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On the mapping class groups of simply connected smooth 4-manifolds

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The mapping class group $M(X)$ of a smooth manifold X is the group of smooth isotopy classes of orientation-preserving diffeomorphisms of X. We prove a number of results about the mapping class groups of compact, simply connected, smooth 4-manifolds. For example, we prove that $M(X)$ is nonfinitely generated for $X = 2n\mathbb{CP}^2 \# 10n\overline{\mathbb{CP}^2}$, where $n \geq 3$ is odd. Let $\Gamma(X)$ denote the group of automorphisms of the intersection lattice of X that can be realised by diffeomorphisms. Then $M(X)$ is an extension of $\Gamma(X)$ by $T(X)$, the Torelli group of isotopy classes of diffeomorphisms that act trivially in cohomology. We prove this extension is split for connected sums of $\mathbb{CP}^2$, but is not split for $2\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$, where $n \geq 11$. We prove that the Nielsen realisation problem fails for certain finite subgroups of $M(p\mathbb{CP}^2 \# q\overline{\mathbb{CP}^2})$ whenever $p+q \geq 4$. Lastly we study the extension $M_1(X) \to M(X)$, where $M_1(X)$ is the group of isotopy classes of diffeomorphisms of X which fix a neighbourhood of a point. When $X = K3$ or $K3 \# (S^2 \times S^2)$ we prove that $M_1(X) \to M(X)$ is a nontrivial extension of $M(X)$ by $\mathbb{Z}_2$. Moreover, we completely determine the extension class of $M_1(K3) \to M(K3)$.

1 Introduction

Let X be a compact, oriented, smooth, simply connected 4-manifold. Define the mapping class group $M(X)$ to be the group of smooth isotopy classes of orientation-preserving diffeomorphisms of X. There is considerable interest in the groups $M(X)$, although little is known about their structure. In this paper we will prove a number of new results concerning the structure of mapping class groups of smooth 4-manifolds.

Recall that the second cohomology group $L_X = H^2(X;\mathbb{Z})$ of X equipped with its intersection form is a unimodular lattice. We let $\mathrm{Aut}(L_X)$ denote the automorphism group of the lattice L_X. The group of orientation-preserving diffeomorphisms of X acts on L_X via $f \mapsto (f^{-1})^*$. This action depends only on this isotopy class and so defines a homomorphism $M(X) \to \mathrm{Aut}(L_X)$. Denoting the image of this map by $\Gamma(X)$ and the kernel by $T(X)$, we obtain a short exact sequence

$$1 \to T(X) \to M(X) \to \Gamma(X) \to 1. \tag{1-1}$$

We call $T(X)$ the *Torelli group* of X. It is the group of isotopy classes of diffeomorphisms of X that act trivially in cohomology. By a result of Quinn, $T(X)$ can also be defined as the group of isotopy classes of diffeomorphisms which are continuously isotopic to the identity [23]. The group $\Gamma(X)$ is the group of automorphisms of L_X that can be realised by diffeomorphisms of X.

Understanding the group $M(X)$ necessitates an understanding of the groups $T(X)$, $\Gamma(X)$ and the extension (1-1). The group $\Gamma(X)$ is known for some classes of 4-manifolds. In particular, a theorem

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of Wall implies that $\Gamma(X) = \mathrm{Aut}(L_X)$ for a large class of 4-manifolds [31]. In contrast, the Torelli group $T(X)$ is poorly understood. Ruberman [25] showed that $T(X)$ is not finitely generated for certain X. However this does not imply that $M(X)$ is not finitely generated, since a finitely generated group can have subgroups which are not finitely generated. Our first main result confirms that $M(X)$ is not finitely generated for certain simply connected 4-manifolds.

Theorem 1.1 *Let $X = 2n\mathbb{CP}^2 \# 10n\overline{\mathbb{CP}^2}$, where $n \geq 3$ is odd. Then $M(X)$ is not finitely generated. More precisely, the following holds:*

(1) *There is an index-2 subgroup $M_+(X)$ of $M(X)$ and a surjective homomorphism $\Phi : M_+(X) \to \mathbb{Z}^\infty$ from $M_+(X)$ to $\mathbb{Z}^\infty$, where $\mathbb{Z}^\infty$ denotes a free abelian group of countably infinite rank.*

(2) *The mod-2 reduction of Φ extends to a surjective homomorphism $\Phi : M(X) \to \mathbb{Z}_2^\infty$.*

As this paper was nearing completion we received a preprint by Hokuto Konno [15] which also proves that the mapping class groups of simply connected 4-manifolds can be nonfinitely generated. Konno's proof uses essentially the same method as ours, however we obtained our proofs completely independently.

Remark 1.2 It is interesting to contrast Theorem 1.1 with finiteness results for mapping class groups in other dimensions. Let X be a compact, simply connected smooth manifold of dimension d and $M(X) = \pi_0(\mathrm{Diff}(X))$ the mapping class group. If $d \neq 4$, then $M(X)$ is finitely generated. For $d \leq 3$, finite generation holds for any compact oriented manifold (see [7] for $d = 2$ and [13] for $d = 3$). If $d \geq 5$ then $M(X)$ is finitely generated [6, Theorem 2.6]. Note that Theorem 2.6 of [6] is only stated for $d \geq 6$, but when X is simply connected, the proof carries over to $d = 5$. In the proof of [6, Theorem 2.6], dimension 6 only enters in the point (i) in the proof, but Cerf's theorem says that in the simply connected case $\pi_0(C^{\mathrm{Diff}}(X)) = 0$, and in [6, Proposition 2.7] where it is not necessary. This follows from specialising Triantafillou [30] to simply connected manifolds, where none of the oversights mentioned in [6, §2.2] cause a problem.[1]

One may also consider the larger group $M'(X)$ consisting of isotopy classes of diffeomorphisms which are not necessarily orientation-preserving. Since $M(X)$ has finite index in $M'(X)$, it follows from Schreier's lemma [28] that if $M(X)$ is not finitely generated then neither is $M'(X)$.

Remark 1.3 Let X be a compact, simply connected smooth 4-manifold and $M^{\mathrm{top}}(X) = \pi_0(\mathrm{Homeo}(X))$ be the topological mapping class group. By work of Freedman [10] and Quinn [23], the natural map $M^{\mathrm{top}}(X) \to \mathrm{Aut}(H^2(X;\mathbb{Z}))$ to the group of automorphisms of the intersection lattice $H^2(X;\mathbb{Z})$ is an isomorphism. By a result of Siegel [29], the automorphism group of any lattice is finitely generated. Hence $M^{\mathrm{top}}(X)$ is finitely generated.

In contrast, we do not know whether the group $\Gamma(X) \subseteq \mathrm{Aut}(H^2(X;\mathbb{Z}))$ is always finitely generated, although we conjecture that it is.

Our next result concerns the question of whether or not the sequence (1-1) admits a splitting.

[1] I thank Alexander Kupers for explaining why [6, Theorem 2.6] works for simply connected 5-manifolds.

Theorem 1.4 (1) *Let $X = n\mathbb{CP}^2$, where $n \geq 1$. Then there exists a splitting $\Gamma(X) \to M(X)$.*

(2) *Let $X = (S^2 \times S^2) \# X'$, where $b_+(X') = 1$, $b_-(X') \geq 10$. Then there does not exist a splitting $\Gamma(X) \to M(X)$.*

More precise information about the failure of a splitting in case (2) is provided by Theorem 5.1.

Remark 1.5 In [5] it is shown when X is a $K3$ surface, there is a splitting $\Gamma(X) \to M(X)$. It is also easy to see that splittings exist for $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Our next result concerns the Nielsen realisation problem. Recall that the Nielsen realisation problem for a smooth manifold X asks whether a subgroup G of the mapping class group of X can be lifted to a subgroup of $\mathrm{Diff}(X)$. Recent results of Baraglia and Konno [5], Farb and Looijenga [9], and Konno [14] show that Nielsen realisation fails for many simply connected spin 4-manifolds. Arabadji and Baykur showed that there are many nonspin 4-manifolds with finite nontrivial fundamental group for which Nielsen realisation fails [1] and Konno, Miyazawa, and Taniguchi gave examples with simply connected indefinite nonspin 4-manifolds [17].

Theorem 1.6 *Let $X = X' \# p\mathbb{CP}^2 \# q\overline{\mathbb{CP}^2}$ where X' is a compact, smooth, simply connected 4-manifold and $p + q \geq 4$. Then $M(X)$ contains a subgroup isomorphic to $\mathbb{Z}_2^4$ which cannot be lifted to $\mathrm{Diff}(X)$.*

In particular, Nielsen realisation fails for $n\mathbb{CP}^2$ for $n \geq 4$. As far as we are aware, these are the first examples of definite, simply connected 4-manifolds where Nielsen realisation fails.

Our last main result concerns a certain extension of $M(X)$. Let $X^{(1)}$ be obtained from X by removing an open ball and let $\mathrm{Diff}(X^{(1)}, \partial X^{(1)})$ denote the group of diffeomorphisms of $X^{(1)}$ which are the identity in a neighbourhood of the boundary. Let $M_1(X) = \pi_0(\mathrm{Diff}(X^{(1)}, \partial X^{(1)}))$ denote the group of components of $\mathrm{Diff}(X^{(1)}, \partial X^{(1)})$. It is known that the map $M_1(X) \to M(X)$ is surjective and that the kernel (which is either trivial or has order 2) is generated by a Dehn twist on the boundary (see Section 7 for more details).

In general it is difficult to determine whether the kernel of $M_1(X) \to M(X)$ is trivial or nontrivial, or equivalently, whether the boundary Dehn twist is trivial or nontrivial. The extension is known to be trivial when X is a connected sum of copies of $S^2 \times S^2$. In contrast we have:

Theorem 1.7 *Let X' be a compact, smooth, simply connected 4-manifold which is homeomorphic to $K3$. Let X be X' or $X' \# (S^2 \times S^2)$. Then the boundary Dehn twist is nontrivial. Moreover, the extension $1 \to \mathbb{Z}_2 \to M_1(X) \to M(X) \to 1$ does not split.*

If $M_1(X) \to M(X)$ is a nontrivial extension, then it is given by an extension class $\xi_X \in H^2(M(X); \mathbb{Z}_2)$ and the above theorem says that $\xi_X \neq 0$ when X is of the stated form. Our final result completely determines ξ_X in the case that X is homeomorphic to $K3$. Let L_X be the intersection lattice of X and $\mathrm{Aut}(L_X)$ the group of automorphisms. Over the classifying space $B\mathrm{Aut}(L_X)$ we have the tautological flat bundle $H = E\mathrm{Aut}(L_X) \times_{\mathrm{Aut}(L_X)} L_X$. Let $H^+ \to B\mathrm{Aut}(L_X)$ be a maximal positive subbundle. This defines a characteristic class $w_2(H^+) \in H^2(\mathrm{Aut}(L_X); \mathbb{Z}_2)$.

Theorem 1.8 *Let X be a smooth 4-manifold which is homeomorphic to $K3$. Then the extension class $\xi_X \in H^2(M(X);\mathbb{Z}_2)$ is the pullback of $w_2(H^+) \in H^2(\mathrm{Aut}(L_X);\mathbb{Z}_2)$ under the map $M(X) \to \mathrm{Aut}(L_X)$.*

1.1 Structure of the paper

The structure of the paper is as follows. In Section 2 we review the Seiberg–Witten invariants for the Torelli group (as in [3; 24; 26]) and show how these invariants can be assembled into cohomology classes on the mapping class group. In Section 3 we use these cohomology classes to show that $M(X)$ is not finitely generated for certain X. In Section 4 we construct a splitting $\Gamma(X) \to M(X)$ when X is a connected sum of copies of $\mathbb{CP}^2$. In Section 5 we prove the nonexistence of splittings $\Gamma(X) \to M(X)$ for certain 4-manifolds. The proof uses families Seiberg–Witten theory and more specifically the main result of [2]. In Section 6 we prove Theorem 1.6. Finally, in Section 7 we study boundary Dehn twists and the extension $M_1(X) \to M(X)$ and we prove Theorems 1.7 and 1.8.

2 Seiberg–Witten invariants for the mapping class group

In this section we define Seiberg–Witten invariants for the mapping class group, extending the Seiberg–Witten invariants on the Torelli group which have previously been considered in [3; 24; 26]. These invariants will be used to show that certain simply connected 4-manifolds have nonfinitely generated mapping class group.

Let X be a compact, smooth, simply connected 4-manifold and let $\mathfrak{s}$ be a spin^c-structure with $d(\mathfrak{s}) = -1$, where

$$d(\mathfrak{s}) = \tfrac{1}{4}(c(\mathfrak{s})^2 - \sigma(X)) - b_+(X) - 1$$

is the expected dimension of the Seiberg–Witten moduli space for $\mathfrak{s}$. Let $\mathcal{S}(X)$ denote the set of all isomorphism classes of spin^c-structures on X for which $d(\mathfrak{s}) = -1$. Since X is assumed to be simply connected, $\mathcal{S}(X)$ can be identified with the set of characteristic elements $c \in L = H^2(X;\mathbb{Z})$ for which $(c^2 - \sigma(X))/4 - b_+(X) = 0$.

Let Π denote the space of pairs (g, η) where g is a Riemannian metric on X and η is a 2-form which is self-dual with respect to g. For any $h \in \Pi$ and any $\mathfrak{s} \in \mathcal{S}(X)$ we may consider the Seiberg–Witten equations on X with respect to the metric g, spin^c-structure $\mathfrak{s}$ and 2-form perturbation η. Let $\mathcal{M}(X,\mathfrak{s},h)$ denote the moduli space of gauge equivalence classes of solutions to the Seiberg–Witten equations for $(X,\mathfrak{s},h)$. Assume $b_+(X) > 2$. We will say that $h \in \Pi$ is regular if $\mathcal{M}(X,\mathfrak{s},h)$ is empty for all $\mathfrak{s} \in S(X)$. Since $b_+(X) > 0$ and the expected dimension of $\mathcal{M}(X,\mathfrak{s},h)$ is negative, the regular elements form a subset of Π of Baire second category with respect to the $\mathcal{C}^\infty$ topology. Let $\Pi^{\mathrm{reg}} \subseteq \Pi$ denote the set of regular elements.

Suppose that $h_0, h_1 \in \Pi^{\mathrm{reg}}$. If $h : [0,1] \to \Pi$ is a path in Π from h_0 to h_1, we can consider the families moduli space, which is the union over $t \in [0,1]$ of the Seiberg–Witten moduli spaces for each $h_t \in \Pi$. For a sufficiently generic path h_t, the moduli space is a compact, smooth, 0-dimensional manifold. A choice of orientation on a maximal positive definite subspace of $H^2(X;\mathbb{R})$ determines an orientation on the moduli space and hence we can count with sign the number of points in the moduli space. Fix a choice of such an orientation. It can be shown [24] that the number of solutions depends on the endpoints h_0, h_1, but not on the choice of generic path h_t. Hence we may denote by $\mathrm{SW}_{\mathfrak{s}}(h_0, h_1) \in \mathbb{Z}$ the signed count of points in the moduli space. From the definition it is clear that this count of points satisfies the following properties:

(1) $\mathrm{SW}_{\mathfrak{s}}(h_0, h_1) + \mathrm{SW}_{\mathfrak{s}}(h_1, h_2) = \mathrm{SW}_{\mathfrak{s}}(h_0, h_2)$.

(2) $\mathrm{SW}_{\mathfrak{s}}(h_0, h_1) = \mathrm{sgn}_+(f)\, \mathrm{SW}_{f(\mathfrak{s})}(f(h_0), f(h_1))$ for any orientation-preserving diffeomorphism f.

In (2), $\mathrm{sgn}_+(f)$ is defined as follows. The space of oriented, maximal positive definite subspaces of $H^2(X;\mathbb{R})$ has two connected components. For an isometry φ of $H^2(X;\mathbb{R})$ we let $\mathrm{sgn}_+(\varphi)$ equal 1 or -1 according to whether φ preserves or exchanges the two components. If f is an orientation-preserving diffeomorphism of X, then $\mathrm{sgn}_+(f)$ denotes $\mathrm{sgn}_+(f_*)$, where $f_* = (f^{-1})^*$ is the isometry of $H^2(X;\mathbb{R})$ induced by f.

Property (1) follows by concatenating a path from h_0 to h_1 with a path from h_1 to h_2. Property (2) follows from diffeomorphism invariance of the Seiberg–Witten equations. In addition, $\mathrm{SW}_{\mathfrak{s}}(h_0, h_1)$ obeys, with respect to charge conjugation, the symmetry

(3) $\mathrm{SW}_{\mathfrak{s}}(h_0, h_1) = (-1)^{b_+(X)/2+1}\, \mathrm{SW}_{\bar{\mathfrak{s}}}(\bar{h}_0, \bar{h}_1)$,

where $\bar{\mathfrak{s}}$ denotes the charge conjugate of $\mathfrak{s}$ and for $h = (g, \eta) \in \Pi$, we set $\bar{h} = (g, -\eta)$. Property (3) is an immediate consequence of the charge conjugation symmetry of the Seiberg–Witten equations.

Let $f \in T(X)$ be an element of the Torelli group. Fix a spin^c-structure $\mathfrak{s}$ with $d(\mathfrak{s}) = -1$. The mapping cylinder of f defines a smooth family $E \to S^1$ over S^1 with fibres diffeomorphic to X. Since f acts trivially on cohomology, it preserves the isomorphism class of $\mathfrak{s}$. It follows easily that there is a unique spin^c-structure on the vertical tangent bundle of E which restricts to $\mathfrak{s}$ on each fibre. Since $b_+(X) > 2$, there is a single chamber for the families Seiberg–Witten equations for E. Furthermore, the families moduli space is oriented and so we obtain an integer-valued invariant $\mathrm{SW}_{\mathfrak{s}}(f) \in \mathbb{Z}$ which depends only on $(X, \mathfrak{s})$ and the isotopy class of f (see [3] for more details). From the definition of $\mathrm{SW}_{\mathfrak{s}}(f)$, it is easy to see that

$$\mathrm{SW}_{\mathfrak{s}}(f) = \mathrm{SW}_{\mathfrak{s}}(h, f(h))$$

for any $h \in \Pi^{\mathrm{reg}}$. It is instructive to see why $\mathrm{SW}_{\mathfrak{s}}(f)$ is independent of the choice of $h \in \Pi^{\mathrm{reg}}$. Let $h' \in \Pi^{\mathrm{reg}}$. Then

$$\begin{aligned} \mathrm{SW}_{\mathfrak{s}}(h', f(h')) &= -\mathrm{SW}_{\mathfrak{s}}(h, h') + \mathrm{SW}_{\mathfrak{s}}(h, f(h)) + \mathrm{SW}_{\mathfrak{s}}(f(h), f(h')) \\ &= -\mathrm{SW}_{\mathfrak{s}}(h, h') + \mathrm{SW}_{\mathfrak{s}}(h, f(h)) + \mathrm{SW}_{f^{-1}(\mathfrak{s})}(h, h') \\ &= \mathrm{SW}_{\mathfrak{s}}(h, f(h)), \end{aligned}$$

where the last line follows from $f^{-1}(\mathfrak{s}) = \mathfrak{s}$, which holds since $f \in T(X)$.

For $f, g \in T(X)$, we have that $\mathrm{SW}_{\mathfrak{s}}(f \circ g) = \mathrm{SW}_{\mathfrak{s}}(f) + \mathrm{SW}_{\mathfrak{s}}(g)$ (this is a special case of Proposition 2.1, proven below). Therefore $\mathrm{SW}_{\mathfrak{s}}$ defines a homomorphism

$$\mathrm{SW}_{\mathfrak{s}} : T(X) \to \mathbb{Z}$$

or equivalently, a cohomology class $\mathrm{SW}_{\mathfrak{s}} \in H^1(T(X);\mathbb{Z})$. These cohomology classes generally do not extend to the full mapping class group $M(X)$, because $\Gamma(X)$ acts nontrivially on the set of spin^c-structures.

Recall that the compactness of the Seiberg–Witten moduli space follows from a priori bounds. These bounds depend on the pair $h \in \Pi$, but not on the spin^c-structure. This argument also works for families over a compact base space, hence for fixed $f \in T(X)$, $\mathrm{SW}_{\mathfrak{s}}(f)$ is nonzero for only finitely many $\mathfrak{s} \in S(X)$. Therefore, we can collect the homomorphisms $\mathrm{SW}_{\mathfrak{s}}$ into a single invariant

$$\mathrm{SW} : T(X) \to \bigoplus_{\mathfrak{s}\in S(X)} \mathbb{Z}, \quad f \mapsto \bigoplus_{\mathfrak{s}} \mathrm{SW}_{\mathfrak{s}}(f).$$

In what follows, we will see that SW can be extended from $T(X)$ to the full mapping class group $M(X)$ as a cohomology class valued in a certain $\Gamma(X)$-module.

Recall that each $\mathfrak{s} \in S(X)$ is determined by the corresponding characteristic element $c(\mathfrak{s}) \in L$. Therefore the group $\Gamma(X)$ acts on $\mathcal{S}(X)$ and hence on $\mathbb{Z}[\mathcal{S}(X)]$, the free abelian group with basis $\mathcal{S}(X)$. Let $\widehat{\mathbb{Z}}$ denote $\mathbb{Z}$ equipped with the action of $\Gamma(X)$ such that $f \in \Gamma(X)$ acts as multiplication by $\mathrm{sgn}_+(f)$. Let $\widehat{\mathbb{Z}}[\mathcal{S}(X)] = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{S}(X)]$. It will be convenient to regard $\widehat{\mathbb{Z}}[\mathcal{S}(X)]$ as the group of functions $\phi : \mathcal{S}(X) \to \mathbb{Z}$ with finite support. Then the action of $f \in \Gamma(X)$ is given by $(f\phi)(\mathfrak{s}) = \mathrm{sgn}_+(f)\phi(f^{-1}(\mathfrak{s}))$. We will show that the families Seiberg–Witten invariant for 1-dimensional families (where $b_+(X) > 2$) can be viewed as an element of $H^1(M(X);\widehat{\mathbb{Z}}[\mathcal{S}(X)])$.

Recall that for a group G and a G-module M, the group $H^1(G;M)$ can be viewed as the set of equivalence classes of twisted homomorphisms $G \to M$. A twisted homomorphism is a map $\phi : G \to M$ such that $\phi(gh) = \phi(g) + g\phi(h)$. A trivial twisted homomorphism is a twisted homomorphism of the form $\phi(g) = gm - m$ for some $m \in M$. Two twisted homomorphisms are considered equivalent if they differ by a trivial twisted homomorphism.

Let $h \in \Pi^{\mathrm{reg}}$. Define a map $\phi_h : \mathrm{Diff}_+(X) \to \widehat{\mathbb{Z}}[\mathcal{S}(X)]$ from the group of orientation-preserving diffeomorphisms to $\widehat{\mathbb{Z}}[\mathcal{S}(X)]$ by setting

$$(\phi_h(f))(\mathfrak{s}) = \mathrm{SW}_{\mathfrak{s}}(h, f(h)).$$

Suppose that $f_0, f_1 \in \mathrm{Diff}_+(X)$ are isotopic. Choose an isotopy f_t. Then

$$\begin{aligned}\phi_h(f_1) &= \mathrm{SW}_{\mathfrak{s}}(h, f_1(h)) \\ &= \mathrm{SW}_{\mathfrak{s}}(h, f_0(h)) + \mathrm{SW}_{\mathfrak{s}}(f_0(h), f_1(h)) \\ &= \phi_h(f_0) + \mathrm{SW}_{\mathfrak{s}}(f_0(h), f_1(h)).\end{aligned}$$

Consider the path $h_t = f_t(h)$ from $f_0(h)$ to $f_1(h)$. By diffeomorphism invariance of the Seiberg–Witten equations, the Seiberg–Witten moduli space for h_t is empty for each $t \in [0,1]$, and hence $\mathrm{SW}_{\mathfrak{s}}(f_0(h), f_1(h)) = 0$ and $\phi_h(f_1) = \phi_h(f_2)$. This shows that ϕ only depends on the underlying isotopy class and so we may view it as a map $\phi_h : M(X) \to \widehat{\mathbb{Z}}[\mathcal{S}(X)]$.

Proposition 2.1 *The map $\phi_h : M(X) \to \widehat{\mathbb{Z}}[\mathcal{S}(X)]$ is a twisted homomorphism. Furthermore, the underlying cohomology class $[\phi_h] \in H^1(M(X);\widehat{\mathbb{Z}}[\mathcal{S}(X)])$ does not depend on the choice of $h \in \Pi^{\mathrm{reg}}$.*

Proof Let $f, g \in M(X)$. Then

$$\begin{aligned}\phi_h(gf)(\mathfrak{s}) &= \mathrm{SW}_{\mathfrak{s}}(h, gf(h)) \\ &= \mathrm{SW}_{\mathfrak{s}}(h, g(h)) + \mathrm{SW}_{\mathfrak{s}}\big(g(h), g(f(h))\big) \\ &= \mathrm{SW}_{\mathfrak{s}}(h, g(h)) + \mathrm{sgn}_+(g)\,\mathrm{SW}_{g^{-1}\mathfrak{s}}(h, f(h)) \\ &= \phi_h(g)(\mathfrak{s}) + \mathrm{sgn}_+(g)\phi_h(f)(g^{-1}\mathfrak{s}) \\ &= (\phi_h(g) + (g\phi_h)(f))(\mathfrak{s}).\end{aligned}$$

Hence ϕ_h is a twisted homomorphism. Next we show that the underlying cohomology class of ϕ_h does not depend on the choice of h. Let $h' \in \Pi^{\mathrm{reg}}$ be another generic pair. Choose a path h_t from $h = h_0$ to $h' = h_1$. Then

$$\begin{aligned}\phi_{h'}(f)(\mathfrak{s}) &= \mathrm{SW}_{\mathfrak{s}}(h_1, f(h_1)) \\ &= \mathrm{SW}_{\mathfrak{s}}(h_0, f(h_0)) - \mathrm{SW}_{\mathfrak{s}}(h_0, h_1) + \mathrm{SW}_{\mathfrak{s}}(f(h_0), f(h_1)) \\ &= \phi_h(f)(\mathfrak{s}) + \mathrm{sgn}_+(f)\,\mathrm{SW}_{f^{-1}\mathfrak{s}}(h_0, h_1) - \mathrm{SW}_{\mathfrak{s}}(h_0, h_1) \\ &= \phi_h(f)(\mathfrak{s}) + (fm - m)(\mathfrak{s}),\end{aligned}$$

where $m(\mathfrak{s}) = \mathrm{SW}_{\mathfrak{s}}(h_0, h_1)$. Hence ϕ_h and $\phi_{h'}$ define the same cohomology class. □

Definition 2.2 We denote by

$$\mathrm{SW} \in H^1(M(X); \widehat{\mathbb{Z}}[\mathcal{S}(X)])$$

the cohomology class $\mathrm{SW} = [\phi_h]$ for any $h \in \Pi^{\mathrm{reg}}$.

Let $M_+(X)$ denote the subgroup of $M(X)$ consisting of all $f \in M(X)$ for which $\mathrm{sgn}_+(f) = 1$. Then $M_+(X)$ has index 1 or 2 in $M(X)$. Observe that $\widehat{\mathbb{Z}}|_{M_+(X)} = \mathbb{Z}$; thus $\mathrm{SW}|_{M_+(X)} \in H^1(M(X); \mathbb{Z}[\mathcal{S}(X)])$. From $\mathrm{SW}|_{M_+(X)}$ we can extract $\mathbb{Z}$-valued cohomology classes as follows: let $\mathcal{O} \subseteq \mathcal{S}(X)$ be a $\Gamma(X)$-invariant subset of $\mathcal{S}(X)$. Then we obtain a morphism $p_{\mathcal{O}} : \mathbb{Z}[\mathcal{S}(X)] \to \mathbb{Z}$ given by $\phi \mapsto \sum_{\mathfrak{s}\in\mathcal{O}} \phi(\mathfrak{s})$. We define $\mathrm{SW}_{\mathcal{O}} \in H^1(M_+(X); \mathbb{Z})$ by setting $\mathrm{SW}_{\mathcal{O}} = p_{\mathcal{O}}(\mathrm{SW}|_{M_+(X)})$. From this definition it follows that

$$\mathrm{SW}_{\mathcal{O}}|_{T(X)} = \sum_{\mathfrak{s}\in\mathcal{O}} \mathrm{SW}_{\mathfrak{s}}.$$

Furthermore, for any $f \in M_+(X)$, we have

$$\mathrm{SW}_{\mathcal{O}}(f) = \sum_{\mathfrak{s}\in\mathcal{O}} \mathrm{SW}_{\mathfrak{s}}(h, f(h)),$$

where $h \in \Pi^{\mathrm{reg}}$.

Remark 2.3 Ruberman [26] defined an invariant $\mathrm{SW}^{\mathrm{tot}}_{\mathfrak{s}}$ which is similar to the definition of $\mathrm{SW}_{\mathcal{O}}$ given above. Namely $\mathrm{SW}^{\mathrm{tot}}_{\mathfrak{s}} : M_+(X) \to \mathbb{Z}$ is given by $\mathrm{SW}^{\mathrm{tot}}_{\mathfrak{s}}(f) = \sum_{\mathfrak{s}'} \mathrm{SW}_{\mathfrak{s}'}(h, f(h))$ where the sum is over all spin^c-structures $\mathfrak{s}'$ such that $\mathfrak{s}' = f^m(\mathfrak{s})$ for some $m \in \mathbb{Z}$. However this invariant is not a group homomorphism and behaves in a complicated manner with respect to composition (see [26, Theorem 3.4]). For this reason, we find it more useful to work with the invariants $\mathrm{SW}_{\mathcal{O}}$.

Let $\mathfrak{s} \in \mathcal{S}(X)$ be a spin^c-structure and let $f \in M(X)$. Suppose that f preserves $\mathfrak{s}$. If $\mathrm{sgn}_+(f) = 1$, then the families Seiberg–Witten moduli space for the mapping cylinder of f with spin^c-structure $\mathfrak{s}$ is oriented, and we obtain an integer families Seiberg–Witten invariant $\mathrm{SW}_{\mathfrak{s}}(f) \in \mathbb{Z}$. It is given by $\mathrm{SW}_{\mathfrak{s}}(f) = \mathrm{SW}_{\mathfrak{s}}(h, f(h))$, for any $h \in \Pi^{\mathrm{reg}}$. If $\mathrm{sgn}_+(f) = -1$ then there is no natural choice of orientation on the families moduli space, hence we only get a mod-2 invariant $\mathrm{SW}_{\mathfrak{s}}(f) \in \mathbb{Z}_2$ which is given by $\mathrm{SW}_{\mathfrak{s}}(f) = \mathrm{SW}_{\mathfrak{s}}(h, f(h)) \pmod 2$, for any $h \in \Pi^{\mathrm{reg}}$ (the value of $\mathrm{SW}_{\mathfrak{s}}(h, f(h))$ depends on h, but its mod-2 reduction does not).

We will make use of the following special case of the gluing formula of [3]:

Proposition 2.4 *Suppose that $X = X' \# (S^2 \times S^2)$, where $b_+(X') > 1$. Let $\mathfrak{s}'$ be a spin^c-structure on X' with $d(\mathfrak{s}') = 0$ and let $\mathfrak{s}_0$ denote the spin structure on $S^2 \times S^2$. Let $\mathfrak{s} = \mathfrak{s}' \# \mathfrak{s}_0$. Let f' be a diffeomorphism on X' that preserves $\mathfrak{s}'$ and ρ a diffeomorphism of $S^2 \times S^2$. Suppose that f' is trivial in a neighbourhood of a point $x_1 \in X'$ and that ρ is trivial in a neighbourhood of a point $x_2 \in S^2 \times S^2$. Set $f = f' \# \rho$, where the connected sum is performed by removing balls around x_1 and x_2 and identifying boundaries. Then*:

(1) *If $\mathrm{sgn}_+(\rho) = 1$, then $\mathrm{SW}_{\mathfrak{s}}(f) = 0 \pmod 2$.*

(2) *If $\mathrm{sgn}_+(\rho) = -1$, then $\mathrm{SW}_{\mathfrak{s}}(f) = \mathrm{SW}(X', \mathfrak{s}') \pmod 2$, where $\mathrm{SW}(X', \mathfrak{s}')$ denotes the ordinary Seiberg–Witten invariant of $(X', \mathfrak{s}')$.*

3 Nonfinitely generated mapping class groups

In this section we prove that $M(X)$ is not finitely generated for certain 4-manifolds.

Theorem 3.1 *Let $X = 2n\mathbb{CP}^2 \# 10n\overline{\mathbb{CP}^2}$, where $n \geq 3$ is odd. Then $M(X)$ is not finitely generated. More precisely, the following holds*:

(1) *There is a surjective homomorphism $\Phi : M_+(X) \to \mathbb{Z}^\infty$ from $M_+(X)$ to $\mathbb{Z}^\infty$, where $\mathbb{Z}^\infty$ denotes a free abelian group of countably infinite rank.*

(2) *The mod-2 reduction of Φ extends to a homomorphism $\Phi : M(X) \to \mathbb{Z}_2^\infty$.*

(3) *$M_+(X)$ has index 2 in $M(X)$.*

(4) *The short exact sequence $1 \to M_+(X) \to M(X) \to \mathbb{Z}_2 \to 0$ splits.*

Proof First note that $X = X' \# (S^2 \times S^2)$, where $X' = (2n-1)\mathbb{CP}^2 \# (10n-1)\overline{\mathbb{CP}^2}$. It follows from [31] that $\Gamma(X) = \mathrm{Aut}(H^2(X; \mathbb{Z}))$. Observe that $d(\mathfrak{s}) = (c(\mathfrak{s})^2 + 8n)/4 - 2n - 1 = c(\mathfrak{s})^2/4 - 1$. Hence $d(\mathfrak{s}) = -1$ if and only if $c(\mathfrak{s})^2 = 0$. For each $k \geq 1$, let $\mathcal{O}_k \subset \mathcal{S}(X)$ denote the set of spin^c-structures whose characteristic c satisfies $c \neq 0$, $c^2 = 0$, and c is k times a primitive element. We will show that the homomorphism

$$\Phi = \bigoplus_{q=1}^{\infty} \mathrm{SW}_{\mathcal{O}_{nq-q-1}} : M_+(X) \to \mathbb{Z}^\infty$$

surjects to a subgroup of $\mathbb{Z}^\infty$ of countably infinite rank.

The decomposition $X = X' \# (S^2 \times S^2)$ yields an orthogonal decomposition $L = L' \oplus H$, where L, L' are the intersection forms of X, X' and $H = H^2(S^2 \times S^2; \mathbb{Z})$ is the hyperbolic lattice. Any characteristic $c \in L$ decomposes as $c = (c_1, c_2)$, where $c_1 \in L'$, $c_2 \in H$ are characteristic. The intersection form L' is odd, hence $c_1 \neq 0$.

We will partition $\mathcal{O}_k$ into two types of subsets:

(1) subsets $\{\mathfrak{s}, \bar{\mathfrak{s}}\}$, where $\mathfrak{s} = \mathfrak{s}' \# \mathfrak{s}_0$ and $c(\mathfrak{s}_0) = 0$,

(2) subsets $\{\mathfrak{s}_1, \mathfrak{s}_2, \bar{\mathfrak{s}}_1, \bar{\mathfrak{s}}_2\}$, where $\mathfrak{s}_1 = \mathfrak{s}' \# \mathfrak{s}''$, $\mathfrak{s}_2 = \mathfrak{s}' \# \bar{\mathfrak{s}}''$ and where $c(\mathfrak{s}'') \neq 0$.

Since $b_+(X) = 2n = 2 \pmod 4$, we have

$$\mathrm{SW}_{\mathfrak{s}}(t) = \mathrm{SW}_{\bar{\mathfrak{s}}}(t)$$

for all $t \in T(X)$. Hence a subset of type of $\mathcal{O}_k$ of type (1) will contribute $2\,\mathrm{SW}_{\mathfrak{s}}(t)$ to $\mathrm{SW}_{\mathcal{O}_k}(t)$ and a subset of type (2) will contribute $2(\mathrm{SW}_{\mathfrak{s}_1}(t) + \mathrm{SW}_{\mathfrak{s}_2}(t))$.

Let $E(n)_q$ be the elliptic surface obtained from $E(n)$ by performing a logarithmic transform of multiplicity $q \geq 1$. Since n is odd, $E(n)_q$ is not spin and its intersection form is diagonal of signature $(2n-1, 10n-1)$. Hence $E(n)_q$ has the same intersection lattice as X'. Furthermore, we have that $E(n)_q \# (S^2 \times S^2)$ is diffeomorphic to $2n\mathbb{CP}^2 \# 10n\overline{\mathbb{CP}^2} = X = X' \# (S^2 \times S^2)$ [12, page 320]. So there is an orientation-preserving diffeomorphism $\psi_q : E(n)_q \# (S^2 \times S^2) \to X' \# (S^2 \times S^2)$. We can choose ψ_q so that it respects the decomposition $H^2(E(n)_q; \mathbb{Z}) \oplus H^2(S^2 \times S^2; \mathbb{Z}) \to H^2(X'; \mathbb{Z}) \oplus H^2(S^2 \times S^2; \mathbb{Z})$. To see this, first let $\psi'_q : E(n)_q \# (S^2 \times S^2) \to X' \# (S^2 \times S^2)$ be any diffeomorphism. Then by [31], every isometry of $H^2(X' \# (S^2 \times S^2); \mathbb{Z})$ can be realised by a diffeomorphism. Hence, composing ψ'_q with a suitable diffeomorphism of X, we obtain the desired diffeomorphism ψ_q.

Let ρ be a diffeomorphism of $S^2 \times S^2$ which acts as -1 on $H^2(S^2 \times S^2; \mathbb{Z})$ and is trivial in a neighbourhood of some point. Such diffeomorphisms can easily be constructed, for example, take the product $r \times r$ of two copies of a reflection on S^2 and then isotopy it to act trivially in a neighbourhood of a point. Define a diffeomorphism $f_0 \in M(X)$ by setting $f_0 = \mathrm{id}_{X'} \# \rho$, where the connected sum is performed by removing a ball of $(S^2 \times S^2)$ on which ρ acts trivially. For each $q \geq 1$, define a diffeomorphism $f_q \in M(X)$ by setting $f_q = \psi_q \circ (\mathrm{id}_{E(n)_q} \# \rho) \circ \psi_q^{-1}$. Note that $\mathrm{sgn}_+(f_0) = \mathrm{sgn}_+(f_q) = -1$. Also $t_q = f_q \circ f_0 \in T(X)$.

We claim that

$$\mathrm{SW}_{\mathcal{O}_{nq-q-1}}(t_q) = 2 \pmod 4 \quad \text{and} \quad \mathrm{SW}_{\mathcal{O}_{nq'-q'-1}}(t_q) = 0 \pmod 4$$

for all $q' > q$. This implies that the elements $\{\Phi(t_q)\}_{q \geq 1}$ are linearly independent. To see this, first note that $\Phi(t_q) \in 2\mathbb{Z}^\infty$ by charge conjugation symmetry and that the image of $\{\Phi(t_q)/2\}_{q \geq 1}$ is a basis for $\mathbb{Z}_2^\infty$, by the above claim. Now suppose $n_1\Phi(t_1) + n_2\Phi(t_2) + \cdots + n_r\Phi(t_r) = 0$ for some $n_1, \ldots, n_r \in \mathbb{Z}$, not all zero. Without loss of generality we can assume that $\gcd(n_1, \ldots, n_r) = 1$. Then $n_1\Phi(t_1)/2 + \cdots + n_r\Phi(t_r)/2 = 0$. But $\{\Phi(t_q)/2\}_{q \geq 1}$ are linearly independent in $\mathbb{Z}_2^\infty$, so $n_1, \ldots, n_r$ are all even, a contradiction.

Now we prove the claim. Let $q, q' \geq 1$ and set $k = nq' - q' - 1$. By partitioning $\mathcal{O}_k$ into subsets of type (1) and (2) as described above, we can then write $\mathrm{SW}_{\mathcal{O}_k}(t_q)$ as a sum of contributions from sets of type (1) and (2). Consider a contribution from a subset $\{\mathfrak{s}, \bar{\mathfrak{s}}\}$ of type (1). So $\mathfrak{s} = \mathfrak{s}' \# \mathfrak{s}_0$. The contribution is $2\,\mathrm{SW}_{\mathfrak{s}}(t_q)$. Since f_q and f_0 both preserve $\mathfrak{s}$, we have that

$$\begin{aligned}\mathrm{SW}_{\mathfrak{s}}(t_q) &= \mathrm{SW}_{\mathfrak{s}}(f_q \circ f_0)\\ &= \mathrm{SW}_{\mathfrak{s}}(f_q) + \mathrm{SW}_{\mathfrak{s}}(f_0) \pmod 2\\ &= \mathrm{SW}(E(n)_q, \mathfrak{s}') \pmod 2,\end{aligned}$$

where the last equality is due to Proposition 2.4. Let $f \in H^2(E(n)_q; \mathbb{Z})$ denote the class of the multiple fibre. Then f is nonzero, primitive and $f^2 = 0$. From the well-known calculation of the Seiberg–Witten invariants for elliptic surfaces [21, Chapter 3], we have that $\mathrm{SW}(E(n)_q, \mathfrak{s}') = 0$ unless $c(\mathfrak{s}')$ is a multiple of f. More precisely, $\mathrm{SW}(E(n)_q, \mathfrak{s}')$ is zero unless $c(\mathfrak{s}') = 2(qk + a)f - (nq - q - 1)f$, where $0 \leq k \leq n - 2$ and $0 \leq a < q$. In such a case $\mathrm{SW}(E(n)_q, \mathfrak{s}') = (-1)^k \binom{n-2}{k}$. In particular, $\mathrm{SW}(E(n)_q, \mathfrak{s}') = \pm 1$ if $c(\mathfrak{s}') = (nq - q - 1)f$ and $\mathrm{SW}(E(n)_q, \mathfrak{s}') = 0$ if $c(\mathfrak{s}) = uf$, $u > nq - q - 1$. Now suppose that $q' \geq q$. We have that $\mathrm{SW}(E(n)_q, \mathfrak{s}') = 0$ unless $c(\mathfrak{s})$ is a multiple of f. But since $\mathfrak{s} = \mathfrak{s}' \# \mathfrak{s}_0 \in \mathcal{O}_k$, this can only happen if $c(\mathfrak{s}') = \pm kf$ (recall that $\mathcal{O}_k$ is the set of spin^c-structures whose characteristic c satisfies $c \neq 0$, $c^2 = 0$, and c is k times a primitive element). Hence if $q' > q$, then every pair $\{\mathfrak{s}, \bar{\mathfrak{s}}\}$ of type (1) contributes 0 mod 4 to $\mathrm{SW}_{\mathcal{O}_k}(t_q)$. If $q' = q$, then there is exactly one pair $\{\mathfrak{s}, \bar{\mathfrak{s}}\}$ of type (1) which contributes 2 mod 4 to $\mathrm{SW}_{\mathcal{O}_k}(t_q)$ and all other pairs are 0 mod 4.

Now consider the contribution from a set $\{\mathfrak{s}_1, \mathfrak{s}_2, \bar{\mathfrak{s}}_1, \bar{\mathfrak{s}}_2\}$ of type (2), where $\mathfrak{s}_1 = \mathfrak{s}' \# \mathfrak{s}''$, $\mathfrak{s}_2 = \mathfrak{s}' \# \bar{\mathfrak{s}}''$ and where $c(\mathfrak{s}'') \neq 0$. As seen above, the contribution is $2(\mathrm{SW}_{\mathfrak{s}_1}(t_q) + \mathrm{SW}_{\mathfrak{s}_2}(t_q))$. We will show that $\mathrm{SW}_{\mathfrak{s}_1}(t_q) + \mathrm{SW}_{\mathfrak{s}_2}(t_q) = 0 \pmod 2$, hence all subsets of type (2) contribute 0 mod 4 and this will prove the claim.

Observe that $\mathfrak{s}_2 = f_q(\mathfrak{s}_1) = f_0(\mathfrak{s}_1)$. Let $h \in \Pi^{\mathrm{reg}}$. Then

$$\begin{aligned}\mathrm{SW}_{\mathfrak{s}_1}(t_q) &= \mathrm{SW}_{\mathfrak{s}_1}(h, t_q(h))\\ &= \mathrm{SW}_{\mathfrak{s}_1}(h, f_q f_0(h))\\ &= \mathrm{SW}_{\mathfrak{s}_1}(h, f_q(h)) + \mathrm{SW}_{\mathfrak{s}_1}(f_q(h), f_q f_0(h))\\ &= \mathrm{SW}_{\mathfrak{s}_1}(h, f_q(h)) - \mathrm{SW}_{\mathfrak{s}_2}(h, f_0(h)).\end{aligned}$$

Similarly, $\mathrm{SW}_{\mathfrak{s}_2}(t_q) = \mathrm{SW}_{\mathfrak{s}_2}(h, f_q(h)) - \mathrm{SW}_{\mathfrak{s}_1}(h, f_0(h))$. Hence

$$\begin{aligned}&\mathrm{SW}_{\mathfrak{s}_1}(t_q) + \mathrm{SW}_{\mathfrak{s}_2}(t_q)\\ &\qquad = \big(\mathrm{SW}_{\mathfrak{s}_1}(h, f_q(h)) + \mathrm{SW}_{\mathfrak{s}_2}(h, f_q(h))\big) - \big(\mathrm{SW}_{\mathfrak{s}_1}(h, f_0(h)) + \mathrm{SW}_{\mathfrak{s}_2}(h, f_0(h))\big)\\ &\qquad = \big(\mathrm{SW}_{\mathfrak{s}_1}(h, f_q(h)) - \mathrm{SW}_{\mathfrak{s}_1}(f_q(h), f_q^2(h))\big) - \big(\mathrm{SW}_{\mathfrak{s}_1}(h, f_0(h)) - \mathrm{SW}_{\mathfrak{s}_1}(f_0(h), f_0^2(h))\big)\\ &\qquad = \mathrm{SW}_{\mathfrak{s}_1}(h, f_q^2(h)) - \mathrm{SW}_{\mathfrak{s}_1}(h, f_0^2(h))\\ &\qquad = \mathrm{SW}_{\mathfrak{s}_1}(f_q^2) - \mathrm{SW}_{\mathfrak{s}_1}(f_0^2)\\ &\qquad = 0 \pmod 2,\end{aligned}$$

where the last line follows from Proposition 2.4. This proves the claim. Hence we have proven that there exists a surjective homomorphism $M_+(X) \to \mathbb{Z}^\infty$.

The fact the mod-2 reduction of Φ extends to $M(X)$ follows by noting that $\frac{1}{2}\,\mathrm{SW}_{\mathcal{O}_k} \in H^1(M_+(X);\mathbb{Z})$ extends to $H^1(M(X);\widehat{\mathbb{Z}})$ and then applying the mod-2 reduction map $\widehat{\mathbb{Z}} \to \mathbb{Z}_2$.

The fact that $M_+(X)$ has index 2 in $M(X)$ follows immediately from $\mathrm{sgn}_+(f_0) = -1$. Furthermore, it is easy to see that f_0^2 is isotopic to a Dehn twist on the neck of the connected sum $X' \,\#\, (S^2 \times S^2)$. By using the circle action on $S^2 \times S^2$, it follows that this is isotopic to the identity. So f_0 defines a splitting $\mathbb{Z}_2 \to M(X)$ of the sequence $1 \to M_+(X) \to M(X) \to \mathbb{Z}_2 \to 0$. □

4 Split extensions

Let $L = \mathbb{Z}^n$ denote the standard diagonal lattice of rank n with orthonormal basis $e_1, \dots, e_n$. The isometry group of L is the hyperoctahedral group H_n, which is also the Coxeter group of type BC_n. This group is easily seen to be generated by permutations of $e_1, \dots, e_n$ and the reflections $r_1, \dots, r_n$ in the hyperplanes orthogonal to $e_1, \dots, e_n$. The reflections generate a normal subgroup isomorphic to $\mathbb{Z}_2^n$ and H_n is the semidirect product $H_n = S_n \ltimes \mathbb{Z}_2^n$.

Let $X = n\mathbb{CP}^2$ be the connected sum of n copies of $\mathbb{CP}^2$. Then $H^2(X;\mathbb{Z})$ is isomorphic to L with $e_1, \dots, e_n$ corresponding to generators of $H^2(\mathbb{CP}^2;\mathbb{Z})$ for the n summands of X. It is not hard to see that $\Gamma(X)$ is equal to the full isometry group of L. This will also follow from the construction given below.

Theorem 4.1 *For $X = n\mathbb{CP}^2$, there is a splitting $\Gamma(X) \to M(X)$.*

Proof We will construct a smooth fibre bundle $\pi : E \to B$ with fibres diffeomorphic to X and such that the monodromy of the local system $R^2\pi_*\mathbb{Z}$ yields an isomorphism $\rho : \pi_1(B) \to \mathrm{Aut}(L)$. The geometric monodromy of the family defines a lift $\widetilde{\rho} : \pi_1(B) \to \pi_0(\mathrm{Diff}(X)) = M(X)$ of ρ to $M(X)$. Then $\widetilde{\rho} \circ \rho^{-1} : \mathrm{Aut}(L) \to M(X)$ is the desired splitting (this also proves that $\Gamma(X) \cong \mathrm{Aut}(L)$).

Let C_m be the space of m-tuples of distinct points on S^4. Clearly C_1 is diffeomorphic to S^4. For $m > 1$ there is a natural map $C_m \to C_{m-1}$ given by forgetting the m-th point. This map gives C_m the structure of a fibre bundle over C_{m-1} with fibre F_{m-1} the 4-sphere with $m-1$ points removed. Since $\pi_1(F_{m-1}) = \pi_0(F_{m-1}) = 1$, the long exact sequence in homotopy yields an isomorphism $\pi_1(C_m) \cong \pi_1(C_m)$. Since $\pi_1(C_1) = \pi_1(S^4) = 1$, it follows by induction that $\pi_1(C_n) = 1$ for all n.

Fix an orientation on S^4. Let $\widetilde{C}_n$ denote the space consisting of an n-tuple $(x_1, \dots, x_n)$ of distinct points of S^4 together with an n-tuple $(I_1, \dots, I_n)$, where I_j is a complex structure on $T_{x_j}S^4$ which induces the given orientation. The forgetful map $\widetilde{C}_n \to C_n$ which forgets the complex structures $I_1, \dots, I_n$ gives $\widetilde{C}_n$ the structure of a fibre bundle over C_n. Since the space of complex structures on $\mathbb{R}^4$ compatible with a given orientation is isomorphic to $\mathrm{SO}(4)/U(2) \cong S^2$, it follows that the fibres of $\widetilde{C}_n \to C_n$ are isomorphic to $(S^2)^n$. The long exact sequence in homotopy implies that $\pi_1(\widetilde{C}_n) = 1$.

Consider the trivial family $\widetilde{E}_0 = \widetilde{C}_n \times S^4 \to \widetilde{C}_n$. This family is equipped with n sections $s_1, \dots, s_n$, where $s_j((x_1, \dots, x_n), (I_1, \dots, I_n)) = x_j$. The normal bundle of s_j is $N_j = T_{x_j}S^4$. The complex structure I_j gives N_j the structure of a complex rank 2 vector bundle. Therefore, we can form a family $\widetilde{E}_n$

by blowing up $\widetilde{E}_0$ along the sections $s_1, \dots, s_n$. More precisely, consider the fibre bundle over $\widetilde{C}_n$ with fibre $\mathbb{CP}^2$ given by the projective bundle $\mathbb{P}(\mathbb{C} \oplus N_j)$. This bundle has a natural section t_j corresponding to the 1-dimensional subbundle $\mathbb{C} \subset \mathbb{C} \oplus N_j$. The normal bundle of t_j is isomorphic to N_j. Since s_j and t_j have isomorphic normal bundles, we can attach $\mathbb{P}(\mathbb{C} \oplus N_j)$ to $\widetilde{E}_0$ by removing tubular neighbourhoods of s_j and t_j and identifying the boundaries.

The hyperoctahedral group $H_n = S_n \ltimes \mathbb{Z}_2^n$ acts on $\widetilde{C}_n$ as follows. The permutation group S_n acts by permuting the points $x_1, \dots, x_n$ as well as the corresponding complex structures $I_1, \dots, I_n$. The group $\mathbb{Z}_2^n$ is generated by reflections $r_1, \dots, r_n$. We let r_j act by fixing $x_1, \dots, x_n$, sending I_j to $-I_j$ and fixing the remaining complex structures. The action of H_n is free and we let $B = \widetilde{C}_n / H_n$ be the quotient. It follows that $\pi_1(B) \cong H_n$. The action of H_n on $\widetilde{C}_n$ lifts to an action on $\widetilde{E}_0 = \widetilde{C}_n \times S^4$ which acts trivially on the S^4 factor. It is not hard to see that $\widetilde{E}_n$ can be constructed in such a way that the action of H_n extends to it. Now let $E = \widetilde{E}_n / H_n$. This is a family $\pi : E \to B$ over B with fibres diffeomorphic to $n\mathbb{CP}^2$. The monodromy of the local system $R^2\pi_*\mathbb{Z}$ is easily seen to yield an isomorphism $\rho : \pi_1(B) \to \mathrm{Aut}(L)$. As explained above, this yields a splitting $\Gamma(X) \to M(X)$. □

5 Nonsplit extensions

Let X be a compact, simply connected, smooth 4-manifold with intersection form $L = H^2(X; \mathbb{Z})$. Let Σ be a compact surface (orientable or nonorientable). Suppose that $\rho : \pi_1(\Sigma) \to \Gamma(X)$ is a homomorphism. Letting $\Gamma(X)$ act on $L_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} L$, we obtain a flat vector bundle $H_\rho \to \Sigma$ which has a covariantly constant bilinear form of signature $(b_+(X), b_-(X))$. Let H_ρ^+ denote a maximal positive definite subbundle of H_ρ. The choice of subbundle H_ρ^+ is not unique, but all such subbundles are isomorphic. In particular the Stiefel–Whitney classes $w_j(H_\rho^+) \in H^j(\Sigma; \mathbb{Z}_2)$ depend only on ρ.

Theorem 5.1 *Let X be a compact, simply connected, smooth 4-manifold with $b_+(X) = 2$ and let $\rho : \pi_1(\Sigma) \to \Gamma(X)$ be a homomorphism. Suppose that $w_2(H_\rho^+) \neq 0$ and suppose there exists a characteristic $c \in L$ which is ρ-invariant and satisfies $c^2 > \sigma(X)$. Then ρ does not lift to a homomorphism $\widetilde{\rho} : \Gamma(X) \to M(X)$.*

Proof Consider first the case that Σ is orientable of genus g. Recall that $\pi_1(\Sigma)$ admits a presentation

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

Suppose that ρ admits a lift $\widetilde{\rho} : \pi_1(\Sigma) \to M(X)$. Let α_j be a diffeomorphism of X whose isotopy class is $\widetilde{\rho}(a_j)$ and let β_j be a diffeomorphism of X whose isotopy class is $\widetilde{\rho}(b_j)$. Then $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$ is isotopic to the identity. The surface Σ can be constructed from a wedge of $2g$ circles by attaching a 2-cell whose attaching map represents $[a_1, b_1] \cdots [a_g, b_g]$ in $\pi_1\big(\bigvee_{i=1}^{2g} S^1\big)$. We will construct a smooth family $\pi : E \to \Sigma$ whose fibres are diffeomorphic to X as follows. Over the 1-skeleton $\bigvee_{i=1}^{2g} S^1$, we take the wedge sum of mapping cylinders associated to the diffeomorphisms $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$. A choice of isotopy from $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$ to the identity allows us to extend this family over the 2-cell and in this way we obtain the family $\pi : E \to \Sigma$. By construction, the local system $R^2\pi_*\mathbb{Z}$ has monodromy ρ. Now suppose that $w_2(H^+_\rho) \neq 0$ and that there exists a characteristic $c \in L$ which is ρ-invariant and satisfies $c^2 > \sigma(X)$. This contradicts [2, Theorem 1.1], hence ρ does not lift to $M(X)$.

The case that Σ is nonorientable is similar. Recall that $\pi_1(\Sigma)$ admits a presentation

$$\pi_1(\Sigma) = \langle a_1, \dots, a_k \mid a_1^2 \cdots a_k^2 \rangle,$$

where Σ has Euler characteristic $1-k$. If ρ lifts to a homomorphism $\widetilde{\rho} : \pi_1(\Sigma) \to M(X)$, then we choose diffeomorphisms $\alpha_1, \dots, \alpha_k$ where the isotopy class of α_j is $\widetilde{\rho}(a_j)$. Then $\alpha_1^2 \cdots \alpha_k^2$ is isotopic to the identity. A choice of such an isotopy allows us to construct a smooth family $\pi : E \to \Sigma$ with fibres diffeomorphic to X and such that the monodromy of $R^2\pi_*\mathbb{Z}$ is ρ. As before, this contradicts [2, Theorem 1.1], hence ρ does not lift to $M(X)$. □

Remark 5.2 A similar argument was used in [16] to prove the nontriviality of the group $T(X)$ for the manifold $X = 2\mathbb{CP} \# n\overline{\mathbb{CP}^2}$, $n \geq 11$.

Corollary 5.3 *Let $X = (S^2 \times S^2) \# X'$, where $b_+(X') = 1$, $b_-(X') \geq 10$. Then there does not exist a splitting $\Gamma(X) \to M(X)$.*

Proof Let $L = H^2(X;\mathbb{Z})$ and $L' = H^2(X';\mathbb{Z})$ denote the intersection lattices of X and X' and let $H = H^2(S^2 \times S^2\ \mathbb{Z})$. So $L \cong H \oplus L'$. Since $X = (S^2 \times S^2) \# X'$, we have that $\Gamma(X) = \mathrm{Aut}(H^2(X;\mathbb{Z}))$ by [31]. Let $x, y \in H$ be a basis with $x^2 = y^2 = 0$, $\langle x, y\rangle = 1$. Since $b_+(X') = 1$ and $\sigma(X') < 0$, it follows that X' is not spin and it follows that $L' \cong H' \oplus E_8 \oplus L''$, where H' has basis x', y', $(x')^2 = (y')^2 = 0$, $\langle x', y'\rangle = 1$, E_8 is the negative definite E_8 lattice and L'' is a diagonal lattice with basis $e_1, \dots, e_m$, where $m = b_-(X') - 9$, with $e_i^2 = -1$ for all i, $\langle e_i, e_j\rangle = 0$ for $i \neq j$.

Let $u = x+y$, $v = x'+y'$. Then $u^2 = v^2 = 2$, $\langle u, v\rangle = 0$. Let r_u be the reflection $r_u(x) = x + \langle x, u\rangle u$ and define r_v similarly. Consider the isometry $f(x) = r_u r_v(x)$. Then $f \in \Gamma(X)$ and $f^2 = 1$. Hence we obtain a homomorphism $\rho : \pi_1(\mathbb{RP}^2) \to \Gamma(X)$ which sends the generator of $\pi_1(\mathbb{RP}^2)$ to f. Since f acts as -1 on the maximal positive definite subspace of $H^2(X;\mathbb{R})$ spanned by u and v, we have that $w_2(H^+_\rho) \neq 0$. Let $c = e_1 + \dots + e_m$. Then c is a characteristic that $c^2 > \sigma(X)$ and $\langle c, u\rangle = \langle c, v\rangle = 0$. Then $r_u(c) = r_v(c) = c$ and hence $f(c) = c$. Then Theorem 5.1 implies that ρ does not lift to $M(X)$. Hence the subgroup $\langle f\rangle \subseteq \Gamma(X)$ does not lift to $M(X)$, in particular, there does not exist a splitting $\Gamma(X) \to M(X)$. □

Remark 5.4 In the above proof $u \in L$ can be realised by an embedded 2-sphere in X, namely the diagonal $S^2 \subset S^2 \times S^2$. By a result of Seidel [27], it follows that r_u can be lifted to an element $\hat{r}_u \in M(X)$ of order 2. Since $\Gamma(X) = \mathrm{Aut}(L)$, it follows that there is a diffeomorphism of X sending u to v. It follows that v can also be realised by an embedded 2-sphere and hence r_v can be lifted to an element $\hat{r}_v \in M(X)$ of order 2. Then $\hat{r}_u\hat{r}_v$ is a lift of f to $M(X)$. If u, v could be represented by *disjoint* embedded 2-spheres, then $\hat{r}_u$, $\hat{r}_v$ commute (since $\hat{r}_u$, $\hat{r}_v$ can be constructed to have disjoint supports) and then $\hat{r}_u\hat{r}_v$ would be an involutive lift of f, contradicting the corollary above. We deduce that u, v can be represented by embedded spheres, but they cannot be represented by disjoint embedded spheres even though $\langle u, v\rangle = 0$.

6 Nielsen realisation

As explained in the introduction, the following result shows that the Nielsen realisation problem fails for $X = X' \# p\mathbb{CP}^2 \# q\overline{\mathbb{CP}^2}$ whenever $p+q \geq 4$.

Theorem 6.1 *Let $X = X' \# p\mathbb{CP}^2 \# q\overline{\mathbb{CP}^2}$ where X' is a compact, smooth, simply connected 4-manifold and $p+q \geq 4$. Then $M(X)$ contains a subgroup isomorphic to $\mathbb{Z}_2^4$ which cannot be lifted to* $\mathrm{Diff}(X)$.

Proof To each summand of $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ in X, there is a corresponding embedded 2-sphere of self-intersection ± 1. Let $E_1, \dots, E_4$ be any four of them. Let $t_1, \dots, t_4 \in M(X)$ be the corresponding Dehn twists around these spheres. Then $t_1, \dots, t_4$ are involutions [27] and they commute since $E_1, \dots, E_4$ are disjoint. Hence the group $G \subseteq M(X)$ generated by $t_1, \dots, t_4$ is isomorphic to $\mathbb{Z}_2^4$. Now suppose that G can be lifted to $\mathrm{Diff}(X)$. Hence we can find commuting diffeomorphisms $\sigma_1, \dots, \sigma_4$ such that the isotopy class of σ_i is t_i.

Consider the fixed point set F of σ_1. Since σ_1 acts on $H^2(X;\mathbb{Z})$ as a reflection in a ± 1 sphere, it follows from [8, Proposition 2.4] that F consists of a single copy of $\mathbb{RP}^2$, together with some isolated points and some 2-spheres. Let G_0 be the subgroup of G generated by σ_2, σ_3, σ_4. Since σ_2, σ_3, σ_4 commute with σ_1, they act on F and in particular must send the copy of $\mathbb{RP}^2$ to itself. Hence G_0 acts on $\mathbb{RP}^2$. We claim that the action is effective. To see this, suppose $f \in G_0$ fixes $\mathbb{RP}^2$ pointwise. Since f is an orientation-preserving involution, it must act on the normal bundle of $\mathbb{RP}^2$ in X as either the identity or multiplication by -1. Hence either f or $\sigma_1 f$ fixes $\mathbb{RP}^2$ pointwise and acts trivially on the normal bundle. For a diffeomorphism of finite order, this can only happen if the diffeomorphism is the identity. Hence f or $\sigma_1 f$ is the identity, but $f \in G_0$, so $f \neq \sigma_1$ and it must be that f is the identity.

A finite group action on $\mathbb{RP}^2$ by diffeomorphisms is conjugate to a subgroup of $\mathrm{PO}(3) \cong \mathrm{SO}(3)$. Since G_0 is abelian, its action on the standard representation of $\mathrm{SO}(3)$ can be simultaneously diagonalised, so G_0 is isomorphic to a subgroup of $\{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3) \in \mathrm{SO}(3)\} \cong \mathbb{Z}_2^2$, which is impossible since $|G_0| = 8$. So G does not lift to $\mathrm{Diff}(X)$. □

7 Boundary Dehn twists

Let $X^{(n)}$ be obtained from X by removing n disjoint open balls. So $X^{(n)}$ is a compact 4-manifold with boundary consisting of n copies of S^3. Let $\mathrm{Diff}(X^{(n)}, \partial X^{(n)})$ denote the group of diffeomorphisms of $X^{(n)}$ which are the identity in a neighbourhood of the boundary. Let $M_n(X) = \pi_0(\mathrm{Diff}(X^{(n)}, \partial X^{(n)})$ denote the group of components of $\mathrm{Diff}(X^{(n)}, \partial X^{(n)})$. It is known that the map $M_n(X) \to M(X)$ is surjective and that the kernel is generated by Dehn twists on the boundary components [11]. More precisely, if $S^3 \to X^{(n)}$ is a boundary component, then $X^{(n)}$ has a tubular neighbourhood $[0,1] \times S^3 \to X$. The Dehn twist on this boundary component is defined by taking a nontrivial loop $\alpha_t : [0,1] \to \mathrm{SO}(4)$ and defining $\phi : [0,1] \times S^3 \to [0,1] \times S^3$ by $\phi(t,x) = (t, \alpha_t(x))$, where $\mathrm{SO}(4)$ acts on S^3 in the standard way. We assume that α_t is smooth and equals the identity in a neighbourhood of $\{0,1\}$, hence ϕ can be extended to an element of $\mathrm{Diff}(X^{(n)}, \partial X^{(n)})$ by taking it to be the identity outside of the tubular neighbourhood.

Let $K_n(X)$ denote the kernel of $M_n(X) \to M(X)$, so we have an short exact sequence

$$1 \to K_n(X) \to M_n(X) \to M(X) \to 1.$$

Furthermore, we have a surjection $\mathbb{Z}_2^n \to K_n(X)$ given by Dehn twists on the boundary components [11, Proposition 3.1].

Proposition 7.1 *Let X be a compact, smooth, simply connected* 4*-manifold.*

(1) *If X is spin, then $K_n(X)$ is either $\mathbb{Z}_2^n$ or $\mathbb{Z}_2^n/\Delta\mathbb{Z}_2$, for all n, where $\Delta\mathbb{Z}_2$ is the diagonal copy of $\mathbb{Z}_2$.*

(2) *If X is not spin, then $K_n(X) = 0$ for all n, hence $M_n(X) \cong M(X)$.*

Proof Part (1) is given by [11, Corollary 2.5] and part (2) by [22, Corollary A.5]. □

In light of Proposition 7.1, boundary Dehn twists are only interesting when X is spin. In this case, we either have $K_n(X) \cong \mathbb{Z}_2^n$ or $K_n(X) \cong \mathbb{Z}_2^n/\Delta\mathbb{Z}_2$. Which of these two cases occurs is completely determined by the $n = 1$ case. We consider this case in more detail. There is a Serre fibration

$$\mathrm{Diff}(X^{(1)}, \partial X^{(1)}) \to \mathrm{Diff}(X) \to \mathrm{Emb}(D^4, X), \tag{7-1}$$

where $\mathrm{Emb}(D^4, X)$ is the space of embeddings of a disc in X which can be extended to a diffeomorphism. Furthermore, there is a homotopy equivalence $\mathrm{Emb}(D^4, X) \cong F(X)$, where $F(X)$ is the oriented frame bundle of X [11]. Since X is simply connected and spin, $\pi_1(F(X)) \cong \mathbb{Z}_2$. Then the fibration (7-1) induces an exact sequence

$$\pi_1(\mathrm{Diff}(X)) \xrightarrow{\phi} \mathbb{Z}_2 \to M_1(X) \to M(X) \to 1.$$

In the absence of a metric we can define the spin bundle of X to be the universal cover $\widetilde{F}(X) \to F(X)$ of $F(X)$. Since $\pi_1(F(X)) \cong \mathbb{Z}_2$, we have $\widetilde{F}(X) \to F(X)$ is a double cover. Since $\mathrm{Emb}(D^4, X) \cong F(X)$, it follows that ϕ is the map that measures whether or not a loop of diffeomorphisms of X lifts to a loop in the spin bundle of X. This leads to an alternative description of the group $M_1(X)$ when X is spin. Let $\mathrm{SpinDiff}(X)$ be the group whose elements consist of a diffeomorphism $f \in \mathrm{Diff}(X)$ and a choice of lift of $f_* : F(X) \to F(X)$ to $\widetilde{F}(X)$. We have a short exact sequence $1 \to \mathbb{Z}_2 \to \mathrm{SpinDiff}(X) \to \mathrm{Diff}(X) \to 1$ and the connecting homomorphism $\pi_1(\mathrm{Diff}(X)) \to \mathbb{Z}_2$ is precisely ϕ. The map $\mathrm{Diff}(X^{(1)}, \partial X^{(1)}) \to \mathrm{Diff}(X)$ admits a lift $\mathrm{Diff}(X^{(1)}, \partial X^{(1)}) \to \mathrm{SpinDiff}(X)$ by taking the unique lift which is the identity over $\partial X^{(1)}$. We then have a commutative diagram

$$\begin{array}{ccccccc}
\pi_1(\mathrm{Diff}(X)) & \xrightarrow{\phi} & \mathbb{Z}_2 & \longrightarrow & M_1(X) & \longrightarrow & M(X) \\
& {\scriptstyle\phi}\searrow & \downarrow & & \downarrow & \nearrow & \\
& & \mathbb{Z}_2 & \longrightarrow & \pi_0(\mathrm{SpinDiff}(X)) & &
\end{array}$$

from which it follows that $M_1(X) \to \pi_0(\mathrm{SpinDiff}(X))$ is an isomorphism. If ϕ is nontrivial, then $K_1(X) = 0$ and $M_1(X) \to M(X)$ is an isomorphism. This happens for $S^2 \times S^2$, as seen by taking a loop of diffeomorphisms given by a circle action which rotates one of the spheres. Similarly, ϕ is nontrivial for $X = S^4$ or for a connected sum of copies of $S^2 \times S^2$. If ϕ is trivial, then $K_1(X) \cong \mathbb{Z}_2$ and $M_1(X) \to M(X)$ is an extension of $M(X)$ by $\mathbb{Z}_2$, hence corresponds to a class $\xi_X \in H^2(M(X); \mathbb{Z}_2)$. It is natural to ask what this class is and in particular, whether or not it is trivial. First, we need some examples of spin 4-manifolds where $\phi = 0$.

Theorem 7.2 *Let X be a compact, smooth, simply connected 4-manifold. If X is homeomorphic to $K3$ then $\phi = 0$. Similarly, if $X = X' \# (S^2 \times S^2)$, where X' is homeomorphic to $K3$, then $\phi = 0$.*

Proof In [4], it is proven that if $E \to S^2$ is a smooth family of $K3$ surfaces over S^2, then $w_2(TE) = 0$. As explained in [18], this implies that the homomorphism ϕ is zero. The same argument works for any X that is homeomorphic to $K3$, since by [20], the Seiberg–Witten invariant of the spin structure of X is odd.

Next, suppose $X = X' \# (S^2 \times S^2)$, where X' is homeomorphic to $K3$. Suppose that ϕ is nonzero. This means that the boundary Dehn twist $\tau \in M_1(X)$ is trivial. But this would imply that the Dehn twist on the neck of $K3 \# X'$ becomes trivial upon connected sum with $S^2 \times S^2$. However this contradicts [19] (in [19] the theorem is stated only for $X' = K3$, but the exact same proof works for any smooth 4-manifold homeomorphic to $K3$). □

Recall that an involution f on a simply connected spin 4-manifold X is called even or odd according to whether or not f lifts to an involution on the spin bundle of X.

Proposition 7.3 *Suppose X is spin and $\phi = 0$, so that the extension class $\xi_X \in H^2(M(X); \mathbb{Z}_2)$ is defined. Suppose that f is an odd involution. Then $\xi_X(f) \neq 0$. In particular, the extension $\mathbb{Z}_2 \to M_1(X) \to M(X)$ is nontrivial.*

Proof As explained above, the extension $1 \to \mathbb{Z}_2 \to M_1(X) \to M(X) \to 1$ is isomorphic to the extension $1 \to \mathbb{Z}_2 \to \pi_0(\mathrm{SpinDiff}(X)) \to M(X) \to 1$. But f defines a class $[f] \in M(X)$ such that $[f]^2 = 1$. But any lift of f to the spin bundle is not an involution. So there is no splitting $M(X) \to M_1(X)$ and more precisely, $\xi_X(f) \neq 0$. □

Corollary 7.4 *If $X = K3$ or $K3 \# (S^2 \times S^2)$, then $\xi_X \in H^2(M(X); \mathbb{Z}_2)$ is nontrivial.*

Proof This is immediate from Theorem 7.2 and Proposition 7.3, since both $K3$ and $K3 \# (S^2 \times S^2)$ admit odd involutions. □

In what follows we will completely determine the class $\xi_X \in H^2(M(X); \mathbb{Z}_2)$ when X is homeomorphic to $K3$.

Proposition 7.5 *Let $\pi : E \to B$ be a smooth fibre bundle, where B is a compact surface and the fibres of E are diffeomorphic to a compact, simply connected, smooth spin 4-manifold X. Then:*

1. *There exists a spin^c-structure $\mathfrak{s}_{E/B}$ on the vertical tangent bundle $TE/B = \mathrm{Ker}(\pi_*)$ whose restriction to each fibre is spin.*
2. *Let $\mathrm{ind}(D) \in K^0(B)$ denote the families index of the Dirac operator D with respect to the spin^c-structure $\mathfrak{s}_{E/B}$. Then*

$$c_1(\mathrm{ind}(D)) = \tfrac{1}{16}\sigma(X) w_2(TE/B) \pmod 2.$$

Proof (1) follows from [2, Proposition 2.1]. The Dirac operator D for the spin^c-structure $\mathfrak{s}_{E/B}$ defines a family of elliptic operators parametrised by B and $\mathrm{ind}(D)$ is the families index. Then $c_1(\mathrm{ind}(D))$ equals $c_1(\mathcal{L})$, where $\mathcal{L} = \det(\mathrm{ind}(D))$ is the determinant line bundle of D. Suppose that the family E is determined by transition function ψ_{ij} valued in $\mathrm{Diff}(X)$. Let $\widetilde{\psi}_{ij}$ be lifts of ψ_{ij} to $\mathrm{SpinDiff}(X)$. Then $\widetilde{\psi}_{ij}\widetilde{\psi}_{jk}\widetilde{\psi}_{ki} = g_{ijk}$, where g_{ijk} is a $\mathbb{Z}_2$-valued cocycle, defining a class $[g_{ijk}] \in H^2(B;\mathbb{Z}_2)$. Clearly $w_2(TE/B) = [g_{ijk}]$. Observe that $c(\mathfrak{s}_{E/B}) \in H^2(B;\mathbb{Z})$ is a lift of $[g_{ijk}]$ to integer coefficients. Therefore we can represent $c(\mathfrak{s}_{E/B})$ as an integer-valued 2-cocycle c_{ijk} such that $c_{ijk} = g_{ijk} \pmod 2$. Choose real-valued smooth functions u_{ij} such that $c_{ijk} = u_{ij} + u_{jk} + u_{ki}$. Set $f_{ij} = e^{2\pi i u_{ij}}$. Then f_{ij} define transition functions for a complex line bundle whose first Chern class is $[c_{ijk}]$. Note that $f_{ij} = h_{ij}^2$, where $h_{ij} = e^{\pi i u_{ij}}$. Then $h_{ij}h_{jk}h_{ki} = (-1)^{g_{ijk}}$. Define $\mathrm{Spin^cDiff}(X) = U(1) \times_{\mathbb{Z}_2} \mathrm{SpinDiff}(X)$. Then $\varphi_{ij} = h_{ij}\widetilde{\psi}_{ij}$ is a 2-cocycle valued in $\mathrm{Spin^cDiff}(X)$.

Consider now the transition functions for the determinant line bundle $\mathcal{L}$. Since $\mathfrak{s}_{E/B}$ restricts to a spin structure on the fibres, the spinor bundles have a quaternionic structure on each fibre. It follows that $\widetilde{\psi}_{ij}$ induces a trivial action on the determinant line. However, the $U(1)$-factor h_{ij} in $\varphi_{ij} = h_{ij}\widetilde{\psi}_{ij}$ acts on the spinor bundles as scalar multiplication which then acts on the determinant line by h_{ij}^d, where d is the virtual rank of $\mathrm{ind}(D)$, which is $d = -\sigma(X)/8$. Therefore $\mathcal{L}$ has transition functions $h_{ij}^{-\sigma(X)/8} = f_{ij}^{-\sigma(X)/16}$. Recalling that f_{ij} are transition functions for a line bundle with Chern class $c(\mathfrak{s}_{E/B})$, it follows that

$$c_1(\mathrm{ind}(D)) = c_1(\mathcal{L}) = -\tfrac{1}{16}\sigma(X)c(\mathfrak{s}_{E/B}) = \tfrac{1}{16}\sigma(X)w_2(TE/B) \pmod 2. \qquad \square$$

Proposition 7.6 *Let $\pi : E \to B$ be a smooth fibre bundle, where the fibres of E are homeomorphic to $K3$. Then $w_2(TE/B) = w_2(H^+)$, where $H^+ \to B$ denotes the bundle whose fibre over b is a maximal positive definite subspace of $H^2(E_b;\mathbb{R})$.*

Proof Since $H^2(B;\mathbb{Z}_2)$ is detected by maps of compact surfaces into B, it suffices to prove the result when B is a compact surface. Then by Proposition 7.5, $w_2(TE/B) = c_1(\mathrm{ind}(D)) \pmod 2$. On the other hand, since the fibres are homeomorphic to $K3$, their Seiberg–Witten with respect to the spin structure is odd [20]. Then by [4, Corollary 1.3], $c_1(\mathrm{ind}(D)) = w_2(H^+)$. $\square$

Let L be a lattice and $A = \mathrm{Aut}(L)$ the group of automorphisms. Over the classifying space BA we have the tautological flat bundle $H = EA \times_A L$. Let $H^+ \to BA$ be a maximal positive subbundle. This defines a characteristic class $w_2(H^+) \in H^2(\mathrm{Aut}(L);\mathbb{Z}_2)$.

Theorem 7.7 *Let X be a smooth 4-manifold which is homeomorphic to $K3$. Let L_X be the intersection lattice of X. Then the extension class $\xi_X \in H^2(M(X);\mathbb{Z}_2)$ is the pullback of $w_2(H^+) \in H^2(\mathrm{Aut}(L_X);\mathbb{Z}_2)$ under the map $M(X) \to \mathrm{Aut}(L_X)$.*

Proof Let B be a compact surface and consider a map $\iota : B \to BM(X)$. This is equivalent to a homomorphism $\rho : \pi_1(B) \to M(X)$. We claim that ρ is the geometric monodromy of a family $E \to B$. We can take B to be given by attaching a 2-cell to a wedge of k circles. Each circle defines a generator $g_i \in \pi_1(B)$ and the 2-cell defines a relation $r = r(g_1, \dots, g_k)$, which is a word in the g_i. Choose a lift $f_i \in \mathrm{Diff}(X)$ of $\rho(g_i) \in M(X)$. Then we can construct a family E_1 over the 1-skeleton on B

as a wedge of mapping cylinders corresponding to the diffeomorphisms $f_1, \dots, f_k$. Since $g_1, \dots, g_k$ satisfy r, it follows that $r(f_1, \dots, f_k)$ is isotopic to the identity. Choosing such an isotopy, we can extend E_1 over the 2-cell, giving the desired family $E \to B$. As explained in [4, Remark 4.20], we can assume that the family $E \to B$ is smooth. Now consider the obstruction to lifting the structure group of E to SpinDiff(X). This is easily seen to coincide with the obstruction to lifting $\rho : \pi_1(B) \to M(X)$ to $M_1(X)$, which is $\iota^*(\xi_X) \in H^2(B; \mathbb{Z}_2)$. On the other hand, the obstruction to lifting the structure group of E to SpinDiff(X) is $w_2(TE/B)$, which by Proposition 7.6 equals $w_2(H^+)$. Since $H^2(M(X); \mathbb{Z}_2)$ is detected by maps of compact surfaces B into $BM(X)$, the result is proven. □

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