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Acylindrical hyperbolicity for Artin groups with a visual splitting

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We establish a criterion that implies the acylindrical hyperbolicity of many Artin groups admitting a visual splitting. This gives a variety of new examples of acylindrically hyperbolic Artin groups, including many Artin groups of FC-type.

Our approach relies on understanding when parabolic subgroups are weakly malnormal in a given Artin group. We formulate a conjecture for when this happens, and prove it for several classes of Artin groups, including all spherical-type, all two-dimensional, and all even FC-type Artin groups. In addition, we establish some connections between several conjectures about Artin groups, related to questions of acylindrical hyperbolicity, weak malnormality of parabolic subgroups, and intersections of parabolic subgroups.

1 Introduction

Background and motivation Artin groups are generalizations of braid groups, with many connections to Coxeter groups. Artin groups remain largely mysterious in general, both from an algebraic and geometric viewpoint, although significant progress has been made in studying specific classes: spherical-type, FC-type, and 2-dimensional Artin groups, etc. (see Section 2 for the definition of each class). For Artin groups outside of these classes, it remains unknown in general whether they have solvable word problem, contain torsion elements, or have nontrivial centers. Geometrically, while no Artin groups other than free groups are hyperbolic (as they otherwise contain $\mathbb{Z}^2$-subgroups), it is expected that they are all CAT(0), although this is still open even for braid groups. Some classes of Artin groups have been shown to satisfy other notions of nonpositive curvature; see, for instance, [21; 23; 25; 26]

Acylindrical hyperbolicity is a notion encapsulating the idea of a group "having hyperbolic directions", a very weak form of hyperbolic behaviour. Many groups of geometric interest are known to be acylindrically hyperbolic, and despite its generality, this notion is strong enough to have important consequences for the structure of the group. (We refer the reader to [37; 38] for a discussion of the consequences.)

In the case of Artin groups, there is a clear conjectural picture of when they are expected to be acylindrically hyperbolic:

Conjecture (acylindrical hyperbolicity conjecture) *Let A_Γ be an irreducible Artin group. Then the central quotient $A_G/Z(A_\Gamma)$ is acylindrically hyperbolic.*

This conjecture essentially states that Artin groups are expected to be acylindrically hyperbolic unless they "clearly cannot be". Indeed, reducible Artin groups cannot be acylidrically hyperbolic as they split as direct products of infinite groups. Spherical-type Artin groups cannot be acylindrically hyperbolic as

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they have an infinite cyclic centre (but the acylindrical hyperbolicity of their central quotient was proved by Calvez and Wiest [8]). Note that Artin groups of nonspherical type are conjectured to have a trivial centre, so in that case the conjecture states that an Artin group of nonspherical type is acylindrically hyperbolic if and only if it is irreducible. This conjecture has been proved for several families of Artin groups already; see Section 2 for details.

Beside being interesting in its own right, the question of acylindrical hyperbolicity for Artin groups has applications to some well-known open problems for these groups. A first possible application is to the centre of these groups, as Artin groups of infinite type are conjectured to have a trivial centre. Since it is known that acylindrically hyperbolic groups have a finite centre, showing this property is a possible first step towards proving the triviality of the centre. Another possible application comes from the isomorphism problem, which asks which labelled graphs produce isomorphic Artin groups. Very little is currently known about the isomorphism problem for Artin groups; see, for instance, [14; 17; 31; 39; 41]. For instance, it is not even known whether being a spherical-type Artin group, or being an irreducible Artin group, is invariant under isomorphism. Since acylindrically hyperbolic groups have finite centres and do not split as direct products of infinite factors, a positive answer to the acylindrical hyperbolicity conjecture would imply that both aforementioned properties are indeed invariant under isomorphism.

A particular family of Artin groups all of whose elements are expected to be acylindrically hyperbolic is the family of Artin groups whose presentation is not a complete graph. Such Artin groups have the useful feature of decomposing as amalgamated products of standard parabolic subgroups (over a standard parabolic subgroup). Such splittings, which we will refer to as *visual splittings* as they can be read directly from the presentation graph, have been used to derive properties of the Artin group from the properties of the corresponding parabolic subgroups; see, for instance, [11; 12; 18; 35]. For groups splitting as amalgamated products, and more generally for groups acting on trees, there is a useful acylindrical hyperbolicity criterion due to Minsayan–Osin [34]. In a nutshell (see Theorem 3.1 for the precise statement), a nonvirtually cyclic group G splitting as an amalgamated product $G = A *_C B$ is acylindrically hyperbolic as soon as the edge group C is *weakly malnormal* in G, i.e., as soon as C intersects one of its conjugates along a finite subgroup. In the case of Artin groups admitting a visual splitting, this amounts to understanding when standard parabolic subgroups are weakly malnormal in the ambient group. We introduce the following conjecture, which provides a complete description of when this is expected to happen:

Conjecture (weak malnormality conjecture) *A proper standard parabolic subgroup of A_Γ is weakly malnormal if and only if it does not contain a standard parabolic subgroup that is a direct factor of A_Γ.*

In particular, if A_Γ is irreducible, then every proper standard parabolic subgroup is weakly malnormal.

In trying to apply the criterion of Minsayan–Osin, we are thus led to study the intersections of standard parabolic subgroups. Such intersections have been heavily studied in recent years in connection with other problems about Artin groups. In particular, the following conjecture is particularly relevant:

Conjecture (intersection conjecture) *The intersection of any two parabolic subgroups in A_Γ is again a parabolic subgroup.*

This conjecture has been proved for a few families of Artin groups but remains open in general. For some families of Artin groups, certain weaker versions have been established; see Section 2 for more details.

In this paper, we study the connections between these three conjectures, and show the acylindrical hyperbolicity of new classes of Artin groups.

Statement of results We now state the main results of this article. Our main theorem is a criterion for showing the acylindrical hyperbolicity of an Artin group admitting a visual splitting, under a mild assumption on the amalgamating subgroup:

Theorem A *Let A_Γ be an irreducible Artin group that splits visually as an amalgamated product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. If the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ, then A_Ω is weakly malnormal in A_Γ. In particular, A_Γ is acylindrically hyperbolic.*

As a consequence of this result, we also obtain the following result showing the connection between the three conjectures at the centre of this article:

Theorem B *Suppose A_Γ is irreducible and Γ is not a clique.*

- *If A_Γ satisfies the intersection conjecture, then it also satisfies the weak malnormality conjecture.*
- *If A_Γ satisfies the weak malnormality conjecture, then it also satisfies the acylindrical hyperbolicity conjecture.*

Theorem A can be used to show the acylindrical hyperbolicity of many new classes of Artin groups. For instance, using the existing results about intersections of parabolic subgroups in Artin groups of FC type [2], we obtain the following:

Corollary C *Even Artin groups of FC type satisfy the acylindrical hyperbolicity conjecture. In addition, any Artin group of FC type that visually splits over a spherical-type parabolic subgroup satisfies the acylindrical hyperbolicity conjecture.*

After this article was first released, Kato–Oguni announced a proof of the acylindrical hyperbolicity conjecture for the class of free-of-infinity Artin groups [28], which contains the above classes.

Note that in Theorem A, the condition on the edge group is strictly weaker than requiring that the whole Artin group A_Γ satisfies the intersection conjecture. In Section 3.3, we give examples of how to check this property in some cases, by using a framework of Godelle–Paris [19] to construct suitable CAT(0) cube complexes associated to Artin groups. As an application, we derive the acylindrical hyperbolicity of some new Artin groups whose underlying graph is a cone (Corollary 3.19), and for which the intersection conjecture is currently unknown.

In some special cases, the main hypothesis of Theorem A may be verified simply by checking that the edge group is weakly malnormal in one of the vertex groups. We thus provide a list of Artin groups for which we know the weak malnormality conjecture, as this allows us to quickly verify this condition, and also allows us to construct new examples of acylindrically hyperbolic Artin groups (see Corollary 4.7):

Theorem D *The weak malnormality conjecture holds for the following classes of groups*:

- *Artin groups satisfying the hypothesis of Theorem A* (*for instance*, *even Artin groups of FC type*),
- *Artin groups of spherical type*,
- *two-dimensional Artin groups.*

Organisation of the paper In Section 2, we recall the terminology and some standard results about Artin groups and their parabolic subgroups. In Section 3, we prove Theorem A and Corollary C by studying in detail the action of these Artin groups on their Bass–Serre trees. We also use CAT(0) cube complexes introduced by Godelle–Paris to prove the acylindrical hyperbolicity of additional classes of Artin groups. In Section 4, we prove Theorems B and D by studying the geometry of the orbits of parabolic subgroups in a suitable complex (depending on the case: Bass–Serre tree of a splitting, or Deligne complex of the group).

2 Preliminaries on Artin groups

Definition 2.1 Let Γ be a graph with vertices labelled by the set S and any edge between s and t labelled by an integer $m_{st} \in \{2, 3, \dots\}$. Define the Artin group by the presentation

$$A_\Gamma = \langle S \mid \underbrace{stst\dots}_{m_{st}\text{ terms}} = \underbrace{tsts\dots}_{m_{st}\text{ terms}} \text{ for all edges in } \Gamma\rangle.$$

Note that in this definition, two vertices s and t which are not joined by an edge in Γ have the free relation and in this case we define $m_{st} = \infty$. The graph Γ is often called the presentation graph. In some literature, a different defining graph is used called the Dynkin diagram, wherein edges with $m_{st} = 2$ are omitted, while edges with $m_{st} = \infty$ are included.

For every Artin group, there is an associated Coxeter group, which is obtained by adding to the Artin presentation the relation $s^2 = 1$ for all generators s. An Artin group is called *spherical-type* or *finite-type* if the corresponding Coxeter group is finite. Such Artin groups have well-understood algebraic and geometric properties in comparison to their infinite-type cousins.

The *dimension* of an Artin group A_Γ is the maximal size of a subgraph $\Omega \subset \Gamma$ such that A_Ω is spherical-type. So for example, a 2-dimensional Artin group is one where the presentation graph Γ is not discrete and for which the only spherical-type parabolic subgroups are either cyclic (1-generator) or dihedral (2-generator) Artin groups.

Definition 2.2 An Artin group A_Γ is said to be

- of *spherical type* if the corresponding Coxeter group W_Γ is finite,
- of *FC type* if for every induced complete subgraph $\Gamma' \subset \Gamma$, the corresponding Coxeter group $W_{\Gamma'}$ is finite,

- *two-dimensional* if Γ is not discrete and for every triangle of Γ with vertices a, b, c, we have
$$\frac{1}{m_{ab}}+\frac{1}{m_{bc}}+\frac{1}{m_{ac}}\leq 1$$
(this is equivalent to requiring that A_Γ has cohomological dimension 2, and also coincides with the notion of dimension of an Artin group introduced above),
- of *even type* if all labels of Γ are even.

2.1 Parabolic subgroups

For an induced subgraph Ω of Γ, the Artin group A_Ω embeds as a subgroup of A_Γ [29]. We call such a subgroup a *standard parabolic subgroup*. A *parabolic subgroup* is a conjugate of some standard parabolic subgroup. Parabolic subgroups play a central role in our understanding of Artin groups. For example, many conjectures about Artin groups can be reduced to the case of parabolic subgroups corresponding to cliques in Γ, that is, if A_Ω satisfies the conjecture for every clique Ω in Γ, then the conjecture also holds for A_Γ [12; 18].

We say that an Artin group A_Γ is *reducible* if there exist two induced disjoint subgraphs Γ_1, Γ_2 of Γ such that $V(\Gamma)=V(\Gamma_1)\cup V(\Gamma_2)$, and every vertex of Γ_1 is connected to every vertex of Γ_2 by an edge labelled 2. In that case, we have that A_Γ decomposes as the direct product $A_{\Gamma_1}\times A_{\Gamma_2}$, and the parabolic subgroups A_{Γ_1}, A_{Γ_2} are called *direct factors* of A_Γ. If no such subgraphs Γ_1, Γ_2 exist, the Artin group A_Γ is called *irreducible*.

The behaviour of parabolic subgroups will be key to the discussion which follows. As noted above, we do not know in general if intersections of parabolic subgroups are parabolic. A useful fact proven by Blufstein and Paris [5], is that if $P\subseteq P'$ are two parabolic subgroups of A_Γ, then P is also a parabolic subgroup of P'. The following lemma will be often used in this article.

Lemma 2.3 *Let A_Γ be an Artin group. Then for every sequence of parabolic subgroups $H_0\subsetneq\cdots\subsetneq H_n$, we have $n\leq|V(\Gamma)|$. In particular, there is an upper bound on the length of chains of parabolic subgroups.*

Proof Say H_i is a conjugate of A_{Ω_i}. By Blufstein–Paris [5], for each i, H_{i-1} is a (proper) parabolic subgroup of H_i, so Ω_{i-1} must be a proper subgraph of Ω_i, that is, $\Omega_0\subsetneq\cdots\subsetneq\Omega_n\subseteq\Gamma$. □

A geometric construction that has played a primary role in the study of Artin groups is the Deligne complex. For an infinite-type Artin group A_Γ, let $\mathcal{P}_\Gamma$ denote the poset consisting of cosets $aA_T\subset A_\Gamma$ such that A_T is a spherical-type parabolic subgroup. Partially order $\mathcal{P}_\Gamma$ by inclusion. The *Deligne complex* D_Γ is the cell complex whose vertices are the elements of $\mathcal{P}_\Gamma$ and whose cells are cubes spanned by intervals $[aA_T, aA_{T'}]$ for pairs $aA_T\subseteq aA_{T'}$. The Artin group acts on the Deligne complex by left multiplication, and the vertex corresponding to the coset aA_T is stabilised by the parabolic subgroup $(A_T)^a:=aA_Ta^{-1}$.

There are two well-known metrics on D_Γ. One is the standard cubical metric; this metric is CAT(0) if and only if A_Γ is FC-type. The other is a piecewise Euclidean metric, called the Moussong metric, in which the metric on a cube $[aA_T, aA_{T'}]$ depends on the shape of the Coxeter cell for $W_{T'}$. It is conjectured that the Moussong metric is CAT(0) for all infinite-type Artin groups. This has been shown to hold for all 2-dimensional Artin groups [11], some 3-dimensional Artin groups [10], and a class known as locally reducible Artin groups [9].

2.2 Visual splittings

Acylindrically hyperbolic groups do not have infinite direct factors. Likewise a group that factors as a direct product clearly has proper subgroups which are not weakly malnormal. Thus, we will focus on irreducible Artin groups in this paper. We will also focus on Artin groups that can be decomposed as amalgamated products.

Definition 2.4 (visual splitting of an Artin group) A *visual splitting* of an Artin group A_Γ is a splitting as an amalgamated free product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ where Γ_1 and Γ_2 are proper, induced subgraphs of Γ and $\Omega = \Gamma_1 \cap \Gamma_2$. This happens precisely when $\Gamma = \Gamma_1 \cup \Gamma_2$. Note that under such assumptions on Γ_1 and Γ_2, the associated splitting is nontrivial, i.e., $A_\Omega \neq A_{\Gamma_1}, A_{\Gamma_2}$.

Note that such a splitting exists if and only if Γ is not a clique. Indeed, assume that the two vertices $s, t \in V(\Gamma)$ are not connected by an edge in Γ. Then one checks that A_Γ splits as the amalgamated product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ with $\Gamma_1 = \Gamma - \{s\}$, $\Gamma_2 = \Gamma - \{t\}$, and $\Omega = \Gamma - \{s, t\}$.

When an Artin group $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ admits a visual splitting, we can sometimes derive properties of A_Γ using properties of A_{Γ_1}, A_{Γ_2}, and A_Ω. This can, for instance, be done to study Artin groups of FC type, as such groups can always be decomposed as a sequence of nested amalgamated free products, where the final splitting has spherical-type edge groups.

2.3 Existing results for the main conjectures

In this section we will review known results for the main conjectures, and prove some elementary implications that we will use later.

Intersection conjecture In 1983, van der Lek showed that the intersection of standard parabolic subgroups $A_{\Gamma_1} \cap A_{\Gamma_2}$ is always parabolic [29]. More recently, the more general intersection conjecture, namely the property that the intersection of any two parabolics is again parabolic, has been proven to hold for the following classes of Artin groups:

- spherical-type Artin groups [15],
- right-angled Artin groups and other graphs of groups [3],
- large type (i.e., all labels satisfy $m_{s,t} \geq 3$) Artin groups [16],
- (2,2)-free two-dimensional Artin group, i.e., Γ does not have two consecutive edges labelled by 2 and the cohomological dimension of A_Γ is 2 [4],
- Euclidean type of the form $\widetilde{A}_n$ and $\widetilde{C}_n$ [22],
- even FC type Artin groups [2].

While this conjecture remains open in general, there are several other classes of Artin groups for which some weaker version of the intersection property is known to hold. For example, in [36] it is shown that in FC-type Artin groups, intersections of two spherical-type parabolics are parabolic, and this was further generalized by Möller, Paris, and Varghese [35] to include the case where just one of the two parabolics is spherical-type.

The proof in [36] that intersections of spherical-type parabolics are parabolic for FC-type Artin groups uses the fact that the cubical metric on the Deligne complex D_Γ is CAT(0) for these groups. This argument can be generalized to other Artin groups acting on CAT(0) spaces where the intersection property is known for the stabilisers of vertices. We include the proof here for completeness.

Proposition 2.5 *Let A_Γ be an Artin group acting on a polyhedral complex X with a piecewise Euclidean* CAT(0) *metric, where each cell stabiliser is a parabolic subgroup of A_Γ. Assume that the action is without inversions, that is, the stabiliser of each cell pointwise fixes the entire cell. Let $\mathcal{P}$ be the collection of parabolic subgroups that appear as stabilisers of the cells of X. Suppose that the stabiliser of every vertex of X satisfies the intersection conjecture. Then the intersection of any two elements of $\mathcal{P}$ is again a parabolic subgroup of A_Γ.*

Proof Let P and P' be two parabolics in $\mathcal{P}$. There exist cells σ and σ' in X such that P, P' are the stabilisers of σ, σ', respectively. Let x, x' be two points in the interior of σ, σ', respectively. Since $P \cap P'$ fixes both σ and σ', it fixes x and x', and hence the unique CAT(0) geodesic γ between them. Moreover, because the action is without inversions, it fixes the subcomplex consisting of the union of all the cells that contain a point of γ in their interior. In particular, it fixes some edge path ρ that contains all vertices of σ and σ'. Thus, $P \cap P'$ is equal to the pointwise stabiliser of ρ. The result now follows from the following:

Claim *Let $v_0, \dots, v_k$ be a combinatorial path in X, with stabilisers $P_0, \dots, P_k$, respectively. Then the intersection $\bigcap_{0 \le i \le k} P_i$ is a parabolic subgroup of A_Γ.*

Let us prove this claim by induction on $k \ge 1$. For $k = 1$, the intersection $P_0 \cap P_1$ is the stabiliser of a single edge. By hypothesis, the stabiliser of the edge is a parabolic subgroup in $\mathcal{P}$.

Now suppose that we have proved the result for some $k \ge 1$. Consider a combinatorial path $v_0, \dots, v_{k+1}$ with stabilisers $P_0, \dots, P_{k+1}$. By the induction hypothesis, we have that $\bigcap_{0 \le i \le k} P_i$ is a parabolic subgroup of A_Γ. Note that this is a subgroup of P_k, and hence a parabolic subgroup of P_k by [5]. We also know that $P_k \cap P_{k+1}$ is the stabiliser of the last edge of this path, and so $P_k \cap P_{k+1}$ is a parabolic subgroup of P_k.

We can thus write

$$\bigcap_{0 \le i \le k+1} P_i = \Bigl(\bigcap_{0 \le i \le k} P_i \Bigr) \cap (P_k \cap P_{k+1}).$$

Since the intersection conjecture holds in P_k by assumption, it follows that $\bigcap_{0 \le i \le k+1} P_i$ is a parabolic subgroup of P_k, and hence of A_Γ. □

In particular, for Artin groups for which the Moussong metric on Deligne complex is known to be CAT(0), the lemma above applies to show that intersections of spherical-type parabolics are parabolic. As noted above, this holds for 2-dimensional Artin groups, some 3-dimensional Artin groups, and locally reducible Artin groups.

Remark 2.6 The above proposition can be generalised to actions on complexes that satisfy other forms of nonpositive curvature, as in [4; 16]. We leave it to the reader to check that the key geometric feature necessary for the proof to carry over is the following property: if an element $g \in A_\Gamma$ fixes two vertices v, v' of X, then it also fixes some combinatorial path of X from v to v'. Such a weak form of convexity is satisfied by many forms of nonpositive curvature.

We can also obtain more examples of groups that satisfy the intersection conjecture by taking products of groups where the intersection conjecture is known.

Lemma 2.7 *Suppose that A_Γ is a reducible Artin group with direct factors $A_\Gamma = A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$. If the intersection conjecture holds for all parabolic subgroups in each direct factor A_{Γ_i} then the intersection conjecture holds for A_Γ.*

Proof Given two parabolics P, Q of A_Γ, one can decompose them as direct products

$$P = P_1 \times \cdots \times P_k, \quad Q = Q_1 \times \cdots \times Q_k$$

with each P_i, Q_i a parabolic subgroup of A_{Γ_i}. We thus have

$$P \cap Q = (P_1 \cap Q_1) \times \cdots \times (P_k \cap Q_k)$$

and the result follows from the fact that each A_{Γ_i} satisfies the intersection conjecture. □

We also add the following examples, which will be used later in this article.

Lemma 2.8 *Suppose that A_Γ is an Artin group with 3 or fewer generators. Then A_Γ satisfies the intersection conjecture.*

Proof The result is clear if A_Γ has one generator. If it has two generators, it is either a free group (in particular, a right-angled Artin group) or a dihedral Artin group (in particular, a spherical-type Artin group), and so the result follows from [3] and [15], respectively. Let us assume that A_Γ has three generators. If A_Γ is spherical-type (i.e., in the case of a triangle Artin group $(2, 3, n)$ for $3 \le n \le 5$ or $(2, 2, n)$ for $n \ge 2$), this holds by [15], so assume it is infinite type. If A_Γ is a free group or the triangle Artin group $(2, 2, \infty)$, then it is a right-angled Artin group, and the result follows from [3]. Otherwise, A_Γ is two-dimensional and $(2, 2)$-free, and the result follows from [4]. □

Acylindrical hyperbolicity In this paragraph, we review previously known results about acylindrically hyperbolic Artin groups. For Artin groups of spherical type, the acylindrical hyperbolicity conjecture was proved by Calvez–Wiest [8], following earlier result for braid groups [6; 33]. Thus, the acylindrical hyperbolicity conjecture reduces to the case of Artin groups of infinite type. Such groups are conjectured to have a trivial centre, so the conjecture asks whether these groups are acylindrically hyperbolic. Currently, the acylindrical hyperbolicity conjecture is known for several classes of Artin groups (we refer the reader to these articles for the definition of some of these classes), which we list below:

- spherical-type Artin groups, by Calvez–Wiest [8],
- right-angled Artin groups, by Osin [37],
- two-dimensional Artin groups, by Vaskou [40], following earlier work for XXL-type Artin groups (i.e., all labels satisfy $m_{s,t} \geq 5$) by Haettel [21], for XL-type Artin groups (i.e., all labels satisfy $m_{s,t} \geq 4$) by Martin–Przytycki [30], and for some two-dimensional Artin groups admitting a specific CAT(0) model by Kato–Oguni [27],
- Euclidean-type Artin groups, by Calvez [7],
- Artin groups whose graph is not a join, by Charney–Morris-Wright [12], following previous work by Chatterji–Martin [13],
- some relatively extra-large type Artin groups, by Goldman [20],
- some locally reducible Artin groups, by Mastrocola [32].

In this article, we add to this list the class of even Artin groups of FC type, among other new examples. Note that after this article was first released, Kato–Oguni announced a proof for the class of free-of-infinity Artin groups [28], which contains in particular the class of (nonspherical) even Artin groups of FC type.

3 Artin groups with visual splittings and acylindrical hyperbolicity

The goal of this section is to obtain new criteria for proving acylindrical hyperbolicity and apply them to get new examples of acylindrically hyperbolic Artin groups. We will focus primarily on the case of Artin groups with a visual splitting. In this case, there is a clear connection between malnormality and acylindricity given by the following theorem of Minasyan and Osin [34].

Theorem 3.1 (see [34]) *Suppose G splits as an amalgamated product of groups $G = A *_C B$ with $A \neq C \neq B$. If C is weakly malnormal in G, then G is either virtually cyclic or acylindrically hyperbolic.*

Thus a key to proving acylindricity for A_Γ is understanding when parabolic subgroups are weakly malnormal.

3.1 The main acylindrical hyperbolicity criterion

The main result of this subsection is the following.

Theorem 3.2 *Let A_Γ be an irreducible Artin group that splits visually as an amalgamated product $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. If the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ, then A_Ω is weakly malnormal in A_Γ. In particular, A_Γ is acylindrically hyperbolic.*

In order to prove this result, we need the following characterisation of normal parabolic subgroups of Artin groups. We start with the irreducible case:

Lemma 3.3 *Let A_Γ be an irreducible Artin group. Then the only normal standard parabolic subgroups of A_Γ are A_Γ and the trivial subgroup.*

Proof Let $\varnothing \subsetneq \Gamma' \subsetneq \Gamma$ be a strict subgraph of Γ. Let Γ_D, Γ'_D be the Dynkin diagrams corresponding to the presentation graph Γ, Γ', respectively (i.e., no edge if $m_{s,t} = 2$, and edges with label ∞ for pairs s, t that are not connected in Γ). Since A_Γ is irreducible, Γ_D is connected and Γ'_D is a strict induced subgraph of Γ_D. Thus, there exists an edge of Γ_D connecting a vertex $s \in \Gamma'_D$ and $t \notin \Gamma'_D$. It follows from van der Lek that

$$A_{\Gamma'} \cap A_{\{s,t\}} = \langle s \rangle.$$

We can also compute that

$$t A_{\Gamma'} t^{-1} \cap A_{\{s,t\}} = t(A_{\Gamma'} \cap A_{\{s,t\}})t^{-1} = t\langle s \rangle t^{-1}.$$

Thus in order to show that $tA_{\Gamma'}t^{-1} \neq A_{\Gamma'}$ it is sufficient to show that t does not normalise $\langle s \rangle$. Suppose by contradiction that $t\langle s \rangle t^{-1} = \langle s \rangle$. Then $\langle s \rangle$ is normal in $A_{\{s,t\}}$ and the corresponding quotient is either trivial or infinite cyclic, depending on the parity of m_{st}. Thus, $A_{\{s,t\}}$ is either cyclic or $\mathbb{Z}$-by-$\mathbb{Z}$, and in particular virtually abelian. But since $m_{st} \geq 3$ as s, t are joined by an edge in the Dynkin diagram, we have that $A_{\{s,t\}}$ contains nonabelian free subgroups, a contradiction. Thus we know that t does not normalise $\langle s \rangle$ and $A_{\Gamma'}$ cannot be a normal subgroup. □

Corollary 3.4 *Let A_Γ be an Artin group. Then a standard parabolic subgroup is normal if and only if it is a product of direct factors of A_Γ.*

Proof Decompose A_Γ as a product of irreducible Artin groups $A_\Gamma = A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$. If $A_{\Gamma'}$ is a product of groups of the form $A_{\Gamma'} = A_{\Gamma_{i_1}} \times \cdots \times A_{\Gamma_{i_k}}$ then $A_{\Gamma'}$ is a direct factor of A_Γ and so must be a normal subgroup.

Conversely, if $A_{\Gamma'}$ is normal in A_Γ then for all i, $A_{\Gamma'} \cap A_{\Gamma_i}$ is normal in A_{Γ_i}. By Lemma 3.3, we see that $A_{\Gamma'} \cap A_{\Gamma_i}$ must be equal to either A_{Γ_i} or to the trivial group. This implies that $A_{\Gamma'}$ is a direct product of factors of A_Γ. □

Proof of Theorem 3.2 Let $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ be a visual splitting and let T be the Bass–Serre tree of this splitting.

Claim 1 *Let e be an edge of T with vertices v and w. The trivial subgroup is the only parabolic subgroup of* $\mathrm{Stab}(e)$ *that is normal in both* $\mathrm{Stab}(v)$ *and* $\mathrm{Stab}(w)$.

Up to conjugation, it is enough to show that the trivial subgroup is the only parabolic subgroup of A_Ω that is normal in both A_{Γ_1} and A_{Γ_2}. Suppose to the contrary that H is normal in both A_{Γ_1} and A_{Γ_2}. Since every element of A_Γ is a product of elements of A_{Γ_1} and A_{Γ_2}, H is also normal in A_Γ. By assumption, A_Γ is irreducible, so applying Corollary 3.4 we conclude that H must be trivial. This proves the claim.

Claim 2 *Let γ be a geodesic segment of T, and let* $\mathrm{Stab}_*(\gamma)$ *be the pointwise stabiliser of γ. If* $\mathrm{Stab}_*(\gamma)$ *is nontrivial, then we can extend γ to a geodesic segment $\gamma' \supsetneq \gamma$ such that* $\mathrm{Stab}_*(\gamma') \subsetneq \mathrm{Stab}_*(\gamma)$.

Let $v_1, \dots, v_n$ be the vertices of γ. Up to the action of A_Γ, we can assume that $\mathrm{Stab}_*(v_1) = A_{\Gamma_1}$ and $\mathrm{Stab}_*(v_2) = A_{\Gamma_2}$. Suppose for every geodesic segment γ' of T extending γ, we have $\mathrm{Stab}_*(\gamma') = \mathrm{Stab}_*(\gamma)$. We first show that for every $g \in A_\Gamma$ such that the translate $g\gamma$ has as its closest-point projection on γ the single point $\{v_n\}$, we also have $\mathrm{Stab}_*(g\gamma) = \mathrm{Stab}_*(\gamma)$ (and in particular, such an element $g \in A_\Gamma$ normalises $\mathrm{Stab}_*(\gamma)$). For such an element g, the geodesic from v_1 to any point of $g\gamma$ contains γ. Thus, the minimal subtree T_g of T containing γ and $g\gamma$ can be written as the union of at most two geodesic segments γ_1, γ_2 extending γ, and thus

$$\mathrm{Stab}_*(\gamma) \cap \mathrm{Stab}_*(g\gamma) = \mathrm{Stab}_*(T_g) = \mathrm{Stab}_*(\gamma_1) \cap \mathrm{Stab}_*(\gamma_2) = \mathrm{Stab}_*(\gamma).$$

Thus, $\mathrm{Stab}_*(\gamma) \subset \mathrm{Stab}_*(g\gamma) = g\mathrm{Stab}_*(\gamma)g^{-1}$. Since $\mathrm{Stab}_*(\gamma)$ is a parabolic subgroup by assumption on the splitting, it follows from Lemma 2.3 that this inclusion is actually an equality, for otherwise the sequence $(g^n \mathrm{Stab}_*(\gamma) g^{-n})_{n \geq 0}$ would form an unbounded strict chain of parabolic subgroups.

Let T' be the subtree of T consisting of all the points of T whose closest-point projection on γ is v_n. Since the action of A_Γ on T is cocompact and T' is an unbounded subtree of T, we can pick an element $g \in A_\Gamma$ such that $g\gamma \subset T'$ and $d(\gamma, g\gamma) > |\gamma|$ (where as usual d denotes the path metric of T and $|\gamma|$ the combinatorial length of γ). These conditions imply that for every $h_1 \in \mathrm{Stab}_*(gv_1) = gA_{\Gamma_1}g^{-1}$ and every $h_2 \in \mathrm{Stab}_*(gv_2) = gA_{\Gamma_2}g^{-1}$, we also have $h_1 g\gamma \subset T'$ and $h_2 g\gamma \subset T'$. Thus, we get that g as well as every element of the form $h_1 g$ or $h_2 g$ normalises $\mathrm{Stab}_*(\gamma)$, for $h_i \in gA_{\Gamma_i}g^{-1}$. Thus, both $gA_{\Gamma_1}g^{-1}$ and $gA_{\Gamma_2}g^{-1}$ normalise $\mathrm{Stab}_*(\gamma)$. Since A_Γ is generated by A_{Γ_1} and A_{Γ_2}, hence by $gA_{\Gamma_1}g^{-1}$ and $gA_{\Gamma_2}g^{-1}$, it follows that $\mathrm{Stab}_*(\gamma)$ is normal in A_Γ. Since A_Γ is irreducible, it follows from Lemma 3.3 that $\mathrm{Stab}_*(\gamma)$ is trivial, which proves the claim.

By Claim 2, we can construct a sequence of geodesic segments $\gamma_0 \subsetneq \gamma_1 \subsetneq \cdots$, such that the sequence of stabilisers $\mathrm{Stab}_*(\gamma_i)$ strictly decreases as long as they are not trivial. By assumption on the splitting, we have that each $\mathrm{Stab}_*(\gamma_i)$ is a parabolic subgroup of A_Γ. It now follows from Lemma 2.3 that $\mathrm{Stab}_*(\gamma_i)$ becomes trivial after finitely many steps.

We thus have a finite length geodesic segment γ with trivial point stabiliser. Let $e_1, \dots, e_n$ be the edges of γ. Since T is a tree, we have $\mathrm{Stab}_*(\gamma) = \mathrm{Stab}(e_1) \cap \mathrm{Stab}(e_n)$, and by assumption $\mathrm{Stab}_*(\gamma) = \{1\}$. Since edge stabilisers are conjugates of A_Ω by construction, it follows that A_Ω is weakly malnormal in A_Γ, hence A_Γ is acylindrically hyperbolic by Theorem 3.1. □

3.2 Applications to some classes of Artin groups

Recall that A_Γ is even FC-type if all edge labels in Γ are even, and all cliques in Γ generate spherical-type parabolics.

Theorem 3.5 *Even FC-type Artin groups satisfy the acylindrical hyperbolicity conjecture.*

Proof Assume that A_Γ is irreducible. If A_Γ is spherical-type, this follows from Calvez–Wiest [8]. If A_Γ is not spherical-type, then it admits a visual splitting. Since even FC-type Artin groups satisfy the intersection property by [2], the intersection of any two conjugates of the edge group for this splitting must be parabolic and so by Theorem 3.2, A_Γ is acylindrically hyperbolic. □

For general FC-type Artin groups, we cannot directly apply Theorem 3.2 as we do not know that they satisfy the intersection conjecture. We therefore ask the following:

Question 3.6 Do FC-type Artin groups satisfy the intersection conjecture?

We can nonetheless prove acylindrical hyperbolicity under more restrictive conditions.

Theorem 3.7 *Let A_Γ be an irreducible Artin group such that the Deligne complex D_Γ admits a piecewise Euclidean* CAT(0) *metric. If A_Γ splits visually over a spherical-type parabolic, then A_Γ is acylindrically hyperbolic.*

Proof Stabilisers of simplices in the Deligne complex are precisely the spherical-type parabolic subgroups. Since spherical-type Artin groups satisfy the intersection conjecture [15], it follows from Proposition 2.5 that the intersection of any two spherical-type parabolic subgroups of A_Γ is again a parabolic subgroup. In particular, the splitting satisfies the hypothesis of Theorem 3.2. □

In particular, Theorem 3.7 applies to FC-type Artin groups, locally reducible Artin groups, and certain Artin groups of dimension 3 (namely those for which all the irreducible three-dimensional parabolic subgroups are isomorphic to the braid group B_4), whenever they split visually over a spherical-type parabolic.

3.3 A weaker version of the intersection property

In Theorem 3.2, the condition on the intersections of conjugates of the edge group A_Ω is a priori weaker than requiring that A_Γ satisfies the intersection conjecture. In this section, we show how to obtain this weaker condition in cases where the intersection conjecture may not be known for A_Γ. We start with the following observation.

Theorem 3.8 *Let A_Γ be an irreducible Artin group with a visual splitting $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. If both A_{Γ_1} and A_{Γ_2} satisfy the intersection conjecture, then the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ. In particular, A_Γ is acylindrically hyperbolic.*

Proof Since A_{Γ_1} and A_{Γ_2} both satisfy the intersection conjecture, the fact that the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ is a direct consequence of Proposition 2.5 applied to the action of A_Γ on the (CAT(0)) Bass–Serre tree of the splitting $A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. □

In the rest of this section, we show how one may understand the intersection of two conjugates of A_Ω even when the vertex groups are not known to satisfy the intersection conjecture. As with the original proofs of the intersection property for some families of Artin groups (see Section 2.3), the idea is to realise the conjugates of A_Ω as stabilisers in a suitable CAT(0) complex and apply Proposition 2.5. We recall a relevant framework of Godelle–Paris to construct such complexes [19].

Definition 3.9 A *complete cover* $\mathcal{U}$ of Γ is a collection of induced subgraphs of Γ that contains every edge of Γ and is stable under taking induced subgraphs (including the empty graph). Given an element $\Gamma' \in \mathcal{U}$, the corresponding standard parabolic subgroup $A_{\Gamma'}$ is called a *$\mathcal{U}$-standard parabolic subgroup*, and a conjugate of $A_{\Gamma'}$ is called a *$\mathcal{U}$-parabolic subgroup*.

Given a complete cover $\mathcal{U}$, one defines the corresponding *Godelle–Paris cube complex* $X_{\mathcal{U}}$ as follows: vertices of $X_{\mathcal{U}}$ correspond to cosets of $\mathcal{U}$-standard parabolic subgroups, and cubes correspond to the intervals (for the inclusion) between gA_{Γ_1} and gA_{Γ_2}, whenever $g \in A_\Gamma$ and $\Gamma_1 \subset \Gamma_2$ are in $\mathcal{U}$.

Note that A_Γ acts on $X_{\mathcal{U}}$ by left multiplication on left cosets. This action is cocompact and without inversion.

For instance, if A_Γ is FC-type and $\mathcal{U}$ consists of all cliques of Γ, we recover the cubical Deligne complex of Charney–Davis [11]. More generally, if $\mathcal{U}$ consists of all the cliques of Γ, one recovers the Godelle–Paris clique complex [19].

There is a complete characterisation of when the standard cubical metric on this complex is CAT(0). We need the following definition:

Definition 3.10 Let $\mathcal{U}$ be a complete cover of Γ. We define a simplicial complex $L_{\mathcal{U}}$ as follows: the vertices of $L_{\mathcal{U}}$ are the vertices of Γ, and a set of vertices $v_0, \dots, v_k$ of $L_{\mathcal{U}}$ span a k-simplex of $L_{\mathcal{U}}$ if and only if the induced subgraph of Γ spanned by $v_0, \dots, v_k$ belongs to $\mathcal{U}$.

Note that $L_{\mathcal{U}}$ is isomorphic to the link of any vertex of $X_{\mathcal{U}}$ corresponding to the empty subgraph of Γ.

Theorem 3.11 [19, Theorem 4.2] *Let $\mathcal{U}$ be a complete cover of Γ. Then $X_{\mathcal{U}}$ is a* CAT(0) *cube complex if and only if $L_{\mathcal{U}}$ is a flag simplicial complex.*

Remark 3.12 In [19], the above theorem is stated with the additional condition that the $\mathcal{U}$-standard parabolic subgroups satisfy the $K(\pi, 1)$-conjecture. However, the reader can follow the proof and check that this assumption is not needed in order to prove that $X_{\mathcal{U}}$ is CAT(0). (Godelle–Paris need this assumption to prove a subsequent theorem showing that an Artin group satisfies the $K(\pi, 1)$-conjecture if all its free-of-infinity standard parabolic subgroups satisfy that conjecture.)

Corollary 3.13 *Let A_Γ be an Artin group. Let $\mathcal{U}$ be a complete cover of Γ. Assume that*

- *$L_{\mathcal{U}}$ is a flag simplicial complex,*
- *for every Γ' in $\mathcal{U}$, the standard parabolic $A_{\Gamma'}$ satisfies the intersection conjecture.*

Then the intersection of any two $\mathcal{U}$-parabolic subgroups is again a parabolic subgroup of A_Γ.

Proof By Theorem 3.11, we have that $X_{\mathcal{U}}$ is a CAT(0) cube complex. By construction, the action is without inversion and the stabilisers of cubes are precisely the $\mathcal{U}$-parabolic subgroups of A_Γ. Thus, the result follows from Proposition 2.5. □

In particular, to show that the intersection of two conjugates of a standard parabolic A_Ω is again a parabolic subgroup, it is enough to include Ω in a suitable complete cover of Γ. We now give an example of a geometric condition on Ω that guarantees that such a cover exists.

Definition 3.14 We say that an induced subgraph Ω is 2*-convex* in Γ if every geodesic path of Γ of length 2 with endpoints in Ω is contained in Ω.

Lemma 3.15 *Let Γ be a simplicial graph, and let Ω be a 2-convex subgraph of Γ. Let $\mathcal{U}$ be the complete cover consisting of all the cliques of Γ and all the induced subgraphs of Ω. Then the simplicial complex $L_{\mathcal{U}}$ is flag.*

Proof Let $T \subset V(\Gamma)$ be a set of vertices that are pairwise connected by edges in $L_{\mathcal{U}}$. To show that T spans a simplex of $L_{\mathcal{U}}$, it is enough to show that either $T \subset \Omega$ or the vertices of T span a clique of Γ.

We can thus assume that T is not contained in Ω, and let us show that T spans a clique of Γ. We decompose T as a disjoint union $T = T_1 \cup T_2$, where $T_1 := T \cap V(\Omega)$ and $T_2 := T - T_1$. Note that by construction of $\mathcal{U}$, two vertices of Γ are adjacent in $L_{\mathcal{U}}$ if and only if they are both contained in Ω or they are connected by an edge of Γ. By assumption, any two vertices of T are adjacent in $L_{\mathcal{U}}$, thus, for every $t \in T_2$ and $t' \in T - \{t\}$, we have that t and t' are connected by an edge in Γ.

It remains to show that any two vertices $t \neq t' \in T_1$ are connected by an edge of Γ. Suppose by contradiction that there exists a pair $t, t' \in T_1$ that is not connected by an edge of Γ. Since T_2 is not empty, we can pick an element $s \in T_2$ and the previous argument shows that t, s, t' forms a path in Γ. Since t, t' are not adjacent in Γ, this path is geodesic. By 2-convexity of Ω, we get that $s \in T_1$, a contradiction. Thus, the vertices of T_1 span a simplex of Γ, and it now follows that T spans a clique of Γ, and hence spans a simplex in $L_{\mathcal{U}}$. □

The following is now a direct consequence of Corollary 3.13:

Corollary 3.16 *Let A_Γ be an Artin group, and let Ω be a 2-convex subgraph of Γ. Assume that A_Ω as well as every clique standard parabolic subgroup of A_Γ satisfy the intersection conjecture. Then the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ.*

Proposition 3.17 *Let A_Γ be an irreducible Artin group that visually splits over a standard parabolic subgroup A_Ω. Assume that*

- *Ω is 2-convex in Γ,*
- *A_Ω and all clique parabolic subgroups of A_Γ satisfy the intersection conjecture.*

Then A_Γ is acylindrically hyperbolic.

Proof It follows from Corollary 3.16 that the intersection of any two conjugates of A_Ω is again a parabolic subgroup of A_Γ. The result thus follows from Theorem 3.2. □

Application We can use this result to obtain new examples of acylindrically hyperbolic Artin groups not covered by the recent results of Charney–Morris-Wright [12].

Definition 3.18 Let C_n denote the graph that is a cycle on n vertices. The *wheel* W_n is the graph obtained from C_n by adding a new vertex (the apex) and connecting it to every vertex of C_n.

Corollary 3.19 *Let A_Γ be an irreducible Artin group whose underlying graph is a wheel W_n with $n \geq 6$. Then A_Γ is acylindrically hyperbolic.*

Proof Since $n \geq 6$, A_{W_n} visually splits over a parabolic subgroup A_Ω where Ω is a geodesic of length 2 containing the apex, that is 2-convex in W_n. Since all 3-generated Artin groups satisfy the intersection conjecture by Lemma 2.8, it follows that A_Ω and all clique parabolic subgroups of A_Γ satisfy the intersection conjecture. The result now follows from Proposition 3.17. □

4 The weak malnormality conjecture

Next we consider the weak malnormality conjecture. The following reduction lemma shows that it is enough to deal with irreducible Artin groups:

Lemma 4.1 *Let $A_{\Gamma_1}, \dots, A_{\Gamma_k}$ be irreducible Artin groups that satisfy the weak malnormality conjecture. Then the direct product $A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$ also satisfies the weak malnormality conjecture.*

Proof Let $A_{\Gamma'}$ be a standard parabolic subgroup of the direct product $A_\Gamma := A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$. It is clear that if $A_{\Gamma'}$ contains one of the (normal) direct factors A_{Γ_i}, then it is not weakly malnormal. Thus, let us assume that $A_{\Gamma'}$ contains no direct factor. Note that $A_{\Gamma'}$ decomposes as the direct product $A_{\Gamma'} = A_{\Gamma'_1} \times \cdots \times A_{\Gamma'_k}$ where for each i, $\Gamma'_i := \Gamma' \cap \Gamma_i$. Since $A_{\Gamma'}$ does not contain any of the A_{Γ_i}, each $A_{\Gamma'_i}$ is a proper parabolic subgroup of A_{Γ_i}, hence weakly malnormal in A_{Γ_i} since by assumption, A_{Γ_i} satisfies the weak malnormality conjecture. Thus, for each i we can pick $g_i \in A_{\Gamma_i}$ such that $A_{\Gamma'_i} \cap A_{\Gamma'_i}^{g_i}$ is finite. Now set $g := g_1 \cdots g_k$. We have

$$A_{\Gamma'} \cap A_{\Gamma'}^{g} = (A_{\Gamma'_1} \cap A_{\Gamma'_1}^{g_1}) \times \cdots \times (A_{\Gamma'_k} \cap A_{\Gamma'_k}^{g_k}),$$

which is finite. Thus, $A_{\Gamma'}$ is weakly malnormal in A_Γ. □

4.1 Connections between the three main conjectures

Proposition 4.2 *Let A_Γ be an irreducible Artin group that visually splits as an amalgamated product over a standard parabolic subgroup A_Ω. If A_Ω is weakly malnormal in A_Γ, then every proper parabolic subgroup of A_Γ is weakly malnormal. Thus A_Γ satisfies the weak malnormality conjecture.*

Proof Let T be the Bass–Serre of the splitting $A_\Gamma = A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$. Let P be a standard parabolic subgroup of A_Γ. Note that we have a splitting $P = P_1 *_{P_\Omega} P_2$, where $P_i := P \cap A_{\Gamma_i}$ and $P_\Omega = P \cap A_\Omega$. The Bass–Serre tree T' of that induced splitting embeds isometrically in T.

Since P is a proper parabolic subgroup, we have $T' \neq T$, hence we can pick a vertex v of T', and an edge $e = [v, w]$ that is not in T'. Since $\mathrm{Stab}(e)$ is conjugated to A_Ω, it is weakly malnormal in A_Γ. Choose $h \in A_\Gamma$ such that $\mathrm{Stab}(e) \cap \mathrm{Stab}(he)$ is finite. Let γ be the geodesic path in T with initial edge e and final edge he. In particular, the pointwise stabiliser of γ is finite.

Claim *There exist elements $g_1, g_2 \in A_\Gamma$ such that the trees $g_1 T'$ and $g_2 T'$ are disjoint, and the unique geodesic between them contains γ.*

If either (or both) endpoints of γ are translates of w, we can extend γ by a single edge at that endpoint to obtain a geodesic path $\gamma' \supseteq \gamma$ both of whose endpoints are translates of v. Say the initial edge of γ' is

$g_1 e$ and the final edge is $g_2 e$. Since e is not contained in T', $g_i e$ is not contained in $g_i T'$ for $i = 1, 2$. Thus γ' intersects $g_i T'$ in a single point. It now follows from standard arguments on the geometry of trees that $g_1 T'$ and $g_2 T'$ are disjoint, and γ' is the unique geodesic between them, which proves the claim.

Since P stabilises the tree T', the conjugate P^{g_i} stabilises the tree $g_i T'$, and it follows that the intersection $P^{g_1} \cap P^{g_2}$ stabilises the unique geodesic between these disjoint trees. Thus it fixes pointwise the path γ'. Since $\gamma' \supseteq \gamma$, it has finite stabiliser and we conclude that $P^{g_1} \cap P^{g_2}$ is finite. Hence P is weakly malnormal in A_Γ. □

Combining the results above, we now conclude:

Corollary 4.3 *Suppose A_Γ is irreducible and Γ is not a clique.*

- *If A_Γ satisfies the intersection conjecture, then it also satisfies the weak malnormality conjecture.*
- *If A_Γ satisfies the weak malnormality conjecture, then it also satisfies the acylindrical hyperbolicity conjecture.*

Proof If Γ is not a clique, A_Γ admits a visual splitting over some standard parabolic A_Ω. The first bullet point is the direct application of Theorem 3.2 and Proposition 4.2. For the second bullet point, since A_Γ is irreducible and satisfies the weak malnormality conjecture, A_Ω is weakly malnormal, so the result follows from Theorem 3.1. □

One might wonder if the converse of these implications also hold. It is possible to obtain a partial converse to the implication in the second bullet point, assuming that A_Γ acts acylindrically on some hyperbolic space such that the geometry of the action is "compatible" with the parabolic subgroups, in the following sense:

Lemma 4.4 *Let A_Γ be an irreducible Artin group, and assume that A_Γ is acylindrically hyperbolic, with a cobounded acylindrical action on a hyperbolic graph X. Suppose that for every proper parabolic subgroup $A_{\Gamma'}$, the following holds: Let $X_{\Gamma'} \subset X$ denote the $A_{\Gamma'}$-orbit of some chosen point of X. Then $X_{\Gamma'}$ is quasiconvex in X and its limit set $\Lambda X_{\Gamma'}$ is a strict subset of the Gromov boundary ∂X.*

Then A_Γ satisfies the weak malnormality conjecture.

Proof The set of limit points of loxodromic elements of A_Γ is dense in ∂X (see, for instance, Theorem 2.6 in [24]), so since $\Lambda X_{\Gamma'}$ is a proper closed subset of ∂X, we can pick a loxodromic element $g \in A_\Gamma$ such that $\Lambda g \cap \Lambda X_{\Gamma'} = \varnothing$. By hyperbolicity of X and quasiconvexity of $X_{\Gamma'}$, there exist constants ℓ and D (that depend only on the space X and the quasiconvexity constants of $X_{\Gamma'}$) such that if two translates of $X_{\Gamma'}$ are at distance at least ℓ, then the diameter of the closest projection of one on the other is bounded above by D. By acylindricity of the action, we can pick a constant L such that if $x, y \in X$ are at distance at least L, there are only finitely many elements $h \in A_\Gamma$ such that $d(x, hx) \leq D$ and $d(y, hx) \leq D$.

Using North-South dynamics of the action, we can now pick a large power $n \geq 0$ such that $X_{\Gamma'}$ and $g^n X_{\Gamma'}$ are disjoint, the diameter of the closest projection on each other is bounded above by D, and such that their distance is greater than L. Let $x \in X_{\Gamma'}$ and $y \in g^n X_{\Gamma'}$ be a pair of points that realises the distance between these two translates. We get in particular that an element $h \in A_{\Gamma'} \cap A_{\Gamma'}^{g^n}$ sends the pair x, y to another pair realising the distance between these two translates. Thus, for every $h \in A_{\Gamma'} \cap A_{\Gamma'}^{g^n}$, we have $d(x, hx) \leq D$ and $d(y, hx) \leq D$. Since $d(x, y) \geq L$ by construction, the acylindricity implies that the set of such h is finite. Thus, $A_{\Gamma'} \cap A_{\Gamma'}^{g^n}$ is finite, and $A_{\Gamma'}$ is weakly malnormal. □

Thus, we ask the following question:

Question 4.5 Let A_Γ be an irreducible Artin group, and assume that A_Γ is acylindrically hyperbolic, with a cobounded acylindrical action on a hyperbolic graph X. Let $A_{\Gamma'}$ be a proper parabolic subgroup of A_Γ, and let $X_{\Gamma'} \subset X$ denote the $A_{\Gamma'}$-orbit of some chosen point of X. Do we have that $X_{\Gamma'}$ is quasiconvex in X, with limit set $\Lambda X_{\Gamma'} \neq \partial X$?

4.2 Artin groups satisfying the weak malnormality conjecture

In this section we will show that the weak malnormality conjecture holds for several classes of Artin groups, which allows us to prove that new classes of Artin groups are acylindrically hyperbolic.

Proposition 4.6 *The weak malnormality conjecture holds for the following classes of groups*:

- *Artin groups satisfying the hypothesis of* Theorem 3.2 (*for instance*, *even Artin group of FC type*),
- *Artin groups of spherical type*,
- *two-dimensional Artin groups.*

Using the criterion of Minasyan–Osin [34], this implies the acylindrical hyperbolicity of many groups admitting a visual splitting:

Corollary 4.7 *Let A_Γ be an Artin group with a visual splitting $A_{\Gamma_1} *_{A_\Omega} A_{\Gamma_2}$ and assume that A_Ω does not contain a direct factor of A_{Γ_1} (which holds in particular if A_{Γ_1} is irreducible). Suppose that A_{Γ_1} is one of the following*:

- *an Artin group satisfying the hypothesis of* Theorem 3.2 (*for instance*, *an even Artin group of FC type*),
- *an Artin group of spherical type*,
- *a two-dimensional Artin group.*

Then A_Γ is acylindrically hyperbolic.

Proof By Proposition 4.6, the hypotheses of the corollary imply that A_Ω is weakly malnormal in A_{Γ_1} and hence also in A_Γ, so the result follows from Theorem 3.1. □

The proof of Proposition 4.6 will occupy the remainder of this section.

Spherical-type Artin groups Although the intersection conjecture is known to hold for spherical-type Artin groups, we cannot apply Corollary 4.3 since the defining graph Γ is always a clique. Nevertheless, we can prove:

Lemma 4.8 *Artin groups of spherical type satisfy the weak malnormality conjecture.*

Proof By Lemma 4.1, it is enough to deal with the irreducible spherical case. Suppose that A_Γ is irreducible and of spherical type, and let $A_{\Gamma'}$ be a proper parabolic subgroup.

If A_Γ is not cyclic or of dihedral type, then by Theorem 3 of [1], A_Γ contains a subgroup isomorphic to $A_{\Gamma'} * \mathbb{Z}$, and where the free factor $A_{\Gamma'}$ is the proper parabolic subgroup under study. A standard argument from actions on trees shows that $A_{\Gamma'}$ is weakly malnormal in $A_{\Gamma'} * \mathbb{Z}$, hence it is weakly malnormal in A_Γ.

If A_Γ is cyclic, there is nothing to prove. Suppose that A_Γ is dihedral, with standard generators s, t, and let us show that $A_{\Gamma'} = \langle s \rangle$ is weakly malnormal. We know from Lemma 3.3 that there exists $g \in A_\Gamma$ such that $\langle s \rangle^g \neq \langle s \rangle$. Let us show by contradiction that $\langle s \rangle^g \cap \langle s \rangle = \{1\}$, which will prove weak malnormality. Let $x \in \langle s \rangle^g \cap \langle s \rangle$ be a nontrivial element, and let $n, m \geq 1$ be such that $x = s^n = gs^m g^{-1}$. By applying the homomorphism $A_\Gamma \to \mathbb{Z}$ sending both generators to 1, we see that $n = m$. Thus, g lies in the centralizer $C(s^n) = C(s)$, the latter equality following, for instance, from Lemma 7 of [14]. It follows that $\langle s \rangle^g = \langle s \rangle$, a contradiction. □

Even Artin groups of FC-type

Lemma 4.9 *Let A_Γ be an Artin group satisfying the hypotheses of Theorem 3.2. Then A_Γ satisfies the weak malnormality conjecture.*

Proof The edge group A_Ω is weakly malnormal in A_Γ by Theorem 3.2, so this is now a direct consequence of Proposition 4.2. □

In particular, we get the following:

Corollary 4.10 *Even FC-type Artin groups satisfy the weak malnormality conjecture.*

Proof By Lemma 4.1, it is enough to assume that A_Γ is irreducible and by Lemma 4.8 we may assume that it is not of spherical-type, that is, Γ is not a clique. The result now follows from Lemma 4.9. □

Note that the previous corollary is also a direct consequence of Corollary 4.3.

Two-dimensional Artin groups An Artin group A_Γ is two-dimensional if Γ has at least one edge (i.e., A_Γ is not a free group) and any three vertices in Γ generate an infinite-type parabolic subgroup. Recall that the intersection conjecture has not yet been proved for two-dimensional Artin groups, with currently the largest subclass for which it has been proved being the class of two-dimensional Artin groups whose presentation graph does not contain two adjacent edges with label 2 [4].

In this section we introduce another strategy for proving the weak malnormality conjecture using an action of A_Γ on the Deligne complex, which we can apply to two-dimensional Artin groups.

Proposition 4.11 *Let A_Γ be an Artin group such that*

- *D_Γ is* CAT(0) *with respect to either the cubical metric or the Moussong metric*,
- *there exists a vertex v of D_Γ with unbounded link and such that* Stab(v) *is weakly malnormal in A_Γ.*

Then A_Γ satisfies the weak malnormality conjecture.

Corollary 4.12 *Two-dimensional Artin groups satisfy the weak malnormality conjecture.*

Proof First suppose A_Γ contains an edge e labelled $k > 2$. Then it cannot be reducible, since e together with any vertex in the opposite direct factor would generate a spherical-type subgroup of rank 3. The Moussong metric on the Deligne complex D_Γ is CAT(0), by Charney–Davis [11]. By Lemma 5.7 of Vaskou [40], there exists vertices a, b in Γ connected by an edge labelled > 2 such that the subgroup $A_{a,b}$ is weakly malnormal in A_Γ. Viewing $A_{a,b}$ as a vertex in D_Γ, it has unbounded link by Proposition E of [40]. Thus, we can apply Proposition 4.11 to conclude that every proper parabolic subgroup in A_Γ is weakly malnormal.

If all edges of Γ are labelled 2 then A_Γ is a RAAG, hence even FC-type, so the result follows from Theorem 3.5. □

Proof of Proposition 4.11 For a proper parabolic subgroup $A_{\Gamma'}$, the Deligne complex $D_{\Gamma'}$ embeds equivariantly as a strict convex subcomplex of the CAT(0) space D_Γ that is stabilised by $A_{\Gamma'}$. (This is easily verified for the cubical metric. For the Moussong metric, see Lemma 5.1 of [9].) We want to construct a translate $gD_{\Gamma'}$ such that the following is satisfied:

- There is a unique geodesic realising the distance between $D_{\Gamma'}$ and $gD_{\Gamma'}$.
- The pointwise stabiliser of that geodesic is finite.

This will imply that $A_{\Gamma'} \cap gA_{\Gamma'}g^{-1}$ is trivial, hence $A_{\Gamma'}$ is weakly malnormal.

Since checking that $A_{\Gamma'}$ is weakly malnormal is equivalent to checking that any of its conjugates is weakly malnormal, we will consider instead a translate $kD_{\Gamma'}$ for some $k \in A_\Gamma$, such that the vertex v from the proposition's statement is not contained in $kD_{\Gamma'}$. We first observe that the projection of $kD_{\Gamma'}$ onto the link, $\mathrm{lk}(v)$, has diameter at most π. To see this, let x, y be two points in $kD_{\Gamma'}$ and let α_x, α_y be the geodesics connecting x to v and y to v. If the angle between α_x and α_y was $\geq \pi$, a standard argument of CAT(0) geometry would imply that the concatenation of α_x and α_y is a geodesic from x to y. Since $kD_{\Gamma'}$ is convex in D_Γ, this geodesic must lie entirely in $kD_{\Gamma'}$. This contradicts our assumption that $v \notin kD_{\Gamma'}$.

Since $\mathrm{Stab}(v)$ is weakly malnormal, there exists a translate w of v such that $\mathrm{Stab}(v) \cap \mathrm{Stab}(w)$ is finite, and hence the geodesic $\gamma = [v, w]$ connecting them has finite pointwise stabiliser. Since the link of v is unbounded and $\mathrm{Stab}(v)$ acts cocompactly on it, we can pick an element $h \in \mathrm{Stab}(v)$ such that the distance in $\mathrm{lk}(v)$ between the projection of $kD_{\Gamma'}$ and $h\gamma$ is at least π. The stabiliser of $h\gamma$ is conjugate to that of γ, hence it is also finite. Next, since the link of hw is also unbounded, we can pick an element $g \in \mathrm{Stab}(hw)$ such that the distance in $\mathrm{lk}(hw)$ between $h\gamma$ and $gh\gamma$ is at least π. And finally, since the distance in $\mathrm{lk}(v)$ between the projection of $kD_{\Gamma'}$ and $h\gamma$ is at least π the same holds for the distance in $\mathrm{lk}(gv)$ the between the projection of $gkD_{\Gamma'}$ and $gh\gamma$. (See Figure 1.)

It follows that for any points $x \in kD_{\Gamma'}$ and $y \in gkD_{\Gamma'}$, the concatenation of the geodesics $[x, v]$, $[v, hw]$, $[hw, gv]$, $[gv, y]$ is the (unique) geodesic from x to y. In particular, taking x to be the nearest point projection of v on $kD_{\Gamma'}$ and y to be the nearest point projection of gv on $gkD_{\Gamma'}$, we obtain a

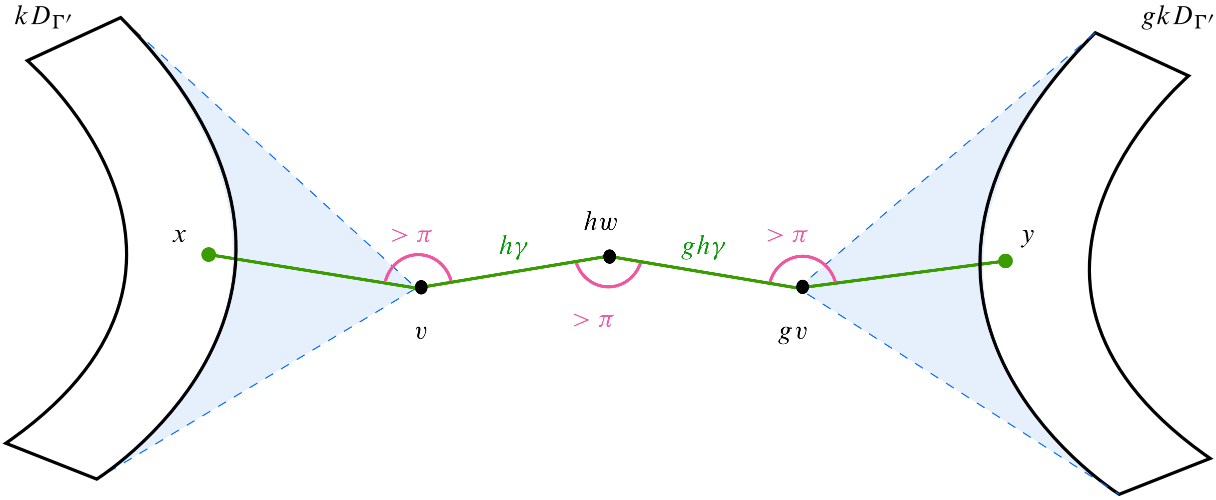

Figure 1: A geodesic (green) between $kD_{\Gamma'}$ and $gkD_{\Gamma'}$, obtained by concatenating several geodesic segments making an angle greater than π at their intersection point. The angles at v between any two points of $kD_{\Gamma'}$ are smaller than π ("visual cone" in light blue).

unique length-minimizing path between $kD_{\Gamma'}$ and $gkD_{\Gamma'}$. Since $kA_{\Gamma'}k^{-1} \cap gkA_{\Gamma'}(gk)^{-1}$ preserves both of these subcomplexes, it must fix this path. In particular, it lies in the pointwise stabilser of $h\gamma$. We conclude that this intersection is finite and hence $A_{\Gamma'}$ is weakly malnormal. $\square$

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