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We establish a new approach to obtain 3-manifold invariants by means of Dehn surgery. In this approach, we introduce skew-racks with good involution and property FR, and define cocycle invariants as 3-manifold invariants.

1 Introduction

Every closed 3-manifold M with orientation can be obtained from a framed link in the 3-sphere S^3 by means of Dehn surgery. Since there is a one-to-one correspondence between closed 3-manifolds and framed links in the S^3 modulo either the Kirby moves [14] or Fenn–Rourke moves [7], any framed link invariant, which is invariant with respect to the moves, is a 3-manifold invariant. For example, in quantum topology, frameworks based on the Chern–Simons theory have produced many 3-manifold invariants, including the concepts of modular categories (see [16; 17]). In contrast, when examined from more classical viewpoints such as algebraic topology, the fundamental groups $\pi_1(M)$ of 3-manifolds contain useful information and are strong invariants. Further, as in the Dijkgraaf–Witten model [6], starting from a finite group G , we can define a certain weight of the set $\text{Hom}(\pi_1(M), G)$ in terms of the group cohomology of G . However, apart from the quantum invariants and fundamental groups, there are relatively few procedures that yield 3-manifold invariants via Dehn surgery.

In this paper, we establish a new approach from Dehn surgery to yield 3-manifold invariants in a classical situation. In our approach, we focus on a class of skew-racks (see Section 3), which is an algebraic system, and a modification of quandles and biracks. As in quandle theory [4; 5; 9; 15], starting from skew-racks, we can define a set of colorings of framed links and weights of a set, where the weights are evaluated by birack 2-cocycles and are called a *cocycle invariant*, as a framed link invariant (see Section 3 for details). The aim of our study is to explore skew-racks such that the cocycle invariant is stable under the Fenn–Rourke moves. To this end, we define the property FR of skew-racks (Definition 4.1) and demonstrate (Theorem 4.2 and Proposition 6.2) that, in some situations, the associated cocycle invariant gives rise to a 3-manifold invariant. In Section 4, we establish several examples of skew-racks with property FR; for instance, from a group G and an involutive automorphism $\kappa : G \rightarrow G$, we can define a skew-rack with property FR (Examples 2.2 and 5.4).

Using the examples of skew-racks, we compute a set of colorings and several cocycle invariants—for example, we determine the invariants of the Brieskorn 3-manifolds as integral homology 3-spheres (Example 5.6). Following the computations, we present a comparison with the Dijkgraaf–Witten invariant

and pose several problems (Problems 6.8 and 7.3). Finally, we attempt to make an application from the skew-racks above; in Section 7, we suggest several elementary approaches to find 3-manifolds, which are not the results of surgery of any knot in S^3 . However, we were ultimately unable to find any examples of their application.

Conventional notation Every 3-manifold is understood to be connected, smooth, oriented, and closed.

2 Symmetric skew-racks and birack cocycle invariants

We introduce skew-racks as a special class of biracks (see [5; 7] for the definition of biracks). We define a skew-rack as a triple of a set X , a binary operation $\triangleleft : X \times X \rightarrow X$, and a bijection $\kappa : X \rightarrow X$ satisfying the following three axioms:

(SR1) For any $a, b \in X$, the equality $\kappa(a \triangleleft b) = \kappa(a) \triangleleft \kappa(b)$ holds.

(SR2) For any $b \in X$, the map $X \rightarrow X$ that sends x to $x \triangleleft b$ is a bijection.

(SR3) For any $a, b, c \in X$, the distributive law $(a \triangleleft b) \triangleleft c = (a \triangleleft \kappa(c)) \triangleleft (b \triangleleft c)$ holds.

As a special case, if $\kappa = \text{id}_X$, the definition of skew-racks coincides with that of racks. We often denote the inverse map $\bullet \triangleleft b$ of the bijection as $\bullet \triangleleft^{-1} b$. Further, as a slight generalization of symmetric quandles in [12; 13], we define a symmetric skew-rack as a pair of a skew-rack $(X, \triangleleft, \kappa)$ and an involution $\rho : X \rightarrow X$ satisfying the following:

(SS1) For any $a, b \in X$, the equalities $(a \triangleleft b) \triangleleft \rho(b) = a$ and $\rho(a) \triangleleft \kappa(b) = \rho(a \triangleleft b)$ hold.

(SS2) The involutivity $\rho \circ \rho = \kappa \circ \kappa = \text{id}_X$ and the commutativity $\rho \circ \kappa = \kappa \circ \rho$ hold.

Such a ρ is called a good involution (as in [12]). If $\kappa = \text{id}_X$ and the equality $a \triangleleft a = a$ holds for any $a \in X$, the definition of symmetric biracks is the same as the original definition of symmetric quandles [12]. A few examples of symmetric skew-racks are as follows.

Example 2.1 Let X be a group G and let $\kappa : G \rightarrow G$ be an involutive automorphism. Define $x \triangleleft y$ by $\kappa(y^{-1})xy$, and $\rho(x)$ by x^{-1} . These maps then define a symmetric skew-rack structure on X .

Example 2.2 Let K be a group and let $f : K \rightarrow K$ be an involutive automorphism. Consider the direct products $X = K \times K$ and $\kappa = f \times f$. Define $(x, a) \triangleleft (y, b)$ by $(f(x)y^{-1}by, f(a))$ and $\rho(x, a) = (f(x), f(a)^{-1})$. Then, these $X, \triangleleft, \kappa, \rho$ define a symmetric skew-rack structure on $K \times K$. As discussed later (Sections 5–6), this skew-rack plays a key role in this paper.

Finally, we conclude this section by defining a bijection $\text{Tw} : X \rightarrow X$ as follows:

Proposition 2.3 Let $(X, \triangleleft, \kappa)$ be a skew-rack satisfying $\kappa^2 = \text{id}_X$, as in (SS2). Define the map $\text{Tw} : X \rightarrow X$ by setting $\text{Tw}(x) = \kappa(x) \triangleleft^{-1} \kappa(x)$. Then, the map is bijective, where the inverse is the map $X \rightarrow X$ that sends x to $\kappa(x) \triangleleft x$.

Proof When we let y be $\text{Tw}(\kappa(x) \triangleleft x)$, we may show $y = x$. Note that $x \triangleleft \kappa(x) = y \triangleleft (x \triangleleft \kappa(x))$, which is equal to

$$((y \triangleleft^{-1} x) \triangleleft x) \triangleleft (x \triangleleft \kappa(x)) = ((y \triangleleft^{-1} x) \triangleleft x) \triangleleft \kappa(x) = y \triangleleft \kappa(x).$$

Thus, by (SS2), we have $y = x$. Similarly, we can easily verify $\kappa(\text{Tw}(x)) \triangleleft \text{Tw}(x) = x$. □

3 Preliminaries: colorings and birack cocycle invariants

Our definition of X -colorings here is a slight modification of the classical X -colorings of quandles or biracks [4; 5; 8]. Let D be a framed link diagram D , and let $(X, \triangleleft, \kappa, \rho)$ be a symmetric skew-rack. Choose orientations o for each component of D , and denote by D^o the diagram with the orientations. In this paper, a *semiarc* of D means a path from a crossing to the next crossing along the diagram. Then, an X -coloring is a map $\mathcal{C} : \{\text{semiarc of } D\} \rightarrow X$ such that, for every crossing τ of D , the semiarcs around τ satisfy $\mathcal{C}(\gamma_\tau) = \kappa(\mathcal{C}(\beta_\tau))$ and $\mathcal{C}(\delta_\tau) = \mathcal{C}(\alpha_\tau) \triangleleft \mathcal{C}(\beta_\tau)$, where $\alpha_\tau, \beta_\tau, \gamma_\tau$, and δ_τ are the semiarcs shown in Figure 1. We denote by $\text{Col}_X(D^o)$ the set of X -colorings of D^o . Then, as a basic fact in quandle theory (see [5; 8]), if two diagrams D^o and $(D')^{o'}$ are related by a Reidemeister move of type II, type III, or a doubled type I, there exists a canonical bijection $\mathcal{B}_{D^o, (D')^{o'}} : \text{Col}_X(D^o) \rightarrow \text{Col}_X((D')^{o'})$. Moreover, thanks to the above axioms (SS1) and (SS2), if $D^{o'}$ is the same diagram D with opposite orientation, the correspondence $a \mapsto \rho(a)$ on the color of each semiarc on the opposite component defines a bijection $\mathcal{B}_{D^o, D^{o'}} : \text{Col}_X(D^o) \rightarrow \text{Col}_X(D^{o'})$. In particular, the set $\text{Col}_X(D^o)$ up to bijections does not depend on the choice of orientations of D . Accordingly, we sometimes use the expression $\text{Col}_X(D)$ instead of $\text{Col}_X(D^o)$. Finally, we should emphasize that the map $\text{Tw}^{\pm 1}$ in Proposition 2.3 corresponds to an addition of a (∓ 1) -framing in an arc, as in the Reidemeister move of type I.

Next, we observe cocycle invariants of a symmetric skew-rack X . According to [4; 5; 9], a map $\phi : X^2 \rightarrow A$ for some abelian group A is called a *birack 2-cocycle* if

$$(1) \quad \phi(a, b) + \phi(a \triangleleft b, c) = \phi(a, \kappa(c)) + \phi(a \triangleleft \kappa(c), b \triangleleft c), \quad \phi(b, c) = \phi(\kappa(b), \kappa(c))$$

hold for any $a, b, c \in X$. Then, we define *the weight (of τ)*, $\Phi(\tau)$, with respect to a crossing τ on D to be $\varepsilon_\tau \phi(\mathcal{C}(\alpha_\tau), \mathcal{C}(\beta_\tau)) \in A$, where ε_τ is the sign τ (as in Figure 1). We further define $\Phi_D(\mathcal{C}) \in A$ to be the sum $\sum_\tau \Phi(\tau)$, where τ runs over every crossing on D . Then, as is known [3; 5], if two diagrams D and D' are related by a Reidemeister move of type II, type III, or a doubled type I move, $\Phi_{D'} \circ \mathcal{B}_{D^o, (D')^{o'}} = \Phi_D$ holds as a map $\text{Col}_X(D^o) \rightarrow A$. In other words, the map $\Phi_D : \text{Col}_X(D^o) \rightarrow A$ up to bijections is an invariant of framed links with orientations. As in [4; 5], we call the map Φ *the (birack) cocycle invariant*.



Figure 1: Positive and negative crossings with eight labeled semiarcs.

Next, as an analogy of symmetric cocycle invariants in [12; 13], we discuss symmetric birack cocycles. We define a birack 2-cocycle $\phi : X^2 \rightarrow A$ to be *symmetric* if

$$\phi(a, b) = -\phi(a \triangleleft b, \rho(b)) = -\phi(\rho(a), \kappa(b)) \in A,$$

for any $a, b \in X$. Then, similarly to the discussion in [13, Theorem 6.3], we can easily confirm that the weight $\Phi(\tau)$ does not depend on the choice of orientations of D ; neither does the map $\Phi_D : \text{Col}_X(D^o) \rightarrow A$ up to bijections. In conclusion, the cocycle invariant $\Phi_D : \text{Col}_X(D^o) \rightarrow A$ up to bijections is an invariant of framed links.

Finally, we briefly review surgery on links and Fenn–Rourke moves [7]. Let us regard a framed link diagram as the surgery on the framed link in the 3-sphere. Conventionally, every closed 3-manifold M can be expressed as the result of S^3 of surgery on a framed link. Furthermore, two framed links in S^3 have orientation-preserving homeomorphic results of surgery if and only if the framed links are related by a finite sequence of Fenn–Rourke moves and isotopies [7], where the Fenn–Rourke move is an operation between the framed links shown in Figures 3 and 4 in Section 4. Throughout this paper, for a framed link diagram D of a link L , we denote by M_D the result of surgery of S^3 on L .

4 Topological invariants from skew-racks with property FR

Our objective is to explore appropriate skew-racks that yield birack cocycle invariants that are invariant with respect to the Fenn–Rourke moves. In this section, we define skew-racks with property FR and the colorings of closed 3-manifolds.

For $\varepsilon \in \{\pm 1\}$ and $a_1, \dots, a_n \in X$, let us consider the bijection

$$A_{a_1, \dots, a_n} : X \rightarrow X, \quad x \mapsto (\dots((x \triangleleft a_1) \triangleleft a_2) \triangleleft \dots) \triangleleft a_n,$$

and define the subsets

$$\begin{aligned} \text{Ann}^{+1}(A_{a_1, \dots, a_n}) &:= \{x \in X \mid \kappa^{n+1}(x) = A_{a_1, \dots, a_n}(x) \triangleleft \kappa^{n+1}(x)\}, \\ (2) \quad \text{Ann}^{-1}(A_{a_1, \dots, a_n}) &:= \{x \in X \mid \kappa^{n+1}(x) \triangleleft \kappa(A_{a_1, \dots, a_n}(x)) = A_{a_1, \dots, a_n}(x)\}. \end{aligned}$$

For the case $n = 0$, we define $\text{Ann}^{\pm 1}(X)$ to be the subset $\{x \in X \mid x \triangleleft \kappa(x) = \kappa(x)\}$. Schematically speaking, as in Figure 2, the set $\text{Ann}(X)^{\pm 1}$ is the set of X -colorings of the unknot of (± 1) -framing.

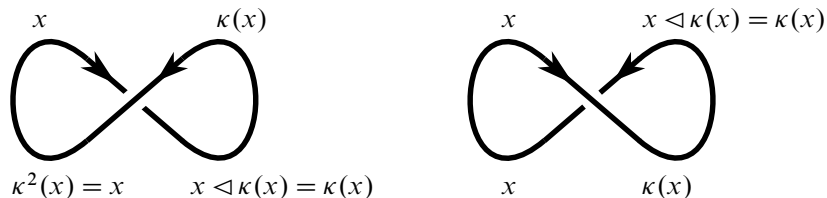


Figure 2: The coloring conditions of unknots of (± 1) -framing.

Definition 4.1 A symmetric skew-rack $(X, \triangleleft, \kappa, \rho)$ is said to have *property FR* if it satisfies the following:

(FR2) The subset $\text{Ann}(X)$ is not empty, and is bijective to the set $\text{Ann}^\varepsilon(A_{a_1, \dots, a_n})$ for arbitrary $n \in \mathbb{Z}$, $a_1, \dots, a_n \in X$ and $\varepsilon \in \{\pm 1\}$.

(FR2) For any $a_1, \dots, a_n \in X$ and $x \in \text{Ann}^{+1}(A_{a_1, \dots, a_n})$, $y \in \text{Ann}^{-1}(A_{a_1, \dots, a_n})$, the equalities

$$(3) \quad \kappa^{n+i}(a_i) = A_{a_1, \dots, a_n}(\kappa^{i+1}(a_i) \triangleleft x),$$

$$(4) \quad \kappa^{n+i}(a_i) \triangleleft \kappa^{n+1}(y) = A_{a_1 \triangleleft \kappa(y), a_2 \triangleleft \kappa^2(y), \dots, a_n \triangleleft \kappa^n(y)}(\kappa^{i+1}(a_i))$$

hold, where $i \leq n$ is arbitrary.

Let us analyze the set of colorings of skew-racks with property FR.

Theorem 4.2 Let $(X, \triangleleft, \kappa, \rho)$ be a symmetric skew-rack with property FR. Suppose that two framed link diagrams D and D' are related by a Fenn–Rourke move (as in Figures 3 and 4) and take orientations on D and D' .

Then, for any coloring $\mathcal{C} \in \text{Col}_X(D)$, there is uniquely another $\mathcal{C}' \in \text{Col}_X(D')$ such that $\mathcal{C}(\alpha_i) = \mathcal{C}'(\alpha'_i)$ and $\mathcal{C}(\beta_i) = \mathcal{C}'(\beta'_i)$ for any $i \leq n$. Furthermore, the map

$$(5) \quad \mathcal{B} : \text{Col}_X(D) \rightarrow \text{Col}_X(D') \times \text{Ann}(X), \quad \mathcal{C} \mapsto (\mathcal{C}', \mathcal{C}(\gamma)),$$

is bijective. Specifically, if X is of finite order, the rational number $|\text{Col}_X(D)|/|\text{Ann}(X)|^{\#D} \in \mathbb{Q}$ gives rise to a topological invariant of closed 3-manifolds.

Proof Take arcs γ, δ, α_i 's, and β_i 's as in Figures 3 and 4. By the properties of good involutions, the coloring conditions are independent of the choices of orientations of D . Thus, we fix the orientations of D and D' as shown in Figures 3 and 4. Given an X -coloring $\mathcal{C} \in \text{Col}_X(D)$, define $a_i := \mathcal{C}(\alpha_i)$, $b_i := \mathcal{C}(\beta_i)$. We now show that the map $\mathcal{C}' : \{\text{semiarc of } D'\} \rightarrow X$ defined by $\mathcal{C}'(\alpha'_i) = a_i$ and $\mathcal{C}'(\beta'_i) = b_i$ gives rise to a unique X -coloring.

First, suppose that \pm is positive and $x = \mathcal{C}(\gamma)$. The coloring condition on the arc δ is

$$\text{Tw}(\kappa^{n+1}(x)) = \kappa^{n+1}(x) \triangleleft^{-1} \kappa^{n+1}(x) = \mathcal{C}(\delta) = A_{a_1, \dots, a_n}(x) \triangleleft \kappa^{n+1}(x);$$

hence, $x \in \text{Ann}^{+1}(A_{a_1, \dots, a_n})$. Notice from (3) that

$$(6) \quad b_i = \kappa(a_i) \triangleleft \kappa^i(x) = \kappa^i(\kappa^{i+1}(a_i) \triangleleft x) = \kappa^i(A_{a_1, \dots, a_n}^{-1}(\kappa^{n+i}(a_i))) = A_{\kappa^i(a_1), \dots, \kappa^i(a_n)}^{-1}(\kappa^n(a_i)).$$

Meanwhile, the coloring condition on the arc δ_i in the right-hand side of Figure 3 is

$$A_{\kappa^i(a_1), \dots, \kappa^i(a_n)}(b_i) = \mathcal{C}(\delta_i) = \kappa^n(a_i),$$

which is equivalent to (6) exactly.

On the other hand, in the negative case of \pm and $y = \mathcal{C}(\gamma)$, the condition on γ is equivalent to $y \in \text{Ann}^{-1}(A_{a_1, \dots, a_n})$ by the definition of (2); notice that the coloring rule in D in Figure 3 implies $\kappa(b_i) = a_i \triangleleft \kappa^i(y)$ by the condition on the arc β_i . The coloring condition from β'_i to α'_i is equivalent

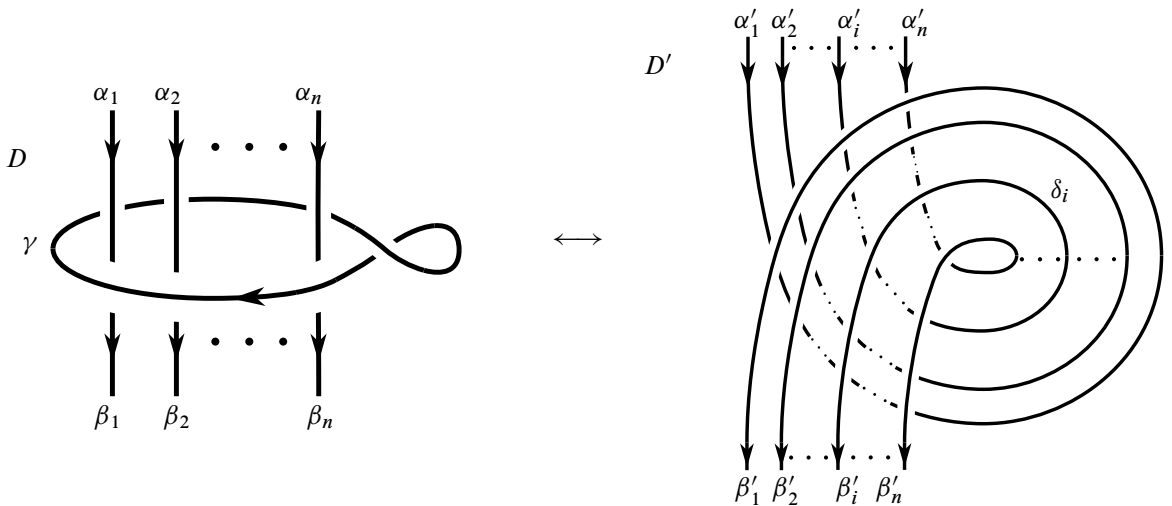


Figure 3: The positive Fenn–Rourke move and labeled semiarcs.

to $\kappa^n(b_i) = A_{\kappa^i(b_1), \dots, \kappa^i(b_n)}(a_i)$, that is, $\kappa^{n+i}(b_i) = A_{b_1, \dots, b_n}(\kappa^i(a_i))$, which directly follows from (4) since $\kappa(b_i) = a_i \triangleleft \kappa^i(y)$ by the condition on a_i .

Conversely, given an X -coloring \mathcal{C}' of $(D')^{o'}$ and $x \in \text{Ann}^{\pm 1}(A_{a_1, \dots, a_n}) \neq \emptyset$, we similarly can define an X -coloring \mathcal{C} of D that sends α_i to $\mathcal{C}'(\alpha'_i)$, β_i to $\mathcal{C}'(\beta'_i)$, and γ to x .

In summary, by construction, the correspondence $\mathcal{C} \mapsto (\mathcal{C}', \mathcal{C}(\gamma))$ gives the required bijection \mathcal{B} . \square

Before moving on to the next section, we briefly discuss triviality of the invariants up to link homotopy. For this, consider the permutation group $\text{Bij}(X)$ of a skew-rack X , and define a subgroup generated by

$$(7) \quad \{(\kappa(\bullet) \triangleleft a) \mid a \in X\} \cup \{(\bullet \triangleleft^{\epsilon_1} a_1) \triangleleft^{\epsilon_2} a_2 \mid a_i \in X, \epsilon_i \in \{\pm 1\}\}.$$

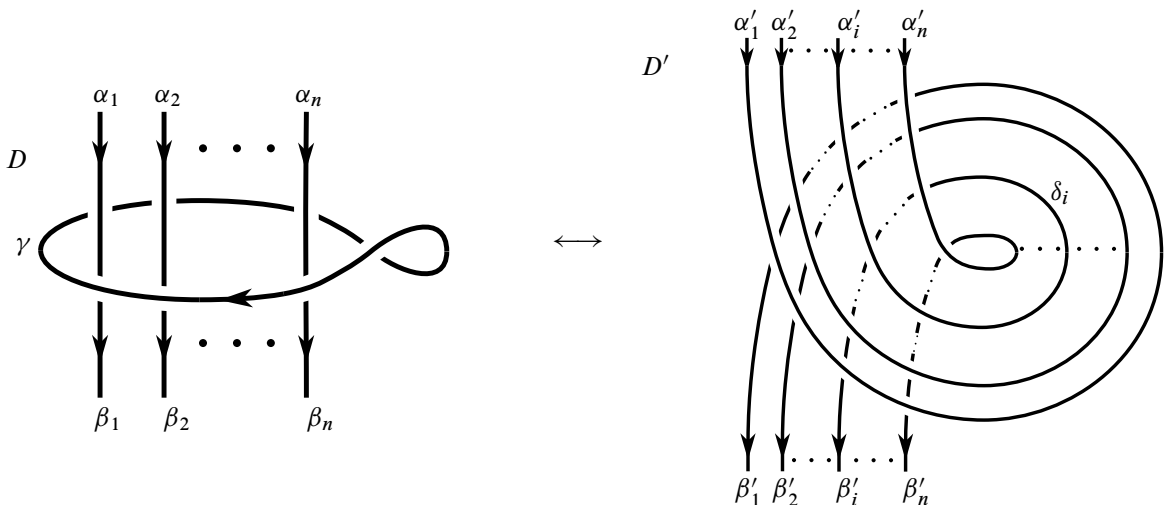


Figure 4: The negative Fenn–Rourke move and labeled semiarcs.

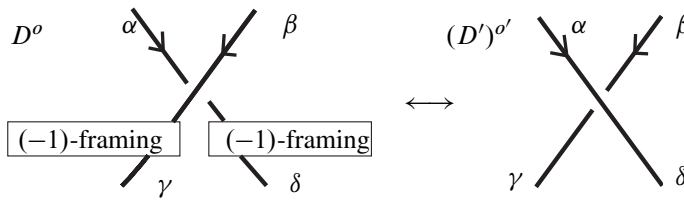


Figure 5: Diagrams D and D' , where all semiarcs lie within a link component.

The subgroup, denoted by $\text{Inn}_\kappa^{\text{even}}(X)$, canonically has the right action on X . We say a skew-rack $(X, \triangleleft, \kappa)$ with property FR is f -link homotopic if $x \triangleleft^\varepsilon \kappa(x) = x \triangleleft^\varepsilon (x \cdot g)$ holds for any $x \in X$, $g \in \text{Inn}_\kappa^{\text{even}}(X)$, $\varepsilon \in \{\pm 1\}$.

Proposition 4.3 Suppose a symmetric skew-rack $(X, \triangleleft, \kappa, \rho)$ with property FR is f -link homotopic. Then, if two framed link diagrams D and D' are transformed by the operation in Figure 5, there is a bijection $\mathcal{B}_f : \text{Col}_X(D^o) \rightarrow \text{Col}_X((D')^{o'})$.

Proof For a coloring $\mathcal{C} \in \text{Col}_X(D^o)$, take $a \in X$ such that $\mathcal{C}(\alpha) = \kappa(a)$. Since α and β lie on the same link-component, there is $g \in \text{Inn}_\kappa^{\text{even}}(X)$ such that $\mathcal{C}(\beta) = a \cdot g$ from definition (7). Since $\text{Tw}^{-1}(x) = x \triangleleft x$ by definition, the rule of colorings implies

$$\mathcal{C}(\gamma) = \text{Tw}^{-1}(\kappa(a \cdot g)) = (a \cdot g) \triangleleft \kappa(a \cdot g), \quad \mathcal{C}(\delta) = \text{Tw}^{-1}(\kappa(a)) \triangleleft (a \cdot g) = (a \triangleleft \kappa(a)) \triangleleft (a \cdot g).$$

Since X is f -link homotopic, $\mathcal{C}(\gamma) = (a \cdot g) \triangleleft^{-1} a$ and $\mathcal{C}(\delta) = a$. Thus, we can define another coloring $\mathcal{B}_f(\mathcal{C})$ of D' by $\mathcal{B}_f(\mathcal{C})(\alpha) = a$ and $\mathcal{B}_f(\mathcal{C})(\beta) = a \cdot g$. Since $\mathcal{C}(\gamma) = (a \cdot g) \triangleleft^{-1} a = \mathcal{B}_f(\mathcal{C})(\gamma)$ and $\mathcal{C}(\delta) = a = \mathcal{B}_f(\mathcal{C})(\delta)$ by definitions, the map $\mathcal{B}_f : \text{Col}_X(D^o) \rightarrow \text{Col}_X((D')^{o'})$ is bijective, as required. \square

Many 3-manifolds can be expressed as the results from S^3 of surgery along various framed knots, so to obtain nontrivial colorings, we consider skew-racks, which are not f -link homotopic.

5 Examples of skew-racks with property FR from groups

Here we provide examples of skew-racks with property FR. Throughout this section, we fix a group G , an automorphism $\kappa : G \rightarrow G$ satisfying $\kappa \circ \kappa = \text{id}_G$, and a map $\delta : G \rightarrow G$ satisfying $\kappa \circ \delta = \delta \circ \kappa$. Consider the binary operation $\triangleleft : G \times G \rightarrow G$ defined by $x \triangleleft y = \kappa(x)\delta(y)$. Then, the twisting map Tw in Proposition 2.3 is given by $\text{Tw}(g) = g\delta(g)^{-1}$.

Lemma 5.1 These operations (\triangleleft, κ) with $x \triangleleft y = \kappa(x)\delta(y)$ define a skew-rack of $X = G$ if and only if, for any $x, y \in G$,

$$(8) \quad \delta(x)\delta(y) = \delta(y)\delta(x\delta(y)) \in G.$$

Let $\rho : G \rightarrow G$ be a good involution. Further, assume that the image $\text{Im}(\delta) \subset G$ is a subgroup of G , and that the cardinality of the preimage $\delta^{-1}(d)$ is constant for any $d \in \text{Im}(\delta)$. Then, the symmetric skew-rack on $X = G$ has property FR.

In addition, if the subgroup $\text{Im}(\delta)$ is commutative, the skew-rack is f -link homotopic.

Proof Since the former part is shown by direct computation, we show only the remaining claims here. We now analyze the set $\text{Ann}^\varepsilon(A_{a_1, \dots, a_n})$ in (2). First, suppose $\varepsilon = +1$. Then, the condition $\kappa^{n+1}(x) = A_{a_1, \dots, a_n}(x) \triangleleft \kappa^{n+1}(x)$ is equivalent to

$$(9) \quad \delta(\kappa^n(a_1))\delta(\kappa^{n-1}(a_2)) \cdots \delta(\kappa(a_n))\delta(\kappa^{n+1}(x)) = 1.$$

Since $\text{Im}(\delta)$ is a subgroup of G by assumption, the set $\text{Ann}^{+1}(A_{a_1, \dots, a_n})$ is nonempty. Moreover, by the second assumption, the cardinality of $\text{Ann}^{+1}(A_{a_1, \dots, a_n})$ does not depend on the choice of a_1, \dots, a_n , that is, X satisfies (FR1). As for (FR2), the equality (3) is shown by

$$A_{a_1, \dots, a_n}(\kappa^{i-1}(a_i) \triangleleft x) = \kappa^{n+i}(a_i)\delta(\kappa^{n+i}(x))\delta(\kappa^{n+i+1}(a_1))\delta(\kappa^{n+i}(a_2)) \cdots \delta(\kappa^i(a_n)) = \kappa^{n+i+1}(a_i).$$

Next, we will show (4) in the case $\varepsilon = -1$. We can easily check $\kappa^{n+1}(x) \triangleleft \kappa(A_{a_1, \dots, a_n}(x)) = A_{a_1, \dots, a_n}(x)$ in (2) is equivalent to (9) exactly. Thus, similarly, the cardinality of $\text{Ann}^{-1}(A_{a_1, \dots, a_n}) \neq \emptyset$ does not depend on the choice of a_1, \dots, a_n . In addition, for $x \in \text{Ann}^{-1}(A_{a_1, \dots, a_n})$, the equality (4) is shown by

$$\begin{aligned} A_{a_1 \triangleleft \kappa(x), a_2 \triangleleft \kappa^2(x), \dots, a_n \triangleleft \kappa^n(x)}(\kappa^{i+1}(a_i)) \\ = \kappa^{i+n+1}(a_i)\delta(\kappa^{n-1}(a_1)\delta(\kappa^n(x)))\delta(\kappa^{n-2}(a_2)\delta(\kappa^n(x))) \cdots \delta(a_n\delta(\kappa^n(x))) \\ = \kappa^{i+n+1}(a_i)\delta(\kappa^n(x))^{-1}\delta(\kappa^{n-1}(a_1))\delta(\kappa^n(a_2)) \cdots \delta(a_n)\delta(\kappa^n(x)) = \kappa^{n+i}(a_i) \triangleleft \kappa^{n+1}(x). \end{aligned}$$

Here, the second and third equalities are obtained from (8) and (9), respectively. Hence, X has property FR, as required.

Finally, we show the last statement. From the definition of the subgroup $\text{Inn}_\kappa^{\text{even}}(X)$, any $g \in \text{Inn}_\kappa^{\text{even}}(X)$ and $a \in G$ admit uniquely $b_1, \dots, b_n \in \text{Im}(\delta)$ such that $a \cdot g = a\delta(b_1) \cdots \delta(b_n) \in G$. Since $\text{Im}(\delta)$ is commutative, (8) means $\delta(a) = \delta(a\delta(b))$. Thus,

$$(z \triangleleft^\varepsilon \kappa(a)) \triangleleft^{-\varepsilon} (a \cdot g) = z\delta(a)^\varepsilon\delta(a\delta(b_1) \cdots \delta(b_n))^{-\varepsilon} = z.$$

Therefore, the skew-rack is f -link homotopic by Proposition 4.3. □

We should point out that (8) comes with a few conditions. For example, if $|G| > 1$, the map δ is not surjective. In fact, if δ is surjective, then (8) with $x = 1$ is equivalent to $z^{-1}\delta(1)z = \delta(z)$ for any $z \in G$, which means that $\text{Im}(\delta)$ is a conjugacy class, and contradicts the surjectivity. However, we provide some examples that satisfy the conditions in Lemma 5.1.

Example 5.2 First, we observe the case where δ is a group homomorphism. Then, we can easily check that (8) is equivalent to that of $\delta \circ \delta = 0$ and the image $\text{Im}(\delta)$ is abelian. If so, the cardinality of $\delta^{-1}(k)$ is constant; thus, if X admits a good involution, the symmetric skew-rack has property FR, and is f -link homotopic by Lemma 5.1.

To avoid f -link homotopic skew-racks, we should focus on δ , which is not a homomorphism.

Example 5.3 (twisted conjugacy classes) Suppose we have a group automorphism $f : G \rightarrow G$, and define $\delta(x) = f(x^{-1})x$. Then, (8) is equivalent to $\text{Im}(\delta \circ \delta) = \{1_G\}$. In general, we can easily check that, for any $g \in \text{Im}(\delta)$, the preimage $\delta^{-1}(g)$ is bijective to the fixed-point subgroup $\{h \in G \mid f(h) = h\}$;

see, e.g., [2]. Thus, to apply Lemma 5.1, the remaining point is to analyze the situation such that the image $\text{Im}(\delta)$ is a subgroup.

The image $\text{Im}(\delta)$ is sometimes called as a *twisted conjugacy class* or *Reidemeister conjugacy class*. Prior works [2; 10] have investigated various conditions requiring that $\text{Im}(\delta)$ be a subgroup and $\text{Im}(\delta \circ \delta) = \{1_G\}$. However, many of the examples in those works satisfy that $\text{Im}(\delta)$ is commutative. Thus, it is difficult to find examples of pairs (G, f) satisfying that the resulting skew-racks are not f -link homotopic.

Example 5.4 Take a group K with a normal subgroup $N \trianglelefteq K$ and an involutive automorphism $f : K \rightarrow K$ satisfying $f(N) \subset N$. Let G be $K \times N$ and κ be $f \times f$. Define $\delta(x, y)$ as $(x^{-1}yx, 1)$, where $x, y \in K$. Next, we check the conditions in Lemma 5.1. Checking (8) is obvious: since $N = \{b^{-1}ab \mid a \in N, b \in K\}$, the image of δ is $N \times 1$ as a subgroup of G . Moreover, for any $(k, 1) \in K \times 1$, the preimage $\delta^{-1}(k, 1)$ is equal to $\{(y^{-1}ky, y) \in G \mid y \in K\}$, which is bijective to K . In conclusion, the symmetric skew-rack on G has property FR, by Lemma 5.1. For example, if $N = K$, the skew-rack on G is exactly equal to that in Example 2.2; here, we should remark $\text{Tw}(x, a) = (a^{-1}x, a)$.

Finally, we compute a few colorings using the above skew-racks with property FR.

Example 5.5 For natural numbers $n, m \in \mathbb{N}$, we first compute colorings of the lens space $L(nm - 1, n)$. Let $X = G$ be a skew-rack with good involution, which satisfies the conditions in Lemma 5.1. Let D be the Hopf link with framing (n, m) . Then, M_D is known to be $L(nm - 1, n)$. We fix two semiarcs α, β in each link-component on D . Then, from the definition of colorings, a coloring $C \in \text{Col}_X(D)$ satisfies

$$(10) \quad \kappa(C(\alpha) \triangleleft C(\beta)) = \text{Tw}^n(C(\alpha)), \quad \kappa(C(\beta) \triangleleft C(\alpha)) = \text{Tw}^m(C(\beta)).$$

Conversely, every $a, b \in X$ satisfying $\kappa(a \triangleleft b) = \text{Tw}^n(a)$ and $\kappa(b \triangleleft a) = \text{Tw}^m(b)$ yield a coloring of D . Since $\text{Tw}^n(a) = a\delta(a)^{-n}$, (10) is equivalent to conditions $C(\beta) = \delta(C(\alpha))^m$ and $\delta(C(\alpha))^{nm-1} = 1$. Hence, $\text{Col}_X(D)$ is bijective to

$$(11) \quad \{(a, b) \in G^2 \mid \delta(a)^{nm-1} = 1, \delta(b) = \delta(a)^n\} \xleftrightarrow{1:1} \{a \in G \mid \delta(a)^{nm-1} = 1\} \times \delta^{-1}(0),$$

where we use a bijection $\delta^{-1}(0) \leftrightarrow \{a \in G \mid \delta(a)^n = c\}$ for any $c \in G$. Therefore, the set $\text{Col}_X(D)$ depends only on nm ; it cannot classify the lens spaces of the forms $L(nm - 1, n)$. In contrast, we later compute various cocycle invariants that can distinguish among different lens spaces (see Example 6.7).

Example 5.6 Next, we observe that the sets of colorings of integral homology 3-spheres seem to be strong invariants, where we consider the skew-rack on $X = K \times K$ in Example 2.2. Let D_n^\pm be the $(2, n)$ -torus knot with framing ± 1 . Then, the resulting 3-manifold $M_{D_n^\pm}$ is the Brieskorn 3-manifold of the form $\Sigma(2, n, 2n \mp 1)$, as an integral homology 3-sphere. For a concrete group K , it is fairly easy to determine the set $\text{Col}_X(D_q^\pm)$ with the help of a computer program. A list of several computations of $|\text{Col}_X(D_q^\pm)|$ is provided in Table 1.

As seen in this example, it is reasonable to focus only on nonabelian groups K . In fact, if K is abelian, $(x, a) \triangleleft (y, b) = (x, a)$; hence, the coloring conditions are trivial; thus, considering the linking matrix of D , we can easily find a one-to-one correspondence $\text{Col}_X(D) \simeq \text{Hom}(H_1(M; \mathbb{Z}), K) \times K^{\#D}$.

p	$ \text{Col}_X(D_3^+) $	$ \text{Col}_X(D_3^-) $	$ \text{Col}_X(D_5^+) $	$ \text{Col}_X(D_5^-) $	$ \text{Col}_X(D_7^+) $	$ \text{Col}_X(D_7^-) $
3	$ K $	$ K $	$ K $	$ K $	$ K $	$ K $
5	$121 K $	$ K $	$121 K $	$ K $	$25 K $	$ K $
7	$ K $	$337 K $	$ K $	$ K $	$ K $	$49 K $
11	$2641 K $	$ K $	$2641 K $	$2641 K $	$ K $	$ K $
13	$ K $	$6553 K $	$ K $	$ K $	$ K $	$ K $

Table 1: Cardinality of $\text{Col}_X(D_n^{\pm 1})$ for various p, n . Here, $K = \text{SL}_2(\mathbb{F}_p)$ of order $p^3 - p$.

6 Cocycle invariants of 3-manifolds

As discussed in [3; 5; 15], there are various procedures to concretely find symmetric birack 2-cocycles. However, the condition that 2-cocycles must have invariance with respect to Fenn–Rourke moves seems strong. Nevertheless, we now investigate 2-cocycle invariants to obtain 3-manifold invariants. Throughout this section, we assume a symmetric skew-rack X with property FR, and a map ϕ from X^2 to an abelian group A .

We first introduce the property FR of birack 2-cocycles as follows.

Definition 6.1 Recall the bijection \mathcal{B} in Theorem 4.2, and denote by 0_A the constant map to A whose image is zero. A symmetric birack 2-cocycle $\phi : X^2 \rightarrow A$ satisfies *property FR*, if $\Phi_D = (\Phi_{D'} \times 0_A) \circ \mathcal{B}$ holds for any diagrams D and D' in Figures 3 and 4. Here, Φ_D is the cocycle invariant explained in Section 3.

Such a ϕ is said to be *f-link homotopic* if X is *f-link homotopic* and, for any $a \in X$ and $g \in \text{Inn}_\kappa^{\text{even}}(X)$,

$$(12) \quad \phi((a \cdot g) \triangleleft \kappa(a), a \cdot g) + \phi(a, a \triangleleft \kappa(a)) = \phi(\kappa(a), a \cdot g) + \phi((a \cdot g) \triangleleft \kappa(a), a).$$

We will see (Proposition 6.2) that symmetric birack 2-cocycles with property FR yield topological invariants of closed 3-manifolds. Take two maps $F : Y \rightarrow A$ and $G : Z \rightarrow A$, where Y and Z are some sets. We call F an *FR-stabilization* of G if there is a bijection $B : Z \rightarrow Y \times \text{Ann}(X)$ such that $g \circ B^{-1} = f \times 0_A$. More generally, F and G are *FR-equivalent* if F and G are related by a finite sequence of FR-(dis-)stabilizations. Then, the following proposition is fairly obvious by definitions.

Proposition 6.2 Let ϕ be a symmetric birack 2-cocycle with property FR. Then, the correspondence $D \mapsto \Phi_D$ up to FR-equivalent relations is an invariant of closed 3-manifolds.

Moreover, if X and ϕ are *f-link homotopic*, and if D and D' are related by the operation in Figure 5, then $\Phi_D = \Phi_{D'} \circ B_f$, where B_f is the bijection $\text{Col}_X(D) \rightarrow \text{Col}_X(D')$ in the proof of Proposition 4.3.

To conclude, to obtain 3-manifold invariants, it is important to find symmetric birack 2-cocycles with concrete expressions. In this context, we discuss Lemmas 6.3 and 6.4 below. Let \tilde{X} be $X \times A$. Define $\tilde{\triangleleft} : \tilde{X} \times \tilde{X} \rightarrow \tilde{X}$ by

$$(x, a) \tilde{\triangleleft} (y, b) = (x \triangleleft y, a + \phi(x, y)), \quad (x, y \in X, a, b \in A),$$

and $\tilde{\kappa} : \tilde{X} \rightarrow \tilde{X}$ by $\tilde{\kappa}(x, a) = (\kappa(x), -a)$.

Lemma 6.3 (see [3, Section 3]) *These maps $\tilde{\triangleleft}, \tilde{\kappa}, \tilde{\delta}$ define a skew-rack on $\tilde{X} = X \times A$ if and only if ϕ is a birack 2-cocycle.*

Proof Describe the distribution law as in (SR3) as

$$\begin{aligned} ((x, a) \tilde{\triangleleft} (y, b)) \tilde{\triangleleft} (z, c) &= ((x \triangleleft y) \triangleleft z, a + \phi(x, y) + \phi(x \triangleleft y, z)), \\ ((x, a) \tilde{\triangleleft} \tilde{\kappa}(z, c)) \tilde{\triangleleft} ((y, b) \tilde{\triangleleft} (z, c)) &= ((x \triangleleft y) \triangleleft z, a + \phi(x, \kappa(z)) + \phi(x \triangleleft \kappa(z), y \triangleleft z)). \end{aligned}$$

Furthermore, (SR1) implies $\phi(\kappa(x), \kappa(y)) = \phi(x, y)$. Hence, the desired claim follows directly from the definition (1) of birack 2-cocycle. □

Lemma 6.4 *Let ϕ be a birack 2-cocycle satisfying $\phi(a, b) = -\phi(\rho(a), \kappa(b))$. Then, the map $\bar{\phi} : X^2 \rightarrow A$ that sends (a, b) to $\phi(a, b) - \phi(a \triangleleft b, \rho(b))$ is a symmetric birack 2-cocycle.*

Proof It is easy to confirm $\bar{\phi}(a, b) + \bar{\phi}(a \triangleleft b, \rho(b)) = 0$. Thus, all that remains is to check the cocycle condition (1) of $\bar{\phi}$. For this, we may show

$$(13) \quad \phi(a \triangleleft b, \rho(b)) + \phi((a \triangleleft b) \triangleleft c, \rho(c)) = \phi(a \triangleleft \kappa(c), \rho(\kappa(c))) + \phi((a \triangleleft b) \triangleleft c, \rho(b \triangleleft c)).$$

Replace $(a \triangleleft b) \triangleleft c, \rho(b \triangleleft c)$, and $\rho(\kappa(c))$ with a, b , and c , respectively. Then, we can easily check that the replacement of (13) coincides with (1). □

Using these lemmas, we provide examples from several skew-racks in Example 5.4. Let $N \trianglelefteq K$ be groups and $f : K \rightarrow K$ be an involutive automorphism satisfying $f(N) \subset N$. Further, take a normalized group 2-cocycle $\theta : K \times K \rightarrow A$, where θ satisfies

$$\theta(x, y) - \theta(x, yz) + \theta(xy, z) - \theta(y, z) = 0, \quad \theta(1_K, x) = \theta(x, 1_K) = 0 \in A,$$

for any $x, y, z \in K$. Then, the product of $\tilde{K} = K \times A$ has a group structure with operation $((x, a), (y, b)) \mapsto (xy, a + b + \theta(x, y))$ as a central extension of K . As is known in group cohomology, every central extension over K with fiber A can be expressed by the product for some θ . Then, from Example 5.4, we can define the symmetric skew-racks on $G = K \times N$ and $\tilde{G} = \tilde{K} \times \tilde{N}$, which have property FR. Moreover, by the definition of \triangleleft on \tilde{G} , we obtain

$$\begin{aligned} ((x, a), (y, b)) \\ \triangleleft ((z, c), (w, d)) &= (\tilde{\kappa}(x, a)(z^{-1}, -c - \theta(z, z^{-1}))(w, d)(z, c), (y, b)) \\ &= ((f(x)z^{-1}wz, f(a) + d + \theta(f(x), z^{-1}) + \theta(f(x)z^{-1}, wz) + \theta(w, z) - \theta(z, z^{-1})), (y, b)) \in \tilde{G}. \end{aligned}$$

Inspired by Lemma 6.4, we obtain the following procedure for producing birack 2-cocycles:

Theorem 6.5 *Let $\lambda : N \rightarrow A$ be a group 1-cocycle. Then, the map*

$$\begin{aligned} \phi_{\lambda, \theta} : G^2 = (K \times N) \times (K \times N) &\rightarrow A, \\ (x, y, z, w) &\mapsto \lambda(y)(\theta(f(x), z^{-1}) + \theta(f(x)z^{-1}, wz) + \theta(z, w) - \theta(z, z^{-1})) \in A, \end{aligned}$$

is a birack 2-cocycle of the skew-rack $G = K \times N$ in Example 2.2. If

$$\lambda(x)\theta(a, b) = \lambda(f(x))\theta(f(a), f(b))$$

hold for any $a, b \in K, x \in N$, the condition in Lemma 6.4 is true. Specifically, the cocycle $\overline{\phi_{\lambda, \theta}}$ mentioned in Lemma 6.4 is a symmetric birack 2-cocycle.

In general, it may seem difficult to find group 2-cocycles θ such that the associated map $\phi_{\lambda, \theta}$ has property FR. However, when K is a cyclic group, we give such examples of birack cocycles with property FR. More precisely, by a direction computation, we can show the following.

Proposition 6.6 Let $p \in \mathbb{Z}$ be an odd prime. Let $K = N = \mathbb{Z}/p$, and take $\varepsilon \in \{\pm 1\}$ such that $f(x) = \varepsilon x$. Define group cocycles λ and θ by setting

$$\lambda(x) = x, \quad \theta(x, y) = \frac{(x + \varepsilon y)^p - x^p - (\varepsilon y)^p}{p} = \sum_{j:1 \leq j < p} j^{-1} x^j (\varepsilon y)^{p-j},$$

respectively, where $x, y \in \mathbb{Z}/p$. Then, $\overline{\phi_{\lambda, \theta}}(x, y, z, w) = 2y\theta(x, w)$, and the symmetric birack 2-cocycle $\overline{\phi_{\lambda, \theta}}$ has property FR and is f -link homotopic.

Example 6.7 Let D be the Hopf link with framings (n, m) , as in Example 5.5. Recall that M_D is the lens space $L(nm - 1, m)$. By (11), if $nm - 1$ is divisible by p and $K = \mathbb{Z}/p$, then $\text{Col}_X(D)$ is bijective to $(\mathbb{Z}/p)^2$. In addition, we can easily show that the cocycle invariant $\Phi_D : (\mathbb{Z}/p)^2 \rightarrow \mathbb{Z}/p$ is equal to the correspondence $(x, y) \mapsto -mx^2$, where we use the 2-cocycle $\overline{\phi_{\lambda, \theta}}$ in Proposition 6.6. For example, the invariant can distinguish between the lens spaces $L(11, 1)$ and $L(11, 3)$, which are not homotopy equivalent.

More generally, consider the lens space $L(p, q)$ and a framed diagram $D_{p,q}$ such that $M_{D_{p,q}} = L(p, q)$. Then, with the help of a computer program, if $p, q < 100$, it is fairly easy to check that the cocycle invariant $\Phi_{D_{p,q}} : (\mathbb{Z}/p)^{1+\#D_{p,q}} \rightarrow \mathbb{Z}/p$ is FR-equivalent to the map $\mathbb{Z}/p \rightarrow \mathbb{Z}/p; x \mapsto -qx^2$.

From this example, it is natural to pose the problem below, together with a relation to the Dijkgraaf–Witten invariant [6, §6]. We first briefly review the invariant. Fix a closed 3-manifold M with fundamental homology 3-class $[M] \in H_3(M; \mathbb{Z}) \cong \mathbb{Z}$. Let K be a group of finite order, and $\psi : K^3 \rightarrow A$ be a group 3-cocycle. Denote by BK the classifying space of K or the Eilenberg–Mac Lane space of type $(K, 1)$, and $c_M : M \rightarrow B\pi_1(M)$ be a classifying map. Then, any group homomorphism $f : \pi_1(M) \rightarrow K$ induces a continuous map $f_* : B\pi_1(M) \rightarrow BK$. Since the (co)homology of BK equals that of K , we can define the pullback $(f_* \circ c_M)^*(\psi)$ as a 3-cocycle of M . Then, the Dijkgraaf–Witten invariant is defined as the map

$$DW_\psi(M) : \text{Hom}(\pi_1(M), K) \rightarrow A, \quad f \mapsto \langle (f_* \circ c_M)^*(\psi), [M] \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Kronecker map.

Problem 6.8 As in Example 2.2, let X be the symmetric skew-rack on $K \times K$. Let $\lambda : K \rightarrow A$ and $\theta : K^2 \rightarrow A$ be group cocycles, and ψ be the cup product $\lambda \smile \theta$ as a group 3-cocycle. Let D be a framed link diagram.

Then, is there a bijection $\mathcal{B} : \text{Col}_X(D) \simeq \text{Hom}(\pi_1(M), K) \times \text{Ann}(X)^{\#D}$? Further, find a condition such that the birack 2-cocycle $\overline{\phi_{\lambda, \theta}}$ in Theorem 6.5 has property FR, and FR-equivalence between the cocycle invariant $\Phi : \text{Col}_X(D) \rightarrow A$ and the Dijkgraaf–Witten invariant $\text{DW}_\psi(M_D)$.

If this problem is correctly solved, we consequently obtain a diagrammatic computation of the Dijkgraaf–Witten invariant via the cocycle invariants and Dehn surgery.

7 Criteria for 3-manifolds that are not the result of surgery of any knot

As an application of the cocycle invariant, we provide two criteria to detect 3-manifolds that are not the result of surgery of any knot in S^3 (see [1, Section 7.1; 11] for the details of such 3-manifolds and other criteria). As in Example 5.4, we fix groups $N \trianglelefteq K$, and $X = K \times N$ with $f = \text{id}_K$; recall that X is a skew-rack by $(x, a) \triangleleft (y, b) = (xy^{-1}by, a)$, and has property FR.

Proposition 7.1 *Suppose $|K| < \infty$ and that a framed link diagram D and a knot diagram of framing zero are related by a sequence of Fenn–Rourke moves and isotopy. Then, the invariant $|\text{Col}_X(D)|/|K|^{\#D} \in \mathbb{Q}$ in Theorem 4.2 is larger than or equal to $|N|$.*

Proof We may suppose that D is a knot diagram of framing zero. For the proof, it is sufficient to construct $|K \times N|$ colorings on D . As in Figure 6, take semiarcs α_i and β_i in D , and denote by $\varepsilon_i \in \{\pm 1\}$ the sign of the crossing between α_i and β_i . For $(g, h) \in K \times N$, we define $C_{g,h}(\alpha_i)$ to be $(h^{\sum_{j=1}^{i-1} \varepsilon_j} g, h) \in X = K \times N$. Since every β_i lies on the same link component, $C_{g,h}(\beta_i) = (h^{n_i} g, h)$ for some $n_i \in \mathbb{Z}$. Hence, we can easily check that $C_{g,h}$ defines an X -coloring as required. □

As a special case, let $K = N = \mathbb{Z}/2$. For $k_1, k_2, k_3 \in \mathbb{Z}/2$, we define a map $\phi_{k_1, k_2, k_3} : X \times X \rightarrow \mathbb{Z}/2$ by setting

$$\phi_{k_1, k_2, k_3}((x, a), (y, b)) = k_1 a + k_2 b + k_3 ab.$$

Then, by direct computation, it is not hard to show the following:

Proposition 7.2 *The map ϕ_{k_1, k_2, k_3} is a symmetric birack 2-cocycle with property FR, and is f -link homotopic. Furthermore, if a framed link diagram D is FR-equivalent to a knot diagram of framing zero, then the symmetric birack 2-cocycle invariant is trivial.*

Unfortunately, we could not find any new examples of framed link diagrams that are not FR-equivalent to any knot diagram of framing zero. We end this paper by presenting problems to be investigated in future work.

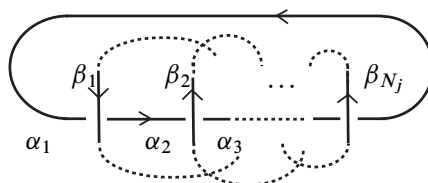


Figure 6: Semiarcs α_i and β_i in the knot diagram D .

Problem 7.3 As applications of the propositions above, find 3-manifolds that are not the surgery of any knot of framing zero. Establish stronger criteria than the propositions above, which are applicable to many framed link diagrams.

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