AGT

Algebraic & Geometric Topology

Volume 26 (2026)

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Algebraic & Geometric Topology **26**:5 (2026) 1749–1780
DOI: 10.2140/agt.2026.26.1749
Published: May 29, 2026

Brauer–Wall groups and truncated Picard spectra of K-theory

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We compute the first two k-invariants of the Picard spectra of KU and KO by analyzing their Picard groupoids and constructing their unit spectra as global sections of sheaves on the category of manifolds. This allows us to determine the $\mathbb{E}_\infty$-structures of their truncations $\mathrm{Pic}(KU)[0,3]$ and $\mathrm{Pic}(KO)[0,2]$. It follows that these truncated Picard spaces represent the Brauer groups of $\mathbb{Z}/2$-graded algebra bundles of Donovan, Karoubi, Moutuou and Maycock; the Brauer groups of super 2-lines; and the K-theory twists of Freed, Hopkins and Teleman. Our results also imply that these spaces represent twists of **String**- and **Spin**-structures on manifolds and can be used to twist tmf-cohomology. Finally, we are able to identify $\mathrm{pic}(KU)[0,3]$ with a cotruncation of the Anderson dual of the sphere spectrum.

1 Introduction

We study the first two k-invariants of the Picard spectra of KO and KU, or equivalently, the infinite loop space structures on the 2-truncation of $\mathrm{Pic}(KO)$ and the 3-truncation of $\mathrm{Pic}(KU)$, which we denote by $\mathrm{Pic}_0^2(KO)$ and $\mathrm{Pic}_0^3(KU)$, respectively. We will also work with their connected covers which we denote by $\mathrm{Pic}_1^2(KO)$ and $\mathrm{Pic}_1^3(KU)$, respectively. These last two spectra are of course equivalent to truncations of $B\mathrm{GL}_1(KO)$ and $B\mathrm{GL}_1(KU)$, respectively. The homotopy types of $\mathrm{Pic}_0^2(KO)$ and $\mathrm{Pic}_0^3(KU)$ are not particularly interesting. Indeed, they both split as products of Eilenberg–Mac Lane spaces:

$$\mathrm{Pic}_0^3(KU) \simeq \mathbb{Z}/2 \times K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3),$$
$$\mathrm{Pic}_0^2(KO) \simeq \mathbb{Z}/8 \times K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2).$$

What we show below however is that neither the first nor second k-invariants of the associated Picard *spectra* $\mathrm{pic}_0^3(KU)$ and $\mathrm{pic}_0^2(KO)$ are trivial. In other words, none of the above splittings are splittings of infinite loop spaces. The first theorems we prove are the computations of these k-invariants:

Theorem *The first k-invariant of $\mathrm{pic}_0^3(KU)$ is $\mathrm{Sq}^2 : H\mathbb{Z}/2 \to \Sigma^2 H\mathbb{Z}/2$. The second k-invariant can be taken to be either of the generators of $H^4(\mathrm{pic}_0^1(KU);\mathbb{Z}) \cong \mathbb{Z}/4$, both of which restrict to the map $\beta \circ \mathrm{Sq}^2 : \Sigma H\mathbb{Z}/2 \to \Sigma^4 H\mathbb{Z}$ upon taking a connected cover.*

MSC2020: 19L50, 55N15, 55P42.

Theorem *The first k-invariant of $\mathrm{pic}_0^2(KO)$ is $\mathrm{Sq}^2 \circ \rho : H\mathbb{Z}/8 \to \Sigma^2 H\mathbb{Z}/2$, where $\rho : H\mathbb{Z}/8 \to H\mathbb{Z}/2$ is the reduction mod-2 map. The second k-invariant is one of two classes in $H^3(\mathrm{pic}_0^1(KO); \mathbb{Z}/2) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, both of which restrict to Sq^2 upon taking connected covers.*

The authors have not been able to find these results in the literature although they do seem to be at least partially known to experts (see, e.g., the proof of [23, Proposition 7.14]). From these theorems we can deduce the group structures of $\mathrm{pic}_0^2(KO)^0(X)$ and $\mathrm{pic}_0^3(KU)^0(X)$, which are nontrivial extensions of ordinary singular cohomology groups:

Corollary *There are bijections of sets*

$$\mathrm{pic}_0^2(KO)^0(X) \cong H^0(X;\mathbb{Z}/8) \times H^1(X;\mathbb{Z}/2) \times H^2(X;\mathbb{Z}/2),$$
$$\mathrm{pic}_0^3(KU)^0(X) \cong H^0(X;\mathbb{Z}/2) \times H^1(X;\mathbb{Z}/2) \times H^3(X;\mathbb{Z}).$$

The group laws on these sets, in the same order, are

$$(a,b,c) + (a',b',c') \mapsto (a+a', b+b', c+c'+b \cup b'),$$
$$(a,b,c) + (a',b',c') \mapsto (a+a', b+b', c+c'+\beta(b \cup b')),$$

where β denotes the Bockstein homomorphism.

Knowing these group structures allows us to identify other roles that these spectra play in algebraic topology, mathematical physics, and the theory of C^*-algebras. Almost all of these manifestations are related to twisted K-theory but in the literature they are not often directly related to Picard spectra and spaces, which are the universal receptacles for twists of any highly structured cohomology theory. Indeed, because of the infinite loop space splittings $B\mathrm{GL}_1(KO) \simeq B\mathrm{GL}_1(KO)[0,2] \times B\mathrm{GL}_1(KO)[3,\infty]$ and $B\mathrm{GL}_1(KU) \simeq B\mathrm{GL}_1(KU)[0,3] \times B\mathrm{GL}_1(KU)[4,\infty]$, any map from a connected space into $\mathrm{Pic}_0^2(KO)$ or $\mathrm{Pic}_0^3(KU)$ gives a purely homotopy-theoretic twist of K-theory by either composing with the inclusion or multiplying by -1 and then including. We now detail the various interpretations of $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$.

The group laws written above imply that the $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$ cohomology groups of a space are isomorphic to well-known Brauer groups of C^*-algebras over that space:

Theorem *If $\mathrm{Br}_{\mathbf{O}}^{\infty}(X)$ and $\mathrm{Br}_{\mathrm{U}}^{\infty}(X)$ are the Brauer groups of (possibly infinite-dimensional) real and complex continuous trace graded C^*-algebras with spectrum X (see [43; 44]) then there are isomorphisms*

$$\mathrm{Br}_{\mathbf{O}}^{\infty}(X) \cong \mathrm{pic}_0^2(KO)^0(X),$$
$$\mathrm{Br}_{\mathrm{U}}^{\infty}(X) \cong \mathrm{pic}_0^3(KU)^0(X).$$

It is a corollary of this fact that $\mathrm{pic}_0^2(KO)^0(X)$ and $\mathrm{Tors}(\mathrm{pic}_0^3(KU)^0(X))$ are also isomorphic to the graded Brauer groups of Donovan and Karoubi [10], where Tors denotes taking the torsion subgroup. Elements of both the C^*-algebra Brauer groups and the Brauer groups of Donovan and Karoubi are known to produce twists of K-theory but our isomorphisms allow one to produce such twists as actual maps of spaces $X \to B\mathrm{GL}_1(KO)$ and $X \to B\mathrm{GL}_1(KU)$. In the case that we replace $\mathrm{pic}_0^3(KU)$ with $\Sigma^3 H\mathbb{Z}$, and use *ungraded* C^*-algebras, the agreement of the two notions of twisted K-theory is shown in [26]. Presumably the same is true in our more general case, but we do not prove it in this paper.

Our computations also imply that $B\mathrm{GL}_1(KO)[0,2]$ and $B\mathrm{GL}_1(KU)[0,3]$ are equivalent, as infinite loop spaces, to certain fibers in the Postnikov tower for $\mathbf{BO}$.

Theorem *Let* $\mathbf{BString} \to \mathbf{BSO}$ *and* $\mathbf{BSpin} \to \mathbf{BO}$ *be the connective covers in the Postnikov tower of* $\mathbf{BO}$. *Then there are equivalences of infinite loop spaces*

$$\mathrm{Pic}_1^2(KO) \simeq \mathrm{fib}(\mathbf{BSpin} \to \mathbf{BO}),$$
$$\mathrm{Pic}_1^3(KU) \simeq \mathrm{fib}(\mathbf{BString} \to \mathbf{BSO}).$$

From this it follows that if M is a manifold (resp. oriented manifold) then the set of **Spin**-structures on M (resp. **String**-structures) is a torsor for $[M, \mathrm{Pic}_1^2(KO)] \cong H^1(M;\mathbb{Z}/2) \ltimes H^2(M;\mathbb{Z}/2)$ (resp. $[M, \mathrm{Pic}_1^3(KU)] \cong H^1(M;\mathbb{Z}/2) \ltimes H^3(M;\mathbb{Z})$). This should be compared with the fact that the set of **Spin**-structures on an oriented manifold is a torsor for $H^2(M;\mathbb{Z}/2)$ and the set of **String**-structures on a **Spin**-manifold is a torsor for $H^3(M;\mathbb{Z})$ (see [45, 2.11, 2.16]). It is claimed in [9, §2.1] that **Spin**-structures are a torsor for $H^1(M;\mathbb{Z}/2) \times H^2(M;\mathbb{Z}/2)$ but a proof is not included. Moreover, because there are equivalences of infinite loop spaces $\mathrm{Pic}_1^3(KU) \simeq B\mathrm{GL}_1(KO[0,1])$ and $\mathrm{Pic}_1^3(KU) \simeq B\mathrm{GL}_1(KU[0,2])$ we can twist **Spin**-structures (resp. **String**-structures) by bundles of $KO[0,1]$ (resp. $KU[0,2]$) modules, which we can interpret as real and complex super 2-line bundles, respectively.

Not surprisingly, our work here is also related to the work on twisted K-theory and mathematical physics by Freed, Hopkins, Teleman and others [9; 16; 17; 20; 21].

Theorem *Let* $\mathrm{cAlg}_{\mathbb{R}}^{\times}$ *and* $\mathrm{cAlg}_{\mathbb{C}}^{\times}$ *be the spectra associated to the Picard* 2*-groupoids of invertible topological* $\mathbb{R}$*- and* $\mathbb{C}$*-superalgebras, respectively. Then there are equivalences of spectra*

$$\mathrm{cAlg}_{\mathbb{R}}^{\times} \simeq \mathrm{pic}_0^2(KO),$$
$$\mathrm{cAlg}_{\mathbb{C}}^{\times} \simeq \mathrm{pic}_0^3(KU).$$

It follows that $\mathrm{gl}_1(KO)[0,1]$ and $\mathrm{gl}_1(KU)[0,2]$ are the Picard spectra of real and complex superlines, respectively. Again using the fact that $\mathrm{gl}_1(KO)[0,2] \simeq \mathrm{gl}_1(KO[0,2])$ and $\mathrm{gl}_1(KU)[0,3] \simeq \mathrm{gl}_1(KU[0,3])$, we have that a bundle of real (resp. complex) superlines on a space X is the same data as a bundle of $KO[0,2]$-lines (resp. $KU[0,3]$-lines). This also identifies $\mathrm{pic}_0^3(KU)$ with the spectrum R_{-1} of [9] whose associated cohomology theory is proposed as the container for the flux of the oriented superstring B-field.

The above theorem can be restated in the context of the K-theory twists of [16, 1.80; 21, Corollary 2.25] assuming we take X to be only a space rather than a topological groupoid.

Theorem *Let* $\pi_0\mathfrak{Twist}_{KU}(X)$ *and* $\pi_0\mathfrak{Twist}_{KO}(X)$ *be the groups of isomorphism classes of KU and KO twists on* X *in the sense of ibid. Then there are group isomorphisms*

$$\pi_0\mathfrak{Twist}_{KO}(X) \cong \mathrm{pic}_0^2(KO)^0(X),$$
$$\pi_0\mathfrak{Twist}_{KU}(X) \cong \mathrm{pic}_0^3(KU)^0(X).$$

In related work by the same authors, the Anderson dual of the sphere spectrum, $I_{\mathbb{Z}}$, often arises (see, for instance, [18, Hypothesis 5.17; 19, Theorem 5.27]). We show that, at least in the complex case, the truncated Picard spectrum is closely related:

Theorem *There is an equivalence of spectra*

$$\Sigma^3(I_{\mathbb{Z}}[-3,\infty)) \simeq \mathrm{pic}_0^3(KU).$$

This suggests a connection between invertible topological field theories (and deformations thereof) and bundles of truncated $KU[0,2]$-lines.

1.1 Conventions and notation

In this paper we will often work with ∞-categories (in the sense of [36]) of *spectra* and ∞-*groupoids*, which we denote by $\mathcal{S}\mathrm{p}$ and $\mathcal{S}$, respectively. There has been some disagreement lately about an efficacious term for the objects of the ∞-category $\mathcal{S}$. We find the term "∞-groupoid" to be too long, the term "anima" to be unpleasant to pluralize, and the term "space" to be far too ambiguous. Therefore, going forward, we will use "h-type" to refer to these structures and propose this as an alternative to the terms listed above. We will still occasionally call them ∞-groupoids when we want to emphasize their use as ∞-categories in which all morphisms are invertible. When we use the term "space" we will specifically be referring to a compactly generated, weakly Hausdorff topological space, the category of which we will denote by Top. Recall that if $E \in \mathcal{S}\mathrm{p}$ is a spectrum then it has an associated cohomology theory E^*. This cohomology theory is defined equally well on $\mathcal{S}$ and Top, so we will apply it to both sorts of objects without comment.

The term "infinite loop space" frequently denotes what, in the language of [38], one might call "grouplike $\mathbb{E}_\infty$-monoids in $\mathcal{S}$." Another commonly used term for such structures, and the one we will employ, is "abelian ∞-group," recognizing that these objects are the higher algebraic analogues of abelian groups. We emphasize however that being an abelian ∞-group is a *structure* rather than a property. If we wish to refer to an h-type which has an $\mathbb{E}_\infty$-structure without concerning ourselves with whether or not it is grouplike, we will often say "$\mathbb{E}_\infty$-type" instead of the unwieldy "$\mathbb{E}_\infty$-h-type."

Many of our constructions will involve a field $\mathbb{F}$. We will always assume that $\mathbb{F}$ has a topology which is accounted for by these constructions, e.g., vector bundles. We will occasionally wish to consider a field with its discrete topology, in which case we will say so.

We will often be interested in truncations and connective covers of spectra and h-types. For integers $n, m \geq 0$ and a spectrum or h-type X we will write $X[n,m]$ for the n-connective cover of the m-truncation of X. Note the use of connective here, as opposed to *connected*, which differs by 1. In the special, and ubiquitous, case that X is $\mathrm{Pic}(R)$ or $\mathrm{pic}(R)$ for a commutative ring spectrum R, we will use the nonstandard notation of Definition 2.2. We do this to keep the names of these frequently used objects compact.

2 Background

In this section we recall, for an $\mathbb{E}_\infty$-ring spectrum R, the construction of the so-called *Picard space* of R along with various spectra and h-types which can be built from it. More detailed constructions of these can be found in [3; 4].

Definition 2.1 Let R be an $\mathbb{E}_\infty$-ring spectrum with symmetric monoidal ∞-category of left modules LMod_R. Then we make the following definitions.

(1) We write $\mathrm{Pic}(R)$ for the maximal ∞-groupoid in LMod_R spanned by modules which are invertible with respect to the tensor product over R. Recall that $\mathrm{Pic}(R)$ is an abelian ∞-group with base point $R \in \mathrm{Pic}(R)$. We denote its infinite delooping by $\mathrm{pic}(R)$, i.e., $\Omega^\infty \mathrm{pic}(R) \simeq \mathrm{Pic}(R)$.

(2) We write $\mathrm{GL}_1(R)$ for the pullback of the projection $\Omega^\infty R \to \pi_0(R)$ along the inclusion $\pi_0(R)^\times \hookrightarrow \pi_0(R)$. Recall that $\mathrm{GL}_1(R)$ is equivalent, as an abelian ∞-group, to $\Omega\,\mathrm{Pic}(R)$. Equivalently, we could take $B\mathrm{GL}_1(R)$ to be the base point component of $\mathrm{Pic}(R)$. We denote the infinite deloopings of $\mathrm{GL}_1(R)$ and $B\mathrm{GL}_1(R)$ by $\mathrm{gl}_1(R)$ and $\mathrm{bgl}_1(R)$, respectively.

Definition 2.2 Given an $\mathbb{E}_\infty$-ring R, we write $\mathrm{Pic}_n^m(R)$ and $\mathrm{pic}_n^m(R)$ for the m-truncated, n-connective covers of $\mathrm{Pic}(R)$ and $\mathrm{pic}(R)$, respectively. In other words, $\mathrm{Pic}_n^m(R)$ and $\mathrm{pic}_n^m(R)$ are equivalent to $\mathrm{Pic}(R)$ and $\mathrm{pic}(R)$, respectively, in homotopy degrees n through m, and have trivial homotopy groups elsewhere. We will use $\mathrm{bgl}_1(R)[0,n]$ and $B\mathrm{GL}_1(R)[0,n]$ interchangeably with $\mathrm{pic}_1^n(R)$ and $\mathrm{Pic}_1^n(R)$. This notation should not be confused with the notation Pic^0 of algebraic geometry, which denotes the identity component of the Picard scheme.

Remark 2.3 Both m-truncating and taking n-connective covers determine symmetric monoidal functors $\mathcal{S} \to \mathcal{S}$ for all n and m, so $\mathrm{Pic}_n^m(R)$ has an abelian ∞-group structure induced by that of $\mathrm{Pic}(R)$.

It will also be useful to have the following definition recorded here, though we will not make frequent use of it. This definition appears, e.g., in [29].

Definition 2.4 Let R be a commutative ring spectrum. Then $\mathrm{LMod}(R)$ is an $\mathbb{E}_\infty$-algebra in the symmetric monoidal ∞-category of ∞-categories, Cat_∞, which in turn has its own category of modules. We write $\mathrm{Br}(R)$ for the abelian ∞-group of invertible $\mathrm{LMod}(R)$-modules and equivalences between them.

3 Nontriviality of k-invariants

In this section we show that the k-invariants of $\mathrm{pic}_0^1(KO)$, $\mathrm{pic}_0^1(KU)$, $\mathrm{pic}_1^2(KO)$ and $\mathrm{pic}_1^3(KU)$ are nontrivial. This will follow from showing that these spectra can be modeled by Picard groupoids, or sheaves thereof, that have nontrivial symmetries.

3.1 Picard groupoids and symmetries

Recall that there is an equivalence between spectral 1-types, i.e., spectra with homotopy groups only in degrees n and $n+1$ for $n \in \mathbb{Z}$, and symmetric monoidal categories in which every morphism is invertible and every object has a tensor inverse, i.e., Picard groupoids (see [30, Proposition B.12; 32]). In particular, Picard groupoids are equivalent to spectra with nontrivial homotopy groups only in degrees 0 and 1 by [32, Theorem 1.5].

Note that the proof of ibid. proceeds by lifting the usual looping/delooping equivalence between groupoids and homotopy 1-types to their categories of grouplike commutative monoids: Picard groupoids and 1-truncated grouplike infinite loop spaces, respectively. It follows immediately that a Picard groupoid, up to equivalence, is entirely determined by the data of two homotopy groups, π_0 and π_1, and a single k-invariant $\alpha \in \mathrm{Map}(H\pi_0, \Sigma^2 H\pi_1)$. However, Picard groupoids are an especially useful model of stable 1-types since, up to symmetric monoidal equivalence, they can always be rigidified to categories that are *permutative* and *skeletal* [32, Theorem 2.2]. In other words we can always assume, up to symmetric monoidal equivalence, that a Picard groupoid is strictly associative and strictly unital and that its only morphisms are automorphisms.

Let us be more explicit about the data by which a Picard groupoid $\mathcal{P}$ is classified. What follows is mostly a restatement of [23, Remark 2.12] but we also direct the reader to [15, Lecture 17, Section 27]. We assume that $\mathcal{P}$ is permutative and skeletal. It is clear from the preceding paragraph that the group of objects of $\mathcal{P}$ must be π_0 of the associated spectrum. Moreover, one can show that for any $x, y \in \mathcal{P}$ there must be an isomorphism $\mathrm{Aut}_{\mathcal{P}}(x) \cong \mathrm{Aut}_{\mathcal{P}}(y)$. By again invoking the description of stable 1-types as grouplike 1-truncated infinite loop spaces, we see that the associated spectrum must have $\pi_1 \cong \mathrm{Aut}_{\mathcal{P}}(x)$ for any choice of basepoint.

A standard diagram chase shows that the symmetry natural isomorphism of $\mathcal{P}$ must be entirely determined by a choice, for each $x \in \mathcal{P}$, of an element $\varepsilon \in \mathrm{Aut}_{\mathcal{P}}(x)$ such that $\varepsilon_x^2 = \mathrm{id}_x$ (since $\mathcal{P}$ can be assumed to have no isomorphic objects which are not identical). This is equivalent to an element of $\varepsilon \in \mathrm{Hom}_{\mathrm{Ab}}(\pi_0 \otimes \mathbb{Z}/2, \pi_1)$. By assuming permutativity, we have no need to define an associativity natural transformation. Therefore, the Picard groupoid is entirely determined by the data of π_0, π_1 and ε. Using the isomorphism

$$\mathrm{Hom}_{\mathrm{Ab}}(\pi_0 \otimes \mathbb{Z}/2, \pi_1) \cong H^2(\pi_0; \pi_1) \cong [H\pi_0, \Sigma^2 H\pi_1] \tag{1}$$

of [12, Section 27], we see that the symmetry datum ε is exactly the k-invariant of the spectrum. Note that if the k-invariant, equivalently the function ε, is trivial, i.e., if the associated spectrum is equivalent to $H\pi_0 \vee \Sigma H\pi_1$, then the symmetry natural isomorphism of $\mathcal{P}$ is the identity transformation.

The above discussion has a somewhat more modern interpretation. A 1-truncated connective spectrum X is equivalent to the data of a connective module over the 1-truncation of the sphere $\mathbb{S}$, which in turn is completely determined by $\pi_0 X$, $\pi_1 X$, and the action of the Hopf element $\eta \in \pi_1 \mathbb{S}$. This is a map

$$\eta \cdot : \pi_0 X \otimes \mathbb{Z}/2 \to \pi_1 X.$$

On the other hand, X is completely determined by its k-invariant in $H^2(\pi_0; \pi_1)$. Therefore the data of two homotopy groups and a k-invariant can be identified with the data of two homotopy groups and an action of η, which is precisely an element of $\mathrm{Hom}_{\mathrm{Ab}}(\pi_0 \otimes \mathbb{Z}/2, \pi_1)$.

In what follows, we show that the first k-invariants of $\mathrm{pic}_0^1(KO)$ and $\mathrm{pic}_0^1(KU)$ are nontrivial. These are both stable 1-types so it suffices to show that the symmetry isomorphisms of the associated Picard groupoids are nontrivial. In fact, it turns out that we only need to consider the symmetry isomorphism of the sphere spectrum. Note that this data is determined at the level of homotopy categories.

Lemma 3.1 *The k-invariant of the Picard spectrum* $\mathrm{pic}_0^1(\mathbb{S})$ *is nontrivial.*

Proof Let $\sigma : \mathbb{S} \otimes \mathbb{S} \to \mathbb{S} \otimes \mathbb{S}$ be the symmetry map of $\mathbb{S}$ in $\mathcal{S}\mathrm{p}$. This map is the stabilization of the "swap" equivalence $S^1 \wedge S^1 \to S^1 \wedge S^1$ in $\pi_2(S^2)^\times$, which is not homotopic to the identity. □

Lemma 3.2 *The functors* $\mathrm{pic}(\mathbb{S}) \to \mathrm{pic}(KO)$ *and* $\mathrm{pic}(\mathbb{S}) \to \mathrm{pic}(KU)$ *given by tensoring with KO and KU, respectively, equivalently the maps induced by applying the Picard spectrum functor to the units $\mathbb{S} \to KO$ and $\mathbb{S} \to KU$, are surjective on π_0 and isomorphisms on π_1.*

Proof We only prove the statement for KO as the proof for KU is identical. Recall from [23] that the objects of $\mathrm{pic}(KO)$ are precisely the shifts $\Sigma^i KO$ for $0 \le i \le 7$, using the eightfold periodicity of KO. Each of these is covered by $\Sigma^i \mathbb{S}$ for $0 \le i \le 7$ under the given functor $\mathrm{pic}(\mathbb{S}) \to \mathrm{pic}(KO)$, so the functor induces a surjection on π_0.

On π_1 this is precisely π_0 of the induced map $\mathrm{gl}_1(\mathbb{S}) \to \mathrm{gl}_1(KO)$. Recall that the unit map $\mathbb{S} \to KO$ is an isomorphism on π_0, for instance because it is a ring map $\mathbb{Z} \to \mathbb{Z}$. Now using the fact that $\mathrm{gl}_1(R)$ is the infinite delooping of the pullback of $\pi_0(R)^\times \to \pi_0 R \leftarrow \Omega^\infty R$ for any commutative ring spectrum R, we have that the induced map $\mathrm{gl}_1(\mathbb{S}) \to \mathrm{gl}_1(KO)$ is also an isomorphism π_0. □

Corollary 3.3 *The first k-invariants of* $\mathrm{pic}(KO)$ *and* $\mathrm{pic}(KU)$ *are nontrivial.*

3.2 Chain bundle models of topological K-theory

In [8, Section 6] it is shown that the commutative ring spectrum *ku* can be recovered as the homotopification of the sheaf of groupoids of complex vector bundles, suitably group completed. The constructions below are similar, but replace vector bundles with $\mathbb{Z}/2$-*graded chain complexes* of vector bundles. This sheaf is still not concordance invariant, so we must localize it with respect to $\mathbb{R}$ (i.e., make it *concordance invariant*). The resulting localized sheaf represents complex topological K-theory by [48, Appendix I]. Moreover, since it is concordance invariant, the results of [1] imply that it is a constant sheaf determined by its value at the point, which we show to be *ku*. This sheaf then has a subsheaf of *invertible* $\mathbb{Z}/2$-graded chain complexes (with respect to the usual tensor product) which evaluates to $\mathrm{gl}_1(ku) \simeq \mathrm{gl}_1(KU)$ on the point.

These sheaves are slightly more complicated than those of [8] but give us better access to the k-invariants of $\mathrm{gl}_1(KU)$ and therefore the k-invariants of $\mathrm{pic}(KU)$. Of course all of our arguments apply equally well to case of real K-theory. It is worth noting that the sheaf constructed in [8] returns $K(\mathbb{C})$ when evaluated at the point before localizing, whereas ours most certainly does not. We suspect that $K(\mathbb{C})$ however, or $K(\mathbb{F})$ in general for a discrete field $\mathbb{F}$, can be recovered by a variant of our construction using principal $\mathrm{GL}(\mathbb{F})$-bundles instead of vector bundles, i.e., by always equipping $\mathbb{F}$ with the discrete topology (see Conjecture 3.38).

Throughout this section, $\mathrm{Ch}_{\mathbb{F}}^{\mathrm{perf}} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Gpd}$ will denote the sheaf whose value at a space is the groupoid of bounded chain complexes of finite-dimensional $\mathbb{F}$-vector bundles on X with homotopy classes of homotopy equivalences between them. We assume all the standard structures of this sheaf, e.g., tensor products and direct sums. For our purposes it will be convenient to work with a slightly different sheaf, which we define below.

Definition 3.4 Let $\mathbb{F}$ be a field. We begin by defining chain complexes of $\mathbb{F}$ vector spaces which are graded by $\mathbb{Z}/2$. We call these *differential super* $\mathbb{F}$*-vector spaces* or DSVs for short.

(1) A differential super $\mathbb{F}$-vector space is a $\mathbb{Z}/2$-graded $\mathbb{F}$-vector space $V = V_0 \oplus V_1$ equipped with two maps $d_0 : V_0 \to V_1$ and $d_1 : V_1 \to V_0$ such that $d_0 d_1 = d_1 d_0 = 0$. A morphism of differential super $\mathbb{F}$-vector spaces is a morphism of $\mathbb{Z}/2$-graded vector spaces which commutes with the differential. These data form a category which we denote by $\mathrm{DSV}_{\mathbb{F}}$.

(2) If $V = (V_0 \oplus V_1, d_0, d_1)$ and $W = (W_0 \oplus W_1, d_0', d_1')$ are DSVs, we will write $V \otimes W$ for the DSV which is $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ in degree zero and $(V \otimes W)_1 = (V_1 \otimes W_0) \oplus (V_0 \otimes W_1)$ in degree one. The differentials are given in the usual way after reducing all indices modulo 2.

(3) For $V = (V_0 \oplus V_1, d_0, d_1)$ a DSV, we define $H_0(V) = \ker(d_0)/\mathrm{im}(d_1)$ and $H_1(V) = \ker(d_1)/\mathrm{im}(d_0)$. We say that a morphism of DSVs is a quasi-isomorphism if it induces isomorphisms of these two homology groups.

(4) Let $V = (V_0 \oplus V_1, d_0, d_1)$ and $W = (W_0 \oplus W_1, d_0', d_1')$ be DSVs. Given two maps $f, g : V \to W$ of DSVs we say that a chain homotopy between f and g is a pair of maps $h_0 : V_0 \to W_1$ and $h_1 : V_1 \to W_0$ such that $f_0 - g_0 = d_1' h_0 + h_1 d_0$ and $f_1 - g_1 = d_0' h_1 + h_0 d_1$. If there is a chain homotopy between f and g we write $f \sim g$.

(5) We say that a map $f : V \to W$ of DSVs is a homotopy equivalence if there exists $g : W \to V$ and chain homotopies $f \circ g \sim \mathrm{id}_W$ and $g \circ f \sim \mathrm{id}_V$.

(6) We write $\varepsilon : \mathrm{Ch}_{\mathbb{F}}^{\mathrm{perf}} \to \mathrm{DSV}_{\mathbb{F}}$ for the functor which takes a bounded and finite-dimensional $\mathbb{F}$-chain complex $(E_\bullet, \partial)$ to the DSV whose graded vector space is $E = \left(\bigoplus_i E_{2i}\right) \oplus \left(\bigoplus_i E_{2i+1}\right)$ and whose differential is the obvious restriction of ∂.

We leave it to the reader to check the following lemmas which are standard arguments in homological algebra.

Lemma 3.5 *A map $f : V \to W$ is a quasi-isomorphism of DSVs if and only if it is a homotopy equivalence.*

Lemma 3.6 *The tensor product of DSVs defines a symmetric monoidal structure on* $\mathrm{DSV}_{\mathbb{F}}$ *with monoidal unit* $(\mathbb{F} \oplus 0, 0, 0)$. *This tensor product distributes over the direct sum of DSVs.*

Lemma 3.7 *The functor ε is symmetric monoidal. Moreover, it takes quasi-isomorphisms to quasi-isomorphisms and chain homotopies to chain homotopies.*

Remark 3.8 As with the classical case, one direction of Lemma 3.5 depends on $\mathbb{F}$ being a field. For a general ring, homotopy equivalences are strictly stronger than quasi-isomorphisms.

Definition 3.9 Let X be a space and $\mathbb{F}$ a field. We write $\mathrm{DSV}_{\mathbb{F}}(X)$ for the groupoid of bundles of DSVs on X whose morphisms are chain homotopy classes of homotopy equivalences. Because bundles can be pulled back along maps of spaces this defines a functor $\mathrm{DSV}_{\mathbb{F}} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Gpd}$. The tensor product and direct sum bundles are defined in the usual way.

Going forward, many of our results will apply equally well to the functors $\mathrm{DSV}_{\mathbb{F}}$ and $\mathrm{Ch}_{\mathbb{F}}^{\mathrm{perf}}$. We will therefore write $\mathscr{G}_{\mathbb{F}}$ to denote either.

The following lemma says that $\mathscr{G}_{\mathbb{F}}$ is a sheaf of *ring groupoids* in the sense of [11], which are a special case of the ring categories of [35]. These are essentially commutative monoids in the category of Picard groupoids. Without loss of generality ring groupoids can always be assumed to have underlying *strict* Picard groupoid.

Lemma 3.10 *The functor $\mathscr{G}_{\mathbb{F}}$ is a presheaf of symmetric monoidal groupoids with respect to direct sum of chain bundles, a presheaf of symmetric monoidal groupoids with respect to tensor product of chain bundles, and the latter structure distributes over the former. Moreover, the natural transformation $\varepsilon(X) : \mathrm{Ch}_{\mathbb{F}}^{\mathrm{perf}}(X) \to \mathrm{DSV}_{\mathbb{F}}(X)$ preserves this structure.*

Proof The lemma follows from standard arguments for bundles extended suitably to chain complexes. □

Remark 3.11 It will be convenient to extend the codomain and restrict the domain of $\mathscr{G}_{\mathbb{F}}$. By taking nerves there is an inclusion $\mathrm{Gpd} \subset \mathcal{S}$ under which the symmetric monoidal structure of Lemma 3.10 makes $L_{\mathbb{R}}\mathscr{G}_{\mathbb{F}}$ into a presheaf of $\mathbb{E}_\infty$-types. For convenience, we will also restrict $\mathscr{G}_{\mathbb{F}}$ to the subcategory of smooth manifolds and smooth maps $\mathrm{Mfd}^{\mathrm{op}} \subset \mathrm{Top}^{\mathrm{op}}$.

Lemma 3.12 *The functor $\mathscr{G}_{\mathbb{F}}$ is a sheaf of symmetric monoidal groupoids on* Mfd *with respect to coverings by families of open embeddings.*

Proof To prove that $\mathscr{G}_{\mathbb{F}}$ is a sheaf, it suffices to show that, for a fixed manifold M, it satisfies descent on the "little" site of open submanifolds of M (indeed by [1, Lemma 3.5.3] it suffices to check only on Euclidean spaces). This follows immediately from the definitions, as bundles themselves are defined locally. Further, $\mathscr{G}_{\mathbb{F}}$ is valued in *symmetric monoidal* ∞-groupoids (i.e., $\mathbb{E}_\infty$-types) as a result of the fact that the inclusion $\mathrm{Gpd} \subset \mathcal{S}$ is symmetric monoidal (with respect to the Cartesian product). □

We recall the $\mathbb{R}$-invariantization functor from Definition 4.2.5 and Proposition 5.1.2 of [1].

Definition 3.13 Let Δ^n_{sm} denote the hyperplane in $\mathbb{R}^{n+1}$ spanned by points $(x_1, x_2, \dots, x_{n+1})$ such that $\sum_{i=1}^{n+1} x_i = 1$, also known as the *smooth n-simplex*. Let $\Delta^\bullet_{\mathrm{sm}}$ denote the cosimplicial manifold which is Δ^n_{sm} is degree n. Its coface maps $\Delta^n_{\mathrm{sm}} \to \Delta^{n+1}_{\mathrm{sm}}$ are the $n+2$ inclusions $\Delta^n_{\mathrm{sm}} \to \Delta^{n+1}_{\mathrm{sm}}$ given by the $n+2$ possible intersections of $\Delta^{n+1}_{\mathrm{sm}}$ with the coordinate hyperplanes. The codegeneracies $\Delta^{n+1}_{\mathrm{sm}} \to \Delta^n_{\mathrm{sm}}$ are given by the n possible ways of adding adjacent coordinates.

Definition 3.14 Let $\mathscr{E} : \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{Top}$ be a presheaf of h-types. Then define $L_{\mathbb{R}}\mathscr{E} : \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{Top}$ to be the presheaf of h-types whose value at X is given by $L_{\mathbb{R}}\mathscr{E}(X) = |\mathscr{E}(X \times \Delta^\bullet_{\mathrm{alg}})|$.

Remark 3.15 If a and a' are objects of $\mathscr{G}_{\mathbb{F}}(X)$ (thought of as a Picard groupoid), b is an object of $\mathscr{G}_{\mathbb{F}}(X \times \Delta^1)$, and we have isomorphisms $d_0(b) \cong a$ and $d_1(b) \cong a'$, where d_0 and d_1 are the face maps, then a and a' are equivalent in $L_{\mathbb{R}}\mathscr{G}_{\mathbb{F}}(X)$. Similar statements hold for the $\mathscr{G}_{\mathbb{F}}(X \times \Delta^n)$ and the relevant higher face maps. This has the effect of making $\mathscr{G}_{\mathbb{F}}$ insensitive to the difference between a space X and its "stabilizations" $X \times \mathbb{R} \simeq X \times \Delta^1$, $X \times \mathbb{R}^2 \simeq X \times \Delta^2$, and so forth.

Remark 3.16 In the Appendix of [48], working with $\mathrm{Ch}_{\mathbb{C}}^{\mathrm{perf}}$, Segal uses an equivalence relation to identify bundles which can be homotoped into one another along $X \times \mathbb{R} \cong X \times \Delta^1$. The above construction can be thought of actually inserting the homotopies (and homotopies between homotopies etc.) between such bundles, forcing the sheaf to be insensitive to deformations of X. However, after taking connected components the resulting group is the same as Segal's (see Proposition 3.18 below).

Lemma 3.17 *The presheaf of $\mathbb{E}_\infty$-types $L_{\mathbb{R}}\mathscr{G}_{\mathbb{F}}$ is a sheaf of grouplike $\mathbb{E}_\infty$-types, i.e., connective spectra.*

Proof It is a general fact that $L_{\mathbb{R}}$ preserves sheaves (see [1, Remark 4.2.6; 8, Proposition 2.6]). The symmetric monoidal structure is now *grouplike* because for any DSVs or chain complex $E_\bullet \to X$ we can always find $E'_\bullet \to X$ such that $E_\bullet \oplus E'_\bullet$ is concordant to an acyclic: that is, there exists an $E''_\bullet \to X \times [0,1]$ whose restriction to $X \times \{0\}$ is isomorphic to $E_\bullet \oplus E'_\bullet$ and whose restriction to $X \times \{1\}$ is acyclic. This is achieved by using the $[0,1]$ coordinate to turn on differentials killing any nontrivial cohomology. □

Proposition 3.18 *There is a group isomorphism $\pi_0 L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(X) \cong KU^0(X)$ which is natural in X.*

Proof By 3.2.3.1 and 1.4.3.9 of [38] the simplicial colimits defining $L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(X)$ and $L_{\mathbb{R}}\mathrm{Ch}_{\mathbb{F}}^{\mathrm{perf}}$ can be taken in spaces. Now, π_0 of a simplicial colimit depends only on the subdiagram (which is a coequalizer) involving the 0- and 1-simplices. From the definition of $L_{\mathbb{R}}\mathrm{Ch}_{\mathbb{C}}^{\mathrm{perf}}(X)$, π_0 of that coequalizer is the set of chain-bundles on X up to concordance. Remark 3.16 therefore implies that [48, Proposition A.I] (in which "concordant" is called "homotopic") gives an isomorphism $\pi_0 L_{\mathbb{R}}\mathrm{Ch}_{\mathbb{C}}^{\mathrm{perf}}(X) \xrightarrow{\cong} KU^0(X)$.

This isomorphism is given by taking a bundle to its Euler characteristic. Specifically, a bundle of chain complexes $E_\bullet$ is taken to the alternating sum of the K-theory classes of each grade, $\Sigma_{i\in\mathbb{Z}}(-1)^i[E^i]$. This clearly factors through $\pi_0(\varepsilon(X))$ so that we have a composite isomorphism

$$\pi_0 L_{\mathbb{R}}\mathrm{Ch}_{\mathbb{C}}^{\mathrm{perf}}(X) \xrightarrow{\varepsilon} \pi_0 L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(X) \to KU^0(X)$$

in which the last morphism forgets the differential. This implies that ε is injective. We have already seen that it is surjective, so it is an isomorphism. □

Theorem 3.19 *There is an equivalence of commutative ring spectra $L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(*) \simeq ku$.*

Proof This follows from [1, Propositions 4.3.1 and I.2] and Proposition 3.18. Specifically, we know that evaluation at a point is an equivalence between concordance invariant sheaves of spectra on Mfd and spectra. The inverse of this equivalence is given by taking the constant sheaf. Moreover, given a spectrum E, the value of $\mathrm{const}(E)$ at a manifold X is $\mathrm{Map}(\Sigma^\infty_+ X, E)$. Hence $L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(X) \simeq \mathrm{Map}(\Sigma^\infty_+ X, L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(*))$ and thus $\pi_0(X, L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(*)) \cong KU^0(X)$ for every manifold X. So $L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(*) \simeq ku$ by Brown representability. □

The above arguments, along with those of [48], can be repeated mutatis mutandis with chain complexes of $\mathbb{R}$-vector spaces rather than $\mathbb{C}$-vector spaces. This leads to:

Theorem 3.20 *There is an equivalence of commutative ring spectra $L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{R}}(*) \simeq ko$.*

Corollary 3.21 *Consider the full symmetric monoidal subgroupoid* $\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(X)$ *of* $\mathrm{DSV}_{\mathbb{C}}(X)$ *on the objects which are invertible in the tensor product monoidal structure. Then* $\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}$ *is a sheaf of connective spectra and* $L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(*) \simeq \mathrm{gl}_1(ku) \simeq \mathrm{gl}_1(KU)$.

Proof The subgroupoid $\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(X)$ is the full subgroupoid of $\mathrm{DSV}_{\mathbb{C}}(X)$ on bundles $E_\bullet$ such that $\dim(E_0) - \dim(E_1) = \pm 1$. The same argument for Lemma 3.12 shows that $\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}$ is a sheaf of connective spectra (but not ring spectra). Let $\mathbb{Z}(-)$ denote the sheaf whose value at X is the discrete groupoid of continuous $\mathbb{Z}$-valued functions on X. Similarly let $\mathbb{Z}/2(X)$ be the discrete groupoid of continuous ± 1-valued functions on X. The inclusion $i_X : \mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(X) \hookrightarrow \mathrm{DSV}_{\mathbb{C}}(X)$ fits (essentially by definition) into a pullback square of h-types

$$\begin{array}{ccc} \mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(X) & \xrightarrow{i_X} & \mathrm{DSV}_{\mathbb{C}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}/2(X) & \hookrightarrow & \mathbb{Z}(X) \end{array}$$

where the vertical maps send a DSV to its graded dimension. This extends to a levelwise pullback square of simplicial h-types

$$\begin{array}{ccc} \mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(X \times \Delta^\bullet_{\mathrm{alg}}) & \xrightarrow{i_X^\bullet} & \mathrm{DSV}_{\mathbb{C}}(X \times \Delta^\bullet_{\mathrm{alg}}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/2(X \times \Delta^\bullet_{\mathrm{alg}}) & \hookrightarrow & \mathbb{Z}(X \times \Delta^\bullet_{\mathrm{alg}}) \end{array}$$

Forgetting the monoidal structure and considering this as just a diagram of h-types, the diagram is still a pullback after applying $L_{\mathbb{R}}$ (see [46, Definition 1.1, Proposition 5.4]). When $X = *$, this gives the pullback diagram of h-types

$$\begin{array}{ccc} \Omega^\infty L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(*) & \longrightarrow & \Omega^\infty L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{C}}(*) \simeq \mathbb{Z} \times BU \\ \downarrow & & \downarrow \\ \mathbb{Z}/2 & \hookrightarrow & \mathbb{Z} \end{array}$$

exhibiting $\Omega^\infty L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(*)$ as $\mathrm{GL}_1\,ku$. Since i_X is symmetric monoidal, there is a natural-in-Y isomorphism of abelian groups

$$L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(*)^0(Y) \cong \mathrm{Map}(\Sigma^\infty_+ Y, L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(*)) \cong ku^0(Y)^\times$$

for any space Y (where the right side has the tensor product abelian group structure). Hence by Brown representability there is an equivalence of spectra $L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}}(*) \simeq \mathrm{gl}_1(ku)$. □

The same arguments apply to the real case, which gives Corollary 3.22. Our arguments also seem likely to apply in the case that $\mathbb{F}$ is a discrete field which we codify in Conjecture 3.38.

Corollary 3.22 *Consider the full symmetric monoidal subgroupoid* $\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{R}}(X)$ *of* $\mathrm{DSV}_{\mathbb{R}}(X)$ *on the objects which are invertible in the tensor product monoidal structure. Then* $\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{R}}$ *is a sheaf of connective spectra and* $L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{R}}(*) \simeq \mathrm{gl}_1(ko) \simeq \mathrm{gl}_1(KO)$.

3.3 The groupoid of $\mathbb{Z}/2$-graded line bundles

We now define a simpler sheaf that will map to $\mathrm{DSV}_{\mathbb{F}}$ and help us to understand its structure. The sheaves $L_{\mathbb{R}}\mathscr{L}^{\mathbb{C}}$ and $L_{\mathbb{R}}\mathscr{L}^{\mathbb{R}}$ that we describe here will end up being truncations of $L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{C}} \simeq \mathrm{gl}_1(KU)$ and $L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{R}} \simeq \mathrm{gl}_1(KO)$ after evaluating at the point. Some of the ideas of this section exist in [16] but we go a step further in relating these structures to the unit spectra of K-theory. This is in contrast to [16, 1.45] in which they are described as truncations of *ko* itself. Our results are arguably more conceptually satisfying given that these structures are in fact used for twisting both real and complex K-theory. Indeed, in [16], Freed remarks that he does not have a conceptual reason for the appearance of *ko* in his constructions.

Definition 3.23 Let X be a space, and $\mathbb{F}$ a (topological) field. Let $\mathscr{L}^{\mathbb{F}}(X)$ be the groupoid of pairs (ξ, n) where $\xi : E \to X$ is an $\mathbb{F}$-line bundle and $n : X \to \mathbb{Z}/2 = \{0,1\}$ is a continuous function. The morphisms between (ξ, m) and (ξ', m') in $\mathscr{L}^{\mathbb{F}}(X)$ will be the empty set if $n \neq m$ and the set of bundle isomorphisms otherwise. We will refer to $\mathscr{L}^{\mathbb{F}}(X)$ as the groupoid of $\mathbb{F}$-superline bundles on X.

Definition 3.24 Given two pairs $(\xi, n), (\xi', m) \in \mathscr{L}^{\mathbb{F}}(X)$, we define a symmetric monoidal structure on $\mathscr{L}^{\mathbb{F}}(X)$ by declaring that:

(1) The tensor product $(\xi, n) \otimes (\xi', m)$ is the object $(\xi \otimes \xi', n+m)$, where the tensor product of the left-hand coordinate is the standard tensor product of principal bundles.

(2) The symmetry isomorphism $(\xi, n) \otimes (\xi', m) \to (\xi', m) \otimes (\xi, n)$ is given on the fiber over x by $(v, w) \mapsto (w, (-1)^{n(x)m(x)}v)$.

Remark 3.25 The above definition is almost identical to the sheaf of graded $\mathbb{T}$-bundles $\mathcal{B}\mathbb{T}^{\pm}$ in [21, Definition 2.1]. Their definition differs from ours only in their definition of the symmetry isomorphism which would be written in our notation as $(v, w) \mapsto (w, v n(x) m(x))$. This makes sense if we take $\mathbb{Z}/2 = \{-1, 1\}$, but then the second coordinate in the tensor product formula, i.e., $n+m$, does not make sense.

Lemma 3.26 *The groupoid* $\mathscr{L}^{\mathbb{F}}(X)$ *does not decompose as a product of symmetric monoidal groupoids.*

Proof Let $\mathbb{Z}/2(X)$ be the discrete symmetric monoidal groupoid of $\mathbb{Z}/2$-valued functions with the pointwise group structure. The sign in the second component of the symmetry isomorphism of $\mathscr{L}^{\mathbb{F}}(X)$ prevents the natural projection $\mathscr{L}^{\mathbb{F}}(X) \to \mathbb{Z}/2(X)$ from having a symmetric monoidal section. □

Remark 3.27 The argument in the proof of Lemma 3.26, along with [32], also shows that the connective spectrum associated to $\mathscr{L}^{\mathbb{F}}(X)$ has nontrivial k-invariant, though we will not need this fact.

Remark 3.28 Definition 3.23 extends to a presheaf of grouplike symmetric monoidal groupoids on Top whose domain, following Remark 3.11, we restrict to Mfd and whose codomain we extend to $\mathcal{S}\mathrm{p}_{\geq 0}$.

Lemma 3.29 *The presheaf* $\mathscr{L}^{\mathbb{F}}$ *of connective spectra on* Mfd *is a sheaf.*

Proof Because the inclusion $\mathrm{Gpd} \subset \mathcal{S}$ is symmetric monoidal (and the category of connective spectra is equivalent to the category of grouplike $\mathbb{E}_\infty$ h-types), it suffices to show that $\mathscr{L}^{\mathbb{F}}$ is a sheaf of grouplike symmetric monoidal groupoids. The fact that it is a sheaf of groupoids, without symmetric monoidal structure, follows immediately from the fact that it decomposes as a sum of presheaves of groupoids which are clearly sheaves. The symmetric monoidal structure glues as well since limits of symmetric monoidal groupoids are computed in Cat. The grouplike condition is certainly satisfied for each $X \in \mathrm{Mfd}$. □

Proposition 3.30 *With respect to the symmetric monoidal structures given by bundle tensor product, there is a natural-in-X symmetric monoidal functor $\ell(X) : \mathscr{L}^{\mathbb{F}}(X) \to \mathrm{DSV}_{\mathbb{F}}(X)$ defined by taking (ξ, n) to the chain bundle which has line bundle ξ concentrated in degree n.*

Proof It suffices to check for trivial bundles on a path connected space. One checks readily that $\ell(X)((\xi,n) \otimes (\xi',m)) \cong \ell(X)(\xi,n) \otimes \ell(X)(\xi',m)$. If $n = m$, so $n + m = 0$, then we obtain the tensor product line bundle $\xi \otimes \xi'$ in degree 0 and the 0-line bundle in degree 1, with the zero differential between them. If $n \neq m$ then we have the reverse situation. The definition of the morphisms in $\mathscr{L}^{\mathbb{F}}(X)$ makes it clear that their tensor product is similarly preserved by ℓ. □

Theorem 3.31 *When $\mathbb{F}$ equals $\mathbb{C}$ the connective spectrum $L_{\mathbb{R}}\,\mathscr{L}^{\mathbb{C}}(*)$ fits into a cofiber sequence $\Sigma^2 H\mathbb{Z} \to L_{\mathbb{R}}\,\mathscr{L}^{\mathbb{C}}(*) \to H\mathbb{Z}/2$ and its k-invariant is nontrivial.*

Proof Consider the simplicial colimit of Definition 3.14 used to define $L_{\mathbb{R}}\,\mathscr{L}^{\mathbb{C}}(*)$. The connective spectra $\mathscr{L}^{\mathbb{C}}(\Delta^k)$ appearing in this colimit have exactly two nonzero homotopy groups, i.e., they are stable 2-types in the sense of [32] (see the discussion preceding Lemma 3.1). These homotopy groups are determined by the group of isomorphism classes of objects of $\mathscr{L}^{\mathbb{C}}(\Delta^k)$ and by the automorphisms of any one of those objects. In this case we have that $\pi_0(\mathscr{L}^{\mathbb{C}}(\Delta^k)) \cong \mathbb{Z}/2$ since Δ^k is connected and $\pi_1(\mathscr{L}^{\mathbb{C}}(\Delta^k)) = \mathrm{Top}(\Delta^k, \mathbb{C}^\times)$ (since that is the group of automorphisms of the trivial bundle on Δ^k), where $\mathbb{C}^\times$ is considered as a topological group with the usual subspace topology inherited from $\mathbb{C}$.

Again as a result of the analysis of [32], we get that the k-invariant is determined by the element of $H^2(H\mathbb{Z}/2; \mathrm{Top}(\Delta^k, \mathbb{C}^\times)) \simeq \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2, \mathrm{Top}(\Delta^k, \mathbb{C}^\times))$ (see (1) and the discussion that follows) corresponding to the function that assigns to an element in each isomorphism class the symmetry isomorphism of its tensor square. In this case this is the function that sends $n \in \mathbb{Z}/2$ to the constant function on Δ^k with value $(-1)^{n^2}$.

The ring $\mathbb{Z}/2$ can be thought of as a Picard groupoid with $\pi_0 \cong \mathbb{Z}/2$ and $\pi_1 = 0$. Therefore there is a forgetful functor of Picard groupoids $(\xi, n) \mapsto n$, corresponding to a map of spectra $\mathscr{L}^{\mathbb{C}}(\Delta^k) \to H\mathbb{Z}/2$ whose fiber is the Eilenberg–Mac Lane spectrum of $\pi_1(\mathscr{L}^{\mathbb{C}}(\Delta^k))$. Therefore there is a cofiber sequence of simplicial spectra

$$\Sigma H\mathrm{Top}(\Delta^\bullet, \mathbb{C}^\times) \to \mathscr{L}^{\mathbb{C}}(\Delta^\bullet) \to H\mathbb{Z}/2 \to \Sigma^2 H\mathrm{Top}(\Delta^\bullet, \mathbb{C}^\times)$$

in which the third term is a constant simplicial diagram and the last map is (the amalgam of) the k-invariants just described.

Note that the leftmost term is equivalent to $\Sigma H\mathrm{Sing}_\bullet(\mathbb{C}^\times)$, the levelwise suspension of the Eilenberg–Mac Lane spectrum of the singular simplicial group of $\mathbb{C}^\times$. Because Ω^∞ preserves geometric realizations we can compute the homotopy groups of this colimit in $\mathcal{S}$, which are trivial except in π_1, where they are $\mathbb{C}^\times$ (with the discrete topology). This implies that the colimit of the above cofiber sequence is the cofiber sequence $\Sigma H\mathbb{C}^\times \to L_\mathbb{R}\,\mathscr{L}^\mathbb{F}(*) \to H\mathbb{Z}/2 \to \Sigma^2 H\mathbb{C}^\times$ with the last map being the element of $H^2(H\mathbb{Z}/2;\mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2,\mathbb{C}^\times)$ sending n to $(-1)^{n^2} \in \mathbb{C}^\times$. Now $\mathbb{C}^\times \simeq B\mathbb{Z}$ so $H\mathbb{C}^\times \simeq \Sigma H\mathbb{Z}$. Thus the preceding cofiber sequence can be rewritten as

$$\Sigma^2 H\mathbb{Z} \to L_\mathbb{R}\,\mathscr{L}^\mathbb{C}(*) \to H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}$$

and the k-invariant is still nontrivial of course (and therefore represented by $\beta\,\mathrm{Sq}^2$). □

Remark 3.32 Although $\mathscr{L}^\mathbb{C}(\Delta^k)$ has homotopy groups concentrated in degrees 0 and 1 for all k, its localization $L_\mathbb{R}\,\mathscr{L}^\mathbb{C}(*)$ has homotopy groups concentrated in degrees 0 and 2.

Corollary 3.33 *When $\mathbb{F}$ equals $\mathbb{C}$ the spectrum $\mathrm{gl}_1(KU)$ splits as*

$$L_\mathbb{R}\,\mathscr{L}^\mathbb{C}(*) \oplus \mathrm{gl}_1(KU)[3,\infty).$$

Proof After applying $L_\mathbb{R}$, the functor of Proposition 3.30 induces a morphism of sheaves of commutative connective ring spectra. By Corollary 3.21, we have a map of spectra

$$L_\mathbb{R}\mathscr{L}^\mathbb{C}(*) \to L_\mathbb{R}\mathrm{gl}_1\,\mathrm{DSV}_\mathbb{C}(*) \simeq \mathrm{gl}_1(KU)$$

after evaluating at the point. This map is an isomorphism on π_0 because π_0 of these spectra can be computed by looking only at the bottom two levels of the simplicial diagram defining $L_\mathbb{R}$ (the coequalizer diagrams). It is of course an isomorphism on π_1 because both spectra have trivial homotopy in degree 1. Even further, it is an isomorphism on π_2. To see this, first note that whenever $\mathcal{F}$ is a concordance invariant sheaf there is an equivalence (as in the proof of Theorem 3.19) $\mathcal{F}(X) \simeq \mathrm{Map}(\Sigma^\infty_+ X, \mathcal{F}(*))$. Therefore the claimed isomorphism on π_2 would follow from an isomorphism $\pi_0 L_\mathbb{R}\,\mathscr{L}^\mathbb{C}(S^2) \simeq \pi_0 L_\mathbb{R}\mathrm{gl}_1\,\mathrm{DSV}_\mathbb{C}(S^2)$. Since the inclusion of spaces induces an isomorphism $\pi_0 L_\mathbb{R}\mathrm{gl}_1\,\mathrm{DSV}_\mathbb{C}(S^2) \to \pi_0 L_\mathbb{R}\mathrm{DSV}_\mathbb{C}(S^2)$ and the latter group is $ku^0(S^2)$, the desired claim follows from the fact that both generators of $ku^0(S^2)$ are represented by line bundles and are therefore in the image of $L_\mathbb{R}\,\mathscr{L}^\mathbb{C}$. Therefore the composite

$$L_\mathbb{R}\,\mathscr{L}^\mathbb{C}(*) \to \mathrm{gl}_1(KU) \to \mathrm{gl}_1(KU)[0,2]$$

is an equivalence, and the lemma follows. □

Corollary 3.34 *The first k-invariant of $\mathrm{gl}_1(KU)$ is nontrivial.*

Similar arguments prove the analogous statements for $L_\mathbb{R}\mathscr{L}^\mathbb{R}$:

Theorem 3.35 *When $\mathbb{F}$ equals $\mathbb{R}$ the connective spectrum $L_\mathbb{R}\,\mathscr{L}^\mathbb{R}(*)$ fits into a cofiber sequence $\Sigma H\mathbb{Z}/2 \to L_\mathbb{R}\,\mathscr{L}^\mathbb{R}(*) \to H\mathbb{Z}/2$ and its k-invariant is nontrivial.*

Corollary 3.36 *When $\mathbb{F}$ equals $\mathbb{R}$ the spectrum $\mathrm{gl}_1(KO)$ splits as*

$$L_{\mathbb{R}}\,\mathscr{L}^{\mathbb{R}}(*) \oplus \mathrm{gl}_1(KO)[2,\infty).$$

Corollary 3.37 *The first k-invariant of $\mathrm{gl}_1(KO)$ is nontrivial.*

Conjecture 3.38 *Let $\mathbb{F}$ be a discrete field. Then there are equivalences $L_{\mathbb{R}}\mathrm{DSV}_{\mathbb{F}}(*) \simeq K(\mathbb{F})$ and $L_{\mathbb{R}}\mathrm{gl}_1\,\mathrm{DSV}_{\mathbb{F}}(*) \simeq \mathrm{gl}_1(K(\mathbb{F}))$. Moreover, there is a splitting $L_{\mathbb{R}}\mathscr{L}^{\mathbb{F}}(*)$ is a split summand of $\mathrm{gl}_1\,K(\mathbb{F})$ and the first and second k-invariants of $\mathrm{pic}(K(\mathbb{F}))$ are nontrivial.*

4 Computations of k-invariants

In this section we compute the possible $\mathbb{E}_\infty$-structures on h-types with the same homotopy groups as $\mathrm{Pic}_0^1(KO)$, $\mathrm{Pic}_0^1(KU)$, $\mathrm{Pic}_1^2(KO)$, and $\mathrm{Pic}_1^3(KU)$. We do this by computing the possible k-invariants of spectra with the same homotopy groups. It will be useful to recall that $\mathrm{Pic}_1^2(KO)$ and $\mathrm{Pic}_1^3(KU)$ are equivalent to $B\mathrm{GL}_1(KO)[0,2]$ and $B\mathrm{GL}_1(KU)[0,3]$, respectively. In most cases we can explicitly name these k-invariants in terms of cohomology operations.

Proposition 4.1 *There are equivalences of h-types*

$$\mathrm{Pic}_0^2(KO) \simeq \mathbb{Z}/8 \times K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$$

and

$$\mathrm{Pic}_0^3(KU) \simeq \mathbb{Z}/2 \times K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3).$$

Proof The k-invariant connecting π_0 to π_1 must be zero, since on each connected component the relevant cohomology group is clearly zero. From [41, Lemma 3.1] we have that the 1-component of $\mathrm{Pic}(KO)$ splits as $K(\mathbb{Z}/2,1) \times \mathbf{BSO}$. The fact that $\mathbf{BSO}[0,2] \simeq K(\mathbb{Z}/2,2)$ completes the proof. The case of $\mathrm{Pic}(KU)$ is essentially identical (the result of [41] applies to both KO and KU). □

4.1 First k-invariants

Now we determine the possible first k-invariants of the associated spectra $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$.

Lemma 4.2 *The first k-invariant of $\mathrm{pic}(KO)$ is either trivial or $\mathrm{Sq}^2 \circ \rho : H\mathbb{Z}/8 \to \Sigma^2 H\mathbb{Z}/2$, where $\rho : H\mathbb{Z}/8 \to H\mathbb{Z}/2$ is the reduction mod-2 map, and the first k-invariant of $\mathrm{pic}(KU)$ is either trivial or $\mathrm{Sq}^2 : H\mathbb{Z}/2 \to \Sigma^2\mathbb{Z}/2$.*

Proof The case for KU is immediate because $H^2(H\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2$. The case of KO follows from the fact that $H^2(H\mathbb{Z}/8;\mathbb{Z}/2) \cong \mathbb{Z}/2$, which follows, e.g., from the argument given after (1). □

Corollary 4.3 *The k-invariants of $\mathrm{pic}_0^1(KO)$ and $\mathrm{pic}_0^1(KU)$ are $\mathrm{Sq}^2 \circ \rho$ and Sq^2, respectively, where $\rho : \mathbb{Z}/8 \to \mathbb{Z}/2$ is the reduction mod-2 map. Therefore $\mathrm{Sq}^2 \circ \rho$ and Sq^2 are also the first k-invariants of $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$.*

Proof This follows from Lemmas 3.1, 3.2 and 4.2. □

Next we wish to determine the *second* k-invariants of $\mathrm{pic}_0^3(KU)$ and $\mathrm{pic}_0^2(KO)$. We begin by determining the k-invariants of their connected covers $\mathrm{bgl}_1(KO)[0,2]$ and $\mathrm{bgl}_1(KU)[0,3]$. This is of course equivalent to determining the k-invariants of $\mathrm{gl}_1(KO)[0,1]$ and $\mathrm{gl}_1(KU)[0,2]$. The first is almost trivial, and we prove it in Proposition 4.4. For the second case, more work is required, and we first prove several lemmas.

Proposition 4.4 *The k-invariant of $\mathrm{pic}_1^2(KO)$ is $\mathrm{Sq}^2 : \Sigma H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}/2$. Equivalently, there are two $\mathbb{E}_\infty$-structures on $K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ and the one on $B\mathrm{GL}_1(KO)[0,2]$ is the one which is not the product structure.*

Proof By considering the Postnikov tower and knowing that $H^1(H\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by Sq^2 we see that the only two possible k-invariants are 0 and Sq^2. The result then follows from Theorem 3.35 and Corollary 3.36. □

Next we determine the k-invariant of $\mathrm{pic}_1^3(KU)$. We begin by determining all possible k-invariants of a spectrum with the same homotopy groups, or equivalently, all possible infinite loop space structures on the space $\Omega^\infty \mathrm{pic}_1^3(KU) \simeq B\mathrm{GL}_1(KU)[0,3]$.

Lemma 4.5 *There are exactly two h-types whose only nontrivial homotopy groups are $\pi_1 \cong \mathbb{Z}/2$ and $\pi_3 \cong \mathbb{Z}$.*

Proof The result follows immediately from the computation $H^4(B\mathbb{Z}/2;\mathbb{Z}) \cong \mathbb{Z}/2$. □

Lemma 4.6 *Let X be the h-type with $\pi_1(X) \cong \mathbb{Z}/2$ and $\pi_3(X) \cong \mathbb{Z}$ and nontrivial k-invariant. Then X does not admit an $\mathbb{E}_\infty$-structure.*

Proof If X admitted an $\mathbb{E}_\infty$-structure then its k-invariant $K(\mathbb{Z}/2,1) \to K(\mathbb{Z},4)$ would be Ω^∞ of a stable k-invariant $\Sigma H\mathbb{Z}/2 \to \Sigma^4 H\mathbb{Z}$ and would therefore induce a map of abelian groups on cohomology. Let $\gamma \in H^4(K(\mathbb{Z}/2;1);\mathbb{Z}) \cong \mathbb{Z}/2$ be the k-invariant of X and let $\alpha \in H^2(K(\mathbb{Z}/2;1);\mathbb{Z})$ be the generator given by the inclusion $B\mathbb{Z}/2 \to BU(1) = B^2\mathbb{Z}$. Then γ must be the cup-square of α.

Now for any h-type Y the cohomology operation induced by γ is the map

$$H^1(Y;\mathbb{Z}/2) \to H^4(Y;\mathbb{Z}), \quad a \mapsto \beta(a)^2,$$

where β is the Bockstein map. Let $Y = K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,1)$. If we take a and b to be the generators of $H^1(Y;\mathbb{Z}/2)$ then we see that the cross term in $\beta(a+b)^2 = \beta(a)^2 + 2\beta(a)\beta(b) + \beta(b)^2$ is nonzero, and therefore the above map is not an abelian group homomorphism. In other words, the k-invariant cannot be Ω^∞ of a stable k-invariant. □

Proposition 4.7 *For any n, $H^{n+3}(\Sigma^n H\mathbb{Z}/2;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$, generated by $\Sigma^n(\beta \circ \mathrm{Sq}^2)$: $\Sigma^n H\mathbb{Z}/2 \to \Sigma^{n+1} H\mathbb{Z}/2 \to \Sigma^{n+3} H\mathbb{Z}$, the appropriate suspension of the composite of the Bockstein and Sq^2.*

Proof It suffices to calculate $H^4(\Sigma H\mathbb{Z}/2;\mathbb{Z}) \simeq H^3(H\mathbb{Z}/2;\mathbb{Z})$. Note that a map $f : X \to \Sigma^3 H\mathbb{Z}$ factors through the Bockstein $\Sigma^3\beta : \Sigma^2 H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}$ if and only if the composite of f with the map $\Sigma^3 2 : \Sigma^3 H\mathbb{Z} \to \Sigma^3 H\mathbb{Z}$ is null, since β is the fiber of multiplication by 2. But the composite of any map $\alpha : H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}$ with $\Sigma^3 2 : \Sigma^3 H\mathbb{Z} \to \Sigma^3 H\mathbb{Z}$ is null because $H^3(H\mathbb{Z}/2;\mathbb{Z})$ is 2-torsion. So every element $\alpha \in H^3(H\mathbb{Z}/2;\mathbb{Z})$ factors as $\beta\alpha'$ for some $\alpha' \in H^2(H\mathbb{Z}/2;\mathbb{Z}/2)$. Therefore $\beta_* : \mathbb{Z}/2 \cong H^2(H\mathbb{Z}/2;\mathbb{Z}/2) \to H^3(H\mathbb{Z}/2;\mathbb{Z})$ is a surjection and $H^3(H\mathbb{Z}/2;\mathbb{Z})$ has only two elements, 0 and $\beta \circ \mathrm{Sq}^2$. Therefore it only remains to check that $\beta \circ \mathrm{Sq}^2$ is nonzero.

Suppose that $\beta \circ \mathrm{Sq}^2$ were null. Then Sq^2 would lift to a map $H\mathbb{Z}/2 \to \Sigma^2 H\mathbb{Z}$, i.e., there would be a factorization, through the fiber of $\Sigma^3\beta$,

$$\begin{array}{ccccc} & & \Sigma^2 H\mathbb{Z}/2 & & \\ & \overset{\omega}{\swarrow} & \downarrow{\scriptstyle \mathrm{Sq}^2} & & \\ \Sigma^2 H\mathbb{Z} & \xrightarrow{\Sigma^2\rho} & \Sigma^2 H\mathbb{Z} & \xrightarrow{\Sigma^3\beta} & \Sigma^3 H\mathbb{Z} \end{array}$$

where ρ is reduction mod 2. But, similarly to above, every class in $H^2(H\mathbb{Z}/2; H\mathbb{Z})$ is 2-torsion and therefore there is another factorization

$$\begin{array}{ccc} H\mathbb{Z}/2 & \xrightarrow{\omega} & \Sigma^2 H\mathbb{Z} \\ {\scriptstyle \mathrm{Sq}^1}\downarrow & \overset{\beta}{\nearrow} & \\ \Sigma H\mathbb{Z}/2 & & \end{array}$$

Therefore we have a commutative diagram

$$\begin{array}{ccc} H\mathbb{Z}/2 & \xrightarrow{\mathrm{Sq}^2} & \Sigma^2 H\mathbb{Z}/2 \\ {\scriptstyle \beta\circ\mathrm{Sq}^1}\downarrow & \overset{\Sigma^2\rho}{\nearrow} & \\ \Sigma^2 H\mathbb{Z} & & \end{array}$$

This however is a contradiction because the composite $\Sigma^2\rho \circ \beta \circ \mathrm{Sq}^1$ must be trivial on cohomology classes in degree greater than 1, whereas Sq^2 is not. Therefore $H^3(H\mathbb{Z}/2;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$ and is generated by $\beta \circ \mathrm{Sq}^2$. □

Proposition 4.8 *Any h-type equivalent to $K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3)$ admits exactly two $\mathbb{E}_\infty$-structures. One of them is the product structure and the other is the one associated to the stable k-invariant $\Sigma(\beta \circ \mathrm{Sq}^2) : \Sigma H\mathbb{Z}/2 \to \Sigma^4 H\mathbb{Z}$.*

Proof By Proposition 4.7 there are at most two $\mathbb{E}_\infty$-structures on $K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3)$, one associated to the coproduct spectrum $\Sigma H\mathbb{Z}/2 \vee \Sigma^3 H\mathbb{Z}$ and the other associated to the spectrum with the same homotopy groups but nontrivial k-invariant $\Sigma(\beta \circ \mathrm{Sq}^2) : \Sigma H\mathbb{Z}/2 \to \Sigma^4 H\mathbb{Z}$. By taking Ω^∞, the latter yields an infinite loop space, distinct from the product infinite loop space, with homotopy groups $\pi_1 \cong \mathbb{Z}/2$ and $\pi_3 \cong \mathbb{Z}$. By Lemma 4.5 this space has either the trivial or nontrivial k-invariant, and by Lemma 4.6 it cannot have the nontrivial k-invariant. Thus each element of $H^4(\Sigma H\mathbb{Z}/2;\mathbb{Z})$ induces a distinct infinite loop space structure on the product $K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3)$. □

Corollary 4.9 *The k-invariant of $\mathrm{pic}_1^3(KU)$ is $\beta \circ \mathrm{Sq}^2$ and therefore the infinite loop space structure on $\mathrm{Pic}_1^3(KU) \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3)$ is not the product structure.*

Proof This follows from Theorem 3.31, Corollary 3.33 and Proposition 4.8. □

The following proposition is not immediately relevant but will be used later and follows naturally from Corollary 4.9. We do not know if the splitting also exists at the level of $\mathbb{E}_\infty$-types.

Proposition 4.10 *There is a splitting of $\mathbb{E}_1$-types $\mathrm{Pic}_0^3(KU) \simeq \mathbb{Z}/2 \times \mathrm{Pic}_1^3(KU)$.*

Proof Consider the cofiber sequence of spectra

$$\mathrm{pic}_1^3(KU) \to \mathrm{pic}_0^3(KU) \to H\mathbb{Z}/2.$$

We will calculate the set of possible k-invariants $[H\mathbb{Z}/2, \Sigma\,\mathrm{pic}_1^3(KU)]$. Consider the second cofiber sequence

$$\Sigma H\mathbb{Z}/2 \xrightarrow{\beta\,\mathrm{Sq}^2} \Sigma^4 H\mathbb{Z} \to \Sigma\,\mathrm{pic}_1^3(KU) \to \Sigma^2 H\mathbb{Z}/2 \xrightarrow{\beta\,\mathrm{Sq}^2} \Sigma^5 H\mathbb{Z}.$$

Applying $[H\mathbb{Z}/2, -]$ produces an exact sequence

$$H^1(H\mathbb{Z}/2;\mathbb{Z}/2) \xrightarrow{\beta\,\mathrm{Sq}^2} H^4(H\mathbb{Z}/2;\mathbb{Z}) \to [\Sigma H\mathbb{Z}/2, \Sigma^2\,\mathrm{pic}_1^3(KU)] \\ \to H^2(H\mathbb{Z}/2;\mathbb{Z}/2) \xrightarrow{\beta\,\mathrm{Sq}^2} H^5(H\mathbb{Z}/2;\mathbb{Z}).$$

In order to compute the two relevant integral cohomology groups of $H\mathbb{Z}/2$, we use the exact sequence $H\mathbb{Z} \to H\mathbb{Z}/2 \xrightarrow{\beta} \Sigma H\mathbb{Z}$ to identify $H^k(H\mathbb{Z}/2;\mathbb{Z})$ as the image of the multiplication-by-Sq^1 map

$$H^{k-1}(H\mathbb{Z}/2;\mathbb{Z}/2) \to H^k(H\mathbb{Z}/2;\mathbb{Z}/2),$$

which in turn can be made explicit via the standard generators of the Steenrod algebra and the Adem relations. We find that our sequence of interest can be rewritten as

$$\mathbb{Z}/2\{\mathrm{Sq}^1\} \xrightarrow{\beta\,\mathrm{Sq}^2} \mathbb{Z}/2\{\beta\,\mathrm{Sq}^2\,\mathrm{Sq}^1\} \to [H\mathbb{Z}/2, \Sigma\,\mathrm{pic}_1^3(KU)] \\ \to \mathbb{Z}/2\{\mathrm{Sq}^2\} \xrightarrow{\beta\,\mathrm{Sq}^2} \mathbb{Z}/2\{\beta\,\mathrm{Sq}^4\}.$$

From this it immediately follows that the first map in the sequence is surjective, and the final map is zero, so the third map gives an isomorphism

$$[H\mathbb{Z}/2, \Sigma\,\mathrm{pic}_1^3(KU)] \cong [H\mathbb{Z}/2, \Sigma^2 H\mathbb{Z}/2] \cong \mathbb{Z}/2\{\mathrm{Sq}^2\}.$$

Now recall that $\mathrm{Sq}^2 : \Sigma H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}/2$ induces the null map on underlying spaces $B\mathbb{Z}/2 \to B^3\mathbb{Z}/2$. Indeed, by the instability relation Sq^2 kills the (degree-1) generator of $H\mathbb{Z}/2^* B\mathbb{Z}/2$. Therefore both infinite-loop maps $B\mathbb{Z}/2 \to B^2\mathrm{Pic}_1^3(KU)$ are null as maps of spaces, and the claim follows. □

4.2 Extending to $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$

So far we have computed the k-invariants of $\mathrm{pic}_0^1(KO)$, $\mathrm{pic}_0^1(KU)$, $\mathrm{pic}_1^2(KO)$ and $\mathrm{pic}_1^3(KU)$. Now we wish to determine how this data can be glued together to understand the k-invariants of $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$. Our computations only determine the second k-invariants of these spectra up to isomorphism. In other words, for KO we determine that the second k-invariant of $\mathrm{pic}_0^2(KO)$ is one of two elements of $H^3(\mathrm{pic}_0^1(KO);\mathbb{Z}/2)\cong\mathbb{Z}/2\times\mathbb{Z}/2$, and for KU it is one of the two generators of $H^4(\mathrm{pic}_0^1(KU))\cong\mathbb{Z}/4$. In the latter case there is an equivalence of Postnikov towers which interchanges the two generators, but in the former case it is less clear which generator one should choose. Ultimately, however, the choice will not matter because both k-invariants work for the applications of Section 5. In particular, both choices satisfy the conclusions of Propositions 4.18 and 4.19.

The proof of Lemma 4.12 was sketched for us by Tyler Lawson. Any mistakes are of course due to our own misunderstanding.

Lemma 4.11 *There is an exact sequence*

$$0\to H^4(H\mathbb{Z}/2;\mathbb{Z})\to H^4(\mathrm{pic}_0^1(KU);\mathbb{Z})\to H^3(H\mathbb{Z}/2;\mathbb{Z})\to 0$$

in which the first group is generated by $\beta\,\mathrm{Sq}^2\,\mathrm{Sq}^1$ *and the last group is generated by* $\beta\,\mathrm{Sq}^2$.

Proof The generators of the first and last groups are standard computations. The fact that

$$H^4(H\mathbb{Z}/2;\mathbb{Z})\to H^4(\mathrm{pic}_0^1(KU);\mathbb{Z})\to H^3(H\mathbb{Z}/2;\mathbb{Z})$$

is exact follows immediately from applying $H^4(-;\mathbb{Z})$ to the cofiber sequence

$$\Sigma H\mathbb{Z}/2\to\mathrm{pic}_0^1(KU)\to H\mathbb{Z}/2$$

from the Postnikov tower of $\mathrm{pic}(KU)$. The fact that the entire sequence is short exact follows on the left from the fact that $H^2(H\mathbb{Z}/2;\mathbb{Z})=0$. For the right-hand side, first recall that the first k-invariant of $\mathrm{pic}(KU)$, i.e., the only k-invariant of $\mathrm{pic}_0^1(KU)$, is $\mathrm{Sq}^2: H\mathbb{Z}/2\to\Sigma^2 H\mathbb{Z}/2$. Therefore there is an exact sequence

$$H^4(\mathrm{pic}_0^1(KU);\mathbb{Z})\to H^3(H\mathbb{Z}/2;\mathbb{Z})\to H^5(H\mathbb{Z}/2;\mathbb{Z})$$

in which the last map is Sq^2. Since $H^3(H\mathbb{Z}/2;\mathbb{Z})\cong\mathbb{Z}/2$ is generated by $\beta\,\mathrm{Sq}^2$ the image of this map is $\beta\,\mathrm{Sq}^2\,\mathrm{Sq}^2$. Taking the quotient by 2, which is an injection $H^*(H\mathbb{Z}/2;\mathbb{Z})\to H^*(H\mathbb{Z}/2;\mathbb{Z}/2)$ (its kernel is the image of multiplication by 2 between $\mathbb{Z}/2$-modules), we get $\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^2$, which is zero by the Adem relations. Therefore the image of $\mathrm{Sq}^2: H^3(H\mathbb{Z}/2;\mathbb{Z})\to H^5(H\mathbb{Z}/2;\mathbb{Z})$ is zero. □

Lemma 4.12 *There is an isomorphism* $H^4(\mathrm{pic}_0^1(KU);\mathbb{Z})\cong\mathbb{Z}/4$.

Proof Consider the cofiber sequence $\mathbb{S}\xrightarrow{2}\mathbb{S}\xrightarrow{q}\mathbb{S}/2$, where $\mathbb{S}/2$ is the mod-2 Moore spectrum. Let $\bar{f}:\mathbb{S}\to\mathrm{pic}_0^1(KU)$ be the generator of $\pi_0(\mathrm{pic}_0^1(KU))\cong\mathbb{Z}/2$. Then since $\pi_0(\mathrm{pic}_0^1(KU))$ is 2-torsion, $\bar{f}$ lifts to a nonzero map $f:\mathbb{S}/2\to\mathrm{pic}_0^1(KU)$. The long exact sequence in homotopy applied to the above cofiber sequence shows that $\pi_0(\mathbb{S}/2) \cong \mathbb{Z}/2$, generated by q and $\pi_1(\mathbb{S}/2) \cong \mathbb{Z}/2$, generated by $q \circ \eta$. By considering the commutative diagram

$$\begin{array}{ccc}
\Sigma^d \mathbb{S} & & \\
{\scriptstyle\phi}\downarrow & \searrow^{q\circ\phi} & \\
\mathbb{S} & \xrightarrow{\;q\;} & \mathbb{S}/2 \\
{\scriptstyle\bar{f}}\downarrow & \swarrow_{f} & \\
\mathrm{pic}_0^1(KU) & &
\end{array}$$

in which $d = 0, 1$ and $\phi = \mathrm{id}, \eta$, respectively, we have that f must be nonzero on π_0 and π_1 and therefore an isomorphism on those two homotopy groups. As a result, the cofiber of f, $\mathrm{cofib}(f)$, is 2-connected. Applying the Hurewicz theorem to $\mathrm{cofib}(f)$, we find that $\pi_3(\mathrm{cofib}(f)) \cong H^3(\mathrm{cofib}(f); \mathbb{Z})$. Applying the long exact sequence in homotopy to the cofiber sequence $\mathbb{S}/2 \to \mathrm{pic}_0^1(KU) \to \mathrm{cofib}(f)$ we see that $\pi_3(\mathrm{cofib}(f)) \cong \pi_2(\mathbb{S}/2)$. Now applying the long exact sequence for homology to the same cofiber sequence, we get $\pi_2(\mathbb{S}/2) \cong H^3(\mathrm{pic}_0^1(KU); \mathbb{Z})$. The universal coefficient theorem tells us that $H^4(\mathrm{pic}_0^1(KU); \mathbb{Z}) \cong \mathrm{Ext}(\pi_2(\mathbb{S}/2), \mathbb{Z}) \cong \pi_2(\mathbb{S}/2)^\vee \cong \pi_2(\mathbb{S}/2)$. We conclude by pointing out that $\pi_2(\mathbb{S}/2) \cong \mathbb{Z}/4$. The authors cannot find this final fact in the published literature, but several sketch proofs of it are provided in [40]. □

Proposition 4.13 *The second k-invariant of $\mathrm{pic}_0^3(KU)$ generates $H^4(\mathrm{pic}_0^1(KU); \mathbb{Z}) \cong \mathbb{Z}/4$.*

Proof By Lemma 4.12 and Theorem 3.31 there are three options for the second k-invariant of $\mathrm{pic}_0^3(KU)$: any of the nontrivial maps $\mathrm{pic}_0^1(KU) \to \Sigma^4 H\mathbb{Z}$. However, by Corollary 4.9, the actual second k-invariant must give $\beta\,\mathrm{Sq}^2$ on $\mathrm{pic}_1^3(KU)$ after taking a connected cover. This corresponds to asking for maps $\mathrm{pic}_0^1(KU) \to \Sigma^4 H\mathbb{Z}$ which restrict to $\beta\,\mathrm{Sq}^2$ when precomposing with the map $\Sigma H\mathbb{Z}/2 \to \mathrm{pic}_0^1(KU)$ in the Postnikov tower of $\mathrm{pic}(KU)$. But by Lemma 4.11, $\mathbb{Z}/4 \cong H^4(\mathrm{pic}_0^1(KU); \mathbb{Z}) \to H^3(\mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/2$ is a surjection, and the codomain is generated by $\beta\,\mathrm{Sq}^2$. Therefore only the two generators of $H^4(\mathrm{pic}_0^1(KU); \mathbb{Z})$ satisfy the necessary property, so one of them of must be the k-invariant of $\mathrm{pic}_0^3(KU)$. □

Theorem 4.14 *The two generators of $\mathbb{Z}/4 \cong H^4(\mathrm{pic}_0^1(KU); \mathbb{Z})$ yield equivalent Postnikov sections, and hence both present $\mathrm{pic}_0^3(KU)$.*

Proof Let a and b be the two generators of $\mathbb{Z}/4 \cong H^4(\mathrm{pic}_0^1(KU); \mathbb{Z})$. Since $a = -b$ there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{pic}_0^1(KU) & \xrightarrow[a]{} & \Sigma^4 H\mathbb{Z} \\
\downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle -1} \\
\mathrm{pic}_0^1(KU) & \xrightarrow{\;b\;} & \Sigma^4 H\mathbb{Z}
\end{array}$$

The induced map between the fibers of the horizontal maps is an equivalence between the Postnikov sections corresponding to the k-invariants a and b. □

Remark 4.15 While we do not have a geometric argument at hand, it seems almost certain that the automorphism used in the proof of Theorem 4.14 corresponds to the complex conjugation automorphism on KU.

Proposition 4.16 *There is an isomorphism* $H^3(\mathrm{pic}_0^1(KO);\mathbb{Z}/2)\cong\mathbb{Z}/2\times\mathbb{Z}/2$. *Moreover, pulling back along the fiber in the Postnikov tower for* $\mathrm{pic}_0^1(KO)$, $\Sigma H\mathbb{Z}/2\to\mathrm{pic}_0^1(KO)$, *induces a surjection* $\mathbb{Z}/2\times\mathbb{Z}/2\to\mathbb{Z}/2\{\mathrm{Sq}^2\}$.

Proof For readability we do not include the entire proof. We only note that it arises from considering the long exact sequence in mod-2 cohomology applied to the Postnikov tower $\Sigma H\mathbb{Z}/2\to\mathrm{pic}_0^1(KO)\to H\mathbb{Z}/8$ and a large number of low-degree cohomology computations for the Eilenberg–Mac Lane spectra $H\mathbb{Z}/8$ and $H\mathbb{Z}/2$. □

Remark 4.17 For the time being, we do not know how to specify the "correct" k-invariant for $\mathrm{pic}_0^2(KO)$, as it could be one of two elements in the preimage of Sq^2. In the case of $\mathrm{pic}(KU)$ the ambiguity is irrelevant up to equivalence (see Theorem 4.14), but a similar approach will not work here. It may be possible to resolve the ambiguity by taking homotopy fixed points of $\mathrm{pic}(KU)$ and comparing the second k-invariant of the resulting fixed point spectrum, via the homotopy fixed points spectral sequence, to the two possibilities given in Proposition 4.16. Luckily, this uncertainty does not effect the group structure on the $\mathrm{pic}_0^2(KO)$-cohomology of a space.

4.3 Group structures

Now that we know the k-invariants of $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$, we can determine the group laws for $\mathrm{pic}_0^2(KO)^0(X)$ and $\mathrm{pic}_0^3(KU)^0(X)$, which we will use in the next section.

Proposition 4.18 *For a space X, the group law on the set*

$$\mathrm{pic}_0^3(KU)^0(X)\cong H^0(X;\mathbb{Z}/2)\times H^1(X;\mathbb{Z}/2)\times H^3(X;\mathbb{Z})$$

is given by $(a,b,c)\boxplus(a',b',c')=(a+a',b+b',c+c'+\beta(b\cup b'))$, *where we abuse notation and use the symbol $+$ to denote the usual addition in* $H^0(X;\mathbb{Z}/2)$, $H^1(X;\mathbb{Z}/2)$ *and* $H^3(X;\mathbb{Z})$.

Proof Proposition 4.10 implies that the first coordinate of the group splits off. Therefore it suffices to prove that the group structure on $\mathrm{pic}_1^3(KU)^0(X)$ is $(b,c)+(b',c')=(b+b',c+c'+\beta(b\cup b'))$. Consider the natural short exact sequence of abelian groups

$$H^3(X;\mathbb{Z})\to\mathrm{pic}_1^3(KU)^0(X)\to H^1(X;\mathbb{Z}/2).$$

The cocycle for that group extension (in the sense of [49, Section 6.6]) is a natural map

$$H^1(X;\mathbb{Z}/2)\times H^1(X;\mathbb{Z}/2)\to H^3(X;\mathbb{Z}),$$

which (since it is natural in X) is represented by some map $K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,1)\to K(\mathbb{Z},3)$. There are only two such maps, the trivial one and $\beta(-\cup-)$. The cocycle cannot be trivial, otherwise $\mathrm{pic}_1^3(KU)$ would be a direct sum of Eilenberg–Mac lane spectra, which contradicts Corollary 4.9. □

The proof of the following proposition is similar.

Proposition 4.19 *For an h-type X, the group law on the set*

$$\mathrm{pic}_0^2(KO)^0(X) \cong H^0(X;\mathbb{Z}/2)\times H^1(X;\mathbb{Z}/2)\times H^2(X;\mathbb{Z}/2)$$

is given by $(a,b,c)\boxplus(a',b',c') = (a+a', b+b', c+c'+b\cup b')$, *where we abuse notation and use the symbol* $+$ *to denote the usual addition in* $H^0(X;\mathbb{Z}/8)$, $H^1(X;\mathbb{Z}/2)$ *and* $H^2(X;\mathbb{Z}/2)$.

5 What $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$ represent

In the following section we describe several algebraic and geometric interpretations of the cohomology theories associated to $\mathrm{pic}_0^2(KO)$ and $\mathrm{pic}_0^3(KU)$.

5.1 Brauer groups

By determining the group structures of $\mathrm{pic}_0^3(KU)^0(X)$ and $\mathrm{pic}_0^2(KO)^0(X)$ for a space X we now have that $\mathrm{Pic}_0^3(KU)$ and $\mathrm{Pic}_0^2(KO)$ represent well-known classical Brauer groups whose elements are Morita classes of bundles of $\mathbb{Z}/2$-graded central simple algebras [10, Theorem 6, Theorem 11]; bundles of $\mathbb{Z}/2$-graded continuous trace C^*-algebras (as described below); and bundles of super (i.e., $\mathbb{Z}/2$-graded) 2-lines (see Corollary 5.8). All of these data were previously known to be isomorphic, at least at the level of folks theorems, but interpreting them in terms of $\mathrm{Pic}_0^3(KU)$ and $\mathrm{Pic}_0^2(KO)$ is new. However, this is consistent with the fact that bundles of graded central simple algebras; bundles of graded C^*-algebras; and bundles of super 2-lines; can all be used to twist K-theory.

Definition 5.1 We let $\mathrm{GBr}\mathbf{O}(X)$ denote the Brauer group of (possibly infinite-dimensional) graded, continuous trace, complex C^*-algebras with spectrum X, as in [44]. We let $\mathrm{GBr}\mathbf{U}(X)$ denote the Brauer group of (possibly infinite-dimensional) graded, continuous trace, real C^*-algebras with spectrum X, as in [43].

The following theorems are proven in [43; 44]:

Theorem 5.2 (Maycock) *If X is a space homotopy equivalent to a CW-complex then*

$$\mathrm{GBr}\mathbf{U}(X) \cong H^0(X;\mathbb{Z}/2)\times H^1(X;\mathbb{Z}/2)\times H^3(X;\mathbb{Z})$$

with group law $(a,b,c)+(a',b',c') = (a+a', b+b', c+c'+\beta(b\cup b'))$ *where* β *is the Bockstein homomorphism.*

Remark 5.3 In [44] the H^0 term is mostly ignored, since Maycock requires that her bundles have isomorphic fibers over every connected component. This assumption is unnecessary, as shown in [43].

Theorem 5.4 (Moutuou) *If X is a space homotopy equivalent to a CW-complex then*

$$\mathrm{GBr}\mathbf{O}(X) \cong H^0(X;\mathbb{Z}/8)\times H^1(X;\mathbb{Z}/2)\times H^2(X;\mathbb{Z}/2)$$

with group law $(a,b,c)+(a',b',c') = (a+a', b+b', c+c'+(b\cup b'))$.

Propositions 4.18 and 4.19 now imply the following corollaries:

Corollary 5.5 *If X is a space homotopy equivalent to a CW-complex then there is an isomorphism*

$$\mathrm{GBr}\mathbf{U}(X) \cong \mathrm{pic}_0^3(KU)^0(X).$$

Corollary 5.6 *If X is a space homotopy equivalent to a CW-complex then there is an isomorphism*

$$\mathrm{GBr}\mathbf{O}(X) \cong \mathrm{pic}_0^2(KO)^0(X).$$

Remark 5.7 The Brauer group of $\mathbb{Z}/2$-graded continuous trace C^*-algebras with spectrum X is equivalent to the Brauer group of $\mathbb{Z}/2$-equivariant C^*-algebras with spectrum X with the property that the induced $\mathbb{Z}/2$-action on X is trivial. Our constructions of $\mathrm{pic}_1^3(KU)$ and $\mathrm{pic}_1^2(KO)$, and the fact that they represent these Brauer groups, should be compared to the construction of the cohomology theory $E_{\mathbb{C},\mathbb{T}}$ in [14]. It is shown therein that $E^0_{\mathbb{C},\mathbb{T}}(X)$ is the group of $\mathbb{T}$-equivariant line bundles on X and that $E^1_{\mathbb{C},\mathbb{T}}(X)$ is the Brauer group of $\mathbb{T}$-equivariant C^*-algebras with spectrum X where X has trivial induced $\mathbb{T}$-action. Our constructions on the other hand show that $\mathrm{pic}_1^3(KU)^{-1}(X)$ is the group of super line bundles on X and that $\mathrm{pic}_1^3(KU)^0(X)$ is the Brauer group of $\mathbb{Z}/2$-equivariant C^*-algebras with spectrum X having trivial induced $\mathbb{Z}/2$-action. Moreover, $\Omega^\infty E_{\mathbb{C},\mathbb{T}} \simeq \mathbb{Z} \times K(\mathbb{Z},2)$ and $\Omega^\infty \Sigma^{-1}\,\mathrm{pic}_1^3(KU) \simeq \mathbb{Z}/2 \times K(\mathbb{Z},2)$. This suggests that Evans and Pennig's $E_{\mathbb{C},\mathbb{T}}$ spectrum is equivalent to $\mathrm{gl}_1(KU_{\mathbb{T}})$, the space of units of $\mathbb{T}$-equivariant complex K-theory.

Propositions 4.18 and 4.19 also give $\mathrm{pic}_0^3(KU)$ and $\mathrm{pic}_0^2(KO)$ the following interpretations in terms of graded 2-line bundles.

Corollary 5.8 *Let X be a smooth manifold. Then*

$$\mathrm{pic}_0^3(KU)^0(X) \cong \mathrm{sLBdl}_{\mathbb{C}}(X) \quad \textit{and} \quad \mathrm{pic}_0^2(KO)^0(X) \cong \mathrm{sLBdl}_{\mathbb{R}}(X),$$

where $\mathrm{sLBdl}_{\mathbb{C}}$ and $\mathrm{sLBdl}_{\mathbb{R}}$ are the Brauer groups of complex and real super 2-line bundles on X as defined in [34].

Proof This follows immediately from either [34, Theorem 4.4] or [42, Theorem 2.2.6]. □

Corollaries 5.5 and 5.8 have the following interpretation in terms of parameterized stable homotopy theory:

Corollary 5.9 *If X is connected then there is an isomorphism between the Brauer group of graded, continuous trace, complex C^*-algebras with spectrum X, equivalently the Brauer group of complex super 2-line bundles on X, and $[X, B(\mathrm{GL}_1(KU[0,2]))]$, the group of $KU[0,2]$-line bundles on X (where the group structure on the latter arises from the abelian ∞-group structure of the target). The same holds in the real case with $KU[0,2]$ replaced by $KO[0,1]$.*

Proof We have a string of equivalences of infinite loop spaces

$$\mathrm{Pic}_1^3(KU) \simeq B\Omega\,\mathrm{Pic}_0^3(KU) \simeq B\mathrm{GL}_1(KU)[0,3] \simeq B(\mathrm{GL}_1(KU)[0,2]).$$

Therefore it suffices to show that $\mathrm{GL}_1(KU)[0,2] \simeq \mathrm{GL}_1(KU[0,2])$, which follows from Lemma 5.12 below. The argument for the real case is identical. □

Remark 5.10 The group structure on $KU[0,2]$-line bundles over X also arises by interpreting it as the set of connected components of the symmetric monoidal slice category ∞-category $\mathrm{Top}_{/B(\mathrm{GL}_1(KU[0,2]))}$. Here, the symmetric monoidal structure is given by [3, Proposition 6.12]. This can be interpreted as applying Lurie's straightening construction to the slice category and equipping the resulting presheaf category with the Day convolution monoidal structure.

Remark 5.11 Corollary 5.9 cannot be stated in terms of all of $\mathrm{Pic}_0^3(KU)$ for the following reason: the spectrum $KU[0,2]$ is no longer 2-periodic, so $\pi_0(\mathrm{Pic}(KU[0,2]))$ will be at least $\mathbb{Z}$. As a result we cannot think of maps $X \to \mathrm{Pic}_0^3(KU)$ as bundles of invertible $KU[0,2]$-modules on X.

Lemma 5.12 *Let R be a connective commutative ring spectrum and $n \in \mathbb{Z}$ a positive integer. Then there is an equivalence of infinite loop spaces*

$$\mathrm{GL}_1(R[0,n]) \simeq \mathrm{GL}_1(R)[0,n].$$

Proof Consider the zigzag of infinite loop maps

$$\mathrm{GL}_1(R[0,n]) \to \mathrm{GL}_1(R[0,n])[0,n] \leftarrow \mathrm{GL}_1(R)[0,n],$$

where the arrow from left to right is the usual map from a space to its truncation and the arrow from right to left is obtained by truncating after applying the functor GL_1 to the ring map $R \to R[0,n]$. Recall that the homotopy groups of $\mathrm{GL}_1(R)$ are isomorphic to those of R except in degree zero where $\pi_0(\mathrm{GL}_1(R)) \cong \pi_0(R)^\times$. Therefore $\mathrm{GL}_1(R[0,n])$ is an n-truncated space and the left to right map from itself to its truncation is an equivalence. Because $R \to R[0,n]$ is an equivalence through homotopy degree n, so is $\mathrm{GL}_1(R) \to \mathrm{GL}_1(R[0,n])$ and therefore the left to right map is also an equivalence. □

Remark 5.13 There is an equivalence of commutative ring spectra $KU[0,2] \simeq ku[0,2]$ which is the truncation of the equivalence $ku \to KU[0,\infty)$. This is essential to the use of Lemma 5.12 in the proof of Corollary 5.9. In particular we must construct $KU[0,2]$ by first taking the connective cover of KU and then truncating, as $KU(-\infty,2]$ is not even a ring spectrum.

Remark 5.14 Recall that there is a C_2-action on KU by complex conjugation whose fixed point spectrum is KO. The C_2-equivariant complex K-theory spectrum is often denoted by $K\mathbb{R}$, following Atiyah. One can show that the second C_2-equivariant Postnikov slice of $K\mathbb{R}$, denoted by $P^2K\mathbb{R}$ (in the sense of [28]) has underlying spectrum $KU[0,2]$ and fixed point spectrum $KO[0,1]$ [27]. Therefore both perspectives are encompassed by the space of *equivariant* units of $K\mathbb{R}$, i.e., $\mathrm{GL}_1(K\mathbb{R})$. In other words, given the correct equivariant generalization of Lemma 5.12, both the real and complex Brauer groups described above are determined by a delooping of $P^2\mathrm{GL}_1(K\mathbb{R})$. This is closely related the operator-theoretic perspective on real K-theory described in [43]. We will return to this question in later work.

5.2 Twisting Spin- and String-structures

Recall from [45] that the set of **String**-structures on a **Spin**-manifold M is a torsor for $H^3(M;\mathbb{Z})$. Our results admit a similar interpretation, except that we are considering **String**-structures on a manifold with a fixed orientation (as opposed to a fixed **Spin**-structure). Our results imply that the set of **String**-structures on M relative to a fixed orientation is a torsor for $\mathrm{pic}_1^3(KU)^0(M) \cong H^1(M;\mathbb{Z}/2) \ltimes H^3(M;\mathbb{Z})$. Similarly, the **Spin**-structures on a real manifold M are a torsor for $\mathrm{pic}_1^2(KO)^0(M) \cong H^1(M;\mathbb{Z}/2) \ltimes H^2(M;\mathbb{Z}/2)$.

Definition 5.15 Let $\mathbf{SO}/\!\!/\mathbf{String}$ denote the abelian ∞-group arising as the fiber of the connective cover $\mathbf{BString} \to \mathbf{BSO}$.

Proposition 5.16 *There is an equivalence of abelian ∞-groups* $\mathbf{SO}/\!\!/\mathbf{String} \simeq \mathrm{Pic}_1^3(KU)$.

Proof Let F be the fiber in $\mathcal{S}\mathrm{p}$ of the connective cover $\mathbf{bstring} \to \mathbf{bso}$. Therefore F has $\pi_1(F) \cong \mathbb{Z}/2$ and $\pi_3(F) \cong \mathbb{Z}$ as its only nontrivial homotopy groups. By Proposition 4.8 and Corollary 4.9 it suffices to show that the k-invariant of F is nontrivial. We will show that the k-invariant of ΣF is nontrivial, which is equivalent.

Note that there is a fiber sequence $\mathbf{bstring} \to \mathbf{bso} \to \mathbf{bso}[0,4] \simeq \Sigma F$. By Lemma 4.6, there is an equivalence of h-types

$$\mathbf{BSO}[0,4] \simeq B(B\mathrm{GL}_1(KU)[0,3]) \simeq K(\mathbb{Z}/2,2) \times K(\mathbb{Z},4)$$

so there is a natural isomorphism of sets $\mathbf{bso}[0,4]^0(X) \cong H^2(X;\mathbb{Z}/2) \oplus H^4(X;\mathbb{Z})$. If the k-invariant of F were trivial, this would be an isomorphism of abelian groups. We will show that is not the case.

Because of the h-type splitting $\mathbf{BSO}[0,4] \simeq K(\mathbb{Z}/2,2) \times K(\mathbb{Z},4)$ described above, there is a projection $\mathbf{BSO}[0,4] \to K(\mathbb{Z}/2,2)$ and the composite of that projection with the truncation $\mathbf{BSO} \to \mathbf{BSO}[0,4]$ must be nontrivial. Therefore the second Stiefel Whitney class $w_2 : \mathbf{BSO} \to K(\mathbb{Z}/2,2)$ can be factored as $\mathbf{BSO} \to \mathbf{BSO}[0,4] \to K(\mathbb{Z}/2,2)$. As a result, the composite $\mathbf{bso}^0(X) \to \mathbf{bso}[0,4]^0(X) \to H^2(X,\mathbb{Z}/2)$ must take an oriented vector bundle V on X to $w_2(V)$.

The composite $p : \mathbf{BSO} \to \mathbf{BSO}[0,4] \to K(\mathbb{Z},4)$ determines *some* integral characteristic class (and is therefore some multiple of the first Pontryagin class p_1). We argue that it must be either p_1 or $-p_1$. First note that by the computations of [12] and the Künneth formula, $H_4(\mathbf{BSO}[0,4];\mathbb{Z})$ splits as $H_4(K(\mathbb{Z}/2,2);\mathbb{Z}) \oplus H_4(K(\mathbb{Z},4);\mathbb{Z}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}$. This implies that $H^4(K(\mathbb{Z}/2,2) \times K(\mathbb{Z},4);\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/4 \oplus \mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$. Thus the projection $\mathbf{BSO}[0,4] \to K(\mathbb{Z},4)$ gives an isomorphism in H^4. The restriction along $\mathbf{BSO} \to \mathbf{BSO}[0,4]$ is surjective on H^4 for connectivity reasons, and hence an isomorphism, so the composite p must be a generator of $H^4(\mathbf{BSO},\mathbb{Z})$, which proves the claim.

Since the natural map $\mathbf{bso}^0(X) \to \mathbf{bso}[0,4]^0(X)$ is a map of abelian groups, if the isomorphism $\mathbf{bso}[0,4]^0(X) \cong H^2(X;\mathbb{Z}/2) \oplus H^4(X;\mathbb{Z})$ were also one of abelian groups then the map which sends an oriented vector bundle V to $(w_2(V), \pm p_1(V))$ would be a map of abelian groups. But the Whitney sum formulas that dictate the behavior of w_2 and p_1 under the direct sum of bundles make this impossible. □

Corollary 5.17 *Let X be a space with an oriented real vector bundle $\xi : X \to \mathbf{BSO}$ which lifts to a string bundle. Then the set of string bundles which lift ξ is a torsor for* $\mathrm{pic}_1^3(KU)^0(X) \simeq \mathrm{bgl}_1(KU[0,2])^0(X)$.

Proof This follows from applying the limit preserving functor $\mathrm{Map}(X,-)$ to the pullback of h-types

$$\begin{array}{ccc} \mathrm{Pic}_1^3(KU) & \longrightarrow & \mathbf{BString} \\ \downarrow & & \downarrow \\ \{*\} & \xrightarrow{\ \xi\ } & \mathbf{BSO} \end{array}$$

□

Remark 5.18 Corollary 5.17 implies that if X is a connected and oriented manifold which admits a string structure then those string structures can be twisted by elements of $\mathrm{bgl}_1^0(KU[0,2])(X)$. These are $KU[0,2]$-line bundles on X and therefore, in light of Corollary 5.8, complex super 2-line bundles.

Remark 5.19 We suspect it is also true that $\mathrm{Pic}_0^3(KU) \simeq \mathrm{fib}(\mathbf{BString} \to \mathbf{BO})$, but we don't have an interesting interpretation of this fact, so we do not investigate it here.

The next proposition can be proven by methods similar to those used in the proof of Proposition 5.16:

Proposition 5.20 *If* $\mathbf{O}/\!\!/\mathbf{Spin}$ *denotes the fiber of the map* $\mathbf{BSpin} \to \mathbf{BO}$ *then there is an equivalence of abelian* ∞*-groups* $\mathbf{O}/\!\!/\mathbf{Spin} \simeq \mathrm{Pic}_1^2(KO) \simeq B\mathrm{GL}_1(KO[0,1])$.

Corollary 5.21 *For an* h*-type* X *with a real vector bundle* $\xi : X \to \mathbf{BO}$ *the set of lifts of* ξ *to* $\mathbf{BSpin}$ *is a torsor for* $\mathrm{bgl}_1^0(KO[0,1])$.

Remark 5.22 In light of the results of [7; 13], Propositions 5.16 and 5.20 imply that $M\mathbf{String} \to M\mathbf{SO}$ and $M\mathbf{Spin} \to M\mathbf{O}$ are Hopf–Galois extensions (or co-Galois extensions) in the sense of [47] with Galois algebras $\mathbb{S}[B\mathrm{GL}_1(KU[0,2])]$ and $\mathbb{S}[B\mathrm{GL}_1(KO[0,1])$, respectively. In other words, there are morphisms of affine spectral schemes $\mathrm{Spec}(M\mathbf{SO}) \to \mathrm{Spec}(M\mathbf{String})$ and $\mathrm{Spec}(M\mathbf{O}) \to \mathrm{Spec}(M\mathbf{Spin})$ which are torsors for the affine commutative group schemes $\mathrm{Spec}(\mathbb{S}[B\mathrm{GL}_1(KU[0,2])])$ and $\mathrm{Spec}(\mathbb{S}[B\mathrm{GL}_1(KO[0,1])]$, respectively. We note however that this in no way implies that either $M\mathbf{String} \to M\mathbf{SO}$ or $M\mathbf{Spin} \to M\mathbf{O}$ is an actual *Galois* extension. These statements could be made more precise with the language of [39] but we leave that for another day.

5.3 Twisting cohomology theories

There are maps of h-types $\mathrm{Pic}_0^3(KU) \to \mathrm{Pic}(KU)$ and $\mathrm{Pic}_0^3(KU) \to \mathrm{Pic}(KO)$, but we do not know if these are maps of abelian ∞-groups because we do not know if there are $\mathbb{E}_\infty$-splittings $\mathrm{Pic}(KU) \simeq \mathrm{Pic}_0^3(KU) \times \mathrm{Pic}_4^\infty(KU)$ or $\mathrm{Pic}(KO) \simeq \mathrm{Pic}_0^2(KO) \times \mathrm{Pic}_3^\infty(KO)$. However, there *are* $\mathbb{E}_\infty$-splittings

$$B\mathrm{GL}_1(KU) \simeq \mathrm{Pic}_1^3(KU) \times B\mathrm{GL}_1(KU)[4,\infty) \quad \text{and} \quad B\mathrm{GL}_1(KO) \simeq \mathrm{Pic}_1^2(KO) \times B\mathrm{GL}_1(KO)[3,\infty)$$

(this is well known, but also follows from our Corollaries 3.33 and 3.36). Therefore for a connected h-type X there are twists of real and complex K-theory by $\mathrm{pic}_1^3(KU)^0(X)$ and $\mathrm{pic}_1^2(KO)^0(X)$.

Question 5.23 *Given a class* $\alpha \in \mathrm{pic}_1^3(KU)^0(X)$ *there is a twisted* K*-theory group* $K^\alpha(X)$. *On the other hand, via the isomorphism of Corollary 5.5,* α *is also a class in* $\mathrm{GBr}\mathbf{U}(X)$, *i.e., a Morita class of graded continuous class* C^**-algebras with spectrum* X. *Then, by* [44, Section 4.1] *there is a twisted* *operator-theoretic KK-theory group* $KK^{\alpha}(X)$*. Are these two groups always isomorphic? This question, for the comparison between ungraded* C^**-algebras and* $H^3(X;\mathbb{Z})$ *is answered in the affirmative in* [26].

More generally, using the results of Section 5.2, we may repeat the constructions of [2] to obtain twists of other cohomology theories that now can be interpreted as coming from $KU[0,2]$ and $KO[0,1]$ line bundles.

Example 5.24 By taking Thom spectra of the fiber sequence $B\mathrm{GL}_1(KO[0,1]) \to \mathbf{BSpin} \to \mathbf{BO}$ of Proposition 5.20 and composing with the $\mathbb{E}_\infty$ Atiyah–Bott–Shapiro orientation of [31], we obtain a composite of maps of $\mathbb{E}_\infty$-ring spectra

$$\Sigma^\infty_+ B\mathrm{GL}_1(KO[0,1]) \to M\mathbf{Spin} \to KO.$$

The $\mathrm{gl}_1 \vdash \Sigma^\infty_+\Omega^\infty$ adjunction then induces a map of abelian ∞-groups $B\mathrm{GL}_1(KO[0,1]) \to \mathrm{GL}_1(KO)$ which deloops to

$$B^2\mathrm{GL}_1(KO[0,1]) \to B\mathrm{GL}_1(KO).$$

Recall that $B^2\mathrm{GL}_1(KO[0,1])$ is the base space component of $\mathrm{Br}(KO[0,1])$ and $B\mathrm{GL}_1(KO[0,1]) \simeq \Omega B\mathrm{GL}_1(KO[0,1])$ classifies super 2-line bundles. We think it reasonable to interpret this map as giving twists of KO-theory by real super 3-line bundles (though there does not appear to be an agreed upon definition of 3-line bundles in the literature).

Remark 5.25 Note that the fiber of the composite $\mathbf{BSpin}^c \to \mathbf{BSO} \to \mathbf{BO}$ is, as an h-type, equivalent to $\mathbb{Z}/2 \times K(\mathbb{Z},2)$. Under the assumption that this fiber has nontrivial $\mathbb{E}_\infty$-structure, Theorem 3.35 implies that it is equivalent as an abelian ∞-group to the h-type classifying complex superline bundles. In other words, $\mathbf{Spin}^c$-structures on manifolds can be twisted by complex superline bundles. We believe that this fact has an interpretation in terms of Clifford algebra bundles which is the subject of joint work of the first two authors and Pacheco-Tallaj.

Example 5.26 Similarly to Example 5.24, we can use the fiber sequence in the proof of Proposition 5.16, $B\mathrm{GL}_1(KU[0,2]) \to \mathbf{BString} \to \mathbf{BSO}$, along with the $\mathbb{E}_\infty$-orientation $M\mathbf{String} \to KU$ of Ando, Hopkins and Rezk [5] to obtain twists

$$B^2\mathrm{GL}_1(KU[0,2]) \to B\mathrm{GL}_1(\mathrm{tmf}).$$

Again one might interpret such twists as twists of tmf-theory by complex super 3-line bundles, or maps to the connected component of $\mathrm{Br}(KU[0,2])$.

Remark 5.27 Recall that, for a commutative ring spectrum R, the base point component of $\mathrm{Br}(R)$ is the space of LMod_R-modules (in Cat_∞) which are equivalent to LMod_R and equivalences between them. Moreover, there is a canonical map of spectra $\mathrm{bgl}_1(R) \to \mathrm{gl}_1(K(R))$, and hence a map of abelian ∞-groups $B^2\mathrm{GL}_1(R) \to B\mathrm{GL}_1(K(R))$, by which we may think of maps $X \to B^2\mathrm{GL}_1(R)$ as $K(R)$-line bundles. Example 5.26 then suggests that our twists $B^2\mathrm{GL}_1(KU[0,2]) \to B\mathrm{GL}_1(\mathrm{tmf})$ are related to $K(KU)$ being a form of elliptic cohomology (see, for instance, [6]).

5.4 Mathematical physics

In this section we describe how our work is connected to work in mathematical physics of Freed, Hopkins and others. In [16, 1.34, 1.38], Freed describes four spectra: $\mathrm{cAlg}^\times_{\mathbb{R}}$, $\mathrm{cAlg}^\times_{\mathbb{C}}$, $\mathrm{Alg}^\times_{\mathbb{R}}$ and $\mathrm{Alg}^\times_{\mathbb{C}}$. These are each Picard spectra of Morita 2-categories of invertible algebras, bimodules between them, and intertwiners between bimodules. In the first two cases Freed requires that the bimodule structures and the intertwiners are all continuous with respect to the topologies of $\mathbb{R}$ and $\mathbb{C}$, respectively. In the second two cases, everything is with respect to the discrete topologies on $\mathbb{R}$ and $\mathbb{C}$. Freed computes the homotopy groups and k-invariants of each of these spectra (implicitly using results which are made concrete in [24; 32]). Each of these have a finite number of nonzero homotopy groups, all of which we exhibit below:

$$\pi_{\{0,1,2,3\}}\mathrm{cAlg}^\times_{\mathbb{C}} \cong \{\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}\}, \quad \pi_{\{0,1,2\}}\mathrm{cAlg}^\times_{\mathbb{R}} \cong \{\mathbb{Z}/8, \mathbb{Z}/2, \mathbb{Z}/2\},$$
$$\pi_{\{0,1,2\}}\mathrm{Alg}^\times_{\mathbb{C}} \cong \{\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{C}^\times\}, \qquad \pi_{\{0,1,2\}}\mathrm{Alg}^\times_{\mathbb{R}} \cong \{\mathbb{Z}/8, \mathbb{Z}/2, \mathbb{R}^\times\}.$$

In the last two cases, $\mathbb{C}^\times$ and $\mathbb{R}^\times$ both have the discrete topology. Freed also computes the k-invariants of these spectra to all be nontrivial. Freed's computations when combined with ours (as well as Conjecture 3.38) yield the following:

Theorem 5.28 *There are equivalences of spectra*

$$\mathrm{cAlg}^\times_{\mathbb{C}} \simeq \mathrm{pic}^3_0(KU), \quad \mathrm{cAlg}^\times_{\mathbb{R}} \simeq \mathrm{pic}^2_0(KO), \quad \mathrm{Alg}^\times_{\mathbb{C}}[1,2] \simeq \mathrm{pic}^2_1(K(\mathbb{C})),^{1} \quad \mathrm{Alg}^\times_{\mathbb{R}}[1,2] \simeq \mathrm{pic}^2_1(K(\mathbb{R})).^{1}$$

Note that the second and last spectra above are essentially the same because there is no nondiscrete topology to put on $\mathbb{Z}/2$.

In [17, 4.3, 4.4] Freed again introduces these spectra, though with different names (and of course ignoring the difference between $\mathrm{cAlg}^\times_{\mathbb{R}}$ and $\mathrm{Alg}^\times_{\mathbb{R}}$). Specifically, he writes R^{-1} for $\mathrm{cAlg}^\times_{\mathbb{C}}$, $R^{-2}_{\mathbb{R}/\mathbb{Z}}$ for $\mathrm{Alg}^\times_{\mathbb{C}}$, and E for $\mathrm{cAlg}^\times_{\mathbb{R}}$. However, Freed is also interested in the connective covers of the one-fold desuspensions of these spectra, which he denotes by R^{-2}, $R^{-3}_{\mathbb{R}/\mathbb{Z}}$ and E^{-1}. This immediately implies the following.

Corollary 5.29 *There are equivalences of spectra*

$$R^{-2} \simeq \mathrm{gl}_1(KU)[0,2], \quad R^{-3}_{\mathbb{R}/\mathbb{Z}} \simeq \mathrm{gl}_1(K(\mathbb{C}))[0,1],^{2} \quad E^{-1} \simeq \mathrm{gl}_1(KO)[0,1].$$

The spectra R^{-2}, $R^{-3}_{\mathbb{R}/\mathbb{Z}}$ and E^{-1} are explained by Freed to be the Picard spectra of the groupoids of complex superlines, *flat* complex superlines, and real superlines. This should not be surprising in light of the identifications made in Corollaries 3.33 and 3.36.

In [17; 16], Freed identifies $\mathrm{cAlg}^\times_{\mathbb{C}}$ with $\Sigma^{-1}ko[1,4]$, which is abstractly equivalent to $\mathrm{pic}^3_0(KU)$, but remarks that he does not have a conceptual reason for this identification. Similarly, for $\mathrm{cAlg}^\times_{\mathbb{R}} \simeq \mathrm{Alg}^\times_{\mathbb{R}}$, Freed mentions that there is not an "off the shelf" spectrum representing it. We claim that the constructions of this paper, and the identifications made in this section, provide spectra which naturally arise in stable

[1] While the nontriviality of the k-invariants of these spectra relies on Conjecture 3.38, they do have the correct homotopy groups in the correct degrees (see [33]).

[2] Again, conjecturally.

homotopy theory (they are "off the shelf") which are the Picard spectra of these Morita categories. Moreover, the constructions of Sections 3.2 and 3.3 essentially prove that the Picard and unit spectra of interest in this paper have the desired geometric interpretations.

5.5 The Anderson dual of $\mathbb{S}$

Recall that there is a spectrum called the *Anderson dual of* $\mathbb{S}$, denoted by $I_{\mathbb{Z}}$, which defines a functor on $\mathcal{S}\mathrm{p}$, $\mathrm{Map}(-, I_{\mathbb{Z}}) : \mathcal{S}\mathrm{p}^{\mathrm{op}} \to \mathcal{S}\mathrm{p}$ (see, e.g., [25, §2; 37, 4.3.9]). Given a spectrum E we will write $I_{\mathbb{Z}}E$ for $\mathrm{Map}(E, I_{\mathbb{Z}})$ and refer to this as the *Anderson dual of* E.

We now show that there is a close relationship between the truncated Picard spectrum $\mathrm{pic}_0^3(KU)$ and the Anderson dual of the sphere spectrum. This is consistent with the results of the prior section as the Anderson dual often appears in the mathematical physics work of Freed, Hopkins and others (see, for instance, [18, Hypothesis 5.17; 19, Ansatz 5.26, Theorem 5.27]).

We will be particularly interested in $\Sigma^n(I_{\mathbb{Z}}[-n, \infty))$, the n-fold suspension of the $-n$-connective cover of $I_{\mathbb{Z}}$. We will simplify notation by writing this as ${}_nI_{\mathbb{Z}}$.

Lemma 5.30 *The spectrum* ${}_3I_{\mathbb{Z}}$ *has homotopy groups* $\pi_0 = \mathbb{Z}/2$, $\pi_1 = \mathbb{Z}/2$, $\pi_2 = 0$, $\pi_3 = \mathbb{Z}$.

Proof The uppermost homotopy groups of $I_{\mathbb{Z}}$ (which is coconnective) are well known, see again [37, 4.3.9], and the result follows by suspending. □

Lemma 5.31 *The unique* k*-invariant of the spectrum* ${}_2I_{\mathbb{Z}}$ *is* $\beta\,\mathrm{Sq}^2$ *and the bottom* k*-invariant of* ${}_3I_{\mathbb{Z}}$ *is* Sq^2.

Proof By [37, 4.2.7(1)] the functor $\mathrm{Map}(-, I_{\mathbb{Z}})$ is a contravariant equivalence on the full subcategory of spectra with finitely many homotopy groups all of which are finitely generated. The first k-invariant of $\mathbb{S}$ is nontrivial and therefore so is the map $\mathrm{Sq}^2 \circ \rho : H\mathbb{Z} \to \Sigma^2 H\mathbb{Z}/2$. This is also of course the k-invariant of $\mathbb{S}[0,1]$. It follows that the uppermost k-invariant of $I_{\mathbb{Z}}$, hence the k-invariant of $I_{\mathbb{Z}}[-2,0]$, or equivalently of ${}_2I_{\mathbb{Z}}$, is a nontrivial map $H\mathbb{Z}/2 \to \Sigma^3 H\mathbb{Z}$, and therefore must be $\beta\,\mathrm{Sq}^2$, the generator of $H^3(H\mathbb{Z}/2;\mathbb{Z}) \cong \mathbb{Z}/2$. By a similar argument the bottom k-invariant of ${}_3I_{\mathbb{Z}}$ is nontrivial and therefore Sq^2. □

Theorem 5.32 *There is an equivalence of spectra* $\mathrm{pic}_0^3(KU) \simeq {}_3I_{\mathbb{Z}}$.

Proof The spectra have the same homotopy groups and both have Sq^2 as their bottom k-invariant. The calculations of Lemmas 4.11 and 4.12 and Propositions 4.13 and 4.14 proceed identically for ${}_3I_{\mathbb{Z}}$ and show that although it has two possible second k-invariants they yield equivalent spectra, both of which must be equivalent to $\mathrm{pic}_0^3(KU)$. □

Corollary 5.33 *There is an equivalence of spectra* $\mathrm{gl}_1(KU[0,2]) \simeq {}_2I_{\mathbb{Z}}$.

We recall the hypothetical characterization of topological field theories given by Freed and Hopkins [16, Ansatz 5.26]:

Ansatz 5.34 *A continuous, invertible, n-dimensional extended topological field theory with symmetry group H_n is a map*

$$\phi : \Sigma^n \mathrm{MTH}_n \to \Sigma^{n+1}(I_{\mathbb{Z}}[-n,0]),$$

where MTH_n *is the Madsen–Tillman spectrum for H_n of* [22].

If we believe Ansatz 5.34 then a topological field theory determines a map, for a manifold X, $\phi^* : \mathrm{MTH}^n_n(X) \to {}_nI^0_{\mathbb{Z}}(X)$. The domain of this map is, by [22], a set of submersions $E \to X$ with n-dimensional fibers, up to cobordism, and therefore essentially a bundle of cobordism classes of n-dimensional manifolds on X. If we let $n = 1, 2$ then Theorem 5.32 and Corollary 5.8 imply that we have a map whose input is bundles of 1- or 2-dimensional manifolds over X and whose output is either super lines bundles on X or super 2-line bundles on X, which is the behavior one would expect of a fully extended, invertible topological field theory (especially in the case that $X = *$). This also suggests that super n-lines in general should be classified by maps into ${}_nI_{\mathbb{Z}}$ (or rather, should be defined as such, as the authors are not aware of any generally accepted definition of super n-line).

The following conjecture is a 2-local and real version of Theorem 5.32, but the indeterminacy of the k-invariants of $\mathrm{pic}^2_0(KO)$ in our calculations prevents us from proving it. Recall that $\pi_{-4}(I_{\mathbb{Z}}) \cong \mathbb{Z}/24$ which becomes $\mathbb{Z}/8$ after 2-localizing. Therefore the left-hand spectrum below has homotopy groups $\{\mathbb{Z}/8, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}_{(2)}\}$, which agree with the homotopy groups of the right-hand side.

Conjecture 5.35 *There is an equivalence of spectra* ${}_4(I_{\mathbb{Z}})_{(2)} \simeq \mathrm{pic}^4_0(KO_{(2)})$.

We conclude with an attempt to collect the ideas of Sections 5.4 and 5.5, at least in the complex case, in the following table:

$\mathrm{gl}_1(KU)[0,2] \simeq \mathrm{gl}_1(KU[0,2])$	$\mathrm{pic}^3_0(KU)$	$\mathrm{br}(KU)[0,4]$
n/a	central simple $\mathbb{C}$-superalgebras	central simple $KU[0,2]$-algebras?
invertible (super) $\mathbb{C}$-modules	invertible $\mathrm{sVect}_{\mathbb{C}}$-modules ($\cong$ invertible $KU[0,2]$-modules?)	invertible $\mathrm{LMod}(KU[0,2])$-modules?
complex superlines	complex super 2-lines	complex super 3-lines?
$\Sigma^2(I_{\mathbb{Z}}[-2,\infty))$	$\Sigma^3(I_{\mathbb{Z}}[-3,\infty))$	$\Sigma^4(I_{\mathbb{Z}}[-4,\infty))$?

Certain entries are conjectural, and therefore ended with question marks. Note that the passage from the first row to the second is obtained by applying the functor $\mathrm{LMod}(-)$ and that the third row is essentially just a renaming of the second row.

Acknowledgements

The authors thank Tyler Lawson, Leon Liu, Natalia Pacheco-Tallaj, Eric Peterson (especially Eric Peterson) and Charles Rezk for helpful conversations regarding this material. They are also grateful to the referee for several very helpful recommendations.

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Received: January 31, 2024 Revised: April 23, 2025

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 26 Issue 5 (pages 1597–1963) 2026