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Skew-rack cocycle invariants of closed 3-manifolds

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We establish a new approach to obtain 3-manifold invariants by means of Dehn surgery. In this approach, we introduce skew-racks with good involution and property FR, and define cocycle invariants as 3-manifold invariants.

1 Introduction

Every closed 3-manifold M with orientation can be obtained from a framed link in the 3-sphere S^3 by means of Dehn surgery. Since there is a one-to-one correspondence between closed 3-manifolds and framed links in the S^3 modulo either the Kirby moves [14] or Fenn–Rourke moves [7], any framed link invariant, which is invariant with respect to the moves, is a 3-manifold invariant. For example, in quantum topology, frameworks based on the Chern–Simons theory have produced many 3-manifold invariants, including the concepts of modular categories (see [16; 17]). In contrast, when examined from more classical viewpoints such as algebraic topology, the fundamental groups $\pi_1(M)$ of 3-manifolds contain useful information and are strong invariants. Further, as in the Dijkgraaf–Witten model [6], starting from a finite group G, we can define a certain weight of the set $\mathrm{Hom}(\pi_1(M), G)$ in terms of the group cohomology of G. However, apart from the quantum invariants and fundamental groups, there are relatively few procedures that yield 3-manifold invariants via Dehn surgery.

In this paper, we establish a new approach from Dehn surgery to yield 3-manifold invariants in a classical situation. In our approach, we focus on a class of skew-racks (see Section 3), which is an algebraic system, and a modification of quandles and biracks. As in quandle theory [4; 5; 9; 15], starting from skew-racks, we can define a set of colorings of framed links and weights of a set, where the weights are evaluated by birack 2-cocycles and are called *a cocycle invariant*, as a framed link invariant (see Section 3 for details). The aim of our study is to explore skew-racks such that the cocycle invariant is stable under the Fenn–Rourke moves. To this end, we define the property FR of skew-racks (Definition 4.1) and demonstrate (Theorem 4.2 and Proposition 6.2) that, in some situations, the associated cocycle invariant gives rise to a 3-manifold invariant. In Section 4, we establish several examples of skew-racks with property FR; for instance, from a group G and an involutive automorphism $\kappa : G \to G$, we can define a skew-rack with property FR (Examples 2.2 and 5.4).

Using the examples of skew-racks, we compute a set of colorings and several cocycle invariants—for example, we determine the invariants of the Brieskorn 3-manifolds as integral homology 3-spheres (Example 5.6). Following the computations, we present a comparison with the Dijkgraaf–Witten invariant

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and pose several problems (Problems 6.8 and 7.3). Finally, we attempt to make an application from the skew-racks above; in Section 7, we suggest several elementary approaches to find 3-manifolds, which are not the results of surgery of any knot in S^3. However, we were ultimately unable to find any examples of their application.

Conventional notation Every 3-manifold is understood to be connected, smooth, oriented, and closed.

2 Symmetric skew-racks and birack cocycle invariants

We introduce skew-racks as a special class of biracks (see [5; 7] for the definition of biracks). We define *a skew-rack* as a triple of a set X, a binary operation $\lhd : X \times X \to X$, and a bijection $\kappa : X \to X$ satisfying the following three axioms:

(SR1) For any $a, b \in X$, the equality $\kappa(a \lhd b) = \kappa(a) \lhd \kappa(b)$ holds.

(SR2) For any $b \in X$, the map $X \to X$ that sends x to $x \lhd b$ is a bijection.

(SR3) For any $a, b, c \in X$, the distributive law $(a \lhd b) \lhd c = (a \lhd \kappa(c)) \lhd (b \lhd c)$ holds.

As a special case, if $\kappa = \mathrm{id}_X$, the definition of skew-racks coincides with that of racks. We often denote the inverse map $\bullet \lhd b$ of the bijection as $\bullet \lhd^{-1} b$. Further, as a slight generalization of symmetric quandles in [12; 13], we define *a symmetric skew-rack* as a pair of a skew-rack $(X, \lhd, \kappa)$ and an involution $\rho : X \to X$ satisfying the following:

(SS1) For any $a, b \in X$, the equalities $(a \lhd b) \lhd \rho(b) = a$ and $\rho(a) \lhd \kappa(b) = \rho(a \lhd b)$ hold.

(SS2) The involutivity $\rho \circ \rho = \kappa \circ \kappa = \mathrm{id}_X$ and the commutativity $\rho \circ \kappa = \kappa \circ \rho$ hold.

Such a ρ is called *a good involution* (as in [12]). If $\kappa = \mathrm{id}_X$ and the equality $a \lhd a = a$ holds for any $a \in X$, the definition of symmetric biracks is the same as the original definition of symmetric quandles [12]. A few examples of symmetric skew-racks are as follows.

Example 2.1 Let X be a group G and let $\kappa : G \to G$ be an involutive automorphism. Define $x \lhd y$ by $\kappa(y^{-1})xy$, and $\rho(x)$ by x^{-1}. These maps then define a symmetric skew-rack structure on X.

Example 2.2 Let K be a group and let $f : K \to K$ be an involutive automorphism. Consider the direct products $X = K \times K$ and $\kappa = f \times f$. Define $(x, a) \lhd (y, b)$ by $(f(x)y^{-1}by, f(a))$ and $\rho(x, a) = (f(x), f(a)^{-1})$. Then, these $X, \lhd, \kappa, \rho$ define a symmetric skew-rack structure on $K \times K$. As discussed later (Sections 5–6), this skew-rack plays a key role in this paper.

Finally, we conclude this section by defining a bijection $\mathrm{Tw} : X \to X$ as follows:

Proposition 2.3 *Let* $(X, \lhd, \kappa)$ *be a skew-rack satisfying* $\kappa^2 = \mathrm{id}_X$, *as in* (SS2). *Define the map* $\mathrm{Tw} : X \to X$ *by setting* $\mathrm{Tw}(x) = \kappa(x) \lhd^{-1} \kappa(x)$. *Then, the map is bijective, where the inverse is the map* $X \to X$ *that sends* x *to* $\kappa(x) \lhd x$.

Proof When we let y be $\mathrm{Tw}(\kappa(x) \lhd x)$, we may show $y = x$. Note that $x \lhd \kappa(x) = y \lhd (x \lhd \kappa(x))$, which is equal to

$$\big((y \lhd^{-1} x) \lhd x\big) \lhd (x \lhd \kappa(x)) = \big((y \lhd^{-1} x) \lhd x\big) \lhd \kappa(x) = y \lhd \kappa(x).$$

Thus, by (SS2), we have $y = x$. Similarly, we can easily verify $\kappa(\mathrm{Tw}(x)) \lhd \mathrm{Tw}(x) = x$. □

3 Preliminaries: colorings and birack cocycle invariants

Our definition of X-colorings here is a slight modification of the classical X-colorings of quandles or biracks [4; 5; 8]. Let D be a framed link diagram D, and let $(X, \lhd, \kappa, \rho)$ be a symmetric skew-rack. Choose orientations o for each component of D, and denote by D^o the diagram with the orientations. In this paper, a *semiarc of* D means a path from a crossing to the next crossing along the diagram. Then, *an* X*-coloring* is a map $\mathcal{C} : \{\text{semiarc of } D\} \to X$ such that, for every crossing τ of D, the semiarcs around τ satisfy $\mathcal{C}(\gamma_\tau) = \kappa(\mathcal{C}(\beta_\tau))$ and $\mathcal{C}(\delta_\tau) = \mathcal{C}(\alpha_\tau) \lhd \mathcal{C}(\beta_\tau)$, where α_τ, β_τ, γ_τ, and δ_τ are the semiarcs shown in Figure 1. We denote by $\mathrm{Col}_X(D^o)$ the set of X-colorings of D^o. Then, as a basic fact in quandle theory (see [5; 8]), if two diagrams D^o and $(D')^{o'}$ are related by a Reidemeister move of type II, type III, or a doubled type I, there exists a canonical bijection $\mathcal{B}_{D^o,(D')^{o'}} : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X((D')^{o'})$. Moreover, thanks to the above axioms (SS1) and (SS2), if $D^{o'}$ is the same diagram D with opposite orientation, the correspondence $a \mapsto \rho(a)$ on the color of each semiarc on the opposite component defines a bijection $\mathcal{B}_{D^o,D^{o'}} : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X(D^{o'})$. In particular, the set $\mathrm{Col}_X(D^o)$ up to bijections does not depend on the choice of orientations of D. Accordingly, we sometimes use the expression $\mathrm{Col}_X(D)$ instead of $\mathrm{Col}_X(D^o)$. Finally, we should emphasize that the map $\mathrm{Tw}^{\pm 1}$ in Proposition 2.3 corresponds to an addition of a (∓ 1)-framing in an arc, as in the Reidemeister move of type I.

Next, we observe cocycle invariants of a symmetric skew-rack X. According to [4; 5; 9], a map $\phi : X^2 \to A$ for some abelian group A is called *a birack* 2*-cocycle* if

$$(1) \qquad \phi(a,b) + \phi(a \lhd b, c) = \phi(a, \kappa(c)) + \phi(a \lhd \kappa(c), b \lhd c), \qquad \phi(b,c) = \phi(\kappa(b), \kappa(c))$$

hold for any $a, b, c \in X$. Then, we define *the weight* (*of* τ), $\Phi(\tau)$, with respect to a crossing τ on D to be $\varepsilon_\tau \phi(\mathcal{C}(\alpha_\tau), \mathcal{C}(\beta_\tau)) \in A$, where ε_τ is the sign τ (as in Figure 1). We further define $\Phi_D(\mathcal{C}) \in A$ to be the sum $\sum_\tau \Phi(\tau)$, where τ runs over every crossing on D. Then, as is known [3; 5], if two diagrams D and D' are related by a Reidemeister move of type II, type III, or a doubled type I move, $\Phi_{D'} \circ \mathcal{B}_{D^o,(D')^{o'}} = \Phi_D$ holds as a map $\mathrm{Col}_X(D^o) \to A$. In other words, the map $\Phi_D : \mathrm{Col}_X(D^o) \to A$ up to bijections is an invariant of framed links with orientations. As in [4; 5], we call the map Φ *the* (*birack*) *cocycle invariant*.

Figure 1: Positive and negative crossings with eight labeled semiarcs.

Next, as an analogy of symmetric cocycle invariants in [12; 13], we discuss symmetric birack cocycles. We define a birack 2-cocycle $\phi : X^2 \to A$ to be *symmetric* if

$$\phi(a,b) = -\phi(a \lhd b, \rho(b)) = -\phi(\rho(a), \kappa(b)) \in A,$$

for any $a, b \in X$. Then, similarly to the discussion in [13, Theorem 6.3], we can easily confirm that the weight $\Phi(\tau)$ does not depend on the choice of orientations of D; neither does the map $\Phi_D : \mathrm{Col}_X(D^o) \to A$ up to bijections. In conclusion, the cocycle invariant $\Phi_D : \mathrm{Col}_X(D^o) \to A$ up to bijections is an invariant of framed links.

Finally, we briefly review surgery on links and Fenn–Rourke moves [7]. Let us regard a framed link diagram as the surgery on the framed link in the 3-sphere. Conventionally, every closed 3-manifold M can be expressed as the result of S^3 of surgery on a framed link. Furthermore, two framed links in S^3 have orientation-preserving homeomorphic results of surgery if and only if the framed links are related by a finite sequence of Fenn–Rourke moves and isotopies [7], where the Fenn–Rourke move is an operation between the framed links shown in Figures 3 and 4 in Section 4. Throughout this paper, for a framed link diagram D of a link L, we denote by M_D the result of surgery of S^3 on L.

4 Topological invariants from skew-racks with property FR

Our objective is to explore appropriate skew-racks that yield birack cocycle invariants that are invariant with respect to the Fenn–Rourke moves. In this section, we define skew-racks with property FR and the colorings of closed 3-manifolds.

For $\varepsilon \in \{\pm 1\}$ and $a_1, \dots, a_n \in X$, let us consider the bijection

$$A_{a_1,\dots,a_n} : X \to X, \quad x \mapsto \big(\cdots((x \lhd a_1) \lhd a_2) \lhd \cdots\big) \lhd a_n,$$

and define the subsets

$$\begin{aligned} &\mathrm{Ann}^{+1}(A_{a_1,\dots,a_n}) := \{x \in X \mid \kappa^{n+1}(x) = A_{a_1,\dots,a_n}(x) \lhd \kappa^{n+1}(x)\}, \\ (2) \qquad &\mathrm{Ann}^{-1}(A_{a_1,\dots,a_n}) := \{x \in X \mid \kappa^{n+1}(x) \lhd \kappa(A_{a_1,\dots,a_n}(x)) = A_{a_1,\dots,a_n}(x)\}. \end{aligned}$$

For the case $n = 0$, we define $\mathrm{Ann}^{\pm 1}(X)$ to be the subset $\{x \in X \mid x \lhd \kappa(x) = \kappa(x)\}$. Schematically speaking, as in Figure 2, the set $\mathrm{Ann}(X)^{\pm 1}$ is the set of X-colorings of the unknot of (± 1)-framing.

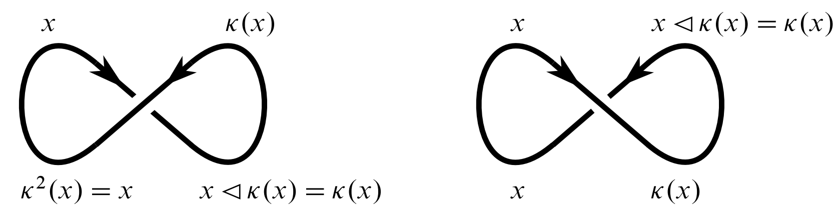

Figure 2: The coloring conditions of unknots of (± 1)-framing.

Definition 4.1 A symmetric skew-rack $(X, \lhd, \kappa, \rho)$ is said to have *property FR* if it satisfies the following:

(FR2) The subset $\mathrm{Ann}(X)$ is not empty, and is bijective to the set $\mathrm{Ann}^{\varepsilon}(A_{a_1,\dots,a_n})$ for arbitrary $n \in \mathbb{Z}$, $a_1,\dots,a_n \in X$ and $\varepsilon \in \{\pm 1\}$.

(FR2) For any $a_1,\dots,a_n \in X$ and $x \in \mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$, $y \in \mathrm{Ann}^{-1}(A_{a_1,\dots,a_n})$, the equalities

$$\kappa^{n+i}(a_i) = A_{a_1,\dots,a_n}(\kappa^{i+1}(a_i) \lhd x), \tag{3}$$

$$\kappa^{n+i}(a_i) \lhd \kappa^{n+1}(y) = A_{a_1 \lhd \kappa(y), a_2 \lhd \kappa^2(y),\dots,a_n \lhd \kappa^n(y)}(\kappa^{i+1}(a_i)) \tag{4}$$

hold, where $i \le n$ is arbitrary.

Let us analyze the set of colorings of skew-racks with property FR.

Theorem 4.2 *Let $(X, \lhd, \kappa, \rho)$ be a symmetric skew-rack with property FR. Suppose that two framed link diagrams D and D' are related by a Fenn–Rourke move (as in Figures 3 and 4) and take orientations on D and D'.*

Then, for any coloring $\mathcal{C} \in \mathrm{Col}_X(D)$, there is uniquely another $\mathcal{C}' \in \mathrm{Col}_X(D')$ such that $\mathcal{C}(\alpha_i) = \mathcal{C}'(\alpha_i')$ and $\mathcal{C}(\beta_i) = \mathcal{C}'(\beta_i')$ for any $i \le n$. Furthermore, the map

$$\mathcal{B} : \mathrm{Col}_X(D) \to \mathrm{Col}_X(D') \times \mathrm{Ann}(X), \quad \mathcal{C} \mapsto (\mathcal{C}', \mathcal{C}(\gamma)), \tag{5}$$

is bijective. Specifically, if X is of finite order, the rational number $|\mathrm{Col}_X(D)|/|\mathrm{Ann}(X)|^{\#D} \in \mathbb{Q}$ gives rise to a topological invariant of closed 3-manifolds.

Proof Take arcs γ, δ, α_i's, and β_i's as in Figures 3 and 4. By the properties of good involutions, the coloring conditions are independent of the choices of orientations of D. Thus, we fix the orientations of D and D' as shown in Figures 3 and 4. Given an X-coloring $\mathcal{C} \in \mathrm{Col}_X(D^o)$, define $a_i := \mathcal{C}(\alpha_i)$, $b_i := \mathcal{C}(\beta_i)$. We now show that the map $\mathcal{C}' : \{\text{semiarc of } D'\} \to X$ defined by $\mathcal{C}'(\alpha_i') = a_i$ and $\mathcal{C}'(\beta_i') = b_i$ gives rise to a unique X-coloring.

First, suppose that $\pm$ is positive and $x = \mathcal{C}(\gamma)$. The coloring condition on the arc δ is

$$\mathrm{Tw}(\kappa^{n+1}(x)) = \kappa^{n+1}(x) \lhd^{-1} \kappa^{n+1}(x) = \mathcal{C}(\delta) = A_{a_1,\dots,a_n}(x) \lhd \kappa^{n+1}(x);$$

hence, $x \in \mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$. Notice from (3) that

$$b_i = \kappa(a_i) \lhd \kappa^i(x) = \kappa^i(\kappa^{i+1}(a_i) \lhd x) = \kappa^i\big(A^{-1}_{a_1,\dots,a_n}(\kappa^{n+i}(a_i))\big) = A^{-1}_{\kappa^i(a_1),\dots,\kappa^i(a_n)}(\kappa^n(a_i)). \tag{6}$$

Meanwhile, the coloring condition on the arc δ_i in the right-hand side of Figure 3 is

$$A_{\kappa^i(a_1),\dots,\kappa^i(a_n)}(b_i) = \mathcal{C}(\delta_i) = \kappa^n(a_i),$$

which is equivalent to (6) exactly.

On the other hand, in the negative case of $\pm$ and $y = \mathcal{C}(\gamma)$, the condition on γ is equivalent to $y \in \mathrm{Ann}^{-1}(A_{a_1,\dots,a_n})$ by the definition of (2); notice that the coloring rule in D in Figure 3 implies $\kappa(b_i) = a_i \lhd \kappa^i(y)$ by the condition on the arc β_i. The coloring condition from β_i' to α_i' is equivalent

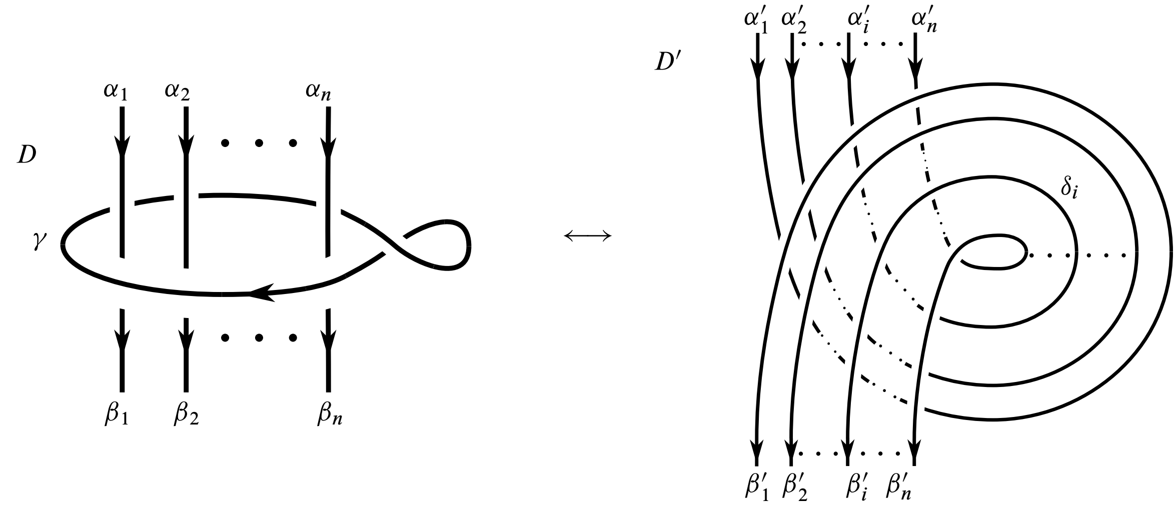

Figure 3: The positive Fenn–Rourke move and labeled semiarcs.

to $\kappa^n(b_i) = A_{\kappa^i(b_1),\dots,\kappa^i(b_n)}(a_i)$, that is, $\kappa^{n+i}(b_i) = A_{b_1,\dots,b_n}(\kappa^i(a_i))$, which directly follows from (4) since $\kappa(b_i) = a_i \lhd \kappa^i(y)$ by the condition on a_i.

Conversely, given an X-coloring $\mathcal{C}'$ of $(D')^{o'}$ and $x \in \mathrm{Ann}^{\pm 1}(A_{a_1,\dots,a_n}) \neq \varnothing$, we similarly can define an X-coloring $\mathcal{C}$ of D that sends α_i to $\mathcal{C}'(\alpha'_i)$, β_i to $\mathcal{C}'(\beta'_i)$, and γ to x.

In summary, by construction, the correspondence $\mathcal{C} \mapsto (\mathcal{C}', \mathcal{C}(\gamma))$ gives the required bijection $\mathcal{B}$. □

Before moving on to the next section, we briefly discuss triviality of the invariants up to link homotopy. For this, consider the permutation group $\mathrm{Bij}(X)$ of a skew-rack X, and define a subgroup generated by

$$\{(\kappa(\bullet) \lhd a) \mid a \in X\} \cup \{(\bullet \lhd^{\epsilon_1} a_1) \lhd^{\epsilon_2} a_2 \mid a_i \in X,\ \epsilon_i \in \{\pm 1\}\}. \tag{7}$$

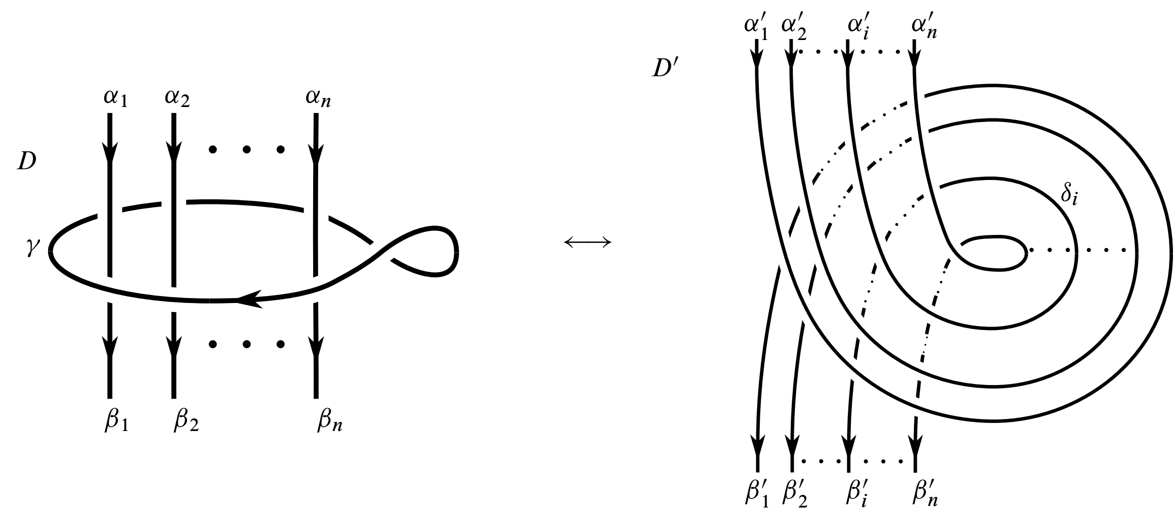

Figure 4: The negative Fenn–Rourke move and labeled semiarcs.

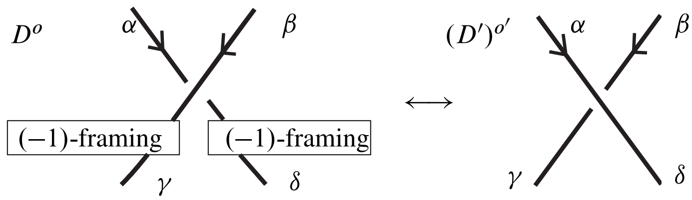

Figure 5: Diagrams D and D', where all semiarcs lie within a link component.

The subgroup, denoted by $\mathrm{Inn}_\kappa^{\mathrm{even}}(X)$, canonically has the right action on X. We say a skew-rack $(X, \lhd, \kappa)$ with property FR is *f-link homotopic* if $x \lhd^\varepsilon \kappa(x) = x \lhd^\varepsilon (x \cdot g)$ holds for any $x \in X$, $g \in \mathrm{Inn}_\kappa^{\mathrm{even}}(X)$, $\varepsilon \in \{\pm 1\}$.

Proposition 4.3 *Suppose a symmetric skew-rack $(X, \lhd, \kappa, \rho)$ with property FR is f-link homotopic. Then, if two framed link diagrams D and D' are transformed by the operation in Figure 5, there is a bijection $\mathcal{B}_f : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X((D')^{o'})$.*

Proof For a coloring $\mathcal{C} \in \mathrm{Col}_X(D^o)$, take $a \in X$ such that $\mathcal{C}(\alpha) = \kappa(a)$. Since α and β lie on the same link-component, there is $g \in \mathrm{Inn}_\kappa^{\mathrm{even}}(X)$ such that $\mathcal{C}(\beta) = a \cdot g$ from definition (7). Since $\mathrm{Tw}^{-1}(x) = x \lhd x$ by definition, the rule of colorings implies

$$\mathcal{C}(\gamma) = \mathrm{Tw}^{-1}(\kappa(a \cdot g)) = (a \cdot g) \lhd \kappa(a \cdot g), \quad \mathcal{C}(\delta) = \mathrm{Tw}^{-1}(\kappa(a)) \lhd (a \cdot g) = (a \lhd \kappa(a)) \lhd (a \cdot g).$$

Since X is f-link homotopic, $\mathcal{C}(\gamma) = (a \cdot g) \lhd^{-1} a$ and $\mathcal{C}(\delta) = a$. Thus, we can define another coloring $\mathcal{B}_f(\mathcal{C})$ of D' by $\mathcal{B}_f(\mathcal{C})(\alpha) = a$ and $\mathcal{B}_f(\mathcal{C})(\beta) = a \cdot g$. Since $\mathcal{C}(\gamma) = (a \cdot g) \lhd^{-1} a = \mathcal{B}_f(\mathcal{C})(\gamma)$ and $\mathcal{C}(\delta) = a = \mathcal{B}_f(\mathcal{C})(\delta)$ by definitions, the map $\mathcal{B}_f : \mathrm{Col}_X(D^o) \to \mathrm{Col}_X((D')^{o'})$ is bijective, as required. □

Many 3-manifolds can be expressed as the results from S^3 of surgery along various framed knots, so to obtain nontrivial colorings, we consider skew-racks, which are not f-link homotopic.

5 Examples of skew-racks with property FR from groups

Here we provide examples of skew-racks with property FR. Throughout this section, we fix a group G, an automorphism $\kappa : G \to G$ satisfying $\kappa \circ \kappa = \mathrm{id}_G$, and a map $\delta : G \to G$ satisfying $\kappa \circ \delta = \delta \circ \kappa$. Consider the binary operation $\lhd : G \times G \to G$ defined by $x \lhd y = \kappa(x)\delta(y)$. Then, the twisting map Tw in Proposition 2.3 is given by $\mathrm{Tw}(g) = g\delta(g)^{-1}$.

Lemma 5.1 *These operations $(\lhd, \kappa)$ with $x \lhd y = \kappa(x)\delta(y)$ define a skew-rack of $X = G$ if and only if, for any $x, y \in G$,*

$$\delta(x)\delta(y) = \delta(y)\delta(x\delta(y)) \in G. \tag{8}$$

Let $\rho : G \to G$ be a good involution. Further, assume that the image $\mathrm{Im}(\delta) \subset G$ is a subgroup of G, and that the cardinality of the preimage $\delta^{-1}(d)$ is constant for any $d \in \mathrm{Im}(\delta)$. Then, the symmetric skew-rack on $X = G$ has property FR.

In addition, if the subgroup $\mathrm{Im}(\delta)$ is commutative, the skew-rack is f-link homotopic.

Proof Since the former part is shown by direct computation, we show only the remaining claims here. We now analyze the set $\mathrm{Ann}^{\varepsilon}(A_{a_1,\dots,a_n})$ in (2). First, suppose $\varepsilon = +1$. Then, the condition $\kappa^{n+1}(x) = A_{a_1,\dots,a_n}(x) \lhd \kappa^{n+1}(x)$ is equivalent to

$$\delta(\kappa^n(a_1))\delta(\kappa^{n-1}(a_2))\cdots\delta(\kappa(a_n))\delta(\kappa^{n+1}(x)) = 1. \tag{9}$$

Since $\mathrm{Im}(\delta)$ is a subgroup of G by assumption, the set $\mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$ is nonempty. Moreover, by the second assumption, the cardinality of $\mathrm{Ann}^{+1}(A_{a_1,\dots,a_n})$ does not depend on the choice of $a_1,\dots,a_n$, that is, X satisfies (FR1). As for (FR2), the equality (3) is shown by

$$A_{a_1,\dots,a_n}(\kappa^{i-1}(a_i) \lhd x) = \kappa^{n+i}(a_i)\delta(\kappa^{n+i}(x))\delta(\kappa^{n+i+1}(a_1))\delta(\kappa^{n+i}(a_2))\cdots\delta(\kappa^{i}(a_n)) = \kappa^{n+i+1}(a_i).$$

Next, we will show (4) in the case $\varepsilon = -1$. We can easily check $\kappa^{n+1}(x) \lhd \kappa\big(A_{a_1,\dots,a_n}(x)\big) = A_{a_1,\dots,a_n}(x)$ in (2) is equivalent to (9) exactly. Thus, similarly, the cardinality of $\mathrm{Ann}^{-1}(A_{a_1,\dots,a_n}) \neq \varnothing$ does not depend on the choice of $a_1,\dots,a_n$. In addition, for $x \in \mathrm{Ann}^{-1}(A_{a_1,\dots,a_n})$, the equality (4) is shown by

$$\begin{aligned}
&A_{a_1 \lhd \kappa(x), a_2 \lhd \kappa^2(x),\dots,a_n \lhd \kappa^n(x)}(\kappa^{i+1}(a_i)) \\
&\quad = \kappa^{i+n+1}(a_i)\delta\big(\kappa^{n-1}(a_1)\delta(\kappa^n(x))\big)\delta\big(\kappa^{n-2}(a_2)\delta(\kappa^n(x))\big)\cdots\delta\big(a_n\delta(\kappa^n(x))\big) \\
&\quad = \kappa^{i+n+1}(a_i)\delta(\kappa^n(x))^{-1}\delta(\kappa^{n-1}(a_1))\delta(\kappa^n(a_2))\cdots\delta(a_n)\delta(\kappa^n(x)) = \kappa^{n+i}(a_i) \lhd \kappa^{n+1}(x).
\end{aligned}$$

Here, the second and third equalities are obtained from (8) and (9), respectively. Hence, X has property FR, as required.

Finally, we show the last statement. From the definition of the subgroup $\mathrm{Inn}^{\mathrm{even}}_{\kappa}(X)$, any $g \in \mathrm{Inn}^{\mathrm{even}}_{\kappa}(X)$ and $a \in G$ admit uniquely $b_1,\dots,b_n \in \mathrm{Im}(\delta)$ such that $a \cdot g = a\delta(b_1)\cdots\delta(b_n) \in G$. Since $\mathrm{Im}(\delta)$ is commutative, (8) means $\delta(a) = \delta(a\delta(b))$. Thus,

$$(z \lhd^{\varepsilon} \kappa(a)) \lhd^{-\varepsilon} (a \cdot g) = z\delta(a)^{\varepsilon}\delta(a\delta(b_1)\cdots\delta(b_n))^{-\varepsilon} = z.$$

Therefore, the skew-rack is f-link homotopic by Proposition 4.3. □

We should point out that (8) comes with a few conditions. For example, if $|G| > 1$, the map δ is not surjective. In fact, if δ is surjective, then (8) with $x = 1$ is equivalent to $z^{-1}\delta(1)z = \delta(z)$ for any $z \in G$, which means that $\mathrm{Im}(\delta)$ is a conjugacy class, and contradicts the surjectivity. However, we provide some examples that satisfy the conditions in Lemma 5.1.

Example 5.2 First, we observe the case where δ is a group homomorphism. Then, we can easily check that (8) is equivalent to that of $\delta \circ \delta = 0$ and the image $\mathrm{Im}(\delta)$ is abelian. If so, the cardinality of $\delta^{-1}(k)$ is constant; thus, if X admits a good involution, the symmetric skew-rack has property FR, and is f-link homotopic by Lemma 5.1.

To avoid f-link homotopic skew-racks, we should focus on δ, which is not a homomorphism.

Example 5.3 (twisted conjugacy classes) Suppose we have a group automorphism $f : G \to G$, and define $\delta(x) = f(x^{-1})x$. Then, (8) is equivalent to $\mathrm{Im}(\delta \circ \delta) = \{1_G\}$. In general, we can easily check that, for any $g \in \mathrm{Im}(\delta)$, the preimage $\delta^{-1}(g)$ is bijective to the fixed-point subgroup $\{h \in G \mid f(h) = h\}$;

see, e.g., [2]. Thus, to apply Lemma 5.1, the remaining point is to analyze the situation such that the image $\mathrm{Im}(\delta)$ is a subgroup.

The image $\mathrm{Im}(\delta)$ is sometimes called as *a twisted conjugacy class* or *Reidemeister conjugacy class*. Prior works [2; 10] have investigated various conditions requiring that $\mathrm{Im}(\delta)$ be a subgroup and $\mathrm{Im}(\delta\circ\delta)=\{1_G\}$. However, many of the examples in those works satisfy that $\mathrm{Im}(\delta)$ is commutative Thus, it is difficult to find examples of pairs (G, f) satisfying that the resulting skew-racks are not f-link homotopic.

Example 5.4 Take a group K with a normal subgroup $N \trianglelefteq K$ and an involutive automorphism $f: K \to K$ satisfying $f(N) \subset N$. Let G be $K \times N$ and κ be $f \times f$. Define $\delta(x, y)$ as $(x^{-1}yx, 1)$, where $x, y \in K$. Next, we check the conditions in Lemma 5.1. Checking (8) is obvious: since $N = \{b^{-1}ab \mid a \in N,\ b \in K\}$, the image of δ is $N \times 1$ as a subgroup of G. Moreover, for any $(k,1) \in K \times 1$, the preimage $\delta^{-1}(k,1)$ is equal to $\{(y^{-1}ky, y) \in G \mid y \in K\}$, which is bijective to K. In conclusion, the symmetric skew-rack on G has property FR, by Lemma 5.1. For example, if $N = K$, the skew-rack on G is exactly equal to that in Example 2.2; here, we should remark $\mathrm{Tw}(x,a) = (a^{-1}x, a)$.

Finally, we compute a few colorings using the above skew-racks with property FR.

Example 5.5 For natural numbers $n, m \in \mathbb{N}$, we first compute colorings of the lens space $L(nm-1, n)$. Let $X = G$ be a skew-rack with good involution, which satisfies the conditions in Lemma 5.1. Let D be the Hopf link with framing (n,m). Then, M_D is known to be $L(nm-1,n)$. We fix two semiarcs α, β in each link-component on D. Then, from the definition of colorings, a coloring $\mathcal{C} \in \mathrm{Col}_X(D)$ satisfies

$$\kappa(\mathcal{C}(\alpha) \lhd \mathcal{C}(\beta)) = \mathrm{Tw}^n(\mathcal{C}(\alpha)), \quad \kappa(\mathcal{C}(\beta) \lhd \mathcal{C}(\alpha)) = \mathrm{Tw}^m(\mathcal{C}(\beta)). \tag{10}$$

Conversely, every $a, b \in X$ satisfying $\kappa(a \lhd b) = \mathrm{Tw}^n(a)$ and $\kappa(b \lhd a) = \mathrm{Tw}^m(b)$ yield a coloring of D. Since $\mathrm{Tw}^n(a) = a\delta(a)^{-n}$, (10) is equivalent to conditions $\mathcal{C}(\beta) = \delta(\mathcal{C}(\alpha))^m$ and $\delta(\mathcal{C}(\alpha))^{nm-1} = 1$. Hence, $\mathrm{Col}_X(D)$ is bijective to

$$\{(a,b) \in G^2 \mid \delta(a)^{nm-1} = 1,\ \delta(b) = \delta(a)^n\} \overset{1:1}{\longleftrightarrow} \{a \in G \mid \delta(a)^{nm-1} = 1\} \times \delta^{-1}(0), \tag{11}$$

where we use a bijection $\delta^{-1}(0) \leftrightarrow \{a \in G \mid \delta(a)^n = c\}$ for any $c \in G$. Therefore, the set $\mathrm{Col}_X(D)$ depends only on nm; it cannot classify the lens spaces of the forms $L(nm-1,n)$. In contrast, we later compute various cocycle invariants that can distinguish among different lens spaces (see Example 6.7).

Example 5.6 Next, we observe that the sets of colorings of integral homology 3-spheres seem to be strong invariants, where we consider the skew-rack on $X = K \times K$ in Example 2.2. Let $D_n^{\pm}$ be the $(2,n)$-torus knot with framing ± 1. Then, the resulting 3-manifold $M_{D_n^{\pm}}$ is the Brieskorn 3-manifold of the form $\Sigma(2, n, 2n \mp 1)$, as an integral homology 3-sphere. For a concrete group K, it is fairly easy to determine the set $\mathrm{Col}_X(D_q^{\pm})$ with the help of a computer program. A list of several computations of $|\mathrm{Col}_X(D_q^{\pm})|$ is provided in Table 1.

As seen in this example, it is reasonable to focus only on nonabelian groups K. In fact, if K is abelian, $(x,a) \lhd (y,b) = (x,a)$; hence, the coloring conditions are trivial; thus, considering the linking matrix of D, we can easily find a one-to-one correspondence $\mathrm{Col}_X(D) \simeq \mathrm{Hom}(H_1(M;\mathbb{Z}), K) \times K^{\sharp D}$.

p	$\lvert\mathrm{Col}_X(D_3^+)\rvert$	$\lvert\mathrm{Col}_X(D_3^-)\rvert$	$\lvert\mathrm{Col}_X(D_5^+)\rvert$	$\lvert\mathrm{Col}_X(D_5^-)\rvert$	$\lvert\mathrm{Col}_X(D_7^+)\rvert$	$\lvert\mathrm{Col}_X(D_7^-)\rvert$
3	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$
5	$121\lvert K\rvert$	$\lvert K\rvert$	$121\lvert K\rvert$	$\lvert K\rvert$	$25\lvert K\rvert$	$\lvert K\rvert$
7	$\lvert K\rvert$	$337\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$49\lvert K\rvert$
11	$2641\lvert K\rvert$	$\lvert K\rvert$	$2641\lvert K\rvert$	$2641\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$
13	$\lvert K\rvert$	$6553\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$	$\lvert K\rvert$

Table 1: Cardinality of $\mathrm{Col}_X(D_n^{\pm 1})$ for various p, n. Here, $K = \mathrm{SL}_2(\mathbb{F}_p)$ of order $p^3 - p$.

6 Cocycle invariants of 3-manifolds

As discussed in [3; 5; 15], there are various procedures to concretely find symmetric birack 2-cocycles. However, the condition that 2-cocycles must have invariance with respect to Fenn–Rourke moves seems strong. Nevertheless, we now investigate 2-cocycle invariants to obtain 3-manifold invariants. Throughout this section, we assume a symmetric skew-rack X with property FR, and a map ϕ from X^2 to an abelian group A.

We first introduce the property FR of birack 2-cocycles as follows.

Definition 6.1 Recall the bijection $\mathcal{B}$ in Theorem 4.2, and denote by 0_A the constant map to A whose image is zero. A symmetric birack 2-cocycle $\phi : X^2 \to A$ satisfies *property FR*, if $\Phi_D = (\Phi_{D'} \times 0_A) \circ \mathcal{B}$ holds for any diagrams D and D' in Figures 3 and 4. Here, Φ_D is the cocycle invariant explained in Section 3.

Such a ϕ is said to be *f-link homotopic* if X is f-link homotopic and, for any $a \in X$ and $g \in \mathrm{Inn}_\kappa^{\mathrm{even}}(X)$,

$$\phi((a \cdot g) \lhd \kappa(a), a \cdot g) + \phi(a, a \lhd \kappa(a)) = \phi(\kappa(a), a \cdot g) + \phi((a \cdot g) \lhd \kappa(a), a). \tag{12}$$

We will see (Proposition 6.2) that symmetric birack 2-cocycles with property FR yield topological invariants of closed 3-manifolds. Take two maps $F : Y \to A$ and $G : Z \to A$, where Y and Z are some sets. We call F *an FR-stabilization* of G if there is a bijection $B : Z \to Y \times \mathrm{Ann}(X)$ such that $g \circ B^{-1} = f \times 0_A$. More generally, F and G are *FR-equivalent* if F and G are related by a finite sequence of FR-(dis-)stabilizations. Then, the following proposition is fairly obvious by definitions.

Proposition 6.2 *Let ϕ be a symmetric birack 2-cocycle with property FR. Then, the correspondence $D \mapsto \Phi_D$ up to FR-equivalent relations is an invariant of closed 3-manifolds.*

Moreover, if X and ϕ are f-link homotopic, and if D and D' are related by the operation in Figure 5, then $\Phi_D = \Phi_{D'} \circ B_f$, where B_f is the bijection $\mathrm{Col}_X(D) \to \mathrm{Col}_X(D')$ in the proof of Proposition 4.3.

To conclude, to obtain 3-manifold invariants, it is important to find symmetric birack 2-cocycles with concrete expressions. In this context, we discuss Lemmas 6.3 and 6.4 below. Let $\widetilde{X}$ be $X \times A$. Define $\widetilde{\lhd} : \widetilde{X} \times \widetilde{X} \to \widetilde{X}$ by

$$(x, a) \mathbin{\widetilde{\lhd}} (y, b) = (x \lhd y, a + \phi(x, y)), \quad (x, y \in X,\ a, b \in A),$$

and $\tilde{\kappa} : \widetilde{X} \to \widetilde{X}$ by $\tilde{\kappa}(x, a) = (\kappa(x), -a)$.

Lemma 6.3 (see [3, Section 3]) *These maps $\widetilde{\lhd}$, $\tilde{\kappa}$, $\tilde{\delta}$ define a skew-rack on $\widetilde{X} = X \times A$ if and only if ϕ is a birack 2-cocycle.*

Proof Describe the distribution law as in (SR3) as

$$\big((x,a) \,\widetilde{\lhd}\, (y,b)\big) \,\widetilde{\lhd}\, (z,c) = \big((x \lhd y) \lhd z, a + \phi(x,y) + \phi(x \lhd y, z)\big),$$
$$\big((x,a) \,\widetilde{\lhd}\, \tilde{\kappa}(z,c)\big) \,\widetilde{\lhd}\, \big((y,b) \,\widetilde{\lhd}\, (z,c)\big) = \big((x \lhd y) \lhd z, a + \phi(x, \kappa(z)) + \phi(x \lhd \kappa(z), y \lhd z)\big).$$

Furthermore, (SR1) implies $\phi(\kappa(x), \kappa(y)) = \phi(x,y)$. Hence, the desired claim follows directly from the definition (1) of birack 2-cocycle. □

Lemma 6.4 *Let ϕ be a birack 2-cocycle satisfying $\phi(a,b) = -\phi(\rho(a), \kappa(b))$. Then, the map $\overline{\phi} : X^2 \to A$ that sends (a,b) to $\phi(a,b) - \phi(a \lhd b, \rho(b))$ is a symmetric birack 2-cocycle.*

Proof It is easy to confirm $\overline{\phi}(a,b) + \overline{\phi}(a \lhd b, \rho(b)) = 0$. Thus, all that remains is to check the cocycle condition (1) of $\overline{\phi}$. For this, we may show

$$\phi(a \lhd b, \rho(b)) + \phi((a \lhd b) \lhd c, \rho(c)) = \phi\big(a \lhd \kappa(c), \rho(\kappa(c))\big) + \phi((a \lhd b) \lhd c, \rho(b \lhd c)). \tag{13}$$

Replace $(a \lhd b) \lhd c$, $\rho(b \lhd c)$, and $\rho(\kappa(c))$ with a, b, and c, respectively. Then, we can easily check that the replacement of (13) coincides with (1). □

Using these lemmas, we provide examples from several skew-racks in Example 5.4. Let $N \trianglelefteq K$ be groups and $f : K \to K$ be an involutive automorphism satisfying $f(N) \subset N$. Further, take a normalized group 2-cocycle $\theta : K \times K \to A$, where θ satisfies

$$\theta(x,y) - \theta(x,yz) + \theta(xy,z) - \theta(y,z) = 0, \quad \theta(1_K, x) = \theta(x, 1_K) = 0 \in A,$$

for any $x, y, z \in K$. Then, the product of $\widetilde{K} = K \times A$ has a group structure with operation $((x,a),(y,b)) \mapsto (xy, a + b + \theta(x,y))$ as a central extension of K. As is known in group cohomology, every central extension over K with fiber A can be expressed by the product for some θ. Then, from Example 5.4, we can define the symmetric skew-racks on $G = K \times N$ and $\widetilde{G} = \widetilde{K} \times \widetilde{N}$, which have property FR. Moreover, by the definition of $\lhd$ on $\widetilde{G}$, we obtain

$$\begin{aligned}
&\big((x,a),(y,b)\big) \\
&\quad \lhd \big((z,c),(w,d)\big) = \big(\tilde{\kappa}(x,a)(z^{-1}, -c - \theta(z,z^{-1}))(w,d)(z,c),\ (y,b)\big) \\
&\quad = \big(\big(f(x)z^{-1}wz,\ f(a) + d + \theta(f(x), z^{-1}) + \theta(f(x)z^{-1}, wz) + \theta(w,z) - \theta(z,z^{-1})\big), (y,b)\big) \in \widetilde{G}.
\end{aligned}$$

Inspired by Lemma 6.4, we obtain the following procedure for producing birack 2-cocycles:

Theorem 6.5 *Let $\lambda : N \to A$ be a group 1-cocycle. Then, the map*

$$\phi_{\lambda,\theta} : G^2 = (K \times N) \times (K \times N) \to A,$$
$$(x,y,z,w) \mapsto \lambda(y)\big(\theta(f(x), z^{-1}) + \theta(f(x)z^{-1}, wz) + \theta(z,w) - \theta(z,z^{-1})\big) \in A,$$

is a birack 2-cocycle of the skew-rack $G = K \times N$ in Example 2.2. If

$$\lambda(x)\theta(a,b) = \lambda(f(x))\theta(f(a), f(b))$$

hold for any $a, b \in K$, $x \in N$, the condition in Lemma 6.4 is true. Specifically, the cocycle $\overline{\phi_{\lambda,\theta}}$ mentioned in Lemma 6.4 is a symmetric birack 2-cocycle.

In general, it may seem difficult to find group 2-cocycles θ such that the associated map $\phi_{\lambda,\theta}$ has property FR. However, when K is a cyclic group, we give such examples of birack cocycles with property FR. More precisely, by a direction computation, we can show the following.

Proposition 6.6 *Let $p \in \mathbb{Z}$ be an odd prime. Let $K = N = \mathbb{Z}/p$, and take $\varepsilon \in \{\pm 1\}$ such that $f(x) = \varepsilon x$, Define group cocycles λ and θ by setting*

$$\lambda(x) = x, \quad \theta(x,y) = \frac{(x+\varepsilon y)^p - x^p - (\varepsilon y)^p}{p} = \sum_{j:1 \leq j < p} j^{-1} x^j (\varepsilon y)^{p-j},$$

respectively, where $x, y \in \mathbb{Z}/p$. Then, $\overline{\phi_{\lambda,\theta}}(x,y,z,w) = 2y\theta(x,w)$, and the symmetric birack 2-cocycle $\overline{\phi_{\lambda,\theta}}$ has property FR and is f-link homotopic.

Example 6.7 Let D be the Hopf link with framings (n,m), as in Example 5.5. Recall that M_D is the lens space $L(nm-1, m)$. By (11), if $nm-1$ is divisible by p and $K = \mathbb{Z}/p$, then $\mathrm{Col}_X(D)$ is bijective to $(\mathbb{Z}/p)^2$. In addition, we can easily show that the cocycle invariant $\Phi_D : (\mathbb{Z}/p)^2 \to \mathbb{Z}/p$ is equal to the correspondence $(x,y) \mapsto -mx^2$, where we use the 2-cocycle $\overline{\phi_{\lambda,\theta}}$ in Proposition 6.6. For example, the invariant can distinguish between the lens spaces $L(11,1)$ and $L(11,3)$, which are not homotopy equivalent.

More generally, consider the lens space $L(p,q)$ and a framed diagram $D_{p,q}$ such that $M_{D_{p,q}} = L(p,q)$. Then, with the help of a computer program, if $p, q < 100$, it is fairly easy to check that the cocycle invariant $\Phi_{D_{p,q}} : (\mathbb{Z}/p)^{1+\#D_{p,q}} \to \mathbb{Z}/p$ is FR-equivalent to the map $\mathbb{Z}/p \to \mathbb{Z}/p; x \mapsto -qx^2$.

From this example, it is natural to pose the problem below, together with a relation to the Dijkgraaf–Witten invariant [6, §6]. We first briefly review the invariant. Fix a closed 3-manifold M with fundamental homology 3-class $[M] \in H_3(M;\mathbb{Z}) \cong \mathbb{Z}$. Let K be a group of finite order, and $\psi : K^3 \to A$ be a group 3-cocycle. Denote by BK the classifying space of K or the Eilenberg–Mac Lane space of type $(K,1)$, and $c_M : M \to B\pi_1(M)$ be a classifying map. Then, any group homomorphism $f : \pi_1(M) \to K$ induces a continuous map $f_* : B\pi_1(M) \to BK$. Since the (co)homology of BK equals that of K, we can define the pullback $(f_* \circ c_M)^*(\psi)$ as a 3-cocycle of M. Then, *the Dijkgraaf–Witten invariant* is defined as the map

$$\mathrm{DW}_\psi(M) : \mathrm{Hom}(\pi_1(M), K) \to A, \quad f \mapsto \langle (f_* \circ c_M)^*(\psi), [M] \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Kronecker map.

Problem 6.8 As in Example 2.2, let X be the symmetric skew-rack on $K \times K$. Let $\lambda : K \to A$ and $\theta : K^2 \to A$ be group cocycles, and ψ be the cup product $\lambda \smile \theta$ as a group 3-cocycle. Let D be a framed link diagram.

Then, is there a bijection $\mathcal{B} : \mathrm{Col}_X(D) \simeq \mathrm{Hom}(\pi_1(M), K) \times \mathrm{Ann}(X)^{\sharp D}$? Further, find a condition such that the birack 2-cocycle $\overline{\phi_{\lambda,\theta}}$ in Theorem 6.5 has property FR, and FR-equivalence between the cocycle invariant $\Phi : \mathrm{Col}_X(D) \to A$ and the Dijkgraaf–Witten invariant $\mathrm{DW}_\psi(M_D)$.

If this problem is correctly solved, we consequently obtain a diagrammatic computation of the Dijkgraaf–Witten invariant via the cocycle invariants and Dehn surgery.

7 Criteria for 3-manifolds that are not the result of surgery of any knot

As an application of the cocycle invariant, we provide two criteria to detect 3-manifolds that are not the result of surgery of any knot in S^3 (see [1, Section 7.1; 11] for the details of such 3-manifolds and other criteria). As in Example 5.4, we fix groups $N \trianglelefteq K$, and $X = K \times N$ with $f = \mathrm{id}_K$; recall that X is a skew-rack by $(x,a) \lhd (y,b) = (xy^{-1}by, a)$, and has property FR.

Proposition 7.1 *Suppose $|K| < \infty$ and that a framed link diagram D and a knot diagram of framing zero are related by a sequence of Fenn–Rourke moves and isotopy. Then, the invariant $|\mathrm{Col}_X(D)|/|K|^{\sharp D} \in \mathbb{Q}$ in Theorem 4.2 is larger than or equal to $|N|$.*

Proof We may suppose that D is a knot diagram of framing zero. For the proof, it is sufficient to construct $|K \times N|$ colorings on D. As in Figure 6, take semiarcs α_i and β_i in D, and denote by $\varepsilon_i \in \{\pm 1\}$ the sign of the crossing between α_i and β_i. For $(g,h) \in K \times N$, we define $\mathcal{C}_{g,h}(\alpha_i)$ to be $(h^{\sum_{j=1}^{i-1}\varepsilon_j} g, h) \in X = K \times N$. Since every β_i lies on the same link component, $\mathcal{C}_{g,h}(\beta_i) = (h^{n_i} g, h)$ for some $n_i \in \mathbb{Z}$. Hence, we can easily check that $\mathcal{C}_{g,h}$ defines an X-coloring as required. □

As a special case, let $K = N = \mathbb{Z}/2$. For $k_1, k_2, k_3 \in \mathbb{Z}/2$, we define a map $\phi_{k_1,k_2,k_3} : X \times X \to \mathbb{Z}/2$ by setting

$$\phi_{k_1,k_2,k_3}((x,a),(y,b)) = k_1 a + k_2 b + k_3 ab.$$

Then, by direct computation, it is not hard to show the following:

Proposition 7.2 *The map ϕ_{k_1,k_2,k_3} is a symmetric birack 2-cocycle with property FR, and is f-link homotopic. Furthermore, if a framed link diagram D is FR-equivalent to a knot diagram of framing zero, then the symmetric birack 2-cocycle invariant is trivial.*

Unfortunately, we could not find any new examples of framed link diagrams that are not FR-equivalent to any knot diagram of framing zero. We end this paper by presenting problems to be investigated in future work.

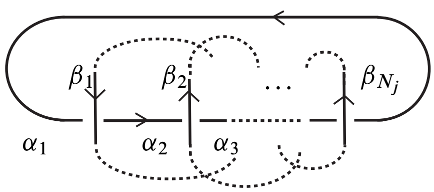

Figure 6: Semiarcs α_i and β_i in the knot diagram D.

Problem 7.3 As applications of the propositions above, find 3-manifolds that are not the surgery of any knot of framing zero. Establish stronger criteria than the propositions above, which are applicable to many framed link diagrams.

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