Algebraic & Geometric Topology

Volume 26 (2026)

Homotopy commutativity in quasitoric manifolds

Sho Hasui, Daisuke Kishimoto, Yichen Tong and Mitsunobu Tsutaya

msp

Algebraic & Geometric Topology **26**:4 (2026) 1549–1564
DOI: 10.2140/agt.2026.26.1549
Published: April 25, 2026

Homotopy commutativity in quasitoric manifolds

Sho Hasui, Daisuke Kishimoto, Yichen Tong and Mitsunobu Tsutaya

We prove that the loop space of a quasitoric manifold is homotopy commutative if and only if the underlying polytope is a product of 3-simplices $(\Delta^3)^n$ and the characteristic matrix is equivalent to a matrix of certain type. Quasitoric manifolds over $(\Delta^3)^n$ include generalized Bott manifolds, and we also construct an infinite family of homotopy nonequivalent generalized Bott manifolds over $(\Delta^3)^n$, only half of them have homotopy commutative loop spaces. In particular, for each $n \geq 2$, there are infinitely many homotopy types of $6n$-dimensional quasitoric manifolds having homotopy (non)commutative loop spaces.

1 Introduction

Quasitoric manifolds were introduced by Davis and Januszkiewicz [8] as a topological counterpart of smooth projective toric varieties. By definition, a quasitoric manifold is a closed manifold of dimension $2n$ equipped with a locally standard action of T^n such that the orbit space M/T^n is isomorphic to an n-dimensional simple polytope as a manifold with corners. Recall that every toric variety is constructed from a fan, a combinatorial object. There is a similar combinatorial construction of quasitoric manifolds, each of which is equivalent (in a precise sense defined in Section 2) to that associated to a simple polytope P and a certain characteristic matrix over P. Here, we remark that our equivalences of quasitoric manifolds are weaker than those in [8] as they respect a fixed isomorphism $M/T^n \cong P$ while ours do not.

It is well known that properties of a toric variety are described in terms of the corresponding fan, which exhibits a fascinating connection between algebraic geometry and combinatorics. Then it may be possible to describe topological properties of a quasitoric manifold in terms of the underlying simple polytope and the characteristic matrix, which also exhibits a fascinating connection between topology and combinatorics. There are examples of such descriptions for quasitoric manifolds, cohomology and Chern classes as in [8].

The understanding of a given space goes often through the study of its loop space. A first question is then whether or not it is commutative, up to homotopy. In this paper, we study the homotopy commutativity of the loop space of a quasitoric manifold. See [1; 2; 9; 10; 14; 15; 16; 19; 25] for other results on the loop spaces of quasitoric manifolds and related spaces. Complex projective spaces are special quasitoric manifolds, and the homotopy commutativity of their loop spaces were determined by Ganea [13]. The first result completely determines whether or not the loop space of any quasitoric manifold is homotopy commutative in terms of the underlying simple polytope and the characteristic matrix. Let Δ^n and E_n denote the n-simplex and the n-dimensional identity matrix.

MSC2020: primary 57S12; secondary 55P35, 55Q15.

Theorem 1.1 *The loop space of a quasitoric manifold over a simple polytope P is homotopy commutative if and only if $P = (\Delta^3)^n$ and the characteristic matrix is equivalent to*

$$\begin{pmatrix} E_3 & a_{11} & & a_{12} & & a_{13} & & a_{1n} \\ & a_{21} & E_3 & a_{22} & & a_{23} & & a_{2n} \\ & a_{31} & & a_{32} & E_3 & a_{33} & & a_{3n} \\ & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ & a_{n1} & & a_{n2} & & a_{n3} & E_3 & a_{nn} \end{pmatrix} \tag{1-1}$$

for $a_{ij} \in \mathbb{Z}^3$ such that

$$a_{ii} = {}^t(1,1,1) \qquad \text{and} \qquad (1,1,1)a_{ij} \equiv 0 \mod 2 \quad (i \neq j), \tag{1-2}$$

where the facets of $(\Delta^3)^n$ are ordered as in Section 4.

Remarks on Theorem 1.1 are in order. First, equivalences of characteristic matrices will be defined in Section 2. Second, the loop spaces of quasitoric manifolds over a common simple polytope have the same homotopy type. Then Theorem 1.1 may indicate that there are quasitoric manifolds whose loop spaces are homotopy equivalent but not H-equivalent, which is verified by Theorem 1.2 below. Third, we can further consider the higher homotopy commutativity of the loop space of a quasitoric manifold if it is homotopy commutative. Actually, by looking at the cohomology of a quasitoric manifold, we can find a nontrivial quadruple higher Whitehead product if its loop space is homotopy commutative. Then if the loop space of a quasitoric manifold is homotopy commutative, it is not a C_4-space in the sense of Williams [24], so it is not very highly homotopy commutative. Fourth, every characteristic matrix over $(\Delta^3)^n$ is equivalent to the matrix (1-1) satisfying the first condition of (1-2) (Lemma 4.1). Then the second condition of (1-2) guarantees that the loop space of a quasitoric manifold over $(\Delta^3)^n$ is homotopy commutative. On the other hand, $\mathbb{C}P^n$ is a quasitoric manifold over Δ^n, and in particular, a characteristic matrix of $\mathbb{C}P^3$ is

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{1-3}$$

Then Theorem 1.1 recovers Ganea's result [13] that the loop space of $\mathbb{C}P^n$ is homotopy commutative if and only if $n = 3$, where $\mathbb{C}P^n$ is a quasitoric manifold over Δ^n. Thus Theorem 1.1 can be thought of as an extension of Ganea's result. See [20; 21] for other extensions of Ganea's result.

As mentioned above, every characteristic matrix over Δ^3 is equivalent to (1-3), so every quasitoric manifold over Δ^3 is equivalent to $\mathbb{C}P^3$ (see [8, Example 1.18]). However, in general, it is quite hard to describe all characteristic matrices over a given simple polytope, and this is the case for $(\Delta^3)^n$ with $n \geq 2$ as in [7]. Then one cannot immediately see how many nonequivalent quasitoric manifolds over $(\Delta^3)^n$ for $n \geq 2$ there are, whose loop spaces are (not) homotopy commutative. For each $n \geq 2$, we construct an infinite family of homotopy nonequivalent quasitoric manifolds over $(\Delta^3)^n$, only half of them have homotopy commutative loop spaces. Let B be as in (1-3), and let

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $k > 0$ and $n \geq 2$, we define a $3n \times 4n$ matrix

$$B(k,n) = \begin{pmatrix} B & kN & & & \\ & B & kN & & \\ & & \ddots & \ddots & \\ & & & B & kN \\ & & & & B \end{pmatrix}.$$

The matrix $B(k,n)$ is a characteristic matrix over $(\Delta^3)^n$, so we get the corresponding quasitoric manifold $M(k,n)$ over $(\Delta^3)^n$. Note that $M(k,n)$ is defined only for $n \geq 2$ and a positive integer k. Observe that Theorem 1.1 shows that the loop space of $M(k,n)$ is homotopy commutative if and only if k is even. The second result concerns the homotopy types of $M(k,n)$, which implies that for each $n \geq 2$, there are infinitely many homotopy types of $6n$-dimensional quasitoric manifolds having homotopy (non)commutative loop spaces.

Theorem 1.2 *The quasitoric manifolds $M(k,n)$ and $M(l,n)$ are not homotopy equivalent for $k \neq l$.*

Remarks on Theorem 1.2 are in order. First, the quasitoric manifold $M(k,n)$ is a so-called generalized Bott manifold, that is, $M(k,n)$ is obtained by iterated "nice" $\mathbb{C}P^3$-bundles starting from a point. Second, we will actually prove that $M(k,n)$ satisfies cohomological rigidity; $M(k,n)$ and $M(l,n)$ are equivalent if and only if their integral cohomology are isomorphic. Third, if M_i is a quasitoric manifold over P_i for $i = 1, \dots, n$, then $M_1 \times \cdots \times M_n$ is a quasitoric manifold over $P_1 \times \cdots \times P_n$. Hence one can construct a quasitoric manifold over $(\Delta^3)^n$ from quasitoric manifolds over $(\Delta^3)^m$ for $m < n$. However, $M(k,n)$ is not equivalent to a product of nontrivial quasitoric manifolds (Proposition 4.4), so it is an "atomic" quasitoric manifold over $(\Delta^3)^n$.

The paper is organized as follows. In Section 2, we recall the basics of quasitoric manifolds, and show a loop space decomposition of quasitoric manifolds. Then by using this decomposition, we deduce that the loop space of a quasitoric manifold is homotopy commutative only if the underlying simple polytope is a product of Δ^3. In Section 3, we extend the method of Barrat, James and Stein [4] for computing Whitehead products. In Section 4, we prove Theorem 1.1 by applying the results in Sections 2 and 3. We also prove Theorem 1.2 by a direct cohomology computation.

2 Loop space decomposition

In this section, we recall the basic properties of quasitoric manifolds, and show a loop space decomposition of a quasitoric manifold. Then by applying this decomposition, we prove that the loop space of a quasitoric manifold is not homotopy commutative unless its underlying simple polytope is $(\Delta^3)^n$.

First, we define characteristic matrices over a simple polytope and equivalences among them. Let P be an n-dimensional convex polytope. A codimension one face of P will be called a facet. We say that P is simple if exactly n facets of P meet at each vertex. For example, simplices are simple polytopes, and a product of simple polytopes is a simple polytope. Suppose that P is simple and has m facets $F_1, \dots, F_m$. A characteristic matrix over P is an integer matrix $(a_1 \cdots a_m)$ for $a_1, \dots, a_m \in \mathbb{Z}^n$ such that $\det(a_{i_1} \cdots a_{i_n}) = \pm 1$ whenever $F_{i_1} \cap \cdots \cap F_{i_n} \neq \varnothing$ for $i_1 < \cdots < i_n$. Since an automorphism of P as a combinatorial polytope permutes facets, it acts on characteristic matrices over P by column permutation. We define that characteristic matrices A and B over P are equivalent if

$$A = \alpha \cdot (QBD) \tag{2-1}$$

for $Q \in \mathrm{GL}_n(\mathbb{Z})$, a diagonal matrix D with diagonal entries ± 1 and an automorphism α of P.

Next, we recall the construction of a quasitoric manifold using a moment-angle complex. Let K be a simplicial complex with vertex set $[m] = \{1, 2, \dots, m\}$, where an ordering of vertices is given. The moment-angle complex for K is defined by

$$Z_K = \bigcup_{\sigma \in K} Z(\sigma),$$

where $Z(\sigma) = X_1 \times \cdots \times X_m$ such that $X_i = D^2$ for $i \in \sigma$ and $X_i = S^1$ for $i \notin \sigma$. Note that the m-dimensional torus T^m acts naturally on Z_K. We will use the following obvious property of a moment-angle complex. For $\varnothing \neq I \subset [m]$, let

$$K_I = \{\sigma \in K \mid \sigma \subset I\}.$$

Lemma 2.1 *For $\varnothing \neq I \subset [m]$, Z_{K_I} is a retract of Z_K.*

Proof We can identify Z_{K_I} with the subspace

$$\{(x_1, \dots, x_m) \in Z_K \mid x_i \text{ is the basepoint for } i \in I\}$$

of Z_K. □

Let P be an n-dimensional simple polytope with m facets. Let $K(P)$ denote the boundary of the dual simplicial polytope of P. Then $K(P)$ is an $(n-1)$-dimensional simplicial sphere with m vertices. Let A be a characteristic matrix over P. Then the kernel of the linear map $A : \mathbb{Z}^m \to \mathbb{Z}^n$ defines a split subtorus $T(A)$ of dimension $m-n$ which acts freely on $Z_{K(P)}$. Let

$$M(A) = Z_{K(P)}/T(A).$$

By [8], we have:

Proposition 2.2 *The orbit space $M(A)$ is a quasitoric manifold over P such that every quasitoric manifold over P is equivalent to $M(A)$ for some characteristic matrix A over P.*

Let M and N be quasitoric manifolds of dimension $2n$. A map $f: M \to N$ is weakly equivariant if there is an automorphism $\theta: T^n \to T^n$ such that

$$f(tx) = \theta(t) f(x)$$

for $t \in T^n$ and $x \in M$. We say that M and N are equivalent if there is a weakly equivariant homeomorphism between them. Note that if M and N are equivalent, their underlying simple polytopes are isomorphic. Then equivalent quasitoric manifolds are essentially the same. As remarked in Section 1, our equivalences of quasitoric manifolds are weaker than those in [8] as Davis and Januszkiewicz demand equivalences to preserve an extra structure, a fixed isomorphism between M/T^n and a simple polytope. By [8], we also have:

Proposition 2.3 *The quasitoric manifolds $M(A_1)$ and $M(A_2)$ over P are equivalent if and only if the characteristic matrices A_1 and A_2 are equivalent as in* (2-1).

Now we prove a loop decomposition of a quasitoric manifold.

Proposition 2.4 *Let M be a quasitoric manifold over an n-dimensional simple polytope P with m facets. Then there is a homotopy equivalence*

$$\Omega M \simeq T^{m-n} \times \Omega Z_{K(P)}.$$

Proof By Proposition 2.2, there is a homotopy fibration $Z_{K(P)} \to M \to BT^{m-n}$, so we get an H-fibration

$$\Omega Z_{K(P)} \to \Omega M \to T^{m-n}.$$

By [6, Theorem 3.4.7], $Z_{K(P)}$ is 2-connected, so $\Omega Z_{K(P)}$ is simply connected. Then the map $\Omega M \to T^{m-n}$ has a section, implying the above H-fibration splits. □

We record an obvious fact about homotopy commutativity.

Lemma 2.5 *Let X, Y be H-groups, and let $f: X \to Y$ be an H-map. If X is not homotopy commutative and f has a left homotopy inverse, then Y is not homotopy commutative.*

Proof Let $g: Y \to X$ be a left homotopy inverse of f. If Y is homotopy commutative, then by definition, the Samelson product $\langle f, f\rangle$ is trivial, implying a contradiction

$$0 \neq \langle 1_X, 1_X\rangle = g \circ f \circ \langle 1_X, 1_X\rangle = g \circ \langle f, f\rangle = 0. \qquad \square$$

Now we consider conditions on the underlying polytope of a quasitoric manifold M that guarantee ΩM is not homotopy commutative. Let K be a simplicial complex. We say that a nonempty subset I of the vertex set of K is a minimal nonface of K if I is not a simplex of K and all proper subsets of I are simplices of K. Equivalently, $K_I = \partial\Delta^{|I|-1}$.

Lemma 2.6 *Let P be a simple polytope. If $K(P)$ has a minimal nonface of cardinality* 2, 3 *or* ≥ 5, *then the loop space of a quasitoric manifold over P is not homotopy commutative.*

Proof By Proposition 2.4 and Lemma 2.5, it suffices to show $\Omega Z_{K(P)}$ is not homotopy commutative. Let I be a minimal nonface of $K(P)$ of cardinality k. Then $Z_{K(P)_I} = Z_{\partial\Delta^{k-1}} = S^{2k-1}$, so by Lemma 2.1, S^{2k-1} is a retract of $Z_{K(P)}$. By [3], the Whitehead product $[1_{S^{2k-1}}, 1_{S^{2k-1}}]$ is nontrivial for $k = 3$ and $k \geq 5$, so by the adjointness of Whitehead products and Samelson products [22], ΩS^{2k-1} is not homotopy commutative for $k = 3$ and $k \geq 5$. Thus by Proposition 2.4 and Lemma 2.5, $\Omega Z_{K(P)}$ is not homotopy commutative either.

Now we suppose $k = 2$. Let M be a quasitoric manifold over P. Since S^3 is a retract of $Z_{K(P)}$, $H^3(Z_{K(P)}; \mathbb{Q})$ has a basis $\{u_1, \dots, u_l\}$ for some $l \geq 1$. By Proposition 2.2, there is a homotopy fibration

$$Z_{K(P)} \to M \to BT^{m-n},$$

where m is the number of facets of P and $n = \dim P$. By [6, Theorem 3.4.7], $Z_{K(P)}$ is 2-connected, so in the Serre spectral sequence of the above homotopy fibration, each u_i is transgressive. Moreover, by [8, Proposition 3.10], $H^*(M; \mathbb{Q})$ is generated by elements of degree two, so the transgression images of $u_1, \dots, u_l$ are linearly independent. Then we get

$$H^*(M; \mathbb{Q}) = \mathbb{Q}[t_1, \dots, t_{m-n}]/(q_1, \dots, q_l), \quad |t_i| = 2,$$

for $* \leq 5$, where q_i is the transgression image of u_i. This readily implies that the minimal Sullivan model for M is given by

$$(\mathbb{Q}[t_1, \dots, t_{m-n}] \otimes \Lambda(x_1, \dots, x_l), d), \quad dt_i = 0, \; dx_i = q_i,$$

in dimension ≤ 4. Since $|q_i| = 4$, q_i is a quadratic polynomial in $t_1, \dots, t_{m-n}$. Thus by Proposition 13.16 of [11], M has nontrivial Whitehead product, implying ΩM is not homotopy commutative. □

Lemma 2.7 *Let P be a simple polytope. If $K(P)$ has intersecting distinct minimal nonfaces, then the loop space of a quasitoric manifold over P is not homotopy commutative.*

Proof Let I_1, I_2 be minimal nonfaces of $K(P)$ with $I_1 \neq I_2$ and $I_1 \cap I_2 \neq \varnothing$. Let $|I_1 \cap I_2| = j > 0$ and $|I_k| = i_k + j$ for $k = 1, 2$. Then for $k = 1, 2$,

$$Z_{K(P)_{I_k}} = S^{2(i_k+j)-1}.$$

Let $\iota_k : Z_{K(P)_{I_k}} \to Z_{K(P)_{I_1 \cup I_2}}$ be the inclusion, and let v_k be a generator of $H^{2(i_k+j)-1}(Z_{K(P)_{I_k}}) \cong \mathbb{Z}$. Then by Lemma 2.1, there is $u_k \in H^{2(i_k+j)-1}(Z_{K(P)_{I_1 \cup I_2}})$ satisfying $\iota_k^*(u_k) = v_k$ for $k = 1, 2$. Now we assume that the Whitehead product $[\iota_1, \iota_2]$ is trivial. Then there is a homotopy commutative diagram

$$\begin{tikzcd}
Z_{K(P)_{I_1}} \vee Z_{K(P)_{I_2}} \arrow[r, "\iota_1+\iota_2"] \arrow[d] & Z_{K(P)_{I_1 \cup I_2}} \arrow[d, equal] \\
Z_{K(P)_{I_1}} \times Z_{K(P)_{I_2}} \arrow[r, "\mu"] & Z_{K(P)_{I_1 \cup I_2}}
\end{tikzcd}$$

Hence $\mu^*(u_1) = v_1 \times 1$ and $\mu^*(u_2) = 1 \times v_2$, so

$$\mu^*(u_1 u_2) = \mu^*(u_1)\mu^*(u_2) = v_1 \times v_2 \neq 0.$$

Thus we get $u_1 u_2 \neq 0$. On the other hand, since $K(P)_{I_1 \cup I_2}$ has at least two minimal nonfaces, it is not a full simplex, implying $\dim Z_{K(P)_{I_1 \cup I_2}} \leq 2(i_1 + i_2 + j) - 1$. Then

$$|u_1 u_2| = 2(i_1 + i_2 + j) - 2 + 2j > 2(i_1 + i_2 + j) - 1 \geq \dim Z_{K(P)_{I_1 \cup I_2}}$$

as $j > 0$, so we get $u_1 u_2 = 0$, a contradiction. Thus the Whitehead product $[\iota_1, \iota_2]$ is nontrivial, so $\Omega Z_{K(P)_{I_1 \cup I_2}}$ is not homotopy commutative. Therefore by Lemma 2.5, $\Omega Z_{K(P)}$ is not homotopy commutative too. □

Now we are ready to prove:

Proposition 2.8 *Let M be a quasitoric manifold over a simple polytope P. If the loop space of M is homotopy commutative, then $P = (\Delta^3)^n$.*

Proof Suppose ΩM is homotopy commutative. Then by Lemmas 2.6 and 2.7, minimal nonfaces of $K(P)$ are of cardinality 4 and pairwise disjoint. Then

$$K(P) = \underbrace{\partial\Delta^3 \star \cdots \star \partial\Delta^3}_{n} \star \Delta^l$$

for some $l \geq -1$, where $\Delta^{-1} = \{\varnothing\}$. Since $K(P)$ is a simplicial sphere, we have $l = -1$, so

$$K(P) = \underbrace{\partial\Delta^3 \star \cdots \star \partial\Delta^3}_{n}.$$

Thus we obtain $P = (\Delta^3)^n$, as stated. □

We further consider a condition equivalent to the loop space of a quasitoric manifold over $(\Delta^3)^n$ being homotopy commutative. As in the proof of Proposition 2.8, if $P = (\Delta^3)^n$, then $K(P)$ is the join of n copies of $\partial\Delta^3$, implying

$$Z_{K(P)} = (S^7)^n.$$

Let M be a quasitoric manifold over $(\Delta^3)^n$. Then by Proposition 2.4, there is a homotopy equivalence

$$\Omega M \simeq (S^1)^n \times (\Omega S^7)^n,$$

which is not necessarily an H-equivalence. For $i = 1, \ldots, n$, let $a_i : S^1 \to \Omega M$ and $b_i : S^6 \to \Omega M$ be the composite maps

$$S^1 \xrightarrow{g_i} (S^1)^n \to \Omega M \quad \text{and} \quad S^6 \xrightarrow{E} \Omega S^7 \xrightarrow{g_i} (\Omega S^7)^n \to \Omega M,$$

where g_i and E denote the i-th inclusion and the suspension map, respectively.

Lemma 2.9 *Let M be a quasitoric manifold over $(\Delta^3)^n$. The loop space of M is homotopy commutative if and only if the Samelson products $\langle a_i, b_j \rangle$ for $i, j = 1, \ldots, n$ are trivial.*

Proof For $i = 1, \dots, n$, let $\bar{b}_i : \Omega S^7 \to \Omega M$ denote the composite of the i-th inclusion $\Omega S^7 \to (\Omega S^7)^n$ and the natural map $(\Omega S^7)^n \to \Omega M$. By [18, Proposition 1], ΩM is homotopy commutative if and only if the Samelson products $\langle a_i, a_j \rangle$, $\langle a_i, \bar{b}_j \rangle$, $\langle \bar{b}_i, \bar{b}_j \rangle$ for $i, j = 1, \dots, n$ are trivial. Clearly, $\langle a_i, a_j \rangle$ are trivial. By [13, Lemma 2.1], $\langle a_i, \bar{b}_j \rangle = 0$ if and only if $\langle a_i, b_j \rangle = 0$, and $\langle \bar{b}_i, \bar{b}_j \rangle = 0$ if and only if $\langle b_i, b_j \rangle = 0$. Note that each $b_i : S^6 \to \Omega M$ lifts to a map $\tilde{b}_i : S^6 \to (\Omega S^7)^n$. Then the Samelson products $\langle b_i, b_j \rangle$ in ΩM lift to the Samelson products $\langle \tilde{b}_i, \tilde{b}_j \rangle$ in $(\Omega S^7)^n$. Hence since $(\Omega S^7)^n$ is homotopy commutative, $\langle \tilde{b}_i, \tilde{b}_j \rangle$ are trivial, implying so are $\langle b_i, b_j \rangle$. □

3 Computation of Whitehead products

In this section, we extend the method of Barrat, James, and Stein [4] computing Whitehead products. The coefficients of cohomology will be the integers $\mathbb{Z}$.

Let X be a simply connected finite complex satisfying a homotopy fibration

$$(S^{2d-1})^n \xrightarrow{\phi} X \xrightarrow{\pi} (\mathbb{C}P^\infty)^n \tag{3-1}$$

for $d \geq 3$ such that

$$H^*(X) = \mathbb{Z}[t_1, \dots, t_n]/(q_1, \dots, q_n), \quad |t_i| = 2,\ |q_i| = 2d,$$

where for $i = 1, \dots, n$, t_i corresponds to the fundamental class of the i-th $\mathbb{C}P^\infty$ in $(\mathbb{C}P^\infty)^n$ and $q_i \in \mathbb{Z}[t_1, \dots, t_n]$ is the transgression image of a generator of $H^{2d-1}(S^{2d-1})$ for the i-th S^{2d-1} in $(S^{2d-1})^n$.

Lemma 3.1 *The sequence $q_1, \dots, q_n$ in $\mathbb{Z}[t_1, \dots, t_n]$ is regular.*

Proof For any field $\mathbb{F}$, $\mathbb{F}[t_1, \dots, t_n]$ is Cohen–Macaulay, and the Krull dimension of $H^*(X) \otimes \mathbb{F}$ is zero as X is a finite complex. Then the sequence $q_1, \dots, q_n$ is regular in $\mathbb{F}[t_1, \dots, t_n]$ for any field $\mathbb{F}$, so the sequence $q_1, \dots, q_n$ is regular in $\mathbb{Z}[t_1, \dots, t_n]$ too, as stated. □

We consider the cofiber Y of the map $\phi : (S^{2d-1})^n \to X$. For $i = 1, \dots, n$, let $\beta_i : S^{2d-1} \to X$ be the composite

$$S^{2d-1} \xrightarrow{i\text{-th incl}} (S^{2d-1})^n \xrightarrow{\phi} X.$$

Then by degree reasons,

$$Y_{4d-2} = X_{4d-2} \cup_{\beta_1} e^{2d} \cup_{\beta_2} \cdots \cup_{\beta_n} e^{2d}, \tag{3-2}$$

where Y_k denotes the k-skeleton of Y.

Lemma 3.2 *For $* \leq 2d+3$,*

$$H^*(Y) = \mathbb{Z}[t_1, \dots, t_n]/(t_i q_j \mid i, j = 1, \dots, n).$$

Proof Let $u_i \in H^{2d-1}((S^{2d-1})^n)$ denote the generator corresponding to the i-th S^{2d-1} in $(S^{2d-1})^n$. By Lemma 3.1, the elements $q_1,\dots,q_n$ of $\mathbb{Z}[t_1,\dots,t_n]$ are linearly independent, so we may assume

$$\tau(u_i)=q_i$$

for $i=1,\dots,n$, where τ denotes the transgression in the Serre spectral sequence for the homotopy fibration (3-1), implying

$$\delta(u_i)=\pi^*(q_i)$$

for the connecting map $\delta : H^{*-1}((S^{2d-1})^n)\to H^*(X,(S^{2d-1})^n)$ of the long exact sequence for the pair $(X,(S^{2d-1})^n)$ and the map

$$\pi^* : H^*((\mathbb{C}P^\infty)^n)\to H^*(X,(S^{2d-1})^n). \tag{3-3}$$

By degree reasons, the kernel of the composite

$$H^{2d}((\mathbb{C}P^\infty)^n)\xrightarrow{\pi^*} H^{2d}(X,(S^{2d-1})^n)\to H^{2d}(X)$$

is generated by $q_1,\dots,q_n$. Then the map (3-3) for $*=2d$ is an isomorphism. Thus since $\widetilde{H}^*(Y)\cong H^*(X,(S^{2d-1})^n)$, it follows from (3-2) that the map (3-3) is an isomorphism for $1\le *\le 2d$. On the other hand, the $(2d+2)$-dimensional part of the ideal $(q_1,\dots,q_n)$ in $\mathbb{Z}[t_1,\dots,t_n]$ is generated by t_iq_j for $i,j=1,\dots,n$. Then by (3-2), the proof is finished. □

Let $\bar{\pi} : Y\to(\mathbb{C}P^\infty)^n$ denote an extension of the map $\pi : X\to(\mathbb{C}P^\infty)^n$. Then by [12, Theorem 1.1], the homotopy fiber of $\bar{\pi} : Y\to(\mathbb{C}P^\infty)^n$ has the homotopy type of the join $(S^1)^n\star(S^{2d-1})^n$. We consider the cofiber $\overline{Y}$ of the fiber inclusion of $\bar{\pi}$. Let $g_i : A\to A^n$ denote the i-th inclusion for $i=1,\dots,n$, and let γ_{ij} denote the composite

$$S^{2d+1}=S^1\star S^{2d-1}\xrightarrow{g_i\star g_j}(S^1)^n\star(S^{2d-1})^n\xrightarrow{\text{incl}} Y.$$

Then the map

$$\bigvee_{i,j=1}^n g_i\star g_j : \bigvee_{i,j=1}^n S^{2d+1}\to(S^1)^n\star(S^{2d-1})^n$$

is an inclusion of the $(2d+1)$-skeleton and has a left homotopy inverse, implying

$$\overline{Y}_{2d+2}=Y_{2d+2}\cup_{\gamma_{11}}e^{2d+2}\cup_{\gamma_{12}}\cdots\cup_{\gamma_{nn}}e^{2d+2}. \tag{3-4}$$

Lemma 3.3 *For* $*\le 2d+2$,

$$H^*(\overline{Y})=\mathbb{Z}[t_1,\dots,t_n].$$

Proof Since $(S^1)^n\star(S^{2d-1})^n$ is homotopy equivalent to a wedge of spheres, all $(2d+3)$-cells of $\overline{Y}$ are attached to Y_{2d+2}. Then by Lemma 3.2 and (3-4), the $(2d+3)$-cells of $\overline{Y}$ do not kill any cohomology class of $\overline{Y}_{2d+2}$, implying $H^*(\overline{Y})=H^*(\overline{Y}_{2d+2})$ for $*\le 2d+2$. Now by Lemma 3.1, t_iq_j for $i,j=1,\dots,n$ are linearly independent in $\mathbb{Z}[t_1,\dots,t_n]$. Then by arguing as in the proof of Lemma 3.2, the statement is proved. □

Lemma 3.4 *The homotopy group $\pi_{2d+1}(Y)$ is a free abelian group generated by γ_{ij} for $i, j = 1, \dots, n$.*

Proof The statement follows from the homotopy exact sequence of the homotopy fibration

$$(S^1)^n \star (S^{2d-1})^n \to Y \to (\mathbb{C}P^\infty)^n,$$

where the $(2d+1)$-skeleton of $(S^1)^n \star (S^{2d-1})^n$ is described as above. □

For $i = 1, \dots, n$, let $\alpha_i : S^2 \to X$ be a map whose Hurewicz image is the dual of t_i, and let $\overline{\beta}_i : (D^{2d}, S^{2d-1}) \to (Y, X)$ denote the obvious extension of $\beta_i : S^{2d-1} \to X$. Then

$$\delta(\overline{\beta}_i) = \beta_i$$

for the connecting homomorphism $\delta : \pi_*(Y, X) \to \pi_{*-1}(X)$. We consider the relative Whitehead product $[\alpha_i, \overline{\beta}_j] \in \pi_{2d+1}(Y, X)$. See [5] for the definition. By [5, (3.5)],

$$\delta([\alpha_i, \overline{\beta}_j]) = -[\alpha_i, \beta_j].$$

Let $\overline{\eta} : (D^{2d+1}, S^{2d}) \to (D^{2d}, S^{2d-1})$ be the obvious extension of the Hopf map $\eta : S^{2d} \to S^{2d-1}$. By [17, Theorem (1.4)] (see [23, (5.8)]), we can compute $\pi_{2d+1}(Y, X)$ as follows.

For a commutative ring R, let $R\{a_1, \dots, a_k\}$ denote the free R-module with a basis $\{a_1, \dots, a_k\}$.

Lemma 3.5 $\pi_{2d+1}(Y, X) = \mathbb{Z}\{[\alpha_i, \overline{\beta}_j] \mid i, j = 1, \dots, n\} \oplus \mathbb{Z}_2\{\overline{\beta}_i \circ \overline{\eta} \mid i = 1, \dots, n\}$.

Let $\beta = \beta_1 \vee \dots \vee \beta_n : (S^{2d-1})^{\vee n} \to X$ and $\overline{\beta} = \overline{\beta}_1 \vee \dots \vee \overline{\beta}_n : ((D^{2d})^{\vee n}, (S^{2d-1})^{\vee n}) \to (Y, X)$. Let $\iota : (A, *) \to (A, B)$ and $\rho : A \to A/B$ denote the inclusion and the pinch map, respectively. There is a commutative diagram

$$\begin{tikzcd}
& \pi_{2d+1}((D^{2d})^{\vee n}, (S^{2d-1})^{\vee n}) \arrow[r, "\delta", "\cong"'] \arrow[d, "\overline{\beta}_*"] & \pi_{2d}((S^{2d-1})^{\vee n}) \arrow[d, "\beta_*"] \\
\pi_{2d+1}(Y) \arrow[r, "\iota_*"] \arrow[d, "\rho_*"] & \pi_{2d+1}(Y, X) \arrow[r, "\delta"] \arrow[d, "\rho_*"] & \pi_{2d}(X) \\
\pi_{2d+1}(Y/X) \arrow[r, equal] & \pi_{2d+1}(Y/X) &
\end{tikzcd} \tag{3-5}$$

in which the middle row is exact. By the homotopy exact sequence for the homotopy fibration (3-1), we can see the map β_* is an isomorphism. Then there is $\epsilon_{ij} \in \pi_{2d+1}((D^{2d})^{\vee n}, (S^{2d-1})^{\vee n})$ such that

$$\beta_* \circ \delta(\epsilon_{ij}) = -[\alpha_i, \beta_j], \tag{3-6}$$

implying

$$\delta([\alpha_i, \overline{\beta}_j] - \overline{\beta}_*(\epsilon_{ij})) = -[\alpha_i, \beta_j] - \beta_* \circ \delta(\epsilon_{ij}) = 0.$$

Hence by Lemma 3.4,

$$[\alpha_i, \overline{\beta}_j] - \overline{\beta}_*(\epsilon_{ij}) = \iota_*(\zeta_{ij}) \tag{3-7}$$

such that ζ_{ij} is a linear combination of $\gamma_{kl} \in \pi_{2d+1}(Y)$ for $k, l = 1, \dots, n$.

Lemma 3.6 *The Whitehead product* $[\alpha_i, \beta_j]$ *vanishes if and only if* $\rho_*(\zeta_{ij}) = 0$, *where* $\rho : Y \to Y/X$ *denotes the pinch map.*

Proof Since β_* in (3-5) is an isomorphism, it follows from (3-6) that $[\alpha_i, \beta_j] = 0$ if and only if $\epsilon_{ij} = 0$. By (3-2),

$$(Y/X)_{4d-2} = \underbrace{S^{2d} \vee \cdots \vee S^{2d}}_{n},$$

so $\rho_* \circ \bar{\beta}_*$ in (3-5) is an isomorphism. Then $\epsilon_{ij} = 0$ if and only if $\rho_* \circ \bar{\beta}_*(\epsilon_{ij}) = 0$. On the other hand, $\rho_*([\alpha_i, \bar{\beta}_j]) = 0$ as $\rho_*(\alpha_i) = 0$, so by (3-7), we get

$$\rho_* \circ \bar{\beta}_*(\epsilon_{ij}) + \rho_*(\zeta_{ij}) = \rho_*([\alpha_i, \bar{\beta}_j]) = 0.$$

Thus $\rho_* \circ \bar{\beta}_*(\epsilon_{ij}) = 0$ if and only if $\rho_*(\zeta_{ij}) = 0$. □

Now we are ready to prove:

Proposition 3.7 *The Whitehead products* $[\alpha_i, \beta_j]$ *are trivial for* $i, j = 1, \dots, n$ *if and only if* $\mathrm{Sq}^2\, q_k = 0$ *in* $\mathbb{Z}_2[t_1, \dots, t_n]$ *for all* k.

Proof By the homotopy fibration (3-1), we can see that $\pi_{2d+1}(X)$ is a finite group, so the map ι_* in (3-5) is injective by Lemma 3.4. In particular, $\operatorname{Im} \iota_*$ is a free abelian group. On the other hand, by Lemma 3.5 and (3-7), the subgroup A of $\pi_{2d+1}(Y, X)$ generated by $\iota_*(\zeta_{ij})$ for $i, j = 1, \dots, n$ is a maximal free abelian subgroup of $\pi_{2d+1}(Y, X)$. Then since $A \subset \operatorname{Im} \iota_*$, we obtain $A = \operatorname{Im} \iota_*$, implying that $\rho_*(\zeta_{ij}) = 0$ for $i, j = 1, \dots, n$ if and only if $\rho_*(\gamma_{ij}) = 0$ for $i, j = 1, \dots, n$.

By Lemma 3.2, the $(2d)$-cells in (3-2) correspond to $q_1, \dots, q_n$, and by (3-2) and (3-4),

$$(\bar{Y}/X)_{2d+2} = (\underbrace{S^{2d} \vee \cdots \vee S^{2d}}_{n}) \cup_{\rho_*(\gamma_{11})} e^{2d+2} \cup_{\rho_*(\gamma_{12})} \cdots \cup_{\rho_*(\gamma_{nn})} e^{2d+2}$$

such that the $(2d{+}2)$-cells may be considered to be corresponding to $t_i q_j$ for $i, j = 1, \dots, n$. Then as the generator of $\pi_{2d+1}(S^{2d}) \cong \mathbb{Z}_2$ is detected by Sq^2, we get that $\rho_*(\gamma_{ij}) = 0$ if and only if $\mathrm{Sq}^2\, q_k$ does not include the terms $t_i q_j$ in $H^*(\bar{Y}/X; \mathbb{Z}_2)$ for $k = 1, \dots, n$. Note that in $H^*(\bar{Y}; \mathbb{Z}_2)$, $\mathrm{Sq}^2\, q_k$ must belong to the ideal $(q_1, \dots, q_n)$ and every degree $2d + 2$ element of $(q_1, \dots, q_n)$ is a linear combination of $t_i q_j$ for $i, j = 1, \dots, n$. Then since the natural map $H^*(\bar{Y}/X; \mathbb{Z}_2) \to H^*(\bar{Y}; \mathbb{Z}_2)$ is injective for $2d \le * \le 2d + 2$, the above condition on $\mathrm{Sq}^2\, q_k$ in $H^*(\bar{Y}/X; \mathbb{Z}_2)$ is equivalent to that $\mathrm{Sq}^2\, q_k = 0$ in $H^*(\bar{Y}; \mathbb{Z}_2) = \mathbb{Z}_2[t_1, \dots, t_n]$ $(* \le 2d + 2)$ for $k = 1, \dots, n$. □

4 Quasitoric manifolds over $(\Delta^3)^n$

In this section, we prove Theorems 1.1 and 1.2. We fix an ordering of the facets of $(\Delta^3)^n$ as in [7] to consider characteristic matrices over $(\Delta^3)^n$. Let F_1, F_2, F_3, F_4 be the facets of Δ^3, where any choice of ordering will do by symmetry. Then facets of $(\Delta^3)^n$ are

$$F_{ij} = (\Delta^3)^{i-1} \times F_j \times (\Delta^3)^{n-i}$$

for $i = 1, \dots, n$ and $j = 1, 2, 3, 4$. We fix an ordering of facets as

$$F_{11}, F_{12}, F_{13}, F_{14}, F_{21}, F_{22}, F_{23}, F_{24}, \dots, F_{n1}, F_{n2}, F_{n3}, F_{n4},$$

where this ordering is used in Theorems 1.1 and 1.2.

Lemma 4.1 *Every characteristic matrix over $(\Delta^3)^n$ is equivalent to a matrix*

$$\begin{pmatrix} E_3 & a_{11} & & a_{12} & & a_{13} & & & a_{1n} \\ & a_{21} & E_3 & a_{22} & & a_{23} & & & a_{2n} \\ & a_{31} & & a_{32} & E_3 & a_{33} & & & a_{3n} \\ & \vdots & & \vdots & & \vdots & \ddots & & \vdots \\ & a_{n1} & & a_{n2} & & a_{n3} & & E_3 & a_{nn} \end{pmatrix}$$

for $a_{ij} \in \mathbb{Z}^3$ such that $a_{ii} = {}^t(1, 1, 1)$ for $i = 1, \dots, n$.

Proof Let $B = (b^1_1\ b^1_2\ b^1_3\ b^1_4\ b^2_1\ b^2_2\ b^2_3\ b^2_4\ \cdots\ b^n_1\ b^n_2\ b^n_3\ b^n_4)$ be a characteristic matrix over $(\Delta^3)^n$, where $b^i_j \in \mathbb{Z}^{3n}$. Since the facets of $(\Delta^3)^n$ except for $F_{14}, F_{24}, \dots, F_{n4}$ meet at a vertex, the matrix $Q = (b^1_1\ b^1_2\ b^1_3\ b^2_1\ b^2_2\ b^2_3\ \cdots\ b^n_1\ b^n_2\ b^n_3)$ is invertible, so B is equivalent to

$$Q^{-1}B = \begin{pmatrix} E_3 & c_{11} & & c_{12} & & c_{13} & & & c_{1n} \\ & c_{21} & E_3 & c_{22} & & c_{23} & & & c_{2n} \\ & c_{31} & & c_{32} & E_3 & c_{33} & & & c_{3n} \\ & \vdots & & \vdots & & \vdots & \ddots & & \vdots \\ & c_{n1} & & a_{n2} & & c_{n3} & & E_3 & c_{nn} \end{pmatrix}$$

for $c_{ij} \in \mathbb{Z}^3$. Since the facets of $(\Delta^3)^n$ except for $F_{14}, \dots, F_{i-1,4}, F_{ij}, F_{i+1,4}, \dots, F_{n4}$ meet at a vertex for $i = 1, \dots, n$ and $j = 1, 2, 3$,

$$\det \begin{pmatrix} E_{3(i-1)} & & \\ & C_{ij} & \\ & & E_{3(n-i)} \end{pmatrix} = \pm 1$$

for $i = 1, \dots, n$ and $j = 1, 2, 3$, where C_{ij} is the 3×4 matrix $(E_3\ \ c_{ii})$ with j-th column removed. Then we get $c_{ii} = {}^t(\pm 1, \pm 1, \pm 1)$. Since multiplying columns and rows of a characteristic matrix by -1 yields an equivalent characteristic matrix, we obtain that $Q^{-1}B$ is equivalent to the matrix in the statement. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 By Propositions 2.2 and 2.8, we only need to consider a quasitoric manifold $M(A)$ over $(\Delta^3)^n$ such that A is a characteristic matrix over $(\Delta^3)^n$ in Lemma 4.1. By [8, Theorem 4.14],

$$H^*(M(A)) = \mathbb{Z}[t_{ij} \mid i = 1, \dots, n,\ j = 1, 2, 3, 4]/I + J, \quad |t_{ij}| = 2,$$

where $I = (t_{i1}t_{i2}t_{i3}t_{i4} \mid i = 1, \dots, n)$ and

$$J = \left(t_{ij} + \sum_{k=1}^{n} a^j_{ik} t_{k4} \;\middle|\; i = 1, \dots, n,\ j = 1, 2, 3 \right),$$

where $a_{ik} = {}^t(a_{ik}^1, a_{ik}^2, a_{ik}^3)$. So we get

$$H^*(M(A)) = \mathbb{Z}[t_1, \dots, t_n]/(q_1, \dots, q_n), \quad |t_i| = 2, \tag{4-1}$$

such that

$$q_i = t_i \prod_{j=1}^{3} \biggl(\sum_{k=1}^{n} a_{ik}^j t_k \biggr),$$

where we put $t_i = t_{i4}$. Now

$$\begin{aligned}\mathrm{Sq}^2 q_i &= \biggl(t_i + \sum_{j=1}^{3} \sum_{k=1}^{n} a_{ik}^j t_k \biggr) q_i = \biggl((1 + a_{ii}^1 + a_{ii}^2 + a_{ii}^3) t_i + \sum_{k \neq i} (a_{ik}^1 + a_{ik}^2 + a_{ik}^3) t_k \biggr) q_i \\ &= \sum_{k \neq i} (a_{ik}^1 + a_{ik}^2 + a_{ik}^3) t_k q_i\end{aligned}$$

because $a_{ii} = {}^t(1,1,1)$. Thus by Lemma 3.1 and Proposition 3.7, the Whitehead products $[\alpha_i, \beta_j]$ are trivial for $i, j = 1, \dots, n$ if and only if $(1,1,1)a_{ij} = a_{ij}^1 + a_{ij}^2 + a_{ij}^3 \equiv 0 \bmod 2$ for all $i \neq j$. On the other hand, by the adjointness of Whitehead products and Samelson products [22], the Whitehead product $[\alpha_i, \beta_j]$ is trivial if and only if the Samelson product $\langle a_i, b_j \rangle$ is trivial, where a_i and b_j are as in Section 3. Therefore by Lemma 2.9, the proof is finished. □

Hereafter, let k be a positive integer. For $n \geq 1$, we define a graded algebra

$$H(k,n) = \mathbb{Z}[t_1, \dots, t_n]/(t_1^4 + k t_1^3 t_2, \dots, t_{n-1}^4 + k t_{n-1}^3 t_n, t_n^4), \quad |t_i| = 2.$$

We need the following properties of $H(k,n)$.

Lemma 4.2 *If $x \in H(k,n)$ satisfies $|x| = 2$ and $x^4 = 0$, then $x = a t_n$ for some $a \in \mathbb{Z}$.*

Proof Since $|x| = 2$, we may put $x = a_1 t_1 + \dots + a_n t_n$ for integers $a_1, \dots, a_n \in \mathbb{Z}$. Note that the set $\{t_{i_1} t_{i_2} t_{i_3} t_{i_4} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq n,\ i_1 < i_4\}$ is a basis of the degree-8 part of $H(k,n)$. We express x^4 as a linear combination of this basis. Then x^4 includes the term $6 a_i^2 a_j^2 t_i^2 t_j^2$ for $i \neq j$, implying $a_i a_j = 0$ for $i \neq j$. This readily implies $x = a_i t_i$ for some $1 \leq i \leq n$. On the other hand, $t_i^4 = 0$ in $H(k,n)$ if and only if $i = n$. □

Lemma 4.3 *If connected graded rings A, B have nontrivial elements of degree two, then $H(k,n)$ is not isomorphic to $A \otimes B$.*

Proof We prove the statement by induction on n. For $n = 1$, the statement holds because the degree-two part of $H(k,1)$ is isomorphic to $\mathbb{Z}$. We assume the $n = m$ case, and prove the $n = m+1$ case. Suppose that there is an isomorphism $f : H(k, m+1) \to A \otimes B$. We may put $f(t_{m+1}) = a + b$ for $a \in A$ and $b \in B$, where $|a| = |b| = 2$. Then

$$0 = f(t_{m+1}^4) = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4.$$

Since $H(k,m+1) \cong A \otimes B$, A and B are isomorphic to polynomial rings in degrees < 8, implying $a = 0$ or $b = 0$. We may assume $b = 0$. Then f induces an isomorphism

$$\bar{f} : H(k,m+1)/(t_{m+1}) \to (A/(a)) \otimes B.$$

Since $H(k,m+1)/(t_{m+1}) \cong H(k,m)$, it follows from the assumption that $A/(a) = \mathbb{Z}$. So $t = f^{-1}(\bar{f}(t_m))$ and t_{m+1} are linearly independent in $H(k,m+1)$, and $t^4 = t_{m+1}^4 = 0$. This is a contradiction by Lemma 4.2, and therefore $H(k,m+1)$ is not isomorphic to $A \otimes B$. □

For the rest of the paper, we set $n \geq 2$. Let $M(k,n)$ denote the quasitoric manifold in Theorem 1.2. Then by (4-1),

$$H^*(M(k,n)) = H(k,n). \tag{4-2}$$

We remark that $M(k,n)$ is a generalized Bott manifold such that $M(k,n+1)$ is the projectivization of a complex vector bundle $E \oplus \underline{\mathbb{C}}^3 \to M(k,n)$, where E is the complex line bundle with total Chern class $c(E) = 1 + kt_1$ and $\underline{\mathbb{C}}$ denotes the trivial bundle. We show atomicity of $M(k,n)$ with respect to products of quasitoric manifolds.

Proposition 4.4 *The quasitoric manifold $M(k,n)$ is not homotopy equivalent to a product of two nontrivial quasitoric manifolds.*

Proof Let $M(k,n) \simeq M \times N$ for nontrivial quasitoric manifolds M, N. Then by the Künneth formula,

$$H(k,n) \cong H^*(M) \otimes H^*(N).$$

Thus the statement follows from Lemma 4.3. □

Now we start to prove Theorem 1.2. The following lemma is immediate from Lemma 4.2.

Lemma 4.5 *Every graded algebra isomorphism $f : H(k,n) \xrightarrow{\cong} H(l,n)$ satisfies*

$$f(t_n) = \pm t_n.$$

For $j = 1, \dots, n$, we define an ideal of $H(k,n)$ by

$$I_j(k,n) = (t_{n-j+1}, t_{n-j+2}, \dots, t_n).$$

Then we get a sequence

$$I_1(k,n) \subset I_2(k,n) \subset \cdots \subset I_n(k,n).$$

Lemma 4.6 *Every graded algebra isomorphism $f : H(k,n) \xrightarrow{\cong} H(l,n)$ satisfies*

$$f(I_j(k,n)) = I_j(l,n)$$

for $j = 1, \dots, n$.

Proof We show $f(I_j(k,n)) = I_j(l,n)$ by induction on j. For $j = 1$, $f(I_1(k,n)) = I_1(l,n)$ by Lemma 4.5. Assume that the statement holds for $j = 1, \dots, p$. Then the map f induces an isomorphism

$$\bar{f} : H(k,n)/I_p(k,n) \xrightarrow{\cong} H(l,n)/I_p(l,n).$$

On the other hand, there is a natural isomorphism

$$H(m,n)/I_p(m,n) \cong H(m,n-p)$$

for any positive integer m such that $I_1(m,n-p)$ in $H(m,n-p)$ lifts to $I_{p+1}(m,n)$ in $H(m,n)$. By the induction hypothesis, $\bar{f}(I_1(k,n-p)) \subset I_1(k,n-p)$ through the above natural isomorphism. Thus we get $f(I_{p+1}(k,n)) = I_{p+1}(l,n)$. ◻

Proposition 4.7 *$H(k,n) \cong H(l,n)$ if and only if $k = l$.*

Proof The if part is trivial, and we consider the only if part. First, we consider the $n = 2$ case. Suppose there is an isomorphism $f : H(k,2) \xrightarrow{\cong} H(l,2)$. By Lemma 4.6,

$$f(t_1) = \epsilon_1(t_1 + ct_2) \quad \text{and} \quad f(t_2) = \epsilon_2 t_2$$

for $\epsilon_1, \epsilon_2 = \pm 1$ and an integer c, so we get

$$0 = f(t_1^4 + kt_1^3t_2) = (4c - l + k\epsilon_1\epsilon_2)t_1^3t_2 + (6c^2 + 3k\epsilon_1\epsilon_2 c)t_1^2t_2^2 + (4c^3 + 3k\epsilon_1\epsilon_2 c^2)t_1t_2^3.$$

Then since $t_1^3t_2$, $t_1^2t_2^2$, $t_1t_2^3$ are linearly independent in $H(l,n)$, we obtain

$$6c^2 + 3k\epsilon_1\epsilon_2 c = 0, \quad 4c^3 + 3k\epsilon_1\epsilon_2 c^2 = 0, \quad 4c - l + k\epsilon_1\epsilon_2 = 0.$$

By the first two equations, we get $c = 0$, so by the third equation, we obtain $k = l$, as desired.

Next, we consider the $n > 2$ case. Let $H(m)$ denote the subalgebra of $H(m,n)$ generated by t_{n-1} and t_n. Then there is a canonical isomorphism

$$H(m) \cong H(m,2).$$

Suppose there is an isomorphism $f : H(k,n) \xrightarrow{\cong} H(l,n)$. Then by Lemma 4.6, the map f restricts to an isomorphism

$$H(k) \xrightarrow{\cong} H(l).$$

Thus by the $n = 2$ case, we get $k = l$. ◻

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2 The first statement follows from Theorem 1.1, and the second statement follows from (4-2) and Proposition 4.7. ◻

Acknowledgements

Kishimoto and Tsutaya were partially supported by JSPS KAKENHI grants JP22K03284 and JP22K03317, respectively. Tong was partially supported by JST SPRING grant JPMJSP2110. The authors are grateful to Jérôme Scherer and anonymous referees for useful comments.

References

[1] **S A Abramyan**, *Iterated higher Whitehead products in the topology of moment-angle complexes*, Sibirsk. Mat. Zh. 60:2 (2019) 243–256 MR

[2] **S A Abramyan**, **T E Panov**, *Higher Whitehead products for moment-angle complexes and substitutions of simplicial complexes*, Tr. Mat. Inst. Steklova 305 (2019) 7–28 MR In Russian; translated in Proc. Steklov Inst. Math. 305:1 (2019), 1-21

[3] **J F Adams**, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) 72 (1960) 20–104 MR

[4] **M G Barratt**, **I M James**, **N Stein**, *Whitehead products and projective spaces*, J. Math. Mech. 9 (1960) 813–819 MR

[5] **A L Blakers**, **W S Massey**, *Products in homotopy theory*, Ann. of Math. (2) 58 (1953) 295–324 MR

[6] **V M Buchstaber**, **T E Panov**, *Torus actions and their applications in topology and combinatorics*, University Lecture Series 24, Amer. Math. Soc., Providence, RI (2002) MR

[7] **S Choi**, **M Masuda**, **D Y Suh**, *Quasitoric manifolds over a product of simplices*, Osaka J. Math. 47:1 (2010) 109–129 MR

[8] **M W Davis**, **T Januszkiewicz**, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62:2 (1991) 417–451 MR

[9] **N E Dobrinskaya**, *Higher commutators in the loop homology of K-products*, Tr. Mat. Inst. Steklova 266 (2009) 97–111 MR In Russian; translated in Proc. Steklov Inst. Math. 266 (2009), 91–104

[10] **N Dobrinskaya**, *Loops on polyhedral products and diagonal arrangements* (2009) arXiv 0901.2871

[11] **Y Félix**, **S Halperin**, **J-C Thomas**, *Rational homotopy theory*, Graduate Texts in Mathematics 205, Springer (2001) MR

[12] **T Ganea**, *A generalization of the homology and homotopy suspension*, Comment. Math. Helv. 39 (1965) 295–322 MR

[13] **T Ganea**, *On the loop spaces of projective spaces*, J. Math. Mech. 16 (1967) 853–858 MR

[14] **J Grbić**, **T Panov**, **S Theriault**, **J Wu**, *The homotopy types of moment-angle complexes for flag complexes*, Trans. Amer. Math. Soc. 368:9 (2016) 6663–6682 MR

[15] **K Iriye**, **D Kishimoto**, *Fat-wedge filtration and decomposition of polyhedral products*, Kyoto J. Math. 59:1 (2019) 1–51 MR

[16] **K Iriye**, **D Kishimoto**, *Whitehead products in moment-angle complexes*, J. Math. Soc. Japan 72:4 (2020) 1239–1257 MR

[17] **I M James**, *On the homotopy groups of certain pairs and triads*, Quart. J. Math. Oxford Ser. (2) 5 (1954) 260–270 MR

[18] **S Kaji**, **D Kishimoto**, *Homotopy nilpotency in p-regular loop spaces*, Math. Z. 264:1 (2010) 209–224 MR

[19] **D Kishimoto**, **T Matsushita**, **R Yoshise**, *Jacobi identity in polyhedral products*, Topology Appl. 312 (2022) art. id. 108079 MR

[20] **D Kishimoto**, **Y Minowa**, **T Miyauchi**, **Y Tong**, *Homotopy commutativity in symmetric spaces*, Bol. Soc. Mat. Mex. (3) 30:2 (2024) art. id. 42 MR

[21] **D Kishimoto**, **M Takeda**, **Y Tong**, *Homotopy commutativity in Hermitian symmetric spaces*, Glasg. Math. J. 64:3 (2022) 746–752 MR

[22] **H Samelson**, *A connection between the Whitehead and the Pontryagin product*, Amer. J. Math. 75 (1953) 744–752 MR

[23] **H Toda**, *On the double suspension E^2*, J. Inst. Polytech. Osaka City Univ. Ser. A 7 (1956) 103–145 MR

[24] **F D Williams**, *Higher homotopy-commutativity*, Trans. Amer. Math. Soc. 139 (1969) 191–206 MR

[25] **E G Zhuravleva**, *Adams–Hilton models and higher Whitehead brackets for polyhedral products*, Tr. Mat. Inst. Steklova 317 (2022) 107–131 MR In Russian; translated in Proc. Steklov Inst. Math. 317 (2022), 94-116

SHO HASUI s.hasui@omu.ac.jp
Department of Mathematics, Osaka Metropolitan University, Osaka, Japan

DAISUKE KISHIMOTO kishimoto@math.kyushu-u.ac.jp
Faculty of Mathematics, Kyushu University, Fukuoka, Japan

YICHEN TONG tongyichen@westlake.edu.cn
Institute for Theoretical Sciences, Westlake University, Hangzhou, China

MITSUNOBU TSUTAYA tsutaya@math.kyushu-u.ac.jp
Faculty of Mathematics, Kyushu University, Fukuoka, Japan

Received: August 15, 2024 Revised: March 30, 2025

Geometry & Topology Publications, an imprint of **mathematical sciences publishers** msp

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Vesna Stojanoska
vesna@illinois.edu
University of Illinois at Urbana-Champaign

BOARD OF EDITORS

Julie Bergner — University of Virginia — jeb2md@eservices.virginia.edu
Steven Boyer — Université du Québec à Montréal — cohf@math.rochester.edu
Tara E Brendle — University of Glasgow — tara.brendle@glasgow.ac.uk
Indira Chatterji — CNRS & Univ. Côte d’Azur (Nice) — indira.chatterji@math.cnrs.fr
Octav Cornea — Université’ de Montreal — cornea@dms.umontreal.ca
Alexander Dranishnikov — University of Florida — dranish@math.ufl.edu
Tobias Ekholm — Uppsala University, Sweden — tobias.ekholm@math.uu.se
Mario Eudave-Muñoz — Univ. Nacional Autónoma de México — mario@matem.unam.mx
David Futer — Temple University — dfuter@temple.edu
John Greenlees — University of Warwick — john.greenlees@warwick.ac.uk
Matthew Hedden — Michigan State University — mhedden@math.msu.edu
Kristen Hendricks — Rutgers University — kristen.hendricks@rutgers.edu
Hans-Werner Henn — Université Louis Pasteur — henn@math.u-strasbg.fr
Kathryn Hess — École Polytechnique Féd. de Lausanne — kathryn.hess@epfl.ch
Daniel Isaksen — Wayne State University — isaksen@math.wayne.edu
Thomas Koberda — University of Virginia — thomas.koberda@virginia.edu
Markus Land — JGU Mainz — mland@uni-mainz.de
Christine Lescop — Université Joseph Fourier — lescop@ujf-grenoble.fr
Norihiko Minami — OCAMI (Osaka Central Adv. Math. Inst.) — norihikominami@gmail.com
Andrés Navas — Universidad de Santiago de Chile — andres.navas@usach.cl
Jessica S Purcell — Monash University — jessica.purcell@monash.edu
Birgit Richter — Universität Hamburg — birgit.richter@uni-hamburg.de
Jérôme Scherer — École Polytech. Féd. de Lausanne — jerome.scherer@epfl.ch
Zoltán Szabó — Princeton University — szabo@math.princeton.edu
Maggy Tomova — University of Iowa — maggy-tomova@uiowa.edu
Daniel T Wise — McGill University, Canada — daniel.wise@mcgill.ca
Lior Yanovski — Hebrew University of Jerusalem — lior.yanovski@gmail.com

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2026 is US $795/year for the electronic version, and $1170/year (+$80, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
https://msp.org/

© 2026 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 26 Issue 4 (pages 1229–1596) 2026