combinatorial constructions. Let Δ be the category with objects the finite ordinals $\{[n] = \{0 < 1 < \cdots < n\} : n \in \mathbb{N}\}$ and morphisms the order-preserving maps. Let $\xi : \mathcal{E}\Delta \to \Delta$ denote the relative nerve [12, Definition 3.2.5.2] of the canonical inclusion $i : \Delta \hookrightarrow s\mathbf{Set}$. Then ξ is a cocartesian fibration classified by i, which is an explicit model for the tautological cocartesian fibration over Δ. Explicitly, an n-simplex $\Delta^n \to \mathcal{E}\Delta$ is given by a sequence $[a_0] \xrightarrow{\alpha_0} [a_1] \xrightarrow{\alpha_1} \cdots \xrightarrow{g_{n-1}} [\alpha_n]$ of order-preserving maps in Δ together with morphisms $\kappa_i : \Delta^{\{0,\dots,i\}} \cong \Delta^i \to \Delta^{a_i}$ which fit into a commutative diagram

$$\begin{array}{ccccccccc}
\Delta^{\{0\}} & \hookrightarrow & \Delta^{\{0,1\}} & \hookrightarrow & \cdots & \hookrightarrow & \Delta^{\{0,\dots,n-1\}} & \hookrightarrow & \Delta^n \\
\downarrow{\scriptstyle \kappa_0} & & \downarrow{\scriptstyle \kappa_1} & & & & \downarrow{\scriptstyle \kappa_{n-1}} & & \downarrow{\scriptstyle \kappa_n} \\
\Delta^{a_0} & \xrightarrow{\alpha_0} & \Delta^{a_1} & \xrightarrow{\alpha_1} & \cdots & \longrightarrow & \Delta^{a_{n-1}} & \xrightarrow{\alpha_{n-1}} & \Delta^{a_n}
\end{array}$$

Let $\mathcal{E}\Delta^{\mathrm{inj}} \subset \mathcal{E}\Delta$ denote the pullback over the subcategory $\Delta^{\mathrm{inj}} \subset \Delta$ of injective order-preserving maps and also denote the structure map of $\mathcal{E}\Delta^{\mathrm{inj}}$ by ξ. Consider the span of marked simplicial sets

$$(\Delta^{\mathrm{inj}})^\sharp \xleftarrow{\xi} {}_\natural(\mathcal{E}\Delta^{\mathrm{inj}}) \xrightarrow{\xi} (\Delta^{\mathrm{inj}})^\sharp,$$

---

[13] We adopt Lurie's terminology in [14]: recall that a map $q : K \to L$ is right cofinal if and only if q^{op} is cofinal.

[14] Here we mark those edges that are locally cocartesian with respect to $\max_{S_0}$ (resp. $\max_S$.)

where we mark the ξ-cocartesian edges in $\mathcal{E}\Delta^{\mathrm{inj}}$. Similar to the definition in [21, Example 2.25] (which considers the source input to be instead a cartesian fibration), let

$$\widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{inj}}}(\mathcal{E}\Delta^{\mathrm{inj}}, -) := \xi_* \xi^*(-) : s\mathbf{Set}^+_{/\Delta^{\mathrm{inj}}} \to s\mathbf{Set}^+_{/\Delta^{\mathrm{inj}}}.$$

Note that with ξ a cocartesian fibration, $\xi_*\xi^*$ is right Quillen with respect to the *cartesian* model structure on $s\mathbf{Set}^+_{/\Delta^{\mathrm{inj}}}$ by the dual of [21, Theorem 2.24], hence takes cartesian fibrations to cartesian fibrations.

3.13 Recollection [12, Corollary 3.2.2.13; 21, Example 2.25] Given an ∞-category B, a cocartesian fibration $\xi : K \to B$, and a cartesian fibration $D \to B$, the *pairing construction* $\widetilde{\mathrm{Fun}}_B(K, D)$ is defined in general as $\xi_*\xi^*(D^\natural)$ and is a cartesian fibration over B whose fibers over $b \in B$ are $\mathrm{Fun}(K_b, D_b)$, and whose functoriality with respect to a morphism $\alpha : b \to b'$ is given by

$$\alpha^* : \mathrm{Fun}(K_{b'}, D_{b'}) \to \mathrm{Fun}(K_b, D_b), \qquad f \mapsto \alpha^* \circ f \circ \alpha_!,$$

where $\alpha_!$ and α^* denote the pushforward functors for K and D as well.

3.14 Definition The ∞-category of *paths*[15] in an ∞-category C is

$$\widehat{\mathrm{Ar}}(C) := \widetilde{\mathrm{Fun}}_{\Delta^{\mathrm{inj}}}(\mathcal{E}\Delta^{\mathrm{inj}}, C \times \Delta^{\mathrm{inj}}).$$

Let $\xi_C : \widehat{\mathrm{Ar}}(C) \to \Delta^{\mathrm{inj}}$ denote the structure map of the cartesian fibration and note that its fiber over $[n] \in \Delta^{\mathrm{inj}}$ is $\mathrm{Fun}(\Delta^n, C)$ and the functoriality is that of restriction in the source variable.

In addition, let $\widehat{\mathrm{Ar}}^{\simeq}(S) \subset \widehat{\mathrm{Ar}}(S)$ be the maximal subright fibration, i.e., the wide subcategory on the ξ_S-cartesian edges over Δ^{inj} (so the fiber of $\widehat{\mathrm{Ar}}^{\simeq}(S)$ over $[n]$ is $\mathrm{Map}(\Delta^n, S)$), and for a functor $p : C \to S$, let

$$\widehat{\mathrm{Ar}}^{\simeq}_S(C) := \widehat{\mathrm{Ar}}^{\simeq}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(C).$$

3.15 Remark (classifying functor for paths) By [8, Proposition 7.3], the cartesian fibration

$$\xi_C : \widehat{\mathrm{Ar}}(C) \to \Delta^{\mathrm{inj}}$$

is classified by the functor

$$(\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathbf{Cat}_\infty, \qquad [n] \mapsto \mathrm{Fun}(\Delta^n, C),$$

where the functoriality is with respect to precomposition in the first variable. It follows that we have an equivalence

$$\widehat{\mathrm{Ar}}^{\simeq}(C) \simeq \Delta^{\mathrm{inj}} \times_{\mathbf{Cat}_\infty} \mathbf{Cat}_\infty^{/C}$$

of right fibrations over Δ^{inj}.

3.16 Remark If $C \to S$ is a categorical fibration, then $\widehat{\mathrm{Ar}}(C) \to \widehat{\mathrm{Ar}}(S)$ is also a categorical fibration by [14, Proposition B.2.7].

[15] For us, a path in C is any n-simplex $\Delta^n \to C$. In contrast, we reserve the term "string" for objects of the barycentric subdivision $\mathrm{sd}(C)$ (see Definition 3.22).

3.17 Remark (explicit description of simplices) By definition, the datum of an n-simplex $\Delta^n \to \widehat{\mathrm{Ar}}(C)$ is given by a map of simplicial sets

$$\Delta^n \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to C.$$

For example, suppose $n = 0$ and $\Delta^0 \to \Delta^{\mathrm{inj}}$ selects the object $[a_0]$. Then we see that 0-simplices of $\widehat{\mathrm{Ar}}(C)$ lying over $[a_0]$ correspond to maps $\Delta^{a_0} \to C$. Indeed, since the fiber $(\mathcal{E}\Delta^{\mathrm{inj}})_{[a_0]}$ is definitionally isomorphic to Δ^{a_0}, we see that the fiber $\widehat{\mathrm{Ar}}(C)_{[a_0]}$ is equivalent to $\mathrm{Fun}(\Delta^{a_0}, C)$, as promised by Remark 3.15.

Now suppose that $n = 1$ and $\Delta^1 \to \Delta^{\mathrm{inj}}$ selects the inclusion $\alpha_1 : [a_0] \subset [a_1]$. Then the data of a map $f : \Delta^1 \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to C$ is equivalent to maps $f_0 : \Delta^{a_0} \to C$, $f_1 : \Delta^{a_1} \to C$, and a natural transformation $f_{01} : f_0 \to f_1 \circ \alpha_1 = f_1|_{[a_0]}$. Moreover, this is a *cartesian* edge in $\widehat{\mathrm{Ar}}(C)$ if and only if f sends cocartesian edges to equivalences, i.e., the natural transformation f_{01} is an equivalence. This is consistent with the functoriality of $\widehat{\mathrm{Ar}}(C)$ as being given by the pullback functor

$$\alpha_1^* : \mathrm{Fun}(\Delta^{a_1}, C) \to \mathrm{Fun}(\Delta^{a_0}, C).$$

3.18 Construction (variant associated to a sieve) Let $\pi : S \to \Delta^1$ be a functor and S_0 the fiber over 0. Let $\widehat{\mathrm{Ar}}(S)_0 \subset \widehat{\mathrm{Ar}}(S)$ be the full subcategory on those objects $\sigma : \Delta^n \to S$ such that $\pi\sigma(0) = 0$ (i.e., *on those paths originating in* S_0), and let $\widehat{\mathrm{Ar}}^{\simeq}(S)_0 := \widehat{\mathrm{Ar}}(S)_0 \cap \widehat{\mathrm{Ar}}^{\simeq}(S)$. Define the "initial segment" functor

$$\lambda_S : \widehat{\mathrm{Ar}}(S)_0 \to \widehat{\mathrm{Ar}}(S_0)$$

by the following rule:

(∗) Suppose $\sigma : \Delta^n \to \widehat{\mathrm{Ar}}(S)_0$ is an n-simplex, which corresponds to a sequence of inclusions

$$\Delta^{a_0} \xhookrightarrow{\alpha_1} \Delta^{a_1} \xhookrightarrow{\alpha_2} \cdots \xhookrightarrow{\alpha_n} \Delta^{a_n}$$

determining a map $a : \Delta^n \to \Delta^{\mathrm{inj}}$ and a functor $f : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ such that for every $0 \le i \le n$, the restriction $f_i : \Delta^{a_i} \to S$ has $f_i(0) \in S_0$. Let $b_i \in \Delta^{a_i}$ be the maximum element such that $f_i(b_i) \in S_0$, and note that a restricts to yield a sequence of inclusions

$$\begin{array}{ccccccc} \Delta^{b_0} & \xhookrightarrow{\beta_1} & \Delta^{b_1} & \xhookrightarrow{\beta_2} & \cdots & \xhookrightarrow{\beta_n} & \Delta^{b_n} \\ \downarrow & & \downarrow & & & & \downarrow \\ \Delta^{a_0} & \xhookrightarrow{\alpha_1} & \Delta^{a_1} & \xhookrightarrow{\alpha_2} & \cdots & \xhookrightarrow{\alpha_n} & \Delta^{a_n} \end{array}$$

because we always have that $\alpha_i(b_{i-1}) \le b_i$ as S_0 is a sieve in S stable under equivalences. Let $b : \Delta^n \to \Delta^{\mathrm{inj}}$ be the map determined by the sequence of upper horizontal inclusions. Then f restricts to yield a map f_0:

$$\begin{array}{ccc} \Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} & \xrightarrow{f_0} & C_0 \\ \downarrow & & \downarrow \\ \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} & \xrightarrow{f} & C \end{array}$$

Define $\lambda_S(\sigma) : \Delta^n \to \widehat{\mathrm{Ar}}(S_0)$ to be the n-simplex determined by f_0. Now observe that this assignment is natural in Δ^n, hence defines a map of simplicial sets.

Observe that λ_S is a retraction of the inclusion $\widehat{\mathrm{Ar}}(S_0) \to \widehat{\mathrm{Ar}}(S)_0$ induced by $S_0 \to S$.

An edge $e : \Delta^1 \to \widehat{\mathrm{Ar}}(S)_0$ is ξ_S-cartesian if and only if the corresponding functor $f : \Delta^1 \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ sends every edge $(i \in [a_0]) \to (\alpha_1(i) \in [a_1])$ to an equivalence, and similarly for ξ_{S_0}-cartesian edges in $\widehat{\mathrm{Ar}}(S_0)$. Therefore, λ_S preserves cartesian edges and restricts to a map

$$\lambda_S : \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \to \widehat{\mathrm{Ar}}^{\simeq}(S_0).$$

3.19 Construction (variant associated to a sieve, relative version) Let $p : C \to S$ be a locally cocartesian fibration and let $p_0 : C_0 \to S_0$ be its fiber over 0. Note that

$$\widehat{\mathrm{Ar}}(C)_0 \cong \widehat{\mathrm{Ar}}(S)_0 \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(C).$$

Let

$$\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0 := \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \times_{\widehat{\mathrm{Ar}}(S)_0} \widehat{\mathrm{Ar}}(C)_0 \cong \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \times_{\widehat{\mathrm{Ar}}^{\simeq}(S)} \widehat{\mathrm{Ar}}^{\simeq}_S(C),$$

so $\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0 \subset \widehat{\mathrm{Ar}}^{\simeq}_S(C)$ is the full subcategory on objects $c : \Delta^n \to C$ with $c(0) \in C_0$. The initial segment functor $\lambda_{(-)}$ fits into a commutative diagram

$$\begin{array}{ccccc} \widehat{\mathrm{Ar}}^{\simeq}(S)_0 & \hookrightarrow & \widehat{\mathrm{Ar}}(S)_0 & \xleftarrow{p} & \widehat{\mathrm{Ar}}(C)_0 \\ \downarrow{\lambda_S} & & \downarrow{\lambda_S} & & \downarrow{\lambda_C} \\ \widehat{\mathrm{Ar}}^{\simeq}(S_0) & \hookrightarrow & \widehat{\mathrm{Ar}}(S_0) & \xleftarrow{p_0} & \widehat{\mathrm{Ar}}(C_0) \end{array}$$

and therefore defines a functor $\lambda_p : \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0 \to \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0)$.

Finally, let $\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}} \subset \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0$ be the full subcategory on those objects $c : \Delta^n \to C$ such that if $i \in \Delta^n$ is the maximum element with $c(i) \in C_0$, then c sends every edge $\{j < j+1\}$, $j \geq i$, to a locally-p cocartesian edge (i.e., a cocartesian edge over Δ^1 in the pullback $\Delta^1 \times_S C$).

The next theorem implies that we can construct a *locally cocartesian pushforward* extending from C_0 to C along paths in the base S that originate in S_0. This will amount to a section of the trivial fibration considered therein.

3.20 Theorem *The map*

$$(\lambda_p, p) : \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}} \to \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0) \times_{p_0, \widehat{\mathrm{Ar}}^{\simeq}(S_0), \lambda_S} \widehat{\mathrm{Ar}}^{\simeq}(S)_0$$

is a trivial fibration of simplicial sets.

Proof We need to solve the lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}} \\ \big\downarrow & \nearrow & \big\downarrow{(\lambda_p, p)} \\ \Delta^n & \longrightarrow & \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0) \times_{\widehat{\mathrm{Ar}}^{\simeq}(S_0)} \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \end{array}$$

Let $a : \Delta^n \to \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \to \Delta^{\mathrm{inj}}$ and $b : \Delta^n \to \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0) \to \Delta^{\mathrm{inj}}$ be as discussed in the definition of λ. This lifting problem transposes to

$$\begin{array}{ccc} \Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \cup_{\partial\Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}} \partial\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} & \longrightarrow & C \\ \Big\downarrow f & \nearrow & \Big\downarrow p \\ \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} & \longrightarrow & S \end{array}$$

Consider $\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ as a marked simplicial set where an edge $(i \in \Delta^{a_k}) \to (j \in \Delta^{a_l})$, $\alpha : \Delta^{a_k} \to \Delta^{a_l}$, $\alpha(i) \le j$, is marked if and only if $k = l$ (so $\alpha = \mathrm{id}$), $b_k \le i$ and $j = i + 1$, and let the domain of f also inherit this marking. Then it suffices to show that f is a trivial cofibration in the locally cocartesian model structure on $s\mathbf{Set}^+_{/S}$, defined by the categorical pattern $\mathfrak{P} = (M_S, T, \varnothing)$ with M_S all of the edges in S and T consisting of the 2-simplices τ in S with the edge $\tau(\{1 < 2\})$ an equivalence. Proceeding by induction on n, by a two-out-of-three argument it suffices to show that the inclusion $f' : \Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ is a trivial cofibration. We define a filtration of the poset inclusion f' as follows:

$(*)$ Let $a_n - b_n = t$. For $0 \le k \le n$, let $\alpha_k : \Delta^{a_k} \to \Delta^{a_n}$ denote the inclusion. Let $P_r \subset \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ be the subposet on those objects $(i \in \Delta^{a_k})$ such that $\alpha_k(i) - b_n \le r$. Note that $P_0 = \Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$, because if $(i \in \Delta^{a_k})$ is such that $i > b_k$, then necessarily $\alpha_k(i) > b_n$, and likewise if $i \le b_k$, then $\alpha_k(i) \le b_n$ (this follows from the definitions of the b_i and that S_0 is a sieve stable under equivalences). Then we have that f' factors as a sequence of poset sieve inclusions $\Delta^n \times_{b,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} = P_0 \subset P_1 \subset \cdots \subset P_t = \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$.

It now suffices to show that $P_i \subset P_{i+1}$ is a trivial cofibration for all $0 \le i < t$. For simplicity, let us suppose $i = 0$ (and $t > 0$ for nontriviality), the other cases being proved similarly. Let $k \in [n]$ be the smallest element such that $b_n + 1 \in \Delta^{a_n}$ is in the image of $\alpha_k : \Delta^{a_k} \to \Delta^{a_n}$. Note then that for all $k \le l \le n$, $\alpha_l(b_l + 1) = b_n + 1$. View the poset $\Delta^{\{k,\dots,n\}} \times \Delta^1$ as a cosieve U in P_1 via the inclusion which sends $(l, 0)$ to $(b_l \in \Delta^{a_l})$ and $(l, 1)$ to $(b_l + 1 \in \Delta^{a_l})$. Then as a marked simplicial set, we have $U = (\Delta^{\{k,\dots,n\}})^\flat \times (\Delta^1)^\sharp$. By [14, B.1.10], the inclusion

$$U \cap P_0 = (\Delta^{\{k,\dots,n\}})^\flat \times \{0\} \to U = (\Delta^{\{k,\dots,n\}})^\flat \times (\Delta^1)^\sharp$$

is $\mathfrak{P}$-anodyne. Noting that P_0 and U together cover P_1, it thus suffices to show that we have a homotopy pushout square of ∞-categories

$$\begin{array}{ccc} U \cap P_0 & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ P_0 & \longrightarrow & P_1 \end{array}$$

as we would then deduce the lower horizontal map to be $\mathfrak{P}$-anodyne. For this, the criterion of Lemma 3.21 is easily verified. □

3.21 Lemma *Suppose P is a poset, $Z \subset P$ is a sieve and $U \subset P$ is a cosieve such that $P = Z \cup U$. Then the commutative square*

$$\begin{array}{ccc} U \cap Z & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & P \end{array}$$

is a homotopy pushout square of ∞-categories if and only if for every $a \notin U$ and $c \notin Z$ such that $a \leq c$, the subposet $P_{a/\!/c} = \{b \in U \cap Z : a \leq b \leq c\}$ is weakly contractible.

Proof Define a map $\pi : P \to \Delta^2$ by

$$\pi(x) = \begin{cases} 0, & x \notin U, \\ 2, & x \notin Z, \\ 1, & x \in U \cap Z. \end{cases}$$

Observe that $P \times_{\Delta^2} \Delta^{\{0,1\}} = Z$, $P \times_{\Delta^2} \Delta^{\{1,2\}} = U$, and $P \times_{\Delta^2} \{1\} = U \cap Z$. We may therefore apply the flatness criterion of [14, B.3.2] to π in order to deduce the criterion in question. □

We now introduce the barycentric subdivision $\mathrm{sd}(S)$.

3.22 Definition An n-simplex $\sigma : \Delta^n \to S$ is a *string* if σ is a conservative functor, i.e., if for every $0 \leq i < j \leq n$, $\sigma(\{i < j\})$ is not an equivalence.[16] The *barycentric subdivision* (or *subdivision*)

$$\mathrm{sd}(S) \subset \widehat{\mathrm{Ar}}^{\simeq}(S)$$

is the full subcategory of $\widehat{\mathrm{Ar}}^{\simeq}(S)$ on the strings in S. Note that the structure map $\xi_S : \widehat{\mathrm{Ar}}^{\simeq}(S) \to \Delta^{\mathrm{inj}}$ restricts to define a right fibration $\xi_S : \mathrm{sd}(S) \to \Delta^{\mathrm{inj}}$.

Given a functor $C \to S$, the *S-relative subdivision* $\mathrm{sd}_S(C)$ is the pullback

$$\mathrm{sd}_S(C) := \mathrm{sd}(S) \times_{\widehat{\mathrm{Ar}}^{\simeq}(S)} \widehat{\mathrm{Ar}}^{\simeq}_S(C) \cong \mathrm{sd}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(C).$$

Similarly, parallel to Constructions 3.18 and 3.19 we may define $\mathrm{sd}(S)_0$, $\mathrm{sd}_S(C)_0$, and $\mathrm{sd}_S(C)_0^{\mathrm{cocart}}$ for a locally cocartesian fibration $C \to S$ and a functor $S \to \Delta^1$. To be specific, let $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$ be the full subcategory on those strings originating in the sieve S_0, let $\mathrm{sd}_S(C)_0 := \mathrm{sd}(S)_0 \times_{\mathrm{sd}(S)} \mathrm{sd}_S(C)$, and let $\mathrm{sd}_S(C)_0^{\mathrm{cocart}} := \mathrm{sd}_S(C)_0 \times_{\widehat{\mathrm{Ar}}^{\simeq}_S(C)_0} \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}}$.

3.23 Observation Suppose that S is the nerve of a category, which we also denote as S. Then $\mathrm{sd}(S)$ is the nerve of the category whose objects are conservative functors $\sigma : \Delta^n \to S$, and where a morphism $[\sigma : \Delta^n \to S] \to [\tau : \Delta^m \to S]$ is given by the data of a map $\alpha : [n] \hookrightarrow [m]$ in Δ^{inj} and a natural equivalence $\sigma \xrightarrow{\simeq} \alpha^* \tau$. In particular, if S is the nerve of a poset P, then $\mathrm{sd}(P)$ is the nerve of the usual barycentric subdivision of P.

On the other hand, the usual definition of the subdivision of an ∞-category [1, Definition 1.15] is as the left Kan extension of the functor $\mathrm{sd} : \Delta \to \mathbf{Cat}_\infty$ along the fully faithful inclusion $\Delta \subset \mathbf{Cat}_\infty$. By

[16] If every retract in S is an equivalence, then it suffices to check that for every $0 \leq i < n$, $\sigma(\{i < i+1\})$ is not an equivalence.

[2, Lemma A.4.8], this recovers $\mathrm{sd}(P)$ for P a poset. In fact, we may transcribe over the proof there to show that $\mathrm{sd}(S) \xleftarrow{\simeq} \mathrm{colim}_{[n]\in\Delta_{/S}} \mathrm{sd}[n]$ for any ∞-category S. Here $\Delta_{/S} := \Delta \times_{\mathbf{Cat}_\infty} (\mathbf{Cat}_\infty)^{/S}$ is the maximal subright fibration in $\widetilde{\mathrm{Fun}}_\Delta(\mathcal{E}\Delta, S \times \Delta)$ (see Remark 3.15).[17] We sketch the argument, leaving routine details to the reader:

(1) First note that for any two strings $\sigma, \tau \in \mathrm{sd}(S)$, every map $[\sigma \Rightarrow \tau] \in \Delta_{/S}$ necessarily lies over Δ^{inj}. Therefore, the inclusion $i : \mathrm{sd}(S) \subset \Delta_{/S}$ is full. Moreover, in view of the factorization system on $\mathbf{Cat}_\infty$ whose right class of maps is given by the conservative functors [9, 11.29], i admits a left adjoint. In particular, i is cofinal, so

$$\mathrm{colim}_{[n]\in\Delta_{/S}} \mathrm{sd}[n] \simeq \mathrm{colim}_{[n]\in\mathrm{sd}(S)} \mathrm{sd}[n].$$

(2) We next observe that the cocartesian fibration $\mathrm{ev}_1 : \mathrm{Ar}(\mathrm{sd}(S)) \to \mathrm{sd}(S)$ is classified by the functor $\mathrm{sd}(S) \to \Delta^{\mathrm{inj}} \subset \Delta \xrightarrow{\mathrm{sd}} \mathbf{Cat}_\infty$. Therefore, $\mathrm{colim}_{[n]\in\mathrm{sd}(S)} \mathrm{sd}[n]$ identifies with the localization of $\mathrm{Ar}(\mathrm{sd}(S))$ at the class of ev_1-cocartesian edges. But this localization also identifies with the source functor $\mathrm{ev}_0 : \mathrm{Ar}(\mathrm{sd}(S)) \to \mathrm{sd}(S)$, yielding the desired equivalence $\mathrm{colim}_{[n]\in\mathrm{sd}(S)} \mathrm{sd}[n] \to \mathrm{sd}(S)$.

We now work towards constructing the "maximum" functor $\mathrm{sd}(S) \to S$. We first define this over $\widehat{\mathrm{Ar}}(S)$:

3.24 Construction Define a *last vertex* map $\max_S : \widehat{\mathrm{Ar}}(S) \to S$ by the following rule:

($*$) Suppose $\sigma : \Delta^n \to \widehat{\mathrm{Ar}}(S)$ is an n-simplex, which corresponds to a sequence of inclusions

$$\Delta^{a_0} \overset{\alpha_1}{\hookrightarrow} \Delta^{a_1} \overset{\alpha_2}{\hookrightarrow} \cdots \overset{\alpha_n}{\hookrightarrow} \Delta^{a_n}$$

determining a map $a : \Delta^n \to \Delta^{\mathrm{inj}}$ and a functor $f : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$. Define a functor $\chi : \Delta^n \to \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ to be the identity on the first component and the n-simplex

$$\begin{array}{ccccccc} \Delta^{\{0\}} & \hookrightarrow & \Delta^{\{0,1\}} & \hookrightarrow & \cdots & \hookrightarrow & \Delta^n \\ \downarrow{\scriptstyle \kappa_0} & & \downarrow{\scriptstyle \kappa_1} & & & & \downarrow{\scriptstyle \kappa_n} \\ \Delta^{a_0} & \overset{\alpha_1}{\hookrightarrow} & \Delta^{a_1} & \overset{\alpha_2}{\hookrightarrow} & \cdots & \overset{\alpha_n}{\hookrightarrow} & \Delta^{a_n} \end{array}$$

of $\mathcal{E}\Delta^{\mathrm{inj}}$ uniquely specified by $\kappa_i(i) = a_i$ on the second component. Then $\max_S(\sigma) = f \circ \chi : \Delta^n \to S$.

In other words, $\max_S$ is the functor induced by precomposing by the section $\Delta^{\mathrm{inj}} \to \mathcal{E}\Delta^{\mathrm{inj}}$ which selects the maximal vertex in every fiber.

The next lemma is obvious when S is a poset, so the reader only interested in that case should feel free to skip its proof.

3.25 Lemma (1) *The functor* $\max_S : \widehat{\mathrm{Ar}}(S) \to S$ *is a categorical fibration.*

(2) *The restricted functor* $\max_S : \widehat{\mathrm{Ar}}^{\simeq}(S) \to S$ *is a locally cocartesian fibration.*

(3) *The restricted functor* $\max_S : \mathrm{sd}(S) \to S$ *is a locally cocartesian fibration.*

[17] Beware that here $\Delta_{/S}$ does *not* denote the nerve of the category of simplices of S regarded as a simplicial set.

Proof (1) We first verify that $\max_S$ is an inner fibration. For this, let $n \geq 2$, $0 < k < n$, and consider the lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \widehat{\mathrm{Ar}}(S) \\ \downarrow & \nearrow & \downarrow{\scriptstyle \max_S} \\ \Delta^n & \longrightarrow & S \end{array}$$

Let $a : \Delta^n \to \Delta^{\mathrm{inj}}$ be the unique extension of the given $\Lambda^n_k \to \Delta^{\mathrm{inj}}$. The lifting problem then transposes to

$$\begin{array}{ccc} \Delta^n \cup_{\Lambda^n_k} \Lambda^n_k \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} & \longrightarrow & S \\ \downarrow & \nearrow & \\ \Delta^n \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} & & \end{array}$$

and it suffices to show the vertical arrow is inner anodyne. Since $\mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^{\mathrm{inj}}$ is a cocartesian fibration, it is in particular a flat inner fibration, and the desired result follows.

We next show that $\max_S$ is a categorical fibration by lifting equivalences from the base. So suppose $e : \Delta^1 \to S$ is an equivalence and $\sigma : \Delta^n \to S$ is an object of $\widehat{\mathrm{Ar}}(S)$ such that $\max_S(\sigma) = \sigma(n) = e(0)$. The restriction of $\max_S$ to $\mathrm{Fun}(\Delta^n, S) \subset \widehat{\mathrm{Ar}}(S)$ is evaluation at $\{n\}$, which is a categorical fibration, so e lifts to an equivalence in $\mathrm{Fun}(\Delta^n, S)$ and hence in $\widehat{\mathrm{Ar}}(S)$.

(2) First observe that since $\widehat{\mathrm{Ar}}^{\simeq}(S) \subset \widehat{\mathrm{Ar}}(S)$ is a subcategory stable under equivalences, the restricted $\max_S$ functor is a categorical fibration by (1). To prove that $\max_S$ is a locally cocartesian fibration, it then suffices to prove that for any edge $e : s \to t$ in S that is *not* an equivalence, the pullback $\max_S(e) : \widehat{\mathrm{Ar}}^{\simeq}(S) \times_S \Delta^1 \to \Delta^1$ is a cocartesian fibration. To this end, we claim that an edge $\tilde{e} : x \to y$ lifting e is $\max_S(e)$-cocartesian if and only if the corresponding data of an inclusion $\alpha : \Delta^{a_0} \to \Delta^{a_1}$ and a functor $f : \Delta^1 \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ is such that in addition $a_1 = a_0 + 1$ and α is the inclusion of the initial segment. Note that given an object $x : \Delta^{a_0} \to S$ with $s = x(a_0)$, such a lift $\tilde{e}$ of e may be defined by "appending" e to x: indeed, let $y : \Delta^{a_0+1} \to S$ be an extension of $x \cup e : \Delta^{a_0} \cup_{a_0, \Delta^0, 0} \Delta^1 \to S$, let

$$r : \Delta^1 \times_{\alpha, \Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^{a_0+1}$$

be the retraction functor which fixes Δ^{a_0+1} and is given by α on Δ^{a_0}, and define $\tilde{e}$ as $y \circ r$. Hence, establishing the claim will complete the proof.

The "only if" direction will follow from the "if" direction together with the stability of cocartesian edges under equivalence. For the "if" direction, fix such an edge $\tilde{e}$. Recall from the definition that $\tilde{e} : x \to y$ is $\max_S(e)$-cocartesian if and only if for all objects $z \in \widehat{\mathrm{Ar}}^{\simeq}(S)$ with $\max_S(z) = t$, the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\widehat{\mathrm{Ar}}^{\simeq}(S)_{\max_S = t}}(y, z) & \xrightarrow{(\tilde{e})^*} & \mathrm{Map}_{\widehat{\mathrm{Ar}}^{\simeq}(S)}(x, z) \\ \downarrow & & \downarrow{\scriptstyle \max_S} \\ \{e\} & \longrightarrow & \mathrm{Map}_S(s, t) \end{array}$$

is a homotopy pullback square. Viewing x as $x : \Delta^{a_0} \to S$, y as $y : \Delta^{a_0+1} \to S$, and z as $z : \Delta^{a_2} \to S$, and computing the mapping spaces in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ as a cartesian fibration over Δ^{inj}, we see that

$$\mathrm{Map}_{\widehat{\mathrm{Ar}}^{\simeq}(S)}(x, z) \simeq \coprod_{\gamma:[a_0]\subset[a_2]} \mathrm{Map}_{\mathrm{Map}(\Delta^{a_0},S)}(x, \gamma^* z).$$

Therefore, it suffices to show that for any *fixed* inclusion $\gamma : \Delta^{a_0} \hookrightarrow \Delta^{a_2}$ with $\gamma(a_0) < a_2$, letting $\beta : \Delta^{a_0+1} \to \Delta^{a_2}$ be the unique extension of γ with $\beta(a_0+1) = a_2$, we have that the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Map}(\Delta^{a_0+1},S)}(y, \beta^* z) & \xrightarrow{\alpha^*} & \mathrm{Map}_{\mathrm{Map}(\Delta^{a_0},S)}(x, \gamma^* z) \\ \downarrow & & \downarrow \\ \{e\} & \longrightarrow & \mathrm{Map}_{\iota S}(x(a_0), z(a_2)) \end{array}$$

is a homotopy pullback square (where the right vertical map sends $x \to \gamma^* z$ to the composite $x(a_0) \to z(\gamma(a_0)) \to z(a_2)$). (Here we implicitly use that maps in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ are natural transformations through equivalences to account for the $\max_S = t$ condition for the upper-left mapping space.) But this follows since $\mathrm{ev}_{a_0+1} : \mathrm{Fun}(\Delta^{a_0+1}, S) \to S$ is a cocartesian fibration with $\bar{x} \to y$ a cocartesian edge lifting e, where $\bar{x}$ is the degeneracy s_{a_0} applied to x (we note that $\mathrm{Map}_{\mathrm{Map}(\Delta^{a_0+1},S)}(\bar{x}, \beta^* z) \simeq \mathrm{Map}_{\mathrm{Map}(\Delta^{a_0},S)}(x, \gamma^* z)$).

(3) This is clear from the description of the locally $\max_S$-cocartesian edges given in (2). □

Finally, we arrive at the main result of this subsection. Lemma 3.25 ensures that the following theorem is well formulated; also note that $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$ is a sublocally cocartesian fibration via $\max_S$ as it is the inclusion of a cosieve stable under equivalences.

3.26 Theorem *Let $p : C \to S$ be a locally cocartesian fibration and $\pi : S \to \Delta^1$ a functor. Let $p_0 : C_0 \to S_0$ be the fiber of p over 0.*

(1) *Restricting the domain and codomain of the map of Theorem 3.20 yields the map*

$$\mathrm{sd}_S(C)_0^{\mathrm{cocart}} \to \mathrm{sd}_{S_0}(C_0) \times_{\mathrm{sd}(S_0)} \mathrm{sd}(S)_0,$$

which is also a trivial fibration of simplicial sets.

(2) *Precomposition by the inclusion $\mathrm{sd}(S_0) \hookrightarrow \mathrm{sd}(S)_0$ defines a trivial fibration of simplicial sets*

$$\mathrm{Fun}_{/S}^{\mathrm{cocart}}(\mathrm{sd}(S)_0, C) \to \mathrm{Fun}_{/S_0}^{\mathrm{cocart}}(\mathrm{sd}(S_0), C_0).$$

For the proof, it will be convenient to introduce an auxiliary construction. Define a functor

$$\delta : \widehat{\mathrm{Ar}}(S) \to \widehat{\mathrm{Ar}}(\widehat{\mathrm{Ar}}(S))$$

by the following rule:

(∗) Suppose $\sigma : \Delta^n \to \widehat{\mathrm{Ar}}(S)$ is an n-simplex, which corresponds to a sequence of inclusions

$$\Delta^{a_0} \xhookrightarrow{\alpha_1} \Delta^{a_1} \xhookrightarrow{\alpha_2} \cdots \xhookrightarrow{\alpha_n} \Delta^{a_n}$$

determining a map $a : \Delta^n \to \Delta^{\mathrm{inj}}$ and a functor $f : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$. Define a map

$$\bar{a} : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^{\mathrm{inj}}$$

on objects by $\bar{a}(i \in \Delta^{a_k}) = \Delta^{\{0,\dots,i\}}$ and on morphisms $(i \in \Delta^{a_k}) \to (j \in \Delta^{a_l})$, $\alpha_{kl} : \Delta^{a_k} \to \Delta^{a_l}$, $\alpha_{kl}(i) \le j$, by restriction of α_{kl} to $\Delta^{\{0,\dots,i\}} \subset \Delta^{a_k}$ (which then is valued in $\Delta^{\{0,\dots,j\}} \subset \Delta^{a_l}$). Then define a functor of categories

$$\phi : (\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}) \times_{\bar{a},\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$$

by sending objects $(i \in \Delta^{a_k}, i' \le i)$ to $(i' \in \Delta^{a_k})$ and morphisms $(i \in \Delta^{a_k}, i' \le i) \to (j \in \Delta^{a_l}, j' \le j)$ (specified by the data of a map $\alpha_{kl} : \Delta^{a_k} \to \Delta^{a_l}$ such that $\alpha_{kl}(i) \le j$ and $\alpha_{kl}(i') \le j'$) to the morphism $(i' \in \Delta^{a_k}) \to (j' \in \Delta^{a_l})$ specified by the same data.

We may then specify a map

$$g : \Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \widehat{\mathrm{Ar}}(S)$$

defined over Δ^{inj} via $\bar{a}$ and the structure map ξ_S as adjoint to the map

$$f \circ \phi : (\Delta^n \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}) \times_{\bar{a},\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S.$$

The map g in turn defines the desired n-simplex $\delta(\sigma) : \Delta^n \to \widehat{\mathrm{Ar}}(\widehat{\mathrm{Ar}}(S))$.

Informally, δ sends paths $s_0 \to s_1 \to \cdots \to s_n$ to their "initial segment parametrization"

$$[s_0] \to [s_0 \to s_1] \to \cdots \to [s_0 \to s_1 \to \cdots \to s_n].$$

Next, using the functor $\max_S$ to make sense of the next statement, we may use δ to define functors

$$\begin{aligned} \delta &: \widehat{\mathrm{Ar}}^{\simeq}(S) \to \widehat{\mathrm{Ar}}^{\simeq}_S(\widehat{\mathrm{Ar}}^{\simeq}(S)) = \widehat{\mathrm{Ar}}^{\simeq}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(\widehat{\mathrm{Ar}}^{\simeq}(S)), \\ \delta &: \mathrm{sd}(S) \to \mathrm{sd}_S(\mathrm{sd}(S)) = \mathrm{sd}(S) \times_{\widehat{\mathrm{Ar}}(S)} \widehat{\mathrm{Ar}}(\mathrm{sd}(S)) \end{aligned}$$

as the identity on the first factor and a restriction of δ on the second factor.

Proof of Theorem 3.26 Item (1) follows from Theorem 3.20 in view of the pullback square

$$\begin{array}{ccc} \mathrm{sd}_S(C)_0^{\mathrm{cocart}} & \longrightarrow & \widehat{\mathrm{Ar}}^{\simeq}_S(C)_0^{\mathrm{cocart}} \\ \downarrow & & \downarrow \\ \mathrm{sd}_{S_0}(C_0) \times_{\mathrm{sd}(S_0)} \mathrm{sd}(S)_0 & \longrightarrow & \widehat{\mathrm{Ar}}^{\simeq}_{S_0}(C_0) \times_{\widehat{\mathrm{Ar}}^{\simeq}(S_0)} \widehat{\mathrm{Ar}}^{\simeq}(S)_0 \end{array}$$

For (2), we need to solve the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S)_0, C) \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0) \end{array}$$

This transposes to

$$\begin{array}{ccc} A\times \mathrm{sd}(S)_0 \cup_{A\times \mathrm{sd}(S_0)} B\times \mathrm{sd}(S_0) & \xrightarrow{G\cup F} & C \\ \big\downarrow & \nearrow & \big\downarrow p \\ B\times \mathrm{sd}(S)_0 & \xrightarrow{\max_S} & S \end{array}$$

The functoriality of $\mathrm{sd}_{S_0}(-)$ in its argument results in a functor

$$\mathrm{sd}_{S_0} : \mathrm{Fun}_{/S_0}(\mathrm{sd}(S_0), C_0) \to \mathrm{Fun}_{/S_0}\big(\mathrm{sd}_{S_0}(\mathrm{sd}(S_0)), \mathrm{sd}_{S_0}(C_0)\big).$$

Given $F : B\times \mathrm{sd}(S_0) \to C_0$, let $\mathrm{sd}_{S_0}(F) : B\times \mathrm{sd}_{S_0}(\mathrm{sd}(S_0)) \to \mathrm{sd}_{S_0}(C_0)$ denote the image. We then define $\overline{F}$ as the composite

$$B\times \mathrm{sd}(S_0) \xrightarrow{\mathrm{id}\times\delta} B\times \mathrm{sd}_{S_0}(\mathrm{sd}(S_0)) \xrightarrow{\mathrm{sd}_{S_0}(F)} \mathrm{sd}_{S_0}(C_0).$$

Also let $\overline{F}'$ denote $\overline{F}$ with codomain $\mathrm{sd}_S(C)_0^{\mathrm{cocart}}$ via the inclusion $\mathrm{sd}_{S_0}(C_0) \subset \mathrm{sd}_S(C)_0^{\mathrm{cocart}}$.

Similarly, given $G : A\times \mathrm{sd}(S)_0 \to C$, we may define $\overline{G}$ as the composite

$$A\times \mathrm{sd}(S)_0 \xrightarrow{\mathrm{id}\times\delta} A\times \mathrm{sd}_S(\mathrm{sd}(S)_0) \xrightarrow{\mathrm{sd}_S(G)} \mathrm{sd}_S(C)_0^{\mathrm{cocart}},$$

where we note that the codomain of $\mathrm{sd}_S(G)$ necessarily lies in $\mathrm{sd}_S(C)_0^{\mathrm{cocart}}$ by definition of the locally $\max_S$-cocartesian edges in $\mathrm{sd}(S)_0$ (here it is essential that we use $\mathrm{sd}(S)$ rather than $\widehat{\mathrm{Ar}}^{\simeq}(S)$). Clearly, $\overline{G}$ and $\overline{F}'$ are compatible on their common domain $A\times \mathrm{sd}(S_0)$ since G and F are. We thereby may factor the square above as

$$\begin{array}{ccccc} A\times \mathrm{sd}(S)_0 \cup_{A\times \mathrm{sd}(S_0)} B\times \mathrm{sd}(S_0) & \xrightarrow{\overline{G}\cup\overline{F}'} & \mathrm{sd}_S(C)_0^{\mathrm{cocart}} & \xrightarrow{\max_C} & C \\ \big\downarrow & \nearrow & \big\downarrow\simeq & & \big\downarrow p \\ B\times \mathrm{sd}(S)_0 & \xrightarrow{(\overline{F}\lambda,\mathrm{pr})} & \mathrm{sd}_{S_0}(C_0)\times_{\mathrm{sd}(S_0)} \mathrm{sd}(S)_0 & \xrightarrow{\max_S} & S \end{array}$$

The dotted lift exists by (1), and postcomposition of such a lift by $\max_C$ defines the desired lift. $\square$

3.2.3 Main results We begin by constructing a factorization system [12, Definition 5.2.8.8] on $\mathrm{sd}(S)$ associated to a sieve-cosieve decomposition of S. To do this, we need a few preparatory lemmas.

3.27 Lemma *Let $p : X \to S$ be a cartesian fibration. Given a functor $\phi : K \to X$, let*

$$\bar{p} : X^{\phi/} = \mathrm{Fun}(K^{\triangleright}, X)\times_{\mathrm{Fun}(K,X)} \{\phi\} \to S^{p\phi/} = \mathrm{Fun}(K^{\triangleright}, X)\times_{\mathrm{Fun}(K,X)} \{p\phi\}$$

be the functor induced by p. Then $\bar{p}$ is a cartesian fibration, and an edge $\bar{e} : \bar{x} \to \bar{y} \in X^{\phi/}$ is $\bar{p}$-cartesian if and only if the underlying edge $e : x \to y \in X$ is p-cartesian.

Proof We may mimic the proof of [12, 3.1.2.1] to prove the lemma, the essential tool being [12, 3.1.2.3]. In more detail, let E be the described collection of edges in $X^{\phi/}$ and suppose we are given a lifting problem in marked simplicial sets of the form

$$\begin{array}{ccc} \Lambda^{n\natural}_n & \longrightarrow & (X^{\phi/}, E) \\ \downarrow & \nearrow & \downarrow \bar{p} \\ \Delta^{n\natural} & \longrightarrow & (S^{p\phi/})^\sharp \end{array}$$

where we mark the edge $\{n-1,n\}$ of Λ^n_n (if $n > 1$) and of Δ^n. This transposes to a lifting problem of the form

$$\begin{array}{ccc} \Lambda^{n\natural}_n \times K^{\rhd} \cup_{\Lambda^{n\natural}_n \times K} \Delta^{n\natural} \times K & \xrightarrow{f} & X^\natural \\ \downarrow i & \nearrow & \downarrow p \\ \Delta^{n\natural} \times K^{\rhd} & \longrightarrow & S^\sharp \end{array}$$

where we mark the p-cartesian edges in X. Note that f is indeed a map of marked simplicial sets: this is by definition of E for f on the edge $\{n-1,n\} \times \{v\}$ ($v \in K^{\rhd}$ the cone point), and by definition of f on $\Delta^n \times K$ as given by $\phi \circ \mathrm{pr}_K$ for the other marked edges. Applying [12, 3.1.2.3], we deduce that i is marked right anodyne, so the dotted lift exists. □

3.28 Lemma *Let $p: X \to S$ be a cartesian fibration. Suppose we have a commutative square in X*

$$\begin{array}{ccc} x & \xrightarrow{h} & z \\ \downarrow f & & \downarrow g \\ y & \xrightarrow{k} & w \end{array}$$

If the edge g is p-cartesian, then we have an equivalence

$$\mathrm{Map}_{x/\!/w}(y,z) \xrightarrow{\simeq} \mathrm{Map}_{px/\!/pw}(py,pz).$$

Proof In the statement of the lemma, the space of factorizations $\mathrm{Map}_{x/\!/w}(y,z)$ may be defined as the pullback

$$\begin{array}{ccc} \mathrm{Map}_{x/\!/w}(y,z) & \longrightarrow & \mathrm{Map}_{x/}(y,z) \\ \downarrow & & \downarrow g_* \\ \{k\} & \longrightarrow & \mathrm{Map}_{x/}(y,w) \end{array}$$

and likewise for $\mathrm{Map}_{px/\!/pw}(py,pz)$.

Now by Lemma 3.27, $\bar{p}: X^{x/} \to S^{px/}$ is a cartesian fibration and g, viewed as an edge $h \to kf$, is a $\bar{p}$-cartesian edge. Therefore, we have a homotopy pullback square of spaces

$$\begin{array}{ccc} \mathrm{Map}_{x/}(y,z) & \xrightarrow{g_*} & \mathrm{Map}_{x/}(y,w) \\ \downarrow p & & \downarrow p \\ \mathrm{Map}_{px/}(py,pz) & \xrightarrow{pg_*} & \mathrm{Map}_{px/}(py,pw) \end{array}$$

Taking fibers over $k \in \mathrm{Map}_{x/}(y,w)$ and $pk \in \mathrm{Map}_{px/}(py,pw)$ yields the claimed equivalence. □

Fix a functor $\pi : S \to \Delta^1$ and let S_i denote the fiber over $i \in \{0,1\}$. We now define a factorization system on $\widehat{\mathrm{Ar}}^{\simeq}(S)$ that will restrict to a factorization system on the full subcategory $\mathrm{sd}(S)$. Recall that the data of a morphism $e : x \to y$ in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ is given by an inclusion $\alpha : \Delta^{a_0} \hookrightarrow \Delta^{a_1}$ and a map $f : \Delta^1 \times_{\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ that restricts to $x : \Delta^{a_0} \to S$ and $y : \Delta^{a_1} \to S$, such that f sends morphisms $(i \in \Delta^{a_0}) \to (\alpha(i) \in \Delta^{a_1})$ to equivalences in S.

3.29 Definition Let $\mathcal{L}$ be the subclass of morphisms $(\alpha, f) : x \to y$ such that for every $i \notin \operatorname{im}\alpha$, we have that $y(i) \in S_0$, and let $\mathcal{R}$ be the subclass of morphisms $(\alpha, f) : x \to y$ such that for every $i \notin \operatorname{im}\alpha$, we have that $y(i) \in S_1$.

3.30 Proposition *$(\mathcal{L}, \mathcal{R})$ defines a factorization system on $\widehat{\mathrm{Ar}}^{\simeq}(S)$ and on $\mathrm{sd}(S)$.*

Proof We will check the assertion concerning $\widehat{\mathrm{Ar}}^{\simeq}(S)$; the second assertion will then be a consequence. We first explain how to factor morphisms. Suppose that $\gamma : \Delta^{a_0} \hookrightarrow \Delta^{a_2}$, $h : \Delta^1 \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to S$ is the data of a morphism in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ from x to z. Let $\Delta^{a_1} \subset \Delta^{a_2}$ be the subset on those $i \in \Delta^{a_2}$ such that $i \in \operatorname{im}\gamma$ or $z(i) \in S_0$. We then obtain a factorization of γ as

$$\Delta^{a_0} \xhookrightarrow{\alpha} \Delta^{a_1} \xhookrightarrow{\beta} \Delta^{a_2}.$$

Define $\bar{a} : \Delta^2 \to \Delta^{\mathrm{inj}}$, extending the given $a : \Delta^{\{0,2\}} \to \Delta^{\mathrm{inj}}$. Let $r : \Delta^2 \times_{\bar{a},\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}} \to \Delta^1 \times_{a,\Delta^{\mathrm{inj}}} \mathcal{E}\Delta^{\mathrm{inj}}$ be the unique retraction which is the identity on Δ^{a_0} and Δ^{a_2} and is given by β on Δ^{a_1}. Let $\bar{h} = h \circ r$. Then $\bar{h}$ is the desired factorization of h, as it corresponds to a factorization

$$x \xrightarrow{f} y \xrightarrow{g} z, \qquad h = g \circ f : x \to z$$

with $y = z \circ \beta : \Delta^{a_1} \to S$ defined so that $y(i) \in S_0$ for all $i \notin \operatorname{im}\alpha$ and $z(j) \in S_1$ for all $j \notin \operatorname{im}\beta$, hence $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

Next, observe that because S_0 and S_1 are closed under retracts, so are $\mathcal{L}$ and $\mathcal{R}$. It only remains to check that $\mathcal{L}$ is left orthogonal to $\mathcal{R}$. For this, suppose we are given a commutative square in $\widehat{\mathrm{Ar}}^{\simeq}(S)$ on the left with $f \in \mathcal{L}$ and $g \in \mathcal{R}$ covering the square in Δ^{inj} on the right:

$$\begin{array}{ccc} x & \xrightarrow{h} & z \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ y & \xrightarrow{k} & w \end{array} \qquad\qquad \begin{array}{ccc} \Delta^a & \xrightarrow{\delta} & \Delta^c \\ {\scriptstyle \alpha}\downarrow & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle \beta} \\ \Delta^b & \xrightarrow{\kappa} & \Delta^d \end{array}$$

Because $\xi_S : \widehat{\mathrm{Ar}}^{\simeq}(S) \to \Delta^{\mathrm{inj}}$ is a right fibration, by Lemma 3.28 it suffices to show $\mathrm{Map}_{\Delta^a /\!/ \Delta^d}(\Delta^b, \Delta^c)$ is contractible. This holds if and only if $\Delta^b \subset \Delta^c$ when viewed as subsets of Δ^d, so that the mapping space is nonempty. Our hypothesis ensures that if $i \notin \operatorname{im}\beta$, then $w(i) \in S_1$, and if $i \in \Delta^b$, either $i \in \operatorname{im}\alpha$ or $y(i) \in S_0$. Therefore, we must have that for every $i \in \Delta^b$ with $i \notin \operatorname{im}\alpha$ that $w(\kappa(i)) \in S_0$, and hence $\kappa(i) \in \operatorname{im}\beta$. We conclude that the dotted lift γ exists. □

Let $\mathrm{Ar}^L(\mathrm{sd}(S)) \subset \mathrm{Ar}(\mathrm{sd}(S))$ denote the full subcategory on those morphisms $x \to y$ in the class $\mathcal{L}$.

3.31 Lemma (1) *The inclusion $i : \mathrm{Ar}^L(\mathrm{sd}(S)) \subset \mathrm{Ar}(\mathrm{sd}(S))$ admits a right adjoint r that on objects sends $h : x \to y$ to $f : x \to z$ where h factors as $g \circ f$ according to the $(\mathcal{L}, \mathcal{R})$ factorization system.*

(2) *$i \dashv r$ defines a relative adjunction with respect to evaluation ev_0 at the source, and therefore for every $x \in \mathrm{sd}(S)$ we obtain an adjunction*

$$\{x\} \times_{\mathrm{sd}(S)} \mathrm{Ar}^L(\mathrm{sd}(S)) \rightleftarrows \mathrm{sd}(S)^{x/}.$$

(3) *The relative adjunction $i \dashv r$ restricts to a relative adjunction*

$$i : \mathrm{Ar}^L(\mathrm{sd}(S)) \times_{\mathrm{ev}_1, \mathrm{sd}(S)} \mathrm{sd}(S)_0 \rightleftarrows \mathrm{Ar}(\mathrm{sd}(S)) \times_{\mathrm{ev}_1, \mathrm{sd}(S)} \mathrm{sd}(S)_0 : r$$

and therefore for every $x \in \mathrm{sd}(S)$ we obtain an adjunction

$$\{x\} \times_{\mathrm{sd}(S)} \mathrm{Ar}^L(\mathrm{sd}(S)) \times_{\mathrm{sd}(S)} \mathrm{sd}(S)_0 \rightleftarrows \mathrm{sd}(S)_0^{x/}.$$

Proof Claim (1) is the dual formulation of [12, 5.2.8.19]. Claims (2) and (3) then follow by the definition of relative adjunction [14, 7.3.2.1] and its pullback property [14, 7.3.2.5]. □

We are now prepared to construct the recollement adjunctions. Note that the hypotheses of the following theorem are satisfied if S is equivalent to a finite poset and $p : C \to S$ is a locally cocartesian fibration such that the fibers admit finite limits and the pushforward functors preserve finite limits.

3.32 Theorem *Let $p : C \to S$ be a locally cocartesian fibration, let $\pi : S \to \Delta^1$ be a functor, and suppose we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{sd}(S)_0 & \xrightarrow{F} & C \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle p} \\ \mathrm{sd}(S) & \xrightarrow{\max_S} & S \end{array}$$

where F preserves locally cocartesian edges. Given $x \in \mathrm{sd}(S_1)$, let

$$J_x = \{x\} \times_{\mathrm{sd}(S)} \mathrm{Ar}^L(\mathrm{sd}(S)) \times_{\mathrm{sd}(S)} \mathrm{sd}(S)_0.$$

Note that $(\max_S \circ \mathrm{ev}_1)|_{J_x}$ is constant at $\max_S(x)$.

(1) *If for every $x \in \mathrm{sd}(S_1)$, the limit of $(F\,\mathrm{ev}_1)|_{J_x} : J_x \to C_{\max_S(x)}$ exists, then the p-right Kan extension G of F along ϕ exists and $G(x) \simeq \varprojlim F|_{J_x}$.*

(2) *If for every $f : s \to t$ in S, the pushforward functor $f_! : C_s \to C_t$ preserves all limits appearing in (1), then G preserves all locally cocartesian edges.*

(3) *If the hypotheses of (1) and (2) hold for all F, then we have an adjunction*

$$\phi^* : \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S)_0, C) : \phi_*.$$

Proof Note that $\mathrm{sd}(S_1) \subset \mathrm{sd}(S)$ is the complementary *sieve* inclusion to the *cosieve* $\mathrm{sd}(S)_0 \subset \mathrm{sd}(S)$. For (1), to show existence of the p-right Kan extension it suffices for every $x \in \mathrm{sd}(S_1)$ to show that the p-limit of $F \circ \mathrm{pr}_1 : \mathrm{sd}(S)_0^{x/} \to \mathrm{sd}(S)_0 \to C$ exists. But by the argument of Corollary 3.12 applied to the adjunction $J_x \rightleftarrows \mathrm{sd}(S)_0^{x/}$ of Lemma 3.31, this follows from the given hypothesis.

For (2), first note that there are no locally $\max_S$-cocartesian edges $e : x \to y$ such that $x \in \mathrm{sd}(S_1)$ and $y \in \mathrm{sd}(S)_0$, or vice versa, so it suffices to handle the case where $e : x \to y$ is a locally $\max_S$-cocartesian edge in $\mathrm{sd}(S_1)$ only. Let $f : \max_S(x) = s \to \max_S(y) = t$ be the edge in $S_1 \subset S$. If f is an equivalence, then e is an equivalence and $G(e)$ is an equivalence, so we may suppose f is not an equivalence. Then by the description of the locally $\max_S$-cocartesian edges in Lemma 3.25, y is obtained from e by appending the edge f. Correspondingly, the functor $J_y \xrightarrow{\simeq} J_x$ defined via sending $y \to z$ to $x \to z$ by precomposing is an equivalence, using that such edges are constrained to only add objects in S_0. Examining how the functoriality of G is obtained from the pointwise existence criterion for Kan extensions, we see that the comparison morphism in C_t,

$$\psi : f_! G(x) \simeq f_!(\varprojlim F\, \mathrm{ev}_1|_{J_x}) \to G(y) \simeq \varprojlim F\, \mathrm{ev}_1|_{J_y},$$

is induced via the functoriality of limits (contravariant in the diagram, covariant in the target) from the commutative diagram

$$\begin{array}{ccc} J_x & \xrightarrow{F\,\mathrm{ev}_1} & C_s \\ \simeq\uparrow & & \downarrow f_! \\ J_y & \xrightarrow{F\,\mathrm{ev}_1} & C_t \end{array}$$

The hypothesis that $f_!$ preserve limits indexed by J_x together with $J_y \simeq J_x$ then proves that ψ is an equivalence.

Finally, for (3) it is clear that if $G : \mathrm{sd}(S) \to C$ preserves locally cocartesian edges, then the restriction $\phi^* G$ of G to $\mathrm{sd}(S)_0$ does as well. Items (1) and (2) establish the same fact for $\phi_* F$. Hence, the characteristic adjunction

$$\phi^* : \mathrm{Fun}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}_{/S}(\mathrm{sd}(S)_0, C) : \phi_*$$

of the p-right Kan extension along ϕ restricts to the full subcategories of functors preserving locally cocartesian edges in order to yield the desired adjunction. □

3.33 Remark Suppose that S is a poset and $x \in S_1 \subset \mathrm{sd}(S_1)$. Then the ∞-category J_x that appears in Theorem 3.32 is the poset whose objects are strings $[a_0 < \cdots < a_n < x]$, $n \geq 0$, with $a_i \in S_0$ and whose morphisms are string inclusions.

3.34 Corollary *Suppose the hypotheses of Theorem 3.32 are satisfied. Let $j : \mathrm{sd}(S_0) \to \mathrm{sd}(S)$ denote the inclusion. Then the functor j^* of restriction along j participates in an adjunction*

$$j^* : \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0) : j_*$$

with fully faithful right adjoint j_*.

Proof Combine Theorems 3.32 and 3.26(2). □

We have a far simpler result concerning the calculation of the left adjoint $j_!$ of j^* (but see Remark 3.41).

3.35 Proposition *Let $p: C \to S$ be a locally cocartesian fibration, let $\pi: S \to \Delta^1$ be a functor, and suppose that for every $s \in S_1$, the fiber C_s admits an initial object $\varnothing$, and for every $[f: s \to t] \in S_1$ the pushforward functors $f_!$ all preserve initial objects. Then j^* admits a fully faithful left adjoint $j_!$ such that for $F: \mathrm{sd}(S_0) \to C_0$, we have $j_! F(x) \simeq \varnothing$ for all $x \in \mathrm{sd}(S_1)$.*

Proof Suppose we have a commutative diagram

$$\begin{array}{ccc} \mathrm{sd}(S)_0 & \xrightarrow{F} & C \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle p} \\ \mathrm{sd}(S) & \xrightarrow{\max_S} & S \end{array}$$

For all $x \in \mathrm{sd}(S_1)$, the fiber product $\mathrm{sd}(S)^{/x} \times_{\mathrm{sd}(S)} \mathrm{sd}(S)_0$ is the empty category. Therefore, under our assumption the p-left Kan extension $\phi_! F$ of F along ϕ exists and is computed by $\phi_! F(x) = \varnothing$ on $\mathrm{sd}(S_1)$. Combining this observation with Theorem 3.26(2), we obtain the desired adjunction

$$j_!: \mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C): j^*. \qquad \square$$

We next turn to the cosieve inclusion $S_1 \subset S$. Note that the inclusion $i: \mathrm{sd}(S_1) \hookrightarrow \mathrm{sd}(S)$ is a sublocally cocartesian fibration with respect to $\max_S: \mathrm{sd}(S) \to S$, and is in addition a *sieve* inclusion, and hence i is a cartesian fibration. In fact, the cosieve inclusion $j: \mathrm{sd}(S)_0 \hookrightarrow \mathrm{sd}(S)$ is complementary to i.

3.36 Proposition *Let $p: C \to S$ be a locally cocartesian fibration, let $\pi: S \to \Delta^1$ be a functor, and suppose the fibers of p admit terminal objects and the pushforward functors preserve terminal objects. Then we have the adjunction*

$$i^*: \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C_1): i_*$$

with i_* fully faithful, where i^* is given by restriction along i and i_* is p-right Kan extension along i. Moreover, for a functor $G: \mathrm{sd}(S_1) \to C_1$, we have $(i_* G)(x) \simeq * \in C_{\max_S(x)}$ for all $x \in \mathrm{sd}(S)_0$.

Proof By Corollary 3.12, using the hypothesis that the fibers of p admit terminal objects we have the adjunction

$$i^*: \mathrm{Fun}_{/S}(\mathrm{sd}(S), C) \rightleftarrows \mathrm{Fun}_{/S_1}(\mathrm{sd}(S_1), C_1): i_*$$

with i^* and i_* as described. Then using that the pushforward functors preserve terminal objects, we see that this adjunction restricts to the one of the proposition. □

3.37 Lemma *Let $p: C \to S$ be a locally cocartesian fibration and suppose that the fibers C_s admit K-(co)limits and the pushforward functors preserve K-(co)limits. Then the category $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ admits K-indexed (co)limits, and for all $\sigma \in \mathrm{sd}(S)$ over $s = \max_S(\sigma)$, the evaluation functor $\mathrm{ev}_\sigma: \mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \to C_s$ preserves K-indexed (co)limits. Moreover, if the fibers C_s are stable ∞-categories and the pushforward functors are exact, then $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ is a stable ∞-category.*

Proof Apply [12, Proposition 5.4.7.11] to the locally cocartesian fibration $\mathrm{sd}(S) \times_S C \to \mathrm{sd}(S)$, with the subcategory of $\widehat{\mathbf{Cat}}_\infty$ either taken to be those ∞-categories that admit K-indexed (co)limits and functors that preserve K-indexed (co)limits, or the subcategory $\mathbf{Cat}_\infty^{\mathrm{stab}}$ of stable ∞-categories and exact functors thereof. □

We encapsulate the assumptions above on existence and preservation of various limits into the following definition (compare with Definition 3.4).

3.38 Definition (standard existence assumptions, left-lax version) Let $p : C \to S$ be a locally cocartesian fibration and let $\pi : S \to \Delta^1$ be a functor. We say that p satisfies the *standard recollement existence assumptions* with respect to π if:

(1) For all $s \in S$, C_s admits finite limits, and for all morphisms $f : s \to t$ in S, the pushforward functors $f_! : C_s \to C_t$ preserves finite limits.
(2) The hypotheses of Theorem 3.32 hold.

Finally, putting everything together, we get:

3.39 Theorem *Let $p : C \to S$ be a locally cocartesian fibration, let $\pi : S \to [1]$ be a functor, and suppose that p satisfies the standard recollement existence assumptions with respect to π. Then the two adjunctions of Corollary 3.34 and Proposition 3.36 combine to exhibit $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ as a recollement of $\mathrm{Fun}^{\mathrm{cocart}}_{/S_0}(\mathrm{sd}(S_0), C_0)$ and $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C_1)$.*

Proof We verify the conditions to be a recollement. By our assumption on p and Lemma 3.37, finite limits in $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ exist and are computed fiberwise. Therefore, the restriction functors j^* and i^* are left exact. By the formula for i_* given in Proposition 3.36, it is clear that $j^* i_*$ is constant at the terminal object. Finally, we check that j^* and i^* are jointly conservative. Suppose given a morphism $\alpha : F \to F'$ in $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ such that $j^*\alpha$ and $i^*\alpha$ are equivalences. Observe that α is an equivalence if and only if for all $x \in S$, $\alpha_x : F(x) \to F'(x)$ is an equivalence (viewing x as an object in $\mathrm{sd}(S)$). Because any object of S lies in either S_0 or S_1, we deduce that α is an equivalence. □

3.40 Remark Suppose that S is a *down-finite* poset P. Let $C \to P$ be a locally cocartesian fibration such that its fiber admits finite limits and its pushforward functors preserve finite limits. Then the hypotheses of Theorem 3.32 automatically hold for every sieve-cosieve decomposition of P. Indeed, the categories J_x that appear there are all finite (see Remark 3.33).

Let us now return to the question of the existence of $j_!$.

3.41 Remark The left adjoint $j_!$ in Proposition 3.35 should exist even if we only suppose that the fibers of C admits initial objects (i.e., we need not suppose that the pushforward functors preserve initial objects). However, in that case $j_!$ will not generally be the p-left Kan extension along the inclusion ϕ, and relatedly, a direct proof of this would appear to be overly cumbersome in our framework. Rather, we can say the following (which covers most cases of practical relevance):

- Suppose that the hypotheses of Theorem 3.39 are satisfied and we have shown that $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C)$ admits an initial object. Then as in any recollement situation, the left adjoint $j_!$ exists and is computed by $j_!(u) = [u, \varnothing \to i^* j_*(u)]$.
- To exhibit the initial object of $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C)$, suppose also that S_1 is a finite poset P. Then using Theorem 3.39 in conjunction with Lemma 2.39, we may proceed by induction on the cardinality of P and repeatedly invoke our assumption that the fibers of C admit an initial object to conclude that $\mathrm{Fun}^{\mathrm{cocart}}_{/S_1}(\mathrm{sd}(S_1), C)$ admits an initial object whose evaluation at every singleton string is also initial.

We conclude this subsection by giving an application of Theorem 3.39 to the presentability of the right-lax limit $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$. First suppose that S is equivalent to a *finite* poset and write $P = S$.

3.42 Proposition *Suppose that the fibers C_s of $p : C \to P$ are presentable and the pushforward functors are left-exact and accessible. Then $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ is presentable, and for all $s \in P$, the evaluation functor $\mathrm{ev}_s : \mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C) \to C_s$ preserves (small) colimits and is accessible.*

Proof The accessibility statements follow from [12, Proposition 5.4.7.11] as in Lemma 3.37, so we only need to show the existence and preservation of small colimits. Our strategy is to proceed by induction on the cardinality of P. If $|P| \leq 1$, then the statement is clear. Suppose for the inductive hypothesis that we have established the statement for all posets Q such that $|Q| < |P|$. Let $b \in P$ be a maximal object and let $\pi : P \to \Delta^1$ be the functor determined by the sieve-cosieve decomposition $P_0 = P \setminus \{b\}$ and $P_1 = \{b\}$. Because the diagrams that appear in Theorem 3.32 are finite, we may apply Theorem 3.39 to decompose $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ as a recollement of $\mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C_0)$ and C_b. By the inductive hypothesis, both these ∞-categories admit all small colimits such that the evaluation functors at objects in P_0 and P_1 are colimit-preserving. By Lemma 2.39, we conclude that $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ admits all small colimits such that the evaluation functors for objects $s \in P$ are colimit-preserving. □

Next, we may use the equivalence (see Observation 3.23)

$$(\star) \qquad \mathrm{sd}(S) \xleftarrow{\simeq} \mathrm{colim}_{[n] \in \Delta_{/S}} \mathrm{sd}([n])$$

to promote Proposition 3.42 to a statement involving arbitrary S.

3.43 Corollary *Suppose the fibers C_s of $p : C \to S$ are presentable and the pushforward functors are left-exact and accessible. Then $\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C)$ is presentable.*

Proof We may simply copy over the proof strategy used to establish [2, Proposition 6.1.6(1)]. By $(\star)$, we have that

$$\mathrm{Fun}^{\mathrm{cocart}}_{/S}(\mathrm{sd}(S), C) \xrightarrow{\simeq} \lim_{[n] \in (\Delta_{/S})^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{cocart}}_{/[n]}(\mathrm{sd}[n], C|_{[n]}).$$

By Proposition 3.42 and Theorem 3.39, for every $[\sigma : [n] \to S] \in \Delta_{/S}$, $\lim^{\mathrm{rlax}} \sigma^* C$ is presentable and the evaluation functors $\{\mathrm{ev}_i : \lim^{\mathrm{rlax}} \sigma^* C \to C_{\sigma(i)}\}_{i=0}^n$ are colimit-preserving and jointly conservative. Note then that for any map $\alpha : [m] \to [n]$, the restriction functor

$$\alpha^* : \lim^{\mathrm{rlax}} \sigma^* C \to \lim^{\mathrm{rlax}} \alpha^* \sigma^* C$$

preserves colimits. Then since $\lim^{\mathrm{rlax}} C$ is a limit of presentable ∞-categories along colimit-preserving functors, it is presentable. □

3.44 Remark We explain a subtle difference between our general approach and the one of [2, §6], which is adapted to the case of locally cocartesian fibrations $p : C \to P$ over a poset P whose fibers are presentable stable ∞-categories and whose pushforward functors are exact and accessible. Suppose one could prove directly that $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ is presentable (for any poset) and that the restriction functor $j^* : \mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/P_0}(\mathrm{sd}(P_0), C)$ preserves colimits, so that it admits a right adjoint j_*. Then without a pointwise formula for j_*, it is generally difficult to show that j_* is fully faithful. However, this would follow if we could also exhibit a fully faithful *left* adjoint $j_!$ to j^*, and this turns out to be easier to analyze (see Proposition 3.35). This is the strategy adopted in the proof of [2, Proposition 6.1.6].

Therefore, if we were only interested in the existence of the recollement on $\mathrm{Fun}^{\mathrm{cocart}}_{/P}(\mathrm{sd}(P), C)$ in the stable presentable case, then we could bypass the work that goes into establishing the pointwise formula of Theorem 3.32. However, our primary motivation for undertaking this work lay precisely in having this pointwise formula. Note also that in the presentable case, the right adjoint j_* exists unconditionally even if it is not describable as a relative right Kan extension.

On the other hand, such tricks are not available in the absence of presentability (though for idempotent-complete small stable ∞-categories, one can pass to their **Ind**-completions as is done in [2, §7.2]). Over a down-finite poset P (see Remark 3.40), our Theorem 3.39 thus allows one to strengthen [2, Theorem A] by removing all of the presentability hypotheses therein.

3.2.4 Symmetric monoidal structure We briefly explain how to promote Theorem 3.39 to a statement involving symmetric monoidal recollements. First recall the notions of left-lax and right-lax morphisms of locally cocartesian fibrations from [2, §A.1 and A.3]:

3.45 Recollection Let $\lambda, \xi : \mathcal{C}, \mathcal{D} \to S$ be locally cocartesian fibrations. A *left-lax* morphism $\lambda \to \xi$ is a functor $F : \mathcal{C} \to \mathcal{D}$ over S (which need not preserve locally cocartesian edges). In contrast, a *right-lax* morphism $\lambda \to \xi$ is defined as in [2, Definition A.3.2] as the “unstraightened” counterpart to a right-lax natural transformation of left-lax functors.

The collection of locally cocartesian fibrations over S and right-lax morphisms thereof assemble into an ∞-category $\mathbf{LocCocart}^{\mathrm{rlax}}_S$ which contains $\mathbf{LocCocart}_S$ as a wide subcategory. Moreover, $\lim^{\mathrm{rlax}}$ extends to a functor over $\mathbf{LocCocart}^{\mathrm{rlax}}_S$ that is right adjoint to the constant functor $\mathrm{const} : \mathcal{E} \mapsto \mathcal{E} \times S$. See [2, Definitions A.3.2 and B.6.1].

In view of the adjunction $\mathrm{const} \dashv \lim^{\mathrm{rlax}}$, $\lim^{\mathrm{rlax}}$ sends commutative monoids in $\mathbf{LocCocart}^{\mathrm{rlax}}_S$ to symmetric monoidal ∞-categories. Moreover, a diagram chase shows that given a commutative monoid structure on $[p : C \to S]$, for any $\alpha : T \to S$ the pullback $[\alpha^* C \to T]$ is a commutative monoid in $\mathbf{LocCocart}^{\mathrm{rlax}}_T$ and the restriction functor $\lim^{\mathrm{rlax}} C \to \lim^{\mathrm{rlax}} \alpha^* C$ is symmetric monoidal. It follows that if the recollement of Theorem 3.39 exists in this situation, then it is symmetric monoidal.

3.46 Remark If $S = \Delta^1$, then a commutative monoid in $\mathbf{LocCocart}^{\mathrm{rlax}}_{\Delta^1}$ is the data of a lax symmetric monoidal functor of symmetric monoidal ∞-categories (see [11, Proposition 2.6]). In general, to endow

$p : C \to S$ with the structure of a commutative monoid entails endowing its fibers with symmetric monoidal structures and its pushforward functors and natural transformations thereof with lax symmetric monoidal structures in a coherent fashion. See [2, §4] for how to produce examples from simpler input.

4 1-generated and extendable objects

Suppose $S = \Delta^2$ and $p : C \to \Delta^2$ is a locally cocartesian fibration classified by a 2-functor

$$\begin{array}{ccc} C_0 & \xrightarrow{\quad H \quad} & C_2 \\ & \Downarrow & \\ {\scriptstyle F} \searrow & & \nearrow {\scriptstyle G} \\ & C_1 & \end{array}$$

Then the data of a functor $\mathrm{sd}(\Delta^2) \to C$ over Δ^2 that preserves locally cocartesian edges can be summarized as follows:

- Objects $c_i \in C_i$ for $i = 0, 1, 2$.
- Morphisms $f : c_1 \to F(c_0)$, $g : c_2 \to G(c_1)$, and $h : c_2 \to H(c_0)$.
- A commutative square

$$\begin{array}{ccc} c_2 & \xrightarrow{h} & H(c_0) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathrm{can}} \\ G(c_1) & \xrightarrow{G(f)} & GF(c_0) \end{array}$$

Furthermore, if the map *can* is an equivalence, then the data of the commutative square and the morphism h are redundant, since then $h \simeq G(f) \circ g$ and compositions in an ∞-category are unique up to contractible choice. More precisely, if we let $\gamma_2 : \mathrm{sd}_1(\Delta^2) \subset \mathrm{sd}(\Delta^2)$ be the subposet on the set $\{[0], [1], [2], [0 < 1], [1 < 2]\}$, then the functor

$$\gamma_2^* : \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^2}(\mathrm{sd}(\Delta^2), C) \to \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^2}(\mathrm{sd}_1(\Delta^2), C)$$

is a trivial fibration onto its image when restricted to objects for which *can* is an equivalence.

Our goal in this section is to generalize this observation to the case where $S = \Delta^n$. We introduce subcategories of 1-generated and extendable objects (Definitions 4.5 and 4.12) and show their equivalence under the restriction functor γ_n^* (Theorem 4.15), given a stability hypothesis on $C \xrightarrow{p} \Delta^n$. This material will play an important role in [19].

4.1 Notation Let $\gamma_n : \mathrm{sd}_1(\Delta^n) \subset \mathrm{sd}(\Delta^n)$ be the subposet on strings $[k]$ and $[k < k+1]$.

We also introduce convenient notation for convex subposets of Δ^n.

4.2 Notation Let $[i : j] \subset \Delta^n$ denote the subposet on $i \le k \le j$.

Via its inclusion into $\mathrm{sd}(\Delta^n)$, we regard $\mathrm{sd}_1(\Delta^n)$ as a simplicial set over Δ^n (i.e., by the functor that takes the maximum) and as a marked simplicial set (so that each edge $[k] \to [k < k+1]$ is marked). We first state the analogue of Theorem 3.39 for sd_1, whose proof is far simpler.

4.3 Proposition *Let $p : C \to \Delta^n$ be a locally cocartesian fibration such that the fibers admit finite limits and the pushforward functors preserve finite limits. Let $0 \le k < n$, so the subcategories $[0:k] \cong \Delta^k$ and $[k+1:n] \cong \Delta^{n-k-1}$ of Δ^n give a sieve-cosieve decomposition. Then we have adjunctions*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:k]}\big(\mathrm{sd}_1([0:k]), C_{[0:k]}\big) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/[k+1:n]}\big(\mathrm{sd}_1([k+1:n]), C_{[k+1:n]}\big)$$

that exhibit $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)$ as a recollement.

Proof Let $j : \mathrm{sd}_1([0:k]) \to \mathrm{sd}_1(\Delta^n)$ and $i : \mathrm{sd}_1([k+1:n]) \to \mathrm{sd}_1(\Delta^n)$ be the inclusions, so j^* and i^* are defined by restriction along j and i. As in the proof of Lemma 3.37, our hypotheses on p ensure that the three ∞-categories admit finite limits and the functors j^* and i^* are left-exact. Moreover, since equivalences are detected on strings $[k]$, j^* and i^* are jointly conservative. The functor i_* is obtained by p-right Kan extension as in the proof of Proposition 3.36, and its essential image consists of functors $F : \mathrm{sd}_1(\Delta^n) \to C$ such that $F(i)$ is a terminal object in C_i for all $0 \le i \le k$, so $j^* i_*$ is the constant functor at the terminal object.

Finally, we show existence of j_*. Let $\mathrm{sd}_1([0:k])^+$ be the subposet of $\mathrm{sd}_1([0:n])$ on all objects in $\mathrm{sd}_1([0:k])$ and $\{[k<k+1]\}$, with marking inherited from $\mathrm{sd}(\Delta^n)$. Then we have a pushout square of marked simplicial sets

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & (\Delta^1)^\sharp \\ \downarrow & & \downarrow \\ \mathrm{sd}_1([0:k]) & \longrightarrow & \mathrm{sd}_1([0:k])^+ \end{array}$$

so the inclusion $\mathrm{sd}_1([0:k]) \subset \mathrm{sd}_1([0:k])^+$ is $\mathfrak{P}$-anodyne for the categorical pattern $\mathfrak{P}$ defining the locally cocartesian model structure on $s\mathbf{Set}^+_{/\Delta^n}$. We thus obtain a trivial fibration

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:k+1]}\big(\mathrm{sd}_1([0:k])^+, C_{[0:k+1]}\big) \to \mathrm{Fun}^{\mathrm{cocart}}_{/[0:k]}\big(\mathrm{sd}_1([0:k]), C_{[0:k]}\big).$$

On the other hand, given a commutative diagram

$$\begin{array}{ccc} \mathrm{sd}_1([0:k])^+ & \xrightarrow{F} & C \\ \downarrow & \overset{G}{\nearrow} & \downarrow p \\ \mathrm{sd}_1([0:k+1]) & \longrightarrow & \Delta^n \end{array}$$

since $\mathrm{sd}_1([0:k])^+ \times_{\mathrm{sd}_1([0:k+1])} \mathrm{sd}_1([0:k+1])_{[k+1]/} \cong \{[k<k+1]\}$, F admits a p-right Kan extension along $\mathrm{sd}_1([0:k])^+ \subset \mathrm{sd}_1([0:k+1])$ and G is a p-right Kan extension of F if and only if G sends the edge $[k+1] \to [k<k+1]$ to an equivalence. Therefore, we may alternate between anodyne extension

and p-right Kan extension along the filtration

$$\mathrm{sd}_1([0:k]) \subset \mathrm{sd}_1([0:k])^+ \subset \mathrm{sd}_1([0:k+1]) \subset \cdots \subset \mathrm{sd}_1([0:n-1])^+ \subset \mathrm{sd}_1(\Delta^n)$$

to define the functor j_*. Moreover, we see that the essential image of j_* consists of those functors $\mathrm{sd}_1(\Delta^n) \to C$ that send the edges $[l+1] \to [l < l+1]$ to equivalences for all $l \geq k$. □

We next wish to introduce a condition on objects of $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)$, which we term 1-*generated*, that indicates that the data of such objects is essentially determined by their restriction to $\mathrm{sd}_1(\Delta^n)$.

4.4 Notation Given a string $\sigma = [i < i+k]$ in $\mathrm{sd}(\Delta^n)$, let $Q_\sigma \subset \mathrm{sd}(\Delta^n)$ be the subposet on all strings $[i < \cdots < i+k]$. Note that Q_σ is a $(k-1)$-dimensional cube lying in the fiber $\mathrm{sd}(\Delta^n)_{\max=i+k}$ with σ as its minimal element.

4.5 Definition Let $C \to \Delta^n$ be a locally cocartesian fibration and $F : \mathrm{sd}(\Delta^n) \to C$ be a functor that preserves locally cocartesian edges. We say that F is 1-*generated* if for all strings $\sigma = [i < i+k]$ in $\mathrm{sd}(\Delta^n)$, $F|_{Q_\sigma}$ is a limit diagram in C_{i+k}.

Let $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{1\text{-gen}}$ be the full subcategory on the 1-generated objects.

4.6 Lemma *Let $C \to \Delta^n$ be a locally cocartesian fibration whose fibers are stable ∞-categories and whose pushforward functors are exact. Then $F : \mathrm{sd}(\Delta^n) \to C$ is 1-generated if and only if for all string inclusions $e : [i < i+k] \to [i < i+1 < i+k]$ in $\mathrm{sd}(\Delta^n)$, $F(e)$ is an equivalence in C_{i+k}.*

Proof We prove the stronger claim that for fixed $k \geq 2$ and all string inclusions $e_{ij} : \sigma_{ij} = [i < i+j] \to [i < i+1 < i+j]$ with $2 \leq j \leq k$, $F|_{Q_{\sigma_{ij}}}$ is a limit diagram for all $Q_{\sigma_{ij}}$ if and only if $F(e_{ij})$ is an equivalence for all e_{ij}.

We proceed by induction on k. For the base case $k = 2$, given a string inclusion $\sigma = [i < i+2] \to [i < i+1 < i+2]$, the edge is the 1-dimensional cube Q_σ, so $F|_{Q_\sigma}$ is a limit diagram if and only if $F(e)$ is an equivalence. Now let $k > 2$ and suppose we have proven the statement for all $l < k$. Note that in proving either direction of the "if and only if" statement, we may suppose that $F|_{Q_{\sigma_{ij}}}$ is a limit diagram *and* $F(e_{ij})$ for all $2 \leq j < k$, so let us do so.

Consider an edge $e : \sigma = [i < i+k] \to [i < i+1 < i+k]$. For $1 < j < k$, let $Q_{\sigma,j} \subset Q_\sigma$ be the subposet on strings excluding vertices $i+j, \ldots, i+k-1$. Then we have a descending filtration of sieve inclusions

$$Q_\sigma := Q_{\sigma,k} \supset Q_{\sigma,k-1} \supset Q_{\sigma,k-2} \supset \cdots \supset Q_{\sigma,2},$$

where $Q_{\sigma,j}$ is a $(j-1)$-dimensional cube and $Q_{\sigma,2}$ consists only of the edge e. Note that if we let $Q'_{\sigma,j} = Q_{\sigma,j+1} \setminus Q_{\sigma,j}$ for $1 < j < k$, then the minimal element of $Q'_{\sigma,j}$ is given by $\sigma_j = [i < i+j < i+k]$, and if we let $\sigma'_j = [i < i+j]$, then $Q'_{\sigma,j}$ is obtained from $Q_{\sigma'_j}$ by concatenating $i+k$. By the inductive hypothesis and using that the pushforward functors are exact, we get that $F|_{Q'_{\sigma,j}}$ is a limit diagram. Taking total fibers of cubes then shows that $F|_{Q_{\sigma,j}}$ is a limit diagram if and only if $F|_{Q_{\sigma,j-1}}$ is a limit diagram. Traversing the filtration, we conclude that $F|_{Q_\sigma}$ is a limit diagram if and only if $F(e)$ is an equivalence. □

4.7 Lemma *Let $Q = \mathrm{sd}(\Delta^n)_{\max=n}$, D a stable ∞-category, and $f : Q \to D$ a functor. Suppose the following condition holds:*

(∗) *For all string inclusions $e : \sigma \to \sigma'$ in Q obtained by concatenating $[i < k] \to [i < i+1 < k]$ by a (possibly empty) suffix τ, $f(e)$ is an equivalence.*

Then f is a limit diagram if and only if $f([n] \to [n-1 < n])$ is an equivalence.

Proof The proof is similar to that of Lemma 4.6. For $0 \le j < n$, let $Q_{\ge j}$ (resp. $Q_{=j}$) be the subposet on strings σ with minimum $\ge j$ (resp. $= j$). Then $Q_{\ge j}$ is an $(n-j)$-dimensional cube, $Q_{=j} = Q_{\ge j} \setminus Q_{\ge j+1}$ is an $(n-j-1)$-dimensional cube, and we have a descending filtration

$$Q = Q_{\ge 0} \supset Q_{\ge 1} \supset Q_{\ge 2} \supset \cdots \supset Q_{\ge n-1}.$$

Observe that $Q_{=j} = Q_{[j<n]}$, so $f|_{Q_{=j}}$ is a limit diagram under our hypotheses by the proof of Lemma 4.6. Therefore, taking total fibers shows that $f|_{Q_{\ge j}}$ is a limit diagram if and only if $f|_{Q_{\ge j+1}}$ is a limit diagram. Traversing the filtration then proves the claim. □

We continue to assume $C \to \Delta^n$ is a locally cocartesian fibration whose fibers are stable ∞-categories and whose pushforward functors are exact. Observe that we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]), C_{[0:n-1]}\big) & \xrightarrow{\gamma^*_{n-1}} & \mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]), C_{[0:n-1]}\big) \\
{\scriptstyle j^*}\uparrow & & \uparrow{\scriptstyle j^*} \\
\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C) & \xrightarrow{\gamma^*_n} & \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) \\
{\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\
C_n & \xrightarrow{\mathrm{id}} & C_n
\end{array}$$

so in particular γ^*_n is a morphism of stable recollements. However γ_n generally fails to be a *strict* morphism of stable recollements, i.e., the natural transformation

$$i^* j_* \to i^* j_* \gamma^*_{n-1}$$

is typically not an equivalence.

4.8 Lemma *Suppose $F : \mathrm{sd}(\Delta^n) \to C$ is 1-generated. Then the comparison map*

$$i^* j_* j^* F = (j_* j^* F)(n) \to i^* j_* \gamma^*_{n-1} j^* F = (j_*(F|_{\mathrm{sd}_1([0:n-1])}))(n)$$

is an equivalence.

Proof Let $K \subset \mathrm{sd}(\Delta^n)$ be the subposet on strings σ with $\max(\sigma) = n$ and $\sigma \ne n$. By the formulas computing j_* given in Theorem 3.32 and Proposition 4.3, we see that the comparison map is given by the canonical map from the limit of $F|_K$ to $F([n-1 < n])$. Since F is 1-generated, by Lemma 4.6 the conditions of Lemma 4.7 are satisfied, so this canonical map is an equivalence. □

4.9 Definition For the functor j_* defined as in Corollary 3.34 with respect to $[0:n-1]$ and $\{n\}$, we say that a functor $F:\mathrm{sd}([0:n-1])\to C_{[0:n-1]}$ is *$+$-1-generated* if both F and j_*F are 1-generated. Let

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]),C_{[0:n-1]}\big)^{+}_{1\text{-gen}}$$

be the full subcategory on the $+$-1-generated objects.

4.10 Lemma *We have adjunctions*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]),C_{[0:n-1]}\big)^{+}_{1\text{-gen}} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n),C)_{1\text{-gen}} \underset{i_*}{\overset{i^*}{\rightleftarrows}} C_n$$

that exhibit $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n),C)_{1\text{-gen}}$ *as a stable recollement.*

Proof Clearly, we may define j_*, i^*, and i_* to be the restrictions of the corresponding functors for the adjunctions of Theorem 3.39. The only subtle point is that given $F:\mathrm{sd}(\Delta^n)\to C$ which is 1-generated, we require that the localization j_*j^*F is also 1-generated. But this holds, since $F\simeq j_*j^*F$ except possibly at $n\in\mathrm{sd}(\Delta^n)$ and the 1-generated condition ignores n. Therefore, we may also define j^* as the restricted functor, and the recollement conditions are then immediate. □

4.11 Corollary *The restriction* $\gamma_n^*:\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n),C)_{1\text{-gen}}\to\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n),C)$ *is a strict morphism of stable recollements with respect to Lemma 4.10 and Proposition 4.3.*

Proof This follows immediately from Lemma 4.8. □

We want to apply Corollary 4.11 to show that γ_n^* is an equivalence (in fact, a trivial fibration) onto its essential image. To understand this image as a condition on objects in the codomain, we introduce the following definition. For $0\le i<j\le n$, let $\tau_i^j:C_i\to C_j$ denote the pushforward functor encoded by the locally cocartesian fibration.

4.12 Definition We say that a functor $f:\mathrm{sd}_1(\Delta^n)\to C$ is *extendable* if for every string $[i<i+1<i+k]$ in $\mathrm{sd}(\Delta^n)$, the canonical map in C_{i+k}

$$\tau_i^{i+k}f(i)\to(\tau_{i+1}^{k}\circ\tau_i^{i+1})f(i)$$

encoded by the locally cocartesian fibration is an equivalence. Let

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n),C)_{\mathrm{ext}}$$

denote the full subcategory on the extendable objects.

4.13 Definition For the functor j_* defined as in Proposition 4.3 with respect to $[0:n-1]$ and $\{n\}$, we say that a functor $f:\mathrm{sd}_1([0:n-1])\to C$ is *$+$-extendable* if both f and j_*f are extendable. Let

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]),C_{[0:n-1]}\big)^{+}_{\mathrm{ext}}$$

be the full subcategory on the $+$-extendable objects.

Note that the extendability condition becomes stronger through considering the additional strings in $\mathrm{sd}(\Delta^n)$; for example, extendability is no condition on $f : \mathrm{sd}_1([0:1]) \to C_{[0:1]}$, but we acquire the condition that the map $\tau_0^2 f(0) \to \tau_1^2 \tau_0^1 f(0)$ is an equivalence upon enlarging to Δ^2. Let us first state the evident counterpart to Lemma 4.10.

4.14 Lemma *We have adjunctions*

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]), C_{[0:n-1]}\big)^+_{\mathrm{ext}} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}} \underset{i_*}{\overset{i^*}{\rightleftarrows}} C_n$$

that exhibit $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}}$ *as a stable recollement.*

Proof This is immediate from restricting the recollement of Proposition 4.3. □

We have assembled all the ingredients needed to prove Theorem 4.15. Note that by Lemma 4.7, γ_n^* of a 1-generated object is extendable, so the functor of Theorem 4.15 is well defined.

4.15 Theorem *Suppose $C \to \Delta^n$ is a locally cocartesian fibration whose fibers are stable ∞-categories and whose pushforward functors are exact. Then the functor*

$$\gamma_n^* : \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{\text{1-gen}} \to \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}}$$

is an equivalence of ∞-categories.

Proof We proceed by induction on n. For the base cases $n = 0$ and $n = 1$, the result is trivial. Let $n > 1$ and suppose we have proven the theorem for all $k < n$. By the inductive hypothesis, γ_{n-1}^* is an equivalence. Observe that γ_{n-1}^* restricts to a functor

$$(\gamma_{n-1}^*)^+ : \mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}([0:n-1]), C_{[0:n-1]}\big)^+_{\text{1-gen}} \to \mathrm{Fun}^{\mathrm{cocart}}_{/[0:n-1]}\big(\mathrm{sd}_1([0:n-1]), C_{[0:n-1]}\big)^+_{\mathrm{ext}}.$$

If we let $(\gamma_{n-1}^*)^{-1}$ be an inverse functor, then by Lemma 4.6, if $f : \mathrm{sd}_1([0:n-1]) \to C_{[0:n-1]}$ is $+$-extendable, then $(\gamma_{n-1}^*)^{-1}(f)$ is $+$-1-generated. Therefore, $(\gamma_{n-1}^*)^+$ is also an equivalence. By Corollary 4.11 (but replacing the codomain there with the recollement of Lemma 4.14) and the two-out-of-three property of equivalences for a strict morphism of stable recollements (Remark 2.7), we deduce that γ_n^* is an equivalence. □

4.16 Observation To make better use of Theorem 4.15, let us further unpack $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)$. Note that we may write $\mathrm{sd}_1(\Delta^n)$ as the union of marked simplicial sets

$$\mathrm{sd}([0:1]) \cup_1 \mathrm{sd}([1:2]) \cup_2 \cdots \cup_n \mathrm{sd}([n-1:n]),$$

so we obtain a fiber product decomposition

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/[0:1]}\big(\mathrm{sd}([0:1]), C_{[0:1]}\big) \times_{C_1} \cdots \times_{C_{n-1}} \mathrm{Fun}^{\mathrm{cocart}}_{/[n-1:n]}\big(\mathrm{sd}([n-1:n]), C_{[n-1:n]}\big).$$

Let $\tau_i^{i+1} : C_i \to C_{i+1}$ be the pushforward functors as before, and with respect to the trivial fibration (induced by the inner anodyne spine inclusion $[0:1] \cup_1 \cdots \cup_{n-1} [n-1:n] \to \Delta^n$)

$$\mathrm{Fun}(\Delta^n, \mathbf{Cat}_\infty) \xrightarrow{\simeq} \mathrm{Fun}([0:1], \mathbf{Cat}_\infty) \times_1 \cdots \times_{n-1} \mathrm{Fun}([n-1:n], \mathbf{Cat}_\infty),$$

let $\tau_\bullet : \Delta^n \to \mathbf{Cat}_\infty$ be a functor lifting the τ_i^{i+1}. Let $C^\vee \to (\Delta^n)^{\mathrm{op}}$ be a cartesian fibration classified by $\tau_\bullet$. Then if we let $[i+1:i] = [i:i+1]^{\mathrm{op}}$, we have that $(C^\vee)_{[i+1:i]} \simeq (C_{[i:i+1]})^\vee$ where the right-hand $(-)^\vee$ denotes the dual cartesian fibration of the cocartesian fibration $C_{[i:i+1]} \to [i:i+1]$. Then by Observation 2.14, we have an equivalences of ∞-categories

$$\mathrm{Fun}^{\mathrm{cocart}}_{/[i:i+1]}\big(\mathrm{sd}([i:i+1]), C_{[i:i+1]}\big) \simeq \mathrm{Fun}_{/[i+1:i]}([i+1:i], C^\vee_{[i+1:i]}) \simeq \mathrm{Ar}(C_{i+1}) \times_{\mathrm{ev}_1, C_{i+1}, \tau_i^{i+1}} C_i.$$

Again using that the spine inclusion is inner anodyne, we obtain the following proposition.

4.17 Proposition *We have equivalences of ∞-categories*

$$\begin{aligned}\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C) &\simeq \mathrm{Fun}_{/(\Delta^n)^{\mathrm{op}}}((\Delta^n)^{\mathrm{op}}, C^\vee) \\ &\simeq \mathrm{Ar}(C_n) \times_{C_n} \mathrm{Ar}(C_{n-1}) \times_{C_{n-1}} \cdots \times_{C_2} \mathrm{Ar}(C_1) \times_{C_1} C_0,\end{aligned}$$

where in the fiber product, the maps $\mathrm{Ar}(C_k) \to C_k$ are given by evaluation at the target, and the maps $\mathrm{Ar}(C_k) \to C_{k+1}$ are given by composing evaluation at the source with $\tau_k^{k+1} : C_k \to C_{k+1}$.

4.18 Notation Let $(\mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0)_{\mathrm{ext}}$ denote the full subcategory of $\mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0$ given by restricting the equivalence of Proposition 4.17 to $\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}_1(\Delta^n), C)_{\mathrm{ext}}$ on the left-hand side.

Then we can also express Theorem 4.15 as

$$\begin{array}{ccc}\mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C)_{1\text{-gen}} & \xrightarrow[\simeq]{\gamma_n^*} & (\mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0)_{\mathrm{ext}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Fun}^{\mathrm{cocart}}_{/\Delta^n}(\mathrm{sd}(\Delta^n), C) & \xrightarrow{\gamma_n^*} & \mathrm{Ar}(C_n) \times_{C_n} \cdots \times_{C_1} C_0\end{array}$$

which is more concrete in practice (e.g., for the example of C_{p^n}-spectra explained in Remark 4.19).

4.19 Remark The type of iterated fiber product occurring in Proposition 4.17 appears in the work of Nikolaus and Scholze when they describe the data of a (genuine) C_{p^n}-spectrum X whose geometric fixed points (except possibly $\Phi^{C_{p^n}} X$ and $\Phi^{C_{p^{n-1}}} X$) are all bounded below; see [17, Remark II.4.8].[18] In fact, Theorem 4.15 together with [2, Theorem E] applies to give a proof of [17, Remark II.4.8] that is independent of the machinery of "coalgebras for endofunctors" developed in [17, §II.5]. We will explain this in more detail in [19, §3.2] as well as prove a dihedral refinement of this assertion. For now, we give an overview of the argument:

[18] Nikolaus and Scholze elide the subtlety involving the lack of bounded-below hypotheses needed on $\Phi^{C_{p^n}} X$ and $\Phi^{C_{p^{n-1}}} X$.

By [2, Theorem E], for any finite group G with subconjugacy poset P there exists a locally cocartesian fibration $\mathbf{Sp}^G_{\phi\text{-locus}} \to P$ whose right-lax limit is canonically equivalent[19] to the ∞-category $\mathbf{Sp}^G$ of (genuine) G-spectra. Furthermore, for every subgroup $H \le G$, $(\mathbf{Sp}^G_{\phi\text{-locus}})_H \simeq \mathbf{Sp}^{hW_G H} = \mathrm{Fun}(BW_G H, \mathbf{Sp})$ where $W_G H = N_G H/H$ is the Weyl group, and the equivalence transports a G-spectrum X to its associated diagram of geometric fixed points $\{\Phi^H X \in \mathbf{Sp}^{hW_G H}\}$. If $G = C_{p^n}$, then we may identify the pushforward functor associated to $[C_{p^k} \le C_{p^m}]$ with the proper Tate construction $(-)^{\tau C_{p^{m-k}}}$ endowed with residual action; in particular, when $m = k+1$, this is the ordinary Tate construction $(-)^{tC_p}$. In addition, under the equivalence $\mathbf{Sp}^{C_{p^n}} \simeq \lim^{\mathrm{rlax}} \mathbf{Sp}^{C_{p^n}}_{\phi\text{-locus}}$ and the isomorphism $P \cong [n]$, the map γ_n^* identifies with the forgetful functor

$$\mathbf{Sp}^{C_{p^n}} \to \mathbf{Sp}^{hC_{p^n}} \times_{(-)^{tC_p}, \mathbf{Sp}^{hC_{p^{n-1}}}, \mathrm{ev}_1} \mathrm{Ar}(\mathbf{Sp}^{hC_{p^{n-1}}}) \times_{(-)^{tC_p}\,\mathrm{ev}_0, \mathbf{Sp}^{hC_{p^{n-2}}}, \mathrm{ev}_1} \mathrm{Ar}(\mathbf{Sp}^{hC_{p^{n-2}}}) \times \cdots \times \mathrm{Ar}(\mathbf{Sp})$$

that sends X to $[\Phi^e X, \Phi^{C_p} X \to (\Phi^e X)^{tC_p}, \ldots, \Phi^{C_{p^n}} X \to (\Phi^{C_{p^{n-1}}} X)^{tC_p}]$ where the maps are the usual ones. The assertion made in [17, Remark II.4.8] is that γ_n^* restricts to an equivalence

$$\mathbf{Sp}_+^{C_{p^n}} \xrightarrow{\simeq} \mathbf{Sp}_+^{hC_{p^n}} \times_{(-)^{tC_p}, \mathbf{Sp}^{hC_{p^{n-1}}}, \mathrm{ev}_1} \mathrm{Ar}'(\mathbf{Sp}^{hC_{p^{n-1}}}) \times_{(-)^{tC_p}\,\mathrm{ev}_0, \mathbf{Sp}^{hC_{p^{n-2}}}, \mathrm{ev}_1} \mathrm{Ar}'(\mathbf{Sp}^{hC_{p^{n-2}}}) \times \cdots \times \mathrm{Ar}(\mathbf{Sp}),$$

where:

- $\mathbf{Sp}_+^{C_{p^n}} \subset \mathbf{Sp}^{C_{p^n}}$ denotes the full subcategory of C_{p^n}-spectra spanned by those objects whose geometric fixed points (except possibly $\Phi^{C_{p^n}}$ and $\Phi^{C_{p^{n-1}}}$) are all bounded below.
- $\mathbf{Sp}_+^{hC_{p^n}} \subset \mathbf{Sp}^{hC_{p^n}}$ denotes the full subcategory of Borel C_{p^n}-spectra spanned by those objects whose underlying spectrum is bounded below.
- Ar' denotes the full subcategory on arrows whose source is bounded below.

To invoke Theorem 4.15 to deduce this, we need to show that for every $X \in \mathbf{Sp}_+^{C_{p^n}}$, X is 1-generated as an object in $\lim^{\mathrm{rlax}} \mathbf{Sp}^{C_{p^n}}_{\phi\text{-locus}}$. If $n = 2$, this is the content of the *Tate orbit lemma* of [17, Lemma I.2.1] once one identifies the fiber of the natural transformation $\mathrm{can} : (-)^{\tau C_{p^2}} \Rightarrow ((-)^{tC_p})^{tC_{p^2}/C_p}$ encoded by $\mathbf{Sp}^{C_{p^2}}_{\phi\text{-locus}}$ with $((-)_{hC_p})^{tC_{p^2}/C_p}$. Proceeding by induction on n, it is then not difficult to verify that the condition of Lemma 4.6 holds for all $X \in \mathbf{Sp}_+^{C_{p^n}}$; we record this as [19, Corollary 3.40].

5 Reconstruction of sheaves on stratified ∞-topoi

We explain how to apply Theorem 3.39 to prove a reconstruction theorem (Theorem 5.13) for sheaves in an ∞-topos stratified by a finite poset P in the sense of Barwick, Glasman, and Haine (Definition 5.5). We then prove a conjecture of Barwick, Glasman, and Haine by establishing an equivalence (Theorem 5.22) between the ∞-category of P-stratified ∞-topoi and that of *toposic* locally cocartesian fibrations over P^{op} (Definition 5.11).

In this section, we will regard the poset P as a topological space via the Alexandroff topology. To begin with, we recall the basic structure theory of recollements of ∞-topoi.

[19] The comparison functor is defined analogously to the functor (1-2) in Theorem B; see [20, Construction 2.43].

5.1 Example Let $\mathcal{X}$ be an ∞-topos and U a (-1)-truncated object. The slice ∞-topos $\mathcal{X}_{/U}$ is said to be an *open subtopos* of $\mathcal{X}$ [12, §6.3.5].[20] Let $\mathcal{X}_{\backslash U} = \{x \in \mathcal{X} : x \times U \xrightarrow{\simeq} U\} \subset \mathcal{X}$. The set $\mathcal{X}_{\backslash U}$ is the *closed subtopos of $\mathcal{X}$ complementary to U* [12, Definition 7.3.2.6]. We then have a diagram of adjunctions

$$\mathcal{X}_{/U} \underset{j_*}{\overset{j_!}{\underset{\longleftarrow j^* \longrightarrow}{\hookrightarrow}}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{X}_{\backslash U}$$

that exhibits $(\mathcal{X}_{/U}, \mathcal{X}_{\backslash U})$ as a recollement of $\mathcal{X}$. Conversely, by [14, Proposition A.8.15], given a left-exact accessible functor $\phi : \mathcal{U} \to \mathcal{Z}$ between ∞-topoi, the fiber product $\mathcal{X} := \mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1, \mathcal{Z}, \phi} \mathcal{U}$ is an ∞-topos and there exists a uniquely determined (-1)-truncated object U such that $\mathcal{U} \simeq \mathcal{X}_{/U}$ and $\mathcal{Z} \simeq \mathcal{X}_{\backslash U}$ compatibly with the adjunctions to $\mathcal{X}$.

In what follows, we will generically use the notation $j_! \dashv j^* \dashv j_*$ and $i^* \dashv i_*$ for these functors arising from a recollement on an ∞-topos.

5.2 Definition A *locale* is a 0-topos, i.e., a poset L such that L admits infinite joins $\bigvee_\alpha x_\alpha$ (so that L is presentable) and infinite joins distribute over finite meets.

5.3 Example Let $\mathcal{X}$ be an ∞-topos. Then its full subcategory $\mathbf{Open}(\mathcal{X})$ of (-1)-truncated objects is a locale. Note that $\mathbf{Open}(\mathcal{X})$ is isomorphic to the poset of open subtopoi of $\mathcal{X}$ (embedded in $\mathcal{X}$ via $j_!$) via the assignment $U \mapsto \mathcal{X}_{/U}$. Also, if X is a topological space, then $\mathbf{Open}(\mathbf{Shv}(X))$ is isomorphic to the poset $\mathbf{Open}(X)$ of open sets in X. If P is a poset equipped with the Alexandroff topology, then these are precisely the cosieves in P.

5.4 Example Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category and suppose there is some regular cardinal κ such that the unit and tensor product restrict to define a symmetric monoidal structure on the full subcategory $\mathcal{C}^\kappa$ of κ-compact objects in $\mathcal{C}$. Then the set of radical thick $\otimes$-ideals in $\mathcal{C}^\kappa$ forms a coherent locale [10, Theorem 3.1.9].

5.5 Definition [5, Definition 8.2.1] Let P be a poset and $\mathcal{X}$ an ∞-topos. A *P-stratification of $\mathcal{X}$* is a geometric morphism $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ of ∞-topoi, or equivalently a geometric morphism $\pi_* : \mathbf{Open}(\mathcal{X}) \to \mathbf{Open}(P)$ of locales. We also say that the data $(\mathcal{X}, \pi_*)$ comprises that of a *P-stratified ∞-topos*.

In the next remark, we consider $P^{\mathrm{op}} \subset \mathbf{Open}(P)$ as a subposet via the map $p \mapsto P^{\geq p}$.

5.6 Remark Via the assignment $\pi_* \mapsto \pi^*|_{P^{\mathrm{op}}}$, geometric morphisms $\pi_* : \mathbf{Open}(\mathcal{X}) \to \mathbf{Open}(P)$ are in bijective correspondence with maps of posets $f : P^{\mathrm{op}} \to \mathbf{Open}(\mathcal{X})$ such that:

(1) $\bigvee_{p \in P} f(p) = \mathbb{1}$.

(2) For every $p, q \in P$, $\bigvee_{r \geq p,q} f(r) \xrightarrow{\cong} f(p) \times f(q)$.

Indeed, given any map of posets $f : P^{\mathrm{op}} \to \mathbf{Open}(\mathcal{X})$, its left Kan extension $F : \mathbf{Open}(P) \to \mathbf{Open}(\mathcal{X})$ admits a right adjoint G defined by $G(U) = \{p \in P : f(p) \leq U\}$, and F is then left-exact if and only if f satisfies conditions (1) and (2).

[20] Lurie uses the terminology "étale geometric morphism".

Furthermore, (2) is equivalent to the following factorization property: for every $p, q \in P$, the square

$$\begin{tikzcd} \mathcal{X}_{/\bigvee_{r \geq p,q} f(r)} \arrow[r, hook, "j_!"] & \mathcal{X}_{/f(p)} \\ \mathcal{X}_{/f(q)} \arrow[u, two heads, "j^*"] \arrow[r, hook, "j_!"] & \mathcal{X} \arrow[u, two heads, "j^*"'] \end{tikzcd}$$

commutes. We thus see that the notion of a P-stratification of $\mathcal{X}$ is the evident toposic analogue of the notion of a P-stratification of a presentable stable ∞-category in the sense of [2, Definition 2.4.3]. Conversely, in view of Example 5.4 one can sometimes give a "localic" reformulation of [2, Definition 2.4.3] (or rather, its symmetric monoidal refinement [2, Definition 4.3.2]).

We now proceed to notate various subtopoi associated to a P-stratified ∞-topos.

5.7 Notation [5, Notation 8.2.3] Let $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ be a P-stratification of $\mathcal{X}$. In what follows, all fiber products are computed in $\mathbf{Top}_\infty$. For any open subset $O \subset P$, we let

$$\mathcal{X}_O := \mathcal{X}_{/\pi^* O} \simeq \mathcal{X} \times_{\mathbf{Shv}(P)} \mathbf{Shv}(O).$$

Dually, for any closed subset $Z \subset P$, we let

$$\mathcal{X}_Z := \mathcal{X}_{\backslash \pi^*(P \backslash Z)} \simeq \mathcal{X} \times_{\mathbf{Shv}(P)} \mathbf{Shv}(Z).$$

For any $p \in P$, we define the *p-th stratum* of $(\mathcal{X}, \pi_*)$ to be

$$\mathcal{X}_p := \mathcal{X} \times_{\mathbf{Shv}(P)} \mathbf{Shv}(\{p\}).$$

5.8 Notation In Notation 5.7, the p-th stratum $\mathcal{X}_p$ is the closed complement of $\mathcal{X}_{P^{>p}}$ in $\mathcal{X}_{P^{\geq p}} = \mathcal{X}_{/\pi^*(p)}$, or alternatively the open complement of $\mathcal{X}_{P^{<p}}$ in $\mathcal{X}_{P^{\leq p}}$. We then have the adjunction

$$\Phi^p : \mathcal{X} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{X}_{/\pi^*(p)} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{X}_p : \rho_p,$$

in which ρ_p is a geometric morphism.

5.9 Remark Let $\pi_* : \mathcal{X} \to \mathbf{Shv}(P)$ be a P-stratification of $\mathcal{X}$ and suppose $p, q \in P$ such that $p \not\geq q$. Then $\Phi^q \rho_p$ is homotopic to the constant map at the final object. Indeed, by Remark 5.6 we have a factorization of $\Phi^q \rho_p$ as

$$\begin{tikzcd} \mathcal{X}_p \arrow[r, hook, "i_*"] & \mathcal{X}_{/\pi^*(p)} \arrow[r, hook, "j_*"] \arrow[d, "j^*"] & \mathcal{X} \arrow[d, "j^*"] \\ & \mathcal{X}_{/\pi^*(P^{\geq p,q})} \arrow[r, hook, "j_*"] & \mathcal{X}_{/\pi^*(q)} \arrow[d, "i^*"] \\ & & \mathcal{X}_q \end{tikzcd}$$

and since $p \notin P^{\geq p,q}$, the composite $j^* i_* : \mathcal{X}_p \to \mathcal{X}_{/\pi^*(P^{\geq p,q})}$ is homotopic to the constant map at the final object.

Given a P-stratified ∞-topos $(\mathcal{X}, \pi_*)$, we may construct its associated *gluing diagram* in the same manner as [2, Definition 2.5.7].

5.10 Construction Let $\mathcal{G}(\mathcal{X}) = \{(x, p) : x \in \mathcal{X}_p\} \subset \mathcal{X} \times P^{\mathrm{op}}$, where $\mathcal{X}_p \subset \mathcal{X}$ via ρ_p. The projection

$$\lambda : \mathcal{G}(\mathcal{X}) \to P^{\mathrm{op}}$$

is then a locally cocartesian fibration with fibers $\mathcal{X}_p$ such that for all $q \leq p$, the corresponding pushforward functor $\Gamma^q_p : \mathcal{X}_p \to \mathcal{X}_q$ is given by $\Phi^q \circ \rho_p$ (see [2, Observation 2.5.6]).

We codify the structure of $\lambda : \mathcal{G}(\mathcal{X}) \to P^{\mathrm{op}}$ by means of the following definition.

5.11 Definition We call a locally cocartesian fibration $\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}$ *toposic* if its fibers are ∞-topoi and its pushforward functors are left-exact and accessible.

If P is finite, we will show that taking the limit in $\mathcal{X}$ furnishes an equivalence $\Theta_P : \lim^{\mathrm{rlax}} \mathcal{G}(\mathcal{X}) \xrightarrow{\simeq} \mathcal{X}$, thereby proving a *reconstruction theorem* for $(\mathcal{X}, \pi_*)$. First, we note:

5.12 Lemma *Let P be a finite poset and $\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}$ a toposic locally cocartesian fibration. Then the right-lax limit $\mathcal{X} = \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \widehat{\mathcal{X}})$ is an ∞-topos. Moreover, any cosieve $O \subset P$ determines a recollement of $\mathcal{X}$ with open subtopos given by the right-lax limit of $\lambda|_{O^{\mathrm{op}}}$ and complementary closed subtopos given by the right-lax limit of $\lambda|_{(P \setminus O)^{\mathrm{op}}}$.*

Proof Given Theorem 3.39 and proceeding by induction on the cardinality of P, the first part follows from the known statement for recollements of ∞-topoi recalled in Example 5.1. The second statement then follows by Theorem 3.39 again. □

Consider now the functor $\Theta_P : \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X})) \to \mathcal{X}$ that sends a functor $f : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})$ to $\lim_{\mathrm{sd}(P^{\mathrm{op}})}(\mathrm{pr}_{\mathcal{X}} \circ f)$.

5.13 Theorem *Suppose P is a finite poset and let $(\mathcal{X}, \pi_*)$ be a P-stratified ∞-topos. Then*

$$\Theta_P : \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X})) \to \mathcal{X}$$

is an equivalence.

Proof To ease notation, let $\mathcal{X}' := \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X}))$. We proceed by induction on the cardinality of P. We may suppose that P is nonempty. Choose a minimal element $b \in P$ and let $O = P \setminus \{b\}$. Let

$$(\pi_O)_* : \mathbf{Open}(\mathcal{X}_{/\pi^*(O)}) \to \mathbf{Open}(O)$$

denote the O-stratification of the open subtopos $\mathcal{X}_{/\pi^*(O)}$ restricted from that of $\mathcal{X}$. Note that $\mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}} \simeq \mathcal{G}(\mathcal{X}_{/\pi^*(O)})$ as locally cocartesian fibrations over O^{op}. Indeed, one observes that for all $p \in O$, the fully faithful inclusion $\rho_p : \mathcal{X}_p \hookrightarrow \mathcal{X}$ factors through $\mathcal{X}_{/\pi^*(O)}$ and identifies $\mathcal{X}_p$ with $(\mathcal{X}_{/\pi^*(O)})_p$ embedded via $(\rho_O)_p$, so the inclusion $\mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}} \subset \mathcal{X} \times O^{\mathrm{op}}$ factors through $\mathcal{X}_{/\pi^*(O)}$ (embedded via j_* in $\mathcal{X}$) and identifies with $\mathcal{G}(\mathcal{X}_{/\pi^*(O)})$.

Let $(\mathcal{X}_{/\pi^*(O)})' := \mathrm{Fun}^{\mathrm{cocart}}_{/O^{\mathrm{op}}}(\mathrm{sd}(O^{\mathrm{op}}), \mathcal{G}(\mathcal{X}_{/\pi^*(O)}))$ and write

$$\Theta_O : (\mathcal{X}_{/\pi^*(O)})' \xrightarrow{\mathrm{pr}_*} \mathrm{Fun}(\mathrm{sd}(O^{\mathrm{op}}), \mathcal{X}_{/\pi^*(O)}) \xrightarrow{\lim} \mathcal{X}_{/\pi^*(O)}.$$

We now show that $\Theta_P : \mathcal{X}' \to \mathcal{X}$ is a morphism of recollements from $((\mathcal{X}_{/\pi^*(O)})', \mathcal{X}_b)$ to $(\mathcal{X}_{/\pi^*(O)}, \mathcal{X}_b)$:

(1) We have a distinguished homotopy making the diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{j^*=\mathrm{res}} & (\mathcal{X}_{/\pi^*(O)})' \\ \downarrow{\scriptstyle \Theta_P} & & \downarrow{\scriptstyle \Theta_O} \\ \mathcal{X} & \xrightarrow{j^*} & \mathcal{X}_{/\pi^*(O)} \end{array}$$

commute as follows: given $[f : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})] \in \mathcal{X}'$, consider the composite

$$g : \mathrm{sd}(P^{\mathrm{op}}) \xrightarrow{f} \mathcal{G}(\mathcal{X}) \xrightarrow{\mathrm{pr}} \mathcal{X} \xrightarrow{j^*} \mathcal{X}_{/\pi^*(O)},$$

whose limit is $j^*\Theta_P(f)$. Then since $\mathcal{X}_b \xrightarrow{i_*=\rho_b} \mathcal{X} \xrightarrow{j^*} \mathcal{X}_{/\pi^*(O)}$ is homotopic to the constant map at the final object, g is a right Kan extension of its restriction g_0 to $\mathrm{sd}(O^{\mathrm{op}})$. But since the limit of g_0 is $\Theta_O j^*(f)$, this supplies an equivalence $j^*\Theta_P(f) \simeq \Theta_O j^*(f)$ that is natural in f.

(2) Likewise, we may construct an equivalence

$$i^*\Theta_P = \Phi^b\Theta_P \simeq \mathrm{ev}_b : \mathcal{X}' \to \mathcal{X}_b$$

as follows: Let $[f : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})] \in \mathcal{X}'$ and consider the composite

$$g : \mathrm{sd}(P^{\mathrm{op}}) \xrightarrow{f} \mathcal{G}(\mathcal{X}) \xrightarrow{\mathrm{pr}} \mathcal{X} \xrightarrow{i^*} \mathcal{X}_b.$$

If $a \not\geq b$, then the composite $\mathcal{X}_a \overset{\rho_a}{\hookrightarrow} \mathcal{X} \overset{\Phi^b}{\twoheadrightarrow} \mathcal{X}_b$ is homotopic to the constant map at the final object by Remark 5.9. Consequently, g is the right Kan extension of its restriction to $\mathrm{sd}((P^{\geq b})^{\mathrm{op}})$. Let $\mathrm{sd}^+((P^{>b})^{\mathrm{op}})$ be the subposet on strings ending at b (in P^{op}) and note that $\mathrm{sd}((P^{>b})^{\mathrm{op}}) \cong \mathrm{sd}^+((P^{>b})^{\mathrm{op}})$ via the "append b" map. We then have a pullback square

$$\begin{array}{ccc} \lim g|_{\mathrm{sd}((P^{\geq b})^{\mathrm{op}})} & \longrightarrow & \lim g|_{\mathrm{sd}((P^{>b})^{\mathrm{op}})} \\ \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} \\ g(b) & \longrightarrow & \lim g|_{\mathrm{sd}^+((P^{>b})^{\mathrm{op}})} \end{array}$$

in which γ is induced by the "append b" homotopy $\mathrm{sd}((P^{>b})^{\mathrm{op}}) \times [1] \hookrightarrow \mathrm{sd}((P^{\geq b})^{\mathrm{op}})$. For all strings $\sigma = [p_1 > \cdots > p_n]$ in $(P^{>b})^{\mathrm{op}}$, letting $\sigma^+ := [p_1 > \cdots > p_n > b]$ we note that $g(\sigma \subset \sigma^+)$ is an equivalence. Therefore, γ and hence γ' is an equivalence, and this is clearly natural in the input f.

We conclude that we have a morphism of recollements

$$\begin{array}{ccccc} (\mathcal{X}_{/\pi^*(O)})' & \xleftarrow{j^*=\mathrm{res}} & \mathcal{X}' & \xrightarrow{i^*=\mathrm{ev}_b} & \mathcal{X}_b \\ \downarrow{\scriptstyle \Theta_O} & & \downarrow{\scriptstyle \Theta_P} & & \downarrow{\scriptstyle =} \\ \mathcal{X}_{/\pi^*(O)} & \xleftarrow{j^*} & \mathcal{X} & \xrightarrow{i^*=\Phi^b} & \mathcal{X}_b \end{array}$$

By the inductive hypothesis, Θ_O is an equivalence. To then deduce that Θ_P is an equivalence, by Remark 2.7 it remains to observe that we have a *strict* morphism of recollements, i.e., that the adjoint square

$$\begin{array}{ccc} (\mathcal{X}_{/\pi^*(O)})' & \xrightarrow{i^* j_*} & \mathcal{X}_b \\ \downarrow{\scriptstyle \Theta_O} & & \downarrow{\scriptstyle =} \\ \mathcal{X}_{/\pi^*(O)} & \xrightarrow{i^* j_*} & \mathcal{X}_b \end{array}$$

commutes. But using that the lower $i^* j_* : \mathcal{X}_{/\pi^*(O)} \to \mathcal{X}_b$ is left-exact, this amounts to our formula for the gluing functor $i^* j_* : (\mathcal{X}_{/\pi^*(O)})' \to \mathcal{X}_b$ of the recollement on $\mathcal{X}'$ that we gave in Theorem 3.32. □

In fact, we can elaborate upon Theorem 5.13 to also reconstruct the P-stratification of $\mathcal{X}$.

5.14 Construction Let P be a finite poset, $\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}$ a toposic locally cocartesian fibration, and $\mathcal{X} = \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \widehat{\mathcal{X}})$ its right-lax limit, which is an ∞-topos by Lemma 5.12. Given a cosieve $O \subset P$, let $\pi^*(O) \in \mathcal{X}$ be the uniquely determined (-1)-truncated object such that $\mathrm{Fun}^{\mathrm{cocart}}_{/O^{\mathrm{op}}}(\mathrm{sd}(O^{\mathrm{op}}), \widehat{\mathcal{X}}|_{O^{\mathrm{op}}}) \simeq \mathcal{X}_{/\pi^*(O)}$. Then we may define a P-stratification of $\mathcal{X}$ by the map of posets

$$\pi^* : \mathbf{Open}(P) \to \mathbf{Open}(\mathcal{X}),$$

as it is clear that π^* preserves joins and meets (e.g., in view of Remark 5.6).

5.15 Corollary *Let P be a finite poset and $(\mathcal{X}, \pi_*)$ a P-stratified ∞-topos. The P-stratification of $\mathcal{X}' := \mathrm{Fun}^{\mathrm{cocart}}_{/P^{\mathrm{op}}}(\mathrm{sd}(P^{\mathrm{op}}), \mathcal{G}(\mathcal{X}))$ given by Construction 5.14 coincides with that of $\mathcal{X}$ under the equivalence Θ_P of Theorem 5.13.*

Proof For every cosieve $O \subset P$, let $(\mathcal{X}_{/\pi^*(O)})' := \mathrm{Fun}^{\mathrm{cocart}}_{/O^{\mathrm{op}}}(\mathrm{sd}(O^{\mathrm{op}}), \mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}})$ and note that as in the proof of Theorem 5.13 that $\mathcal{G}(\mathcal{X})|_{O^{\mathrm{op}}} \simeq \mathcal{G}(\mathcal{X}_{/\pi^*(O)})$. By Theorem 5.13, $\Theta_O : (\mathcal{X}_{/\pi^*(O)})' \to \mathcal{X}_{/\pi^*(O)}$ is an equivalence. To then see that $(\mathcal{X}_{/\pi^*(O)})'$ identifies with the open subtopos $\mathcal{X}_{/\pi^*(O)}$ under the equivalence Θ_P, it remains to observe that the square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{j^*} & (\mathcal{X}_{/\pi^*(O)})' \\ \downarrow{\scriptstyle \Theta_P} & & \downarrow{\scriptstyle \Theta_O} \\ \mathcal{X} & \xrightarrow{j^*} & \mathcal{X}_{/\pi^*(O)} \end{array}$$

commutes. We may proceed by induction on the cardinality of $P \setminus O$.[21] If $O = P$ or $O = P \setminus \{b\}$, we are

[21] Of course, we could also adapt the proof of Theorem 5.13 to show this directly.

done by the proof of Theorem 5.13. If not, let $b \in P \setminus O$ be a minimal element. We have a factorization

$$\begin{array}{ccccc} \mathcal{X}' & \xrightarrow{j^*} & (\mathcal{X}_{/\pi^*(P\setminus\{b\})})' & \xrightarrow{j^*} & (\mathcal{X}_{/\pi^*(O)})' \\ \downarrow{\scriptstyle \Theta_P} & & \downarrow{\scriptstyle \Theta_{P\setminus\{b\}}} & & \downarrow{\scriptstyle \Theta_O} \\ \mathcal{X} & \xrightarrow{j^*} & \mathcal{X}_{/\pi^*(P\setminus\{b\})} & \xrightarrow{j^*} & \mathcal{X}_{/\pi^*(O)} \end{array}$$

By the inductive hypothesis, both the inner squares commute, hence the outer square commutes. □

5.16 Remark By Corollary 5.15, it follows that given a sheaf $x \in \mathcal{X}$, under the equivalence of Theorem 5.13 x corresponds to a functor $f_x : \mathrm{sd}(P^{\mathrm{op}}) \to \mathcal{G}(\mathcal{X})$ that sends $[p]$ to $\Phi^p(x)$. The equivalence $x \simeq \Theta_P(f_x)$ then "reconstructs" x from its stratumwise values $\Phi^p(x)$ and gluing data thereof.

We next turn to questions of functoriality in the P-stratified ∞-topos.

5.17 Observation Continuing from Example 5.1, we explain how recollements of topoi are functorial in geometric morphisms. In one direction, suppose we are given a commutative square

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\phi} & \mathcal{Z} \\ {\scriptstyle (f_U)_*}\downarrow & & \downarrow{\scriptstyle (f_Z)_*} \\ \mathcal{U}' & \xrightarrow{\phi'} & \mathcal{Z}' \end{array}$$

of ∞-topoi, where $(f_U)_*$, $(f_Z)_*$ are geometric morphisms and ϕ, ϕ' are left-exact accessible functors. Let $\mathcal{X}$ and $\mathcal{X}'$ be the ∞-topoi $\mathrm{Ar}(\mathcal{Z}) \times_{\mathrm{ev}_1,\mathcal{Z},\phi} \mathcal{U}$ and $\mathrm{Ar}(\mathcal{Z}') \times_{\mathrm{ev}_1,\mathcal{Z}',\phi'} \mathcal{U}'$. Then the induced functor $f_* : \mathcal{X} \to \mathcal{X}'$ admits a left adjoint f^* induced by the mate $(f_Z)^*\phi' \Rightarrow \phi(f_U)^*$; explicitly,

$$f^*[u', z' \to \phi'(u')] = [(f_U)^*(u'), (f_Z)^*(z') \to (f_Z)^*\phi'(u') \to \phi(f_U)^*(u')].$$

Moreover, since $(f_U)^*$, $(f_Z)^*$, ϕ, ϕ' are left-exact and $(j^*, i^*) : \mathcal{X} \to \mathcal{U} \times \mathcal{Z}$ creates finite limits, we see that f^* is left-exact. We conclude that f_* is a geometric morphism. Moreover, f_* is a strict morphism of recollements whose left adjoint f^* is a (not necessarily strict) morphism of recollements. Note also that if we identify $\mathcal{U} \simeq \mathcal{X}_{/U}$ and $\mathcal{U}' \simeq \mathcal{X}_{/U'}$ for (-1)-truncated objects U, U', then $f^*(U') \simeq U$.

Conversely, let $\mathcal{X}$ and $\mathcal{X}'$ be ∞-topoi decomposed by recollements $(\mathcal{U}, \mathcal{Z})$ and $(\mathcal{U}', \mathcal{Z}')$ with gluing functors ϕ and ϕ', and suppose $f_* : \mathcal{X} \to \mathcal{X}'$ is a geometric morphism such that both f^* and f_* are morphisms of recollements. Then f_* is necessarily a strict morphism of recollements, and we obtain a commutative square $(f_Z)_*\phi \simeq \phi'(f_U)_*$ as above.

Finally, the theory of recollements implies that these constructions are mutually inverse.

5.18 Definition [5, 8.2.2] A *geometric morphism of P-stratified ∞-topoi* $(\mathcal{X}, \pi_*) \to (\mathcal{Y}, \rho_*)$ is a geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ subject to the condition that the induced diagram of posets

$$\begin{array}{ccc} \mathbf{Open}(\mathcal{X}) & \xrightarrow{\quad f_* \quad} & \mathbf{Open}(\mathcal{Y}) \\ & {\scriptstyle \pi_*}\searrow \quad \swarrow{\scriptstyle \rho_*} & \\ & \mathbf{Open}(P) & \end{array}$$

commutes, i.e., for all cosieves $O \subset P$, $f^*\rho^*(O) \cong \pi^*(O)$.

The collection of P-stratified ∞-topoi and geometric morphisms thereof assembles into an ∞-category $\mathbf{StrTop}_{\infty,P}$. Note also that $\mathbf{StrTop}_{\infty,P} \simeq \mathbf{Top}_\infty \times_{\mathbf{Top}_0} (\mathbf{Top}_0)_{/\mathbf{Open}(P)}$.

5.19 Definition [5, Remark 8.2.7] A *geometric morphism of toposic locally cocartesian fibrations* from $[\lambda : \widehat{\mathcal{X}} \to P^{\mathrm{op}}]$ to $[\xi : \widehat{\mathcal{Y}} \to P^{\mathrm{op}}]$ is a functor $F : \widehat{\mathcal{X}} \to \widehat{\mathcal{Y}}$ over P^{op} such that:

(1) F preserves locally cocartesian edges.

(2) For all $p \in P$, the fiber $F_p : \widehat{\mathcal{X}}_p \to \widehat{\mathcal{Y}}_p$ is a geometric morphism of ∞-topoi.

The collection of toposic locally cocartesian fibrations and geometric morphisms thereof assembles into an ∞-category $\mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}}$ (Barwick, Glasman, and Haine label this ∞-category as $\mathbf{LocCocart}^{\mathrm{lex,top}}_{P^{\mathrm{op}}}$).

5.20 Observation Let $f_* : (\mathcal{X}, \pi_*) \to (\mathcal{Y}, \rho_*)$ be a geometric morphism of P-stratified ∞-topoi. Then for all cosieves $O \subset P$, f_* is a strict morphism of recollements with respect to $(\mathcal{X}_{/\pi^*(O)}, \mathcal{X}_{\backslash \pi^*(O)})$ and $(\mathcal{Y}_{/\rho^*(O)}, \mathcal{Y}_{\backslash \rho^*(O)})$. Moreover, for all maps of posets $Q \to P$, restriction along $\mathbf{Shv}(Q) \to \mathbf{Shv}(P)$ (in $\mathbf{Top}_\infty$) defines a geometric morphism $f'_* : \mathbf{Shv}(Q) \times_{\mathbf{Shv}(P)} \mathcal{X} \to \mathbf{Shv}(Q) \times_{\mathbf{Shv}(P)} \mathcal{Y}$ of Q-stratified ∞-topoi. Consequently, for all $p \in P$, f_* sends the stratum $\mathcal{X}_p$ into $\mathcal{Y}_p$ (with respect to the embeddings ρ_p of Notation 5.8) and we may thus restrict $f_* \times \mathrm{id} : \mathcal{X} \times P^{\mathrm{op}} \to \mathcal{Y} \times P^{\mathrm{op}}$ to obtain a functor

$$\mathcal{G}(f_*) : \mathcal{G}(\mathcal{X}) \to \mathcal{G}(\mathcal{Y})$$

over P^{op} that preserves locally cocartesian edges. We may thereby promote Construction 5.10 to a functor

$$\mathcal{G} : \mathbf{StrTop}_{\infty,P} \to \mathbf{LocCocart}^{\mathrm{lex,top}}_{P^{\mathrm{op}}}.$$

Conversely, suppose P is a finite poset and let $F : \widehat{\mathcal{X}} \to \widehat{\mathcal{Y}}$ be a geometric morphism of toposic locally cocartesian fibrations. Let $\mathcal{X} = \lim^{\mathrm{rlax}} \widehat{\mathcal{X}}$ and $\mathcal{Y} = \lim^{\mathrm{rlax}} \widehat{\mathcal{Y}}$. Let $f_* : \mathcal{X} \to \mathcal{Y}$ denote the functor induced by F. Then by Observation 5.17, Theorem 3.39, and proceeding by induction on the cardinality of P, we see that f_* is a geometric morphism such that for every cosieve $O \subset P$, f_* is a strict morphism of recollements from $(\lim^{\mathrm{rlax}} \widehat{\mathcal{X}}|_{O^{\mathrm{op}}}, \lim^{\mathrm{rlax}} \widehat{\mathcal{X}}|_{(P \backslash O)^{\mathrm{op}}})$ to $(\lim^{\mathrm{rlax}} \widehat{\mathcal{Y}}|_{O^{\mathrm{op}}}, \lim^{\mathrm{rlax}} \widehat{\mathcal{Y}}|_{(P \backslash O)^{\mathrm{op}}})$. It follows that f_* is a geometric morphism of P-stratified ∞-topoi with respect to the P-stratifications of Construction 5.14. Therefore, $\lim^{\mathrm{rlax}}$ promotes to a functor

$$\lim^{\mathrm{rlax}} : \mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}} \to \mathbf{StrTop}_{\infty,P}.$$

Our remaining goal is to prove that $\mathcal{G}$ and $\lim^{\mathrm{rlax}}$ define an adjoint equivalence of ∞-categories. For the proof, we will need to use the following deep result in $(\infty, 2)$-category theory:

5.21 Theorem [2, Lemma B.5.7] *Let $\mathcal{C}, \mathcal{D} \to P^{\mathrm{op}}$ be locally cocartesian fibrations. Then the space $\mathrm{Map}^{\mathrm{llax},R}_{/P^{\mathrm{op}}}(\mathcal{C}, \mathcal{D})$ of left-lax morphisms whose fibers are right adjoints is naturally equivalent to the space $\mathrm{Map}^{\mathrm{rlax},L}_{/P^{\mathrm{op}}}(\mathcal{D}, \mathcal{C})$ of right-lax morphisms whose fibers are left adjoints, with the equivalence implemented fiberwise by passage to adjoints.*

5.22 Theorem *Let P be a finite poset. Then $\mathcal{G}$ and $\lim^{\mathrm{rlax}}$ participate in an adjoint equivalence*

$$\lim^{\mathrm{rlax}} : \mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}} \rightleftarrows \mathbf{StrTop}_{\infty,P} : \mathcal{G}.$$

Proof We proceed as in the proof of [2, Theorem 6.2.6]. Suppose $[\lambda : \widehat{\mathfrak{X}} \to P^{\mathrm{op}}]$ is a toposic locally cocartesian fibration and $(\mathfrak{Y}, \rho_*)$ is a P-stratified ∞-topos. In view of the adjunction const $\dashv \lim^{\mathrm{rlax}}$, we first note that we have a natural equivalence[22]

$$\psi : \mathrm{Map}_{\mathbf{Cat}}(\mathfrak{Y}, \lim^{\mathrm{rlax}} \widehat{\mathfrak{X}}) \xrightarrow{\simeq} \mathrm{Map}^{\mathrm{rlax}}_{/P^{\mathrm{op}}}(\mathfrak{Y} \times P^{\mathrm{op}}, \widehat{\mathfrak{X}}).$$

Since the evaluation functors $\lim^{\mathrm{rlax}} \widehat{\mathfrak{X}} \to \widehat{\mathfrak{X}}_p$ at each $p \in P$ are all left adjoints, ψ restricts to the equivalence ψ' in the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{Pr}^L}(\mathfrak{Y}, \lim^{\mathrm{rlax}} \widehat{\mathfrak{X}}) & \underset{\simeq}{\xrightarrow{\psi'}} & \mathrm{Map}^{\mathrm{rlax},L}_{/P^{\mathrm{op}}}(\mathfrak{Y} \times P^{\mathrm{op}}, \widehat{\mathfrak{X}}) \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ \mathrm{Map}_{\mathbf{Pr}^R}(\lim^{\mathrm{rlax}} \widehat{\mathfrak{X}}, \mathfrak{Y}) & \underset{\simeq}{\xrightarrow{\psi''}} & \mathrm{Map}^{\mathrm{llax},R}_{/P^{\mathrm{op}}}(\widehat{\mathfrak{X}}, \mathfrak{Y} \times P^{\mathrm{op}}) \end{array}$$

We then have the vertical equivalences (with the right-hand one given by Theorem 5.21), yielding the equivalence ψ'' in which a right-adjoint functor $f_* : \lim^{\mathrm{rlax}} \widehat{\mathfrak{X}} \to \mathfrak{Y}$ transports to a functor $F : \widehat{\mathfrak{X}} \to \mathfrak{Y} \times P^{\mathrm{op}}$ such that for all $p \in P$, the fiber $F_p : \widehat{\mathfrak{X}}_p \to \mathfrak{Y}$ is the right adjoint to the composite

$$\mathfrak{Y} \xrightarrow{f^*} \lim^{\mathrm{rlax}} \widehat{\mathfrak{X}} \xrightarrow{\mathrm{ev}_p} \widehat{\mathfrak{X}}_p.$$

We now observe that f_* is a geometric morphism of P-stratified ∞-topoi if and only if for all $p \in P$, F_p is a geometric morphism, F_p factors through $\mathfrak{Y}_p$, and the resulting map $F : \widehat{\mathfrak{X}} \to \mathcal{G}(\mathfrak{Y})$ preserves locally cocartesian edges. Indeed, the "only if" implication follows from the first half of Observation 5.20, while for the "if" implication, we note that f_* factors as the composite

$$\lim^{\mathrm{rlax}} \widehat{\mathfrak{X}} \xrightarrow{\lim^{\mathrm{rlax}} F} \lim^{\mathrm{rlax}} \mathcal{G}(\mathfrak{Y}) \xrightarrow{\Theta_P} \mathfrak{Y},$$

which respect P-stratifications by the second half of Observation 5.20 and Corollary 5.15, respectively. Therefore, ψ'' restricts to the desired natural equivalence

$$\psi''' : \mathrm{Map}_{\mathbf{StrTop}_{\infty,P}}(\lim^{\mathrm{rlax}} \widehat{\mathfrak{X}}, \mathfrak{Y}) \simeq \mathrm{Map}_{\mathbf{LocCocart}^{\mathrm{top}}_{P^{\mathrm{op}}}}(\widehat{\mathfrak{X}}, \mathcal{G}(\mathfrak{Y})).$$

We conclude that $\lim^{\mathrm{rlax}} \dashv \mathcal{G}$. Furthermore, unpacking this equivalence of mapping spaces shows that Θ_P is the counit of the adjunction. Since Θ_P is an equivalence by Theorem 5.13, it remains to show that the unit η is an equivalence. But the compatibility of the equivalence ψ''' with restriction in the base P shows that η_p is homotopic to the identity for all $p \in P$, hence η is an equivalence. □

5.23 Remark Theorem 5.22 should be viewed as the unstable counterpart to [2, Theorem A], which sets up a similar equivalence between P-stratified stable presentable ∞-categories [2, Definition 2.4.3] and locally cocartesian fibrations fibered in such with exact accessible pushforward functors.

[22] Here, **Cat** refers to the ∞-category of large ∞-categories, so that $\mathbf{Pr}^L$ and $\mathbf{Pr}^R$ are subcategories of **Cat**.

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