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Equivariant preimage theory for G-maps

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Let X and Y be closed G-manifolds and $B \subset Y$ a closed invariant nonempty subset where G is a finite group. For any G-map $f : X \to Y$ and for every subgroup $H \leq G$, we introduce a Nielsen type number $N(f^H, B^H)$ which is a lower bound for the number of connected components of WH-orbits of $(f^H)^{-1}(B^H)$. This theory generalizes existing Nielsen type numbers for various G and B with an application to the Nielsen Borsuk–Ulam theory for the minimal number of coincidences of $f(x) = f\tau(x)$ where $f : X \to Y$ and τ a free involution on X.

1 Introduction

It is well known that the Brouwer fixed point theorem is equivalent to the Borsuk–Ulam theorem. The former says that for any (continuous) self-map $f : X \to X$, the fixed point set $\mathrm{Fix}\, f = \{x \in X \mid f(x) = x\}$ is nonempty when $X = D^n$ is the closed n-disk. The latter is equivalent to the following: for any $\mathbb{Z}_2$-equivariant map $\varphi : S^n \to \mathbb{R}^n$, the preimage $\varphi^{-1}(\{0\})$ is nonempty, where the $\mathbb{Z}_2$-actions on S^n and on $\mathbb{R}^n$ are the usual antipodal actions. Lefschetz generalized Brouwer's result to coincidences of two maps $f, g : X \to Y$ between closed orientable manifolds of the same dimension. If we let $F : X \to Y \times Y$ be given by $F(x) = (f(x), g(x))$ and $\Delta = \{(y, y) \mid y \in Y\}$, then the coincidence set $C(f, g) = \{x \in X \mid f(x) = g(x)\} = F^{-1}(\Delta)$. Thus the coincidence theorem of Lefschetz asserts that $F^{-1}(\Delta) \neq \varnothing$ if the Lefschetz coincidence number $L(f, g)$ is nonzero. Subsequently, many authors study the general problem of determining whether $\Phi^{-1}(B) \neq \varnothing$ for a mapping $\Phi : W \to Z$ and $B \subset Z$ a closed subspace (e.g., Dobreńko [4], Frolkina [7], Gonçalves and Wong [14], Ha and Lee [15], and Liu and Zhao [16]). In other words, fixed point and coincidence point problems, as well as Borsuk–Ulam type theorems can be formulated as a preimage problem $\varphi^{-1}(B)$ for $\varphi : X \to Y \supset B$, where the latter is under the presence of a group action.

While the above-mentioned problems study whether the preimage set $\varphi^{-1}(B)$ is nonempty or not, the results do not give any information about the *size* of $\varphi^{-1}(B)$. Nielsen (fixed point or coincidence point) theory gives a geometric count of the number of connected components in $\varphi^{-1}(B)$. Such a theory was developed by Dobreńko and Kucharski [5], who introduced a Nielsen type number for the number of preimages of a map $f : X \to Y \supset B$. Under appropriate dimension conditions, a minimality theorem was established. From the (co)homological aspect, the algebraic size of $\varphi^{-1}(B)$ has been studied by Gonçalves and Wong [14] using local coefficients, and various cohomological index theories by Conner and Floyd [1], Fadell and Husseini [6], and Yang [23], among others. There is a vast literature on

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Borsuk–Ulam and Bourgin–Yang type results and their applications to nonlinear analysis (see, e.g., Mawhin and Willem [18]).

The main objective of this paper is to give a geometric approach, a la Nielsen, similar to the nonequivariant setting of [5] and of [7], and to introduce a Nielsen type number that yields a lower bound for the number of connected components of $\varphi^{-1}(B)$ for a G-equivariant map $\varphi: X \to Y \supset B$ between G-spaces. Here B is a closed G-invariant subset and G is a finite group. We develop this theory using both the universal coverings and the Hopf coverings, generalizing the previous works on nonequivariant settings in [5; 7; 15; 16]. The algebraic approach here establishes an equivariant Reidemeister number which is an upper bound for the equivariant Nielsen preimage number for $\varphi^{-1}(B)$.

This equivariant preimage theory also generalizes existing works. If $X = Y$ and $B = \Delta_X$ is the diagonal with $f = 1 \times h$, $h: X \to X$, then we recover the equivariant Nielsen fixed point theory of [22]. Similarly if $B = \{a\}$ is a point in Y^G, we recover the equivariant Nielsen root theory of [20]. On the other hand, when G is trivial, our setting reduces to the nonequivariant Nielsen and Reidemeister settings of [5] and of [15; 16].

If $G = \mathbb{Z}_2 = \langle\tau\rangle$ is generated by a free involution τ on X, our equivariant Nielsen equivalence coincides with the Nielsen coincidence equivalence for Borsuk–Ulam coincidences studied by Cotrim, de Melo and Vendrúscolo [2; 3; 19]. In this setting of a free involution τ on X, our work provides a Reidemeister number for the Nielsen Borsuk–Ulam theory in [2; 3], and this should facilitate computation in future work in this direction. It is easy to see that one can generalize the Borsuk–Ulam coincidences to the study of the set of points $x \in X$ such that the orbit $\{x, \tau(x), \dots, \tau^{k-1}(x)\}$ is mapped to the same value under f, i.e., $f(x) = f(\tau(x)) = \cdots = f(\tau^{k-1}(x))$ for a free $\mathbb{Z}_k = \langle\tau\rangle$ action on X and a map $f: X \to Z$.

This paper is organized as follows. In Section 2, we introduce the concept of G-preimage classes utilizing the framework of universal covering. In Section 3, we define G-Nielsen preimage classes using the geometric essentiality for such classes. Furthermore, we provide an interpretation of the G-Nielsen preimage classes as the nonempty G-preimage classes established in the previous section. In Section 4, we present an algebraic approach to derive an upper bound for the number of essential G-Nielsen preimage classes via the universal cover. When using a Hopf cover, we also obtain in Section 5 a sharper upper bound. Finally, in Section 6, we conclude the paper by showcasing the practical application of our invariants in the context of the Nielsen Borsuk–Ulam theory of [2; 3; 19].

Throughout this paper, X and Y are connected, locally pathwise-connected and semilocally simply connected spaces. For any group G, a G-space X is assumed to have an effective G-action, i.e., if $g \cdot x = x$ for all $x \in X$ then $g = 1_G$.

2 G-preimage classes

Let X and Y be G-spaces where G is a finite group, $\varnothing \neq B \subset Y$ be a G-invariant closed subset of Y, and $f: X \to Y$ be a G-map. In this section, we use liftings to the universal cover to define the G-preimage classes.

In [7], a preimage problem is denoted by $f : X \to Y \supset B$. Our setting will be referred as a G-preimage problem and it will be denoted by $f : X \to_G Y \supset B$.

If $x_0 \in f^{-1}(B)$ then the orbit of x_0, $G \cdot x_0 = \{g \cdot x_0 \mid g \in G\}$, is also contained in $f^{-1}(B)$, that is, $f^{-1}(B)$ is a G-invariant subspace of X. In what follows, the set $f^{-1}(B)$ will be partitioned into the so-called *G-preimage classes*.

Let $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$ be the universal coverings of X and Y, respectively. It is well known from the nonequivariant preimage theory (see, e.g., [15] or [16]) that

$$f^{-1}(B) = \bigcup_{\tilde{f},\widetilde{B}} \eta_X(\tilde{f}^{-1}(\widetilde{B})),$$

where $\tilde{f}$ ranges over all liftings of f with respect to universal coverings $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$, and $\widetilde{B}$ ranges over all path components of $\eta_Y^{-1}(B)$. Each subset $\eta_X(\tilde{f}^{-1}(\widetilde{B}))$ of $f^{-1}(B)$ is referred to as a *preimage class*. Since we assume that f is an equivariant map and that B is a G-invariant subset, it follows that the group G acts on the set of preimage classes. More precisely, for each $g \in G$, the set

$$g \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$$

is itself a preimage class. To see this, consider a lifting $\widetilde{\psi}_g : \widetilde{Y} \to \widetilde{Y}$ of the homeomorphism $\psi_g : Y \to Y$ given by multiplication by g, i.e., $y \mapsto g \cdot y$. It is straightforward to verify that $\widetilde{\psi}_g(\widetilde{B})$ is also a path component of $\eta_Y^{-1}(B)$. Furthermore, one can check that

$$g \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B})) = \eta_X(\tilde{f}_0(\widetilde{B}_0)),$$

where

$$\tilde{f}_0 = \widetilde{\psi}_g \circ \tilde{f} \circ \tilde{\tau}_{g^{-1}},$$

with $\widetilde{\psi}_g : \widetilde{Y} \to \widetilde{Y}$ and $\tilde{\tau}_{g^{-1}} : \widetilde{X} \to \widetilde{X}$ being liftings of the maps $Y \to Y$, $y \mapsto g \cdot y$, and $X \to X$, $x \mapsto g^{-1} \cdot x$, respectively. Also, we define $\widetilde{B}_0 = \widetilde{\psi}_g(\widetilde{B})$. Since $\tilde{f}_0$ is a lifting of f, it follows that $g \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$ is indeed a preimage class of f.

As an immediate consequence of the above observation, we obtain the following result.

Lemma 2.1 *The preimage set*

$$f^{-1}(B) = \bigcup_{\tilde{f},\widetilde{B}} G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B})),$$

where $\tilde{f}$ ranges over all liftings of f with respect to universal coverings $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$, and $\widetilde{B}$ ranges over all path components of $\eta_Y^{-1}(B)$.

Following [16], a pair $(\tilde{f}, \widetilde{B})$ as in Lemma 2.1 is called a *lifting data pair* for preimage of f at B.

Lemma 2.2 *For any two lifting data pairs $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ of f at B, either*

$$G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) = G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) \quad \text{or} \quad G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = \varnothing.$$

Proof Suppose $x_0 \in G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$. Then, $x_0 = g_1 \cdot \eta_X(a) = g_2 \cdot \eta_X(b)$, for some $a \in \tilde{f}_1^{-1}(\widetilde{B}_1)$, $b \in \tilde{f}_2^{-1}(\widetilde{B}_2)$, and $g_1, g_2 \in G$. Therefore, $\eta_X(a) = g \cdot \eta_X(b)$, where $g = g_1^{-1} g_2$.

Let $\tau_g : X \to X$ be the homeomorphism given by $x \mapsto g \cdot x$ and let $\tilde{\tau}_g : \widetilde{X} \to \widetilde{X}$ be a lifting of τ_g. Then

$$\eta_X(\tilde{\tau}_g(b)) = g \cdot \eta_X(b) = \eta_X(a).$$

Hence, there exists $\alpha \in \mathrm{Cov}(\eta_X)$ such that $a = \alpha(\tilde{\tau}_g(b))$.

Let $\psi_{g^{-1}} : Y \to Y$ be the homeomorphism given by $y \mapsto g^{-1} \cdot y$, and let $\widetilde{\psi}_{g^{-1}} : \widetilde{Y} \to \widetilde{Y}$ be a lifting of $\psi_{g^{-1}}$. Then

$$\begin{aligned}
\eta_Y(\widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g(b)) &= \psi_{g^{-1}}\big(\eta_Y(\tilde{f}_1(\alpha(\tilde{\tau}_g(b))))\big) \\
&= g^{-1} \cdot f(\eta_X(\alpha(\tilde{\tau}_g(b)))) \\
&= g^{-1} \cdot f(\eta_X(\tilde{\tau}_g(b))) \\
&= g^{-1} \cdot f(g \cdot \eta_X(b)) \\
&= f(\eta_X(b)) \\
&= \eta_Y(\tilde{f}_2(b)).
\end{aligned}$$

Hence, there exists $\beta \in \mathrm{Cov}(\eta_Y)$ such that

$$\beta(\widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g(b)) = \tilde{f}_2(b). \tag{2-1}$$

Since $\beta \circ \widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g$ and $\tilde{f}_2$ are liftings of f that coincide at a point, they are the same, that is,

$$\beta \circ \widetilde{\psi}_{g^{-1}} \circ \tilde{f}_1 \circ \alpha \circ \tilde{\tau}_g = \tilde{f}_2. \tag{2-2}$$

Thus, $\tilde{\delta} = \beta \circ \widetilde{\psi}_{g^{-1}}$ is a lifting of $\psi_{g^{-1}}$, $\tilde{\sigma} = \alpha \circ \tilde{\tau}_g$ is a lifting of τ_g, and

$$\tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma} = \tilde{f}_2. \tag{2-3}$$

Moreover, since $\tilde{f}_1(a) \in \widetilde{B}_1$ and

$$\tilde{\delta}(\tilde{f}_1(a)) = \beta \circ \widetilde{\psi}_{g^{-1}}(\tilde{f}_1(a)) = \beta \circ \widetilde{\psi}_{g^{-1}}(\tilde{f}_1(\alpha(\tilde{\tau}_g(b)))) = \tilde{f}_2(b) \in \widetilde{B}_2,$$

we have

$$\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2. \tag{2-4}$$

Thus,

$$\begin{aligned}
\eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) &= \eta_X((\tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma})^{-1}(\widetilde{B}_2)) \\
&= \eta_X(\tilde{\sigma}^{-1}(\tilde{f}_1^{-1}(\tilde{\delta}^{-1}(\widetilde{B}_2)))) \\
&= \eta_X(\tilde{\sigma}^{-1}(\tilde{f}_1^{-1}(\widetilde{B}_1))) \subset g \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)).
\end{aligned}$$

Therefore, $G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1))$.

Similarly, by using that $\tilde{f}_1 = \tilde{\delta}^{-1} \circ \tilde{f}_2 \circ \tilde{\sigma}^{-1}$, we conclude that

$$G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)).$$

Therefore, $G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1))$. □

Definition 2.3 Lemma 2.1 asserts that the preimage $f^{-1}(B)$ is a disjoint union of subsets of the form $G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. Each one of such subsets is called a *G-preimage class* of f at B.

Following the observation made before Lemma 2.1, each G-preimage class is a union of ordinary nonequivariant preimage classes.

The next result follows immediately from the proof of the Lemma 2.2.

Corollary 2.4 *Two lifting data pairs, $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$, define the same G-preimage class of f at B if and only if*

$$\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1},$$

where $\tilde{\delta} : \widetilde{Y} \to \widetilde{Y}$ is a lifting of the homeomorphism $\psi_g : Y \to Y$, $\psi_g(y) = g \cdot y$, and $\tilde{\sigma} : \widetilde{X} \to \widetilde{X}$ is a lifting of the homeomorphism $\tau_g : X \to X$, $\tau_g(x) = g \cdot x$, for some $g \in G$, and $\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$.

3 *G*-Nielsen preimage classes

We now introduce an equivariant analog of the Nielsen equivalence of [5] (see also [7]).

Definition 3.1 Two points $x_0, x_1 \in f^{-1}(B)$ are said to be G-Nielsen equivalent, denoted by $x_0 \sim_G x_1$, if

(i) $x_0 = g \cdot x_1$ for some $g \in G$ or

(ii) there exists a path γ in X from x_0 to $g \cdot x_1$ and a path β in B from $f(x_0)$ to $f(g \cdot x_1)$ such that $f \circ \gamma \sim \beta$ relative to the endpoints, for some $g \in G$.

The above relation splits $f^{-1}(B)$ into equivalence classes, the so-called *G-Nielsen preimage classes* of f at B.

In nonequivariant preimage theory, two points $x_0, x_1 \in f^{-1}(B)$ are said to be Nielsen related with respect to the subset B, $x_0 \sim x_1$, if there exists a path γ in X from x_0 to x_1 and a path β in B from $f(x_0)$ to $f(x_1)$ such that $f \circ \gamma \sim \beta$ relative to the end points (see [5, Definition 1.2]). Consequently, we have that $x_0, x_1 \in f^{-1}(B)$ are G-Nielsen equivalent if and only if x_0 and $g \cdot x_1$ are Nielsen equivalent (in the sense of standard preimage theory), for some $g \in G$. Analogously to [21, Theorem 2.1, page 32], one can show that two points $x_0, x_1 \in f^{-1}(B)$ are Nielsen equivalent with respect to the subset B if and only if there is a lifting data pair $(\tilde{f}, \widetilde{B})$ such that $x_0, x_1 \in \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. In other words, the set of Nielsen preimage classes coincides with the set of nonempty preimage classes.

Furthermore, the same relationship holds for G-Nielsen preimage classes and G-preimage classes: every G-Nielsen preimage class is a nonempty G-preimage class, as we now show.

Proposition 3.2 *Let $x_0, x_1 \in f^{-1}(B)$. Then $x_0 \sim_G x_1$ if and only if $x_0, x_1 \in G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$ for some lifting data pair $(\tilde{f}, \widetilde{B})$.*

Proof Let $x_0, x_1 \in f^{-1}(B)$ be related G-Nielsen preimage points with respect to the subset B. As we commented, it means that x_0 is Nielsen related to $g \cdot x_1$ (in the standard sense of preimage theory), for some $g \in G$. In turn, this is equivalent to the existence of a lifting data pair $(\tilde{f}, \widetilde{B})$ such that $x_0, g \cdot x_1 \in \eta_X(\tilde{f}^{-1}(\widetilde{B}))$, as we pointed out above. Therefore, $x_0, x_1 \in G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$.

On the other hand, let $(\tilde{f}, \widetilde{B})$ be a lifting data pair such that $x_0, x_1 \in G \cdot \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. Suppose that $x_0 = g_0 \cdot z_0$ and $x_1 = g_1 \cdot z_1$, where $g_0, g_1 \in G$ and $z_0, z_1 \in \eta_X(\tilde{f}^{-1}(\widetilde{B}))$. Since every nonempty preimage class is a Nielsen preimage class, there exists a path γ from z_0 to z_1 such that $f \circ \gamma$ is homotopic, relative to the endpoints, to a path β in B. Now, consider the path $g_0\gamma$, defined by $t \mapsto g_0 \cdot \gamma(t)$. This is a path from $x_0 = g_0 \cdot z_0$ to $g_0 \cdot z_1$ such that $f \circ (g_0\gamma) = g_0(f \circ \gamma)$ is homotopic to $g_0\beta$ relative to the endpoints. Since B is G-invariant, the path $g_0\beta$ also lies in B, implying that $x_0 \sim_G z_1$. Consequently, we obtain $x_0 \sim_G g_1 \cdot z_1 = x_1$, as desired. □

3.1 Topological essentiality of a G-Nielsen preimage class

Definition 3.3 Let $\{f_t : X \to Y\}$ be a G-homotopy of $f_0 = f$. A preimage point $x_0 \in f^{-1}(B)$ of f at B is $\{f_t\}_G$-related to a preimage point $x_1 \in f_1^{-1}(B)$ of f_1 at B, denoted by $x_0\{f_t\}_G\, x_1$, if x_0 is $\{f_t\}$-related to $g \cdot x_1$, for some $g \in G$. This means that there exist paths γ in X from x_0 to $g \cdot x_1$ and β in B from $f_0(x_0)$ to $f_1(g \cdot x_1)$ such that $\{f_t(\gamma(t))\} \sim \beta$ relative to the endpoints.

Similar to the nonequivariant case, the $\{f_t\}_G$ relation above induces a one-to-one correspondence between the G-preimage classes of $f = f_0$ and the G-preimage classes of f_1, as it is stated below. The proof is straightforward.

Lemma 3.4 *Let $\{f_t : X \to Y\}$ be a G-homotopy of f, and let $x_0 \in f^{-1}(B)$ and $x_1 \in f_1^{-1}(B)$ be such that $x_0\{f_t\}_G\, x_1$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the G-preimage classes of f and f_1, respectively, such that $x_i \in \mathcal{A}_i$, $i = 0, 1$. Then $x_0' \in \mathcal{A}_0$ if and only if $x_0'\{f_t\}_G x_1$, and $x_1' \in \mathcal{A}_1$ if and only if $x_0\{f_t\}_G x_1'$.*

In other words, the relation $x_0\{f_t\}_G x_1$ induces a correspondence from $\mathcal{A}_0$ to $\mathcal{A}_1$ under $\{f_t\}$, which is denoted by $\mathcal{A}_0\{f_t\}_G\mathcal{A}_1$.

Definition 3.5 A G-Nielsen preimage class of f at B is *essential* if given any G-homotopy $\{f_t : X \to Y\}$ from f it is $\{f_t\}_G$-related to a G-Nielsen preimage class of f_1 at B. Otherwise, it is called *inessential*. The *G-Nielsen preimage number* of f at B is defined as the number of essential G-preimage classes; it is denoted by $N_G(f; B)$. If X is a compact space, then $0 \le N_G(f; B) < \infty$.

This Nielsen number $N_G(f, B)$ has the usual properties that it is a G-homotopy invariant and is a lower bound for the number of connected components of the preimages. We have the following.

Proposition 3.6 *Given a G-preimage problem $f : X \to_G Y \supset B$,*

(1) *if $f : X \to Y$ is G-homotopic to h then $N_G(h, B) = N_G(f, B)$,*

(2) $N_G(f, B) \le \pi_0(f^{-1}(B))$.

4 Classes of lifting data pairs

Let $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$ be the universal covering of X and Y, respectively.

Since X is a G-space, each $g \in G$ can be associated to the homeomorphism of X, $\tau_g : X \to X$, given by $\tau_g(x) = g \cdot x$. The same for the G-space Y, where we will denote by $\psi_g : Y \to Y$ the homeomorphism given by $\psi_g(y) = g \cdot y$.

We will consider the following groups:

$$\pi_X = \{\tilde{\alpha} \in \mathrm{Homeo}(\widetilde{X}) \mid \eta_X \circ \tilde{\alpha} = \eta_X\} = \mathrm{Cov}(\eta_X),$$
$$\hat{\pi}_X = \{\tilde{\tau}_g \in \mathrm{Homeo}(\widetilde{X}) \mid \eta_X \circ \tilde{\tau}_g = \tau_g \circ \eta_X \text{ for some } g \in G\},$$

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\tau}_g} & \widetilde{X} \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ X & \xrightarrow{\tau_g} & X \end{array}$$

Analogously:

$$\pi_Y = \{\tilde{\gamma} \in \mathrm{Homeo}(\widetilde{Y}) \mid \eta_Y \circ \tilde{\gamma} = \eta_Y\} = \mathrm{Cov}(\eta_Y),$$
$$\hat{\pi}_Y = \{\widetilde{\psi}_g \in \mathrm{Homeo}(\widetilde{Y}) \mid \eta_Y \circ \widetilde{\psi}_g = \psi_g \circ \eta_Y \text{ for some } g \in G\},$$

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\widetilde{\psi}_g} & \widetilde{Y} \\ {\scriptstyle \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ Y & \xrightarrow{\psi_g} & Y \end{array}$$

Note that $\hat{\pi}_X$ and $\hat{\pi}_Y$ are extensions of π_X and π_Y, respectively. The elements in π_X and π_Y are the ones in $\hat{\pi}_X$ and $\hat{\pi}_Y$, respectively, that cover the identity $\mathrm{Id} = \tau_e$, where $e \in G$ is the identity.

Remark 4.1 In general, the short exact sequence $1 \to \pi_X \to \hat{\pi}_X \to G \to 1$ does not split so G does not act on π_X unless π_X is abelian. Moreover, every $g \in G$ induces a homeomorphism $\theta_g : X \to X$, which in turn induces an isomorphism $\pi_1(X, x_0) \to \pi_1(X, gx_0)$ but gx_0 need not be the same as x_0.

Let

$$\Gamma = \{(\tilde{\delta}, \tilde{\sigma}) \in \hat{\pi}_Y \times \hat{\pi}_X \mid \eta_Y \circ \tilde{\delta} = \psi_g \circ \eta_Y \text{ and } \eta_X \circ \tilde{\sigma} = \tau_g \circ \eta_X \text{ for some } g \in G\}.$$

Definition 4.2 Two lifting data pairs $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ are said to be equivalent if there is $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$ such that $\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1}$ and $\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$.

Lemma 4.3 *Let $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ be two lifting data pairs.*

(1) *If the two pairs are equivalent, then $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) = G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$.*

(2) *If the two pairs are not equivalent, then $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = \varnothing$.*

Proof (1) Since $(\tilde{f}_1, \widetilde{B}_1)$ and $(\tilde{f}_2, \widetilde{B}_2)$ are equivalent,

$$\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1} \quad \text{and} \quad \tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$$

for some $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$, which means that $\eta_Y \circ \tilde{\delta} = \psi_g \circ \eta_Y$ and $\eta_X \circ \tilde{\sigma} = \tau_g \circ \eta_X$ for some $g \in G$. Thus,

$$\tilde{f}_2^{-1}(\widetilde{B}_2) = (\tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1})^{-1}(\tilde{\delta}(\widetilde{B}_1)) = \tilde{\sigma}(\tilde{f}_1^{-1}(\widetilde{B}_1)).$$

Therefore,

$$\eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2)) = \eta_X(\tilde{\sigma}(\tilde{f}_1^{-1}(\widetilde{B}_1))) = \tau_g(\eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1))) \subset G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)).$$

By Lemma 2.2, $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) = G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$.

(2) Suppose on the contrary that $G \cdot \eta_X(\tilde{f}_1^{-1}(\widetilde{B}_1)) \cap G \cdot \eta_X(\tilde{f}_2^{-1}(\widetilde{B}_2))$ contains a point x_0. Then, by following the proof of Lemma 2.2, there is $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$ such that $\tilde{\delta}(\widetilde{B}_1) = \widetilde{B}_2$ and $\tilde{f}_2 = \tilde{\delta} \circ \tilde{f}_1 \circ \tilde{\sigma}^{-1}$. □

Next, we define the so-called G-Reidemeister preimage number $R_G(f, B)$ of f at B. Such number is an upper bound for the number $N_G(f; B)$ — the G-Nielsen preimage number of f at B — defined previously. There are two possible approaches: either by using universal covering or by using Hopf covering. First, we use the universal covering to define $R_G(f, B)$.

4.1 G-Reidemeister preimage number via universal covering

Once and for all, let us fix a lifting data pair $(\tilde{f}, \widetilde{B})$ of f at B. Note that given an arbitrary data pair $(\tilde{f}_1, \widetilde{B}_1)$,

$$\tilde{f}_1^{-1}(\widetilde{B}_1) = (\alpha \circ \tilde{f})^{-1}(\widetilde{B})$$

for some $\alpha \in \mathrm{Cov}(\eta_Y)$. Thus,

$$f^{-1}(B) = \bigcup_{\alpha \in \mathrm{Cov}(\eta_Y)} G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B}))$$

and, from what was shown before, given $\alpha, \beta \in \mathrm{Cov}(\eta_Y)$,

$$G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B})) = G \cdot \eta_X((\beta \circ \tilde{f})^{-1}(\widetilde{B}))$$

if and only if there is $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma$ such that $\tilde{\delta}(\widetilde{B}) = \widetilde{B}$ and $\beta \circ \tilde{f} = \tilde{\delta} \circ (\alpha \circ \tilde{f}) \circ \tilde{\sigma}^{-1}$.

If $\tilde{\alpha} \in \pi_X$ then $\tilde{f} \circ \tilde{\alpha}$ is a lifting of f. Therefore, there exists a unique element $\tilde{f}_\pi(\tilde{\alpha}) \in \pi_Y$ such that

$$\tilde{f} \circ \tilde{\alpha} = \tilde{f}_\pi(\tilde{\alpha}) \circ \tilde{f} \tag{4-1}$$

and, consequently, $\tilde{f}_\pi : \pi_X \to \pi_Y$ is a group homomorphism.

Similarly, given $\tilde{\alpha} \in \hat{\pi}_X$, there exists a unique element $\Phi(\tilde{\alpha}) \in \hat{\pi}_Y$ such that

$$\tilde{f} \circ \tilde{\alpha} = \Phi(\tilde{\alpha}) \circ \tilde{f}. \tag{4-2}$$

Therefore, $\Phi : \hat{\pi}_X \to \hat{\pi}_Y$ is a group homomorphism.

Let $\Gamma_{\widetilde{B}}$ be the subgroup of Γ given by

$$\Gamma_{\widetilde{B}} = \{(\tilde{\delta}, \tilde{\sigma}) \in \Gamma \mid \tilde{\delta}(\widetilde{B}) = \widetilde{B}\}.$$

Then $\Gamma_{\widetilde{B}}$ acts on π_Y via: given $\alpha \in \pi_Y$ and $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}$,

$$(\tilde{\delta}, \tilde{\sigma}) \cdot \alpha = \tilde{\delta}\, \alpha\, \Phi(\tilde{\sigma})^{-1} \in \pi_Y.$$

Such action splits π_Y into disjoint orbit sets: given $\alpha \in \pi_Y$, the orbit of α is the set

$$\{\tilde{\delta}\, \alpha\, \Phi(\tilde{\sigma})^{-1} \mid (\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}\}.$$

The orbit set, $\pi_Y / \Gamma_{\widetilde{B}}$, will be denoted by $\mathcal{R}_G[f, B]$, its cardinality will be denoted by $R_G(f, B)$, and we call $R_G(f, B)$ the *G-Reidemeister preimage number* of f at B.

Theorem 4.4 *Let $\alpha, \beta \in \mathrm{Cov}(\eta_Y)$. Then*

$$G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B})) = G \cdot \eta_X((\beta \circ \tilde{f})^{-1}(\widetilde{B}))$$

if and only if

$$\beta = (\tilde{\delta}, \tilde{\sigma}) \cdot \alpha = \tilde{\delta}\, \alpha\, \Phi(\tilde{\sigma})^{-1} \quad \textit{for some } (\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}.$$

Proof Let $\alpha, \beta \in \mathrm{Cov}(\eta_Y)$. From Lemma 4.3,

$$G \cdot \eta_X((\alpha \circ \tilde{f})^{-1}(\widetilde{B})) = G \cdot \eta_X((\beta \circ \tilde{f})^{-1}(\widetilde{B}))$$

if and only if $(\alpha \circ \tilde{f}, \widetilde{B})$ and $(\beta \circ \tilde{f}, \widetilde{B})$ are equivalent lifting data pairs, which means that

$$\beta \circ \tilde{f} = \tilde{\delta} \circ (\alpha \circ \tilde{f}) \circ \tilde{\sigma}^{-1},$$

for some $(\tilde{\delta}, \tilde{\sigma}) \in \Gamma_{\widetilde{B}}$. By the definition of the group homomorphism $\Phi : \hat{\pi}_X \to \hat{\pi}_Y$,

$$\tilde{f} \circ \tilde{\sigma}^{-1} = \Phi(\tilde{\sigma})^{-1} \circ \tilde{f}.$$

Therefore,

$$\beta \circ \tilde{f} = \tilde{\delta} \circ \alpha \circ \Phi(\tilde{\sigma})^{-1} \circ \tilde{f},$$

and so

$$\beta = \tilde{\delta} \circ \alpha \circ \Phi(\tilde{\sigma})^{-1}.$$ □

Corollary 4.5 *The number of G-preimage classes of f at B is the G-Reidemeister preimage number of f at B. Therefore, $N_G(f, B) \le R_G(f, B)$.*

Remark 4.6 Theorem 4.4 reduces to the classical (nonequivariant) Reidemeister action in [15] or [16]. The G-action induces an action on $\pi_Y \equiv \pi_1(Y)$ by the group $\Gamma_{\widetilde{B}}$ and thus a G-(Reidemeister or Nielsen) preimage class is a finite union of nonequivariant (Reidemeister or Nielsen) preimage classes.

5 G-Reidemeister preimage number via Hopf covering

In [20], a Nielsen root theory for G-maps via an equivariant analog of the approach of Brooks using Hopf lifts was developed. In this section, we will develop the analogous construction to the case of an equivariant preimage problem, generalizing that of [7] in the nonequivariant case.

The map $f : X \to Y$ induces a homomorphism $f_\sharp : \pi_1(X) \to \pi_1(Y)$ on fundamental groups, and there exists a covering $\eta : \widehat{Y} \to Y$ such that $\eta_\sharp \pi_1(\widehat{Y}) = f_\sharp \pi_1(X)$, so we can lift f through η to $\hat{f} : X \to \widehat{Y}$, that is, $f = \eta \circ \hat{f}$. The map $\hat{f}$ is called a Hopf lifting of f, and η a Hopf covering for f.

Frolkina [7] proved the following (nonequivariant setting):

Theorem 5.1 [7, Theorem 2] *Let $(\widehat{Y}, p)$ and $\hat{f}$ be a Hopf covering and a Hopf lift for $f : X \to Y \supset B$. Let $\{f_t\} : X \to Y$ be a homotopy from $f_0 = f$ to f_1 and $\{\hat{f}_t\} : X \to \widehat{Y}$ its lift such that $\hat{f}_0 = \hat{f}$. Then:*

1. *Two preimage points $x_0, x_1 \in f^{-1}(B)$ are Nielsen equivalent if and only if the points $\hat{f}(x_0)$ and $\hat{f}(x_1)$ lie in the same path component of the set $p^{-1}(B)$.*
2. *Nielsen classes of $f : X \to Y \supset B$ are precisely nonempty sets of the form $\hat{f}^{-1}(C)$, where C is a path component of the set $p^{-1}(B)$.*
3. *A point $x_0 \in f_0^{-1}(B)$ is $\{f_t\}$-related to a point $x_1 \in f_1^{-1}(B)$ if and only if the points $\hat{f}_0(x_0)$, $\hat{f}_1(x_1)$ are contained in the same path component of the set $p^{-1}(B)$.*
4. *A preimage class $A_0 \subset f_0^{-1}(B)$ is $\{f_t\}$-related to a class $A_1 \subset f_1^{-1}(B)$ if and only if the sets $\hat{f}_0(A_0)$ and $\hat{f}_1(A_1)$ are contained in one path component of the set $p^{-1}(B)$.*
5. *A preimage class $A_0 \subset f_0^{-1}(B)$ is $\{f_t\}$-related to a class $A_1 \subset f_1^{-1}(B)$ if and only if A_0 and A_1 are 0- and 1-sections of some preimage class of $F : X \times I \to Y \supset B$, where $F(x,t) = f_t(x)$.*

In [20], it was shown that in the setting of $f : X \to Y$ being a G-map, $(\widehat{Y}, \eta)$ a Hopf covering of f and $\hat{f}$ a Hopf lifting of f, there is an action of G on $\widehat{Y}$ under which $\hat{f} : X \to \widehat{Y}$ and $\eta : \widehat{Y} \to Y$ are G-maps, among other properties, as we recall below.

Denote by $\mathfrak{D}(\eta) = \{\delta \in \mathrm{Homeo}(\widehat{Y}) \mid \eta\delta = \eta\}$ the group of deck transformations of η. Let

$$\Gamma_G(\widehat{Y}) = \{\hat{g} \in \mathrm{Homeo}(\widehat{Y}) \mid \eta\hat{g} = g\eta \text{ for some } g \in G\},$$

where g can be regarded as a homeomorphism of Y (previously, we denoted such homeomorphism by ψ_g). Now, there is a short exact sequence

$$1 \to \mathfrak{D}(\eta) \xhookrightarrow{i} \Gamma_G(\widehat{Y}) \xrightarrow{p} G \to 1$$

where i is the inclusion and $p(\hat{g}) = g$ is the projection (the projection is well defined because the action of G on Y is supposed to be effective).

The map f induces a group homomorphism $\varphi : G \to \Gamma_G(\widehat{Y})$ as follows. Pick a point $x_0 \in X$, and let $\hat{f}(x_0) \in \widehat{Y}$. There is a unique lift $\varphi(g)$ of g such that

$$\varphi(g)\hat{f}(x_0) = \hat{f}(gx_0).$$

Now, $\hat{f}g, \varphi(g)\hat{f} : X \to \widehat{Y}$ are both liftings of the same map $fg = gf : X \to Y$, and they agree at x_0. Therefore, $\hat{f}g = \varphi(g)\hat{f}$. The map φ defined under such construction depends on f, η and $\hat{f}$.

Lemma 5.2 [20, Lemma 3.6] *The map $\varphi : G \to \Gamma_G(\widehat{Y})$ is a group homomorphism and is a section to p, i.e., $p \circ \varphi = 1_G$. In particular, $\Gamma_G(\widehat{Y}) = \mathfrak{D}(\eta) \rtimes G$.*

Remark 5.3 When η is a regular cover, we have $\mathfrak{D}(\eta) = \pi_1(Y)/f_\sharp(\pi_1(X)$.

Lemma 5.4 [20, Lemma 3.7] *The maps $\hat{f} : X \to \widehat{Y}$ and $\eta : \widehat{Y} \to Y$ are equivariant maps.*

Theorem 5.5 [20, Theorem 3.8] *If $f' : X \to Y$ is G-homotopic to f, then they induce the same action on the Hopf covering space $\widehat{Y}$ of Y.*

Consider the restriction of the G-action on $\widehat{Y}$ given by φ on the set $\eta^{-1}(B)$. Since η is equivariant and B is G-invariant, $\eta^{-1}(B)$ becomes a G-set.

We now prove an equivariant analog of Theorem 5.1.

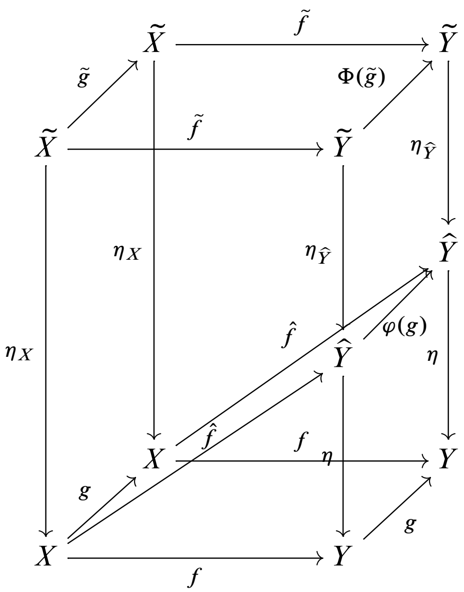
Theorem 5.6 *Let $(\widehat{Y}, \eta)$ and $\hat{f}$ be a Hopf covering and a Hopf lift for $f : X \to_G Y \supset B$. Let $\{f_t\} : X \to Y$ be a G-homotopy from $f_0 = f$ to f_1 and $\{\hat{f}_t\} : X \to \widehat{Y}$ its lift such that $\hat{f}_0 = \hat{f}$. Then:*

1. *Two preimage points $x_0, x_1 \in f^{-1}(B)$ are G-Nielsen equivalent if and only if the points $\hat{f}(x_0)$, $g \cdot \hat{f}(x_1)$ lie in the same path component of the set $\eta^{-1}(B)$, for some $g \in G$.*
2. *The G-Nielsen preimage classes of $f : X \to_G Y \supset B$ are precisely the nonempty sets of the form $G \cdot \hat{f}^{-1}(C)$, where C is a path component of the set $\eta^{-1}(B)$; and a class $G \cdot \hat{f}^{-1}(C)$ is essential if and only if $\hat{f}_1^{-1}(C) \neq \varnothing$ for any G-homotopy $\{\hat{f}_t\}$ beginning at $\hat{f}_0 = \hat{f}$.*
3. *A point $x_0 \in f_0^{-1}(B)$ is $\{f_t\}_G$-related to a point $x_1 \in f_1^{-1}(B)$ if and only if $\hat{f}_0(x_0)$ and $g \cdot \hat{f}_1(x_1)$ are contained in the same path component of the set $\eta^{-1}(B)$, for some $g \in G$.*
4. *A G-preimage class $\mathcal{A}_0 \subset f_0^{-1}(B)$ is $\{f_t\}_G$-related to a class $\mathcal{A}_1 \subset f_1^{-1}(B)$ if and only if the sets $\hat{f}_0(\mathcal{A}_0)$ and $g \cdot \hat{f}_1(\mathcal{A}_1)$ are contained in one path component of the set $\eta^{-1}(B)$.*
5. *A G-preimage class $\mathcal{A}_0 \subset f_0^{-1}(B)$ is $\{f_t\}_G$-related to a G-preimage class $\mathcal{A}_1 \subset f_1^{-1}(B)$ if and only if $\mathcal{A}_0$ and $\mathcal{A}_1$ are the 0- and 1-sections of some G-preimage class of $F : X \times I \to_G Y \supset B$, where $F(x,t) = f_t(x)$, and the action of G on $X \times I$ is given by $g \cdot (x,t) = (g \cdot x, t)$.*

Proof Similar to [7, Theorem 2], one can note that (3) $\Longrightarrow$ (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). So, it is sufficient to prove (3).

By definition, a point $x_0 \in f_0^{-1}(B)$ is $\{f_t\}_G$-related to a point $x_1 \in f_1^{-1}(B)$ if and only if x_0 is $\{f_t\}$-related to $g \cdot x_1$, for some $g \in G$, which is, by Theorem 5.1, equivalent to $\hat{f}(x_0)$ and $\hat{f}(g \cdot x_1)$ lying in the same path component of $\eta^{-1}(B)$. Since $\hat{f}(g \cdot x_1) = g \cdot \hat{f}(x_1)$, the result follows. □

Consider the commutative diagram



where $g : X \to X$ denotes the homeomorphism $x \mapsto g \cdot x$, $g : Y \to Y$ denotes the homeomorphism $y \mapsto g \cdot y$, and $\hat{f} : X \to \widehat{Y}$ is a Hopf lifting of f and $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ is a lifting of $\hat{f}$ with respect to the universal coverings $\eta_X : \widetilde{X} \to X$ and $\eta_Y = \eta \circ \eta_{\widehat{Y}} : \widetilde{Y} \to Y$, where $\eta_{\widehat{Y}} : \widetilde{Y} \to \widehat{Y}$ is a universal covering of $\widehat{Y}$. Let $\widehat{B}$ be a path component of $\eta^{-1}(B)$ and $\widetilde{B}$ a path component of $\eta_{\widehat{Y}}^{-1}(\widehat{B})$, so $\widetilde{B}$ is a path component of $\eta_Y^{-1}(B)$. Then:

(1) The G-preimage classes are of the form $\{G \cdot (\hat{\alpha}\hat{f})^{-1}(\widehat{B})\}$ where $\widehat{B}$ is a path component of $\eta^{-1}(B)$ and $\hat{\alpha} \in \mathfrak{D}(\eta)$. Indeed, from Theorem 5.6(2), the G-Nielsen preimage classes of $f : X \to_G Y \supset B$ are precisely the nonempty sets of the form $G \cdot \hat{f}^{-1}(C)$, where C is a path component of $\eta^{-1}(B)$; fix base points $b_0 \in B$, $\hat{b}_0 \in \widehat{B}$ and $c_0 \in C$ such that $\hat{b}_0, c_0 \in \eta^{-1}(b_0)$. Let $\hat{\alpha} \in \mathfrak{D}(\eta)$ be such that $\hat{\alpha}(\hat{b}_0) = c_0$. Thus, $\hat{\alpha}(\widehat{B}) = C$. Therefore, $(\hat{\alpha}\,\hat{f})^{-1}(\widehat{B}) = \hat{f}^{-1}(\hat{\alpha}^{-1}(\widehat{B})) = \hat{f}^{-1}(C)$. For a general G-preimage class (eventually an empty one), we have the following.

(2) Suppose $\tilde{\alpha} \in \pi_Y$ covers $\hat{\alpha}$ and $\widetilde{B}$ covers $\widehat{B}$. Then

$$G \cdot (\hat{\alpha}\hat{f})^{-1}(\widehat{B}) = G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}).$$

In fact, let $x = \eta_X(\tilde{x})$, with $\tilde{x} \in (\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B})$ be arbitrary. Then

$$\hat{\alpha}\,\hat{f}(x) = \hat{\alpha}\,\hat{f}(\eta_X(\tilde{x})) = \hat{\alpha}\,\eta_{\widehat{Y}}\,\tilde{f}(\tilde{x}) = \eta_{\widehat{Y}}\,\tilde{\alpha}\,\tilde{f}(\tilde{x}) \in \widehat{B} \quad \text{since } \tilde{\alpha}\tilde{f}(\tilde{x}) \in \widetilde{B}.$$

Therefore,

$$G \cdot \eta_X(\tilde{\alpha}\tilde{f})^{-1}(\widetilde{B}) \subset G \cdot (\hat{\alpha}\hat{f})^{-1}(\widehat{B}).$$

On the other hand, let $x \in (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B})$ be arbitrary. And let $\tilde{\alpha} \in \pi_Y$ be an arbitrary element such that $\eta_{\widehat{Y}}\,\tilde{\alpha} = \hat{\alpha}\,\eta_{\widehat{Y}}$, i.e., $\tilde{\alpha}$ covers $\hat{\alpha}$. Let $\tilde{x} \in \widetilde{X}$ be such that $\eta_X(\tilde{x}) = x$. Therefore,

$$\eta_{\widehat{Y}}(\tilde{\alpha}\tilde{f}(\tilde{x})) = \hat{\alpha}\eta_{\widehat{Y}}\tilde{f}(\tilde{x}) = \hat{\alpha}\hat{f}\eta_X(\tilde{x}) = \hat{\alpha}\hat{f}(x) \in \widehat{B},$$

so $\tilde{\alpha}\tilde{f}(\tilde{x}) \in \eta_{\widehat{Y}}^{-1}(\widehat{B})$. Let C be the path component of $\eta_{\widehat{Y}}^{-1}(\widehat{B})$ such that $\tilde{\alpha}\tilde{f}(\tilde{x}) \in C$. Then $x \in \eta_X((\tilde{\alpha}\tilde{f})^{-1}(C))$. Let $\tilde{b} \in \widetilde{B}$ be such that $\eta_{\widehat{Y}}(\tilde{b}) = \hat{\alpha}\hat{f}(x)$ and let $\beta \in \mathfrak{D}(\eta_{\widehat{Y}})$ be the unique element such that $\beta(\tilde{\alpha}\tilde{f}(\tilde{x})) = \tilde{b}$. Then $\beta\tilde{\alpha} \in \pi_Y$, $\beta\tilde{\alpha}$ covers $\hat{\alpha}$, and $x \in G \cdot \eta_X((\beta\tilde{\alpha})\tilde{f})^{-1}(\widetilde{B})$.

Therefore,

$$G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B}) = \bigcup_{\tilde{\alpha} \text{ covers } \hat{\alpha},\, \tilde{\alpha}(\widetilde{B}) = \widehat{B}} G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B}).$$

With the above equality established, we conclude the following:

(a) If $G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B}) = \varnothing$ then $G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B}) = \varnothing$, for any $\tilde{\alpha} \in \pi_Y$ that covers $\hat{\alpha}$ and $\tilde{\alpha}(\widetilde{B}) = \widehat{B}$.

(b) If $G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$ then

$$G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B}) = G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B})$$

because, in this case, both $G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B})$ and $G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B})$ are G-Nielsen preimage classes and $G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B}) \subset G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B})$.

(c) If $G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$ for some $\tilde{\alpha} \in \pi_Y$ that covers $\hat{\alpha}$, then $G \cdot \eta_X(\tilde{\beta}\,\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$ for any $\tilde{\beta} \in \pi_Y$ that covers $\hat{\alpha}$. Indeed, suppose $x = \eta_X(\tilde{x})$, with $\tilde{x} \in (\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B})$. Let $\tilde{\beta} \in \pi_Y$ be any element that covers $\hat{\alpha}$. Then $\eta_{\widehat{Y}}(\tilde{\beta}\,\tilde{f}(\tilde{x})) = \eta_{\widehat{Y}}(\tilde{\alpha}\,\tilde{f}(\tilde{x})) \in \widehat{B}$, i.e., $\tilde{\beta}\,\tilde{f}(\tilde{x}) \in \eta_{\widehat{Y}}^{-1}(\widehat{B})$, but not necessarily $\tilde{\beta}\,\tilde{f}(\tilde{x}) \in \widetilde{B}$. Anyway, let $b_0 = \tilde{\beta}\,\tilde{f}(\tilde{x})$ and $b_1 = \tilde{\alpha}\,\tilde{f}(\tilde{x}) \in \widetilde{B}$. Let $\gamma : I \to \widetilde{Y}$ be a path with $\gamma(0) = b_0$ and $\gamma(1) = b_1$. Then $\eta_{\widehat{Y}} \circ \gamma$ is a loop in $\widehat{Y}$ with base point $\hat{b} = \hat{\alpha}\,\hat{f}(x)$. Therefore, $\eta(\eta_{\widehat{Y}}\gamma)$ is a loop in Y with base point $\eta(\hat{b}) = f(x)$. Since $\eta_\sharp(\pi_1(Y)) = f_\sharp(\pi_1(X))$, there is a loop $\rho : I \to X$ with base point x such that $f\rho \sim \eta(\eta_{\widehat{Y}}\gamma)$ rel $\{0, 1\}$. Let $\tilde{\rho} : I \to \widetilde{X}$ be a lifting of ρ such that $\tilde{\rho}(0) = \tilde{x}$. Now,

$$\eta\eta_{\widehat{Y}}(\tilde{\beta}\,\tilde{f}\,\tilde{\rho}) = \eta\eta_{\widehat{Y}}\,\tilde{f}\,\tilde{\rho} = f\,\eta_X\,\tilde{\rho} = f\rho \sim \eta\eta_{\widehat{Y}}\gamma.$$

Since $\tilde{\beta}\,\tilde{f}\,\tilde{\rho}(0) = \tilde{\beta}\,\tilde{f}(\tilde{x}) = \gamma(0)$, it follows from [17, Lemma 3.3, page 152] that

$$\tilde{\beta}\,\tilde{f}\,\tilde{\rho}(1) = \gamma(1) = b_1 \in \widetilde{B}.$$

Therefore, $(\tilde{\beta}\,\tilde{f})^{-1}(\widetilde{B}) \neq \varnothing$.

(d) From the above items (a)–(c), it follows that if $\tilde{\alpha} \in \pi_Y$ covers $\hat{\alpha}$ and $\widetilde{B}$ covers $\widehat{B}$ then

$$G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B}) = G \cdot \eta_X(\tilde{\alpha}\,\tilde{f})^{-1}(\widetilde{B}).$$

(3) Moreover, $G \cdot (\hat{\alpha}\,\hat{f})^{-1}(\widehat{B}) = G \cdot (\hat{\beta}\,\hat{f})^{-1}(\widehat{B})$ if and only if

$$\hat{\beta} = \hat{\delta}\hat{\alpha}\varphi(g)^{-1} \tag{5-1}$$

for some $\hat{\delta} \in \Gamma_G(\widehat{Y})$, $\hat{\delta}(\widehat{B}) = \widehat{B}$ and $g \in G$ such that $\hat{\delta}$ covers g.

Let $\Gamma_h = \{\hat{\delta} = (\hat{\delta}', g) \in \Gamma_G(\hat{y}) \mid \hat{\delta} \text{ covers } g \text{ and } \hat{\delta}(\widehat{B}) = \widehat{B}\}$.

Definition 5.7 Equation (5-1) defines a (Hopf–Reidemeister) action of Γ_h on $\mathfrak{D}(\eta)$. We define the equivariant G-Hopf–Reidemeister number $R_{G,h}(f,B)$ to be the cardinality of the set of orbits of the action $\hat{\alpha}\mapsto\hat{\delta}\hat{\alpha}\varphi(g)^{-1}$ given by (5-1). Furthermore, by Theorem 5.6, $N_G(f,B)\le R_{G,h}(f,B)$.

From the previous section, we have defined $R_G(f,B)$. Next, we will relate these two Reidemeister numbers by showing that $R_{G,h}(f,B)\le R_G(f,B)$.

Note that for any homeomorphism $\zeta:\widetilde{Y}\to\widetilde{Y}$ that belongs to $\hat{\pi}_Y$, i.e.,

$$(\eta\circ\eta_{\widehat{Y}})\circ\zeta=g\cdot(\eta\circ\eta_{\widehat{Y}})\quad\text{for some element } g\in G,$$

there is a unique homeomorphism $\hat{\zeta}:\widehat{Y}\to\widehat{Y}$ that ζ covers it (also, $\hat{\zeta}$ belongs to $\Gamma_G(\widehat{Y})$). In fact, let $\tilde{y}_0\in\widetilde{Y}$ be a base point and let $\hat{y}_0=\eta_{\widehat{Y}}(\tilde{y}_0)$. Then

$$\eta\,\eta_{\widehat{Y}}\,\zeta(\tilde{y}_0)=g\cdot\eta\,\eta_{\widehat{Y}}(\tilde{y}_0)=g\cdot\eta(\hat{y}_0)=\eta(g\cdot\hat{y}_0).$$

Therefore, there is a unique element $\delta\in\mathfrak{D}(\eta)$ such that

$$\delta\,(\eta_{\widehat{Y}}\,\zeta(\tilde{y}_0))=g\cdot\hat{y}_0.$$

Let $\hat{\zeta}:\widehat{Y}\to\widehat{Y}$ be given by $\hat{\zeta}(y)=\delta^{-1}(g\cdot y)$. Then:

(a) $\hat{\zeta}$ belongs to $\Gamma_G(\widehat{Y})$:

$$\eta\,\hat{\zeta}(y)=\eta\,\delta^{-1}(g\cdot y)=\eta(g\cdot y)=g\cdot\eta(y).$$

(b) ζ covers $\hat{\zeta}$: Let $\beta:\widetilde{Y}\to\widetilde{Y}$ be the unique lifting of $\hat{\zeta}$ such that

$$\beta(\tilde{y}_0)=\zeta(\tilde{y}_0).$$

It easy to see that β belongs to $\hat{\pi}_Y$ and, consequently, $\beta=\zeta$.

Now, we let $\Lambda([\alpha])=\langle\hat{\alpha}\rangle$ where $[\cdot]$ denotes the classes using the universal cover and $\langle\cdot\rangle$ denotes the classes using Hopf coverings. To see that this is well defined, let $[\alpha]=[\beta]$. Thus, $\beta=\tilde{\delta}\alpha\Phi(\tilde{\sigma})^{-1}$. The corresponding map $\hat{\beta}$ is given by $\hat{\delta}\hat{\alpha}\varphi(g)^{-1}$. This can be verified using the commutative diagram above. Also, note that $\tilde{\delta}(\widetilde{B})=\widetilde{B}$. Since $\eta_{\widehat{Y}}\tilde{\delta}=\hat{\delta}\eta_{\widehat{Y}}$, it follows that $\hat{\delta}(\widehat{B})=\widehat{B}$ where $\widehat{B}=\eta_{\widehat{Y}}(\widetilde{B})$. We have just shown that $\Lambda:[\cdot]\to\langle\cdot\rangle$ is surjective.

Now we have:

Proposition 5.8 $$N_G(f,B)\le R_{G,h}(f,B)\le R_G(f,B).$$

Remark 5.9 It should be pointed out that both $R_G(f,B)$ and $R_{G,h}(f,B)$ are well defined and independent of the lifts $\tilde{f}$ and $\hat{f}$ or the Hopf covering $\widehat{Y}$. Moreover, when $B=\{a\}$ is a singleton where $a\in Y^G$, our equivariant Nielsen preimage theory reduces to that of [20]. The equivariant Reidemeister root number defined in [20] using Hopf liftings coincides with $R_{G,h}(f,B)$ for $B=\{a\}$.

Given a G-space Z, we say that Z is *G-connected* if for any subgroup H, Z^H is connected. If G is a finite group and $f : X \to Y$ is a G-map between two G-connected spaces then for each subgroup $H \le G$, $f^H : X^H \to Y^H$ is a WH-map between WH-spaces X^H and Y^H, where f^H is the restriction of f on the fixed point set X^H and $WH = NH/H$ is the Weyl group of H in G where NH is the normalizer of H in G. Thus, for each $H \le G$, the previous sections will yield the invariants $N_{WH}(f^H, B^H)$, $R_{WH}(f^H, B^H)$ and $R_{WH,h}(f^H, B^H)$.

6 Application to Nielsen Borsuk–Ulam theory

In recent years, a theory called the "Nielsen Borsuk–Ulam theory" has been developed (see [2; 3; 19]). This theory is not only to consider the question of the existence of Borsuk–Ulam type coincidences but also to study the minimum number of such coincidences using methods inspired by Nielsen theory for fixed points and coincidences. In what follows, we will show that this theory is a special case of the "equivariant preimage theory for G-maps".

6.1 Borsuk–Ulam property (BUP)

In the literature, several authors have been studying the so-called Borsuk–Ulam property (see, for example, [8; 9; 10; 11; 13]).

The classical Borsuk–Ulam theorem states that for every continuous function $f : S^n \to \mathbb{R}^n$, there exists a point $z \in S^n$ such that $f(z) = f(-z)$, where $-z$ is the antipode of z on the sphere S^n. This result leads to a more general question: given a free involution τ on a space X, does every continuous function $f : X \to Y$ have the property that $f(x) = f(\tau(x))$ for some $x \in X$? If the answer is affirmative, it is said that the triple (X, τ, Y) has the Borsuk–Ulam property or, briefly, BUP. More generally, one can replace the sphere S^n with a topological space X equipped with a free $\mathbb{Z}_p$-action, where p is prime, and Euclidean space $\mathbb{R}^n$ with a topological space Y. In this setup, one possible question is: given $f : X \to Y$ a continuous function, does there exist $x \in X$ such that $f(x) = f(g \cdot x) = f(g^2 \cdot x) = \cdots = f(g^{p-1} \cdot x)$, where $\mathbb{Z}_p = \langle g \rangle$? Another way to pose this question is as follows: Let $\tau : X \to X$ be defined by $\tau(x) = g \cdot x$. Thus, the $\mathbb{Z}_p$-action on X is determined by the homeomorphism τ, and vice versa. One can ask if, given a continuous function $f : X \to Y$, the set of coincidences among the multiple maps $f, f \circ \tau, \dots, f \circ \tau^{p-1}$,

$$\mathrm{Coin}(f, f \circ \tau, f \circ \tau^2, \dots, f \circ \tau^{p-1}) = \{x \in X \mid f(x) = f(\tau(x)) = \cdots = f(\tau^{p-1}(x))\},$$

is nonempty. When the answer is positive for every continuous function from X to Y, we will say that the triple (X, τ, Y) has the Borsuk–Ulam property (briefly, BUP). Also, a homotopy class $\beta \in [X, Y]$ is said to have the BUP with respect to τ when for every continuous function $f : X \to Y$ representing β, there exists a point $x \in X$ such that $f(x) = f(\tau(x)) = \cdots = f(\tau^{p-1}(x))$.

The problem of determining the BUP for a triple (X, τ, Y) can be translated into an equivariant context as follows: let X be a topological space equipped with a free $\mathbb{Z}_p$-action, and let Y be a topological space.

As before, let $\tau : X \to X$ be defined by $\tau^p(x) = x$. Consider the $\mathbb{Z}_p$-action on Y^p determined by the homeomorphism $\tau' : Y^p \to Y^p$ given by

$$\tau'(y_1, y_2, \dots, y_p) = (y_2, y_3, \dots, y_p, y_1).$$

Let $\Delta_p(Y) = \{(y_1, \dots, y_p) \in Y^p \mid y_1 = y_2 = \cdots = y_p\}$ be the diagonal in Y^p. Note that this construction produces two types of isotropy subgroups: if $y \in Y^p \setminus \Delta_p(Y)$, then the isotropy subgroup of $\mathbb{Z}_p$ at y is Id; if $y \in \Delta_p(Y)$, then the isotropy subgroup of $\mathbb{Z}_p$ at y is $\mathbb{Z}_p$.

There is a bijection between the set of continuous functions from X to Y and the set of equivariant maps from X to Y^p: given a continuous function $f : X \to Y$, define $\varphi_f : X \to Y^p$ as

$$\varphi_f(x) = (f(x), f(\tau(x)), \dots, f(\tau^{p-1}(x))).$$

Thus, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ \tau\ } & X \\ {\scriptstyle\varphi_f}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_f} \\ Y^p & \xrightarrow{\ \tau'\ } & Y^p \end{array}$$

In other words, φ_f is equivariant. On the other hand, if $g : X \to Y^p$, $g(x) = (g_1(x), g_2(x), \dots, g_p(x))$, is an equivariant map, then $\varphi_{g_1} = g$.

Note that, under this bijection, given a continuous function $f : X \to Y$, there exists $x \in X$ such that $f(x) = f(\tau(x)) = \cdots = f(\tau^{p-1}(x))$ if and only if $\varphi_f^{-1}(\Delta_p(Y)) \neq \varnothing$.

Furthermore, $\{f_t\}$ is a homotopy between f_0 and f_1 if and only if $\{\varphi_{f_t}\}$ is a $\mathbb{Z}_p$-homotopy between φ_{f_0} and φ_{f_1} where $\varphi_{f_t}(x) = (f_t(x), f_t(\tau x), \dots, f_t(\tau^{p-1}x))$. Thus two continuous functions $f, f' : X \to Y$ are homotopic if and only if φ_f and $\varphi_{f'}$ are $\mathbb{Z}_p$-homotopic. Also,

$$\operatorname{Coin}(f, f \circ \tau, \dots, f \circ \tau^{p-1}) = \varphi_f^{-1}(\Delta_p(Y)).$$

Therefore, the study of the set of Borsuk–Ulam type coincidences, $\operatorname{Coin}(f, f \circ \tau, \dots, f \circ \tau^{p-1})$, is equivalent to the study of the $\mathbb{Z}_p$-preimage problem for φ_f at $\Delta_p(Y)$.

Because of the above observation, we will use the following nomenclature and notation: given X a $\mathbb{Z}_p$-space and $f : X \to Y$ a continuous map, a $\mathbb{Z}_p$-Nielsen preimage class for $\varphi_f : X \to_{\mathbb{Z}_p} Y^p \supset \Delta_p(Y)$ will be called a Borsuk–Ulam class of f (compare with [19]), $N_{\mathbb{Z}_p}(\varphi_f; \Delta_p(Y))$ will be denoted by $N_{\mathrm{BU}}(f)$, $R_{\mathbb{Z}_p}(\varphi_f, \Delta_p(Y))$ by $R_{\mathrm{BU}}(f)$ and $R_{\mathbb{Z}_p,h}(\varphi_f, \Delta_p(Y))$ by $R_{\mathrm{BU},h}(f)$.

6.2 Borsuk–Ulam coincidences as a $\mathbb{Z}_p$-preimage problem

As before, let X and Y be connected, locally pathwise-connected and semilocally simply connected spaces. Suppose X is a free $\mathbb{Z}_p$-space and let $\tau : X \to X$ be the homeomorphism given by $\tau(x) = g \cdot x$, where $\mathbb{Z}_p = \langle g \rangle$. Given a continuous function $f : X \to Y$, a point $x \in X$ such that $f(x) = f(\tau^i(x))$, $i = 1, \dots, p-1$, will be referred as a Borsuk–Ulam type coincidence for (f, τ).

On the cartesian product Y^p, consider the $\mathbb{Z}_p$-action determined by the homeomorphism $\tau' : Y^p \to Y^p$ given by

$$\tau'(y_1, y_2, \dots, y_p) = (y_2, y_3, \dots, y_p, y_1)$$

and let $B = \Delta_p(Y) = \{(y_1, \dots, y_p) \in Y^p \mid y_1 = y_2 = \cdots = y_p\}$ be the thin diagonal in Y^p. As we pointed above, B is invariant with respect to τ'. Also, the map $\varphi_f : X \to Y^p$ given by

$$\varphi_f(x) = (f(x), f(\tau(x)), \dots, f(\tau^{p-1}(x)))$$

is a $\mathbb{Z}_p$-equivariant map.

Let $\eta_X : \widetilde{X} \to X$ and $\eta_Y : \widetilde{Y} \to Y$ be universal coverings of X and Y, respectively. Then

$$\eta = \eta_Y \times \cdots \times \eta_Y : \widetilde{Y}^p \to Y^p$$

is a universal covering of Y^p.

Proposition 6.1 *Let $\widetilde{B}$ be the path component of $\eta^{-1}(\Delta_p(Y))$ that contains the thin diagonal $\Delta_p(\widetilde{Y})$. Then $\widetilde{B} = \Delta_p(\widetilde{Y})$.*

Proof Let $\widetilde{B}$ be the path component of $\eta^{-1}(\Delta_p(Y))$ that contains the thin diagonal $\Delta_p(\widetilde{Y})$. Let $(\tilde{y}_1, \dots, \tilde{y}_p) \in \widetilde{B}$ be an arbitrary point and consider $\lambda : I \to \widetilde{B}$ a continuous path from $(\tilde{y}, \dots, \tilde{y})$ to $(\tilde{y}_1, \dots, \tilde{y}_p)$. Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_p(t))$. Since $\widetilde{B} \subset \eta^{-1}(\Delta_p(Y))$,

$$\eta(\lambda(t)) = (\eta_Y(\lambda_1(t)), \dots, \eta_Y(\lambda_p(t))) \in \Delta_p(Y) \quad \text{for all } t \in I,$$

that is,

$$\eta_Y(\lambda_1(t)) = \eta_Y(\lambda_2(t)) \quad \text{for all } t \in I.$$

Since $\lambda_1(0) = \cdots = \lambda_p(0) = \tilde{y}$, it follows that $\lambda_1(1) = \cdots = \lambda_p(1)$, that is, $\tilde{y}_1 = \cdots = \tilde{y}_p$.

Hence, $\widetilde{B} \subset \Delta_p(\widetilde{Y})$. Therefore, $\widetilde{B} = \Delta_p(\widetilde{Y})$. □

In the special case of $p = 2$, one can show that

$$\hat{\pi}_{Y\times Y} = \pi_{Y\times Y} \rtimes \mathbb{Z}_2.$$

Indeed, consider the covering $\tilde{\tau}' : \widetilde{Y} \times \widetilde{Y} \to \widetilde{Y} \times \widetilde{Y}$ of τ' given by

$$\tilde{\tau}'(\tilde{y}_1, \tilde{y}_2) = (\tilde{y}_2, \tilde{y}_1).$$

Then the short exact sequence

$$1 \longrightarrow \pi_{Y\times Y} \hookrightarrow \hat{\pi}_{Y\times Y} \xrightarrow{\text{proj}} \mathbb{Z}_2 \longrightarrow 1$$

splits

$$1 \longrightarrow \pi_{Y\times Y} \longrightarrow \hat{\pi}_{Y\times Y} \underset{s}{\overset{\text{proj}}{\rightleftarrows}} \mathbb{Z}_2 \longrightarrow 1$$

where $s : \mathbb{Z}_2 \to \hat{\pi}_{Y\times Y}$ is the homomorphism such that $s(\bar{1}) = \tilde{\tau}'$.

Therefore, $\hat{\pi}_{Y\times Y} = \pi_{Y\times Y} \rtimes \mathbb{Z}_2$.

Remark 6.2 The equivariant Nielsen theory developed in this paper can also be applied to the Nielsen Borsuk–Ulam setting even when the $\mathbb{Z}_p$-action is not free. Furthermore, one can develop a Nielsen Borsuk–Ulam type theory for an arbitrary finite group G and arbitrary G-invariant subspace B. For instance, G can be taken to be the symmetric group and B to be the *fat* diagonal. Such a Borsuk–Ulam problem has already been studied in [12]. The applications to these various Borsuk–Ulam type settings will be further developed in a forthcoming work.

6.3 Maps to a topological group

In [9] and [19], the authors considered self-maps of the torus. We now give a different proof of some of their results.

Proposition 6.3 *Let $f : T^2 \to T^2$ be a continuous function and $\tau : T^2 \to T^2$ a free involution on the 2-torus T^2. If τ is an orientation preserving map then all Borsuk–Ulam classes of f (with respect to τ) are inessential. Consequently, $N_{\mathrm{BU}}(f) = 0$, which gives the existence of $f' \sim f$ such that $\mathrm{Coin}(f', f' \circ \tau) = 0$; that means, $\beta = [f]$ does not have the BUP.*

Proof Let F be an essential Borsuk–Ulam class of f. Thus, F is a finite disjoint union of ordinary coincidence classes of f and $f \circ \tau$. From the classical coincidence theory, the ordinary coincidence classes of f and $f \circ \tau$ have the same coincidence index. Since τ is orientation-preserving, it follows from [3, Definition 2.5] that the BU-index of F has the same sign as that of an ordinary coincidence class. This implies that either $N_{\mathrm{BU}}(f) = 0$ when $\mathrm{ind}(f; F) = 0$, or equivalently the Lefschetz coincidence number $L(f, f \circ \tau) = 0$, or all BU-classes are essential, or equivalently when $\mathrm{ind}(f; F) \neq 0$. Denote by A_f and A_τ the matrices associated to the map f and to the map τ, respectively. Then $L(f, f \circ \tau) = 0$ if and only if $\det(A_f - A_f A_\tau) = \det A_f \cdot \det(I - A_\tau)$ vanishes. Since τ is orientation-preserving, it follows from [9] that τ is equivalent to a map that lifts to the map $\tau_1 : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (x + 1/2, y)$. Thus, we conclude that $\det(I - A_\tau) = 0$. □

Corollary 6.4 *Let $\tau : T^2 \to T^2$ be a free involution that preserves orientation. If $\beta \in [T^2, T^2]$ is a homotopy class then β does not have the Borsuk–Ulam property with respect to τ.*

Proof Let $f : T^2 \to T^2$ be an arbitrary self-map on the 2-torus. Therefore, from Proposition 6.3, there is $f' \sim f$ such that f' has no Borsuk–Ulam coincidences. Hence, for any homotopy class $\beta \in [T^2, T^2]$, β does not have BUP with respect to τ. □

The above result was already proved in [9, Theorem 1] and in [19, Theorem 4.1] using different techniques.

We end this paper with the following slight generalization of the setting of self-maps of the torus. Let X be a closed connected manifold with a free involution τ and $f : X \to K$ a map where K is a compact connected topological group. The inversion $\mu : K \to K$ given by $\mu(k) = k^{-1}$ is an involution on K. Define $\varphi_f : X \to K$ by

$$\varphi_f(x) = f(\tau(x)) \cdot [f(x)]^{-1} = f(\tau(x)) \cdot \mu(f(x)).$$

Then

$$\varphi_f(\tau(x)) = f(\tau^2(x)) \cdot \mu(f(\tau(x)) = f(x) \cdot [f(\tau(x))]^{-1} = [f(\tau(x)) \cdot [f(x)]^{-1}]^{-1} = \mu(\varphi_f(x)). \tag{6-1}$$

It follows that φ_f is a $\mathbb{Z}_2$-equivariant map where $\mathbb{Z}_2 \cong \langle\tau\rangle \cong \langle\mu\rangle$. Moreover, f is homotopic to f' if and only if φ_f is $\mathbb{Z}_2$-homotopic to $\varphi_{f'}$. Now,

$$C(f, f\tau) = \{x \in X \mid f(x) = f\tau(x)\} = \varphi_f^{-1}(e),$$

where $e \in K$ is the unit element of the group K. Thus, we are in the equivariant root problem as in [20].

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