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On local fibrations of $(\infty, 2)$-categories

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We provide a model independent notion of local fibrations of $(\infty, 2)$-categories which generalises the well-known theory of locally cocartesian fibrations of $(\infty, 1)$-categories. Based on previous work, we construct a model structure which serves as a specific combinatorial model for this type of fibration. Our main result is a generalisation of the locally cocartesian straightening and unstraightening construction of Lurie which yields for any scaled simplicial set S an equivalence of $(\infty, 2)$-categories between the $(\infty, 2)$-category $\mathbb{F}\mathrm{ib}_{0,1}(S)$ of $(0, 1)$-fibrations over S and the $(\infty, 2)$-category of functors $\mathrm{Fun}(S, \mathbb{B}\mathrm{icat}_\infty)$. Given an $(\infty, 2)$-category $\mathbb{B}$ our Grothendieck construction can be specialised to produce an equivalence between the $(\infty, 2)$-category of local fibrations over $\mathbb{B}$ and the $(\infty, 2)$-category of oplax normalised functors with values in $\mathbb{B}\mathrm{icat}_\infty$. Finally, as an application of our results we provide a version of the Yoneda lemma for $(\infty, 2)$-categories.

1 Introduction

The theory of $(\infty, 2)$-categories is enjoying in recent years a rapid and extensive development. Several fundamental constructions such as Gray tensor products [11], partially lax colimits [5; 8; 10] and a 2-dimensional theory of fibrations [3; 6; 13] are already available and ready to be used in the study of homotopy coherent structures. Even more remarkably, we are starting to see specific examples [9; 17] of how these techniques can be used to categorify existing areas of study and how they can put into perspective known constructions. In this paper, we continue our study of fibrations of $(\infty, 2)$-categories

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and provide an additional piece of technology which will be relevant for future applications: a theory of local fibrations.

In the homotopy-coherent world, fibrations are an essential tool that facilitates the construction of functors in a context where an infinite amount of coherent-data must be specified. More precisely, given an $(\infty,2)$-category $\mathbb{C}$ the so-called "Grothendieck construction" (also known in the higher-categorical world as the straightening-unstraightening equivalence due to Lurie [18]) states that there exists an equivalence between the $(\infty,2)$-category of $(0,1)$-fibrations[1] over $\mathbb{C}$ and the $(\infty,2)$-category of functors $F:\mathbb{C}\to\mathbb{C}\mathrm{at}_{(\infty,2)}$, with values in $(\infty,2)$-categories. This result has been realised in several models [3; 7; 19; 20; 21] and it is even available for general fibrations (and functors) of (∞,n)-categories.

However, in many situations we would like to have a version of the Grothendieck construction that is flexible enough to accommodate the notion of a (op)lax normalised functor, i.e., a functor which only preserves composition up to noninvertible coherent-data but preserves identity morphisms. This is achieved by means of the notion of a local fibration of $(\infty,2)$-categories as demonstrated by our main result.[2]

Theorem *Let $\mathbb{S}$ be an $(\infty,2)$-category. Then the straightening-unstraightening adjunction*

$$\mathbb{S}\mathrm{t}^{\mathrm{oplax}}_{\mathbb{S}}:\mathbb{L}\mathrm{Fib}(\mathbb{S})\rightleftarrows \mathrm{Fun}^{\mathrm{oplax}}(\mathbb{S},\mathbb{C}\mathrm{at}_{(\infty,2)}):\mathbb{U}\mathrm{n}^{\mathrm{oplax}}_{\mathbb{S}}$$

yields an equivalence of $(\infty,2)$-categories between the $(\infty,2)$-category of local $(0,1)$-fibrations over $\mathbb{S}$ and the $(\infty,2)$-category of oplax normalised functors with values in $(\infty,2)$-categories.

This result had already been proved for functors with values in $(\infty,1)$-categories by Lurie [19] and in full generality (in the model of 2-fold Segal spaces) by Ayala, Mazel-Gee and Rozenblyum [7]. Our theorem can be seen as a direct extension of Lurie's result. The main motivation for proving the general version of this result in the scaled simplicial model is due to the tractable description that the Gray tensor product admits in this setting [11] which can be used to provide further versions of the Grothendieck construction which are adapted to handle (op)lax natural transformations between functors, as seen in [2]. In the aforementioned work, we extensively exploit this result to produce a general calculus of mates, as well as providing formulas for computing partially lax (co)limits of functors with values in $\mathbb{C}\mathrm{at}_{(\infty,2)}$.

Another advantage of our construction in the scaled simplicial model is that we have full control over the "laxness" of our functors. This is materialised in the following result.

Theorem *Let $\mathbb{S}$ be an $(\infty,2)$-category and let $\mathcal{U}=\{(f_i,g_i)\}_{i\in I}$ be a collection of composable morphisms in $\mathbb{S}$. Then the straightening-unstraightening adjunction*

$$\mathbb{S}\mathrm{t}^{\mathcal{U}\mathrm{oplax}}_{\mathbb{S}}:\mathbb{L}\mathrm{Fib}^{\mathcal{U}}(\mathbb{S})\rightleftarrows \mathrm{Fun}^{\mathcal{U}\mathrm{oplax}}(\mathbb{S},\mathbb{C}\mathrm{at}_{(\infty,2)}):\mathbb{U}\mathrm{n}^{\mathcal{U}\mathrm{oplax}}_{\mathbb{S}}$$

yields an equivalence of $(\infty,2)$-categories between the $(\infty,2)$-category of $\mathcal{U}$-local $(0,1)$-fibrations over $\mathbb{S}$ and the $(\infty,2)$-category consisting in those oplax normalised functors which preserve the composites in $\mathcal{U}$.

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[1] By an (i,j)-fibration we mean a 2-categorical version of the notion of a (co)cartesian fibration which we will define later in the introduction.

[2] For more detailed statements and references to the relevant theorems, see the following subsection in the introduction.

When we specialise to the case where $\mathcal{U}$ is precisely the collection of *all* pairs of composable morphisms the definition a $\mathcal{U}$-local $(0,1)$-fibration collapses to that of an ordinary $(0,1)$-fibration and we recover the usual Grothendieck construction for $(\infty,2)$-categories.

Local fibrations of $(\infty,2)$-categories

Let $p:\mathbb{X}\to\mathbb{S}$ be a functor of $(\infty,2)$-categories. We say that p is a (i,j)-fibration where $i,j\in\{0,1\}$ if:

(F1) For every $a,b\in\mathbb{X}$ the induced map $\mathbb{X}(a,b)\to\mathbb{S}(p(a),p(b))$ on mapping $(\infty,1)$-categories is a cocartesian fibration if $j=0$ or a cartesian fibration if $j=1$.

(F2) For every $a,b,c\in\mathbb{X}$ the composition functors

$$\mathbb{X}(a,b)\times\mathbb{X}(b,c)\to\mathbb{X}(a,c)$$

preserve cocartesian edges if $j=0$ (resp. cartesian edges if $j=1$).

(C1) Let $i=0$. Given an object $a\in\mathbb{X}$ and a morphism $e:p(a)\to y$ in $\mathbb{S}$, there exists an edge $\hat{e}:a\to\hat{y}$ over e with the following property: for every $z\in\mathbb{X}$ precomposition with $\hat{e}$ induces a pullback of $(\infty,1)$-categories

$$\begin{tikzcd}
\mathbb{X}(\hat{y},z) \arrow[r] \arrow[d] & \mathbb{X}(a,z) \arrow[d] \\
\mathbb{S}(y,p(z)) \arrow[r] & \mathbb{S}(p(a),p(z))
\end{tikzcd}$$

We say that $\hat{e}$ is a $(0,j)$-cartesian lift of e. If $i=1$ one defines a dual condition (which generalises the $(\infty,1)$-notion of cartesian edge) and obtains the definition of a $(1,j)$-cartesian edge.

Let us remind the reader that $(0,j)$-fibrations appear in the literature [6; 13] under the name of outer cocartesian when $j=0$ (resp. inner cocartesian if $j=1$) and similarly $(1,j)$-fibrations are called outer cartesian fibrations if $j=0$ and inner cartesian fibrations if $j=1$.

In [3; 6] we gave a systematic analysis of the theory of $(1,0)$-fibrations and provided the corresponding Grothendieck construction which identifies this kind of fibrations with contravariant functors $\mathcal{F}:\mathbb{S}^{\mathrm{op}}\to\mathbb{C}\mathrm{at}_{(\infty,2)}$. However, the models employed to study these fibrations in the aforementioned works do not generalise the theory of locally cocartesian fibrations developed in [18; 19]. In order to realise the local theory of fibrations in the scaled simplicial model [19], we will be working in this paper with the $(0,1)$-variance. Needless to say, in model independent terms these problems do not arise and we can give the following general definition.

Let $p:\mathbb{X}\to\mathbb{S}$ be a functor of $(\infty,2)$-categories. We say that p is a local (i,j)-fibration if these hold:

- Conditions (F1) and (F2) are satisfied. In this case, we say that p is *cocartesian*-enriched if $j=0$ and *cartesian*-enriched if $j=1$.
- For every morphism $e:\Delta^1\to\mathbb{S}$ the pullback $\mathbb{X}\times_{\mathbb{S}}\{e\}\to\Delta^1$ is an (i,j)-fibration. We say that an edge $\hat{e}:\Delta^1\to\mathbb{X}$ is a local $(0,1)$-cartesian edge if it is $(0,1)$-cartesian after taking the corresponding pullback.

In order to access the model-independent results mentioned above, we resort to the model of marked-biscaled simplicial sets constructed in [6], which provides us with a robust combinatorial framework to model local $(0,1)$-fibrations and to implement the desired Grothendieck construction. A marked-biscaled simplicial set denoted as $(X, E, T_X \subseteq C_X)$ consists of a simplicial set X, together with a subset of edges $E \subset X_1$ containing all degenerate 1-simplices, and two collections of 2-simplices (or triangles), $T_X \subseteq C_X \subseteq X_2$, such that T_X (and hence C_X) contains every degenerate 2-simplex. The collection E is used to model the cocartesian 1-morphisms of our fibrations, while C_X models the cartesian 2-morphisms. The subcollection T_X is used to represent the invertible 2-morphisms which are always cartesian. Equipped with this formalism we establish as the first result in this paper the existence a model structure whose fibrant objects we refer to as U-local $(0,1)$-fibrations over a scaled simplicial set (S, U) (see Theorems 3.25 and 3.34).

Theorem 1 *Let (S, U) be a scaled simplicial set. Then there exists a left proper, combinatorial, simplicial model structure*[3] *on $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$, which is characterised uniquely by the following properties:*

(C) *A morphism $f : X \to Y$ is a cofibration if and only if f induces a monomorphism on the underlying simplicial sets.*

(F) *An object $p : X \to S$ is fibrant if and only if it is a U-local $(0,1)$-fibration.*

Moreover, if $S = \Delta^0$ then this model structure is Quillen equivalent to Lurie's model structure (see Theorem 4.27 in [19]) on $\mathrm{Set}_{\Delta}^{\mathbf{sc}}$, the category of scaled simplicial sets

In the case where $\mathbb{S} = (S, T_S)$ is a fibrant scaled simplicial set and thus models an $(\infty,2)$-category (here T_S consists in those 2-simplices representing commuting triangles in $\mathbb{S}$), we can consider a subcollection of triangles $M_S \subseteq T_S$ depicted visually as

$$\begin{tikzcd}
& b \arrow[dr, "v"] \arrow[d, phantom, "\Uparrow \simeq"] & \\
a \arrow[ur, "u"] \arrow[rr] & {} & c
\end{tikzcd}$$

where either u or v are equivalences. In Theorem 4.22, we specialise the previous result to (S, M_S) to obtain a model independent interpretation of the theory of M_S-local (or simply local) $(0,1)$-fibrations over (S, M_S). In order to achieve this, we construct (see Definition 4.4) from every M_S-local $(0,1)$-fibration $p : (X, E_X, T_X \subseteq C_X) \to (S, \sharp, M_S \subseteq \sharp)$ a fibration among fibrant scaled simplicial sets $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ satisfying (F1), (F2) and (C1) above which we call the *bicategorical interpretation* of p.

Theorem 2 *Let (S, T_S) be a fibrant scaled simplicial set and let $p : (X, E_X, T_X \subseteq C_X) \to (S, \sharp, M_S \subseteq \sharp)$ be an object of $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,M_S\subset\sharp)}$. Then p defines a fibrant object if and only if its bicategorical interpretation $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ is a local $(0,1)$-fibration of $(\infty,2)$-categories.*

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[3] Here "$\sharp$" denotes the maximal collection consisting in *all* 1-simplices (or all 2-simplices).

It is well known (see Proposition 2.4.2.8 in [18]) that a locally cocartesian fibration whose locally cocartesian edges compose must be a cocartesian fibration. We extend this analysis to the $(\infty,2)$-categorical case by considering a fibrant scaled simplicial set (S,T_S) and a collection of triangles $M_S \subseteq U \subseteq T_S$ which allows us to make the following definition:

- A local $(0,1)$-fibration $p:\mathbb{X}\to\mathbb{S}$ is said to be $\mathcal{U}$-local if local $(0,1)$-cartesian edges compose along triangles lying over U.

It then follows from Theorem 4.29 that $\mathcal{U}$-local fibrations can also be characterised as fibrant objects in our model structure.

Theorem 3 *Let* (S,T_S) *be a fibrant scaled simplicial and let* U *be a collection of triangles such that* $M_S\subseteq U\subseteq T_S$. *Then an object* $p:X\to S$ *is fibrant in* $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ *if and only if its bicategorical interpretation* $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ *is a local* $(0,1)$*-fibration of* $(\infty,2)$*-categories which is in addition* $\mathcal{U}$*-local.*

Once the basics of the local theory of fibrations of $(\infty,2)$-categories are established we focus our attention into providing the expected Grothendieck construction which will allow us to interpret our fibrations as functors with values in $\mathbb{B}\mathrm{icat}_\infty$ the ∞-bicategory (i.e., fibrant scaled simplicial set) of ∞-bicategories.

Let $(S,U)=S_U$ be a scaled simplicial and denote by $\mathbb{F}\mathrm{ib}_{0,1}(S_U)$ the ∞-bicategory of U-local $(0,1)$-fibrations over S and by $\mathrm{Fun}(S_U,\mathbb{B}\mathrm{icat}_\infty)$ the functor ∞-bicategory. Our main construction is a generalisation of the straightening-unstraightening equivalence of Lurie [19] to the setting of $(0,1)$-fibrations whose fibres are ∞-bicategories. Combining Theorem 5.52 and Remark 5.54 we obtain:

Theorem 4 *Let* $S_U=(S,U)$ *be a scaled simplicial set. Then the straightening-unstraightening adjunction*

$$\mathbb{S}\mathrm{t}_{S_U}:\mathbb{F}\mathrm{ib}_{0,1}(S_U)\rightleftarrows \mathrm{Fun}(S_U,\mathbb{B}\mathrm{icat}_\infty):\mathbb{U}\mathrm{n}_{S_U}$$

yields an equivalence of ∞*-bicategories between the* ∞*-bicategory of* U*-local* $(0,1)$*-fibrations over* S *and the* ∞*-bicategory of covariant functors with values in* ∞*-bicategories.*

The general nature of the theorem above allows us to consider special cases which are of special interest. Namely, let $\mathbb{S}=(S,T_S)$ be an ∞-bicategory and let $S_{M_S}=(S,M_S)$. We can now consider $\mathbb{L}\mathrm{Fib}(\mathbb{S}):=\mathbb{F}\mathrm{ib}_{0,1}(S_{M_S})$ and apply our Grothendieck construction to obtain an equivalence between the ∞-bicategory of local $(0,1)$-fibrations over $\mathbb{S}$ and the category $\mathrm{Fun}(S_{M_S},\mathbb{B}\mathrm{icat}_\infty)$. The latter can be interpreted using the work of Gagna, Harpaz and Lanari [11] as a model for the ∞-bicategory of oplax normalised functors with values in ∞-bicategories. Our main theorem then specialises (Corollary 5.57) to the results presented at the beginning of the introduction.

The ∞-bicategorical Yoneda lemma

As an application of our results we give a fibrational proof of the Yoneda lemma for ∞-bicategories. We would like to stress that such a result was already present in the work of Hinich [16] where a general Yoneda lemma for enriched ∞-categories is established.

In this work we provide a proof of the Yoneda lemma as a direct application of the Grothendieck construction. Given an ∞-bicategory $\mathbb{C}$ we consider the $(0,1)$-fibration $\mathrm{ev}_1 : \mathrm{Fun}^{\mathrm{opgr}}(\Delta^1, \mathbb{C}) \to \mathbb{C}$ (see Proposition 2.2.6 in [13] for more details) which is sometimes referred as the oplax arrow category of $\mathbb{C}$. Carefully unwinding this construction reveals that the fibres of this map come equipped with maps

$$\mathbb{C}_{\nearrow c} = \mathrm{Fun}^{\mathrm{opgr}}(\Delta^1, \mathbb{C}) \times_{\mathbb{C}} \{c\} \to \mathbb{C},$$

which are in turn $(1,0)$-fibrations with ∞-categorical fibres. Applying the Grothendieck construction in two steps we obtain a functor $\mathcal{Y}_{\mathbb{C}} : \mathbb{C} \to \mathrm{Fun}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty)$. Moreover, it follows from previous work (Theorem 3.17 in [4]) that $\mathbb{C}_{\nearrow c}$ corresponds under the Grothendieck construction to the representable functors $\mathbb{C}(-,c)$. We prove in Theorem 6.16 the final result of this paper.

Theorem 5 *For every ∞-bicategory $\mathbb{C}$ the Yoneda embedding*

$$\mathcal{Y}_{\mathbb{C}} : \mathbb{C} \to \mathrm{Fun}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty), \quad c \mapsto \mathbb{C}(-,c),$$

is fully faithful. Moreover, given a functor $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}_\infty$ there is a equivalence of $\mathbb{C}\mathrm{at}_\infty$-valued functors

$$\mathrm{Nat}_{\mathbb{C}^{\mathrm{op}}}(\mathcal{Y}_{\mathbb{C}}(-), \mathcal{F}) \xRightarrow{\simeq} \mathcal{F},$$

(*where* $\mathrm{Nat}_{\mathbb{C}^{\mathrm{op}}}(-,-)$ *is the mapping ∞-category in* $\mathrm{Fun}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty)$) *which is natural in $\mathcal{F}$.*

2 Preliminaries

In this section we will mainly gather the main definitions of the theory of scaled simplicial sets as presented by Lurie [19].

Definition 2.1 A scaled simplicial set (X, T_X) consists in a simplicial set X together with a collection of 2-simplices (also called triangles) T_X which contains *every degenerate* triangle. We call the elements of T_X the *thin triangles* of X. A morphism of scaled simplicial sets $f : (X, T_X) \to (Y, T_Y)$ is a map of simplicial sets $f : X \to Y$ such that $f(T_X) \subseteq T_Y$. We denote the corresponding category of scaled simplicial sets by $\mathrm{Set}_\Delta^{\mathbf{sc}}$.

Notation 2.2 Given a simplicial set A we have two canonical ways of viewing it as a scaled simplicial set:

- We define $A_\flat = (A, \flat)$ where $\flat$ is the collection consisting only in the degenerate triangles of A.
- We define $A_\sharp = (A, \sharp)$ where $\sharp$ is the collection consisting in every triangle of A.

Definition 2.3 The set of *generating scaled anodyne maps* $\mathbf{S}$ is the set of maps of scaled simplicial sets consisting of

(i) the inner horns inclusions

$$\left(\Lambda_i^n, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}\right), \quad n \geq 2,\ 0 < i < n;$$

(ii) the map

$$(\Delta^4, T) \to (\Delta^4, T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \overset{\mathrm{def}}{=} \{\Delta^{\{0,2,4\}}, \Delta^{\{1,2,3\}}, \Delta^{\{0,1,3\}}, \Delta^{\{1,3,4\}}, \Delta^{\{0,1,2\}}\};$$

(iii) the set of maps

$$(\Lambda^n_0 \amalg_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}) \to (\Delta^n \amalg_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}), \quad n \geqslant 3.$$

A general map of a scaled simplicial set is said to be *scaled anodyne* if it belongs to the weakly saturated closure of $\mathbf{S}$.

Definition 2.4 A scaled simplicial set (X, T_X) is said to be an ∞-bicategory if it has the right lifting property against the class of scaled anodyne maps in Definition 2.3. In this case, we view the 2-simplices of T_X as the collection of commuting triangles.

Definition 2.5 Given an ∞-bicategory (X, T_X) we can construct an ∞-category $X^{\leqslant 1}$ by considering the subsimplicial set of X given by those n-simplices $\sigma : \Delta^n \to X$ such that each 2-dimensional face belongs to T_X. We call $X^{\leqslant 1}$ the underlying ∞-category of (X, T_X). We similarly define $X^{\simeq}$ as the underlying ∞-groupoid of $X^{\leqslant 1}$.

Definition 2.6 We denote by Cat_Δ^+ the category of Set_Δ^+-enriched categories (i.e., categories enriched in marked simplicial sets). We note that we can view the category of (strict) 2-categories $2\,\mathrm{Cat}$ as a full subcategory of Cat_Δ^+ by applying the nerve functor Hom-wise and marking the equivalences in each mapping category.

Proposition 2.7 *There is a left-proper, combinatorial model structure on* Cat_Δ^+ *such that:*

(W) *The weak equivalences are those enriched functors which are essentially surjective on homotopy categories and induce weak equivalences on all mapping marked simplicial sets.*[4]

(C) *The cofibrations are the smallest weakly saturated class containing* $\varnothing \to [0]_{\mathrm{Set}_\Delta^+}$, *and each inclusion* $[1]_A \to [1]_B$ *where* $A \to B$ *is a generating cofibration for* Set_Δ^+.

Proof This is a special case of [18, A.3.2.4]. □

Definition 2.8 Let I be a linearly ordered finite set. We define a 2-category $\mathbb{O}^I$ as follows:

- The objects of $\mathbb{O}^I$ are the elements of I.
- The category $\mathbb{O}^I(i,j)$ of morphisms between objects $i, j \in I$ is defined as the poset of finite sets $S \subseteq I$ such that $\min(S) = i$ and $\max(S) = j$ ordered by inclusion.
- The composition functors are given, for $i, j, l \in I$, by

$$\mathbb{O}^I(i,j) \times \mathbb{O}^I(j,l) \to \mathbb{O}^I(i,l), \quad (S,T) \mapsto S \cup T.$$

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[4] See [18, Section 3.1] for a model structure on marked simplicial sets.

When $I = [n]$, we denote $\mathbb{O}^I$ by $\mathbb{O}^n$. Note that the $\mathbb{O}^n$ form a cosimplicial object in 2Cat, which we denote by $\mathbb{O}^\bullet$.

Definition 2.9 The map

$$\Delta \xrightarrow{\mathfrak{C}} \mathrm{Cat}_\Delta^+,$$

which sends $[n]$ to $\mathbb{O}^n$ gives us a cosimplicial object in Cat_Δ^+. We can moreover send the thin 2-simplex $\Delta^2_\sharp$ to $\mathfrak{C}[\Delta^2]$ equipped with maximally marked mapping spaces. The usual machinery of nerve and realisation then gives us adjoint functors

$$\mathfrak{C}^{\mathrm{sc}} : \mathrm{Set}_\Delta^{\mathbf{sc}} \rightleftarrows \mathrm{Cat}_\Delta^+ : \mathrm{N}^{\mathbf{sc}},$$

which we will call the *scaled nerve* and *scaled rigidification*.

Theorem 2.10 *There is a left proper, combinatorial model structure on* $\mathrm{Set}_\Delta^{\mathbf{sc}}$ *in which:*

- (W) *The weak equivalences are the morphisms* $f : A \to B$ *such that* $\mathfrak{C}^{\mathrm{sc}}[f] : \mathfrak{C}^{\mathrm{sc}}[A] \to \mathfrak{C}^{\mathrm{sc}}[B]$ *is an equivalence in* Cat_Δ^+.
- (C) *The cofibrations are the monomorphisms.*

Moreover, the fibrant objects in this model structure are the ∞*-bicategories, and the adjunction*

$$\mathfrak{C}^{\mathrm{sc}} : \mathrm{Set}_\Delta^{\mathbf{sc}} \rightleftarrows \mathrm{Cat}_\Delta^+ : \mathrm{N}^{\mathbf{sc}}$$

is a Quillen equivalence.

Proof This is [19, Theorem A.3.2.4]. The characterisation of fibrant objects is [12, Theorem 5.1]. □

Definition 2.11 We say that a map of scaled simplicial sets is a *bicategorical equivalence* if it is a weak equivalence in the model structure given in Theorem 2.10. Similarly, call the fibrations in the model structure of scaled simplicial sets *bicategorical fibrations*.

Remark 2.12 In [12], the authors characterise the model structure on scaled simplicial sets as a Cisinski–Olschok model structure. This in turn implies that a map between fibrant scaled simplicial sets is a bicategorical fibration if and only if it is an isofibration and it has the right lifting property against the class of scaled anodyne maps.

Definition 2.13 Given a pair of scaled simplicial sets X, Y we denote by $\mathrm{Fun}(X, Y)$ the scaled simplicial set determined by the universal property

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{sc}}}(K, \mathrm{Fun}(X, Y)) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{sc}}}(K \times X, Y),$$

where $K \times X$ denotes the cartesian product of scaled simplicial sets.

Definition 2.14 Let $\mathbb{C}$ be an ∞-bicategory. Given an object $y \in \mathbb{C}$ we define a scaled simplicial set $\mathbb{C}_{\nearrow y}$ as follows. The data of an n-simplices $\Delta^n \to \mathbb{C}_{\nearrow y}$ is given by a map $\sigma : \Delta^{n+1} \to \mathbb{C}$ such that $\sigma(n+1) = y$. The inclusion $d_{n+1} : \Delta^n \to \Delta^{n+1}$ induces a map

$$\pi : \mathbb{C}_{\nearrow y} \to \mathbb{C},$$

which we use to declare a triangle in $\mathbb{C}_{\nearrow y}$ to be thin if and only if its image under π is thin in $\mathbb{C}$. It follows from [12, Proposition 2.33] that the fibre of π at an object $x \in \mathbb{C}$ is a model for $\mathbb{C}(x,y)$, the mapping ∞-category.

3 The model structure

Definition 3.1 A *marked biscaled* simplicial set (**MB** simplicial set) is given by

- a simplicial set X,
- a collection of edges $E_X \in X_1$ containing all degenerate edges,
- a collection of triangles $T_X \in X_2$ containing all degenerate triangles,
- a collection of triangles $C_X \in X_2$ such that $T_X \subseteq C_X$,

where we will refer to the elements of T_X as *thin triangles*, and we will refer to the elements of C_X as *lean triangles*. We will denote such objects as triples $(X, E_X, T_X \subseteq C_X)$. A map $(X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ is given by a map of simplicial sets $f : X \to Y$ compatible with the collections of edges and triangles above. We denote by $\mathrm{Set}_\Delta^{\mathbf{mb}}$ the category of **MB** simplicial sets.

Notation 3.2 Let $(X, E_X, T_X \subseteq C_X)$ be an **MB** simplicial set. Suppose that the collection E_X consists only of degenerate edges. Then we fix the notation $(X, E_X, T_X \subseteq C_X) = (X, \flat, T_X \subseteq E_X)$ and do similarly for T_X. If C_X consists only of degenerate triangles we fix the notation $(X, E_X, T_X \subseteq C_X) = (X, E_X, \flat)$. In an analogous fashion we will use the symbol "$\sharp$" to denote a collection containing all edges (or all triangles). Finally if $T_X = C_X$ then we will employ the notation (X, E_X, T_X).

Remark 3.3 We will often abuse notation when defining the collections E_X (resp. T_X or C_X) and just specify its nondegenerate edges (resp. triangles).

Definition 3.4 The set of *generating marked-biscaled-anodyne maps* **MB** is the set of maps of **MB** simplicial sets consisting of

(A1) the inner horn inclusions

$$(\Lambda^n_i, \flat, \flat \subset \{\Delta^{\{i-1,i,i+1\}}\}) \to (\Delta^n, \flat, \flat \subset \{\Delta^{\{i-1,i,i+1\}}\}), \quad n \geqslant 2,\ 0 < i < n;$$

(A2) the map

$$(\Delta^4, \flat, \flat \subset T) \to (\Delta^4, \flat, \flat \subset T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \overset{\text{def}}{=} \{\Delta^{\{0,2,4\}}, \Delta^{\{1,2,3\}}, \Delta^{\{0,1,3\}}, \Delta^{\{1,3,4\}}, \Delta^{\{0,1,2\}}\};$$

(A3) the set of maps

$$(\Lambda^n_0, \{\Delta^{\{0,1\}}\}, \{\Delta^{\{0,1,n\}}\}) \to (\Delta^n, \{\Delta^{\{0,1\}}\}, \{\Delta^{\{0,1,n\}}\}), \quad n \geqslant 2;$$

(A4) the inclusion of the initial vertex

$$(\Delta^0, \sharp, \sharp) \to (\Delta^1, \sharp, \sharp);$$

(S1) the map

$$(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{1,2\}}\}, \sharp) \to (\Delta^2, \sharp, \sharp);$$

(S2) the map

$$(\Delta^2, \flat, \flat \subset \sharp) \to (\Delta^2, \flat, \sharp);$$

(E) for every Kan complex K, the map

$$(K, \flat, \sharp) \to (K, \sharp, \sharp).$$

A map of **MB** simplicial sets is said to be **MB**-anodyne if it belongs to the weakly saturated closure of **MB**.

Remark 3.5 Let $p : (X, E_X, T_X \subseteq C_X) \to (S, E_S, T_S \subseteq C_S)$ be a morphism of **MB** simplicial sets. We will informally think of the collection of E_X as representing p-cocartesian edges. The collection C_X is understood to represent cartesian 2-morphisms, while the collection T_X is simply interpreted as representing commuting triangles. Equipped with this intuition let us clarify the meaning of our anodyne maps:

(A1) Having the right lifting property against this class guarantees the existence of enough cartesian 2-morphisms.

(A2) This class expresses a saturation property of cartesian 2-morphisms.

(A3) This class guarantees the marked morphisms to be p-cocartesian with respect to the given thin triangles.

(A4) This guarantees the existence of enough p-cocartesian lifts.

(S1) This class enforces that p-cocartesian morphisms compose across thin triangles.

(S2) This expresses that lean triangles lying over thin triangles are themselves thin. In other words, a cartesian 2-morphism lying over an invertible 2-morphism is itself invertible.

(E) This class simply expresses that equivalences must always be marked.

Remark 3.6 In [6] we also introduced the notion of a **MB**-anodyne map when dealing with the theory of $(1,0)$-fibrations. We would like to point out that both classes of maps are different but we are using the same name to avoid the overly cumbersome notation $\mathbf{MB}^{01}$-anodyne.

Definition 3.7 A map of **MB** simplicial sets is said to be an **MB**-*fibration* if it has the right lifting property against the class of **MB**-anodyne maps.

Lemma 3.8 *Let* $p : X \to S$ *be an* **MB***-fibration. Then for every* $s \in S$ *the fibre, over* s*,*

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "s"] & S
\end{tikzcd}$$

is of the form (X_s, E_s, T_s) *(see Notation 3.2), where* (X_s, T_s) *is an* ∞*-bicategory and where* E_s *is precisely given by the equivalences.*

Proof Observe that in X_s the lean and thin triangles coincide since in a **MB**-fibration a lean triangle lying over a thin triangle is itself thin. It follows that X_s has the right lifting property against the class of scaled anodyne maps and thus it is an ∞-bicategory. Note that by definition the equivalences must be marked in X_s. Moreover, since X_s lifts against the class of maps (A3) one checks easily that marked morphisms are equivalences. □

Lemma 3.9 *The morphism of* **MB***-simplicial sets* $(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{0,2\}}\}, \sharp) \to (\Delta^2, \sharp, \sharp)$ *is* **MB***-anodyne.*

Proof The proof is dual to [6, Lemma 3.11]. □

Lemma 3.10 *The morphism of* **MB** *simplicial sets*

$$\iota : (\Delta^3, \Delta^{\{0,1\}}, \{\Delta^{\{0,1,3\}}\} \subset U_0) \to (\Delta^3, \Delta^{\{0,1\}}, \{\Delta^{\{0,1,3\}}\} \subset \sharp)$$

is **MB***-anodyne where* U_0 *is the collection of all* 2*-faces except* $\Delta^{\{1,2,3\}}$*.*

Proof Let $S = (S, E_S, T_S \subset \sharp)$ be an **MB** simplicial set and let $p : X \to S$ be an **MB**-fibration. We will show that p has the right lifting property against the map ι. Once this claim is established we will factor ι as an **MB**-anodyne morphism followed by an **MB**-fibration where B is of the form $(S, E_S, T_S \subset \sharp)$, that is,

$$\iota : (\Delta^3, \Delta^{\{0,1\}}, \{\Delta^{\{0,1,3\}}\} \subset U_0) := A \xrightarrow{\alpha} X \xrightarrow{p} (\Delta^3, \Delta^{\{0,1\}}, \{\Delta^{\{0,1,n\}}\} \subset \sharp) = B.$$

It will then follow that we can produce a solution to the lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\iota"] & X \arrow[d] \\
B \arrow[r, "\mathrm{id}_B"] \arrow[ur, dotted] & B
\end{tikzcd}$$

which exhibits ι as a retract of the **MB**-anodyne map α and thus concluding the proof.

In order to complete the proof we must show the claim. Suppose we are given a lifting problem

$$\begin{tikzcd}
A \arrow[r, "\sigma"] \arrow[d, "\iota"] & X \arrow[d, "p"] \\
B \arrow[r] & S
\end{tikzcd}$$

Let $\sigma(1 \to 2) = u$, $\sigma(2 \to 3) = v$ and $\sigma(1 \to 3) = \omega$. Since p is an **MB**-fibration we can solve the lifting problem

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^2 \arrow[r, "d_0(p(\sigma))"'] \arrow[ur, dotted, "\varphi"] & S
\end{tikzcd}$$

and produce a lean triangle φ. We consider a subsimplicial set of $Q \subset \Delta^4$ consisting in the

- the face missing the vertex 2,
- the face missing the vertex 4,
- the 2-dimensional face $\Delta^{\{2,3,4\}}$.

We then produce a map $\theta : Q \to X$ as follows:

- We map the face missing the vertex 2 via σ.
- We map the face missing the vertex 4 via $s_1(d_3(\sigma))$.
- We map $\Delta^{\{2,3,4\}}$ via φ.

We equip Q with the induced decorations. It follows that we can extend Q to a map $\Xi : \Delta^4 \to X$ lying over $s_1(p(\sigma))$. Since p has the right lifting property against the morphism (A2) we see that $d_0(\sigma)$ is lean if and only if the image of $\Delta^{\{1,2,4\}}$ under Ξ is thin in X.

We observe that in $d_1(\Xi)$ every face is lean scaled except possibly the face missing the vertex 2. Again, as a consequence of (A2) it follows that this face must also be lean scaled. Moreover this face lies over a thin simplex of S so it must be itself thin. From now on we can focus our attention to $d_3(\Xi)$.

In $d_3(\Xi) = \rho$ every face is thin scaled except possibly the face missing the vertex 0 and the edge $0 \to 1$ is marked. One checks easily that the pullback of p along the simplex thin $d_2(p(\sigma))$ simplex yields a fibration of ∞-bicategories $X^{'} \to (\Delta^2, \sharp)$ where we can identify the image of $d_0(\rho)$ with a morphism in the mapping ∞-category of $X^{'}$. One easily shows that this morphism is an equivalence and thus it must be thin. □

Definition 3.11 We say that a map of **MB** simplicial sets is a cofibration if its underlying map of simplicial sets is a monomorphism. One can easily verify that the class of cofibrations is generated by

(C1) the boundary inclusions $(\partial\Delta^n, \flat, \flat) \to (\Delta^n, \flat, \flat)$ for $n \geqslant 0$,
(C2) the map $(\Delta^1, \flat, \flat) \to (\Delta^1, \sharp, \flat)$,
(C3) the map $(\Delta^2, \flat, \flat) \to (\Delta^2, \flat, \flat \subset \sharp)$,
(C4) the map $(\Delta^2, \flat, \flat \subset \sharp) \to (\Delta^2, \flat, \sharp)$.

Proposition 3.12 *Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subset C_Y)$ be a cofibration of **MB** simplicial sets and $g : (A, E_A, T_A \subseteq C_A) \to (B, E_B, C_B \subseteq T_B)$ be an **MB**-anodyne morphism. Then the pushout-product*

$$f \wedge g : X \times B \amalg_{X \times A} A \times Y \longrightarrow Y \times B$$

*is again **MB**-anodyne.*

Proof The proof is almost identical to the proof of [6, Proposition 3.14] and left as an exercise. □

Remark 3.13 Observe that given a pair of **MB** simplicial sets X, Y we can produce a functor **MB** simplicial set $\mathrm{Fun}^{\mathbf{mb}}(X,Y)$ in an obvious way via the isomorphism

$$\mathrm{Hom}(A,\mathrm{Fun}^{\mathbf{mb}}(X,Y)) \simeq \mathrm{Hom}(A\times X,Y),$$

where Hom denotes the mapping set in the category $\mathrm{Set}_{\Delta}^{\mathbf{mb}}$.

Corollary 3.14 *Let $p:Y\to S$ be an **MB**-fibration. Then for every **MB** simplicial set X the induced map*

$$\mathrm{Fun}^{\mathbf{mb}}(X,Y) \longrightarrow \mathrm{Fun}^{\mathbf{mb}}(X,S)$$

*is an **MB**-fibration.*

Proof It follows from Proposition 3.12 after looking at the adjoint lifting problems. □

Definition 3.15 Let $p:Y\to S$ be an **MB**-fibration and consider a map of **MB** simplicial sets $q:X\to S$. We define an ∞-bicategory of functors over S as the pullback

$$\begin{tikzcd}
\mathrm{Map}_S(X,Y) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathbf{mb}}(X,Y) \arrow[d] \\
\Delta^0 \arrow[r, "q"] & \mathrm{Fun}^{\mathbf{mb}}(X,S)
\end{tikzcd}$$

Definition 3.16 Given a scaled simplicial set (S,U) we define the category $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ of **MB** simplicial sets over $(S,\sharp,U\subset\sharp)$ as follows:

- The objects are maps $p:(X,E_X,T_X\subset C_X)\to(S,\sharp,U\subset\sharp)$.
- A morphism from $p:(X,E_X,T_X\subset C_X)\to(S,\sharp,U\subset\sharp)$ to $q:(Y,E_Y,T_Y\subset C_Y)\to(S,\sharp,U\subset\sharp)$ is given by a map $f:(X,E_X,T_X\subseteq C_X)\to(Y,E_Y,T_Y\subset C_Y)$ such that $q\circ f=p$.

An object of $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ is said to be a U-local $(0,1)$-fibration if the corresponding map of **MB** simplicial sets is an **MB**-fibration.

Definition 3.17 A morphism $f:A\to B$ in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ is said to be

- a cofibration if its underlying map of **MB** simplicial sets is a cofibration;
- a weak equivalence if for every U-local $(0,1)$-fibration $p:X\to S$ the associated map of ∞-bicategories

$$f^*:\mathrm{Map}_S(B,X) \xrightarrow{\simeq} \mathrm{Map}_S(A,X)$$

is a bicategorical equivalence;

- a trivial fibration if it has the right lifting property against every cofibration in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$;
- a trivial cofibration if it is *both* a weak equivalence and a cofibration.

Lemma 3.18 *Let $f:A\to B$ be a morphism over S in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ such that f is a trivial fibration. Then f is a weak equivalence.*

Proof Note that since f is a trivial fibration we can construct a section $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. Moreover, we can further produce a marked homotopy $A \times (\Delta^1)^\sharp \to A$ between the identity morphism on A and the composite $g \circ f$ which is compatible with the projection map from A to S. This pair of homotopy inverse morphisms thus define the desired equivalence on mapping ∞-bicategories

$$f^* : \mathrm{Map}_S(B, X) \xrightarrow{\simeq} \mathrm{Map}_S(A, X)$$

and thus f is a weak equivalence. □

Proposition 3.19 *Let $f : A \to B$ be a morphism in $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$. Given a U-local $(0,1)$-fibration $p : X \to S$ let us consider the induced functor on mapping ∞-bicategories*

$$f^* : \mathrm{Map}_S(B, X) \longrightarrow \mathrm{Map}_S(A, X).$$

Then it follows that

(i) *if f is **MB**-anodyne then f^* is a trivial fibration of scaled simplicial sets*;

(ii) *if f is a cofibration then f^* is a fibration of ∞-bicategories*;

(iii) *if f is a trivial cofibration then f^* is a trivial fibration of scaled simplicial sets.*

Proof The first statement follows directly from Proposition 3.12. To see that (ii) holds we observe that again by Proposition 3.12 that f^* has the right lifting property against all scaled anodyne maps. Since the marked morphisms in the mapping ∞-bicategories are equivalences it follows that f^* is an isofibration. Therefore by Remark 2.12 it follows that f^* is a fibration of scaled simplicial sets. The final claim follows from (ii) together with the definition of the class of weak equivalences. □

Lemma 3.20 *Let us consider a pushout diagram, in $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$,*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "v"] & B \arrow[d] \\
C \arrow[r, "i"] & P
\end{tikzcd}$$

where u is a weak equivalence and v is a cofibration. Then $i : B \to C$ is also a weak equivalence.

Proof Given a U-local $(0,1)$-fibration $p : X \to S$ we observe that since v is a cofibration we obtain a pullback diagram of ∞-bicategories

$$\begin{tikzcd}
\mathrm{Map}_S(P, X) \arrow[r, "i^*"] \arrow[d] & \mathrm{Map}_S(C, X) \arrow[d, "v^*"] \\
\mathrm{Map}_S(B, P) \arrow[r, "u^*"] & \mathrm{Map}_S(A, X)
\end{tikzcd}$$

which shows that i^* is a bicategorical equivalence and consequently we see that $i : B \to C$ is a weak equivalence in $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$. □

Proposition 3.21 *An object $p : X \to S$ in $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ has the right lifting property against the class of trivial cofibrations if and only if it is a U-local $(0,1)$-fibration.*

Proof Observe that due to (i) in Proposition 3.19 it follows that every **MB**-anodyne morphism is a trivial cofibration. Therefore, any object having the right lifting property against trivial cofibrations must be a U-local $(0,1)$-fibration. To check the converse we consider a U-local $(0,1)$-fibration $p : X \to S$ and a trivial cofibration $f : A \to B$. Then, in order to produce a solution to the lifting problem

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

we observe that since the map $f^* : \mathrm{Map}_S(B, X) \to \mathrm{Map}_S(A, X)$ is a trivial fibration and in particular surjective, we can pick a preimage of $\alpha \in \mathrm{Map}_S(A, X)$ which yields the solution to our problem. □

Definition 3.22 Let $p : X \to S$ be a U-local $(0,1)$-fibration. Consider an object $A \to S$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/(S,\sharp,U\subset\sharp)}$. We set the following notation:

(1) We denote by $\mathrm{Map}_S^{\mathrm{th}}(A, X)$ the underlying ∞-category (cf. Definition 2.5) of the mapping ∞-bicategory.

(2) We denote by $\mathrm{Map}_S^{\simeq}(A, X)$ the underlying groupoid of the mapping ∞-bicategory.

Proposition 3.23 *Let* $f : X \to Y$ *be a morphism in* $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/(S,\sharp,U\subset\sharp)}$ *where both* X *and* Y *are* U*-local* $(0,1)$*-fibrations. Then the following are equivalent*:

(i) *For every* U*-local* $(0,1)$*-fibration* $Z \to S$ *the induced map* $f^* : \mathrm{Map}_S(Y, Z) \to \mathrm{Map}_S(X, Z)$ *is an equivalence of* ∞*-bicategories.*

(ii) *For every* U*-local* $(0,1)$*-fibration* $Z \to S$ *the induced map* $f^* : \mathrm{Map}_S^{\leq 1}(Y, Z) \to \mathrm{Map}_S^{\leq 1}(X, Z)$ *is an equivalence of* ∞*-categories.*

(iii) *For every* U*-local* $(0,1)$*-fibration* $Z \to S$ *the induced map* $f^* : \mathrm{Map}_S^{\simeq}(Y, Z) \to \mathrm{Map}_S^{\simeq}(X, Z)$ *is a homotopy equivalence of groupoids.*

(iv) *The exists a morphism* $g : Y \to X$ *over* S*, which is a homotopy inverse to* f*.*

(v) *For every* $s \in S$ *the induced morphism on fibres* $f_s : X_s \to Y_s$ *is a bicategorical equivalence.*

Proof The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear. We commence the proof by showing that (iii) $\Longrightarrow$ (iv). We consider the homotopy equivalence $\mathrm{Map}_S^{\simeq}(Y, X) \to \mathrm{Map}_S^{\simeq}(X, X)$ and pick an object $g \in \mathrm{Map}_S^{\simeq}(Y, X)$ such that $g \circ f \simeq \mathrm{id}_X$. To show that g is the desired homotopy inverse to f we need to show that $f \circ g \simeq \mathrm{id}_Y$. To see this we see that the map $\mathrm{Map}_S^{\simeq}(Y, Y) \to \mathrm{Map}_S^{\simeq}(X, Y)$ maps both $f \circ g$ and id_Y to morphisms which are equivalent to f. Consequently we see that $f \circ g \simeq \mathrm{id}_Y$.

Observe that (iv) $\Longrightarrow$ (v) follows from the fact that since our homotopies are fibrewise they descend to equivalences on the corresponding fibres.

In order to exhibit that (v) $\Longrightarrow$ (i) we use the small object argument to factor $f : X \to Y$ as a composite $X \to \widehat{X} \xrightarrow{\hat{f}} Y$ where the first morphism is **MB**-anodyne (and therefore a weak equivalence) and the second morphism has the right lifting property against the class of **MB**-anodyne morphisms.

In particular it follows that $\widehat{X} \to S$ is again a U-local $(0,1)$-fibration and that the induced maps on fibres $\hat{f}_s : \widehat{X}_s \to Y_s$ are bicategorical equivalences. These assumptions imply that $\hat{f}$ is itself a trivial fibration (see Proposition 4.24 for a proof) and so the claim follows. □

Lemma 3.24 *Given an object $A \to S$ in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$, the projection map $\pi_A : A \times (\Delta^n, \sharp, \sharp) \to A$ is a weak equivalence.*

Proof Let $\iota : \Delta^0 \to (\Delta^n, \sharp, \sharp)$ be the inclusion of the initial vertex. It is an easy exercise to show that ι is **MB**-anodyne. It then follows from Proposition 3.12 that $\iota_A : A \to A \times (\Delta^n, \sharp, \sharp)$ is also **MB**-anodyne. We conclude the proof by observing that $\pi_A \circ \iota_A = \mathrm{id}_A$. □

Theorem 3.25 *Let (S, U) be a scaled simplicial set. Then there exists a left proper combinatorial simplicial model structure on $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$, which is characterised uniquely by the following properties*:

- (C) *A morphism $f : X \to Y$ in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ is a cofibration if and only if f induces a monomorphism on the underlying simplicial sets.*
- (F) *An object $p : X \to S$ in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ is fibrant if and only if it is a U-local $(0,1)$-fibration.*

Proof The proof is totally analogous to the proof of Theorem 3.42 in [6] where we verify that the conditions of [18, A.2.6.10] are satisfied. □

Remark 3.26 We will refer to the model structure in the previous theorem as model structure on U-local $(0,1)$-cartesian fibrations over (S, U).

Definition 3.27 Let $K = (K, E_K) \in \mathrm{Set}_{\Delta}^{+}$ and let $p : X \to S$ be an object in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$. We define the tensor $K \otimes X$ as $I(K) \times X \to X \to S$ where $I(K) = (K, E_K, \sharp)$. Similarly, we define the cotensor X^K by declaring that a map **MB**-simplicial sets $\varphi : Y \to X^K$ over $\bar{\varphi} : Y \to S$ to be equivalent to the data of a commutative diagram

$$\begin{tikzcd}
(K, E_K, \sharp) \times Y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
Y \arrow[r, "\bar{\varphi}"] & S
\end{tikzcd}$$

Theorem 3.28 *The model category $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ is a $\mathrm{Set}_{\Delta}^{+}$-enriched model category.*

Proof It is clear that the construction $\mathrm{Map}_S^{\leqslant 1}(-,-)$ equips $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ with the structure of a $\mathrm{Set}_{\Delta}^{+}$-enriched model category. Since the tensor preserves colimits in both variables separately it will be enough to show that given $i : L \to K$ a cofibration in $\mathrm{Set}_{\Delta}^{+}$ and a cofibration $f : X \to Y$ in $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$ the corresponding pushout-product map

$$i \wedge f : L \otimes Y \amalg_{L \otimes X} K \otimes X \to K \otimes Y$$

is again cofibration which is a weak equivalence whenever i or f is. Note that $i \wedge f$ is clearly a cofibration so we can focus our attention in proving the weak equivalence part of the claim.

First, let us recall that given an anodyne morphism of marked simplicial sets $A \to B$ it follows that $I(A) \to I(B)$ is **MB**-anodyne. It then follows as a consequence Proposition 3.12 that we can assume that i and f are morphisms among fibrant objects in the corresponding model structures.

We note that given a pair of fibrant objects L and $p : X \to S$ it follows that $L \otimes X$ is again fibrant. To finish the proof we assume that $f : X \to Y$ is a weak equivalence (the case for i is totally analogous). Then it follows that for every $s \in S$ the map $(L \otimes X)_s \to (L \otimes Y)_s$ is identified with the map

$$L \times X_s \xrightarrow{\simeq} L \times Y_s,$$

which is a bicategorical equivalence by assumption. It follows that the map $L \otimes X \to L \otimes Y$ is a weak equivalence in $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}$. We can now consider a pushout diagram

$$\begin{tikzcd}
L \otimes X \arrow[r, "\simeq"] \arrow[d] & L \otimes Y \arrow[d] \\
K \otimes X \arrow[r, "\simeq"] & L \otimes Y \amalg_{L\otimes X} K \otimes X
\end{tikzcd}$$

Moreover, using a similar argument as before we see that $K \otimes X \xrightarrow{\simeq} K \otimes Y$ is also a weak equivalence. The claim now follows from two-out-of-three. □

Proposition 3.29 *Let $f : (S,U) \to (T,V)$ be a map of scaled simplicial sets. Then postcomposition with f induces a left Quillen functor*

$$f_! : \left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)} \rightleftarrows \left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(T,\sharp,V\subset\sharp)} : f^*,$$

which is left adjoint to the pullback functor f^*.

Proof It is clear that $f_!$ preserves cofibrations. To finish the proof we only need to show that $f_!$ preserves weak equivalences. Given $\iota : A \to B$ and a fibrant object $p : Y \to S'$ we observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_S(B, f^*Y) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{Map}_S(A, f^*Y) \arrow[d, "\simeq"] \\
\mathrm{Map}_T(i_!B, Y) \arrow[r] & \mathrm{Map}_T(i_!A, Y)
\end{tikzcd}$$

so the conclusion holds by two-out-of-three. □

We finish this section by comparing the model structure in Theorem 3.25 with the model structure constructed in Theorem 3.2.6 in [19], and in the case where $(S,U) = (\Delta^0, \sharp)$ with the model structure on scaled simplicial sets given in Theorem 2.10.

Definition 3.30 Let (S,U) be a scaled simplicial set and consider the category $\left(\mathrm{Set}_\Delta^{+}\right)_{/(S,U)}$ of marked simplicial sets over S. We have a functor

$$R : \left(\mathrm{Set}_\Delta^{+}\right)_{/(S,U)} \to \left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,U\subset\sharp)}, \quad (X, E_X) \mapsto (X, E_X, T_X \subset \sharp),$$

where T_X consists in those triangles in X lying over thin triangles in S.

Theorem 3.31 *Let (S,U) be a scaled simplicial set and consider an object (X,E_X) in $\left(\mathrm{Set}_\Delta^+\right)_{/(S,U)}$. Then $X=(X,E_X)$ is $\mathfrak{P}_S$-fibred (see Definition 3.2.1 and Example 3.2.9 in [19]) if and only if $R(X)$ is a U-local $(0,1)$-fibration. Moreover, if Y is a U-local $(0,1)$-fibration over S such that every triangle of Y is lean then there exists a $\mathfrak{P}_S$-fibred object T such that $R(T)=Y$.*

Proof Let us suppose that X is $\mathfrak{P}_S$-fibred and let us show that $R(X)$ defines a U-local $(0,1)$-fibration. We will show that $R(X)$ has the right lifting property against the class of **MB**-anodyne morphisms. To this end we recall that an object in $\left(\mathrm{Set}_\Delta^+\right)_{/S}$ is $\mathfrak{P}_S$-fibred if and only if it has the right lifting property against the class of $\mathfrak{P}_S$-anodyne morphisms described in Definition 3.2.10 in [19]. We first check that $R(X)$ has the right lifting property against the class of maps given in (E). Indeed given a map from a Kan complex $K\to R(X)$ we can use the morphisms of type (A_1) in [19, Definition 3.2.10] to see that every morphism of K maps to a marked edge in $R(X)$. Similarly, our construction of R guarantees the $R(X)$ has the right lifting property against morphisms of type (S2). The rest of the lifting problems follow immediately from the definition of the class of $\mathfrak{P}_S$-anodyne morphisms.

The converse follows by a similar argument. To finish the proof we suppose that we are given U-local $(0,1)$-fibration of the form $Y=(Y,E_Y,T_Y\subset\sharp)$. Then (S2) implies that T_Y is simply the collection of triangles lying over thin triangles in S. Therefore, it we can consider $\widehat{Y}=(Y,E_Y)$ and observe that $R(\widehat{Y})=Y$. The previous part of the proof shows that $\widehat{Y}$ must be $\mathfrak{P}_S$-fibred. □

Proposition 3.32 *Let $p:X\to S$ be a U-local $(0,1)$-fibration. Then for every $s\in S$ the fibre X_s is an ∞-category if and only if every triangle of X is lean.*

Proof It is clear that if every triangle of X is lean the fibres must be ∞-categories. For the converse, let us assume that for every $s\in S$ the fibre X_s is an ∞-category. Let $\sigma:\Delta^2\to X$ and assume that $p(\sigma)=s_1(p(f))$ for some edge $\Delta^1\to S$. We pick a marked edge lying over $p(f)$ and construct a 3-simplex $\theta:\Delta^3\to X$ lying over $s_1(p(\sigma))$ such that $\theta(0\to 1)$ is our chosen marked edge and such that every face is lean except possibly $d_1(\theta)=\sigma$. We conclude that σ is lean since p has the right lifting property against morphisms of type (A2).

For the general case we take a marked edge lying over $p(1\to 2)$ and construct another 3-simplex $\Xi:\Delta^3\to X$ such that $\Xi(1\to 2)$ is our chosen marked edge and such that $d_2(\Xi)=\sigma$. It follows that every face of Ξ is lean except possibly the face skipping the vertex 1 and the face missing the vertex 2. We conclude that the face missing the vertex 1 is lean since it falls in the previous case. It then follows that σ is lean. □

Definition 3.33 Let $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(\Delta^0,\sharp,\sharp)}=\mathrm{Set}_\Delta^{\mathbf{mb}}$. We consider a functor $R:\mathrm{Set}_\Delta^{\mathbf{mb}}\to\mathrm{Set}_\Delta^{\mathbf{sc}}$ which sends an **MB** simplicial set $(X,E_X,T_X\subseteq C_X)$ to the scaled simplicial set $R(X,E_X,T_X\subseteq C_X)=(X,C_X)$. This functor has a left adjoint $L:\mathrm{Set}_\Delta^{\mathbf{sc}}\to\mathrm{Set}_\Delta^{\mathbf{mb}}$ which is given by $L(Y,T_Y)=(Y,\flat,\flat\subset T_Y)$.

Theorem 3.34 *The functor $L:\mathrm{Set}_\Delta^{\mathbf{sc}}\to\mathrm{Set}_\Delta^{\mathbf{mb}}$ is a left Quillen equivalence.*

Proof It is clear that L preserves cofibrations and colimits so in order to show that L is a left Quillen functor it will be enough to show that L preserves weak equivalences. Observe that given an object

$(X, E_X, T_X \subset C_X)$ we have an anodyne morphism $(X, E_X, T_X \subset C_X) \to (X, E_X, C_X)$ since every triangle in Δ^0 is thin. Using this observation it is easy to see that L maps scaled anodyne morphisms to trivial cofibrations in $\mathrm{Set}_\Delta^{\mathbf{mb}}$. This shows that it is enough to show that L preserves weak equivalences among fibrant scaled simplicial sets. However, this is clear since a weak equivalence between fibrant scaled simplicial sets has an inverse up to homotopy.

To conclude that L is a left Quillen equivalence we first observe that $R \circ L = \mathrm{id}$ by definition. This shows that it is only left to show that for any fibrant object $(Y, E_Y, T_Y \subseteq C_Y) \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ the counit map $LR(Y) \to Y$ is a weak equivalence. Note that since our chosen **MB** simplicial set is fibrant $C_Y = T_Y$ and (Y, T_Y) is an ∞-bicategory. In particular, we see that $LR(Y) \to Y$ is the weakly saturated class of morphisms of type (E) and (S2) in Definition 3.4. □

3.1 Marked-scaled simplicial sets

We saw in Theorem 3.34 that the model structure of **MB** simplicial sets over Δ^0 is Quillen equivalent to the model structure on scaled simplicial sets given in [19]. While doing this, we observed that the collection of lean and thin triangles becomes redundant. To deal with this issue we introduce a third model structure on simplicial sets equipped with a collection of marked edges and *one* collection of triangles which we call marked-scaled simplicial sets.

Definition 3.35 A marked-scaled simplicial set denoted by (X, E_X, T_X) is given by

(1) a simplicial set X,

(2) a collection of edges $E_X \subseteq X_1$ which contains *all degenerate* edges,

(3) a collection of triangles $T_X \subseteq X_2$ which contains *all degenerate* triangles,

where we refer to the elements of E_X as *marked* edges, and we refer to the elements of T_X as *thin* triangles. A morphism of marked-scaled simplicial sets $f : (X, E_X, T_X) \to (Y, E_Y, T_Y)$ is given by a map of simplicial sets such that $f(E_X) \subseteq E_Y$ and $f(T_X) \subseteq T_Y$. We denote by $\mathrm{Set}_\Delta^{\mathbf{ms}}$ the category of marked-scaled simplicial sets.

Definition 3.36 The set of *generating marked-scaled anodyne maps* **MS** is the set of maps of marked-scaled simplicial sets consisting of

(M1) the inner horn inclusions

$$\bigl(\Lambda^n_i, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\bigr) \to \bigl(\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\bigr), \quad n \geq 2,\ 0 < i < n;$$

(M2) the map

$$(\Delta^4, \flat, T) \to (\Delta^4, \flat, T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \overset{\mathrm{def}}{=} \{\Delta^{\{0,2,4\}}, \Delta^{\{1,2,3\}}, \Delta^{\{0,1,3\}}, \Delta^{\{1,3,4\}}, \Delta^{\{0,1,2\}}\};$$

(M3) the set of maps

$$(\Lambda^n_0, \{\Delta^{\{0,1\}}\}, \{\Delta^{\{0,1,n\}}\}) \to (\Delta^n, \{\Delta^{\{0,1\}}\}, \{\Delta^{\{0,1,n\}}\}), \quad n \geq 2;$$

(M4) the inclusion of the initial vertex

$$(\Delta^0, \sharp, \sharp) \to (\Delta^1, \sharp, \sharp);$$

(MS1) the map

$$(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{1,2\}}\}, \sharp) \to (\Delta^2, \sharp, \sharp);$$

(ME) for every Kan complex K, the map

$$(K, \flat, \sharp) \to (K, \sharp, \sharp),$$

which requires that every equivalence is a marked morphism.

A map of **MS** simplicial sets is said to be **MS**-anodyne if it belongs to the weakly saturated closure of **MS**.

Remark 3.37 Observe that (X, E_X, T_X) has the right lifting property against the class of **MS**-anodyne morphisms if and only if X is an ∞-bicategory, E_X is the collection of equivalences in X and T_X is the collection of thin triangles. Consequently, we might call such marked-scaled simplicial sets ∞-bicategories as well.

Definition 3.38 We say that a morphism of marked-scaled simplicial sets is a cofibration if its underlying map of simplicial sets is a cofibration.

Proposition 3.39 *Let $f : X \to Y$ be a cofibration in $\mathrm{Set}_\Delta^{\mathbf{ms}}$ and $g : A \to B$ be an **MS**-anodyne morphism. Then the pushout-product*

$$X \times B \amalg_{X \times A} Y \times A \to Y \times B$$

*is again **MS**-anodyne.*

Corollary 3.40 *Given a marked-biscaled simplicial set X which has the right lifting property against the class of **MS**-anodyne morphisms it follows that $\mathrm{Fun}^{\mathbf{ms}}(A, X)$ (compare to Remark 3.13) has the right lifting property against the class **MS**-anodyne morphisms for every $A \in \mathrm{Set}_\Delta^{\mathbf{ms}}$.*

Definition 3.41 A morphism of marked-scaled simplicial sets $i : A \to B$ is said to be a *weak equivalence* if for every ∞-bicategory X the induced map

$$i^* : \mathrm{Fun}^{\mathbf{ms}}(B, X) \xrightarrow{\simeq} \mathrm{Fun}^{\mathbf{ms}}(A, X)$$

is a bicategorical equivalence.

Remark 3.42 In a similar way as before given $K \in \mathrm{Set}_\Delta^{+}$ and $X \in \mathrm{Set}_\Delta^{\mathbf{ms}}$ we define the tensor $K \times X := I(K) \times X$ where $I(K) = (K, E_K, \sharp)$. Similarly, we define the cotensor $X^K := \mathrm{Fun}^{\mathbf{ms}}(I(K), X)$.

Theorem 3.43 *There exists a left proper combinatorial simplicial model category on $\mathrm{Set}_\Delta^{\mathbf{ms}}$, which is characterised uniquely by the following properties*:

(C) *A morphism $f : X \to Y$ in $\mathrm{Set}_\Delta^{\mathbf{ms}}$ is a cofibration if and only if f induces a monomorphism on the underlying simplicial set.*

(F) *An object X in $\mathrm{Set}_\Delta^{\mathbf{ms}}$ is fibrant if and only if it was the right lifting property against the class of marked-anodyne morphisms.*

Moreover the tensor and cotensor in Remark 3.42 equips $\mathrm{Set}_\Delta^{\mathbf{ms}}$ with the structure of a Set_Δ^{+}-enriched model category.

Theorem 3.44 *The functor $L\colon \mathrm{Set}_\Delta^{\mathbf{sc}} \to \mathrm{Set}_\Delta^{\mathbf{ms}}$ sending a scaled simplicial set (X, T_X) to the marked-scaled simplicial set $(X, \flat, T_X)$ is a left Quillen equivalence.*

Proof The proof is almost identical to the proof of Theorem 3.34 and thus omitted. □

4 Local fibrations

In this section we will show that the model independent definition of a locally cocartesian fibration of $(\infty,2)$-categories, or in our terminology a local $(0,1)$-fibration, given in the introduction can be realised within our framework of **MB** simplicial sets as follows:

Given a fibrant scaled simplicial set (S, T_S) we will construct a subcollection $M_S \subseteq T_S$ and show in Theorem 4.22 that a local $(0,1)$-fibration is precisely given by an M_S-local $(0,1)$-fibration. This perspective will allow us to give an easy proof of Proposition 4.24, as promised in Proposition 3.23.

Definition 4.1 Let (S, T_S) be an ∞-bicategory we define $M_S \subseteq T_S$ as the subcollection of triangles consisting in those *thin* triangles such that the edge $\Delta^{\{0,1\}}$ or the edge $\Delta^{\{1,2\}}$ is an equivalence in S. We call the elements of M_S the invertible 2-morphisms of S.

Remark 4.2 The collection M_S has been studied in [11] to give a definition of an oplax normalised functor of ∞-bicategories.

Notation 4.3 We will use boldface letters $\mathbb{S} := (S, T_S)$ to describe fibrant scaled simplicial sets.

Definition 4.4 Let $\mathbb{S} = (S, T_S)$ be an ∞-bicategory and let $p\colon X \to S$ be an M_S-local (see Definition 4.1) $(0,1)$-fibration. We define a scaled simplicial set $\mathbb{X}$ whose underlying simplicial set is the underlying simplicial set of X and where a triangle is declared to be thin if it is lean in X and its image under p belongs to T_S. We denote the resulting map of scaled simplicial sets by $\mathfrak{p}\colon \mathbb{X} \to \mathbb{S}$ and call it the *bicategorical interpretation* of p.

Proposition 4.5 *Let $\mathbb{S} = (S, T_S)$ be an ∞-bicategory and let $p\colon X \to S$ be an M_S-local $(0,1)$-fibration. Then its bicategorical interpretation $\mathfrak{p}\colon \mathbb{X} \to \mathbb{S}$ (see Definition 4.4) is a bicategorical fibration.*

Proof It is clear that $\mathfrak{p}\colon \mathbb{X} \to \mathbb{S}$ has the right lifting property against the class of scaled anodyne maps and thus it follows that $\mathbb{X}$ is itself an ∞-bicategory. To finish the proof we will need to show that $\mathfrak{p}$ is an isofibration (see Remark 2.12). Given an equivalence $e\colon \Delta^1 \to \mathbb{S}$ and a lift of the source $\Delta^0 \to \mathbb{X}$ we can produce a marked edge $\hat{e}\colon \Delta^1 \to X$ lying over e. We observe that the 2-simplices needed for exhibiting e as an equivalence in $\mathbb{S}$ are contained in M_S; in particular this can be used to show that $\hat{e}$ is an equivalence in $\mathbb{X}$. □

Proposition 4.6 *Let $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ be a bicategorical interpretation as in Definition 4.4. Then for every pair of objects $a, b \in \mathbb{X}$ the induced functor on mapping ∞-categories*

$$\mathbb{X}(a,b) \longrightarrow \mathbb{S}(\mathfrak{p}(a), \mathfrak{p}(b))$$

is a cartesian fibration. Moreover, the composition functors $\mathbb{X}(a,b) \times \mathbb{X}(b,c) \to \mathbb{X}(a,c)$ preserve cartesian edges.

Proof We pick a model for the mapping ∞-category discussed in Definition 2.14. Given a morphism $\alpha : \Delta^1 \to \mathbb{S}(\mathfrak{p}(a), \mathfrak{p}(b))$ and a lift of the target $g : \Delta^0 \to \mathbb{X}(a,b)$ we can consider a morphism of type (A1) to produce an edge in $\mathbb{X}(a,b)$ which is lean in the **MB** simplicial set X. We can similarly translate lifting problems of the form

$$\begin{tikzcd}
\Lambda^n_n \arrow[r] \arrow[d] & \mathbb{X}(a,b) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \mathbb{S}(\mathfrak{p}(a), \mathfrak{p}(b))
\end{tikzcd}$$

where the last edge in the top horizontal morphism is mapped to a lean 2-simplex to lifting problems of the form

$$\begin{tikzcd}
\Lambda^{n+1}_n \arrow[r] \arrow[d] & X \arrow[d] \\
\Delta^{n+1} \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

where the triangle $\Delta^{\{n-1,n,n+1\}}$ is mapped to a lean triangle in X and thus admits a solution. We conclude that $\mathbb{X}(a,b) \to \mathbb{S}(\mathfrak{p}(a), \mathfrak{p}(b))$ is a cartesian fibration. To finish the proof we consider the composition functor

$$\mathbb{X}(a,b) \times \mathbb{X}(b,c) \longrightarrow \mathbb{X}(a,c).$$

Given a pair of cartesian edges $\alpha : f \to g$ and $\beta : u \to v$ the composition map yields a commutative diagram in $\mathbb{X}(a,c)$ of the form

$$\begin{tikzcd}
f \circ u \arrow[r] \arrow[d] & f \circ v \arrow[d] \\
g \circ u \arrow[r] & g \circ v
\end{tikzcd}$$

which tells us that it will suffice to check that precomposition and postcomposition with a 1-morphism preserves cartesian 2-morphisms. This follows from the fact that $\mathfrak{p}$ has the right lifting property against morphisms of type (A2). ☐

Definition 4.7 A bicategorical fibration (see Notation 4.3) $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ of ∞-bicategories is said to be *cartesian-enriched* if:

- For every $a, b \in \mathbb{X}$ the morphisms $\mathbb{X}(a,b) \to \mathbb{S}(p(a), p(b))$ are cartesian fibrations.
- For every $a, b, c \in \mathbb{X}$ the composite maps $\mathbb{X}(a,b) \times \mathbb{X}(b,c) \to \mathbb{X}(a,c)$ preserve cartesian edges.

Given two cartesian-enriched bicategorical fibrations $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ and $\mathfrak{q}:\mathbb{Y}\to\mathbb{S}$ we say that a functor $\mathfrak{f}:\mathbb{X}\to\mathbb{Y}$ is cartesian-enriched if it preserves cartesian edges in the mapping categories.

Definition 4.8 Let $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ be a bicategorical fibration of ∞-bicategories. An edge $e:a\to b$ in $\mathbb{X}$ is said to be locally $(0,1)$-cartesian (or a local $(0,1)$-cartesian edge) if:

(i) Given $g:a\to c$ in $\mathbb{X}$ and a commutative diagram (represented by a thin simplex), in $\sigma:\Delta^2\to\mathbb{S}$,

$$\begin{tikzcd} & \mathfrak{p}(b) \arrow[dr, "\alpha"] & \\ \mathfrak{p}(a) \arrow[ur, "\mathfrak{p}(e)"] \arrow[rr, "\mathfrak{p}(g)"'] & & \mathfrak{p}(c) \end{tikzcd}$$

such that α is an equivalence, there exists a morphism $\widehat{\alpha}:b\to c$ such that $\mathfrak{p}(\widehat{\alpha})=\alpha$ and a thin 2-simplex $\widehat{\sigma}$ exhibiting $e\circ\widehat{\alpha}\simeq g$ such that $\mathfrak{p}(\widehat{\sigma})=\sigma$.

(ii) Given any $\phi:b\to c$ such that $\phi\circ e\simeq g$ and such that $\mathfrak{p}(\phi)=\alpha$ as above, for any other $\varphi:b\to c$, precomposition along e induces a pullback diagram of spaces

$$\begin{tikzcd} \mathrm{Map}_{\mathbb{X}(b,c)}(\phi,\varphi) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{X}(a,c)}(\phi\circ e,\varphi\circ e) \arrow[d] \\ \mathrm{Map}_{\mathbb{S}(\mathfrak{p}(b),\mathfrak{p}(c))}(\mathfrak{p}(\phi),\mathfrak{p}(\varphi)) \arrow[r] & \mathrm{Map}_{\mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(c))}(\mathfrak{p}(\phi\circ e),\mathfrak{p}(\varphi\circ e)) \end{tikzcd}$$

Remark 4.9 Given a bicategorical fibration $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ of ∞-bicategories, we observe that a local $(0,1)$-cartesian edge lying over an equivalence in $\mathbb{S}$ is necessarily an equivalence in $\mathbb{X}$. Moreover, the composition of a local $(0,1)$-cartesian edge with an equivalence is again locally $(0,1)$-cartesian.

Definition 4.10 Let $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ be a bicategorical fibration of ∞-bicategories. An edge $e:a\to b$ is $(0,1)$-cartesian if for every $c\in\mathbb{X}$ precomposition along e induces a pullback diagram of ∞-categories

$$\begin{tikzcd} \mathbb{X}(b,c) \arrow[r] \arrow[d] & \mathbb{X}(a,c) \arrow[d] \\ \mathbb{S}(\mathfrak{p}(b),\mathfrak{p}(c)) \arrow[r] & \mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(c)) \end{tikzcd}$$

Proposition 4.11 *Let $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ be a bicategorical fibration of ∞-bicategories and suppose further that $\mathfrak{p}$ is cartesian-enriched. Given an edge $e:a\to b$ in $\mathbb{X}$, the following are equivalent*:

(i) *The edge e is locally $(0,1)$-cartesian.*

(ii) *For every $\ell\in\mathbb{X}$ such that $\mathfrak{p}(\ell)=\mathfrak{p}(b)$ we have a pullback diagram of ∞-categories*

$$\begin{tikzcd} \mathbb{X}_{\mathfrak{p}(b)}(b,\ell) \arrow[r, "(-)\circ e"] \arrow[d] & \mathbb{X}(a,\ell) \arrow[d] \\ \Delta^0 \arrow[r, "\mathfrak{p}(e)"] & \mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(b)) \end{tikzcd}$$

where $X_{\mathfrak{p}(b)}$ is the fibre of $\mathfrak{p}$ over $\mathfrak{p}(b)$.

(iii) *The edge e is $(0,1)$-cartesian in the pullback along $\mathfrak{p}(e)$, $\mathbb{X}_{\mathfrak{p}(e)}=\mathbb{X}\times_{\mathbb{S}}\Delta^1\to\Delta^1$.*

Proof To show that (i) $\implies$ (ii) we observe that since

$$\mathbb{X}(a,\ell) \to \mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(b))$$

is a cartesian fibration of ∞-categories it follows that the strict fibre over $\mathfrak{p}(e)$ is already a model for the ∞-categorical pullback. Therefore it suffices to show that the map

$$e^* : \mathbb{X}_{\mathfrak{p}(b)}(b,\ell) \to \mathbb{X}_{\mathfrak{p}(e)}(a,\ell)$$

is an equivalence of ∞-categories. Observe that the first condition in Definition 4.8 guarantees that e^* is essentially surjective. Given a pair of objects $\phi,\varphi \in \mathbb{X}_{\mathfrak{p}(b)}(b,\ell)$ it follows from condition (ii) in Definition 4.8 that we have a pullback diagram of spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathbb{X}(b,\ell)}(\phi,\varphi) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{X}(a,\ell)}(\phi\circ e,\varphi\circ e) \arrow[d] \\
\mathrm{Map}_{\mathbb{S}(\mathfrak{p}(b),\mathfrak{p}(b))}(\mathfrak{p}(\phi),\mathfrak{p}(\varphi)) \arrow[r] & \mathrm{Map}_{\mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(b))}(\mathfrak{p}(\phi\circ e),\mathfrak{p}(\varphi\circ e))
\end{tikzcd}$$

so in particular after taking fibres we have a homotopy equivalence of spaces

$$\mathrm{Map}_{\mathbb{X}(b,\ell)}(\phi,\varphi)_{\mathrm{id}} \xrightarrow{\simeq} \mathrm{Map}_{\mathbb{X}(a,\ell)}(\phi\circ e,\varphi\circ e)_{\mathfrak{p}(e)},$$

which we identify with the action of e^* on mapping spaces which shows our claim.

Note that (ii) $\iff$ (iii) follows immediately from the definition so it will suffice to show that (ii) $\implies$ (i).

First let us remark that since $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ is a bicategorical fibration, it is in particular an isofibration. So, in order to show that e is locally $(0,1)$-cartesian, we specialise the conditions in Definition 4.8 to the case where $\mathfrak{p}(\alpha)$ is a degenerate edge. Let $\ell\in\mathbb{X}$ such that $\mathfrak{p}(\ell)=\mathfrak{p}(b)$ and consider an edge $u:a\to\ell$ such that $\mathfrak{p}(u)\simeq\mathfrak{p}(e)$ in $\mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(b))$. Since $\mathfrak{p}$ is cartesian enriched we can pick an equivalence in $\mathbb{X}(a,\ell)$, $\hat{u}\simeq u$, such that $\mathfrak{p}(\hat{u})=\mathfrak{p}(e)$. Then our assumptions guarantee the existence of an object $\phi\in\mathbb{X}(b,\ell)$ such that $\mathfrak{p}(\phi)$ is degenerate and such that

$$\phi\circ e\simeq\hat{u}\simeq u.$$

This shows that the first condition in Definition 4.8 holds. Let $\phi:b\to\ell$ in $\mathbb{X}$ such that $\mathfrak{p}(\phi)$ is degenerate on $\mathfrak{p}(b)$. To show that we have the necessary pullback square of spaces it will be enough to show that for every $\Xi\in\mathrm{Map}_{\mathbb{S}(\mathfrak{p}(b),\mathfrak{p}(b))}(\mathfrak{p}(\phi),\mathfrak{p}(\varphi))$ the associated morphism on fibres

$$\mathrm{Map}_{\mathbb{X}(b,\ell)}(\phi,\varphi)_{\Xi} \xrightarrow{\simeq} \mathrm{Map}_{\mathbb{X}(a,\ell)}(\phi\circ e,\varphi\circ e)_{\Xi\circ\mathfrak{p}(e)}$$

is a homotopy equivalence. Note that our assumptions guarantee that this holds whenever Ξ is the identity morphism. Since the map $\mathbb{X}(b,\ell)\to\mathbb{S}(\mathfrak{p}(b),\mathfrak{p}(b))$ is a cartesian fibration we pick a cartesian lift of Ξ in $\mathbb{X}(b,\ell)$ which we denote by $i:\hat{\varphi}\to\varphi$ and we note that the enrichment of $\mathfrak{p}$ implies that $i\circ e$ is again a cartesian morphism in $\mathbb{X}(a,\ell)$. The fact that i is cartesian allows us to construct a commutative diagram of spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathbb{X}(b,\ell)}(\phi,\widehat{\varphi})_{\mathrm{id}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathbb{X}(a,\ell)}(\phi\circ e,\widehat{\varphi}\circ e)_{\mathfrak{p}(e)} \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathbb{X}(b,\ell)}(\phi,\varphi)_{\Xi} \arrow[r] & \mathrm{Map}_{\mathbb{X}(a,\ell)}(\phi\circ e,\varphi\circ e)_{\Xi\circ\mathfrak{p}(e)}
\end{tikzcd}$$

so the conclusion follows by two-out-of-three. □

Definition 4.12 A bicategorical fibration $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ of ∞-bicategories is said to be a local $(0,1)$-fibration if the following conditions hold:

1. The map $\mathfrak{p}$ is cartesian-enriched (see Definition 4.7).
2. For every $a\in\mathbb{X}$ and every $e:\mathfrak{p}(a)\to b$ in $\mathbb{S}$ there exists a local $(0,1)$-cartesian edge $\hat{e}:a\to\hat{b}$ such that $\mathfrak{p}(\hat{e})=e$.

We say that a commutative diagram

$$\begin{tikzcd}
\mathbb{X} \arrow[rr, "\mathfrak{f}"] \arrow[dr, "\mathfrak{p}"'] & & \mathbb{Y} \arrow[dl, "\mathfrak{q}"] \\
& \mathbb{S} &
\end{tikzcd}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are local $(0,1)$-fibrations is a morphism of local $(0,1)$-fibrations if $\mathfrak{f}$ is a morphism of cartesian-enriched fibrations and it maps local $(0,1)$-cartesian edges in $\mathfrak{p}$ to local $(0,1)$-cartesian edges in $\mathfrak{q}$.

Lemma 4.13 *Let* $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ *be a local* $(0,1)$*-fibration of* ∞*-bicategories and consider a commutative diagram, in* $\mathbb{X}$*,*

$$\begin{tikzcd}
a \arrow[r, "\simeq"] \arrow[d, "f"'] & u \arrow[d, "g"] \\
b \arrow[r, "\simeq"] & v
\end{tikzcd}$$

such that the horizontal morphisms are equivalences in $\mathbb{X}$*. Then* f *is a local* $(0,1)$*-cartesian edge if and only if* g *is.*

Proof Left as an exercise. □

Theorem 4.14 *Let us consider a commutative diagram of* ∞*-bicategories*

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\simeq"] \arrow[d, "\mathfrak{p}"'] & \mathbb{C} \arrow[d, "\mathfrak{q}"] \\
\mathbb{S} \arrow[r, "\simeq"] & \mathbb{D}
\end{tikzcd}$$

where the vertical morphisms are bicategorical fibrations and the horizontal morphisms are bicategorical equivalences. Then $\mathfrak{p}$ *is a local* $(0,1)$*-fibration if and only if* $\mathfrak{q}$ *is.*

Proof Observe that we can view our diagram as an injectively fibrant-cofibrant diagram in the arrow category of $\mathrm{Set}_{\Delta}^{\mathbf{sc}}$. This guarantees the existence weak equivalences $\mathbb{C} \to \mathbb{X}$ and $\mathbb{D} \to \mathbb{S}$ making the obvious diagram commute. Therefore we might assume without loss of generality that $\mathfrak{q}$ is a local $(0,1)$-fibration. After inspecting the associated diagram in mapping categories we learn that $\mathfrak{p}$ must be cartesian enriched. To finish the proof we need to show that $\mathfrak{p}$ has a sufficient supply of local $(0,1)$-cartesian edges.

First, we observe that our commutative diagram is in fact a pullback diagram of ∞-bicategories. Therefore we have a weak equivalence $\varphi : \mathbb{X} \to \mathbb{P}$ where $\hat{\mathfrak{q}} : \mathbb{P} \to \mathbb{S}$ denotes the strict pullback of $\mathfrak{q}$ along the bottom horizontal morphism. It follows that $\hat{\mathfrak{q}}$ is a local $(0,1)$-fibration. Factoring φ as a cofibration (which is necessary a trivial cofibration) and a trivial fibration we might assume without loss of generality that φ is a trivial cofibration. We conclude that we have a section $\xi : \mathbb{P} \to \mathbb{X}$ such that $\xi \circ \varphi = \mathrm{id}$.

Given $a \in \mathbb{X}$ and an edge $e : \mathfrak{p}(a) \to b$ in $\mathbb{S}$ we pick a lift in $\mathbb{P}$ of e with source $\varphi(a)$ which we denote by $\hat{e}$. We finally consider $\xi(\hat{e}) = \tau$. We claim that τ is a local $(0,1)$-cartesian edge of $\mathfrak{p}$. To see this we note that due to Lemma 4.13 we have that $\varphi(\tau)$ is a local $(0,1)$-cartesian edge of $\mathbb{P}$. The existence of a section ξ and the fact that $\varphi(\tau)$ is locally $(0,1)$-cartesian shows that condition (i) in Definition 4.8 holds. The second condition follows immediately from the fact that φ is a bicategorical equivalence. □

Lemma 4.15 *Let $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ be a local $(0,1)$-fibration of ∞-bicategories. Let $\sigma : \Delta^2 \to \mathbb{X}$ be a 2-simplex whose associated 2-morphism in $\mathbb{X}(\sigma(0), \sigma(2))$ is cartesian. Given a lifting problem*

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\varphi"] \arrow[d] & \mathbb{X} \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathbb{S} \end{tikzcd}$$

such that restriction of φ to $\Delta^{\{i-1,i,i+1\}}$ equals σ, the dotted arrow exists.

Proof In virtue of Theorem 4.14 we might assume that our functor is of the form $\mathfrak{p} : \mathrm{N}^{\mathbf{sc}}(\mathbb{C}) \to \mathrm{N}^{\mathbf{sc}}(\mathbb{D})$. Then it follows that our lifting problem is equivalent to

$$\begin{tikzcd} \mathfrak{C}^{\mathbf{sc}}[\Lambda^n_i](0,n) \arrow[r] \arrow[d] & \mathbb{C}(\varphi(0), \varphi(n)) \arrow[d] \\ \mathfrak{C}^{\mathbf{sc}}[\Delta^n](0,n) \arrow[r] \arrow[ur, dotted] & \mathbb{D}(\mathfrak{p}\varphi(0), \mathfrak{p}\varphi(n)) \end{tikzcd}$$

To show that the dotted arrow exists it will suffice to show that the left-hand side can be written as an iterated pushout of anodyne morphisms in the cartesian model structure. This follows from the (more general) argument given in [3, Lemma 3.41] by forgetting the scaling. □

Definition 4.16 Let $\pi : \mathcal{C} \to \mathcal{D}$ be a fibration of ∞-categories. We say that an object $x \in \mathcal{C}$ is *p-initial* if for every $y \in \mathcal{C}$ the functor π yields a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \mathrm{Map}_{\mathcal{D}}(\pi(x), \pi(y)).$$

Remark 4.17 An object $x \in \mathcal{C}$ as above is p-initial if and only if for every $n \geqslant 1$ the lifting problems

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\varphi"] \arrow[d] & \mathcal{C} \arrow[d, "\pi"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \mathcal{D}
\end{tikzcd}$$

admit a solution provided $\varphi(0) = x$.

Lemma 4.18 *Let $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ be a local $(0,1)$-fibration of ∞-bicategories. Then an edge $e : \Delta^1 \to \mathbb{X}$ is locally $(0,1)$-cartesian if and only if for every $n \geqslant 2$ the lifting problems of the form*

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\varphi"] \arrow[d] & \mathbb{X} \arrow[d, "\mathfrak{p}"] \\
\Delta^n \arrow[r, "\phi"] \arrow[ur, dotted] & \mathbb{S}
\end{tikzcd}$$

admit a solution provided $\varphi(0 \to 1) = e$, $\varphi(0 \to 1 \to n)$ is thin whenever $n \geqslant 2$ and $\phi(1 \to n)$ is an equivalence in $\mathbb{S}$. If $n = 2$ we require $\phi(0 \to 1 \to n)$ to be thin, $\phi(1 \to n)$ to be an equivalence in $\mathbb{S}$ and that our solution is a thin simplex in $\mathbb{X}$.

Proof First let us remark that if we can produce solutions to those lifting problems, e must be a $(0,1)$-cartesian edge once restricted to Δ^1 and the claims follow from Proposition 4.11. We now prove the converse.

The case $n = 2$ is precisely the first condition in Definition 4.8. To tackle the cases $n \geqslant 3$ we will assume once more that $\mathfrak{p} : \mathrm{N}^{\mathrm{sc}}(\mathbb{C}) \to \mathrm{N}^{\mathrm{sc}}(\mathbb{D})$. Since $\varphi(0 \to 1 \to n)$ is thin we can solve the lifting problem

$$\begin{tikzcd}
\mathfrak{C}^{\mathrm{sc}}[\Lambda^n_0](0,n) \arrow[r] \arrow[d] & \mathbb{C}(\varphi(0), \varphi(n)) \arrow[d] \\
\mathfrak{C}^{\mathrm{sc}}[\Delta^n](0,n) \arrow[r] \arrow[ur, dotted] & \mathbb{D}(\mathfrak{p}\varphi(0), \mathfrak{p}\varphi(n))
\end{tikzcd}$$

To conclude the proof we must show that we can produce the dotted arrow in

$$\begin{tikzcd}
\mathfrak{C}^{\mathrm{sc}}[\Lambda^n_0](1,n) \arrow[r] \arrow[d] & \mathbb{C}(\varphi(1), \varphi(n)) \arrow[d, "\phi"] \\
\mathfrak{C}^{\mathrm{sc}}[\Delta^n](1,n) \arrow[r] \arrow[ur, dotted] & \mathbb{C}(\varphi(0), \varphi(n) \times_{\mathbb{D}(\mathfrak{p}\varphi(0), \mathfrak{p}\varphi(n))} \mathbb{D}(\mathfrak{p}\varphi(1), \mathfrak{p}\varphi(n))
\end{tikzcd}$$

However, by (ii) in Definition 4.8 the object $1n$ on the left-hand side gets mapped to a ϕ-initial object in $\mathbb{C}(\varphi(1), \varphi(n))$. Since the left-most vertical map can be obtained as an iterated pushout along boundary inclusions $\partial\Delta^n \to \Delta^n$, where the initial object is always $1n$, we conclude that the dotted arrow above can be constructed. □

Lemma 4.19 *Let $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ be a local $(0,1)$-fibration of ∞-bicategories. Suppose that we are given a simplex $\sigma : \Delta^4 \to \mathbb{X}$ such that the collection of triangles*

$$T = \{\Delta^{\{0,2,4\}}, \Delta^{\{1,2,3\}}, \Delta^{\{0,1,3\}}, \Delta^{\{1,3,4\}}, \Delta^{\{0,1,2\}}\}$$

gets mapped to 2*-simplices representing cartesian* 2*-morphisms in the corresponding mapping categories. Then the triangles* $\Delta^{\{0,1,4\}}$ *and* $\Delta^{\{0,3,4\}}$ *also represent cartesian* 2*-morphisms in* $\mathbb{X}(\sigma(0),\sigma(4))$.

Proof As usual, we will assume that our functor is of the form $\mathfrak{p} : \mathrm{N}^{\mathrm{sc}}(\mathbb{C}) \to \mathrm{N}^{\mathrm{sc}}(\mathbb{D})$. This allows us to reduce our problem to show that certain edges in $\mathcal{P} = \mathfrak{C}^{\mathrm{sc}}[\Delta^4](0,4)$ get mapped to cartesian edges in $\mathbb{C}(\sigma(0),\sigma(4))$. More specifically, we view $\mathcal{P}$ as the poset

$$\begin{tikzcd}
& 0134 \arrow[rr, "\circ" marking] & & 01234 \\
034 \arrow[ur, "\circ" marking] \arrow[rr] & & 0234 \arrow[ur, "\circ" marking] & \\
& 014 \arrow[uu, "\circ" marking] \arrow[rr] & & 0124 \arrow[uu] \\
04 \arrow[uu] \arrow[ur] \arrow[rr, "\circ" marking] & & 024 \arrow[uu] \arrow[ur, "\circ" marking] &
\end{tikzcd}$$

where the circled arrows are mapped by assumption to cartesian edges (note that to see this it is crucial to use that $\mathfrak{p}$ is cartesian-enriched) and we wish to show that $04 \to 014$ and $04 \to 034$ are mapped to cartesian edges in $\mathbb{C}(\sigma(0),\sigma(4))$.

Since $\pi : \mathbb{C}(\sigma(0),\sigma(4)) \to \mathbb{D}(\mathfrak{p}\sigma(0),\mathfrak{p}\sigma(4))$ is a cartesian fibration this is equivalent to require that certain morphisms are equivalences in the fibre over $\pi(04) = \alpha$. Using the functoriality of π we can move the diagram above to a diagram in the fibre over α where now the circled arrow are equivalences. It is easy to see that we can produce now an inverse as in Proposition 3.1.13 in [19]. $\square$

Definition 4.20 Let $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ be a local $(0,1)$-fibration of ∞-bicategories and let $\mathbb{X} = (X, T_X)$ and $\mathbb{S} = (S, T_S)$. We define an **MB**-simplicial set $\mathrm{F}_{\mathfrak{p}}$ as follows:

- The underlying simplicial set of $\mathrm{F}_{\mathfrak{p}}$ is $\mathbb{X}$.
- An edge is declared to be marked if an only if it is a local $(0,1)$-cartesian edge in $\mathbb{X}$.
- A triangle is declared to be lean if its associated 2-morphism is a cartesian edge in $\mathbb{X}(a,b)$.
- A triangle is declared to be thin if it is lean and its image in $\mathbb{S}$ belongs to M_S.

This definition clearly yields a map $\pi_{\mathfrak{p}} : \mathrm{F}_{\mathfrak{p}} \to (S, \sharp, M_S \subseteq \sharp)$ which we call the **MB**-model of $\mathfrak{p}$.

Lemma 4.21 *Let* $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ *be a local* $(0,1)$*-fibration of* ∞*-bicategories. Then its associated* ***MB****-model (see Definition 4.20)* $\pi_{\mathfrak{p}} : \mathrm{F}_{\mathfrak{p}} \to (S, \sharp, M_S \subseteq \sharp)$ *defines a fibrant object in* $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)_{/(S,\sharp,M_S\subset\sharp)}$.

Proof We need to show that $\pi_{\mathfrak{p}}$ has the right lifting property against the class of morphisms in Definition 3.4. It follows from Lemmas 4.15 and 4.18 that $\pi_{\mathfrak{p}}$ has the RLP property with respect to the morphisms of type (A1) and (A3) in Definition 3.4. Lemma 5.27 shows $\pi_{\mathfrak{p}}$ has the right lifting property against the class of morphisms of type (A2). The rest of the lifting problems follow immediately and thus our result is proved. $\square$

Theorem 4.22 *Let $\mathbb{S}=(S,T_S)$ be a fibrant scaled simplicial set and let $p:X\to S$ be a fibrant object in $\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,M_S\subset\sharp)}$. Then its bicategorical interpretation (see Definition 4.4) $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ is a local $(0,1)$-fibration. Moreover, this assignment admits an inverse given by sending each local $(0,1)$-fibration of ∞-bicategories $\mathfrak{q}:\mathbb{Y}\to\mathbb{S}$ to its **MB**-model as in Definition 4.20.*

Proof Let $p:X\to S$ be a fibrant object in $\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,M_S\subset\sharp)}$. Then it follows from Propositions 4.5 and 4.6 that $p:\mathbb{X}\to\mathbb{S}$ is cartesian-enriched. To show that our map in question defines a local $(0,1)$-fibration it will be enough to show that the marked edges in X are local $(0,1)$-cartesian edges. However, one easily sees that a marked edge e is $(0,1)$-cartesian over Δ^1 and thus the claim holds.

Note that Lemma 4.21 shows that the **MB**-model of a local $(0,1)$-fibration of ∞-bicategories defines a fibrant object in $\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,M_S\subset\sharp)}$. To complete the proof, we must show that these two assignments are mutually inverse. Note that this simply amounts to verifying that the decorations of our simplicial sets remain unchanged after applying both procedures. This follows by virtue of the following observations:

- Let $p:X\to S$ be a fibrant object in $\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,M_S\subset\sharp)}$. Then a 2-simplex $\sigma:\Delta^2\to X$ is lean if and only if the associated 2-morphism in its bicategorical interpretation $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ is cartesian.
- Let $\mathfrak{q}:\mathbb{Y}\to\mathbb{S}$ be a local $(0,1)$-fibration of ∞-bicategories. Then a 2-simplex $\sigma:\Delta^2\to\mathbb{Y}$ is thin if and only if its associated 2-morphism is cartesian and its image in $\mathbb{S}$ is invertible. □

Proposition 4.23 *Let $\mathfrak{f}:\mathbb{X}\to\mathbb{Y}$ be a morphism of local $(0,1)$-fibrations. Then the following are equivalent:*

(i) *The map $\mathfrak{f}$ is a bicategorical equivalence.*

(ii) *For every $s\in\mathbb{S}$ the map $\mathfrak{f}$ induces an equivalence on fibres $\mathfrak{f}_s:\mathbb{X}_s\xrightarrow{\simeq}\mathbb{Y}_s$.*

Proof Let us suppose that $\mathfrak{f}$ is a bicategorical equivalence and let $s\in\mathbb{S}$. First we show that $\mathfrak{f}_s$ is essentially surjective. Given $y_s\in\mathbb{Y}_s$, we use that $\mathfrak{f}$ is essentially surjective to get some $x\in\mathbb{X}$ such that $\mathfrak{f}(x)\simeq y$. We denote by u the image of the equivalence $\mathfrak{f}(x)\simeq y$ under $\mathfrak{q}:\mathbb{Y}\to\mathbb{S}$ and use the fact that $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ is an isofibration to get an equivalence $v:x\to x_s$ where $\mathfrak{p}(x_s)=s$. It follows that $\mathfrak{f}(v)$ is again an equivalence and therefore defines a local $(0,1)$-cartesian edge in $\mathbb{Y}$ which allows to construct an equivalence $\mathfrak{f}(x_s)\simeq y_s$ lying over the identity on s. To show fully faithfulness of $\mathfrak{f}_s$ we consider $a,b\in\mathbb{X}_s$ and observe that we have a map of cartesian fibrations

$$\begin{tikzcd}
\mathbb{X}(a,b) \arrow[rr] \arrow[dr] & & \mathbb{Y}(\mathfrak{f}(a),\mathfrak{f}(b)) \arrow[dl] \\
& \mathbb{S}(s,s) &
\end{tikzcd}$$

which is an equivalence by assumption. It then follows that we have an equivalence after taking the fibre over the identity map on s. This morphism is then identified using Proposition 4.11 with the map

$$\mathbb{X}_s(a,b)\xrightarrow{\simeq}\mathbb{Y}_s(\mathfrak{f}(a),\mathfrak{f}(b)),$$

which shows that $\mathfrak{f}_s$ is fully faithful.

To show that the converse holds we note that by assumptions $\mathfrak{f}$ is already essentially surjective. It will then be enough to show that for every $a, b \in \mathbb{X}$ and every $\alpha : \mathfrak{p}(a) \to \mathfrak{p}(b)$ the induced morphism on fibres

$$\mathbb{X}(a,b)_\alpha \to \mathbb{Y}(\mathfrak{f}(a),\mathfrak{f}(b))_\alpha$$

is a categorical equivalence. By picking a cartesian lift $e : a \to \hat{b}$ such that $\mathfrak{p}(e) = \alpha$ we can again use Proposition 4.11 to produce a commutative diagram

$$\begin{tikzcd}
\mathbb{X}_{\mathfrak{p}(b)}(\hat{b},b) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathbb{Y}_{\mathfrak{p}(b)}(\mathfrak{f}(\hat{b}),\mathfrak{f}(b)) \arrow[d, "\simeq"] \\
\mathbb{X}(a,b)_\alpha \arrow[r] & \mathbb{Y}(\mathfrak{f}(a),\mathfrak{f}(b))_\alpha
\end{tikzcd}$$

where we use two-out-of-three to conclude that the bottom horizontal morphism is a weak equivalence and thus our claim holds. □

Proposition 4.24 *Let (T,U) be a scaled simplicial set and consider a morphism $f : X \to Y$ between fibrant objects in $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(T,\sharp,U\subset\sharp)}$. Suppose further that the following conditions hold*:

(1) *The map f has the right lifting property against the class of **MB**-anodyne morphisms.*

(2) *For every $t \in T$ the induced morphism $f_t : X_t \to Y_t$ is a bicategorical equivalence.*

*Then f is a trivial fibration of **MB**-simplicial sets.*

Proof We claim that f is a trivial fibration of **MB**-simplicial sets if and only if for every minimally scaled simplex $\Delta^n_\flat$ and every morphism $\sigma : \Delta^n_\flat \to T$ the restricted morphism

$$f_{|\sigma} : X \times_{\Delta^n_\flat} \Delta^n_\flat \to Y \times_{\Delta^n_\flat} \Delta^n_\flat$$

is a trivial fibration of **MB**-simplicial sets. One direction is obviously true. Let us assume that $f_{|\sigma}$ is always a trivial fibration. Then it is clear that f has the right-lifting property against the morphisms

- $(\partial\Delta^n,\flat,\flat) \to (\Delta^n,\flat,\flat)$,
- $(\Delta^2,\flat,\flat) \to (\Delta^2,\flat,\flat\subset\sharp)$,
- $(\Delta^1,\flat,\sharp) \to (\Delta^1,\sharp,\sharp)$.

Since a thin triangle in X is just a lean triangle lying over a thin triangle in T it follows that f also detects thin triangles and the claim holds.

Let us assume without loss of generality that $T = \Delta^n_\flat$ and consider the associated diagram (see Theorem 4.22) of local (0, 1)-fibrations ∞-bicategories

$$\begin{tikzcd}
\mathbb{X} \arrow[rr, "\mathfrak{f}"] \arrow[dr, "\mathfrak{p}"'] & & \mathbb{Y} \arrow[dl, "\mathfrak{q}"] \\
& \Delta^n &
\end{tikzcd}$$

First, let us show that $\mathfrak{f}$ is a trivial fibration of ∞-bicategories. Observe that our assumptions together with Proposition 4.23 imply that $\mathfrak{f}$ is a bicategorical equivalence. Moreover, $\mathfrak{f}$ has the right lifting property against the class of scaled anodyne maps given in Definition 2.3. It will then suffice by Remark 2.12 to show that $\mathfrak{f}$ is an isofibration. Given an equivalence in $e:\Delta^1\to\mathbb{Y}$ it follows that its image in Δ^n must be degenerate. Since this lifting problem is occurring in a fibre and by our assumptions the maps $\mathfrak{f}_t:\mathbb{X}_t\to\mathbb{Y}_t$ are trivial fibrations of scaled simplicial sets it follows that $\mathfrak{f}$ is an isofibration.

To finish the proof, we must show that $\mathfrak{f}$ detects local $(0,1)$-cartesian edges. Given an edge $e:\Delta^1\to\mathbb{X}$ such that $\mathfrak{f}(e)$ is a local $(0,1)$-cartesian edge we consider a local $(0,1)$-cartesian edge $u:\Delta^1\to\mathbb{X}$ such that $u(0)=e(0)$ and such that $\mathfrak{f}(u)=\mathfrak{f}(e)$. It follows that we have an edge $\alpha:u(1)\to e(1)$ such that $\alpha\circ u\simeq e$. Moreover, $\mathfrak{f}(\alpha)$ is an equivalence and lies over a degenerate morphism in Δ^n. We see then that α must be an equivalence in $\mathbb{X}$ and consequently e is a local $(0,1)$-cartesian edge. □

Definition 4.25 Let $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ be bicategorical fibration of ∞-bicategories and let $\sigma:\Delta^2\to\mathbb{S}$ be a thin triangle. We say that an edge $e:a\to b$ in $\mathbb{X}$ lying over $\sigma(0\to1)$ is σ-local if the following conditions hold:

(i) For every $g:a\to c$ in $\mathbb{X}$ lying over $\sigma(0\to2)$ there exists some $\hat{\alpha}:b\to c$ and a thin simplex θ exhibiting $e\circ\hat{\alpha}\simeq g$ such that $\mathfrak{p}(\theta)=\sigma$.

(ii) For any $\phi:b\to c$ such that $e\circ\phi\simeq g$ with associated simplex τ such that $\mathfrak{p}(\tau)=\sigma$ and for any $\varphi:b\to c$, precomposition along e induces a pullback diagram of spaces

$$\begin{tikzcd}
\mathrm{Map}_{\mathbb{X}(b,c)}(\phi,\varphi) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{X}(a,c)}(\phi\circ e,\varphi\circ e) \arrow[d] \\
\mathrm{Map}_{\mathbb{S}(\mathfrak{p}(b),\mathfrak{p}(c))}(\mathfrak{p}(\phi),\mathfrak{p}(\varphi)) \arrow[r] & \mathrm{Map}_{\mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(c))}(\mathfrak{p}(\phi\circ e),\mathfrak{p}(\varphi\circ e))
\end{tikzcd}$$

Remark 4.26 Definition 4.25 implies that an edge is $(0,1)$-cartesian if and only if it is σ-local for every thin simplex $\sigma:\Delta^2\to\mathbb{S}$. Similarly, an edge is locally $(0,1)$-cartesian if and only if it is σ-local for every invertible 2-morphism (see Definition 4.1).

Proposition 4.27 *Let $\mathfrak{p}:\mathbb{X}\to\mathbb{S}$ be a local $(0,1)$-fibration of ∞-bicategories and let $\sigma:\Delta^2\to\mathbb{S}$ be a thin triangle. Then a local $(0,1)$-cartesian edge $e:a\to b$ is σ-local if and only if for every local $(0,1)$-cartesian edge $u:b\to c$ such that the composite $v\simeq u\circ e$ lies over σ then v is also a local $(0,1)$-cartesian edge.*

Proof Let us assume that e is σ-local and suppose that we have u and v as above. Let us suppose that we have $h:a\to d$ and let us show that condition (i) in Definition 4.8 is satisfied.

Since $\mathfrak{p}$ is a local $(0,1)$-fibration and, in particular, cartesian-enriched, we only need to show that v is $(0,1)$-cartesian once after pulling back along $\mathfrak{p}(v)$. Therefore, we can assume without loss of generality that $\mathfrak{p}(h)=\mathfrak{p}(v)$. We observe that since e is σ-local we can obtain a morphism $\alpha:b\to d$ such that $\alpha\circ e\simeq h$. Furthermore, we can use that u is a local $(0,1)$-cartesian edge to get a morphism $\phi:c\to d$ such that $\alpha\simeq\phi\circ u$. It follows that $h\simeq v\circ\phi$ and so the first condition holds.

Given $\phi : c \to d$ as above and another $\varphi : c \to d$ such that $\mathfrak{p}(\varphi) = \mathrm{id}$ we construct the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathbb{X}(c,d)}(\phi,\varphi) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{X}(b,d)}(\phi \circ u, \varphi \circ u) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{X}(a,d)}(\phi \circ v, \varphi \circ v) \arrow[d] \\
\mathrm{Map}_{\mathbb{S}(\mathfrak{p}(c),\mathfrak{p}(d))}(\mathrm{id},\mathrm{id}) \arrow[r] & \mathrm{Map}_{\mathbb{S}(\mathfrak{p}(b),\mathfrak{p}(d))}(\mathfrak{p}(u),\mathfrak{p}(u)) \arrow[r] & \mathrm{Map}_{\mathbb{S}(\mathfrak{p}(a),\mathfrak{p}(d))}(\mathfrak{p}(v),\mathfrak{p}(v))
\end{tikzcd}$$

We observe that the outer commutative diagram is obtained by pasting two pullback diagrams so it must be itself a pullback diagram. It follows that v is a local $(0,1)$-cartesian edge.

We wish now to show that the converse holds. Let $h : a \to d$ be an edge over $\sigma(0 \to 2)$. We take a local $(0,1)$-cartesian lift $u : b \to c$ of $\sigma(1 \to 2)$. Since by assumption $v \simeq u \circ e$ is again local $(0,1)$-cartesian we obtain a certain $\Xi : c \to d$ such that $h \simeq \Xi \circ v$. We can then set $\phi = \Xi \circ u$. It is then clear that $\phi \circ e = \Xi \circ u \circ e \simeq \Xi \circ v \simeq h$ and thus condition (i) in Definition 4.25 holds. Let $\phi : b \to d$ as above and assume we are given any other $\varphi : b \to d$. We wish to show that the associated commutative diagram (see (ii) in Definition 4.25) of spaces is cartesian. We note that a totally analogous argument as in Proposition 4.11 shows that it is enough to show that the associated map of fibres

$$\mathrm{Map}_{\mathbb{X}(b,d)}(\phi,\varphi)_{\mathfrak{p}(u)} \xrightarrow{\simeq} \mathrm{Map}_{\mathbb{X}(a,e)}(\phi \circ e, \varphi \circ e)_{\mathfrak{p}(v)}$$

is an equivalence whenever $\mathfrak{p}(\phi) = \mathfrak{p}(\varphi)$. Since u is a local $(0,1)$-cartesian edge we can find morphisms $\tilde{\phi}, \tilde{\varphi} : c \to d$ such that $\tilde{\phi} \circ u \simeq \phi$ and $\tilde{\varphi} \circ u = \varphi$. We can then produce the morphisms

$$\mathrm{Map}_{\mathbb{X}(c,d)}(\tilde{\phi},\tilde{\varphi})_{\mathrm{id}} \xrightarrow{\simeq} \mathrm{Map}_{\mathbb{X}(b,d)}(\phi,\varphi)_{\mathfrak{p}(u)} \to \mathrm{Map}_{\mathbb{X}(a,e)}(\phi \circ e, \varphi \circ e)_{\mathfrak{p}(v)}.$$

We conclude the proof by noting that the composite map must also be a weak equivalence since v is by assumption a local $(0,1)$-cartesian edge. □

Definition 4.28 Let $\mathbb{S}$ be an ∞-bicategory and let U be a subcollection of the thin triangles in $\mathbb{S}$ which contains the invertible 2-morphisms of $\mathbb{S}$ (see Definition 4.1). We say that a local $(0,1)$-fibration $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ is $\mathcal{U}$-local if given a pair of local $(0,1)$-cartesian edge $u, v : \Delta^1 \to \mathbb{X}$ and a thin 2-simplex σ pictured as

$$\begin{tikzcd}
 & b \arrow[dr, "v"] & \\
a \arrow[ur, "u"] \arrow[rr, "w"] & & c
\end{tikzcd}$$

such that $p(\sigma) \in \mathcal{U}$ then we have that w is also locally $(0,1)$-cartesian. If $\mathcal{U}$ consists in all thin triangles we say that $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ is a $(0,1)$-cartesian fibration.

Theorem 4.29 *Let (S, T_S) be a fibrant scaled simplicial set and let $U \subset T_S$ be a subset containing all invertible 2-morphisms. Given a fibrant object $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/(S,\sharp,U\subset\sharp)}$, its bicategorical interpretation $\mathfrak{p} : \mathbb{X} \to \mathbb{S}$ defines a $\mathcal{U}$-local fibration. Conversely any $\mathcal{U}$-local fibration defines canonically a fibrant object in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/(S,\sharp,U\subset\sharp)}$.*

Proof Let us assume that we are given a fibrant object in $\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,U\subset\sharp)}$. In particular, we can use Theorem 4.22 to obtain a local $(0,1)$-fibration $\mathfrak{p}: \mathbb{X} \to \mathbb{S}$ of ∞-bicategories. Since our original object has the right lifting property against the class (S1) it follows that our local $(0,1)$-cartesian edges compose across triangles which lie over triangles in $\mathcal{U}$ and the claim follows.

We now show the converse. Note that due to Proposition 4.27 our local $(0,1)$-cartesian edges are σ-local with respect to the elements of $\mathcal{U}$. The only thing that we need to prove is that given a $\mathcal{U}$-local fibration we can produce the dotted arrow in

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "f"] \arrow[d] & \mathbb{X} \arrow[d, "\mathfrak{p}"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & \mathbb{S}
\end{tikzcd}$$

where $f(0 \to 1)$ is locally $(0,1)$-cartesian and $f(0 \to 1 \to n)$ lands in $\mathcal{U}$. The proof of this fact is essentially the same as the proof in Lemma 4.18 and therefore left as an exercise. $\square$

5 The Grothendieck construction

5.1 The Quillen adjunction

Let $S = (S, T_S)$ be a scaled simplicial set and let $\mathfrak{C}^{\mathbf{sc}}[S]$ denote the scaled rigidification (Definition 2.9) of (S, T_S). The goal of this section is to prove the following theorem.

Theorem 6 *Let S be a scaled simplicial set. Then there exists a Quillen equivalence*

$$\mathbb{S}\mathrm{t}_S : \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S\subset\sharp)} \rightleftarrows \mathrm{Fun}(\mathfrak{C}^{\mathbf{sc}}[S], \mathrm{Set}_\Delta^{\mathbf{ms}}) : \mathbb{U}\mathrm{n}_S$$

between the model structure on $(0,1)$-cartesian fibrations over S and the projective model structure of Set_Δ^+-enriched functors with values in marked-scaled simplicial sets.

Our first order of business will be to define the left adjoint $\mathbb{S}\mathrm{t}_S$ which will be given by a 2-categorical enhancement of the straightening functor constructed in Section 3 of [19]. Before we present our main construction, we need to give some preliminary definitions.

Definition 5.1 Let $X, Y \in \mathrm{Set}_\Delta^{\mathbf{ms}}$. We define the Gray tensor product $X \otimes Y \in \mathrm{Set}_\Delta^{\mathrm{sc}}$ (see Definition 4.1.1 in [10]) as follows:

(1) The underlying simplicial set of $X \otimes Y$ is given by $X \times Y$, the cartesian product of the underlying simplicial sets.

(2) Given a simplex $\sigma : \Delta^2 \to X \otimes Y$ let us denote by σ_X and σ_Y the projections to the corresponding factors in the cartesian product. We say that σ is scaled in $X \otimes Y$ if and only if the following conditions hold:

- (i) The projection of the simplex σ is both scaled in X and in Y.
- (ii) The restriction $\sigma_X(1 \to 2)$ is marked in X *or* the restriction $\sigma_Y(0 \to 1)$ is marked in Y.

Given marked scaled simplicial sets X, C we define marked scaled simplicial sets $\mathrm{Fun}^{\mathrm{gr}}(X, C)$ and $\mathrm{Fun}^{\mathrm{opgr}}(X, C)$ by means of the universal properties,

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathrm{sc}}}(A, \mathrm{Fun}^{\mathrm{gr}}(X, C)) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathrm{sc}}}(A \otimes X, C),$$
$$\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathrm{sc}}}(A, \mathrm{Fun}^{\mathrm{opgr}}(X, C)) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathrm{sc}}}(X \otimes A, C).$$

Definition 5.2 Let $n \geqslant 0$. We define a poset P_n as follows:

- The objects are given by subsets $S \subseteq [n]$ such $S \neq \varnothing$ and $\max(S) = n$.
- We define a partial order on P_n by declaring $S \leqslant T$ whenever $\min(S) \leqslant \min(T)$ and there exists some U such that $\min(U) = \min(S)$ and $\max(U) = \min(T)$ and such that $S \subseteq U \cup T$.

Remark 5.3 In the definition above, $U \leqslant V$ if and only if $\min(U) \leqslant \min(V)$ and for every $x \in U$ such that $x \geqslant \min(V)$ then $x \in V$. Moreover, we can identify those inequalities $U < V$ in P_n which cannot be decomposed as $U < W < V$ as:

(O1) We have $U < V$ with $\min(U) = \min(V)$ and $V = U \cup \{s\}$.

(O2) We have $U < V$ with $V = U \setminus \min(U)$.

Remark 5.4 Given $S, T \in P_n$ such that $S \leqslant T$ we can have several subsets U as above such that $S \subseteq U \cup T$. Moreover, we can order such subsets by inclusion and define $U_{S,T}$ to be the minimal subset such that $S \subseteq U_{S,T} \cup T$. Let $\min S = s$ and let $\min T = t$. We can then write $U_{S,T} = \{s, t\} \cup \{s < i < t \mid i \in S\}$.

Definition 5.5 Let $\mathcal{P}_n = \mathrm{N}(P_n)$. We promote $\mathcal{P}_n$ to a scaled simplicial set as follows. Given a 2-simplex σ represented by $S \leqslant T \leqslant W$ we declare σ to be thin if $U_{S,W} = U_{S,T} \cup U_{T,W}$.

Remark 5.6 Let $\Delta^n_\flat = (\Delta^n, \flat, \flat)$ and let $\Delta^1 \otimes_\flat \Delta^n = \Delta^1_\flat \otimes \Delta^n_\flat$. We consider $\mathfrak{C}^{\mathbf{sc}}[\Delta^1 \otimes_\flat \Delta^n]$. Recall that given $(i, j) \leqslant (k, \ell)$ in $\Delta^1 \times \Delta^n$ we have that $\mathfrak{C}^{\mathbf{sc}}[\Delta^1 \otimes_\flat \Delta^n]((i, j), (k, \ell))$ is given by the nerve of the poset of chains C,

$$(i, j) = (i_0, j_0) < (i_1, j_1) < \cdots < (i_{\alpha-1}, j_{\alpha-1}) < (i_\alpha, j_\alpha) = (k, \ell),$$

ordered by refinement. Let us suppose that $i = 0$ and that $k = 1$. Then, given a chain $C = \{(i_\alpha, j_\alpha)\}_{\alpha \in A}$ we can define m_C to be the biggest index in A such that $i_{m_C} = 0$. This allows us to define a map

$$\pi_{j,\ell} : \mathfrak{C}^{\mathbf{sc}}[\Delta^1 \otimes_\flat \Delta^n]\,((0, j), (1, \ell)) \to \mathcal{P}_{[j,\ell]}, \quad C \mapsto \bigcup_{\alpha \geqslant m_C} j_\alpha.$$

This assignment is clearly a map of posets which sends marked edges in our mapping simplicial set to identities in $\mathcal{P}_{[j,l]}$. We use the map $\pi_{j,\ell}$ to equip the left-hand side with the scaling induced by $\mathcal{P}_{[j,l]}$.

Definition 5.7 We define a colimit preserving functor $\Pi : \mathrm{Set}_\Delta^{\mathbf{mb}} \to \mathrm{Cat}_\Delta^{\mathbf{ms}}$ with values in the category of $\mathrm{Set}_\Delta^{\mathbf{ms}}$-enriched categories by specifying its values on the generators under colimits of $\mathrm{Set}_\Delta^{\mathbf{mb}}$ as follows:

(1) Given a minimally marked and biscaled simplex $\Delta^n_\flat = (\Delta^n, \flat, \flat)$ we define $\Pi(\Delta^n)$ to have as underlying Set_Δ^+-category the scaled rigidification of the Gray tensor product $\Delta^1 \otimes_\flat \Delta^n$ studied in Remark 5.6. Given $(i, j) < (k, \ell)$ in $\Pi(\Delta^n_\flat)$ we equip the mapping simplicial sets with a scaling by

declaring every triangle to be scaled if $i \neq 0$ and $k \neq 1$. If $i = 0$ and $k = 1$ we scale $\Pi(\Delta^n_\flat)((0,j),(1,\ell))$ according to Remark 5.6.

(2) Given a lean scaled 2-simplex, i.e., $\Delta^2_\dagger = (\Delta^2, \flat, \flat \subset \sharp)$ we define $\Pi(\Delta^2_\dagger)$ from $\Pi(\Delta^2_\flat)$ by scaling every triangle in the mapping simplicial sets.

(3) Given a thin scaled 2-simplex, i.e., $\Delta^2_\sharp = (\Delta^2, \flat, \sharp)$ we define $\Pi(\Delta^2_\sharp)$ from $\Pi(\Delta^2_\dagger)$ by additionally marking every morphism in $\Pi(\Delta^2_\sharp)((0,0),(1,2))$ which gets maps under the map in Remark 5.6 to the morphism $02 \to 012$ in $\mathcal{P}_2$.

(4) Given a marked edge $(\Delta^1)^\sharp = (\Delta^1, \sharp, \sharp)$ we can identify $\Pi((\Delta^1)^\sharp) = \mathfrak{C}^{\mathbf{sc}}[\Delta^1 \times \Delta^1]$.

One easily checks that our choice of decorations is compatible with composition and thus our definition yields well-defined $\mathrm{Set}^{\mathbf{ms}}_\Delta$-enriched categories and that our definition is functorial on the set of generators of $\mathrm{Set}^{\mathbf{mb}}_\Delta$. Since $\mathrm{Cat}^{\mathbf{ms}}_\Delta$ is cocomplete our functor can be extended by colimits and the definition is complete.

Definition 5.8 Let $j : \mathrm{Cat}^+_\Delta \to \mathrm{Cat}^{\mathbf{ms}}_\Delta$ be the functor that scales every 2-simplex in the mapping simplicial sets. Given a scaled simplicial set S we define a functor

$$\Pi_S : \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)_{/(S,\sharp,T_S \subset \sharp)} \to \mathrm{Cat}^{\mathbf{ms}}_\Delta, \quad X \mapsto \Pi(X) \amalg_{j \circ \mathfrak{C}^{\mathrm{sc}}[X]} j \circ \mathfrak{C}^{\mathbf{sc}}[S],$$

where $\mathfrak{C}^{\mathbf{sc}}[X]$ denotes the scaled rigidification of the *underlying* scaled simplicial set of X and where the morphism $j \circ \mathfrak{C}^{\mathbf{sc}}[X] \to \Pi(X)$ is given by the inclusion of $\Delta^{\{1\}} \times X$.

We define a further functor

$$C_S : \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)_{/(S,\sharp,T_S \subset \sharp)} \to \mathrm{Cat}^{\mathbf{ms}}_\Delta, \quad X \mapsto \Pi_S(X) \amalg_{j \circ \mathfrak{C}^{\mathrm{sc}}[X]} \Delta^0,$$

where the morphism $j \circ \mathfrak{C}^{\mathbf{sc}}[X] \to \Pi_S(X)$ is induced by the inclusion $\Delta^0 \times X \to \Delta^1 \otimes X$.

Remark 5.9 From this point on we will drop the notation $j \circ \mathfrak{C}^{\mathbf{sc}}$ and we will view Set^+_Δ-enriched categories as a full subcategory of $\mathrm{Set}^{\mathbf{ms}}_\Delta$-enriched categories consisting in those enriched categories whose mapping simplicial sets are fully scaled.

Remark 5.10 Let $f : S \to S'$ be a map of scaled simplicial sets. Given $p : X \to S$ in $\left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)_{/(S,\sharp,T_S \subset \sharp)}$ we claim that we have an isomorphism of $\mathrm{Set}^{\mathbf{ms}}_\Delta$-enriched categories

$$C_S(X) \amalg_{\mathfrak{C}^{\mathrm{sc}}[S]} \mathfrak{C}^{\mathbf{sc}}[S'] \xrightarrow{\simeq} C_{S'}(f_! X),$$

where $f_! X$ denotes the value of the functor C'_S at the object $f \circ p : X \to S'$. The isomorphism on the underlying Set^+_Δ-categories is clear. The only thing to show is that the scaling on mapping simplicial sets of the form $C_{S'}(f_! X)((0,x),(1,s'))$ is the same for both $\mathrm{Set}^{\mathbf{ms}}_\Delta$-enriched categories. This follows after a direct inspection since the scaling on those simplicial sets (which does not factor through some mapping simplicial set between objects $(1,s)$ and $(1,s')$) is independent of the base.

Definition 5.11 Let us denote by v the collapsed point in the definition of $C_S(X)$. Then for every $p : X \to S$ and every morphism of Set^+_Δenriched categories $\phi : \mathfrak{C}^{\mathbf{sc}}[S] \to \mathfrak{C}$ we can define a functor

$$\mathbb{S}\mathrm{t}_\phi(X) : \mathfrak{C} \to \mathrm{Set}^{\mathbf{ms}}_\Delta, \qquad c \mapsto \mathrm{Map}_{C_\phi(X)}(v,c), \quad \text{where } C_\phi(X) := C_S(X) \amalg_{\mathfrak{C}^{\mathrm{sc}}[S]} \mathfrak{C},$$

which we call the *straightening* of $p : X \to S$. This definition extends to a functor

$$\mathbb{S}\mathrm{t}_\phi : \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S \subset \sharp)} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta^{\mathbf{ms}})$$

with values in the category of Set_Δ^+-enriched functors. If ϕ is an isomorphism we will use the notation $\mathbb{S}\mathrm{t}_S$.

Definition 5.12 Let $\mathcal{F} : \mathcal{C} \to \mathrm{Set}_\Delta^{\mathbf{mb}}$ where $\mathcal{C}$ is a Set_Δ^+-enriched category. We define $T_{\mathcal{F}} \in \mathrm{Cat}_\Delta^{\mathbf{ms}}$ as follows:

- The objects of $T_{\mathcal{F}}$ are those of $\mathcal{C}$ in addition to an object v which we call the cone point.
- The mapping marked-scaled simplicial sets are as follows: We declare $T_{\mathcal{F}}(x, v) = \varnothing$ unless $x = v$ in which case $T_{\mathcal{F}}(v, v) = \Delta^0$. Let $c, c' \in \mathcal{C}$. We declare $T_{\mathcal{F}}(v, c) = \mathcal{F}(c)$ and $T_{\mathcal{F}}(c, c') = \mathcal{C}(c, c')$ (cf. Remark 5.9).
- Given $a, b, c \in T_{\mathcal{F}}$ such that $a, b, c \neq v$, the composition rule is that of $\mathcal{C}$. If $a = v$, then the composition rule is given by functoriality of $\mathcal{F}$, i.e., $\mathcal{F}(b) \times \mathcal{C}(b, c) \to \mathcal{F}(c)$.

Proposition 5.13 *The straightening functor given in Definition 5.11 admits a right adjoint*

$$\mathbb{U}\mathrm{n}_\phi : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta^{\mathbf{ms}}) \to \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S \subset \sharp)},$$

which we call the ***unstraightening*** *functor.*

Proof By the adjoint functor theorem it suffices to show $\mathbb{S}\mathrm{t}_\phi$ preserves all colimits in $\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S \subset \sharp)}$. To see this, we observe that by construction the functor

$$C_\phi : \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S \subset \sharp)} \to \mathrm{Cat}_\Delta^{\mathbf{ms}}$$

preserves all colimits since it is built out of colimit-preserving functors. We now observe that a functor of $\mathrm{Set}_\Delta^{\mathbf{ms}}$-categories $C_\phi(X) \to T_{\mathcal{F}}$ (see Definition 5.12) which preserves cone points and restricts to the identity on $\mathcal{C}$ is precisely the data of a natural transformation $\mathbb{S}\mathrm{t}_\phi(X) \Rightarrow \mathcal{F}$. The claim now follows since C_ϕ preserves colimits. □

Proposition 5.14 *Let $f : S \to S'$ be a map of scaled simplicial sets and consider the commutative diagram of Set_Δ^+-enriched categories*

$$\begin{tikzcd}
\mathfrak{C}^{\mathrm{sc}}[S] \arrow[r, "\mathfrak{C}^{\mathrm{sc}}[f]"] \arrow[d, "\phi"] & \mathfrak{C}^{\mathrm{sc}}[S'] \arrow[d, "\phi'"] \\
\mathcal{C} \arrow[r, "\psi"] & \mathcal{C}'
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S \subset \sharp)} \arrow[r, "\mathbb{S}\mathrm{t}_\phi"] \arrow[d, "f_!"] & \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta^{\mathbf{ms}}) \arrow[d, "\psi_!"] \\
\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S',\sharp,T_{S'} \subset \sharp)} \arrow[r, "\mathbb{S}\mathrm{t}_{\phi'}"] & \mathrm{Fun}(\mathcal{C}', \mathrm{Set}_\Delta^{\mathbf{ms}})
\end{tikzcd}$$

commutes up to invertible natural transformation where $\psi_!$ is the left adjoint to the restriction functor ψ^*.

Proof Let $\theta=\psi\circ\phi$. We will show that $\mathbb{S}\mathrm{t}'_{\phi}\circ f_! \simeq \mathbb{S}\mathrm{t}_{\theta}$ and that $\psi_!\circ\mathbb{S}\mathrm{t}_{\phi}\simeq\mathbb{S}\mathrm{t}_{\theta}$. We proceed case by case:

- To show that $\mathbb{S}\mathrm{t}'_{\phi}\circ f_! \simeq \mathbb{S}\mathrm{t}_{\theta}$, we consider the diagram of $\mathrm{Set}_{\Delta}^{\mathbf{ms}}$-categories

$$\begin{tikzcd}
\mathfrak{C}^{\mathrm{sc}}[S] \arrow[r] \arrow[d] & \mathfrak{C}^{\mathrm{sc}}[S'] \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d] \\
C_S(X) \arrow[r] & C_{S'}(f_!X) \arrow[r] & C_{\phi'}(f_!X)
\end{tikzcd}$$

The left-most square is a pushout by Remark 5.10 and right-most is also a pushout square by definition. We conclude that $C_{\phi'}(f_!X)\cong C_{\theta}(X)$. Since our constructions are natural the claim holds.

- We now show that $\psi_!\circ\mathbb{S}\mathrm{t}_{\phi}\simeq\mathbb{S}\mathrm{t}_{\theta}$. We will show that for every $\varphi:\mathfrak{C}^{\mathrm{sc}}[S]\to\mathcal{C}$, we have that $\mathbb{S}\mathrm{t}_{\varphi}\cong\varphi_!\circ\mathbb{S}\mathrm{t}_S$ which immediately implies the claim. Let $\mathcal{F}:\mathcal{C}\to\mathrm{Set}_{\Delta}^{\mathbf{ms}}$ and recall the construction $T_{\mathcal{F}}$ given in Definition 5.12. Then it follows that commutative diagrams of the form

$$\begin{tikzcd}
\mathfrak{C}^{\mathrm{sc}}[S] \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \arrow[ddr, "\iota"] & \\
C_S(X) \arrow[r] \arrow[drr] & C_{\varphi}(X) \arrow[dr, dashed] & \\
 & & T_{\mathcal{F}}
\end{tikzcd}$$

where $\iota:\mathcal{C}\to T_{\mathcal{F}}$ is the obvious fully faithful functor, correspond to natural transformations $\mathbb{S}\mathrm{t}_S(X)\Rightarrow\varphi^*\mathcal{F}$. Since $C_{\varphi}(X)$ is a pushout we conclude that this data is equivalent to natural transformations $\mathbb{S}\mathrm{t}_{\varphi}(X)\Rightarrow\mathcal{F}$ which provides the desired isomorphism $\mathbb{S}\mathrm{t}_{\varphi}(X)\cong\varphi_!\,\mathbb{S}\mathrm{t}_S(X)$. Our constructions are natural and so the claim holds. □

Remark 5.15 In the situation above passing to right adjoints we obtain an isomorphism of functors $\mathbb{U}\mathrm{n}_{\phi}\circ\psi^*\cong f^*\circ\mathbb{U}\mathrm{n}_{\phi'}$.

Lemma 5.16 *For any simplicial set (S,T_S) the functor $\mathbb{S}\mathrm{t}_S$ preserves cofibrations.*

Proof Since $\mathbb{S}\mathrm{t}_S$ preserves colimits it will be enough to prove the claim on the generating class of cofibrations given in Definition 3.11. Moreover, given a cofibration $\alpha:A\to B$ we can use Proposition 5.14 to reduce to the case where $S=B$. The case $\varnothing\to\Delta^0$ is obviously true. For the rest of the generators we have that $B=(\Delta^n,\flat)$ for $n\geqslant 1$ or $B=(\Delta^2,\sharp)$ and that the map $\mathbb{S}\mathrm{t}_B\,A(i)\to\mathbb{S}\mathrm{t}_B\,B(i)$ is the identity except when $i=n$ in which case the map is a cofibration. It is immediate to see that the map in this situation $\mathbb{S}\mathrm{t}_B\,A\to\mathbb{S}\mathrm{t}_B\,B$ has the left lifting property against the class of trivial fibrations. □

Remark 5.17 Let $\iota:\mathrm{Set}_{\Delta}^{+}\to\mathrm{Set}_{\Delta}^{\mathbf{ms}}$ be the functor defined by $\iota(X,E_X)=(X,E_X,\sharp)$. Given a scaled simplicial set (S,T_S) let $\iota_*\,\mathrm{St}_S:\left(\mathrm{Set}_{\Delta}^{+}\right)_{/S}\to\mathrm{Fun}(\mathfrak{C}^{\mathrm{sc}}[S],\mathrm{Set}_{\Delta}^{\mathbf{ms}})$ be the straightening functor given in Definition 3.5.4 in [19] postcomposed with the enriched functor ι. Recalling the definition of the functor R

(see Definition 3.30), we can then define a functor

$$\mathbb{S}\mathrm{t}_S \circ R : \left(\mathrm{Set}_\Delta^+\right)_{/S} \to \mathrm{Fun}(\mathfrak{C}^{\mathrm{sc}}[S], \mathrm{Set}_\Delta^{\mathbf{ms}}).$$

It follows that $\iota_* \mathrm{St}_S$ and $\mathbb{S}\mathrm{t}_S \circ R$ differ only in the scaling and thus induce a natural transformation $\eta_S : \mathbb{S}\mathrm{t}_S \circ R \Rightarrow \iota_* \mathrm{St}_S$.

Proposition 5.18 *Let (S, T_S) be a scaled simplicial set. Then $\eta_S : \mathbb{S}\mathrm{t}_S \circ R \Rightarrow \iota_* \mathrm{St}_S$ is an object-wise weak equivalence.*

Proof It is clear that both functors preserve colimits. Moreover, a totally analogous proof to that of Lemma 5.16 shows that $\iota_* \mathrm{St}_S$ preserves cofibrations. It is also not hard to verify that $\iota_* \mathrm{St}_S$ satisfies similar base change properties as those in Proposition 5.14. We conclude that it will be enough to show that $\eta_k = \eta_{\Delta^k_\flat}(\Delta^k)$ is an equivalence for $k \geqslant 0$ and similarly for $\eta_1^\sharp = \eta_{(\Delta^1)^\sharp}((\Delta^1)^\sharp)$. It is easy to check that η_0, η_1, η_2 and $\eta_1^\sharp$ are all isomorphisms.

For $k \geqslant 3$ let us define $\mathbb{L}^k = \Pi_{\Delta^k_\flat}((\Delta^k, \flat, \flat \subset \sharp))$ and $\mathbb{L}^k_\sharp$ by scaling every triangle in the mapping simplicial categories of $\mathbb{L}^k$. It is not hard to see that our problem can be reduced to showing that the map

$$\varphi_k : \mathbb{L}^k \to \mathbb{L}^k_\sharp$$

is an equivalence of $\mathrm{Set}_\Delta^{\mathbf{ms}}$-enriched categories. Using induction on k we can assume that the mapping simplicial sets $\mathbb{L}^k((0,i),(1,j))$ are maximally scaled except if $i = 0$ and $j = k$. Let $\mathbb{L}^k((0,0),(1,k)) = A^k$ and similarly $\mathbb{L}^k_\sharp((0,0),(1,k)) = A^k_\sharp$. We observe that for every face $d_i : \Delta^{k-1} \to \Delta^k$ we have a commutative diagram

$$\begin{tikzcd}
A^{k-1} \arrow[r, "\simeq"] \arrow[d, "\alpha_i"] & A^{k-1}_\sharp \arrow[d] \\
A^k \arrow[r] & A^k_\sharp
\end{tikzcd}$$

where the top horizontal morphism is a weak equivalence. We can therefore assume inductively that the triangles of A^k which are in the image of the maps α_i for $i = 0, \dots, k$ are all scaled.

Let $\sigma : C_0 \subset C_1 \subset C_2$ be a triangle in A^k and let us fix the notation $C_j = \{(\varepsilon_i^j, a_i^j\}_{i=0}^{\ell}$. Let $e = C_0 \subset C_1$ be an edge in A^k. We define $S(e)$ as the set of nondegenerate simplices with initial vertex C_0 and final vertex C_1. We finally set $|e| = \max\{\dim(\phi) \mid \phi \in S(e)\}$. Note that this a well-defined number. Given a 2-simplex σ as above we define $|\sigma| = |d_1(\sigma)|$. We will show that we can scale σ using a pushout along a **MS**-anodyne morphism using induction on $|\sigma| = \ell$. The case $\ell = 2$ follows easily after direct inspection. So we will assume from this point on that $\ell > 1$.

Let us suppose that the claim holds for those 2-simplices τ such that $|d_2(\tau)| = 1$ and for those 2-simplices σ such that $|\sigma| < \ell - 1$. Then given $\sigma : C_0 \subset C_1 \subset C_2$ such that $|d_2(\sigma)| > 1$ and such that $|\sigma| = \ell$ we can construct a 3-simplex $\rho : C_0 \subset D \subset C_1 \subset C_2$ such that the following holds:

(1) We have $|d_2 d_3(\rho)| = 1$; in particular, $d_i(\rho)$ is thin scaled for $i = 2, 3$.

(2) We have $|d_0(\rho)| = \ell - 1$ and is therefore scaled.

It follows that we can fully scale ρ using a pushout along a morphism of type (M2) in Definition 3.36. Therefore, we have reduced our problem to proving the claim above.

Let $\sigma : C_0 \subset C_1 \subset C_2$ such that $|d_2(\sigma)| = 1$. We observe that unless $C_0 = (0,0) < (1,k)$ then σ is scaled. Otherwise we could express σ as a certain composition in the category $\mathbb{L}^k$ and it would follow from the induction hypothesis that σ is thin.

We can now see that $C_1 = (0,0) < (\varepsilon, a) < (1,k)$ which leads us to consider cases depending on the parameter $\varepsilon \in \{0,1\}$.

($\boldsymbol{\varepsilon = 1}$) Then we can assume without loss of generality that C_2 contains an element of the form $(0,x)$ in C_2 with $x \neq 0$. This is true since otherwise the maps $\pi_{0,k}$ in Remark 5.6 would show that σ is already scaled. Moreover, we can further assume that there is only element of the form $(0,x)$ in C_2. Indeed, if we had some $(0,y) < (0,x)$ then we can produce a 3-simplex $\rho : C_0 \to C_1 \to \widetilde{C}_2 \to C_2$ where $\widetilde{C}_2 = C_2 \setminus \{(0,y)\}$. Since $d_0(\rho)$ and $d_1(\rho)$ must be scaled by definition it follows that we can scale $d_2(\rho)$ if and only if we can scale $d_3(\rho)$ and thus the claims follows. Additionally, we note that if $x \neq 1$ then σ factors through one of the morphisms $\alpha_j : A^{k-1} \to A^k$ above. We finally see that in this case $\pi_{0,k}(\sigma)$ is given by a simplex of the form $0n \to 0an \to S$ with $\min(S) = 1$ and it is consequently scaled in $\mathcal{P}_k$.

($\boldsymbol{\varepsilon = 0}$) Observe that if C_2 contains an element of the form $(0,x)$ with $x < a$ we can define $\widetilde{C}_2$ as above an produce a 3-simplex $\rho : C_0 \to C_1 \to \widetilde{C}_2 \to C_2$ which shows that we can scale $\sigma = d_2(\rho)$ if and only if we can scale $d_3(\rho)$. In a totally analogous way as in the case $\varepsilon = 1$ we can assume that $a = 1$. If C_2 does not contain any element of the form $(0,z)$ with $z > 1$ then σ must be already scaled. Moreover, we can assume without loss of generality that C_2 only contains one element of the form $(0,z)$ using a similar argument as before by constructing a certain $\widetilde{C}_2$. We can assume that in this case $z = 2$ since otherwise, the simplex factors through a certain morphism $\alpha_j : A^{k-1} \to A^k$. If C_2 does not contain an element of the form $(1,s)$ with $s \neq k$ it follows by direct inspection that σ is already scaled. If this is not the case we consider D which is obtained from C_2 by discarding every element of the form $(1,s)$ with $s \neq k$. Then we get a 3-simplex $\Xi : C_0 \to C_1 \to D \to C_2$. One easily checks that every face of Ξ is scaled except possibly $d_2(\Xi) = \sigma$ and thus our result follows. □

Corollary 5.19 *Let (S, T_S) be a scaled simplicial set. Consider a weak equivalence $u : (A, E_A) \to (B, E_B)$ in $(\mathrm{Set}_\Delta^+)_{/S}$. Then the functor $\mathbb{St}_S$ sends the morphism $R(u) : (A, E_A, T_A \subset \sharp) \to (B, E_B, T_B \subset \sharp)$ (see Definition 3.30) to a weak equivalence in $\mathrm{Fun}(\mathfrak{C}^{\mathrm{sc}}[S], \mathrm{Set}_\Delta^{\mathrm{ms}})$.*

Remark 5.20 Let $i : A \to B$ be a cofibration of **MB** simplicial sets which we view as a morphism in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/B}$ and recall the definition of Π_B in Definition 5.8. We define $\Pi_B(A)^{\uparrow}$ as the pushout

$$\begin{tikzcd}
j \circ \mathfrak{C}^{\mathrm{sc}}[A] \times \Delta^{\{0\}} \arrow[r] \arrow[d] & j \circ \mathfrak{C}^{\mathrm{sc}}[B] \times \Delta^{\{0\}} \arrow[d] \\
\Pi_B(A) \arrow[r] & \Pi_B(A)^{\uparrow}
\end{tikzcd}$$

It follows from our definition that in order to check that $\mathbb{S}\mathrm{t}_B(A) \Rightarrow \mathbb{S}\mathrm{t}_B(B)$ is a pointwise weak equivalence it suffices to check that the induced map $\Pi_B(A)^{\uparrow} \to \Pi_B(B)$ is a weak equivalence of $\mathrm{Set}_{\Delta}^{\mathbf{ms}}$-categories.

Remark 5.21 Let $\theta : \Delta^{n+1} \to \Delta^1 \times \Delta^n$ be a nondegenerate simplex and let $i \in \Delta^{n+1}$ be the biggest element such that $\theta(i) = (i,0)$. We will use the notation $\theta = \sigma_i$ and give an order in the set of nondegenerate simplices in $\Delta^1 \times \Delta^n$ of maximal dimension by declaring $\sigma_i < \sigma_j$ if $i < j$.

Definition 5.22 Let $\mathbb{K}_n = \Pi_{\Delta^n_\flat}((\Delta^n,\flat,\flat))$ and let $K_n = \mathbb{K}_n((0,0),(1,n))$. For every σ_i as in Remark 5.21, define K_n^i as the subposet (with the inherited decorations) of K_n consisting in those chains whose elements are in the image of σ_i. Observe that K_n^i is isomorphic as a simplicial set to $C_n = (\Delta^1)^n$.

Given $0 < s < n+1$ we define $d_s(K_n^j)$ as the simplicial set of K_n^j consisting in those chains whose elements are in the image of $d_s(\sigma_j)$.

Lemma 5.23 *Let $0 < j < n$ and let $d_{j+1}(K_n^j) \subset K_n^j$ be the simplicial subset* (*with the induced decorations*) *consisting in those chains whose elements factor through $d_{j+1}(\sigma_j)$. We view $\Delta^1 \times d_{j+1}(K_n^j)$ as marked-scaled simplicial set as follows*:

- *The marking consists in those marked edges in the cartesian product together with the edges contained in $\Delta^1 \times \{e\}$ where $e : C_0 \to C_1$ is a marked edge in K_n^j such that C_0 contains the element $(0,j)$.*
- *The scaling is given by the usual scaling on the cartesian product.*

Then we have an isomorphism of marked-scaled simplicial sets $\Delta^1 \times d_{j+1}(K_n^j) \simeq K_n^j$

Proof We only need to check that the decorations agree on each side. To show the claim regarding the marking we note that the marked edges of K_n^j always factor through a face of the cube C_n (see Definition 5.22) and thus the conclusion follows after direct inspection.

To see that the scaling of K_n^j is given by the product scaling we consider a triangle $\sigma : C_0 \to C_1 \to C_2$ and we define $D_i = C_i \setminus \{(1,j)\}$ which yields another triangle $\varphi : D_0 \to D_1 \to D_2$. We claim that σ is scaled in K_n^j if and only if φ is. Observe that since φ lies always in $d_{j+1}(K_n^j)$ this will be enough to show the claim regarding the scaling.

We set the notation $\pi_{0,n}(C_i) = S_i$ and $\pi_{0,n}(D_i) = T_i$. Then we have the following:

(1) We have that $S_i = T_i$ if $(0,j) \in C_i$ and $T_i = S_i \setminus \{j\}$ otherwise.

(2) We have $\min(S_i) = \min(T_i)$.

(3) Given $C \in d_{j+1}(K_n^j)$ and setting $V = \pi_{0,n}(C)$, it follows that $\min(V) \leqslant j$.

Our final claim is that for $i < j$ we have that (see Remark 5.4) $U_{S_i,S_j} = U_{T_i,T_j}$. Observe that $j \in U_{S_i,S_j}$ if and only if $j \in \{\min(S_i), \min(S_j)\}$ or if $j \in S_i$ and $\min(S_i) < j < \min(S_j)$. However by (3) above this cannot be the case. □

Lemma 5.24 *Let $(A, E_A, T_A) \subset (B, E_B, T_B)$ be an inclusion of marked-scaled simplicial sets such that $E_A = E_B$. Suppose that there exists some vertex $v \in A$ with the following property*:

- *For every simplex $\sigma : \Delta^n \to B$ which does not factor through A, v is the final vertex of σ.*

Let M_A *(resp.* M_B*) be the collection of marked edges in the cartesian product (of marked-scaled simplicial sets)* $\Delta^1 \times A$ *(resp.* $\Delta^1 \times B$*) together with the edge* $\Delta^1 \times \{v\}$*. Then the induced morphism*

$$j : \Delta^1 \times A \amalg_{\Delta^{\{1\}} \times A} B \to \Delta^1 \times B,$$

where both simplicial sets are equipped with the product scaling and the marking given by M_A *and* M_B*, respectively, is a trivial cofibration in* $\mathrm{Set}_{\Delta}^{\mathbf{ms}}$*.*

Proof First let us assume that the claim holds for $E_A = E_B = \flat$. Then it follows that for a general marking the map j is obtained as a pushout of the map $j_\flat$ where the latter map is the inclusion associated to the minimal marking. This shows that the general result will follow.

Working simplex by simplex we can reduce the problem to the cases

(i) $(\partial\Delta^n, \flat, \flat) \to (\Delta^n, \flat, \flat)$ for $n \geqslant 1$,

(ii) $(\Delta^2, \flat, \flat) \to (\Delta^2, \flat, \sharp)$,

where v is given by the final vertex.

To check (i) we use an argument analogous to Lemma 3.5.12 in [19] which tells us that in this case the map j above is in the weakly saturated class of morphisms of type (M1) in Definition 3.36 and of type

($*$) $(\Lambda^n_n, \Delta^{\{n-1,n\}}, \Delta^{\{0,n-1,n\}}) \to (\Delta^n, \Delta^{\{n-1,n\}}, \Delta^{\{0,n-1,n\}})$

and thus the claim holds. To prove (ii) we note that we can scale the remaining simplices using pushouts along morphisms of type (M2) in Definition 3.36 together with the morphism

($\diamond$) $(\Delta^3, \Delta^{\{n-1,n\}}, U_3) \to (\Delta^3, \Delta^{\{n-1,n\}}, \sharp)$,

where U_3 is the collection of all triangles except $\Delta^{\{0,1,2\}}$. □

Lemma 5.25 *Let* $i : A \to B$ *be a morphism of type (A1) in Definition 3.4. Then the induced natural transformation* $\mathbb{S}\mathrm{t}_B(A) \Rightarrow \mathbb{S}\mathrm{t}_B(B)$ *is a pointwise weak equivalence.*

Proof Since $\mathbb{S}\mathrm{t}_B$ preserves colimits and cofibrations it will be enough to prove the claim in the specific case where $A = (\Lambda^n_i, \flat, \flat \subset \Delta^{\{i-1,i,i+1\}})$ and $B = (\Delta^n, \flat, \flat \subset \Delta^{\{i-1,i,i+1\}})$. We will show according to Remark 5.20 that the map $\Pi_B(A)^{\uparrow} \to \Pi_B(B)$ is an equivalence of $\mathrm{Set}_{\Delta}^{\mathbf{ms}}$-categories.

We observe that the induced morphism of marked-scaled simplicial sets $\Pi_B(A)^{\uparrow}(x, y) \to \Pi_B(B)(x, y)$ is an isomorphism except when $x = (0, 0)$ and $y = (1, n)$. Recall the definition of K_n in Definition 5.22 and equip this marked-scaled simplicial set with the decorations induced from $\Pi_B((0, 0), (1, n))$. We similarly define $\Lambda^n_i K = \Pi_B(A)^{\uparrow}((0, 0), (1, n))$. We define a filtration

$$\Lambda^n_i K = A_{-1} \to A_0 \to \cdots \to A_{n-1} \to A_n = K_n,$$

where A_s is the subsimplicial set of K_n containing every simplex which factors through K^j_n for $j \leqslant s$ (see Definition 5.22 for a definition of K^j_n). We will show that each of the steps in this filtration is a weak equivalence. The case $n = 2$ follows easily by a direct computation. For the rest of the proof we will assume that $n \geqslant 3$.

For $0 \leqslant j \leqslant n$ we consider the pullback-pushout diagram

$$\begin{tikzcd}
Q_n^j \arrow[r] \arrow[d] & K_n^j \arrow[d] \\
A_{j-1} \arrow[r] & A_j
\end{tikzcd}$$

We will show that the top horizontal morphism is a trivial cofibration. First we consider the case $0 \leqslant j < n$. We describe Q_n^j as a simplicial subset of K_n^j which contains every face of the cube C_n, except those that factor through $d_\alpha(K_n^j)$ for $\alpha \notin \Phi(i)$ where

$$\Phi(i) = \begin{cases} \{j+1, i+1\} & \text{if } j < i, \\ \{j+1\} & \text{if } j = i, \\ \{j+1, i\} & \text{if } j > i. \end{cases}$$

We produce a 2-step filtration $Q_n^j \to Z_n^j \to K_n^j$ where Z_n^j is obtained from Q_n^j by attaching the simplices in $d_\beta(K_n^j)$ where $\beta \neq j+1$ and $\beta \in \Phi(i)$.

We observe that for $n \geqslant 3$ we have that every marked edge in $d_{j+1}(K_n^j)$ factors through Z_n^j. Therefore we can use Lemma 5.24 with $B = d_{j+1}(K_n^j)$ and $A = (Z_n^j)_{|d_{j+1}(K_n^j)}$ to obtain a trivial cofibration of marked-scaled simplicial sets

$$\varphi : \Delta^1 \times (Z_n^j)_{|d_{j+1}(K_n^j)} \to \Delta^1 \times d_{j+1}(K_n^j).$$

As a consequence of Lemma 5.23 we see that the scaling of $\Delta^1 \times A$ and the scaling of Z_n^j coincide except possibly in those triangles coming from $\Delta^{\{i-1,i,i+1\}}$ and similarly for $\Delta^1 \times B$ and K_n^j. We further note that every marked edge of K_n^j factors through Z_n^j. After direct inspection we observe that we can produce a pushout diagram

$$\begin{tikzcd}
\Delta^1 \times (Z_n^j)_{|d_{j+1}(K_n^j)} \arrow[r] \arrow[d] & \Delta^1 \times d_{j+1}(K_n^j) \arrow[d] \\
Z_n^j \arrow[r] & K_n^j
\end{tikzcd}$$

Therefore we can add the remaining decorations via a pushout of φ along a cofibration which shows that the last step in the filtration is a trivial cofibration. To show that $Q_n^j \to Z_n^j$ is a trivial cofibration we consider a pushout-pullback diagram

$$\begin{tikzcd}
Z_{n-1}^u \arrow[r] \arrow[d] & d_\beta(K_n^j) = K_{n-1}^u \arrow[d] \\
Q_n^j \arrow[r] & Z_n^j
\end{tikzcd}$$

and conclude by the previous argument or by a direct computation if $n = 3$.

To finish the proof we will show that $Q_n^n \to K_n^n$ is a trivial cofibration. In this case Q_n^n contains every simplex that factors through a face of C_n except those factoring through $d_i(K_n^n)$. For every $k \in [n]$ we define a chain

$$D_k = (0,0) < (0,1) < \cdots < (0,k) < (1,n).$$

We observe that using morphisms of type (M2) in Definition 3.36 and morphisms of type ($\diamond$) as in the proof of Lemma 5.24 we can scale the triangle $D_{i-1} \subset D_i \subset D_{i+1}$ in A_{n-1} and in A_n. Recall that for every chain $C = \{(i_\alpha, j_\alpha)\}_{\alpha \in A}$ we defined m_C to be the biggest element in A such that $i_{m_C} = 0$. We can use this parameter to define a map

$$r_n : K_n^n \to (\Delta^n, \flat, \Delta^{\{i-1,i,i+1\}}), \quad C \mapsto j_{m_C}.$$

Moreover r_n admits a section s_n which sends j to D_j as defined above. It follows that there exists a marked homotopy between $s_n \circ r_n$ and the identity map on K_n^n. Furthermore, r_n restricts to a map $\hat{r}_n : Q_n^n \to \Lambda_i^n$. One checks that since Q_n^n and Λ_i^n can be expressed as iterated pushouts along cofibrations indexed by $(n-1)$-dimensional faces of Q_n^n and the map r_n restricts to an equivalence in each of its faces, that $\hat{r}_n$ is also a weak equivalence. We conclude that we have a commutative diagram

$$\begin{tikzcd}
Q_n^n \arrow[r] \arrow[d, "\simeq"] & K_n^n \arrow[d, "\simeq"] \\
(\Lambda_i^n, \flat, \Delta^{\{i-1,i,i+1\}}) \arrow[r, "\simeq"] & (\Delta^n, \flat, \Delta^{\{i-1,i,i+1\}})
\end{tikzcd}$$

and so the result follows by two-out-of-three. □

Lemma 5.26 *Let* $i : A \to B$ *be a morphism of type* (A3) *in* Definition 3.4. *Then the induced natural transformation* $\mathbb{S}\mathrm{t}_B(A) \Rightarrow \mathbb{S}\mathrm{t}_B(B)$ *is a pointwise weak equivalence.*

Proof The proof will mirror the strategy of the previous lemma. Again, we observe that the induced morphism of mapping simplicial sets $\Pi_B(A)^{\uparrow}(x,y) \to \Pi_B(B)(x,y)$ is an isomorphism except when $x = (0,0)$ and $y = (1,n)$ or when $x = (0,1)$ and $y = (1,n)$. However we observe that since the edge $(0,0) \to (0,1)$ will be collapsed in order to define the value of the functor $\mathbb{S}\mathrm{t}_B$ it will suffice to construct the analogous filtration

$$\Lambda_0^n K = A_{-1} \to A_0 \to \cdots \to A_{n-1} \to A_n = K_n$$

and show that each step is a trivial cofibration. As before, we will leave the case $n = 2$ as an easy exercise and focus our attention to the cases $n \geqslant 3$. Note that in this case, we decorations coming from the marked edge $0 \to 1$ and the thin triangle $0 \to 1 \to n$ are already contained in $\Lambda_0^n K$.

For $0 \leqslant j < n$ we consider pullback-pushout diagrams

$$\begin{tikzcd}
Q_n^j \arrow[r] \arrow[d] & K_n^j \arrow[d] \\
A_{j-1} \arrow[r] & A_j
\end{tikzcd}$$

where Q_n^j is the simplicial subset of K_n^j which contains every face of C_n except the $d_{j+1}(K_n^j)$ and if $j > 0$ the face C_n^0 consisting in those chains that have the element $(0,1)$. The proof at this point is totally analogous to the proof of Lemma 5.25. We construct Z_n^j by adding to Q_n^j the face C_n^0 and conclude by Lemma 5.24 that each step in the filtration $Q_n^j \to Z_n^j \to K_n^j$ is given by a trivial cofibration.

To show that $A_{n-1} \to A_n$ is a trivial cofibration we need to work a little bit harder. First we consider a commutative diagram where we are using circled arrows to represent marked morphisms

$$\begin{tikzcd}
(0,0) < (1,0) < (1,n) \arrow[r, "\circ" marking] & (0,0) < (1,0) < (1,1) < (1,n) \\
(0,0) < (1,n) \arrow[r] \arrow[u, "\circ" marking] \arrow[d] & (0,0) < (1,1) < (1,n) \arrow[u, "\circ" marking] \arrow[d, "\circ" marking] \\
(0,0) < (0,1) < (1,n) \arrow[r, "\circ" marking] & (0,0) < (0,1) < (1,1) < (1,n)
\end{tikzcd}$$

We note that every 2-simplex in this diagram is scaled and therefore we can mark every morphism via a pushout along a trivial cofibration. Let us remark that we can mark this morphisms in both A_{n-1} and A_n since $n \geqslant 3$. Recall the definition of D_i in Lemma 5.25 and consider a 3-simplex

$$\rho_W : D_0 \subset D_1 \subset (0,0) < (0,1) < (0,n) < (1,n) \subset (0,0) < (0,1) < (0,n) < (1,n) \cup W,$$

where W is any chain starting at $(0,1)$ and ending at $(0,n)$. It follows that every face of ρ_W is scaled except possibly $d_2(\rho_W)$. Therefore we might scale that face using a pushout along a morphism of type (M2) in Definition 3.36. Note that any possible ρ_W factors through A_{n-1} except in the case where W is the maximal chain. A similar argument as in the previous lemma shows that we have a commutative diagram

$$\begin{tikzcd}
A_{n-1} \arrow[r] \arrow[d, "\simeq"] & A_n \arrow[d, "\simeq"] \\
(\Lambda_0^n, \{\Delta^{\{0,1\}}\}, \{\Delta^{\{0,1,n\}}\}) \arrow[r, "\simeq"] & (\Delta^n, \{\Delta^{\{0,1\}}\}, \{\Delta^{\{0,1,n\}}\})
\end{tikzcd}$$

and the claim follows from two-out-of-three. □

Lemma 5.27 *Let $i : A \to B$ be a morphism of type (M2) in (M2). Then the induced natural transformation $\mathbb{S}\mathrm{t}_B(A) \Rightarrow \mathbb{S}\mathrm{t}_B(B)$ is a pointwise weak equivalence.*

Proof Let $L = \Pi_B(A)^{\uparrow}((0,0),(1,4))$ and let $K_4 = \Pi_B(B)((0,0),(1,4))$. The only thing we need to show is that the map $L \to K_4$ is a weak equivalence. Using the notation from the previous proofs it follows that the missing scaled simplices in L are all contained in K_4^4. Therefore if we denote by L_4^4 the restriction of K_4^4 to L we see that it will be enough to show that the induced map $L_4^4 \to K_4^4$ is a weak equivalence. However, one can easily construct a commutative diagram

$$\begin{tikzcd}
L_4^4 \arrow[r] \arrow[d, "\simeq"] & K_4^4 \arrow[d, "\simeq"] \\
(\Delta^4, \flat, T) \arrow[r, "\simeq"] & (\Delta^4, \flat, T')
\end{tikzcd}$$

where vertical maps are weak equivalences and the scaling in the bottom horizontal map is that of (A2) in Definition 3.4. □

Proposition 5.28 *Let (S,T_S) be a scaled simplicial set and let $i: A \to B$ be an* **MB***-anodyne morphism in $\big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S\subset\sharp)}$. Then $\mathbb{S}\mathrm{t}_S(i)$ is a weak equivalence in $\mathrm{Fun}(\mathfrak{C}^{\mathrm{sc}}[S],\mathrm{Set}_\Delta^{\mathbf{ms}})$.*

Proof Using Proposition 5.14 we can assume that $S=B$. The rest of the proof will consist in verifying that the claim holds for each of the generators given in Definition 3.4. We proceed case by case:

(A1) This follows from Lemma 5.25.

(A2) This follows from Lemma 5.27.

(A3) This follows from Lemma 5.26.

(A4) This follows from Corollary 5.19.

(S1) This follows from Corollary 5.19.

(S2) This follows by explicit verification.

(E) This follows from Corollary 5.19. □

Proposition 5.29 *Let (S,T_S) be a scaled simplicial set and let $\phi:\mathfrak{C}^{\mathrm{sc}}[S]\to\mathcal{C}$ be a functor of Set_Δ^+-enriched categories. Then the straightening functor*

$$\mathbb{S}\mathrm{t}_\phi : \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/(S,\sharp,T_S\subset\sharp)} \to \mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta^{\mathbf{ms}})$$

is a left Quillen functor.

Proof We will show that $\mathbb{S}\mathrm{t}_\phi$ preserves cofibrations and weak equivalences. First, we point out that due to Proposition 5.14 it will be enough to consider the case $\phi=\mathrm{id}$. In this case, we saw in Lemma 5.16 that our functor preserves cofibrations.

To address the claim regarding weak equivalences, we see that Proposition 5.28 implies $\mathbb{S}\mathrm{t}_S$ preserves **MB**-anodyne morphisms. We can therefore restrict our attention to showing that $\mathbb{S}\mathrm{t}_S$ preserves weak equivalences between fibrant objects. To this end it will be enough to show the following:

($*$) Let $f,g: X\to Y$ be morphisms between fibrant objects such that $\mathbb{S}\mathrm{t}_S(f)$ is a weak equivalence. Then given a homotopy $H: X\times(\Delta^1)^\sharp\to Y$ between f and g it follows that $\mathbb{S}\mathrm{t}_S(g)$ is also a weak equivalence.

The claim follows after noting that we have an anodyne morphism $X\times\Delta^{\{0\}}\to X\times(\Delta^1)^\sharp$ due to Proposition 3.12 which implies that $\mathbb{S}\mathrm{t}_S(H)$ is a weak equivalence as well as the map induced by the projection onto X, $\mathbb{S}\mathrm{t}_S(p):\mathbb{S}\mathrm{t}_S(X\times(\Delta^1)^\sharp)\to\mathbb{S}\mathrm{t}_S(X)$. □

5.2 Straightening over a point

The goal of this section is to prove the following result.

Proposition 5.30 *The straightening-unstraightening adjunction over the point*

$$\mathbb{St}_* : \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/\Delta^0} \rightleftarrows \mathrm{Set}_\Delta^{\mathbf{ms}} : \mathbb{Un}_*$$

is a Quillen equivalence.

To do this we will construct a left Quillen equivalence

$$L_* : \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/\Delta^0} \to \mathrm{Set}_\Delta^{\mathbf{ms}}$$

and a natural transformation $\alpha : \mathbb{St}_* \Rightarrow L$ which is pointwise a weak equivalence of marked-scaled simplicial sets.

Proposition 5.31 *Let $L_* : \big(\mathrm{Set}_\Delta^{\mathbf{mb}}\big)_{/\Delta^0} \to \mathrm{Set}_\Delta^{\mathbf{ms}}$ be the functor that assigns to an **MB** simplicial set $(X, E_X, T_X \subseteq C_X)$ the marked scaled simplicial set (X, E_X, C_X). Then L_* is a left Quillen equivalence.*

Proof The functor L_* admits a right adjoint R_* which is given by $R_*(X, E_X, T_X) = (X, E_X, T_X)$. We observe that $L_* \circ R_* = \mathrm{id}$ and that the unit $\mathrm{id} \Rightarrow R_* \circ L_*$ is given by $(X, E_X, T_X \subset C_X) \to (X, E_X, C_X)$ which is in the weakly saturated class of morphisms of type (S2) in Definition 3.4.

To finish the proof we observe that L_* preserves cofibrations and maps **MB**-anodyne morphisms to **MS**-anodyne morphisms. It is then easy to see that L_* preserves weak equivalences between fibrant objects and the result follows. □

Recall the definition of the maps $\pi_{0,n}$ in Remark 5.6. Then postcomposing this map with the morphism $m_n : \mathcal{P}_n \to (\Delta^n, \flat)$ which assigns to every $S \in \mathcal{P}_n$ the value $m_n(S) = \min(S)$ we obtain a map of marked scaled simplicial sets

$$\alpha_n^\flat : \mathbb{St}_*((\Delta^n, \flat, \flat)) \to (\Delta^n, \flat, \flat).$$

One can easily produce marked variants of this maps $\alpha_2^\sharp$ and $\alpha_1^\sharp$ associated to the **MB** simplicial sets $(\Delta^2, \flat, \sharp)$ and $(\Delta^1, \sharp, \sharp)$. We would like to remark that $\mathbb{St}_*(\Delta^2, \flat, \flat \subset \sharp) = \mathbb{St}_*(\Delta^2, \flat, \sharp)$ which justifies why we are defining only one $\alpha_2^\sharp$. Since our definitions are functorial with respect to monotone morphisms $[k] \to [n]$ this collection of maps assemble into a natural transformation $\alpha : \mathbb{St}_* \Rightarrow L_*$.

Proposition 5.32 *The natural transformation $\alpha : \mathbb{St}_* \Rightarrow L_*$ is a pointwise weak equivalence.*

Proof Since both functors are left Quillen and both model categories are left proper, a simplex by simplex argument shows that it will be enough to show that the components $\alpha_n^\flat$, $\alpha_2^\sharp$ and $\alpha_1^\sharp$ are weak equivalences. We further observe $\alpha_2^\sharp$ and $\alpha_1^\sharp$ can be obtained from their undecorated countermarks via pushouts along cofibrations. This shows that we can restrict our attention to $\alpha_n^\flat$ for $n \geqslant 0$.

Let $S = (\Delta^n, \flat)$ and consider $\Pi_S(\Delta^n, \flat, \flat)$ (see Definition 5.8). Denote by Φ_n the $\mathrm{Set}_\Delta^{\mathbf{ms}}$-category obtained from $\Pi_S(\Delta^n, \flat, \flat)$ by marking every edge in the mapping simplicial sets of the form

$$\Pi_S(\Delta^n, \flat, \flat)((1, a), (1, b)).$$

We will show that the map $\hat{\alpha}_n^\flat : K_n = \Phi_n((0, n), (1, n)) \to (\Delta^n, \flat)$ is a weak equivalence. Note that $\alpha_n^\flat$ is obtained from $\hat{\alpha}_n^\flat$ after identifying certain simplices. It is easy to see that we can mark every edge in K_n

whose image under $\widehat{\alpha}_n^\flat$ becomes degenerate using pushouts along **MB**-anodyne morphisms. We consider a filtration

$$A_{-1} = K_n^0 \to A_1 \to A_2 \to \cdots \to A_{n-1} \to A_n = K_n,$$

where A_i is obtained from A_{i-1} by attaching those simplices contained in K_n^i (see Definition 5.22) where K_n^i has the decorations induced from Φ_n. We further denote by $\overline{A}_i$ the image of A_i under the collapse map in the definition of $\mathbb{S}\mathrm{t}_*((\Delta^n,\flat,\flat))(*)$ and similarly denote $\overline{K}_n^i$.

We will show that the restriction of $\alpha_n^\flat$ to each $\overline{A}_i$ defines a weak equivalence

$$\alpha_{n,i}^\flat : \overline{A}_i \to (\Delta^{[0,i]},\flat,\flat).$$

Since weak equivalences are stable under filtered colimits this will imply the result. Assume that $\alpha_{n,j}^\flat$ is a weak equivalence for $j \leqslant i-1$ and consider the pullback-pushout square

$$\begin{tikzcd}
Q_n^i \arrow[r] \arrow[d] & K_n^i \arrow[d] \\
A_{i-1} \arrow[r] & A_i
\end{tikzcd}$$

Observe that $\widehat{\alpha}_n^\flat$ induces a commutative diagram

$$\begin{tikzcd}
Q_n^i \arrow[r] \arrow[d] & K_n^i \arrow[d, "r_i"] \\
\Delta^{[0,i-1]} \arrow[r] & \Delta^{[0,i]}
\end{tikzcd}$$

We claim that the vertical morphisms are weak equivalences. Note that we can define, for $0 \leqslant j \leqslant i$,

$$C_j = (0,0) < (0,1) \cdots < (0,j) < (1,i) < \cdots < (1,n),$$

which provides us with a section $s_i : \Delta^{[0,i]} \to K_n^i$ sending j to C_j. One checks that $r_i \circ s_i = \mathrm{id}$ and that there is a marked homotopy between the identity of K_n^i and $s_i \circ r_i$. Moreover, the section s_i and the homotopy restrict to Q_n^i. It is immediate to see that both the section an the homotopy can be factored through the quotient simplicial sets $\overline{K}_n^i$ and $\overline{Q}_n^i$ which shows that $\alpha_{n,i}^\flat$ is again a weak equivalence. □

Corollary 5.33 *Let S be a scaled simplicial set. Let $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ be a Set_Δ^+-enriched functor. Assume that ϕ is essentially surjective, and let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a map between fibrant objects of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta^{\mathbf{ms}})$. Then the following conditions are equivalent:*

(1) *The map α is a weak equivalence in $(\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathcal{C}}$.*

(2) *For every $C \in \mathcal{C}$ the induced map $\alpha_C : \mathcal{F}(C) \to \mathcal{F}'(C)$ is a weak equivalence.*

(3) *For every vertex $s \in S$ the induced map on fibres*

$$\mathbb{U}\mathrm{n}_\phi(\mathcal{F})_s \to \mathbb{U}\mathrm{n}_\phi(\mathcal{F}')_s$$

is a bicategorical equivalence.

(4) *The map $\mathbb{U}\mathrm{n}_\phi(\alpha)$ is a weak equivalence in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/(S,\sharp,T_S\subset\sharp)}$.*

Proof The equivalence (1) $\iff$ (2) is immediate from the definition. Since both $\mathcal{F}$ and $\mathcal{F}'$ are fibrant it follows from Proposition 5.29 that $\mathbb{U}\mathrm{n}_{\phi}(\alpha)$ is a map between fibrant objects in $(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/(S,\sharp,T_S\subset\sharp)}$. Then the equivalence (3) $\iff$ (4) follows from (v) in Proposition 3.23. To finish the proof we need to show that (2) $\iff$ (3). However, the fact that ϕ is surjective allows us to reduce to the case where $S=\Delta^0$ and conclude by Remark 5.15 and Proposition 5.30. $\square$

5.3 Straightening over a simplex

In this section we establish the key element in the proof of our main theorem.

Proposition 5.34 *Denote by* $\Delta^n_\flat=(\Delta^n,\flat)$ *the minimally scaled* n*-simplex. Then the straightening-unstraightening adjunction, over* $\Delta^n_\flat$,

$$\mathbb{S}\mathrm{t}_{\Delta^n_\flat}:(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{(\Delta^n,\sharp,\flat\subset\sharp)}\rightleftarrows \mathrm{Fun}(\mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat],\mathrm{Set}_{\Delta}^{\mathbf{ms}}):\mathbb{U}\mathrm{n}_{\Delta^n_\flat}$$

is a Quillen equivalence.

Let us comment on the general structure of the proof before we dive into the details. First, let us observe that Corollary 5.33 shows that $\mathbb{U}\mathrm{n}_{\Delta^n_\flat}$ detects weak equivalences between fibrant objects and thus it descends to a conservative right adjoint on homotopy categories. Consequently, in order to prove the result we only need to show that given an object $X\in(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{(\Delta^n,\sharp,\flat\subset\sharp)}$ and an equivalence of $\mathrm{Set}_{\Delta}^{+}$-enriched functors

$$\mathbb{S}\mathrm{t}_{\Delta^n}(X)\xrightarrow{\simeq}\mathcal{F},$$

where $\mathcal{F}$ is a fibrant functor, the adjoint map $X\to\mathbb{U}\mathrm{n}_{\Delta^n}(\mathcal{F})$ is a weak equivalence in $(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{(\Delta^n,\sharp,\flat\subset\sharp)}$. Using two-out-of-three we can assume without loss of generality that $p:X\to\Delta^n$ is a fibrant object. Now, in order to check that our adjoint map is a weak equivalence it suffices by Proposition 3.23 to check that the induced morphisms on fibres

$$\varphi_i^n:X_i\xrightarrow{\simeq}\mathbb{U}\mathrm{n}_{\Delta^n}(\mathcal{F})_i$$

are bicategorical equivalences for $0\leqslant i\leqslant n$. We will use induction on n, the dimension of our simplex $\Delta^n_\flat$. Let us assume that we have proved Proposition 5.34 for $0\leqslant k\leqslant n-1$ and note the base case was already shown in Proposition 5.30. We claim that for every $0\leqslant i\leqslant n-1$ the map φ_i^n is a bicategorical equivalence.

We consider the morphism $\alpha:\Delta^{[0,n-1]}_\flat\to\Delta^n_\flat$ and denote by $\overline{X}$ the pullback of X along α. Similarly we denote by $\overline{\mathcal{F}}=j^*\mathcal{F}$ the restriction of $\mathcal{F}$ along $\mathfrak{C}^{\mathbf{sc}}[\alpha]=j$. We observe the following:

(1) As a direct consequence of the point-wise formula for left (enriched) Kan extensions in terms of weighted colimits (see [18, Proposition A.3.3.7]) and fully faithfulness of j we see that for every functor $\mathcal{H}:\mathfrak{C}^{\mathbf{sc}}[\Delta^{n-1}_\flat]\to\mathrm{Set}_{\Delta}^{\mathbf{ms}}$ we have an isomorphism $j^*j_!\mathcal{H}\simeq\mathcal{H}$. This implies that for every $\mathcal{G}:\mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat]\to\mathrm{Set}_{\Delta}^{\mathbf{ms}}$ we have that $j_!j^*\mathcal{G}(i)\to\mathcal{G}(i)$ is an isomorphism for $0\leqslant i<n$.

(2) For every $p:X\to\Delta^n$ we have that the map $\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(\overline{X})(i)\to\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X)(i)$ is an isomorphism for $0\leqslant i<n$. This follows from the previous point and Proposition 5.14 since $\mathbb{S}\mathrm{t}_{\Delta^n_\flat}\overline{X}\simeq j_!\,\mathbb{S}\mathrm{t}_{\Delta^{n-1}_\flat}\overline{X}$.

(3) Using Proposition 5.14 and Remark 5.15 we obtain a commutative diagram

$$\begin{tikzcd}
j_! \, \mathbb{S}\mathrm{t}_{\Delta^{n-1}_\flat} \overline{X} \arrow[r] \arrow[d] & j_! j^*(\mathcal{F}) \arrow[d] \\
\mathbb{S}\mathrm{t}_{\Delta^n_\flat} X \arrow[r] & \mathcal{F}
\end{tikzcd}$$

We conclude from (1)–(3) above that $\mathbb{S}\mathrm{t}_{\Delta^{n-1}_\flat} \overline{X} \to \overline{\mathcal{F}}$ is a weak equivalence. It follows from our induction hypothesis that $\overline{X} \to \mathbb{U}\mathrm{n}_{\Delta^{n-1}_\flat}(\overline{\mathcal{F}})$ is a weak equivalence and thus the maps φ^i_n are equivalences for $0 \leqslant i < n$.

We have reduced our problem to showing that φ^n_n is a weak equivalence. We claim that is enough to show the following.

$(*)$ The map $\mathbb{S}\mathrm{t}_{\Delta^n}(X_n)(n) \to \mathbb{S}\mathrm{t}_{\Delta^n}(X)(n)$ is a weak equivalence.

Indeed, let $r^n : \Delta^0 \to \Delta^n_\flat$ denote the inclusion of the terminal vertex. Given $K \in \mathrm{Set}^{\mathbf{ms}}_\Delta$ it follows that we have an isomorphism $r^n_!(K)(n) \simeq K$ which in turn shows that the adjoint morphism to

$$r^n_!(\mathbb{S}\mathrm{t}_* \, X_n) \simeq \mathbb{S}\mathrm{t}_{\Delta^n}(X_n) \to \mathbb{S}\mathrm{t}_{\Delta^n}(X) \to \mathcal{F},$$

which is given by $\mathbb{S}\mathrm{t}_* \, X_n \to \mathcal{F}(n)$ is a weak equivalence. We can now use Proposition 5.30 to conclude that we have a bicategorical equivalence

$$X_n \xrightarrow{\simeq} \mathbb{U}\mathrm{n}_{\Delta^n}(\mathcal{F})_n.$$

Therefore, we will devote the rest of this section to the proof of the claim $(*)$ above.

Definition 5.35 Let I be a finite linearly ordered set and let $i \in I$. We define simplicial set as the nerve of a poset $\mathcal{O}^I_{i\nearrow}$ whose elements are given by subsets $S \subseteq I$ such that $S \neq \varnothing$ and such that $\min(S) = i$. We declare $S \leqslant T$ if $S \subseteq T$. We observe that we have a map

$$\pi_I : \mathcal{O}^I_{i\nearrow} \to \Delta^I, \quad S \mapsto \max(S).$$

We upgrade $\pi_I : \mathcal{O}^I_{i\nearrow} \to \Delta^I$ to an object of $\left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)_{/\Delta^I_\flat}$ as follows:

- We declare an edge $S \to T$ to be marked if $T = S \cup \max(T)$.
- We declare every triangle of $\mathcal{O}^I_{i\nearrow}$ to be lean.
- We declare a triangle to be thin if its image in Δ^I is degenerate.

Definition 5.36 To ease the notation we set $\mathbb{O}^n = \mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat]$ (see Definition 2.8).

Remark 5.37 Let $I = [n]$. For every $i \leqslant j$ we view $\mathbb{O}^n(i,j)$ as an **MB** simplicial set by declaring every triangle to be thin scaled and only degenerate edges to be marked. We further note that we have functors

$$\mathbb{O}^n(i,j) \times \mathcal{O}^n_{j\nearrow} \to \mathcal{O}^n_{i\nearrow}, \quad (S,T) \mapsto S \cup T,$$

which preserve the decorations.

Definition 5.38 Let $\mathcal{F} : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ be a Set^+_Δ-enriched functor. We define an **MB** simplicial set $\mathcal{M}(\mathcal{F})$ over $\Delta^n_\flat$ as follows:

(1) Let $i \in [n]$. We upgrade the marked-scaled simplicial set $\mathcal{F}(i)$ to an **MB** simplicial set by declaring the collection of thin triangles and lean triangles to coincide.

(2) We define $\mathcal{M}(\mathcal{F})$ as the coequaliser of the diagram in the category of **MB** simplicial sets

$$\coprod_{i<j} \mathcal{F}(i) \times \mathbb{O}^n(i,j) \times \mathcal{O}^n_{j\nearrow} \rightrightarrows \coprod_{i} \mathcal{F}(i) \times \mathcal{O}^n_{i\nearrow}, \quad i,j \in [n],$$

where the maps in the diagram are given by

$$\mathcal{F}(i) \times \mathbb{O}^n(i,j) \times \mathcal{O}^n_{j\nearrow} \to \mathcal{F}(j) \times \mathcal{O}^n_{j\nearrow}, \quad \mathcal{F}(i) \times \mathbb{O}^n(i,j) \times \mathcal{O}^n_{j\nearrow} \to \mathcal{F}(i) \times \mathcal{O}^n_{i\nearrow}.$$

(3) The maps $\mathcal{F}(i) \times \mathcal{O}^n_{i\nearrow} \to \mathcal{O}^n_{i\nearrow} \to \Delta^n$ assemble into a functor $\mathcal{M}(\mathcal{F}) \to \Delta^n$.

Remark 5.39 Given $0 \leqslant i < n$ we have a map (actually an isomorphism) of simplicial sets

$$\Xi^i : \Delta^1 \times \mathcal{O}^{n-1}_{i\nearrow} \to \mathcal{O}^n_{i\nearrow}, \qquad \Xi^i(\varepsilon, S) = \begin{cases} S & \text{if } \varepsilon = 0, \\ S \cup \{n\} & \text{if } \varepsilon = 1, \end{cases}$$

which we use to equip $\Delta^1 \times \mathcal{O}^{n-1}_{i\nearrow}$ with the induced decorations from $\mathcal{O}^n_{i\nearrow}$.

Given a functor $\mathcal{F} : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ let us denote by $\overline{\mathcal{F}}$ the restriction of $\mathcal{F}$ to $\mathfrak{C}^{\mathbf{sc}}[\Delta^{n-1}_\flat]$ along the map $i : \Delta^{[0,n-1]}_\flat \to \Delta^n_\flat$. We can use the maps Ξ^i to construct a map of simplicial sets

$$\Xi_{\mathcal{F}} : \Delta^1 \times \mathcal{M}(\overline{\mathcal{F}}) \to \mathcal{M}(\mathcal{F}).$$

Finally, we equip $\Delta^1 \times \mathcal{M}(\overline{\mathcal{F}})$ with the decorations induced (via $\Xi_{\mathcal{F}}$) from $\mathcal{M}(\mathcal{F})$ and denote the resulting **MB** simplicial set by $\Delta^1 \mathbin{\overline{\otimes}} \mathcal{M}(\overline{\mathcal{F}})$.

Remark 5.40 The construction of $\mathcal{M}(\mathcal{F})$ defines a colimit-preserving functor

$$\mathcal{M}(-) : \mathrm{Fun}(\mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat], \mathrm{Set}^{\mathbf{ms}}_\Delta) \to \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)_{/\Delta^n_\flat},$$

which enjoys several properties:

(i) For every $j \in [n]$ we have an isomorphism $\mathcal{M}(\mathcal{F})_j \simeq \mathcal{F}(j)$.

(ii) Given a functor $\mathcal{F} : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ let us denote by $\overline{\mathcal{F}}$ as in Remark 5.39. Then we have a pushout diagram

$$\begin{tikzcd}
\Delta^{\{1\}} \times \mathcal{M}(\overline{\mathcal{F}}) \arrow[r] \arrow[d] & \mathcal{F}(n) \arrow[d] \\
\Delta^1 \mathbin{\overline{\otimes}} \mathcal{M}(\overline{\mathcal{F}}) \arrow[r] & \mathcal{M}(\mathcal{F})
\end{tikzcd}$$

of **MB** simplicial sets.

(iii) The functor $\mathcal{M}(-)$ preserves cofibrations.

Lemma 5.41 *Let* $A \to \Delta^n_\flat$ *be an object of* $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{(\Delta^n,\sharp,\flat\subset\sharp)}$. *Define an* ***MB*** *simplicial set* $\Delta^1 \overline{\otimes} A \to \Delta^{n+1}_\flat$ *as follows:*

- *The underlying simplicial set is given by the cartesian product.*
- *The projection map* $\Delta^1 \times A \to \Delta^{n+1}_\flat$ *is induced by the map*
$$r : \Delta^1 \times \Delta^n \to \Delta^{n+1}, \qquad r(\varepsilon, i) = \begin{cases} i & \text{if } i = 0, \\ n & \text{if } i = 1. \end{cases}$$
- *Let* $(e_1, e_A) : \Delta^1 \to \Delta^1 \times A$ *be marked if* e_A *is marked in* A *and* $e_1 = 0 \to 0$ ***or*** *if* e_A *is degenerate.*
- *A triangle is lean if and only if it is lean in* A.
- *A triangle is thin if and only if it is lean and its image in* Δ^{n+1} *is degenerate.*

Then the map $\Delta^{\{0\}} \times A \to \Delta^1 \overline{\otimes} A$ *is* ***MB****-anodyne.*

Proof The claim follows from a standard simplex by simplex argument and is left as an exercise to the reader. □

Remark 5.42 The scaling of $\Delta^1 \overline{\otimes} \mathcal{M}(\overline{\mathcal{F}})$ given in Remark 5.39 is precisely that of Lemma 5.41 so in particular we obtain an anodyne morphism $\mathcal{M}(\overline{\mathcal{F}}) \to \Delta^1 \overline{\otimes} \mathcal{M}(\overline{\mathcal{F}})$. Moreover, in the particular case where $\mathcal{F}$ is the corepresentable functor on the object 0 it follows $\mathcal{M}(\mathcal{F}) = \mathcal{O}^n_{0\nearrow}$ so applying Lemma 5.41 n times we obtain an anodyne morphism
$$\Delta^0 \xrightarrow{\simeq} \mathcal{O}^n_{0\nearrow},$$
where the map above selects the subset $\{0\}$.

Definition 5.43 Let $\mathcal{F} : \mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat] \to \mathrm{Set}_\Delta^{\mathbf{ms}}$ be a Set_Δ^+-enriched functor. We define a scaled simplicial set $\mathbb{M}(\mathcal{F})$ whose underlying simplicial set is given by that of $\mathcal{M}(\mathcal{F})$ and whose thin triangles are precisely the lean triangles of $\mathcal{M}(\mathcal{F})$.

Lemma 5.44 *Let* $\mathcal{F} : \mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat] \to \mathrm{Set}_\Delta^{\mathbf{ms}}$ *be a fibrant* Set_Δ^+*-enriched functor. Given a* $\flat$*-local* $(0,1)$*-fibration* $p : X \to \Delta^n_\flat$ *and a morphism, over* $\Delta^n_\flat$,
$$f : \mathcal{M}(\mathcal{F}) \to X$$
such that for every $j \in [n]$ *the map* f *induces bicategorical equivalences* $\mathcal{F}(j) \simeq X_j$, *it follows that* f *is a weak equivalence in* $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{(\Delta^n,\sharp,\flat\subset\sharp)}$.

Proof Since $p : X \to \Delta^n_\flat$ is a $\flat$-local $(0,1)$-fibration it follows that we can construct its associated bicategorical interpretation (see Definition 4.4) $\mathfrak{p} : \mathbb{X} \to \Delta^n$ by declaring lean triangles to be scaled. We claim that is enough to show the following:

($\diamond$) The associated map $\mathfrak{f} : \mathbb{M}(\mathcal{F}) \to \mathbb{X}$ is a bicategorical equivalence.

Indeed, given a $\flat$-local $(0,1)$-fibration $q : Z \to \Delta^n_\flat$ with associated bicategorical interpretation $\mathfrak{q} : \mathbb{Z} \to \Delta^n$ it follows from our claim that we have a bicategorical equivalence

$$\phi : \mathrm{Fun}^{\mathbf{sc}}(\mathbb{X},\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathbf{sc}}(\mathbb{M}(\mathcal{F}),\mathbb{Z}).$$

Moreover, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}^{\mathbf{sc}}_{\Delta^n}(\mathbb{X},\mathbb{Z}) \arrow[r, "\psi"] \arrow[d] & \mathrm{Map}^{\mathbf{sc}}_{\Delta^n}(\mathbb{M}(\mathcal{F}),\mathbb{Z}) \arrow[d] \\
\mathrm{Fun}^{\mathbf{sc}}(\mathbb{X},\mathbb{Z}) \arrow[r, "\phi"] & \mathrm{Fun}^{\mathbf{sc}}(\mathbb{M}(\mathcal{F}),\mathbb{Z})
\end{tikzcd}$$

where $\mathrm{Map}^{\mathbf{sc}}_{\Delta^n}(-,-)$ denotes the full subcategory on maps which preserve the marked edges and commute with the projection maps. We observe that higher simplices in the aforementioned scaled simplicial set are also compatible with the projection maps. Since a simplex in $\mathcal{M}(\mathcal{F})$ is thin if and only if it is lean and its image is thin in $\Delta^n_\flat$ we see that if ψ is a bicategorical equivalence it will follow that f is a weak equivalence $\left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)_{(\Delta^n,\sharp,\flat\subset\sharp)}$. It is clear by construction that ψ is fully faithful so it will suffice to show that it is essentially surjective.

Given $u \in \mathrm{Map}^{\mathbf{sc}}_{\Delta^n}(\mathbb{M}(\mathcal{F}),\mathbb{Z})$ we can find some $v \in \mathrm{Fun}^{\mathbf{sc}}(\mathbb{X},\mathbb{Z})$ such that $\phi(v) \simeq u$. Consequently, it will be enough to show that v factors through $\mathrm{Map}^{\mathbf{sc}}_{\Delta^n}(\mathbb{X},\mathbb{Z})$. Let $x \in X$ such that $p(x) = i$ and pick an equivalence $f(y) \xrightarrow{\simeq} x$. We then see that

$$v(f(y)) \simeq u(y), \quad v(f(y)) \simeq v(x) \implies q(v(x)) = i.$$

Therefore, we can focus our attention into proving the statement $(\diamond)$ above. Let $x_i \in \mathbb{M}(\mathcal{F})$ be an object represented by a pair $(a_i, \{i\})$ in $\mathcal{F}(i) \times \mathcal{O}^n_{i\nearrow}$. We consider a marked morphism $f : x_i \to \hat{x}_i$ given by $(a_i,\{i\}) \to (a_i,\{in\})$. Given an object x_n, lying over n, we claim the following:

$(\star)$ Restriction along f induces a weak equivalence of marked simplicial sets

$$\mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\mathcal{F})](\hat{x}_i,x_n) \xrightarrow{\simeq} \mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\mathcal{F})](x_i,x_n).$$

It is easy to see that $(\diamond)$ follows from $(\star)$ together with a routine inductive argument.

We observe that we have cofibrations of Set^+_Δ-enriched categories

$$\mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\overline{\mathcal{F}})] \to \mathfrak{C}^{\mathbf{sc}}[\Delta^1 \times \mathbb{M}(\overline{\mathcal{F}})], \quad \mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\overline{\mathcal{F}})] \to \mathfrak{C}^{\mathbf{sc}}[\Delta^1] \times \mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\overline{\mathcal{F}})]$$

and a diagram

$$\begin{tikzcd}
\mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\overline{\mathcal{F}})] \arrow[r] \arrow[d] & \mathfrak{C}^{\mathbf{sc}}[\Delta^1 \times \mathbb{M}(\overline{\mathcal{F}})] \arrow[r] \arrow[d] & \mathfrak{C}^{\mathbf{sc}}[\Delta^1] \times \mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\overline{\mathcal{F}})] \arrow[d] \\
\mathfrak{C}^{\mathbf{sc}}[\mathcal{F}(n)] \arrow[r] & \mathfrak{C}^{\mathbf{sc}}[\mathbb{M}(\mathcal{F})] \arrow[r] & \mathbb{P}(\mathcal{F})
\end{tikzcd}$$

where both squares are pushouts. It is not hard to see using the fact $\mathfrak{C}^{\mathbf{sc}}[-]$ is a left Quillen equivalence that the top right-most horizontal map is a weak equivalence. We conclude that the bottom right-most horizontal morphism is also a weak equivalence. It is easy to see by direct inspection that the analogous claim to $(\star)$ holds for $\mathbb{P}(\mathcal{F})$. □

Proposition 5.45 *Let $p : X \to \Delta^n_\flat$ be a $\flat$-local $(0,1)$-fibration. Then there exists a projectively fibrant-cofibrant functor $\mathcal{F} : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ and a weak equivalence $\mathcal{M}(\mathcal{F}) \to X$ in $\big(\mathrm{Set}^{\mathbf{mb}}_\Delta\big)_{(\Delta^n,\sharp,\flat\subset\sharp)}$.*

Proof Using Lemma 5.44 it will be enough to construct a map $\mathcal{M}(\mathcal{F}) \to X$ inducing categorical equivalences on fibres. We proceed using induction on n, the case $n = 0$ being clear. Let us suppose that the claim holds for $n-1$ and let $i : \Delta^{[0,n-1]}_\flat \to \Delta^n_\flat$. We denote by $\overline{X}$ the restriction of X along i. Using our induction hypothesis we obtain a projectively fibrant-cofibrant functor $\overline{\mathcal{F}} : \mathfrak{C}^{\mathbf{sc}}[\Delta^{n-1}_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ and fibrewise equivalence $\mathcal{M}(\overline{\mathcal{F}}) \to \overline{X}$. We use Remark 5.42 to provide a solution to the lifting problem

$$\begin{tikzcd}
\mathcal{M}(\overline{\mathcal{F}}) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \overline{\otimes}\, \mathcal{M}(\overline{\mathcal{F}}) \arrow[r] \arrow[ur, dotted] & \Delta^n
\end{tikzcd}$$

which provides us with a map $\mathcal{M}(\overline{\mathcal{F}}) \to X_n$. We factor this map as

$$\mathcal{M}(\overline{\mathcal{F}}) \xrightarrow{u} \widehat{X}_n \xrightarrow{v} X_n,$$

where u is a cofibration and v is a trivial fibration. Note that it follows that $\widehat{X}_n$ is also an ∞-bicategory. We can use the map u to extend $\overline{\mathcal{F}}$ to a fibrant-cofibrant functor $\mathcal{F} : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ such that $\mathcal{F}(n) = \widehat{X}_n$. □

Proposition 5.46 *Let $p : X \to \Delta^n_\flat$ be a $\flat$-local $(0,1)$-fibration. Then there exists an equivalence of marked-scaled simplicial sets*

$$\mathbb{S}\mathrm{t}_{\Delta^n}(X_n)(n) \xrightarrow{\simeq} \mathbb{S}\mathrm{t}_{\Delta^n}(X)(n),$$

where X_n denotes the fibre over n of X.

Proof We note that due to Proposition 5.45 we have a fibrant-cofibrant functor $\mathcal{F} : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ and a weak equivalence $\mathcal{M}(\mathcal{F}) \to X$. We will show that for every projectively cofibrant functor $\mathcal{G}$ the map

$$\eta_{\mathcal{G}} : \mathbb{S}\mathrm{t}_{\Delta^n}(\mathcal{G}(n))(n) \xrightarrow{\simeq} \mathbb{S}\mathrm{t}_{\Delta^n}(\mathcal{M}(\mathcal{G}))(n)$$

is a weak equivalence of marked-scaled simplicial sets. We observe that we have a pair of functors

$$L_i : \mathrm{Fun}(\mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat], \mathrm{Set}^{\mathbf{ms}}_\Delta) \to \mathrm{Set}^{\mathbf{ms}}_\Delta, \quad i = 1,2,$$

given by $L_1(\mathcal{G}) = \mathbb{S}\mathrm{t}_{\Delta^n}(\mathcal{G}(n))(n)$ and $L_2 = \mathbb{S}\mathrm{t}_{\Delta^n}(\mathcal{M}(\mathcal{G}))(n)$ which preserve colimits and cofibrations together with a natural transformation $\eta : L_1 \Rightarrow L_2$. We say that a functor $\mathcal{G}$ is *good* if $\eta_{\mathcal{G}}$ is a weak equivalence. To finish the proof we need to show that every cofibrant functor is good.

For every $-1 \leqslant j \leqslant n$ let $i_j : \Delta^{[0,j]}_\flat \to \Delta^n_\flat$ be the obvious inclusion with the convention $\Delta^{[0,-1]} = \varnothing$. Given a functor $\mathcal{G}$ we define $\mathcal{G}_j$ as the result of first restricting $\mathcal{G}$ along $\mathfrak{C}^{\mathbf{sc}}[i_j] = f^j$ and then applying the left Kan extension along f^j. We set $\mathcal{G}_{-1}$ to be the initial functor. We further denote by $r^j : \mathfrak{C}^{\mathbf{sc}}[\Delta^0] \to \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat]$ the functor that picks the object j for $0 \leqslant j \leqslant n$.

Note that given a projectively cofibrant functor $\mathcal{G}$, it follows that the canonical map $\mathcal{G}_{j-1}(j) \to \mathcal{G}(j)$ is a cofibration for $0 \leqslant j \leqslant n$. We further note that we have a pushout diagram

$$\begin{tikzcd} r_!^j \mathcal{G}_{j-1}(j) \arrow[r] \arrow[d] & r_!^j \mathcal{G}(j) \arrow[d] \\ \mathcal{G}_{j-1} \arrow[r] & \mathcal{G}_j \end{tikzcd}$$

where $r_!^j$ denotes the left Kan extension functor along r^j. Since the top horizontal map is a cofibration it follows that in order to show that $\mathcal{G}_j$ is good it is enough to show that $r_!^j \mathcal{G}_{j-1}(j)$, $r_!^j \mathcal{G}(j)$ and $\mathcal{G}_{j-1}$ are good. We note the following:

- If $j > 0$ it follows that $r_!^j A(i) = \varnothing$ for $i < j$. In particular, it follows that $\mathcal{M}(r_!^j A)$ factors through $\Delta_\flat^{[j,n]}$. We can now induct on n to see that $r_!^j A$ is good for $j > 0$.

Finally, we can use induction on j to reduce our problem to show that for every $K \in \mathrm{Set}_\Delta^{\mathbf{ms}}$ the functor

$$\underline{K} : \mathfrak{C}^{\mathbf{sc}}[\Delta_\flat^n] \to \mathrm{Set}_\Delta^{\mathbf{ms}}, \quad i \mapsto K \times \mathbb{O}^n(0,i),$$

is good. Note that we can use further simplify our computation to the cases where $K = \Delta_\flat^k$ for $k \geqslant 0$, $\Delta_\sharp^2$ and $(\Delta^1)^\sharp$. We will only show the case $K = \Delta_\flat^k$ for $k \geqslant 0$ the other cases will follow by a totally analogous argument.

We can identify $\mathbb{St}_\Delta(\Delta^k \times \mathcal{O}^n_{0\nearrow})(n)$ as a quotient of the (decorated) poset of chains of $\Delta^1 \times \Delta^k \times \mathcal{O}^n_{0\nearrow}$ starting at $(0,0,\{0\})$ at ending at some element $(1,\ell,S)$ with $\max(S) = n$. We define a map

$$\psi : \mathbb{St}_\Delta(\Delta_\flat^k \times \mathcal{O}^n_{0\nearrow})(n) \to \Delta_\flat^k \times \mathbb{O}^n(0,n)$$

by sending a chain $C = \{(\varepsilon_i, k_i, S_i)\}_{i=0}^{i=\ell}$ to $(k_{i_{m_C}}, S_{m_C} \cup \{\max(S_j)\}_{j \geqslant m_C})$ where m_C is the biggest index such that $\varepsilon_i = 0$. We consider the map $p_0 : \Delta^k \to \Delta_\flat^n$ where Δ^k has the minimal decorations and where p_0 is constant on the vertex 0. We can now look at the commutative diagram

$$\begin{tikzcd} \mathbb{St}_{\Delta^n}(\Delta^k)(n) \arrow[r, "\varphi"] \arrow[dr, "u"'] & \mathbb{St}_\Delta(\Delta_\flat^k \times \mathcal{O}^n_{0\nearrow})(n) \arrow[d, "\psi"] & \mathbb{St}_{\Delta^n}(\Delta^k \times \mathbb{O}^n(0,n))(n) \arrow[l, "\phi"'] \arrow[dl, "v"] \\ & \Delta_\flat^k \times \mathbb{O}^n(0,n) & \end{tikzcd}$$

and make the following observations:

(1) It follows from Lemma 5.41, Remark 5.42 and Proposition 3.12 that φ is a weak equivalence.

(2) The map $u : \mathbb{St}_{\Delta^n}(\Delta^k)(n) \simeq \mathbb{St}_*(\Delta^k) \times \mathbb{O}^n(0,n) \to \Delta_\flat^k \times \mathbb{O}^n(0,n)$ can be identified with a product of the natural transformation α at Δ^k considered in the proof of Proposition 5.30 and the identity map on $\mathbb{O}^n(0,n)$ and its a consequently a weak equivalence. It follows from 1 that ψ is also a weak equivalence.

(3) The map $v:\mathbb{S}\mathrm{t}_{\Delta^n}(\Delta^k\times\mathbb{O}^n(0,n))(n)\simeq\mathbb{S}\mathrm{t}_*(\Delta^k\times\mathbb{O}^n(0,n))\to\Delta^k_\flat\times\mathbb{O}^n(0,n)$ can also be identified with the component of the natural transformation α and thus it is a weak equivalence. We conclude that ϕ is a weak equivalence. $\square$

Our final claim is established. Therefore, Proposition 5.34 is proved.

5.4 The main theorem

Let $\mathrm{Set}^{\mathbf{sc}}_\Delta$ be the category of scaled simplicial sets and observe that we have a pair of functors

$$F_1,F_2:(\mathrm{Set}^{\mathbf{sc}})^{\mathrm{op}}_\Delta\to\mathrm{Set}^+_\Delta\text{–}\mathrm{Cat},\quad F_1(S)=\mathrm{Fun}^{\mathrm{o}}(\mathfrak{C}^{\mathbf{sc}}[S],\mathrm{Set}^{\mathbf{ms}}_\Delta),\quad F_2(S)=\left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)^{\mathrm{o}}_{/(S,\sharp,T_S\subset\sharp)},$$

which values in the category of Set^+_Δ-enriched categories and where the superscript “o” denotes the full (enriched) subcategory on fibrant-cofibrant objects. Let us remind the reader that it then follows that for every $S\in\mathrm{Set}^{\mathbf{sc}}_\Delta$ we have fibrant Set^+_Δ-enriched categories $F_i(S)$ for $i=1,2$.

We claim that the unstraightening construction $\mathbb{U}\mathrm{n}_{(-)}$ defines a natural transformation. In virtue of Remark 5.15 it will be enough to show that for every scaled simplicial set S we have that $\mathbb{U}\mathrm{n}_S$ defines a Set^+_Δ-enriched functor. Given a fibrant-cofibrant functors $\mathcal{F},\mathcal{G}:\mathfrak{C}^{\mathbf{sc}}[S]\to\mathrm{Set}^{\mathbf{ms}}_\Delta$ let $\mathrm{Nat}(\mathcal{F},\mathcal{G})$ denote the corresponding mapping marked simplicial set. Then a map $K\to\mathrm{Nat}(\mathcal{F},\mathcal{G})$ is precisely the data of an enriched natural transformation $K\otimes\mathcal{F}\Rightarrow\mathcal{G}$ where

$$K\otimes\mathcal{F}(s)=K\times\mathcal{F}(s).$$

We can consequently define a map $K\to\mathrm{Map}_S(\mathbb{U}\mathrm{n}_S(\mathcal{F}),\mathcal{G})$ as the composite

$$K\times\mathbb{U}\mathrm{n}_S(\mathcal{F})\to\mathrm{Un}_*(K)\times\mathbb{U}\mathrm{n}_S(\mathcal{F})\simeq\mathbb{U}\mathrm{n}_S(K\times\mathcal{F})\to\mathbb{U}\mathrm{n}_S(\mathcal{G}),$$

which shows that $\mathbb{U}\mathrm{n}_S$ defines a Set^+_Δ-enriched functor. The main goal of this section is to show that for every scaled simplicial set S, it follows that $\mathbb{U}\mathrm{n}_S$ is an equivalence of Set^+_Δ-enriched categories.

Proposition 5.47 *Let S be a scaled simplicial set. Then the following are equivalent*:

(i) *The functor $\mathbb{U}\mathrm{n}_S$ is a right Quillen equivalence.*

(ii) *The functor $\mathbb{U}\mathrm{n}_S$ defines an equivalence of fibrant Set^+_Δ-enriched categories after restriction to the full subcategories of fibrant-cofibrant objects.*

Proof It follows from Proposition 3.1.10 in [18] and our previous discussion that (ii) $\Longrightarrow$ (i). To show that (i) $\Longrightarrow$ (ii) we show that given a marked simplicial set K, $\mathbb{U}\mathrm{n}_S$ induces an isomorphism in the homotopy category of Set^+_Δ between $[K,\mathrm{Nat}(\mathcal{F},\mathcal{G})]\simeq[K,\mathrm{Map}_S(\mathbb{U}\mathrm{n}_S(\mathcal{F},\mathbb{U}\mathrm{n}_S(\mathcal{G}))]$. This follows from the chain of isomorphisms

$$[K,\mathrm{Nat}(\mathcal{F},\mathcal{G})]\simeq[K\otimes\mathcal{F},\mathcal{G}]^{\mathrm{Fun}}\simeq[\mathbb{U}\mathrm{n}_*(K)\times\mathbb{U}\mathrm{n}_S(\mathcal{F}),\mathbb{U}\mathrm{n}_S(\mathcal{G})]^{\mathrm{Fib}}\simeq[K,\mathrm{Map}_S(\mathbb{U}\mathrm{n}_S(\mathcal{F}),\mathbb{U}\mathrm{n}_S(\mathcal{G}))],$$

where the second isomorphism is a consequence of (i). $\square$

Remark 5.48 Let S be a scaled simplicial set and $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ an equivalence of Set_Δ^+-enriched categories. Observe that it follows from Proposition 5.14 that $\mathbb{U}\mathrm{n}_\phi$ is a right Quillen equivalence if and only if $\mathbb{U}\mathrm{n}_S$ is a right Quillen equivalence. Therefore for the rest of the section we will let $\phi = \mathrm{id}$.

Corollary 5.49 *Let $\Delta^2_\sharp = (\Delta^2, \sharp)$ denote a maximally scaled 2-simplex. Then the straightening–unstraightening adjunction*

$$\mathbb{S}\mathrm{t}_{\Delta^2_\sharp} : \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)_{(\Delta^2,\sharp,\sharp)} \rightleftarrows \mathrm{Fun}(\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\sharp], \mathrm{Set}^{\mathbf{ms}}_\Delta) : \mathbb{U}\mathrm{n}_{\Delta^2_\sharp}$$

is a Quillen equivalence.

Proof Observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{o}}(\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\sharp], \mathrm{Set}^{\mathbf{ms}}_\Delta) \arrow[r, "\mathbb{U}\mathrm{n}_{\Delta^2_\sharp}"] \arrow[d] & \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)^{\mathrm{o}}_{(\Delta^2,\sharp,\sharp)} \arrow[d] \\
\mathrm{Fun}^{\mathrm{o}}(\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\flat], \mathrm{Set}^{\mathbf{ms}}_\Delta) \arrow[r, "\mathbb{U}\mathrm{n}_{\Delta^2}"] & \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)^{\mathrm{o}}_{/(\Delta^2,\sharp,\flat\subset\sharp)}
\end{tikzcd}$$

where the vertical maps are fully faithful functors. We conclude that $\mathbb{U}\mathrm{n}_{\Delta^2_\sharp}$ is fully faithful. It follows from Proposition 5.47 that it will be enough to show that $\mathbb{U}\mathrm{n}_{\Delta^2_\sharp}$ is essentially surjective.

Let $p : X \to \Delta^2_\sharp$ be a fibrant object and pick a fibrant-cofibrant functor $\mathcal{F} : \mathfrak{C}^{\mathrm{sc}}[\Delta^2_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ such that $\mathbb{U}\mathrm{n}_{\Delta^2}(\mathcal{F}) \simeq X$. To finish the proof we need to show that $\mathcal{F}$ factors through $\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\sharp]$. Since $\mathbb{U}\mathrm{n}_{\Delta^2}(\mathcal{F})$ is equivalent to X it follows that the composition of local $(0,1)$-cartesian edges in this fibration remains $(0,1)$-cartesian. Direct inspection reveals that our functor must factor through $\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\sharp]$. □

Remark 5.50 It follows from Lemma A.3.6.17 and Corollary A.3.6.18 in [18] that F_1 sends homotopy colimits of scaled simplicial sets to homotopy limits of Set_Δ^+-enriched categories.

Lemma 5.51 *Let $f : S_0 \to S$ be a cofibration of scaled simplicial sets. Then the functor*

$$f^* : \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)^{\mathrm{o}}_S \to \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)^{\mathrm{o}}_{S_0}$$

is a fibration of Set_Δ^+-enriched categories.

Proof Since both Set_Δ^+-enriched categories are fibrant it follows from Theorem A.3.2.24 in [18] that it will be enough to show the following:

(∗) Given a pair of fibrant objects $X, Y \in \left(\mathrm{Set}^{\mathbf{mb}}_\Delta\right)^{\mathrm{o}}_S$, the induced morphism on mapping ∞-categories

$$\mathrm{Map}^{\leq 1}_S(X, Y) \to \mathrm{Map}^{\leq 1}_{S_0}(f^*X, f^*Y)$$

is a fibration of marked simplicial sets.

More generally we consider a pair of adjoint lifting problems

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{Map}_S(X,Y) \arrow[d] & & A\times X \amalg_{A\times f^*X} B\times f^*X \arrow[r] \arrow[d] & Y \arrow[d] \\
B \arrow[r] \arrow[ur, dotted] & \mathrm{Map}_{S_0}(f^*X, f^*Y) & \leftrightsquigarrow & B\times X \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where $A \to B$ is **MB**-anodyne. Since $f : S_0 \to S$ is a cofibration it follows that the canonical map $f^*X \to X$ is a cofibration so we can use Proposition 3.12 to conclude that we can produce the desired solution to the lifting problem. □

Theorem 5.52 *Let S be a scaled simplicial set. Then the functor $\mathbb{U}\mathrm{n}_S$ induces an equivalence of Set_Δ^+-enriched categories*

$$\mathbb{U}\mathrm{n}_S : \mathrm{Fun}^{\mathrm{o}}(\mathfrak{C}^{\mathrm{sc}}[S], \mathrm{Set}_\Delta^{\mathbf{ms}}) \to \left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)^{\mathrm{o}}_{/(S,\sharp,T_S\subset\sharp)}.$$

Proof We say that a scaled simplicial set S is good if the conclusion of the theorem holds. In virtue of Proposition 5.47 we know that Proposition 5.34 shows that the scaled simplicial sets $(\Delta^n,\flat)$ are good for $n \geqslant 0$. Moreover, it follows from Corollary 5.49 that $(\Delta^2,\sharp)$ is also good.

Recall that every scaled simplicial set S can be expressed as a filtered colimit over the natural numbers

$$S_0 \to S_1 \to \cdots \to S_k \to \cdots$$

such that each map $S_i \to S_{i+1}$ is a cofibration and such that S_0 is a disjoint union of points. Moreover, a simplex by simplex argument shows that each step in this filtration can be obtained via pushouts along the cofibrations

- $(\partial\Delta^n,\flat) \to (\Delta^n,\flat)$ for $n \geqslant 0$,
- $(\Delta^2,\flat) \to (\Delta^2,\sharp)$.

We saw in Remark 5.50 that F_1 maps homotopy colimits to homotopy limits. We see that in order to finish the proof it will be enough to show that F_2 maps the colimits appearing in our filtration to homotopy limits. Using Lemma 5.51 we reduce the problem to verifying that F_2 maps those colimits to ordinary limits which follows from direct inspection. □

Corollary 5.53 *Let S be a scaled simplicial set and let $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}$ be an equivalence of Set_Δ^+-enriched categories. Then the straightening-unstraightening adjunction*

$$\mathbb{S}\mathrm{t}_\phi : \left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/(S,\sharp,T_S\subset\sharp)} \rightleftarrows \mathrm{Fun}(\mathfrak{C}, \mathrm{Set}_\Delta^{\mathbf{ms}}) : \mathbb{U}\mathrm{n}_\phi$$

is a Quillen equivalence.

Remark 5.54 Let N^{sc} be as in Definition 2.9. Given a scaled simplicial set S we define $\mathbb{F}\mathrm{ib}_{0,1}(S) = \mathrm{N}^{\mathrm{sc}}((\mathrm{Set}_\Delta^{\mathbf{mb}})^{\mathrm{o}}_{/(S,\sharp,T_S\subset\sharp)})$. We also define $\mathbb{B}\mathrm{icat}_\infty = \mathrm{N}^{\mathrm{sc}}((\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathrm{o}})$ and observe that an analogous discussion to that of [3] shows that Theorem 5.52 shows that we have an equivalence of ∞-bicategories

$$\mathbb{U}\mathrm{n}_S : \mathrm{Fun}(S, \mathbb{B}\mathrm{icat}_\infty) \to \mathbb{F}\mathrm{ib}_{0,1}(S).$$

Definition 5.55 Let $\mathbb{S}=(S,T_S)$ be an ∞-bicategory and let $S_{M_S}=(S,M_S)$ (see Definition 4.1). We denote by $\mathbb{L}\mathrm{Fib}(\mathbb{S})=\mathbb{F}\mathrm{ib}_{0,1}(S_{M_S})$ the ∞-bicategory of local $(0,1)$-fibrations over $\mathbb{S}$. Similarly, given another ∞-bicategory $\mathbb{D}$ we define $\mathrm{Fun}^{\mathrm{oplax}}(\mathbb{S},\mathbb{D})=\mathrm{Fun}(S_{M_S},\mathbb{D})$.

Remark 5.56 In [11], the definition of $\mathrm{Fun}^{\mathrm{oplax}}(\mathbb{S},\mathbb{D})$ is proposed as a model for oplax normalised functors. In [1], we establish an equivalence between the aforementioned notion of an oplax normalised functor and that given in [14; 15].

Corollary 5.57 *Let $\mathbb{S}$ be an ∞-bicategory. Then the straightening-unstraightening adjunction, associated to the scaled simplicial set (S,M_S),*

$$\mathbb{S}\mathrm{t}^{\mathrm{oplax}}_{\mathbb{S}}:\mathbb{L}\mathrm{Fib}(\mathbb{S})\rightleftarrows \mathrm{Fun}^{\mathrm{oplax}}(\mathbb{S},\mathbb{B}\mathrm{icat}_\infty):\mathbb{U}\mathrm{n}^{\mathrm{oplax}}_{\mathbb{S}}$$

yields an equivalence of ∞-bicategories between the ∞-bicategory of local $(0,1)$-fibrations over $\mathbb{S}$ and the ∞-bicategory of oplax normalised functors with values in ∞-bicategories.

Proof This follows immediately from Theorem 5.52 and Remark 5.54. □

Definition 5.58 Let $\mathbb{S}=(S,T_S)$ be an ∞-bicategory and let $S_{\mathcal{U}}=(S,\mathcal{U})$ where $M_S\subset\mathcal{U}\subset T_S$. We denote by $\mathbb{L}\mathrm{Fib}^{\mathcal{U}}(\mathbb{S})=\mathbb{F}\mathrm{ib}_{0,1}(S_{\mathcal{U}})$ the ∞-bicategory of $\mathcal{U}$-local $(0,1)$-fibrations over $\mathbb{S}$ (see Definition 4.28 and Theorem 4.29). Similarly, given another ∞-bicategory $\mathbb{D}$ we define $\mathrm{Fun}^{\mathcal{U}\mathrm{oplax}}(\mathbb{S},\mathbb{D})=\mathrm{Fun}(S_{\mathcal{U}},\mathbb{D})$.

Corollary 5.59 *Let $\mathbb{S}$ be an ∞-bicategory. Then the straightening-unstraightening adjunction associated to the scaled simplicial set $(S,\mathcal{U})$ (see Definition 5.58)*

$$\mathbb{S}\mathrm{t}^{\mathcal{U}\mathrm{oplax}}_{\mathbb{S}}:\mathbb{L}\mathrm{Fib}^{\mathcal{U}}(\mathbb{S})\rightleftarrows \mathrm{Fun}^{\mathcal{U}\mathrm{oplax}}(\mathbb{S},\mathbb{B}\mathrm{icat}_\infty):\mathbb{U}\mathrm{n}^{\mathcal{U}\mathrm{oplax}}_{\mathbb{S}}$$

yields an equivalence of ∞-bicategories between the ∞-bicategory of $\mathcal{U}$-local $(0,1)$-fibrations over $\mathbb{S}$ and the ∞-bicategory consisting in those oplax normalised functors which preserve the composites in $\mathcal{U}$.

6 The ∞-bicategorical Yoneda embedding

In this section, we present a proof of the Yoneda lemma for $(\infty,2)$-categories. We would like to point out that this result has already appeared in the work of Hinich [16] in the context of enriched ∞-category theory. Throughout this section we fix an ∞-bicategory $\mathbb{C}$.

Definition 6.1 Let $\mathbb{C}$ be an ∞-bicategory and let $\mathbb{F}(\mathbb{C})=\mathrm{Fun}^{\mathrm{opgr}}(\Delta^1,\mathbb{C})$ (see Definition 5.1). We observe that evaluation at 0 and evaluation at 1 induce maps $\mathrm{ev}_i:\mathbb{F}(\mathbb{C})\to\mathbb{C}$ for $i=0,1$. It follows from Proposition 2.2.6 in [13] that evaluation at 0 yields a $(1,0)$-fibration. One can similarly show that the map ev_1 defines a $(0,1)$-fibration.

Remark 6.2 Let us recall that a morphism in $\mathbb{F}(\mathbb{C})$ is $(0,1)$-cartesian if and only if it is sent to an equivalence under ev_0. Similarly, a 2-morphism is cartesian in $\mathbb{F}(\mathbb{C})(x,y)$ if and only if its image in $\mathbb{C}$ under ev_0 is an invertible. The same description holds for the $(1,0)$-cartesian morphisms by replacing ev_0 with ev_1.

Definition 6.3 Let $\mathbb{C}$ be an ∞-bicategory. We define an ∞-bicategory $\mathbb{F}\mathrm{ib}_{1,0}(\mathbb{C})$ in a manner entirely analogous to Remark 5.54, that is, as the nerve (see Definition 2.9) of the fibrant $\mathrm{Set}_{\Delta}^{+}$-enriched category $\left(\mathrm{Set}_{\Delta}^{\mathbf{mb}}\right)^{\mathrm{o}}_{/(\mathbb{C},\sharp,T_{\mathbb{C}}\subset\sharp)}$ (see Section 5.4) associated to the model structure on $(1,0)$-fibrations (see [6]).

Remark 6.4 For the rest of this section we will work simultaneously with $(0,1)$- and $(1,0)$-fibrations and we will consequently introduce some notation to avoid confusion. Given an (i,j)-fibration, incarnated as a fibrant **MB** simplicial set $p:(X,E_X,T_X\subseteq C_X)\to(S,\sharp,\sharp)$ and any other **MB** simplicial set $(A,E_A,T_A\subseteq E_A)$, we will denote by $\mathrm{Map}_S^{i,j}(A,X)$ the ∞-bicategory obtained (see Definition 3.15) as the pullback

$$\begin{tikzcd}
\mathrm{Map}_S^{i,j}(A,X) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathbf{mb}}(A,X) \arrow[d] \\
\Delta^0 \arrow[r, "q"] & \mathrm{Fun}^{\mathbf{mb}}(A,S)
\end{tikzcd}$$

Note that the underlying marked simplicial set of $\mathrm{Map}_{\mathbb{C}}^{i,j}(-,-)$ is precisely the mapping marked simplicial set in $\mathbb{F}\mathrm{ib}_{i,j}(\mathbb{C})$.

Remark 6.5 Let $f:\mathbb{A}\to\mathbb{C}$ be a functor of ∞-bicategories and consider a pullback diagram

$$\begin{tikzcd}
\mathbb{F}(\mathbb{C})\times_{\mathbb{C}}\mathbb{A} \arrow[r] \arrow[d] & \mathbb{F}(\mathbb{C}) \arrow[d, "\mathrm{ev}_1"] \\
\mathbb{A} \arrow[r, "f"] & \mathbb{C}
\end{tikzcd}$$

It follows that evaluation at 0 induces a map $\mathbb{F}(\mathbb{C})\times_{\mathbb{C}}\mathbb{A}\to\mathbb{C}$ which is a $(1,0)$-fibration by Proposition 3.8 in [4]. We observe that we have a commutative diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[rr] \arrow[dr, "f"'] & & \mathbb{F}(\mathbb{C})\times_{\mathbb{C}}\mathbb{A} \arrow[dl, "\mathrm{ev}_0"] \\
& \mathbb{C} &
\end{tikzcd}$$

where the horizontal morphism is induced by the map $\Delta^1\otimes\mathbb{A}\to\Delta^0\otimes\mathbb{A}\simeq\mathbb{A}$. Moreover, we see as a consequence of Theorem 3.17 in [4] that for every $(1,0)$-fibration $\pi:\mathbb{X}\to\mathbb{C}$ we have a trivial fibration of ∞-bicategories

$$\mathrm{Map}_{\mathbb{C}}^{1,0}(\mathbb{F}(\mathbb{C})\times_{\mathbb{C}}\mathbb{A},\mathbb{X})\to\mathrm{Fun}_{\mathbb{C}}(\mathbb{A},\mathbb{X}),$$

where we are denoting by $\mathrm{Map}_{\mathbb{C}}^{1,0}(-,-)$ the mapping ∞-bicategory (as in Remark 6.4) between $(1,0)$-fibrations and by $\mathrm{Fun}_{\mathbb{C}}(\mathbb{A},\mathbb{X})$ the ∞-bicategory obtained via the pullback

$$\begin{tikzcd}
\mathrm{Fun}_{\mathbb{C}}(\mathbb{A},\mathbb{X}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathbb{A},\mathbb{X}) \arrow[d, "\pi_*"] \\
\Delta^0 \arrow[r, "f"] & \mathrm{Fun}(\mathbb{A},\mathbb{C})
\end{tikzcd}$$

We are now ready to define the Yoneda embedding. Let us consider the $(0,1)$-fibration $\mathrm{ev}_1 : \mathbb{F}(\mathbb{C}) \to \mathbb{C}$ and observe that we have a commutative diagram, over $\mathbb{C}$,

$$\begin{tikzcd}
\mathbb{F}(\mathbb{C}) \arrow[rr, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \arrow[dr, "\mathrm{ev}_1"'] & & \mathbb{C} \times \mathbb{C} \arrow[dl, "\pi_2"] \\
& \mathbb{C} &
\end{tikzcd}$$

where π_2 is the projection onto the second factor. It follows from our definitions that the map $\mathrm{ev}_0 \times \mathrm{ev}_1$ can be seen as a map of $(0,1)$-fibrations where $\mathbb{C} \times \mathbb{C}$ is classified by the constant functor with value $\mathbb{C}$. Let $\mathcal{F}$ be the functor classified by $\mathbb{F}(\mathbb{C}) \to \mathbb{C}$. We make the following observations:

(1) For every $c \in \mathbb{C}$ we have a functor $p_c : \mathcal{F}(c) \to \mathbb{C}$ and for every morphism $u : c \to c'$ we have a commutative diagram $p_{c'} \circ \mathcal{F}(u) = p_c$.

(2) For every $c \in \mathbb{C}$ it follows that we have a morphism

$$\mathbb{F}(\mathbb{C}) \times_{\mathbb{C}} * = \mathbb{C}_{\nearrow c} \to \mathbb{C},$$

which is a $(1,0)$-fibration by Remark 6.5. Furthermore, we note that for every $c \in \mathbb{C}$ it follows that we have a morphism

$$\mathbb{F}(\mathbb{C}) \times_{\mathbb{C}} * = \mathbb{C}_{\nearrow c} \to \mathbb{C},$$

which is a $(1,0)$-fibration by Remark 6.5 whose fibres are ∞-categories. Careful inspection reveals that for every $u : c \to c'$ the induced morphism $u_* : \mathbb{C}_{\nearrow c} \to \mathbb{C}_{\nearrow c'}$ preserves $(1,0)$-cartesian edges.

(3) Combining (1) and (2) we see that $p_c : \mathcal{F}(c) \to \mathbb{C}$ is a $(1,0)$-fibration and that $\mathcal{F}(u)$ is a functor of $(1,0)$-fibrations.

We conclude that $\mathcal{F}$ can be expressed as a composite

$$\mathcal{F} : \mathbb{C} \to \mathbb{F}\mathrm{ib}_{1,0}(\mathbb{C})^{\text{1-fib}} \to \mathbb{B}\mathrm{icat}_\infty,$$

where the second functor is the obvious projection and the superscript "1-fib" denotes the ∞-bicategory spanned by $(1,0)$-fibrations whose fibres are ∞-categories. Using a dual version of our main result or equivalently Corollary 3.90 in [3] we obtain a functor

$$\mathcal{Y}_{\mathbb{C}} : \mathbb{C} \to \mathrm{Fun}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty),$$

which we call the bicategorical Yoneda embedding. Dually, using the map $\mathrm{ev}_0 : \mathbb{F}(\mathbb{C}) \to \mathbb{C}$ we obtain the co-Yoneda embedding

$$\mathcal{Y}^{\mathrm{co}}_{\mathbb{C}} : \mathbb{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathbb{C}, \mathbb{C}\mathrm{at}_\infty).$$

Definition 6.6 Let $f : (A, E_A, T_A \subset C_A) \to (\mathbb{C}, \sharp, T_{\mathbb{C}} \subset \sharp)$ be a map of **MB** simplicial sets. We define an **MB** simplicial set $(Q(f), E_f, T_f \subset C_f)$ over $(\mathbb{C}, \sharp, T_{\mathbb{C}} \subset \sharp)$ as follows:

- The underlying map of simplicial sets is the composite

$$\mathbb{F}(\mathbb{C}) \times_{\mathbb{C}} A \to \mathbb{F}(\mathbb{C}) \xrightarrow{\mathrm{ev}_0} \mathbb{C},$$

where the pullback is taken along to the map $\mathrm{ev}_1 : \mathbb{F}(\mathbb{C}) \to \mathbb{C}$.

- An edge is marked if its associated map $\Delta^1 \otimes \Delta^1 \to \mathbb{C}$ factors through $\Delta^1 \times \Delta^1$ and the restriction to $\Delta^1 \times \Delta^{\{1\}}$ is marked in A.
- A triangle is lean if its restriction $\Delta^{\{1\}} \otimes \Delta^2$ is lean in A.
- A triangle is thin if it is lean and its image in $\mathbb{C}$ is thin.

The proofs in [4, Theorem 3.17, Corollary 3.20] are of an entirely combinatorial nature and under close inspection one sees we always have an **MB** anodyne morphism $(A, E_A, T_A \subset C_A) \to (Q(f), E_f, T_f \subset C_f)$ (in the model structure for $(1,0)$-fibrations) whose definition is induced by $\Delta^1 \otimes A \to A$ as in Remark 6.5.

Proposition 6.7 *There exists a functor* $\mathbb{I} : \mathbb{F}\mathrm{ib}_{1,0}(\mathbb{C})^{1\text{-fib}} \to \mathbb{F}\mathrm{ib}_{1,0}(\mathbb{C})^{1\text{-fib}}$ *which sends a* $(1,0)$*-fibration* $p : \mathcal{G} \to \mathbb{C}$ *with* ∞*-categorical fibres to the* $(1,0)$*-fibration* $\mathbb{I}(\mathcal{G})$ *which is defined as the* ***MB*** *simplicial set characterised uniquely by the isomorphism*

$$\mathrm{Map}^{1,0}_{\mathbb{C}}(A, \mathbb{I}(\mathcal{G})) \cong \mathrm{Map}^{1,0}_{\mathbb{C}}(Q(f), \mathcal{G}),$$

where $f : (A, E_A, T_A \subseteq C_A) \to (\mathbb{C}, \sharp, T_{\mathbb{C}} \subset \sharp)$ *is a map of* ***MB*** *simplicial sets and* $Q(f)$ *is defined as in Definition 6.6.*

Proof Note that since $Q(-)$ is functorial on $\left(\mathrm{Set}^{\mathbf{mb}}_{\Delta}\right)_{/(\mathbb{C},\sharp,T_{\mathbb{C}}\subset\sharp)}$ it follows that our definition yields an **MB** simplicial set over $(\mathbb{C}, \sharp, T_{\mathbb{C}} \subset \sharp)$.

There are two main things to prove: we need to show that $\mathbb{I}(\mathcal{G}) \to \mathbb{C}$ is a $(1,0)$-fibration with ∞-categorical fibres and that the construction $\mathbb{I}(-)$ is functorial. We will start by first proving the second assertion. Note that by Definition 6.3, it suffices to verify that $\mathbb{I}(-)$ yields a functor of Set^+_{Δ}-enriched categories.

Let $K \in \mathrm{Set}^+_{\Delta}$ (which we view as having the maximal scaling) and consider a map $K \to \mathrm{Map}^{1,0}_{\mathbb{C}}(\mathcal{G}, \mathcal{H})$. We will construct a morphism $K \to \mathrm{Map}^{1,0}_{\mathbb{C}}(\mathbb{I}(\mathcal{G}), \mathbb{I}(\mathcal{H}))$ as follows:

Given a simplex $\Delta^n \xrightarrow{\sigma \times \bar{\varphi}} \mathbb{I}(\mathcal{G}) \times K$ we consider a morphism

$$\mathbb{F}(\mathbb{C}) \times_{\mathbb{C}} \Delta^n \xrightarrow{f_\sigma \times \varphi} \mathcal{G} \times K \to \mathcal{H},$$

where φ is given by the composite $\mathbb{F}(\mathbb{C}) \times_{\mathbb{C}} \Delta^n \to \Delta^n \xrightarrow{\bar{\varphi}} K$. This map is clearly compatible with the projection and functorial a thus yields a map $\mathbb{I}(\mathcal{G}) \times K \to \mathbb{I}(\mathcal{H})$.

To finish the proof we will show that $\mathbb{I}(\mathcal{G}) \in \mathbb{F}\mathrm{ib}_{1,0}(\mathbb{C})^{1\text{-fib}}$. First let us observe that given $c \in \mathbb{C}$ we have a canonical isomorphism

$$\mathbb{I}(\mathcal{G}) \times_{\mathbb{C}} \{c\} \simeq \mathrm{Map}^{1,0}_{\mathbb{C}}(\mathbb{C}_{\nearrow c}, \mathcal{G}),$$

which shows that the fibres of $\mathbb{I}(\mathcal{G})$ are in fact ∞-categories. Let us consider a commutative diagram of **MB** simplicial sets

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "u"'] & & B \arrow[dl, "v"] \\ & \mathbb{C} & \end{tikzcd}$$

where i is an anodyne morphism in the model structure for $(1,0)$-fibrations developed in [6]. It follows from Definition 6.6 that we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\simeq"] \arrow[d, "\simeq"] & Q(u) \arrow[d] \\ B \arrow[r, "\simeq"] & Q(v) \end{tikzcd}$$

where the horizontal morphisms are weak equivalences. It then follows by two-out-of-three that the right-most vertical morphism is also a weak equivalence. We conclude that $\mathbb{I}(\mathcal{G})$ is a $(1,0)$-fibration. □

Remark 6.8 By the previous proposition the fibration $\mathbb{I}(\mathcal{G})$ corresponds under the straightening equivalence to a functor $\mathbb{C}^{\mathrm{op}} \to \mathbb{C}\mathrm{at}_\infty$ mapping an object c to

$$\mathrm{Map}^{1,0}_{\mathbb{C}}(\mathbb{C}_{\nearrow c}, \mathcal{G}) \simeq \mathrm{Nat}_{\mathbb{C}^{\mathrm{op}}}(\mathbb{C}(-,c), \mathbb{S}\mathrm{t}_{\mathbb{C}}(\mathcal{G})),$$

where $\mathrm{Nat}_{\mathbb{C}^{\mathrm{op}}}(-,-)$ is the mapping ∞-category in $\mathrm{Fun}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty)$. We will omit the explicit verification of the fact that there exists an equivalence of contravariant functors

$$\mathbb{S}\mathrm{t}_{\mathbb{C}}(\mathbb{I}(\mathcal{G})) \simeq \mathrm{Nat}_{\mathbb{C}^{\mathrm{op}}}(\mathcal{Y}_{\mathbb{C}}(-), \mathbb{S}\mathrm{t}_{\mathbb{C}}(\mathcal{G})).$$

Remark 6.9 The coming proofs will involve using the straightening-unstraightening equivalence for $(0,1)$- and $(1,0)$-fibrations. We will employ the notation $\mathbb{S}\mathrm{t}^{i,j}_S$ to distinguish between both variances.

Definition 6.10 Let $\mathbb{C}$ be an ∞-bicategory and pick a fibrant replacement $s : \mathfrak{C}^{\mathbf{sc}}[\mathbb{C}] \xrightarrow{\simeq} \mathbb{C}$. We consider a functor of fibrant Set_Δ^+-enriched categories

$$\mathbb{Y}_{\mathbb{C}} : \mathfrak{C}^{\mathbf{sc}}[\mathbb{C}] \xrightarrow{J} \mathrm{Fun}_{\mathrm{Set}_{\Delta^+}}(\mathfrak{C}^{\mathbf{sc}}[\mathbb{C}]^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty) \xrightarrow{\mathbb{U}\mathrm{n}^{1,0}_{\mathbb{C}}} \mathbb{F}\mathrm{ib}_{1,0}(\mathbb{C}) \to \mathbb{B}\mathrm{icat}_\infty,$$

where the first functor sends each $c \in \mathfrak{C}^{\mathbf{sc}}[\mathbb{C}]$ to the functor $\mathbb{C}(s(-), s(c))$. Now we can consider the Grothendieck construction and obtain a $(0,1)$-fibration $p : \mathfrak{Y}_{\mathbb{C}} \to \mathbb{C}$.

The definition above yields another reasonable candidate for the Yoneda embedding. In the next theorem, we will identify our purely fibrational approach to the Yoneda embedding with the previous definition, which relies heavily on the model of Set_Δ^+-enriched categories. Before proving the main results of the section we will need some preliminary work.

Construction 6.11 Let $\mathbb{C}$ be an ∞-bicategory and let $\mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] = \mathbb{O}^n$. Let us denote by $\iota_n : \Delta^n_\flat \to \mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n)$ (see Definition 2.9) the canonical trivial cofibration adjoint to the identity map. Then for every $\sigma : \Delta^n_\flat \to \mathbb{C}$ we can pick an extension $E_\sigma : \mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n) \to \mathbb{C}$ such that $E_\sigma \circ \iota_n = \sigma$. We claim that we can make a choice $\{E_\sigma\}_\sigma$ where σ runs over the nondegenerate simplices of $\mathbb{C}$ which is in addition functorial. This means that if we are given a map $\tau : \Delta^\ell_\flat \to \Delta^n_\flat$ then $E_\sigma \circ \mathfrak{C}^{\mathbf{sc}}[\tau] = E_{\sigma \circ \tau}$.

Since $\mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n) = \Delta^n_\flat$ for $n = 0, 1$, the choices are already fixed for dimensions $n \leqslant 1$. Suppose we have made a compatible choice for simplices up to dimension $k - 1$ and consider $\Delta^k_\flat \to \mathbb{C}$. Then we can consider the homotopy pushout diagram

$$\begin{tikzcd}
\partial\Delta^k_\flat \arrow[r, "\simeq"] \arrow[d] & \operatorname{colim}_{\partial\Delta^k} \mathrm{N}^{\mathbf{sc}}(\mathbb{O}^{k-1}) \arrow[d] \\
\Delta^n_\flat \arrow[r, "\simeq"] & P
\end{tikzcd}$$

where top horizontal (and hence the bottom) map is a weak equivalence the colimits involved are homotopy colimits. We obtain by the universal property of the pushout a map $P \to \mathbb{C}$. Moreover, we have a factorisation of ι_n as $\Delta^n_\flat \to P \xrightarrow{j} \mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n)$, so we conclude by two-out-of-three that j is also a weak equivalence. The solution of the lifting problem

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "j"] & \mathbb{C} \\
\mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n) \arrow[ur, dotted] &
\end{tikzcd}$$

provides the desired functorial extension.

Definition 6.12 We define a functor $\mathbb{P}^n : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \mathrm{Set}^{\mathbf{ms}}_\Delta$ as follows:

- For every $i \in [n]$ we declare the value $\mathbb{P}^n(i)$ to be the scaled poset $\mathcal{P}_{[0,i]}$ given in Definitions 5.2 and 5.5 (equipped with the minimal marking on 1-simplices).
- For every $S \subset [n]$ with $\min(S) = i$ and $\max(S) = j$ we consider the functor $\mathcal{P}_{[0,i]} \to \mathcal{P}_{[0,j]}$ sending $U \in \mathcal{P}_{[0,i]}$ to $S \cup U$. Similarly given an inclusion $S \subset T$ we consider the natural transformation
$$\mathcal{P}_{[0,i]} \times \Delta^1 \to \mathcal{P}_{[0,j]},$$
with components given by $S \cup U \subset T \cup V$ (see Remark 5.3).

Remark 6.13 We have functors $\mathcal{P}_{[0,j]} \to \mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n)$ which are uniquely determined by the assignment given by $S \mapsto \min(S)$, $S \leqslant T \mapsto U_{S,T}$ (see Definition 5.5 and Remark 5.4 to see why this definition is compatible with the scaling). These functors allow us to produce a lift $\mathbb{P}^n : \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat] \to \big(\mathrm{Set}^{\mathbf{mb}}_\Delta\big)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)}$ by picking the maximal lean scaling.

Lemma 6.14 *Let* $\phi : \mathfrak{C}^{\mathbf{sc}}[\mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n)] \to \mathbb{O}^n$ *be the counit map associated to the Quillen equivalence* $\mathfrak{C}^{\mathbf{sc}} \dashv \mathrm{N}^{\mathbf{sc}}$ *(see Definition 2.9) where* $\mathbb{O}^n = \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat]$*. Then there exists a natural transformation* $\alpha_n : \mathbb{P}^n \Rightarrow \mathbb{U}\mathrm{n}^{\mathbf{1,0}}_\phi \circ \mathbb{T}$ *where* $\mathbb{T}^n(i) = \mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat](-, i)$*.*

Proof By adjunction, it will enough to produce natural transformation $\mathbb{S}\mathrm{t}^{\mathbf{1,0}}_\phi \circ \mathbb{P}^n \Rightarrow \mathbb{T}$. Let $i \in \mathbb{O}^n$ and let $\mathcal{P}^{\triangleright}_{[0,i]}$ the scaled simplicial set obtained from the join $P^{\triangleright}_{[0,i]} = P_{[0,i]} * \Delta^0$, by scaling those simplices of the form $u(\sigma)$ where $u : P_{[0,i]} \to P^{\triangleright}_{[0,i]} = P_{[0,i]} * \Delta^0$ is the canonical map, and σ is scaled in $\mathcal{P}_{[0,i]}$. We consider a pushout diagram

$$\begin{tikzcd}
\mathfrak{C}^{\mathrm{sc}}[\mathcal{P}_{[0,i]}] \arrow[r] \arrow[d, "\Phi"] & \mathfrak{C}^{\mathrm{sc}}[\mathcal{P}^{\triangleright}_{[0,i]}] \arrow[d] \\
\mathbb{O}^n \arrow[r, "r"] & C_\phi(\mathbb{P}(i))
\end{tikzcd}$$

where Φ is the adjoint to the map given in Remark 6.13. Let $*$ be the cone point of $\mathfrak{C}^{\mathrm{sc}}[\mathcal{P}^{\triangleright}_{[0,i]}]$. Then it follows that $C_\phi(\mathbb{P}(i))(r(-), *) = \mathbb{St}^{1,0}_{\phi}(\mathbb{P}(i))$ as constructed in [3, Section 3]. Let $S \in \mathcal{P}_{[0,i]}$ with $\min(S) = s$. We consider the map

$$\bar{\alpha}^n_i(S) : \mathfrak{C}^{\mathrm{sc}}[\mathcal{P}^{\triangleright}_{[0,i]}](S, *) \to \mathbb{O}^n(s, i), \quad \{S = S_0 < S_1 < \cdots < S_n < *\} \mapsto S_n \cup \bigcup_{i=0}^{n} U_{S_i, S_{i-1}},$$

which is compatible with the marking of $\mathfrak{C}^{\mathrm{sc}}[\mathcal{P}_{[0,i]}](S, *)$ and descends to a map

$$\alpha^n_i(s) : \mathbb{St}^{1,0}_{\phi}(\mathbb{P}(i))(s) \to \mathbb{O}^n(s, i).$$

Note that if we are given $S < T$ with $\min(S) = s$ and $\min(T) = t$, then we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{C}^{\mathrm{sc}}[\mathcal{P}^{\triangleright}_{[0,i]}](T, *) \arrow[r, "\bar{\alpha}^n_i(T)"] \arrow[d] & \mathbb{O}^n(t, i) \arrow[d, "\cup U_{S,T}"] \\
\mathfrak{C}^{\mathrm{sc}}[\mathcal{P}^{\triangleright}_{[0,i]}](S, *) \arrow[r, "\bar{\alpha}^n_i(S)"] & \mathbb{O}^n(s, i)
\end{tikzcd}$$

which guarantees that the maps $\{\alpha^n_i(s)\}_{s \in \mathbb{O}^n}$ assemble into a natural transformation $\alpha^n_i : \mathbb{St}^{1,0}_{\phi}(\mathbb{P}^n(i)) \Rightarrow \mathbb{O}^n(-, i)$. It is straightforward to verify that the maps $\{\alpha^n_i\}_{i \in \mathbb{O}^n}$ also assemble into the desired natural transformation $\alpha^n : \mathbb{St}^{1,0}_{\phi} \circ \mathbb{P}^n \Rightarrow \mathbb{T}$. □

Proposition 6.15 *Let* $\mathbb{C}$ *be an* ∞*-bicategory. Then there exists an equivalence of* $(0,1)$*-fibrations, over* $\mathbb{C}$,

$$\begin{tikzcd}
\mathbb{F}(\mathbb{C}) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}_1"'] & & \mathfrak{Y}_{\mathbb{C}} \arrow[dl, "p"] \\
& \mathbb{C} &
\end{tikzcd}$$

Proof We observe that by construction there exists a map of $(0,1)$-fibrations $\mathfrak{Y}_{\mathbb{C}} \to \mathbb{C} \times \mathbb{C}$ such that for every $c \in \mathbb{C}$ the induced map on fibres

$$\mathfrak{Y}_{\mathbb{C}} \times_{\{c\}} \mathbb{C} \to \mathbb{C}$$

defines a $(1,0)$-fibration whose fibres are ∞-categories. Since the map $\mathbb{C} \to \mathbb{F}(\mathbb{C})$ is a trivial cofibration (this is the dual to Theorem 3.17 in [4]) it follows that to produce a map $\mathbb{F}(\mathbb{C}) \to \mathfrak{Y}_{\mathbb{C}}$ it suffices to construct a section of p.

Let $\sigma:\Delta^n_\flat\to\mathbb{C}$. Our goal is to produce an n-simplex in the pullback

$$\begin{tikzcd}
\mathfrak{Y}_n \arrow[r] \arrow[d] & \mathfrak{Y}_{\mathbb{C}} \arrow[d] \\
\Delta^n_\flat\times\Delta^n_\flat \arrow[r, "\sigma\times\sigma"] & \mathbb{C}\times\mathbb{C}
\end{tikzcd}$$

in a functorial way. Let $\mathbb{St}^{\mathbf{0,1}}_{\Delta^n_\flat}(\Delta^n_\flat):\mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat]\to\mathrm{Set}^{\mathbf{ms}}_\Delta$ be the straightening of the identity functor $\Delta^n_\flat\to\Delta^n_\flat$. The maps defined in Remark 5.6 provide us with a natural transformation $\mathbb{St}^{\mathbf{1,0}}_{\Delta^n_\flat}(\Delta^n_\flat)\Rightarrow\mathbb{P}^n$ with $\mathbb{P}^n$ as in Definition 6.12. It then follows from Remark 6.13 that $\mathbb{St}^{(0,1)}_{\Delta^n_\flat}(\Delta^n_\flat)$ admits a lift to $(\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)}$.

Note that since we have a map $\mathfrak{C}^{\mathbf{sc}}[\sigma]:\mathfrak{C}^{\mathbf{sc}}[\Delta^n_\flat]\to\mathfrak{C}^{\mathbf{sc}}[\mathbb{C}]\xrightarrow{\simeq}\mathbb{C}$, it follows that we have a commutative diagram, for every $i\in\mathbb{O}^n$,

$$\begin{tikzcd}
\mathbb{Un}^{1,0}_{\phi}(\mathbb{T}(i)) \arrow[r] \arrow[d] & \mathbb{Un}^{1,0}_{\mathbb{C}}\big(J(\mathfrak{C}^{\mathbf{sc}}[\sigma](i))\big) \arrow[d] \\
\mathrm{N}^{\mathbf{sc}}(\mathbb{O}^n) \arrow[r, "E_\sigma"] & \mathbb{C}
\end{tikzcd}$$

where E_σ was given in Construction 6.11 and $\mathbb{T}$ in Lemma 6.14. This family of maps is natural in $i\in\mathbb{O}$, thus yielding a natural transformation $\mathbb{Un}^{(1,0)}_{\phi}\circ\mathbb{T}\Rightarrow\mathbb{Un}^{(1,0)}_{\mathbb{C}}\circ J\circ\mathfrak{C}^{\mathbf{sc}}[\sigma]$. Finally, let us observe that

$$\mathbb{St}^{\mathbf{0,1}}_{\Delta^n_\flat}(\Delta^n_\flat)\Rightarrow\mathbb{P}^n\Rightarrow\mathbb{Un}^{(1,0)}_{\phi}\circ\mathbb{T}\Rightarrow\mathbb{Un}^{(1,0)}_{\mathbb{C}}\circ J\circ\mathfrak{C}^{\mathbf{sc}}[\sigma]$$

yields the desired simplex in $T(\sigma):\Delta^n_\flat\to\mathfrak{Y}_n$. Note that the functoriality of our lifts E_σ (Construction 6.11) guarantees that the assignment $\sigma\mapsto T(\sigma)$ is functorial. It is also straightforward to verify that our assignment is compatible with the decorations thus yielding the desired map $\mathbb{F}(\mathbb{C})\to\mathfrak{Y}_{\mathbb{C}}$ of $(0,1)$-fibrations over $\mathbb{C}$.

To finish the proof we need to show that for every $c\in\mathbb{C}$ the induced map on fibres

$$\begin{tikzcd}
\mathbb{C}_{\nearrow c} \arrow[rr, "f"] \arrow[dr, "\mathrm{ev}_0"'] & & \mathfrak{Y}_{\mathbb{C}}\times_{\{c\}}\mathbb{C} \arrow[dl] \\
& \mathbb{C} &
\end{tikzcd}$$

is a weak equivalence. We note that by construction the horizontal map above sends the object $i:\Delta^0\to\mathbb{C}_{\nearrow c}$, represented to the identity morphism to an object in $\mathfrak{Y}_{\mathbb{C}}\times_{\{c\}}\mathbb{C}$ also represented by the identity morphism. We claim that we have a sequence of maps

$$\Delta^0\xrightarrow{i}\mathbb{C}_{\nearrow c}\xrightarrow{f}\mathfrak{Y}_{\mathbb{C}}\times_{\{c\}}\mathbb{C}$$

such that i and $f\circ i$ are weak equivalences of in the model structure of $(1,0)$-fibrations. The first map is a weak equivalence by [4, Theorem 3.17]. Since $\mathfrak{Y}_{\mathbb{C}}\times_{\{c\}}\mathbb{C}\simeq\mathbb{Un}^{1,0}_{\mathbb{C}}\big(\mathbb{C}(s(-),s(c))\big)$ (see Definition 6.10), we see that $f\circ i$ is adjoint to the natural transformation $\mathfrak{C}^{\mathbf{sc}}[\mathbb{C}](-,c)\Rightarrow\mathbb{C}(s(-),s(c))$ which is a pointwise weak equivalence. The result follows by two-out-of-three. $\square$

Theorem 6.16 *For every ∞-bicategory $\mathbb{C}$ the Yoneda embedding*

$$\mathcal{Y}_{\mathbb{C}} : \mathbb{C} \to \mathrm{Fun}(\mathbb{C}^{\mathrm{op}}, \mathfrak{Cat}_\infty), \quad c \mapsto \mathbb{C}(-,c),$$

is fully faithful. Moreover, given a functor $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathfrak{Cat}_\infty$ there is a equivalence of $\mathfrak{Cat}_\infty$-valued functors

$$\mathrm{Nat}_{\mathbb{C}^{\mathrm{op}}}(\mathcal{Y}_{\mathbb{C}}(-), \mathcal{F}) \overset{\simeq}{\Longrightarrow} \mathcal{F},$$

which is natural in $\mathcal{F}$.

Proof It follows from Proposition 6.15 that the embedding $\mathcal{Y}_{\mathbb{C}}$ is fully faithful if and only if the functor J in Definition 6.10 induces a weak equivalence of the corresponding mapping marked simplicial sets. This is equivalent to showing that the composite

$$\mathfrak{C}^{\mathrm{sc}}[\mathbb{C}] \xrightarrow{s} \mathbb{C} \xrightarrow{j} \mathrm{Fun}_{\mathrm{Set}_\Delta^+}(\mathbb{C}^{\mathrm{op}}, \mathrm{Set}_\Delta^+) \xrightarrow{s^*} \mathrm{Fun}_{\mathrm{Set}_\Delta^+}(\mathfrak{C}^{\mathrm{sc}}[\mathbb{C}]^{\mathrm{op}}, \mathrm{Set}_\Delta^+)$$

induces a weak equivalence of the corresponding mapping marked simplicial sets. This follows easily as the functor s is an equivalence of Set_Δ^+-enriched categories, j is the enriched Yoneda embedding (which is fully faithful in the enriched sense) and s^* is also an equivalence of Set_Δ^+-enriched categories by [18, A.3.3.8]. We conclude that $\mathcal{Y}_{\mathbb{C}}$ is fully faithful.

To prove the final claim we show that the functor $\mathbb{I}$ in Proposition 6.7 is naturally equivalent to the identity. We construct a natural transformation $\mathbb{I} \Rightarrow \mathbb{1}$ to the identity functor which is induced by the canonical map $A \to Q(f)$ (see Definition 6.6). We observe that the induced map on fibres

$$\mathrm{Map}^{1,0}_{\mathbb{C}}(\mathbb{C}_{\nearrow c}, \mathcal{F}) \xrightarrow{\simeq} \mathcal{F}(c)$$

is given by restriction along the anodyne map $\Delta^0 \to \mathbb{C}_{\nearrow c}$ selecting the identity morphism and thus is a weak equivalence. □

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