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Negative-definite spin filling and branched double covers

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We investigate the negative-definite spin fillings of branched double covers of alternating knots. We derive obstructions to the existence of such fillings and characterize special alternating knots using these results.

1 Introduction

Given a nonsplit link K, let $\Sigma(S^3, K)$ denote the branched double cover of S^3 along K. A filling of $\Sigma(S^3, K)$ is a 4-manifold X with $\partial X = \Sigma(S^3, K)$. A common approach to constructing fillings of $\Sigma(S^3, K)$ involves taking a spanning surface F of K and forming the branched double cover of D^4 over F^+, where F^+ is obtained by pushing the interior of F inside D^4. We use the term spanning filling to distinguish fillings that can be constructed through this method. One of the most important facts about spanning fillings of branched double covers of links is due to Gordon and Litherland [10]. They proved that the intersection form of $\Sigma(D^4, F^+)$ is equal to the Goeritz form of F. We call a spanning surface F positive- or negative-definite if the Goeritz form of F (or equivalently $\Sigma(D^4, F^+)$) is positive- or negative-definite, respectively.

A standard choice for a spanning surface of K comes from a checkerboard coloring of the regions in a knot diagram. Considering all of the white (resp. black) regions in S^2 and adding twisted bands between them around each crossing will result in a spanning surface of K. We refer to this surface as the white (resp. black) Tait surface and denote it by F_W (resp. F_B). Note that generally Tait surfaces are not invariants of the knot and depend on the choice of diagram.

A link diagram is alternating if its crossings alternate over and under around each link component, and a link is alternating if it admits an alternating diagram. One can prove that a diagram is alternating if and only if the associated Tait surfaces are definite and of opposite signs (see Proposition 4.1 of [14]). Greene [14] proved that the existence of definite spanning surfaces gives a topological characterization of alternating links as follows:

Theorem 1.1 [14, Theorem 1.1] *If F_P and F_N are positive- and negative-definite spanning surfaces for a nonsplit link K in S^3, then K is an alternating link, and it has an alternating diagram whose Tait surfaces are isotopic to F_P and F_N.*

In the rest of this paper, we only focus on nonsplit alternating links. Whenever we consider an alternating link K, we are in fact working with an arbitrary reduced alternating diagram of K. For convenience we do not differentiate between the link and its diagram. Furthermore, to fix a standard checkerboard coloring of an alternating diagram, we assume that the black Tait surface is a negative-definite spanning surface.

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We call an oriented alternating diagram *special* if the associated black Tait surface is orientable. Seifert's algorithm outputs this surface when applied to such a diagram. An oriented alternating link is special if it has a special alternating diagram. One can see that the orientability of F_B is equivalent to evenness of its Goeritz form (see Section 2 for more details). In this paper, whenever we consider a special alternating link K, we are working with a special alternating diagram of K.

Building on the preceding discussion, for an alternating link K, the 4-manifolds $\Sigma(D^4, F_W^+)$ and $\Sigma(D^4, F_B^+)$ are definite fillings of $\Sigma(S^3, K)$, which we refer to as the white and black Tait fillings, respectively. Furthermore, if K is a special alternating link, then $\Sigma(D^4, F_B^+)$ is also a spin filling of the branched double cover.

The existence and properties of definite fillings have been the subject of extensive research, including works by Ozsváth and Szabó [19], Scaduto [22], Choe and Park [3], Golla and Scaduto [8], and Aceto, McCoy, and Park [1]. These works also establish bounds on the Betti numbers using tools from Heegaard Floer and Seiberg–Witten theories. In this paper, we discuss how definite spin fillings can detect special alternating links among all alternating links. This is described in Theorem 1.2.

Before we state Theorem 1.2, note that, in the rest of the paper, we assume that all of the diagrams are decorated, i.e., they have a marked arc between two crossings. We refer to the two regions separated by the marked arc as marked (adjacent) regions. We use m to represent the number of unmarked white regions in a reduced alternating diagram.

Theorem 1.2 *Let K be a nonsplit alternating link and m be the number of unmarked white regions in a reduced alternating diagram. If X is a simply connected negative-definite spin filling of $\Sigma(S^3, K)$, then*

$$b_2(X) \le m.$$

Furthermore, equality is achieved if and only if K is special alternating.

The existence of simply connected spin negative-definite fillings of the branched double cover of nonspecial alternating knots is not trivial. Using inequalities from Heegaard Floer and Seiberg–Witten theories, we develop several obstructions to the existence of such fillings. The main ones are in the form of Theorems 1.3 and 1.4. First, we need to explain some notation.

Consider a reduced alternating diagram of link K. The white (resp. black) Tait graph, denoted by W (resp. B), is constructed by considering a vertex for each white (resp. black) region in a diagram and drawing an edge between two regions if and only if they have a common crossing on their boundary. Note that W and B are planar duals. Let $\widetilde{W}$ be the reduced white Tait graph; i.e., the white Tait graph W with the vertex associated with the marked region deleted. Both reduced and unreduced Tait graphs are generally undirected multigraphs with no self-loops or degree-one vertices. This follows from the assumption that the diagram is reduced.

In this paper, we use V_G to denote the vertex set of a graph G and $E_G(\cdot,\cdot)$ to denote the set of edges between two disjoint subsets of V_G. A subgraph C of $\widetilde{W}$ is called *characteristic* if it satisfies

$$e_W(v, C) \equiv \deg_W(v) \mod 2 \qquad \text{for all } v \in V_{\widetilde{W}},$$

where $e_W(v, C)$ is defined by the formula

$$e_W(v,C) = \begin{cases} |E_W(\{v\}, V_C)| + \deg_W(v) & \text{if } v \in C, \\ |E_W(\{v\}, V_C)| & \text{if } v \notin C. \end{cases} \tag{1-1}$$

Using W, we will build a surgery diagram for the black Tait filling of $\Sigma(S^3, K)$ in Section 2. One can see that $e_W(\cdot, \cdot)$ is a reformulation of the intersection form of this filling.

Let $\mathcal{C}_{\widetilde{W}}$ denote the set of characteristic subgraphs of $\widetilde{W}$. We will see that these subgraphs classify spin structures on $\Sigma(S^3, K)$ (see Theorem 4.2). The empty subgraph is characteristic if and only if all of the vertices of W have even degrees. In this case, the dual plane graph B will be bipartite. The 2-coloring of B induces an orientation on F_B which means K is special alternating. As a result, if K is a nonspecial alternating knot, a characteristic subgraph can not be empty.

Theorem 1.3 *Let K be a nonspecial alternating knot. If*

$$\min_{C \in \mathcal{C}_{\widetilde{W}}} |E_W(V_C, V_W \setminus V_C)| \geq |V_W| - 1,$$

then $\Sigma(S^3, K)$ does not have a simply connected negative-definite spin filling.

Theorem 1.4 *Let K be a nonspecial alternating link. If*

$$\min_{C \in \mathcal{C}_{\widetilde{W}}} |E_W(V_C, V_W \setminus V_C)| \geq 9(|V_W| - 1),$$

then $\Sigma(S^3, K)$ does not have a simply connected negative-definite spin filling.

While investigating Theorem 1.4, we identify obstructions preventing 4-manifolds from having chainmail Kirby diagrams (see Section 4). This leads to Corollary 4.11 which states that any closed 4-manifold with a chainmail Kirby diagram is either spin or has a characteristic embedded sphere.

Theorems 1.3 and 1.4 turn out to be generalized versions of a known obstruction of negative-definite spin fillings given by the Neumann–Siebenmann invariant. See Remark 4.10.

We should point out that both of these theorems result in a *twisting phenomenon*. Consider a coherent twist region R in an alternating diagram of the link K. By coherent, we mean a twist region corresponding to a family of parallel edges between two vertices in the white Tait graph (see Figure 1). Let $K_{(R,i)}$ be the link obtained by adding i full twists to the twist region R. We call this operation *enlarging R in K*. In terms of the white Tait graph, the enlarging operation can be defined as adding $2i$ parallel edges.

The enlarging operation can increase the left-hand side of the inequalities of Theorems 1.3 and 1.4 while it does not change the right-hand side, i.e., $|V_W|$. Hence if you enlarge twist regions of the knot diagram, you will end up with links whose branched double cover does not have a negative-definite spin filling.

The main condition of Theorem 1.3 can be seen as an analogue of Elkies's condition [6, Theorem 1] (i.e., nonexistence of short characteristic vectors) for the *Goeritz lattice*. This is the content of Corollary 1.5. See Remark 3.2 for a detailed explanation.

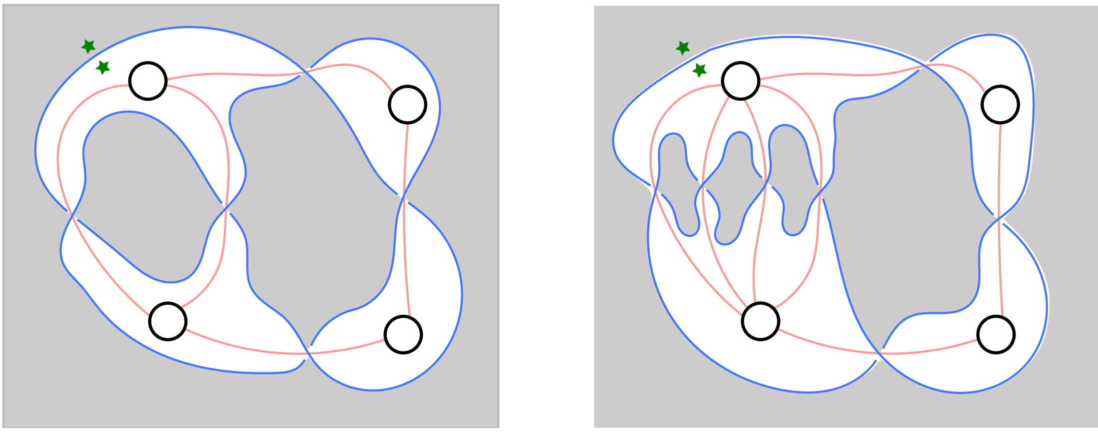

Figure 1: Enlarging a twist region.

Corollary 1.5 *Let K be a nonspecial alternating knot. Also let Λ be the lattice defined by the Goeritz form of F_B, and let* $\mathrm{Char}(\Lambda)$ *be the set of characteristic vectors of Λ. If*

$$\min_{w \in \mathrm{Char}(\Lambda)} |\langle w, w\rangle| \geq |V_W| - 1,$$

then $\Sigma(S^3, K)$ does not have a simply connected negative-definite spin filling.

Note that the only difference between Corollary 1.5 and the main result of Elkies is that the Goeritz lattice is not unimodular.

Generally, there are nonspecial alternating knots with branched double covers that admit simply connected negative-definite spin fillings. Examples of such knots are provided in Section 5. However, it turns out that by further restricting the search to the class of plumbed 4-manifolds, one can prove a nonexistence result explained in Theorem 1.7. First, we need to explain some notation.

Theorem 1.7 is about algebraic (or arborescent) links which are defined as follows.

Definition 1.6 The following algorithm associates a link to a planar weighted tree. Let T be a planar tree with weight $w(a_i) \in \mathbb{Z}$ associated with any vertex $a_i \in V_T$. For each $a_i \in V_T$ consider a twisted band F_i with $w(a_i)$ half twists. If two vertices are connected by an edge in T, plumb the two corresponding bands together. Let F be the resulting surface and consider the link ∂F. If a link can be constructed using this algorithm, we call it *algebraic* (*or arborescent*).

Following the work of Siebenmann [23], it is known that these are the only links whose branched double covers admit plumbed filling. We call an algebraic link *excessive* if it is constructed from a plumbing tree T with weight function w, satisfying

$$w(a_i) \leq \min\{-2, -\deg_T(a_i)\} \quad \text{for all } a_i \in V_T.$$

Excessiveness is a technical condition defined by Murasugi [16]. We use it to ensure that the algebraic link is alternating and relate the tree T to the Tait graph (see Lemma 5.4).

Theorem 1.7 *Let K be an excessive algebraic alternating knot. Then $\Sigma(S^3, K)$ admits a simply connected spin negative-definite plumbed filling if and only if K is special.*

This paper is organized as follows. In Section 2, we set our basic notation and recall some of the theorems from the literature. This section will include the construction of a Kirby diagram for the black Tait filling, some facts about the Heegaard–Floer homology of $\Sigma(S^3, K)$, and some inequalities about fillings of rational homology spheres and closed spin 4-manifolds. In Section 3, we discuss proofs of Theorems 1.2 and 1.3 which come from a formula for the correction term of the branched double cover. In Section 4, we discuss an algorithm of Kaplan which helps us to construct spin fillings and combine it with Furuta's $\frac{10}{8}$ theorem to prove Theorem 1.4. In Section 5, we will discuss Neumann's plumbing calculus and use it to prove Theorem 1.7.

2 Background and notation

Let $K \subset S^3$ be an alternating knot and let D be an alternating diagram of K in the plane. Since the diagram is alternating, one can construct a checkerboard coloring of the diagram such that all the crossings have $\mu = -1$, using the notation of Gordon and Litherland [10]. The coloring will look like Figure 2 around each crossing.

In this setting, one can define *white and black Tait graphs and Tait surfaces*. Tait graphs are constructed by considering regions with the same color as vertices and drawing an edge between two regions if and only if they have a common crossing on their boundary. We use the notation W and B for the graphs and F_W and F_B for the spanning surfaces. In this paper, we assume that diagrams are always decorated, i.e, they have a marked arc between two crossings. We refer to the two regions separated by the marked arc as marked (adjacent) regions. We refer to the graphs resulting from deleting the vertices associated with the marked regions as the reduced Tait graphs and denote them by $\widetilde{B}$ and $\widetilde{W}$. To define the Tait surfaces, let us first recall the definition of median construction.

Definition 2.1 Given a plane graph G, consider a thickening of $G \subset \mathbb{R}^2$. This thickening has a disc centered around each vertex of G, together with a band along each edge. Apply a right-handed (resp. left-handed) half-twist on each of the bands, which gives us a surface (in S^3). This construction is called the *positive* (*resp. negative*) median construction.

Now we define the white (resp. black) Tait surface, denoted by F_W (resp. F_B), as the result of negative (resp. positive) median construction on the Tait graph W (resp. B).

Figure 2: Standard coloring of a crossing in an alternating link.

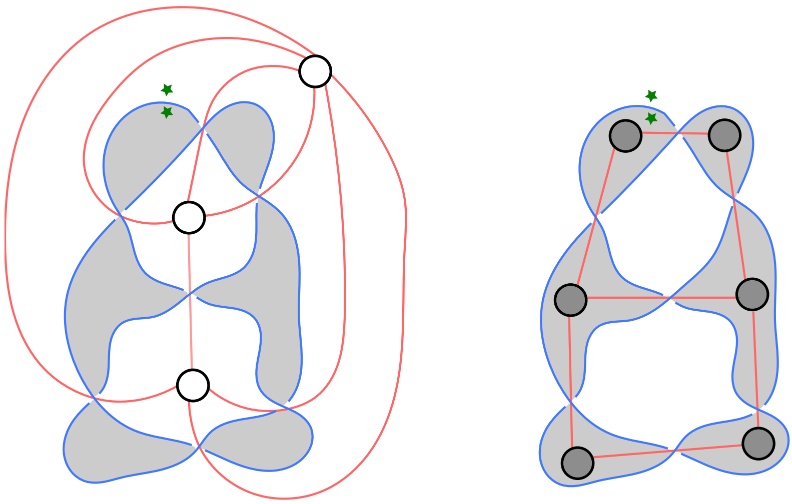

Figure 3: A special alternating knot and its Tait graphs.

An alternating knot is called *special* if the black Tait surface is orientable, i.e., a Seifert surface. This is equivalent to the black Tait graph being bipartite. Since black and white Tait graphs are dual planar graphs, this definition is also equivalent to the white Tait graph having no vertex with odd degree. An example of a special alternating knot and its Tait graphs can be seen in Figure 3.

As noted in the introduction and in [10], the branched double cover of D^4 over the black Tait surface F_B, which is denoted by $\Sigma(D^4, F_B)$, is a negative-definite filling of $\Sigma(S^3, K)$. The intersection form of $\Sigma(D^4, F_B)$ turns out to be the Goeritz form of F_B. There is a combinatorial description of the Goeritz form of the black Tait surface of an alternating knot in terms of the white Tait graph. Enumerate the vertices of $\widetilde{W}$ by $v_1, \dots, v_m$ and set

$$g_{ij} := |E_W(v_i, v_j)| \quad \text{for } i \neq j \text{ and } g_{ii} = -\deg_W(v_i).$$

Then the white Goeritz matrix $G_W := (g_{ij})$ represents the Goeritz form. Note that this is the definition of the Laplacian matrix of the graph W with the row and column associated with the marked vertex deleted. For the example presented in Figure 3, we have

$$G_W = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}.$$

Now we are going to describe a Kirby diagram of $\Sigma(D^4, F_B^+)$ which also acts as a surgery diagram for the $\Sigma(S^3, K)$. In the rest of the article, we refer to this construction as the *Tait surgery diagram*. This diagram originates from work of Ozsváth and Szabó [20].

We consider an unknot component with framing g_{ii} centered around each $v_i \in V_{\widetilde{W}}$ and then add a positive clasp between the unknot components corresponding to v_i and v_j for each edge $e \in E_{\widetilde{W}}$ between v_i and v_j. The intersection form of this Kirby diagram is clearly the same as G_W. Applying this to the example in Figure 3 gives us the surgery diagram in Figure 4.

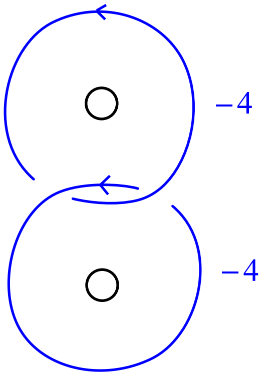

Figure 4: Surgery diagram of the branched double cover.

Greene [13] derives a Heegaard triple $\mathcal{H}_1$ subordinate to this surgery diagram and in combination with another Heegaard triple $\mathcal{H}_2$ originating from the Montesinos trick, he gives a combinatorial description of $\widehat{\mathrm{HF}}(\Sigma(S^3, K))$. We only need some of Greene's results about alternating links which we will summarize in the following.

Given a Kauffman state x for K, we induce an orientation on the white graph W in the following way. Given an edge $e \in E_W$, consider the crossing c to which it corresponds, as well as the white region which abuts c and lies on the same side of the over-strand as $x(c)$. We direct e to point towards the vertex corresponding to this white region. This process is illustrated in Figure 5.

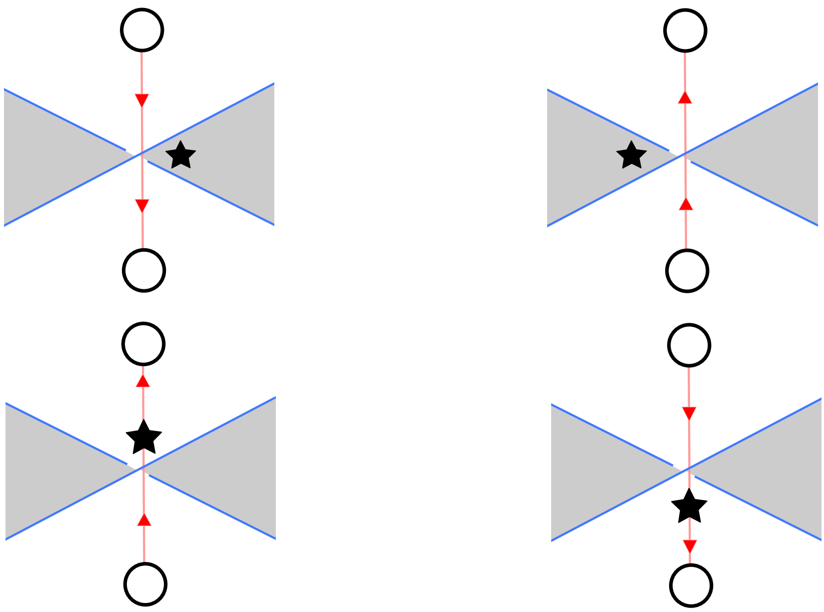

Figure 5: Orientation induced by a Kauffman state on the white graph. The Kauffman marker is depicted by the black star.

At a vertex $v \in V_{\widetilde{W}}$, we compute the signed degree $d_x(v)$ as

$$d_x(v) = d_x^+(v) - d_x^-(v),$$

where $d_x^+(v)$ (resp. $d_x^-(v)$) denotes the number of edges directed into (resp. out of) v with respect to this orientation on W.

Define $v_x^W := (d_x(v_1), \dots, d_x(v_m))^T$. Define the quadratic form $q(v)$ as

$$q(v) := v^T G_W^{-1} v \quad \text{for all } v \in \mathbb{Z}^m.$$

A *characteristic covector* $v \in \mathbb{Z}^m$ is defined by the condition that

$$v_i \equiv (G_W)_{ii} \mod 2 \qquad \text{for all } 1 \le i \le m.$$

The characteristic covectors are useful in studying the space of $\mathrm{Spin}^{\mathbb{C}}$ structures of $\Sigma(S^3, K)$. We know that $\mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K))$ is an affine space over $H^2(\Sigma(S^3, K))$. Using Poincaré duality we have

$$H^2(\Sigma(S^3, K)) \cong H_1(\Sigma(S^3, K)).$$

We can compute this homology using the Tait surgery diagram and Mayer–Vietoris sequence as follows.

Let $L = \bigcup_{i=1}^m L_i$ be the underlying m-component link of the Tait surgery diagram. The Tait surgery diagram and the definition of Dehn surgery give us the decomposition

$$\Sigma(S^3, K) = (S^3 \setminus N(L)) \cup_\phi \bigsqcup_{i=1}^m H_i.$$

In this decomposition $H_i = D^2 \times S^1$ is the solid torus glued in a tubular neighborhood of L_i using the gluing map

$$\phi : \bigsqcup_{i=1}^m \partial H_i \to \partial(S^3 \setminus N(L)).$$

Let μ_i and λ_i respectively be the meridian and longitude of L_i in S^3. Also let μ_i' and λ_i' respectively be the meridian and longitude of ∂H_i. Then the gluing map ϕ induces a map ϕ_* on the first homology such that

$$\phi_*([\mu_i']) = g_{ii} \cdot [\mu_i] + [\lambda_i].$$

Now the Mayer–Vietoris sequence gives us

$$H_1\Big(\bigsqcup_{i=1}^m \partial H_i\Big) \to H_1(S^3 \setminus N(L)) \oplus H_1\Big(\bigsqcup_{i=1}^m H_i\Big) \to H_1(\Sigma(S^3, K)) \to 0.$$

We can rewrite this sequence as

$$\bigoplus_{i=1}^m \langle [\mu_i'], [\lambda_i'] \rangle \hookrightarrow \langle [\mu_1], \dots, [\mu_m] \rangle \oplus \bigoplus_{i=1}^m \langle [\lambda_i'] \rangle \twoheadrightarrow H_1(\Sigma(S^3, K)).$$

The first map in the sequence is defined, for all $i = 1, \dots, m$, as

$$[\mu_i'] \to g_{ii} \cdot [\mu_i] + [\lambda_i] = g_{ii} \cdot [\mu_i] + \sum_{\substack{1 \le j \le m \\ j \ne i}} \mathrm{lk}(L_i, L_j) \cdot [\mu_j] = \sum_{j=1}^{m} g_{ji} \cdot [\mu_j] \quad \text{and} \quad [\lambda_i'] \to [\lambda_i'].$$

As a result, we have

$$H_1(\Sigma(S^3, K)) \cong \frac{\mathbb{Z}^m}{\mathrm{im}(G_W)} \cong \mathrm{coker}(G_W).$$

The combination of these results gives us an identification of $H^2(\Sigma(S^3, K))$ with $\mathrm{coker}(G_W)$.

Now we go back to studying $\mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K))$. Greene [13] proved that $\mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K))$ can be identified with $2 \cdot \mathrm{im}(G_W)$-orbits of characteristic covectors. The identification comes from the first Chern class

$$c_1 : \mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K)) \to H^2(\Sigma(S^3, K)) \cong \mathrm{coker}(G_W).$$

Now as before we assume that K is a nonsplit alternating link. Let $\mathcal{T}$ be the set of Kauffman states of K. Greene [13] also proved that there is a one-to-one correspondence

$$x \in \mathcal{T} \longleftrightarrow \mathfrak{t}(x) = v_x^W + 2 \cdot \mathrm{im}(G_W) \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K)).$$

Furthermore, when $\det(K)$ is odd, the first Chern class c_1 is a canonical identification of $\mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K))$ and $\mathrm{coker}(G_W)$, and hence we have

$$c_1(\mathfrak{t}(x)) = [v_x^W] \in \mathrm{coker}(G_W).$$

We can now finally state Greene's computation of the Heegaard Floer homology of $\Sigma(S^3, K)$ in Theorem 2.2.

Theorem 2.2 [13] *Let K denote a nonsplit alternating link. Then*

$$\widehat{\mathrm{HF}}(\Sigma(S^3, K)) = \bigoplus_{x \in \mathcal{T}} \widehat{\mathrm{HF}}(\Sigma(S^3, K), \mathfrak{t}(x)) = \bigoplus_{x \in \mathcal{T}} \mathbb{Z}.$$

The correction term can be computed by the formula

$$d(\Sigma(S^3, K), \mathfrak{t}(x)) = \max_{v \in \mathfrak{t}(x)} \frac{q(v) + m}{4} = \frac{q(v_x^W) + m}{4},$$

where $m = |V_W| - 1$.

Remark 2.3 The notation v_x^W might be a bit confusing. Note that v_x^W is the degree vector of W restricted to the unmarked vertices $\{v_1, \dots, v_m\}$ which are also the vertices of $\widetilde{W}$.

We are going to use these formulas to obstruct branched double covers from having spin negative-definite fillings. To accomplish this, we are going to use known inequalities about fillings of rational homology spheres and closed spin 4-manifolds. We recall some of the theorems that we are going to use later.

Theorem 2.4 [19, Theorem 9.6] *Let Y be a rational homology three-sphere, and fix a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{t}$ over Y. Then, for each smooth, negative-definite filling X of Y, and for each $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$ with $\mathfrak{s}|_Y = \mathfrak{t}$, we have that*

$$c_1(\mathfrak{s})^2 + b_2(X) \le 4d(Y, \mathfrak{t}).$$

Theorem 2.5 [7, Theorem 1] *If M is a closed spin manifold with indefinite intersection form, then*

$$b_2(M) \ge \tfrac{10}{8}|\sigma(M)| + 2.$$

3 Bounds from correction terms

Now we are ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2 Assume X is a simply connected, spin, negative-definite filling of $Y = \Sigma(S^3, K)$. Recall that $\mathrm{Spin}^{\mathbb{C}}$ structures of X correspond to integral lifts of the second Stiefel–Whitney class under the second map in the exact sequence

$$H^2(X;\mathbb{Z}) \xrightarrow{\times 2} H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2) \xrightarrow{\beta} \cdots.$$

The first Chern class of a $\mathrm{Spin}^{\mathbb{C}}$ structure is equal to this integral lift of the Stiefel–Whitney class. Since X is spin, the second Stiefel–Whitney class vanishes and, as a result, one can find a trivial lift $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$ with $c_1(\mathfrak{s}) = 0$. Due to Theorem 2.2, there is a Kauffman state x such that $[v_x^W] \in \mathrm{coker}(G_W)$ is identified with $\mathfrak{s}|_Y \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K))$. Using Theorem 2.4, we can write

$$b_2(X) \le 4d(Y, \mathfrak{s}|_Y) = q(v_x^W) + m \le m.$$

The last inequality follows from the fact that q is a negative-definite form as its defined using inverse of Goeritz matrix.

Now we are going to prove that the last inequality is sharp if K is not special. This again comes from Theorem 2.2. We only need to show that, for all Kauffman states x on a nonspecial alternating knot, $q(v_x^W) < 0$. Since q is negative-definite, we only need to prove that $v_x^W \neq 0$. This follows from the fact that, for nonspecial knots, W contains at least two vertices with odd degrees, since its dual can not be bipartite. As a result, there exist $v_i \in V_{\widetilde{W}}$ such that $\deg_W(v_i)$ is odd. On the other hand, v_x^W is a characteristic covector; i.e.,

$$v_x^W = (d_x(v_i)) = (d_x^+(v_i) - d_x^-(v_i)) \equiv (\deg_W(v_i)) = -g_{ii} \mod 2.$$

Finally, we need to show that, for special alternating knots, there exists a simply connected, spin, and negative-definite filling of $\Sigma(S^3, K)$ with $b_2 = m$. The 4-manifold $X = \Sigma(D^4, F_B^+)$ satisfies these conditions. Using the Tait surgery diagram (which is a Kirby diagram of X), one can see that X is simply connected with $b_2(X) = m$ and negative-definite with intersection form $Q_X = G_W$. Furthermore Q_X is even as $g_{ii} = -\deg_W(v_i)$ which is even since K is special. Hence, X is spin as well. □

Proof of Theorem 1.3 This proof is similar to the previous one. Due to Theorem 2.2, there is a Kauffman state x such that $[v_x^W] \in \operatorname{coker}(G_W)$ is identified with $\mathfrak{s}|_Y \in \operatorname{Spin}^{\mathbb{C}}(\Sigma(S^3,K))$. Using Theorem 2.4, we can write

$$b_2(X) \le 4d(Y,\mathfrak{s}|_Y) = q(v_x^W) + m.$$

Note that $\mathfrak{s}$ is induced by a Spin structure on X, and, as a result, $\mathfrak{s}|_Y$ is also the $\operatorname{Spin}^{\mathbb{C}}$ structure induced by the unique Spin structure on $\Sigma(S^3,K)$. The uniqueness of the Spin structure follows from the assumption that K is a knot. It is known that for a knot K, $\det(K) = |H_1(\Sigma(S^3,K);\mathbb{Z})|$ is odd. Consequently, $H_1(\Sigma(S^3,K);\mathbb{Z})$ has no 2-torsion, and $H^1(\Sigma(S^3,K);\mathbb{Z}_2)$ vanishes. Using this argument and the one made in the first lines of the proof of Theorem 1.2, we can deduce that $c_1(\mathfrak{s}|_Y) = 0$.

We will show that the inequality in the statement of Theorem 1.3 results in $d(Y,\mathfrak{s}|_Y)$ being negative. We know that $\operatorname{Spin}^{\mathbb{C}}(\Sigma(S^3,K))$ is identified with $\operatorname{coker}(G_W)$ through the first Chern class. As a result

$$[v_x^W] = [0] = [c_1(\mathfrak{s}|_Y)] \in \operatorname{coker}(G_W),$$

which means that there exists $y \in \mathbb{Z}^m$ such that $G_W y = v_x^W$. Note that we can rewrite

$$q(v_x^W) = (v_x^W)^T G_W^{-1} v_x^W = y^T G_W^T y = y^T G_W y.$$

We know that v_x^W is a characteristic covector and by definition we have

$$(v_x^W)_i \equiv -g_{ii} \mod 2. \tag{3-1}$$

We can rewrite (3-1) as

$$(v_x^W)_i = (G_W y)_i = \Sigma_j g_{ij} y_j \equiv \deg_W(v_i) = -g_{ii} \mod 2.$$

Let $y' \in \mathbb{Z}^m$ be the vector defined by $y'_i = (y_i \bmod 2) \in \{0,1\}$. Let J be the support of y'. Let C be the subgraph of $\widetilde{W}$ induced by v_j for $j \in J$.

Now one can see that

$$\deg_W(v_i) \equiv (G_W y)_i \equiv (G_W y')_i \equiv e_W(v_i, C) \mod 2.$$

The last equality follows from the properties of the Laplacian matrix. Indeed, we have that

$$(G_W y')_i = \sum_{j\in J} g_{ij} = \sum_{j\in J-\{i\}} g_{ij} + \sum_{i\in J} g_{ii} = \sum_{j\in J} |E(\{v_i\},\{v_j\})| - \sum_{i\in J} \deg_W(v_i),$$

which is equal to $e(v_i, C) \bmod 2$. As a result, C is a characteristic subgraph. Note that, since we assume the knot to be nonspecial, C can not be empty.

We can also use the interpretation of G_W as a submatrix of the Laplacian of W to reformulate $y^T G_W y$ as the following sum. Assume that L_W is the full Laplacian of W with the $(m+1)$-st (last) row and column associated to the distinguished vertex and set $y_{m+1} = 0$. Then we have

$$y^T G_W y = [y^T \ 0] L_W \begin{bmatrix} y \\ 0 \end{bmatrix} = -\sum_{\{v_i,v_j\}\in E_W} (y_i - y_j)^2. \tag{3-2}$$

Due to the definition of C, we will have $y_i \neq y_j$ for $v_i \in V_C$ and $v_j \in V_W \setminus V_C$. Combined with (3-2),

$$q(v_x^W) = y^T G_W y \leq -|E(V_C, V_W \setminus V_C)|.$$

Combining this with the statement of Theorem 1.3, we have that $q(v_x^W) \leq -m$, which gives us the result that we want. □

Remark 3.1 As we will mention in the next section, the definition of characteristic subgraph is an analogue of the definition of characteristic sublink in a surgery diagram (see Definition 4.1). Characteristic sublinks in a surgery diagram of Y are in one-to-one correspondence with $H^1(Y;\mathbb{Z}_2)$ and, as a result, in the setting of Theorem 1.3, there is only one characteristic sublink in the white Tait graph.

Remark 3.2 We can rephrase the main inequality of Theorems 1.3 and 1.4 in terms of the length of characteristic vectors in *Goeritz lattice* as follows.

Same as before let $m = |V_W| - 1$. The integral *Goeritz lattice* $\Lambda \subset \mathbb{R}^m$ is defined with the symmetric bilinear form

$$\langle x, y\rangle = x^T G_W y \quad \text{for all } x, y \in \mathbb{Z}^m.$$

A vector $w \in \mathbb{Z}^m$ is called *characteristic* if for all $x \in \mathbb{Z}^m$ we have

$$\langle w, x\rangle \equiv \langle x, x\rangle \mod 2.$$

Let $\mathrm{Char}(\Lambda)$ be the set of all characteristic vectors in Λ.

This definition is directly related to our definition of characteristic subgraph. First, let w' be the mod 2 reduction of w, i.e., for all $i \in \{1, \dots, m\}$ we have $w'_i = (w_i \bmod 2) \in \{0, 1\}$. Note that w is characteristic if and only if w' is characteristic. Let $J \subseteq \{1, \dots, m\}$ be the support of w', and let C be the subgraph of $\widetilde{W}$ induced by v_j for $j \in J$. One can see that w is a characteristic vector if and only if C is a characteristic subgraph.

Furthermore, similar to (3-2), for all $x \in \mathbb{Z}^m$ we have

$$\langle x, x\rangle = x^T G_W x = -\sum_{\{v_i, v_j\} \in E_W} (x_i - x_j)^2.$$

Now note that for any edge $\{v_i, v_j\} \in E_W(V_C, V_W \setminus V_C)$ we have

$$w_i \equiv w'_i = 1 \mod 2 \quad \text{and} \quad w_j \equiv w'_j = 0 \mod 2 \qquad \Longrightarrow \qquad (w_i - w_j)^2 \geq 1.$$

As a result,

$$\langle w, w\rangle = -\sum_{\{v_i, v_j\} \in E_W} (w_i - w_j)^2 \leq -|E_W(V_C, V_W \setminus V_C)| = \langle w', w'\rangle.$$

Finally we can conclude that

$$\min_{C \in \mathcal{C}_{\widetilde{W}}} |E_W(V_C, V_W \setminus V_C)| = \min_{w \in \mathrm{Char}(\Lambda)} |\langle w, w\rangle|.$$

As a result, we can conclude Corollary 1.5 from Theorem 1.4.

4 Bound from Furuta's $\frac{10}{8}$ theorem

It is well known that the third spin cobordism group vanishes. This means that any spin 3-manifold $(Y,\mathfrak{t})$ has a spin filling; i.e., there exists a spin 4-manifold $(W,\mathfrak{s})$ such that $\partial W = Y$ and $\mathfrak{s}|_Y = \mathfrak{t}$. In fact, Kaplan [15] built an algorithm that turns any surgery diagram of Y to a Kirby diagram of a spin filling through Kirby calculus. We will recall this algorithm from its exposition by Gompf and Stipsicz [9]. First, let us define the notion of a characteristic sublink of a framed link.

Definition 4.1 Let L be a framed link. A sublink $L' \subseteq L$ is called *characteristic* if and only if for any sublink $L_i \subseteq L$ we have

$$\operatorname{lk}(L', L_i) \equiv \operatorname{lk}(L_i, L_i) \mod 2.$$

Now assume that we have a surgery diagram L of a 3-manifold Y. Considering L as a Kirby diagram gives us a handlebody filling X_L of Y. For any component L_i of L, we get a class $[H_i] \in H_2(X_L;\mathbb{Z}_2)$ from capping off core of the 2-handle attached along L_i. These homology classes give a basis for $H_2(X_L;\mathbb{Z}_2)$. As a result, we have a bijection between sublinks of L and elements of $H_2(X_L;\mathbb{Z}_2)$ as

$$L' \subseteq L \longleftrightarrow \sum_{L_i \subseteq L'} [H_i] \in H_2(X_L;\mathbb{Z}_2).$$

Due to Poincaré duality we have $H_2(X_L;\mathbb{Z}_2) \cong H^2(X_L, Y;\mathbb{Z}_2)$. Combining these two facts, we end up with a bijection between sublinks of L and $H^2(X_L, Y;\mathbb{Z}_2)$.

Now we are ready to state the necessary tools from Gompf and Stipsicz [9] in Theorems 4.2 and 4.3.

Theorem 4.2 [9, Proposition 5.7.11] *For any* Spin *structure* $\mathfrak{t}$ *on* Y, *the nonvanishing of the relative Stiefel–Whitney class* $w_2(X_L,\mathfrak{t}) \in H^2(X_L, Y;\mathbb{Z})$ *serves as an obstruction for extending* $\mathfrak{t}$ *to* X. *The map*

$$\mathfrak{t} \in \operatorname{Spin}(Y) \to w_2(X_L,\mathfrak{t}) \in H^2(X_L, Y;\mathbb{Z})$$

gives a bijection between the set of Spin *structures on* Y *and characteristic sublinks of* L.

Proposition 4.3 [9, Theorem 5.7.14] *Assume that we have a surgery diagram* L *of a* 3*-manifold* Y. *Fix any* Spin *structure* $\mathfrak{t}$ *on* Y *and assume that* L' *is the corresponding characteristic sublink. The following steps will lead to a Kirby diagram of a spin filling of* $(Y,\mathfrak{t})$:

(1) *Slide one component* L'_0 *of* L' *over the rest of the components of* L'. *The characteristic sublink corresponding to* $\mathfrak{t}$ *in the new Kirby diagram will be the sublink consisting of one component* L'_0.

(2) *Unknot* L'_0 *using blow-ups. The blow-up circles can be imagined as connected sum of two small meridian circles* m_{i1}, m_{i2} *along a band* b_i *forming* D_i *as in Figure 6, left.*

The characteristic sublink will be the union of L'_0 *and all the blown-up circles.*

(3) *One can again use blow-ups to change the crossings between the bands and other components of* L, *without changing the characteristic sublink*; *see Figure 6, right. Use an isotopy to turn* L'_0 *into a circle in the plane and then use the operation of Figure 6, right, to turn the characteristic sublink into Figure 7, left.*

Figure 6: Left: Step 2 in Kaplan's algorithm [9, Figure 5.48]. Right: Step 3 in Kaplan's algorithm [9, Figure 5.49].

(4) *The operation shown in Figure 7, right, can be done using a blow-up. Use this operation to turn the characteristic sublink into an unlink.*

(5) *Consider each component of the characteristic sublink one by one. Blowing up its meridians turns the framing to* ± 1*. Then, by blowing down the characteristic sublink, one can turn it into the empty link.*

In the end, you will have a Kirby diagram with even framings. This 4-manifold with its unique Spin *structure is a spin filling of* $(Y, \mathfrak{t})$.

We are going to show that, in the setting of Theorem 1.4, Kaplan's algorithm simplifies and, as a result, one can compute the change in the signature and second Betti number and prove the obstruction of Theorem 1.4. We will use Lemma 4.5 in the proof.

Before we state the lemma we need to introduce some notation. We call a framed link a *chainmail link* if it is constructed in the following way. Let D be a weighted and signed plane multigraph. Assigned to each $v_i \in V_D$, there is an integer weight $w_i \in \mathbb{Z}$ and assigned to each $e_k \in E_D$, there is a sign $\mu_k \in \{+, -\}$. The framed link L_D is constructed by considering an unknot component L_i oriented counterclockwise and with framing w_i centered around each $v_i \in V_D$ and then add a left-handed (resp. right-handed) clasp between the unknot components corresponding to v_i and v_j for each edge $e_k \in E_D$ between v_i and v_j with $\mu_k = +$ (resp. $-$). This definition generalizes the construction of Tait surgery diagram explained in Section 2. Further details can be found in the work of Polyak [21].

Now we are going to explain a modification of the first step of Kaplan's algorithm. This procedure is called $MK1$ and is defined in Definition 4.4.

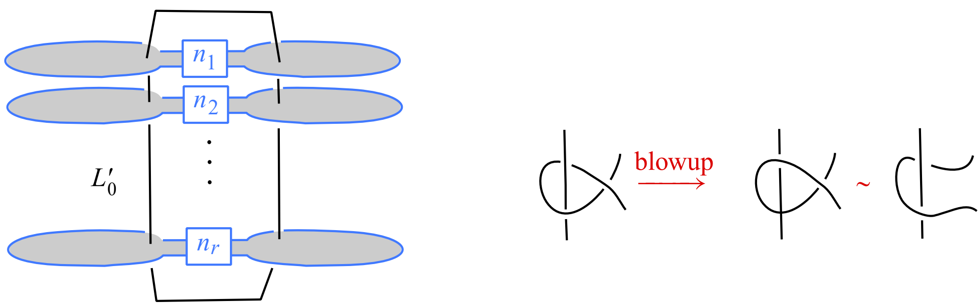

Figure 7: Left: Step 3 in Kaplan's algorithm [9, Figure 5.50]. Right: Step 4 in Kaplan's algorithm [9, Figure 5.51].

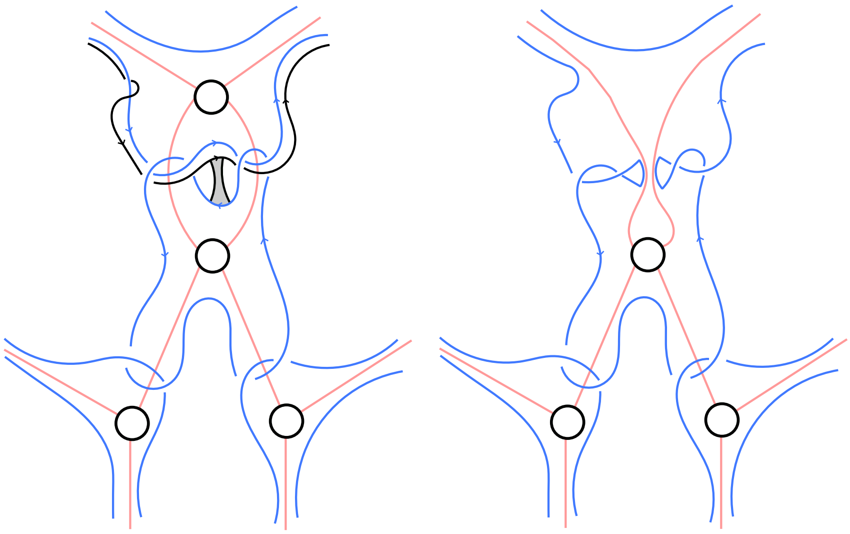

Figure 8: Handle slides in MK1.

Definition 4.4 Let L_D be the chainmail link based on the connected plane graph D and T be a spanning tree of D. Fix an arbitrary orientation (on each edge) and a total order $<$ on E_T. Consider the following procedure:

(1) Take the maximal edge e_m of T in the ordering and assume it is directed from v_p to v_t. Slide L_p over L_t with an orientation-preserving band.

(2) Contract e in T. The two vertices at two ends of e will form a new vertex which we denote by v_p (this labeling will be important in repeating step (1)).

(3) Repeat the process until $|V_T| = 1$.

The knot corresponding to the remaining vertex is denoted by $\mathrm{MK1}(L_D, T)$. See Figure 8.

Lemma 4.5 *Let L_D be the chainmail link based on the connected plane graph D. There exist a spanning rooted tree T with a total ordering and direction on edges such that* $\mathrm{MK1}(L_D, T)$ *is an unknot.*

Proof of Lemma 4.5 We construct T by induction on the number of vertices in D. For two adjacent vertices v_i, v_j in D, let $\mathrm{ER}(v_i, v_j)$ be the maximal bounded region in the plane enclosed by the edges between v_i and v_j with respect to inclusion. If only one edge connects v_i and v_j, let $\mathrm{ER}(v_i, v_j)$ be that single edge (as a subset of plane). Among all pairs of adjacent vertices, pick $\{v_r, v_s\}$ such that $\mathrm{ER}(v_r, v_s)$ is minimal, meaning it does not contain $\mathrm{ER}(v_i, v_j)$ for any other adjacent pair $\{v_i, v_j\}$. An example of such a minimal pair is a pair of adjacent vertices which only have one connecting edge.

Picking this minimal pair guarantees that all the other vertices lie in $\mathbb{R}^2 \setminus \mathrm{ER}(v_r, v_s)$ and this gives a standard model (see Figure 9) for the configuration of the clasps between L_r and L_s with respect to the other clasps involving L_r or L_s. We use this to control the result of sliding L_r over L_s.

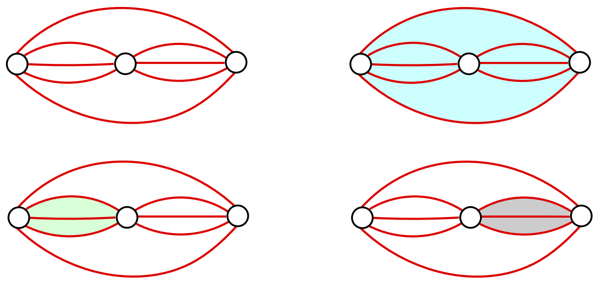

Figure 9: Examples of $\mathrm{ER}(v_i, v_j)$. The green and gray (shaded) regions in the bottom row are minimal.

Without loss of generality assume that $r = 1$ and $s = 2$. Figure 8 shows this sliding operation. It is clear that after a number of R1 moves, L_1 will be an unknot around v_1 and v_2 and there will be clasps associated with each edge between v_1 or v_2, and any v_j for $j \neq 1, 2$. This means that, after sliding L_1 over L_2 and deleting L_2, we will have a chainmail link on the plane graph $D/\{v_1, v_2\}$, which is the plane graph coming from contracting v_1 and v_2 to one vertex which we will denote by $v_{1,2}$. This decreases the size of the vertex set by one. Using induction, we know that there exists a spanning tree T' such that $\mathrm{MK1}(L_{D/\{v_1,v_2\}}, T')$ is an unknot. Spanning tree T can be constructed by replacing $v_{1,2}$ with v_1 and v_2 and the edge between them, directing the edge from v_1 to v_2 and putting it as the new maximal edge in the total ordering. □

Remark 4.6 As mentioned, the operation MK1 is designed to be a modification of the first step of Kaplan's algorithm for a chainmail link. We explain this in a bit more detail in the following. Note that we are using the notation of Theorems 4.2 and 4.3.

The main goal of the first step of the Kaplan algorithm is to construct a surgery diagram of Y such that this characteristic sublink only has one component. This can be achieved through handle slides. Note that the handle slides does not change the isomorphism type of the filling X_L. Sliding a component L'_i of L' over another component L'_j can be seen as a change of basis of $H^2(X_L, Y; \mathbb{Z}_2)$. As a result, for the purpose of the Kaplan's algorithm, we only need to find the expansion of $w_2(X_L, \mathfrak{t})$ in this new basis. This expansion shows that the characteristic sublink associated to $\mathfrak{t}$ in the new surgery diagram (after handle slide) is just $L' \setminus \{L'_j\}$.

Due to this fact, one can turn the characteristic sublink into a knot with handle slides in the first step of Kaplan's algorithm. The same reasoning shows that MK1 can replace the first step of Kaplan's algorithm when L' is a chainmail link.

Remark 4.7 The operation MK1 is only defined for chainmail links based on connected planar graphs. However, we can easily generalize MK1 and Lemma 4.5 to all chainmail links as follows.

If L_D is a chainmail link based on a disconnected graph $D = \bigsqcup_{i=1}^{k} D_i$, then L_D is also a split link which splits to chainmail sublinks as

$$L_D = \bigsqcup_{i=1}^{k} L_{D_i}.$$

We apply Lemma 4.5 to the each of L_{D_i}. For each $i = 1, \dots, k$, there exists a spanning rooted tree T_i in D_i such that $\mathrm{MK1}(L_{D_i}, T_i)$ is an unknot. Applying all of these operations turns L_D to the unlink $\bigsqcup_{i=1}^{k} \mathrm{MK1}(L_{D_i}, T_i)$. We can then slide one of the components of this unlink over the other components such that the final result is an unknot.

Due to Remark 4.6, this generalized operation can also replace the first step of Kaplan's algorithm.

Remark 4.8 The linking between the components of the link changes under the handle slides. We can update the linking matrix at each step with the rule

$$\mathrm{lk}(L_{1,2}, L_i) = \mathrm{lk}(L_1, L_i) + \mathrm{lk}(L_2, L_i).$$

Now we are ready to prove Theorem 1.4. In fact, we are going to prove a stronger result stated in Theorem 4.9.

Theorem 4.9 *Let K be a nonspecial alternating link, and $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(S^3, K))$. Let C be the characteristic subgraph associated to $\mathfrak{t}$. If*

$$|E_W(V_C, V_W \setminus V_C)| \geq 9(|V_W| - 1),$$

then $(\Sigma(S^3, K), \mathfrak{t})$ does not have a simply connected negative-definite spin filling.

Proof of Theorem 4.9 Assume that $(X, \mathfrak{s})$ is a simply connected negative-definite spin filling of $Y = (\Sigma(S^3, K), \mathfrak{t})$. Let L be the Kirby diagram of $\Sigma(D^4, F_B)$ described in Section 2. Let L' be the characteristic sublink of L associated with $\mathfrak{t}$. We are going to apply the modified version of Kaplan's algorithm on L using L' which will result in a simply connected spin filling $(X', \mathfrak{s}')$ of $(\Sigma(S^3, K), \mathfrak{t})$. We can take $-X'$ and build the closed 4-manifold $W = X \cup_Y (-X')$ which is also spin since $\mathfrak{s}|_Y = \mathfrak{s}'|_Y = \mathfrak{t}$. Using Furuta's inequality (Theorem 2.5) on W gives

$$b_2(X) + b_2(-X') \geq \tfrac{10}{8}|\sigma(X) + \sigma(-X')| + 2. \tag{4-1}$$

The right-hand side comes from Novikov additivity. Now we need to compute $b_2(-X')$ and $\sigma(-X')$.

Using Lemma 4.5 and Remark 4.6, we know that Steps 2, 3, 4 of Kaplan's algorithm will not be needed in our setting, and we can easily compute the change of b_2 and σ. First note that in each of the described handle slides, the framings change in the following way. If we assume that framing of L_p, L_s are r_p, r_s, respectively, then the framing of the L_p after the sliding will be $r_p + r_s + 2\,\mathrm{lk}(L_p, L_r)$. Using induction and Remark 4.8, we can see that framing on the final component of the characteristic sublink (after finishing Step 1) is equal to

$$\sum_{v_i \in C} g_{ii} + 2 \sum_{\substack{i \neq j \\ v_i, v_j \in C}} \mathrm{lk}(L_i, L_j) = \sum_{v_i \in C} g_{ii} + 2 \sum_{\substack{i \neq j \\ v_i, v_j \in C}} g_{ij}.$$

Since $g_{ii} = -\deg_W(v_i)$ and $g_{ij} = |E(v_i, v_j)|$, we will have that

$$\begin{aligned}\sum_{v_i \in C} g_{ii} + 2 \sum_{\substack{i \neq j \\ v_i, v_j \in C}} g_{ij} &= -\sum_{v_i \in C} \deg_W(v_i) + 2 \sum_{\substack{i \neq j \\ v_i, v_j \in C}} |E(v_i, v_j)| \\ &= -|E_W(V_C, V_W \setminus V_C)| - 2|E_C| + 2|E_C| = -|E_W(V_C, V_W \setminus V_C)|.\end{aligned}$$

This is the right-hand side of the inequality stated in Theorem 1.4. We denote this value by $-f$. Note that in Step 1, we only use handle slides and isotopies which means that the filling will not change. To calculate the change in b_2 and signature, we only need to look at Step 5. In this step, we blow up $f-1$ meridians in order to turn the characteristic sublink into an unknot with framing -1 and then blow down this unknot. These increase b_2 and σ by $f-2$ and f, respectively. Now we only need to use this information in Furuta's inequality. Note that since X is negative-definite $\sigma(X)=-b_2(X)$. Assuming that $f \geq 9m$, which is the assumption of Theorem 1.4 (where $m=|V_W|-1$), we can rewrite (4-1),

$$\begin{aligned} b_2(X)+m+f-2 \geq \tfrac{10}{8}|-b_2(X)+m-f|+2 &\iff b_2(X)+m+f-2 \geq \tfrac{10}{8}(b_2(X)+f-m)+2 \\ &\iff \tfrac{18}{8}m \geq \tfrac{2}{8}f+\tfrac{2}{8}b_2(X)+4. \end{aligned}$$

The final inequality is a clear contradiction to $f \geq 9m$.

Note that the equivalence between the first and second inequality follows from the assumption that $f \geq 9m$ which means that $b_2(X)+f-m>0$. □

Remark 4.10 This procedure gives a generalization of a corollary of Ue [24, Theorem 1]. Recall that, for a plumbed 3-manifold Y, the Neumann–Siebenmann invariant $\bar{\mu}$ is defined as follows. Assume that Γ is the plumbing tree and $P(\Gamma)$ is the 4-manifold constructed from plumbing sphere bundles based on Γ. We know that $Y=\partial P(\Gamma)$. Let w_s be the indicator vector of the characteristic sublink associated with a Spin structure $\mathfrak{s}$ on Y. Then

$$\bar{\mu}(Y,\mathfrak{s}) := \tfrac{1}{8}\big(\sigma(P(\Gamma))-\langle w_s,w_s\rangle\big),$$

where $\langle\cdot,\cdot\rangle$ represents the intersection pairing. Ue proves that a Seifert homology sphere Y with Spin structure $\mathfrak{s}$ bounding a negative-definite 4-manifold X with Spin structure $\mathfrak{s}_X$ must satisfy

$$-\tfrac{8}{9}\bar{\mu}(Y,\mathfrak{s}) \leq b_2(X) \leq -8\bar{\mu}(Y,\mathfrak{s}).$$

Now an obstruction to the existence of simply connected negative-definite spin fillings is $\bar{\mu}(Y,\mathfrak{s})>0$. In cases when $\widetilde{W}$ is a star-shaped tree (which is the plumbing tree of a Seifert homology S^3) one can apply this obstruction to our problem. Whenever $\widetilde{W}$ is a tree, any characteristic subgraph C will be a disjoint union of isolated vertices. As a result, if $w_C \in \mathbb{Z}^m$ is the indicator vector of V_C, then

$$\langle w_C,w_C\rangle = \sum_{v_i\in C} -g_{ii} = \sum_{v_i\in C} -\deg_W(v_i) = -|E_W(V_C,V_W\setminus V_C)|. \tag{4-2}$$

This comes from the fact that $E_C=\varnothing$. Combining (4-2) and definition of $\bar{\mu}(Y,\mathfrak{s})$ gives us

$$8\bar{\mu}(Y,\mathfrak{s}) = -m+|E_W(V_C,V_W\setminus V_C)|.$$

This means that Theorem 1.4 generalizes the obstruction $\bar{\mu}(Y,\mathfrak{s})>0$.

This simplification of Kaplan's algorithm and the fact that one can build a characteristic unknot without blowing up or down is of independent interest. The following corollary easily follows from this observation.

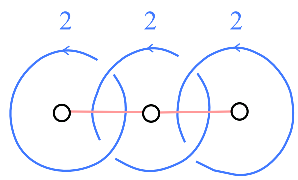

Figure 10: Kirby diagram of P_4.

Corollary 4.11 *If a closed 4-manifold X has a chainmail Kirby diagram, then it is either spin or has a characteristic sphere.*

This can act as an obstruction for a manifold to have a chainmail diagram. The following is an explicit example of this. This example was pointed out to us by Marco Golla.

Example 4.12 Let X be the Akhmedov–Park exotic $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ [2]. We prove that X does not have a characteristic sphere as follows.

Note that X is symplectic and minimal. Any embedded sphere $S \subset X$ satisfies $S \cdot S \le -2$. Adjunction inequality gives us that $S \cdot S \le 0$, and due to the minimality of X there is no embedded sphere with self-intersection -1. Let $S \cdot S = -m$. We use P_m to denote the 4-manifold constructed by negative linear plumbing of $m-1$ disk bundles with Euler number 2 over S^2. We can construct a Kirby diagram of P_m by considering the path graph on $m-1$ vertices and applying the chainmail construction (with all the weights equal to 2, and all the signs equal to $-$). We can see an example of this for P_4 in Figure 10.

There is an orientation-preserving diffeomorphism between ∂P_m and $\partial N(S)$. Using the mentioned Kirby diagram as a surgery diagram for ∂P_m, and the iterated slam-dunk move (see [18]), we see that $\partial P_m = L(m, m-1)$ since

$$\frac{m}{m-1} = [\underbrace{2, \dots, 2}_{m-1}]^- = 2 - \cfrac{1}{\cdots - \cfrac{1}{2 - \frac{1}{2}}}.$$

Finally, $\partial N(S)$ is the circle bundle over S^2 with Euler number $-m$. This means that $\partial N(S) \simeq -L(m, 1)$. We also have that $L(m, m-1) \simeq -L(m, 1)$. The composition of these two gives us the desired diffeomorphism.

Now one can form the closed 4-manifold $M = (X \setminus N(S)^{\circ}) \cup P_m$. We are going to prove that M is spin. Note that $X \setminus N(S)$ is spin since S is characteristic. Let $\mathfrak{s}$ be the Spin structure induced on $\partial(X \setminus N(S)^{\circ})$. We need to prove that P_m induces the same Spin structure on $\partial P_m = L(m, m-1)$. If m is odd, then $L(m, m-1)$ has a unique Spin structure and we are done. If m is even, then $L(m, m-1)$ has two different Spin structures $\mathfrak{s}, \mathfrak{s}'$. Furthermore when m is even, $N(S)$ is spin. However, since X is not spin, the induced Spin structure on $\partial N(S)$ must be $\mathfrak{s}'$. Now we only need to show that P_m and $N(S)$ induce different Spin structures on $L(m, m-1)$ which is equivalent to $P_m \cup -N(S)$ not being spin. The closed 4-manifold $P_m \cup -N(S)$ is positive-definite and $b_2(P_m \cup -N(S)) = m$, hence it can not be spin due to Donaldson's diagonalization theorem [4].

Now M is a closed, simply connected, spin 4-manifold. Using Novikov additivity we can deduce that $\sigma(M) = m-1$. Furthermore, $b_2^-(M) = 1$ since the negative-definite part of $H_2(M)$ lies inside $X \setminus N(S)$. Donaldson's theorem B [5] tell us that $\sigma(M) = 0$ which is a contradiction.

5 Spin negative-definite plumbed fillings

The previous results might lead one to ask if there are any nonspecial alternating knots K such that $\Sigma(S^3, K)$ has a simply connected, spin and negative-definite filling. A result such as the following theorem might further support this.

Theorem 5.1 *A nonspecial alternating link K does not have a spanning filling which is spin and negative-definite.*

Proof Assume $\Sigma(D^4, F^+)$ is a negative-definite spin filling. We know that the intersection form of $\Sigma(D^4, F^+)$ is the Goeritz form of the surface, which means that G_F is a negative-definite spanning surface. Using Theorem 1.1, we can conclude that F must be the black Tait surface in a diagram of K. Then we know, that for G_F to be even, the knot needs to be special, which contradicts the assumption. □

For general fillings, this is far from the truth. We present an example of a nonspecial alternating knot K with a spin negative-definite filling of $\Sigma(S^3, K)$. The main tool for the construction of this example is the lens space realization problem and one can generate a family of examples in the same way. We must mention that part of the inspiration for the example comes from Aceto, McCoy, and Park [1] which addresses negative-definite fillings of lens spaces with minimal b_2. The main difference is that we need to use the lens space fillings that emerge as the trace of a surgery on a knot instead of rational homology ball fillings. We use the notation K^m to denote the result of Dehn surgery on S^3 along K with slope m. We also use $\mathrm{Tr}(K^m)$ to denote the trace of this surgery. Using this notation we have

$$\partial \mathrm{Tr}(K^m) = K^m.$$

Example 5.2 The knot K will be the alternating knot in Figure 11. The white Tait graph is also drawn in the figure. The reduced white Tait graph will be a single path with framings $(-4, -2, -5, -2)$. Plumbing along this path with these framings will give us the standard negative-definite filling of $\Sigma(S^3, K)$. We use the notation $P(a_1, \dots, a_n)$ to denote the linear plumbing with framings $a_1, \dots, a_n$. For example,

$$\partial P(-4, -2, -5, -2) = \Sigma(S^3, K).$$

We have a standard embedding of $P(-2, -5, -2)$ in $P(-4, -2, -5, -2)$ which comes from the embedding of the plumbing graph of the first 4-manifold in the second. In other words, $P(-4, -2, -5, -2)$ can be constructed from $P(-2, -5, -2)$ by attaching a 2-handle with framing -4 to its boundary. The integers $(-2, -5, -2)$ also arise in the fractional expansion

$$\tfrac{16}{9} = 2 - \cfrac{1}{5 - \frac{1}{2}}.$$

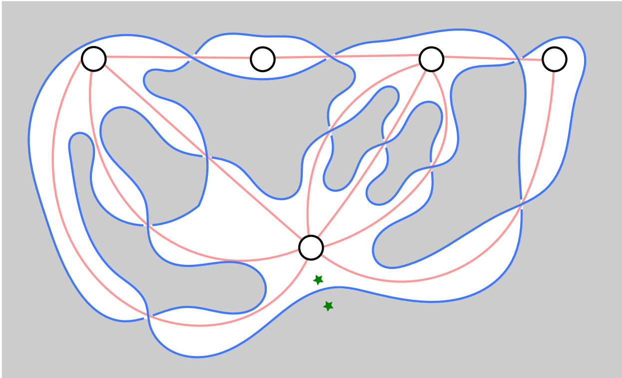

Figure 11: Alternating knot K of Example 5.2.

This means that $P(-2,-5,-2)$ bounds $L(16,9)$. We claim that $L(16,9)$ also has a filling in the form of the trace of a knot surgery, i.e.,

$$L(16,9) = \partial \mathrm{Tr}(K^{-16}).$$

This comes from the description of the Berge knots of type $I_\pm$ (see [12]). Picking $i,k \in \mathbb{Z}$ such that $\gcd(i,k)=1$ and setting $p = ik \pm 1$ and $q \equiv -k^2 \bmod p$, leads to the lens space $L(p,q)$ which can be realized by a positive surgery on a knot. Setting $i=3$ and $k=5$, gives us $p=16$ and $q \equiv -25 \bmod 16$, which means that there exists a knot K such that $L(16,7) = K^{+16}$, which in turn means that $L(16,7) = \partial \mathrm{Tr}(K^{+16})$. By reversing the orientation, we have $L(16,9) = -L(16,7) = \partial \mathrm{Tr}(K^{-16})$.

Let $i : P(-2,-5,-2) \hookrightarrow P(-4,-2,-5,-2) = X$ be the aforementioned embedding. Now we construct a 4-manifold X' by deleting the interior of $\mathrm{Im}(i)$ from X and gluing $\mathrm{Tr}(K^{-16})$ in its place. Then X' will be the result of attaching a -4-framed 2-handle to $\mathrm{Tr}(K^{-16})$, which means it is simply connected and has $b_2 = 2$, as it has a handle decomposition with two 2-handles. The intersection form is even as the framing of both 2-handles is even, and hence, X' is spin. Using Novikov additivity we can also prove that X' is negative-definite. This is again due to the fact that, while constructing X', we delete a submanifold of X with signature -3 and replace it with one with signature -1. This finally gives us the example we need.

Although there is no general nonexistence result for simply connected negative-definite spin fillings, by imposing suitable combinatorial conditions on the Kirby diagram of the filling we can prove such results. The first result of this type is Theorem 1.7. This result is directly rooted in Neumann's plumbing calculus. In the following theorem we recall the facts we need from [17]. Note that the branched double covers of links with nonzero determinant are rational homology spheres. As a result, they can be realized as plumbings of disk bundles over surfaces when the base surfaces are all spheres and the plumbing graph is a tree. We can describe these plumbings with a tree with integer weights on vertices and $\pm$ signs on edges. This means that we do not need Neumann's plumbing calculus in its full generality. In the following theorem, we only recall the facts we need.

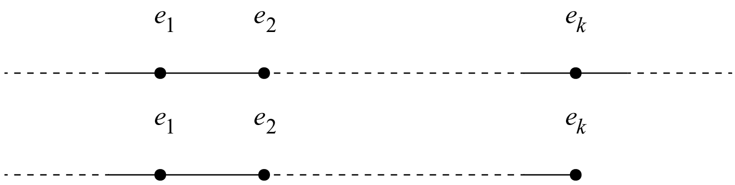

Figure 12: A chain in a plumbing graph [17].

In the rest of the paper, we use the term chain to refer to path subgraphs which contain at most one leaf of T (see Figure 12).

Theorem 5.3 [17, Theorem 4.1] *Any plumbing tree T can be reduced to a unique normal form using the following moves while keeping the boundary unchanged.*

- (R0) *Reverse the sign of all the edges adjacent to a vertex v.*
- (R1a) *Delete a component consisting of an isolated vertex with weight ± 1.*
- (R1b–R3) *These are the moves which are described in Figure 13.*

The normal form is defined by the following properties:

- (N1) *None of the operations can be applied, except that T might contain a component like Figure 14 with $k \geq 1$ and $e_i \leq -2$ for all i.*
- (N2) *The weights e_i on all chains of T satisfy $e_i \leq -2$ for all i.*
- (N3) *No portion of T has the form shown in Figure 15, top, unless it is in a component of the form shown in Figure 15, bottom, with $k \geq 1$ and $e_i \leq -2$ for all i.*

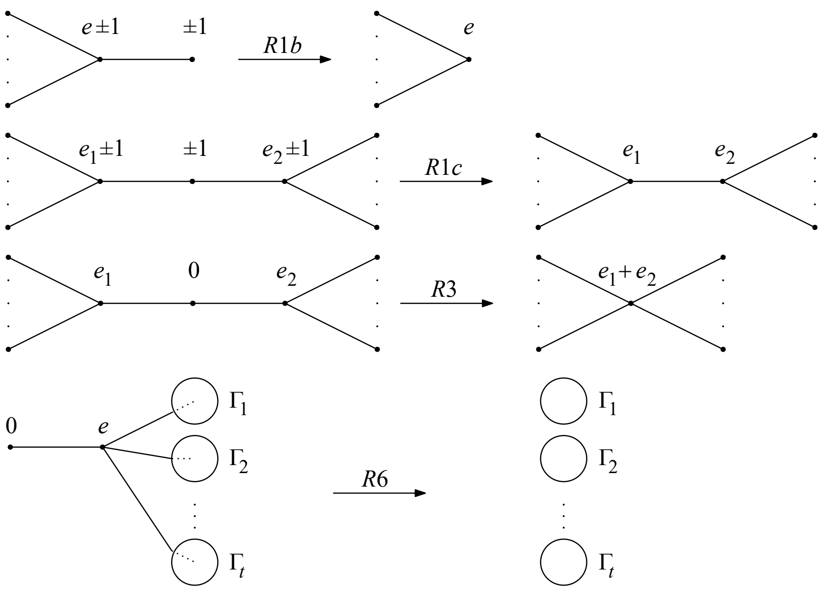

Figure 13: Neumann moves [17].

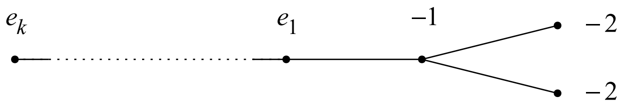

Figure 14: Property N1 of Neumann normal form [17].

You might notice that the moves described in Theorem 5.3 do not describe the change in the edge signs. When we are dealing with trees, the move R0 gives us that the edge signs do not matter.

Before we proceed with proving Theorem 1.7, we need to define the excessive property. This definition comes from Murasugi [16].

Recall that a link is called algebraic (or arborescent) if it can be constructed as the boundary of a plumbing of twisted bands according to a tree (see Definition 1.6). For a weighted tree T, we denote the algebraic link constructed from a plumbing based on T by $l(T)$. The tree T is called *negative excessive* if

$$w(a_i) \leq \min\{-2, -\deg_T(a_i)\} \quad \text{for all } a_i \in V_T.$$

The following lemma is proved by Murasugi.

Lemma 5.4 [16, Propositions 3.3 and 4.1] *For a negative excessive tree T, the link $l(T)$ is alternating. Furthermore, there is an alternating diagram of $l(T)$ such that T is isomorphic to the reduced white Tait graph $\widetilde{W}$. This isomorphism takes the weights of T to g_{ii} (diagonal of the Goeritz matrix).*

With this information, we can proceed with proving Theorem 1.7.

Proof of Theorem 1.7 We start by proving that any simply connected negative-definite spin plumbed filling is automatically in normal form. Due to the spin condition, we will not have framing ± 1 on any vertex as all framings are even. Due to the negative-definite condition, we can not have any vertex with framing 0 as all framings are negative. This means that conditions N1 and N2 are satisfied. To show that N3 is satisfied, we use the assumption that K is a knot and hence has an odd determinant. We know that the determinant of the knot is equal to $|H_1(\Sigma(S^3, K); \mathbb{Z})|$. When the determinant is odd, the 2-torsion vanishes and, as a result, $H^1(\Sigma(S^3, K); \mathbb{Z}_2) = 0$, which means that $\Sigma(S^3, K)$ has a unique Spin structure. We now use Proposition 4.3 to deduce that the number of characteristic sublinks of the Kirby diagram is equal to 1. This in turn means that the plumbing tree T has a unique characteristic subgraph.

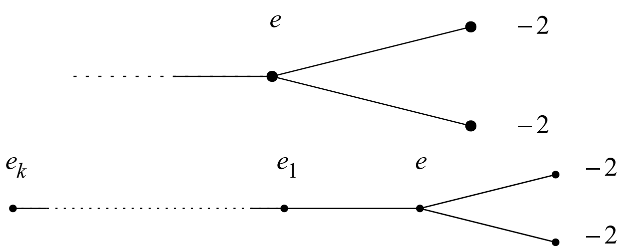

Figure 15: Property N3 of Neumann normal form [17].

We use proof by contradiction. Assume T violates condition N3, which means it contains the forbidden subgraph of Figure 15, top. A characteristic subgraph $C \subseteq T$ can not contain the parent vertex of the -2-framed leaves since the number of edges between a -2-framed leaf and C must be even (due to the definition of characteristic sublink). Let us use the names $L = \{l_1, l_2\}$ and p to denote the -2 framed leaves and their parent vertex. Also define $A := C \cap \{l_1, l_2\}$. Now we consider the subgraph C' defined as

$$C' = (C - A) \cup (L - A).$$

This subgraph is also characteristic. The only change happens with taking the complement of $C \cap L$ on L, which means that $E(v, C)$ and $E(v, C')$ are only different for $v \in \{p, l_1, l_2\}$. In all three cases, the parity of $|E(v, C)|$ and $|E(v, C')|$ are the same as $|E(p, C')| = |E(p, C)| - |A| + (2 - |A|)$ and $|E(l_i, C)| = |E(l_i, C')| \pm 2$. This construction builds a fixed-point-free bijection on the set of characteristic subgraphs, which means that the size of this set must be even. This gives us a contradiction with the argument in the previous paragraph. As a result, T must satisfy condition N3.

Let $\widetilde{W}$ be the reduced white Tait graph of K. By Lemma 5.4, We know that $\widetilde{W}$ is isomorphic to the plumbing tree associated to K. Using the Tait surgery diagram, we can see that $\Sigma(D^4, F_B^+)$ is a plumbed filling. We are going to prove that this plumbed filling is also in normal form. The excessive condition forces all weights to be ≤ -2 and as a result N1 and N2 are satisfied. Using the same argument as the previous paragraph, we can prove that condition N3 is also satisfied.

Now using the uniqueness of Neumann normal form, one can deduce that if a simply connected negative-definite spin plumbed filling exists, then its plumbing tree is exactly the reduced white Tait graph. This means that the framings in the white Tait graph; i.e., g_{ii}, must be all even, which is equivalent to the knot being special. □

The main idea behind Theorem 1.7 can be generalized to some other types of fillings.

Definition 5.5 We call a filling X of a 3-manifold Y a chainmail filling if and only if there exist a Kirby diagram of X which is a chainmail link

Following the discussion in Section 2, the 4-manifold $\Sigma(D^4, F_B^+)$ always gives a chainmail filling of the branched double cover. Unfortunately, there are no known normal forms for chainmail Kirby diagrams in the literature so the proof of Theorem 1.7 can not be replicated, but we can use the trick described here which is inspired by Murasugi [16].

Definition 5.6 A weighted planar graph is called *accessible* if it can be realized as the white Tait graph of an alternating link K such that the weights are equal to diagonal entries of the Goeritz matrix of K. We call a chainmail filling accessible if it has a chainmail Kirby diagram which is based on an accessible planar graph.

The main examples of accessible planar graphs come from the following example:

Example 5.7 Let G be a 2-connected plane graph such that all vertices are adjacent to the unbounded region, i.e., the boundary of the unbounded region includes all of the vertices of G. Furthermore, assume that G is negative excessive; i.e., weights satisfy

$$w(v_i) \le \min\{-2, -\deg_G(v_i)\} \quad \text{for all } v_i \in V_G.$$

In this setting, one can add a vertex $\hat{v}$ in the unbounded region and connect it to all $v_i \in V_G$ such that

$$|E(\hat{v}, v_i)| + \deg_G(v_i) = |w(v_i)|.$$

The median construction on $G \cup \{\hat{v}\}$ gives an alternating link such that the reduced white Tait graph is isomorphic to G and the weights of G will become the diagonal entries of Goeritz matrix.

Theorem 5.8 *Let K be an alternating link. Then $\Sigma(S^3, K)$ admits a simply connected negative-definite spin accessible filling if and only if K is special alternating.*

Proof Assume such a filling X exists and it has a chainmail diagram based on an accessible plane graph like G. Let K' be an alternating link with $\widetilde{W}_{K'} = G$. This means that the chainmail Kirby diagram based on G is also a surgery diagram for $\Sigma(S^3, K')$, which means that branched double covers of K and K' are diffeomorphic. By a result of Greene [11, Theorem 1.1], we can deduce that K and K' are mutants. Planar mutation of alternating knots preserves the number of white regions of the diagram and as a result

$$b_2(X) = |V_G| = |V_{\widetilde{W}_{K'}}| = |V_{\widetilde{W}_K}|.$$

Using Theorem 1.2, we can deduce that K is special alternating. □

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References

[1] **P Aceto**, **D McCoy**, **J Park**, *Definite fillings of lens spaces*, J. Differential Geom. 131:1 (2025) 1–41 MR

[2] **A Akhmedov**, **B D Park**, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. 181:3 (2010) 577–603 MR

[3] **D H Choe**, **K Park**, *On intersection forms of definite 4-manifolds bounded by a rational homology 3-sphere*, Topology Appl. 238 (2018) 59–75 MR

[4] **S K Donaldson**, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. 18:2 (1983) 279–315 MR

[5] **S K Donaldson**, *Connections, cohomology and the intersection forms of 4-manifolds*, J. Differential Geom. 24:3 (1986) 275–341 MR

[6] **N D Elkies**, *A characterization of the $\mathbb{Z}^n$ lattice*, Math. Res. Lett. 2:3 (1995) 321–326 MR

[7] **M Furuta**, *Monopole equation and the $\frac{11}{8}$-conjecture*, Math. Res. Lett. 8:3 (2001) 279–291 MR

[8] **M Golla**, **C Scaduto**, *On definite lattices bounded by integer surgeries along knots with slice genus at most 2*, Trans. Amer. Math. Soc. 372:11 (2019) 7805–7829 MR

[9] **R E Gompf**, **A I Stipsicz**, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, Amer. Math. Soc., Providence, RI (1999) MR

[10] **C M Gordon**, **R A Litherland**, *On the signature of a link*, Invent. Math. 47:1 (1978) 53–69 MR

[11] **J E Greene**, *Lattices, graphs, and Conway mutation*, Invent. Math. 192:3 (2013) 717–750 MR

[12] **J E Greene**, *The lens space realization problem*, Ann. of Math. (2) 177:2 (2013) 449–511 MR

[13] **J E Greene**, *A spanning tree model for the Heegaard Floer homology of a branched double-cover*, J. Topol. 6:2 (2013) 525–567 MR

[14] **J E Greene**, *Alternating links and definite surfaces*, Duke Math. J. 166:11 (2017) 2133–2151 MR

[15] **S J Kaplan**, *Constructing framed 4-manifolds with given almost framed boundaries*, Trans. Amer. Math. Soc. 254 (1979) 237–263 MR

[16] **K Murasugi**, *On the Alexander polynomial of alternating algebraic knots*, J. Austral. Math. Soc. Ser. A 39:3 (1985) 317–333 MR

[17] **W D Neumann**, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. 268:2 (1981) 299–344 MR

[18] **B Owens**, **S Strle**, *Dehn surgeries and negative-definite four-manifolds*, Selecta Math. (N.S.) 18:4 (2012) 839–854 MR

[19] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173:2 (2003) 179–261 MR

[20] **P Ozsváth**, **Z Szabó**, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. 194:1 (2005) 1–33 MR

[21] **M Polyak**, *From 3-manifolds to planar graphs and cycle-rooted trees*, lecture slides (2014) https://www.fields.utoronto.ca/programs/scientific/14-15/arnoldconf/slides/polyak.pdf

[22] **C Scaduto**, *On definite lattices bounded by a homology 3-sphere and Yang–Mills instanton Floer theory*, Geom. Topol. 28:4 (2024) 1587–1628 MR

[23] **L Siebenmann**, *On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres*, from "Topology Symposium" (Siegen, 1979) (U Koschorke, W D Neumann, editors), Lecture Notes in Math. 788, Springer (1980) 172–222 MR

[24] **M Ue**, *Constraints on intersection forms of spin 4-manifolds bounded by Seifert rational homology 3-spheres in terms of $\bar{\mu}$ and κ invariants* (2022) arXiv 2206.05412

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