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We define real topological Hochschild homology of separated log schemes with involutions. We show that real topological Hochschild homology is $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariant, which leads to the definition of the motivic real topological Hochschild spectrum living in a certain $\mathbb{Z}/2$-equivariant logarithmic motivic category. We explore properties of real topological Hochschild homology that can be deduced from the logarithmic motivic homotopy theory. We also define the motivic real topological cyclic spectrum.

1 Introduction

Topological Hochschild homology THH and its cousin TC have a deep connection with algebraic K-theory via the cyclotomic trace of Bökstedt, Hsiang, and Madsen [8]. The Dundas–Goodwillie–McCarthy theorem [11] provided a computational tool for algebraic K-theory. It was also discovered that THH contains arithmetic data. Bhatt, Morrow, and Scholze [3] studied filtrations on the S^1-homotopy fixed point spectrum TC^- of THH, whose graded pieces are closely related to the prismatic cohomology of Bhatt and Scholze according to [4, §13].

Hesselholt and Madsen [15] defined real algebraic K-theory, which refines both algebraic and hermitian K-theories in a uniform manner via equivariant homotopy theory. They also defined real topological Hochschild homology THR of rings. For further developments on THR, we refer to [10; 18; 27]. A forthcoming work of Harpaz, Nikolaus, and Shah will show a real refinement of the Dundas–Goodwillie–McCarthy theorem, which would justify the real trace method as a computational tool for real or hermitian K-theory.

One feature of THH different from algebraic K-theory is that THH is not $\mathbb{A}^1$-invariant even for regular schemes, i.e., the induced map $\mathrm{THH}(X) \to \mathrm{THH}(X \times \mathbb{A}^1)$ is not an equivalence of spectra for every nonempty scheme X. The definition of THH of schemes is due to Geisser and Hesselholt [13]. Hence the motivic methods in $\mathbb{A}^1$-homotopy theory initiated by Morel and Voevodsky [23] are not directly applicable to THH.

Non-$\mathbb{A}^1$-invariance of THH is related to the fact that the sequence of spectra

$$\mathrm{THH}(Z) \xrightarrow{i_*} \mathrm{THH}(X) \xrightarrow{j^*} \mathrm{THH}(X - Z)$$

is not a fiber sequence when $i : Z \to X$ is a closed immersion of regular schemes and $j : X - Z \to X$ is its open complement. On the other hand, the localization sequence in algebraic K-theory can be read as the fiber sequence of spectra

$$\mathrm{K}(Z) \xrightarrow{i_*} \mathrm{K}(X) \xrightarrow{j^*} \mathrm{K}(X - Z).$$

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Hesselholt and Madsen [16] and Rognes, Sagave, and Schlichtkrull [30] studied localization sequences for THH in the logarithmic setting.

The author's joint work with Binda and Østvær [7] introduced the ∞-category of logarithmic motivic spectra $\mathrm{logSH}(S)$ for fs log schemes S, which aims to incorporate various non-$\mathbb{A}^1$-invariant cohomology theories into the motivic framework using logarithmic geometry. We refer to Ogus's book [25] for logarithmic geometry. For a closed immersion of schemes $Z \to X$, let (X, Z) denote the log scheme with the underlying scheme X and with the compactifying log structure associated with the open immersion $X - Z \to X$. In the construction of $\mathrm{logSH}(S)$ when S is a scheme (with the trivial log structure), the interval $\mathbb{A}^1$ is replaced with the set of log schemes $(\mathbb{P}^n, \mathbb{P}^{n-1})$ for all integers $n \geq 1$, and the Nisnevich topology is replaced with the strict Nisnevich topology in [6]. To construct a motivic spectrum in $\mathrm{logSH}(S)$ that represents an existing cohomology theory of schemes, one can take the following two steps:

Step 1. Extend the cohomology theory to fs log schemes.

Step 2. Show that the extension is $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariant for all integers $n \geq 1$ and satisfies strict Nisnevich descent.

Rognes [29] defines THH of log rings. According to [7, §8], it is possible to define THH and TC of log schemes based on Rognes' definition and to construct a $\mathbb{P}^1$-spectrum **THH** and **TC** (written as **logTHH** and **logTC** in loc. cit.) representing THH and TC. The author's joint work with Binda, Lundemo, and Østvær [5, §8] constructed a $\mathbb{P}^1$-spectrum representing Hochschild homology.

In this article, we employ this strategy for THR. We define THR of log rings with involutions, where an involution means an automorphism σ such that $\sigma \circ \sigma = \mathrm{id}$. With the aid of the author's joint work with Hornbostel [19], we define THR of separated log schemes with involutions. Then we prove that THR is $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariant for all integers $n \geq 1$. This allows us to apply motivic methods in logarithmic motivic homotopy theory to THR. We also consider real topological cyclic homology, which is a variant of THR. One way to obtain TCR is to impose the real cyclotomic structure on THR, which is due to Quigley and Shah [27]. The following two results are examples of this.

Theorem 1.1 (special case of Theorem 6.8) *Let $Z \to X$ be a closed immersion of smooth schemes over a noetherian separated scheme S. Then there exist fiber sequences of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(\mathrm{Th}(\mathrm{N}_Z X)) \to \mathrm{THR}(X) \to \mathrm{THR}(\mathrm{Bl}_Z X, E),$$
$$\mathrm{TCR}(\mathrm{Th}(\mathrm{N}_Z X)) \to \mathrm{TCR}(X) \to \mathrm{TCR}(\mathrm{Bl}_Z X, E),$$

where E is the exceptional divisor.

We refer to (6-1) for the notation $\mathrm{THR}(\mathrm{Th}(\mathrm{N}_Z X))$. Proposition 6.5 implies that $\mathrm{THR}(\mathrm{Th}(\mathrm{N}_Z X))$ can be written in terms of THR of schemes. This sequence can be considered as the *localization sequence for* THR. The map $\mathrm{THR}(\mathrm{Th}(\mathrm{N}_Z X)) \to \mathrm{THR}(X)$ is called the *Gysin map*.

Theorem 1.2 (Theorem 6.10) *Let $Z \to X$ be a closed immersion of smooth schemes over a finite-dimensional noetherian separated scheme S. Then the induced squares of $\mathbb{Z}/2$-spectra*

$$\begin{array}{ccc} \mathrm{THR}(X) & \longrightarrow & \mathrm{THR}(Z) \\ \downarrow & & \downarrow \\ \mathrm{THR}(Z \times_X \mathrm{Bl}_Z X) & \longrightarrow & \mathrm{THR}(\mathrm{Bl}_Z X) \end{array} \qquad \begin{array}{ccc} \mathrm{TCR}(X) & \longrightarrow & \mathrm{TCR}(Z) \\ \downarrow & & \downarrow \\ \mathrm{TCR}(Z \times_X \mathrm{Bl}_Z X) & \longrightarrow & \mathrm{TCR}(\mathrm{Bl}_Z X) \end{array}$$

are cartesian, where $\mathrm{Bl}_Z X$ denotes the blow-up of X along Z.

The statement in this theorem involves no log schemes, but the proof uses log schemes.

Hu, Kriz, and Ormsby [20] considered $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-spectra instead of $\mathbb{P}^1$-spectra to define a motivic $\mathbb{Z}/2$-spectrum, where $\mathbb{P}^\sigma$ is the scheme $\mathbb{P}^1$ with the involution given by $[x:y] \mapsto [y:x]$. In Definition 6.12, we analogously introduce the ∞-category of prelogarithmic motivic $\mathbb{Z}/2$-spectra $\mathrm{prelogSH}^{\mathbb{Z}/2}(S)$ for any scheme S, which is helpful for constructing logarithmic motivic spectra using the fixed point functor

$$(-)^{\mathbb{Z}/2} : \mathrm{prelogSH}^{\mathbb{Z}/2}(S) \to \mathrm{logSH}(S)$$

in Definition 6.14. We also construct the "forgetful" functor

$$i^* : \mathrm{prelogSH}^{\mathbb{Z}/2}(S) \to \mathrm{logSH}(S)$$

in Definition 6.16. The reason why we add "pre" in the notation $\mathrm{prelogSH}^{\mathbb{Z}/2}(S)$ is that a further localization is desired to obtain a better behaved ∞-category of logarithmic motivic $\mathbb{Z}/2$-spectra; see Remark 6.13 for an explanation.

Together with the $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-periodicity of THR in [19, Proposition 5.1.5], we can define the periodic $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-spectra

$$\mathbf{THR} := (\mathrm{THR}^{\mathbb{Z}/2}, \mathrm{THR}^{\mathbb{Z}/2}, \dots) \in \mathrm{prelogSH}^{\mathbb{Z}/2}(S),$$
$$\mathbf{TCR} := (\mathrm{TCR}^{\mathbb{Z}/2}, \mathrm{TCR}^{\mathbb{Z}/2}, \dots) \in \mathrm{prelogSH}^{\mathbb{Z}/2}(S)$$

for every finite-dimensional noetherian separated scheme S. The construction of **THR** resembles the construction of the motivic real K-theory $\mathbb{Z}/2$-spectrum [20, §4.3]. We obtain new $\mathbb{P}^1$-spectra $\mathbf{THR}^{\mathbb{Z}/2}$ and $\mathbf{TCR}^{\mathbb{Z}/2}$, while we show $i^*\mathbf{THR} \simeq \mathbf{THH}$ and $i^*\mathbf{TCR} \simeq \mathbf{TC}$ in Proposition 6.17.

Organization of the article

To define the $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-spectrum **THR**, we need the following ingredients:

(1) the definition of THR of log rings with involutions,

(2) isovariant étale descent property for THR,

(3) invariance of THR under passing to the associated log structure,

(4) $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-periodicity of THR,

(5) $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariance of THR.

Due to [19], we have (2) and (4). In Section 3, we deal with (1). This requires dihedral replete bar constructions in Section 2, which is an equivariant analogue of Rognes' replete bar constructions [29, Definition 3.16]. In Section 4, we show (3) for separated log schemes. The strategy is to work in a sufficiently local situation. In Section 5, we review the notion of real cyclotomic spectra following Quigley and Shah [27]. Then we generalize many results in the previous sections to real cyclotomic spectra so that we can define TCR of finite-dimensional noetherian separated log schemes. In Section 6, we show (5) by providing an explicit computation of $\mathrm{THR}(\mathbb{P}^n, \mathbb{P}^{n-1})$ using cubes in ∞-categories. Then we discuss properties of THR that can be deduced from [7], and we construct the $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-spectra **THR** and **TC**.

2 Dihedral replete bar constructions

Rognes [29, Definition 3.16] defined the replete bar construction of a commutative monoid, which is a key input to define topological Hochschild homology of log rings. In this section, we discuss a dihedral refinement of this construction.

We refer to [19, Example 4.1.3] for a review of real simplicial sets and dihedral sets, which are defined using crossed simplicial groups of Fiedorowicz and Loday [12]. A *real simplicial set* X is a simplicial set equipped with involutions $w_q : X_q \to X_q$ (i.e., an automorphism such that $w_q \circ w_q = \mathrm{id}$) for all simplicial degrees q satisfying the relations

$$d_i w_q = w_{q-1} d_{q-i}, \quad s_i w_q = w_{q+1} s_{q-i}$$

for $0 \le i \le q$. Note that a real simplicial set is different from a simplicial set with involution, i.e., a simplicial object in the category of sets with involution. A *dihedral simplicial set* X is a real simplicial set equipped with automorphisms $t_q : X_q \to X_q$ for all simplicial degrees q satisfying certain relations.

Let $\oplus$ denote the coproduct in the category of commutative monoids. For commutative monoids P and Q, the coproduct $P \oplus Q$ is naturally isomorphic to the product $P \times Q$.

Definition 2.1 Let Δ^1_σ be the real simplicial set whose underlying simplicial set is Δ^1 and whose involution $w : (\Delta^1)_q \to (\Delta^1)_q$ in simplicial degree q sends the q-simplex $a_0 \cdots a_q$ to $(1-a_0)\cdots(1-a_q)$. For a commutative monoid P with involution w, let $P \otimes \Delta^1_\sigma$ be the real simplicial set whose underlying simplicial set is the tensor product $P \otimes \Delta^1$ defined by

$$(P \otimes \Delta^1)_q := \bigoplus_{i \in (\Delta^1)_q} P$$

and whose involution is given by $(x_0, \dots, x_{q+1}) \to (w(x_{q+1}), \dots, w(x_0))$ in simplicial degree q, where the indices $0, 1, \dots, q+1$ correspond to the q-simplices $0\cdots 00, 0\cdots 01, \dots, 1\cdots 11$ in Δ^1.

Proposition 2.2 *Let P be a commutative monoid with involution. Then the map of real simplicial sets*

$$P \otimes \Delta^1_\sigma \to P$$

given by $(x_0, \dots, x_q) \mapsto x_0 + \cdots + x_q$ in simplicial degree q is a $\mathbb{Z}/2$-weak equivalence.

Proof Let sd_σ be the Segal subdivision functor [33]. We have an isomorphism of simplicial sets with involutions

$$\mathrm{sd}_\sigma(\Delta^1_\sigma) \simeq \Delta^1 \amalg_{\{1\}} \Delta^1,$$

where the involution on the right-hand side switches the factors. This induces an isomorphism of simplicial sets with involutions

$$\mathrm{sd}_\sigma(P \otimes \Delta^1_\sigma) \simeq (P \otimes \Delta^1) \oplus_P (P \otimes \Delta^1),$$

where the involution on the right-hand side is obtained by the formula $(x, y) \mapsto (w(y), w(x))$ with w in Definition 2.1. Since Δ^1 is contractible, the map $P \to P \otimes \Delta^1$ induced by $\{1\} \to \Delta^1$ is a homotopy equivalence. This yields a $\mathbb{Z}/2$-homotopy equivalence

$$P \xrightarrow{\simeq} (P \otimes \Delta^1) \oplus_P (P \otimes \Delta^1),$$

which implies the claim. □

We review the ∞-categorical formulation of equivariant homotopy theory due to Bachmann and Hoyois [2, §9]. See [19, §A.1] for a more detailed review. Let FinGpd denote the 2-category of finite groupoids. Bachmann and Hoyois constructed the functor

$$\mathrm{SH} : \mathrm{Span}(\mathrm{FinGpd}) \to \mathrm{CAlg}(\mathrm{Cat}_\infty), \quad (X \xleftarrow{f} Y \xrightarrow{p} Z) \mapsto p_\otimes f^*;$$

see [2, §C] for the notation Span. For a morphism f in FinGpd, f^* admits a right adjoint f_*. If f is a finite covering, then f^* admits a left adjoint $f_\sharp$.

For a finite groupoid X, let $\mathrm{NAlg}(\mathrm{SH}(X))$ be the ∞-category of normed X-spectra [2, Definition 9.14], and let $\mathrm{CAlg}(\mathrm{SH}(X))$ be the ∞-category of $\mathbb{E}_\infty$-rings in $\mathrm{SH}(X)$. We have the forgetful functor $\mathrm{NAlg}(\mathrm{SH}(X)) \to \mathrm{CAlg}(\mathrm{SH}(X))$. We use the notation $\wedge$ for the coproduct in $\mathrm{NAlg}(\mathrm{SH}(X))$ and $\mathrm{CAlg}(\mathrm{SH}(X))$. For a morphism of finite groupoids $f : X \to S$, we have the induced adjoint functors

$$\mathrm{CAlg}(\mathrm{SH}(S)) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{SH}(X)).$$

If $f : X \to S$ is a finite covering of finite groupoids, we have the induced sequence of adjoint functors

$$\mathrm{NAlg}(\mathrm{SH}(S)) \begin{array}{c} \xleftarrow{f_\sharp} \\ \xrightarrow{f^*} \\ \xleftarrow[f_*]{} \end{array} \mathrm{NAlg}(\mathrm{SH}(X)).$$

On the other hand, if f has connected fibers, then the induced functor

$$f_\otimes : \mathrm{CAlg}(\mathrm{SH}(X)) \to \mathrm{CAlg}(\mathrm{SH}(S))$$

preserves colimits.

For a finite group G, the *∞-category of G-spectra* is

$$\mathrm{Sp}^G := \mathrm{SH}(BG).$$

We also set $\mathrm{NAlg}^G := \mathrm{NAlg}(\mathrm{SH}(BG))$ and $\mathrm{CAlg}^G := \mathrm{CAlg}(\mathrm{SH}(BG))$. We omit the superscripts G in this notation if G is trivial. There is an equivalence of ∞-categories $\mathrm{CAlg} \simeq \mathrm{NAlg}$.

Consider the obvious functors $\mathrm{pt} \xrightarrow{i} B(\mathbb{Z}/2) \xrightarrow{p} \mathrm{pt}$. Observe that i is a finite covering and p has connected fibers. We often use the alternative notation

$$(-)^{\mathbb{Z}/2} := p_*, \quad N^{\mathbb{Z}/2} := i_\otimes, \quad \Phi^{\mathbb{Z}/2} := p_\otimes.$$

The functor $N^{\mathbb{Z}/2}$ is the *norm functor* of Hill, Hopkins, and Ravenel [17], and the functor $\Phi^{\mathbb{Z}/2}$ is the *geometric fixed point functor*. We have the induced functors

$$\begin{aligned} &i^* : \mathrm{Sp}^{\mathbb{Z}/2} \to \mathrm{Sp}, && N^{\mathbb{Z}/2}, i_\sharp, i_* : \mathrm{Sp} \to \mathrm{Sp}^{\mathbb{Z}/2}, \\ &i^* : \mathrm{NAlg}^{\mathbb{Z}/2} \to \mathrm{CAlg}, && N^{\mathbb{Z}/2}, i_\sharp, i_* : \mathrm{CAlg} \to \mathrm{NAlg}^{\mathbb{Z}/2}, \\ &p^* : \mathrm{Sp} \to \mathrm{Sp}^{\mathbb{Z}/2}, && \Phi^{\mathbb{Z}/2}, (-)^{\mathbb{Z}/2} : \mathrm{Sp}^{\mathbb{Z}/2} \to \mathrm{Sp}, \\ &p^* : \mathrm{CAlg} \to \mathrm{CAlg}^{\mathbb{Z}/2}, && \Phi^{\mathbb{Z}/2}, (-)^{\mathbb{Z}/2} : \mathrm{CAlg}^{\mathbb{Z}/2} \to \mathrm{CAlg}. \end{aligned}$$

We refer to [19, Proposition A.2.7] for fundamental relations among these functors. We note that the pair of functors $(i^*, \Phi^{\mathbb{Z}/2})$ is conservative.

Let $f : X \to S$ be a morphism of finite groupoids. For every map $R \to A$ in $\mathrm{NAlg}(\mathrm{SH}(S))$ and map $f^*R \to B$ in $\mathrm{NAlg}(\mathrm{SH}(X))$, we have the natural map

$$A \wedge_R f_* B \to f_*(f^* A \wedge_{f^* R} B)$$

given by the composite

$$A \wedge_R f_* B \xrightarrow{\mathrm{ad}} f_* f^* (A \wedge_R f_* B) \xrightarrow{\simeq} f_*(f^* A \wedge_{f^* R} f^* f_* B) \xrightarrow{\mathrm{ad}'} f_*(f^* A \wedge_{f^* R} B),$$

where ad (resp. ad$'$) denotes the map obtained by the unit (resp. counit).

Lemma 2.3 *Let $r : \mathbb{Z}/2 \to \mathrm{pt}$ be the obvious morphism of finite groupoids. Then the natural map*

$$A \wedge_R r_* B \to r_*(r^* A \wedge_{r^* R} B)$$

is an equivalence for every map $R \to A$ in CAlg and map $r^* R \to B$ in $\mathrm{NAlg}(\mathrm{SH}(\mathbb{Z}/2))$.

Proof As observed in [2, Example 9.15], we have an equivalence of ∞-categories $\mathrm{NAlg}(\mathrm{SH}(\mathbb{Z}/2)) \simeq \mathrm{CAlg}(\mathrm{SH}(\mathbb{Z}/2))$. We additionally have an equivalence of ∞-categories $\mathrm{SH}(\mathbb{Z}/2) \simeq \mathrm{SH}(\mathrm{pt}) \times \mathrm{SH}(\mathrm{pt})$ by [2, Lemma 9.6]; hence $\mathrm{CAlg}(\mathrm{SH}(\mathbb{Z}/2)) \simeq \mathrm{CAlg} \times \mathrm{CAlg}$. The functors $i_0^*, i_1^* : \mathrm{NAlg}(\mathrm{SH}(\mathbb{Z}/2)) \to \mathrm{CAlg}$ induced by the inclusions $i_0, i_1 : \mathrm{pt} \to \mathbb{Z}/2$ can be identified with the two projections $\mathrm{CAlg} \times \mathrm{CAlg} \to \mathrm{CAlg}$. Using $ri_0 = ri_1$, we see that the functor $r^* : \mathrm{CAlg} \to \mathrm{NAlg}(\mathrm{SH}(\mathbb{Z}/2))$ can be identified with the diagonal functor $\mathrm{CAlg} \to \mathrm{CAlg} \times \mathrm{CAlg}$, and the functor $r_* : \mathrm{NAlg}(\mathrm{SH}(\mathbb{Z}/2)) \to \mathrm{CAlg}$ can be identified with the direct sum functor $\oplus : \mathrm{CAlg} \times \mathrm{CAlg} \to \mathrm{CAlg}$. Use these explicit descriptions to show the claim. □

Lemma 2.4 *For $R, A \in \mathrm{NAlg}^{\mathbb{Z}/2}$ and $B \in \mathrm{CAlg}$, the natural map*

(2-1) $$A \wedge_R i_* B \to i_*(i^* A \wedge_{i^* R} B)$$

is an equivalence.

Proof It suffices to show that (2-1) becomes equivalences after applying i^* and $\Phi^{\mathbb{Z}/2}$. With the aid of the forgetful functor $\mathrm{NAlg}^{\mathbb{Z}/2} \to \mathrm{CAlg}^{\mathbb{Z}/2}$, we obtain $\Phi^{\mathbb{Z}/2}(A \wedge_R i_* B) \simeq \Phi^{\mathbb{Z}/2} A \wedge_{\Phi^{\mathbb{Z}/2} R} \Phi^{\mathbb{Z}/2} i_* B$. By [19, Proposition A.2.7(3), (5)], $\Phi^{\mathbb{Z}/2} i_* \simeq 0$. Hence both sides of (2-1) vanish after applying $\Phi^{\mathbb{Z}/2}$.

Apply [19, Proposition A.1.9] to the cartesian square

$$\begin{array}{ccc} \mathbb{Z}/2 & \xrightarrow{r} & \mathrm{pt} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle i} \\ \mathrm{pt} & \xrightarrow{i} & B(\mathbb{Z}/2) \end{array}$$

to obtain a natural equivalence $r_* r^* \simeq i^* i_*$. We have natural equivalences

$$r_* r^*((-) \wedge_{(-)} (-)) \simeq r_*(r^*(-) \wedge_{r^*(-)} r^*(-)) \simeq (-) \wedge_{(-)} r_* r^*(-),$$

where the second one is due to Lemma 2.3. We have the induced commutative diagram

$$\begin{array}{ccccccc} i^*(A \wedge_R i_* B) & \xrightarrow{\mathrm{ad}} & i^* i_* i^*(A \wedge_R i_* B) & \xrightarrow{\simeq} & i^* i_*(i^* A \wedge_{i^* R} i^* i_* B) & \xrightarrow{\mathrm{ad}'} & i^* i_*(i^* A \wedge_{i^* R} B) \\ \simeq\downarrow & & & & \downarrow\simeq & & \downarrow\simeq \\ i^* A \wedge_{i^* R} i^* i_* B & & \xrightarrow{\mathrm{ad}} & & r_* r^*(i^* A \wedge_{i^* R} i^* i_* B) & \xrightarrow{\mathrm{ad}'} & r_* r^*(i^* A \wedge_{i^* R} B) \\ & \searrow^{\mathrm{ad}} & & & \downarrow\simeq & & \downarrow\simeq \\ & & & & i^* A \wedge_{i^* R} r_* r^* i^* i_* B & \xrightarrow{\mathrm{ad}'} & i^* A \wedge_{i^* R} r_* r^* B \end{array}$$

Hence to show that (2-1) becomes an equivalence after applying i^*, it suffices to show that the composite of the upper vertical maps in this commutative diagram is an equivalence:

$$\begin{array}{ccccc} i^* i_* B & \xrightarrow{\mathrm{ad}} & r_* r^* i^* i_* B & \xrightarrow{\mathrm{ad}'} & r_* r^* B \\ \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq \\ r_* r^* B & \xrightarrow{\mathrm{ad}} & r_* r^* r_* r^* B & \xrightarrow{\mathrm{ad}'} & r_* r^* B \end{array}$$

This is true since the composite of the lower horizontal maps is an equivalence by the counit-unit identity. □

For a simplicial set (resp. real simplicial set) X, we use the notation $\mathbb{S}[X] := \Sigma^\infty X_+$, which is a spectrum (resp. $\mathbb{Z}/2$-spectrum). If P is a commutative monoid, then $\mathbb{S}[P]$ is an object of CAlg. If P is a commutative monoid with involution, then $\mathbb{S}[P]$ is a commutative monoid in the model category of orthogonal $\mathbb{Z}/2$-spectrum in the sense of [19, Definition A.2.2]. Due to [19, Remark A.2.6], we can regard $\mathbb{S}[P]$ as an object of $\mathrm{NAlg}^{\mathbb{Z}/2}$.

For a commutative monoid P with involution, we denote by $\mathrm{N}^{\mathrm{di}} P$ the *dihedral nerve of* P in [19, Definition 4.2.2], and by $\mathrm{B}^{\mathrm{di}} P$ its dihedral geometric realization. In simplicial degree q, we have $(\mathrm{N}^{\mathrm{di}} P)_q := P^{\times(q+1)}$. We obtain the cyclic nerve $\mathrm{N}^{\mathrm{cy}} i^* P$ [34, §2.3] of $i^* P$ if we forget the involution structure on $\mathrm{N}^{\mathrm{di}} P$. There is a map of dihedral sets

$$\mathrm{N}^{\mathrm{di}} P \to P \tag{2-2}$$

sending $(x_0, \dots, x_q)$ to $x_0 + \cdots + x_q$ in simplicial degree q.

Definition 2.5 Let $\mathcal{C}$ be an ordinary category. An *object of $\mathcal{C}$ with involution* is an object of the category $\mathcal{C}_{\mathbb{Z}/2} := \mathrm{Fun}(B(\mathbb{Z}/2), \mathcal{C})$. Let

$$i^* : \mathrm{Fun}(B(\mathbb{Z}/2), \mathcal{C}) \to \mathcal{C}$$

be the forgetful functor, and let $i_\sharp$ (resp. i_*) be its left adjoint (resp. right adjoint) if it exists.

Example 2.6 If P is a commutative monoid, then $i_* P$ is the commutative monoid $P \times P$ with the involution w given by $w(x,y) := (w(y), w(x))$. Furthermore, there is a natural isomorphism $i_\sharp P \simeq i_* P$. We have a similar description of $i_* A$ for any commutative ring A, but $i_\sharp A \not\simeq i_* A$.

If X is a scheme, then $i_\sharp X$ is the scheme $X \amalg X$ with the involution switching the two components.

Lemma 2.7 *Let $P \to Q$ be a map of commutative monoids with involutions. Then there is a natural equivalence of $\mathbb{Z}/2$-spectra*

$$\mathbb{S}[P \amalg P] \wedge_{\mathbb{S}[P]} \mathbb{S}[Q] \simeq \mathbb{S}[(P \amalg P) \oplus_P Q],$$

where the involution on $P \amalg P$ in the formulation switches the components.

Proof There is a natural isomorphism of sets with involutions

$$(P \amalg P) \oplus_P Q \simeq Q \amalg Q,$$

where the involution on the right-hand side switches the components. By the explicit description of the functors i^* and i_* in terms of orthogonal spectra in [19, Construction A.2.4], we have natural equivalences

$$\mathbb{S}[P \amalg P] \simeq i_* i^* \mathbb{S}[P] \quad \text{and} \quad \mathbb{S}[Q \amalg Q] \simeq i_* i^* \mathbb{S}[Q].$$

Lemma 2.4 yields a natural equivalence

$$i_* i^* \mathbb{S}[P] \wedge_{\mathbb{S}[P]} \mathbb{S}[Q] \simeq i_*(i^* \mathbb{S}[P] \wedge_{i^* \mathbb{S}[P]} i^* \mathbb{S}[Q]).$$

Combine what we have discussed above to obtain the desired equivalence. □

Proposition 2.8 *Let P be a commutative monoid with involution. Then there is a natural equivalence of $\mathbb{Z}/2$-spectra*

$$\mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \simeq \mathbb{S}[P] \wedge_{\mathbb{S}[i_\sharp i^* P]} \mathbb{S}[P],$$

where the homomorphism $i_\sharp i^* P \to P$ in the formulation is obtained by the counit of the adjunction pair $(i_\sharp, i^*)$.

Proof There is an isomorphism of real simplicial sets

$$(P \otimes \Delta^1_\sigma) \oplus_{i_\sharp i^* P} P \simeq \mathrm{N}^{\mathrm{di}} P, \tag{2-3}$$

where the map $i_\sharp i^* P \to P \otimes \Delta^1_\sigma$ is given by $(x,y) \mapsto (x, 0, \ldots, 0, y)$ in simplicial degree q. The composite

$$i_\sharp i^* P \to P \otimes \Delta^1_\sigma \to P$$

coincides with the counit homomorphism. Since $P \otimes \Delta^1_\sigma$ is degreewise the disjoint union of finitely many copies of $i_\sharp i^* P$ and $i_\sharp i^* P \amalg i_\sharp i^* P$ with the switching involution as an $i_\sharp i^* P$-set, Lemma 2.7 yields a natural equivalence

$$\mathbb{S}[P \otimes \Delta^1_\sigma] \wedge_{\mathbb{S}[i_\sharp i^* P]} \mathbb{S}[P] \simeq \mathbb{S}[(P \otimes \Delta^1_\sigma) \oplus_{i_\sharp i^* P} P].$$

Use Proposition 2.2 and (2-3) to finish the proof. $\square$

For a commutative monoid P, let P^{gp} denote its group completion.

Definition 2.9 Let P be a commutative monoid with involution. The *dihedral replete nerve of P* is the dihedral set

$$\mathrm{N}^{\mathrm{drep}} P := \mathrm{N}^{\mathrm{di}} P^{\mathrm{gp}} \times_{P^{\mathrm{gp}}} P, \tag{2-4}$$

where the map $\mathrm{N}^{\mathrm{di}} P^{\mathrm{gp}} \to P^{\mathrm{gp}}$ in this formulation is given by (2-2) for P^{gp}. The *dihedral replete bar construction of P*, denoted by $\mathrm{B}^{\mathrm{drep}} P$, is the dihedral geometric realization of $\mathrm{N}^{\mathrm{drep}} P$.

We obtain the replete bar construction $\mathrm{B}^{\mathrm{di}} i^* P$ [29, Definition 3.16] of $i^* P$ if we forget the involution structure on $\mathrm{B}^{\mathrm{drep}} P$.

Proposition 2.10 *Let P and Q be commutative monoids with involutions. Then there is a natural isomorphism of dihedral sets*

$$\mathrm{N}^{\mathrm{drep}}(P \times Q) \simeq \mathrm{N}^{\mathrm{drep}} P \times \mathrm{N}^{\mathrm{drep}} Q.$$

Proof By [19, Proposition 4.2.4], we have a natural isomorphism of dihedral sets

$$\mathrm{N}^{\mathrm{di}}(P^{\mathrm{gp}} \times Q^{\mathrm{gp}}) \times_{P^{\mathrm{gp}} \times Q^{\mathrm{gp}}} (P \times Q) \simeq (\mathrm{N}^{\mathrm{di}} P^{\mathrm{gp}} \times \mathrm{N}^{\mathrm{di}} Q^{\mathrm{gp}}) \times_{P^{\mathrm{gp}} \times Q^{\mathrm{gp}}} (P \times Q).$$

The left side is isomorphic to $\mathrm{N}^{\mathrm{drep}}(P \times Q)$, and the right side is isomorphic to $\mathrm{N}^{\mathrm{drep}} P \times \mathrm{N}^{\mathrm{drep}} Q$. $\square$

For a commutative monoid P with involution, let $N^\sigma P$ denote the *real nerve of P* [19, Definition 4.2.1].

Proposition 2.11 *Let P be a commutative monoid with involution. Then there is a natural isomorphism of dihedral sets*

$$\mathrm{N}^{\mathrm{drep}} P \simeq P \times \mathrm{N}^\sigma P^{\mathrm{gp}}.$$

Proof Immediate from [19, Proposition 4.2.6] and (2-4). $\square$

Definition 2.12 For a commutative monoid P with involution w, let $(i_\sharp i^* P)^{\mathrm{ex}}$ denote the commutative monoid $P \oplus P^{\mathrm{gp}}$ with the involution w given by

$$w(x, y) := (w(x), w(x) - w(y)).$$

We have the commutative triangle

$$\begin{array}{ccc} & (i_\sharp i^* P)^{\mathrm{ex}} & \\ {\scriptstyle\gamma}\nearrow & & \searrow{\scriptstyle\mu^{\mathrm{ex}}} \\ i_\sharp i^* P & \xrightarrow{\ \mu\ } & P \end{array}$$

such that $\mu(x,y) := x+y$, $\mu^{\mathrm{ex}}(x,y) := x$, and $\gamma(x,y) := (x+y,y)$ for $x,y \in P$. We note that μ is the counit homomorphism. The construction of $(i_\sharp i^* P)^{\mathrm{ex}}$ is an equivariant analogue of the exactification in [25, Proposition I.4.2.19].

Proposition 2.13 *Let P be a commutative monoid with involution. Then there is a natural equivalence of $\mathbb{Z}/2$-spectra*

$$\mathbb{S}[\mathrm{B}^{\mathrm{drep}} P] \simeq \mathbb{S}[P] \wedge_{\mathbb{S}[(i_\sharp i^* P)^{\mathrm{ex}}]} \mathbb{S}[P].$$

Proof Let $\mathrm{E}P^{\mathrm{gp}}$ denote the total simplicial set of P^{gp}, and let Q be the real simplicial set whose underlying simplicial set is $P \times \mathrm{E}P^{\mathrm{gp}}$, and whose involution is given by

$$(x, g_0, \dots, g_q) \in P \times (P^{\mathrm{gp}})^{\times(q+1)} \mapsto (w(x), w(x) - w(g_q), \dots, w(x) - w(g_0))$$

in simplicial degree q for every integer $q \geq 0$. Consider the maps of real simplicial sets

$$(i_\sharp i^* P)^{\mathrm{ex}} \to Q \to P \times \mathrm{N}^{\sigma} P^{\mathrm{gp}}$$

given by $(x,g) \mapsto (x, g, \dots, g)$ and $(x, g_0, \dots, g_q) \mapsto (x, g_1 - g_0, \dots, g_q - g_{q-1})$ in simplicial degree q.

There is a real simplicial isomorphism

$$Q \oplus_{(i_\sharp i^* P)^{\mathrm{ex}}} P \simeq P \times \mathrm{N}^{\sigma} P^{\mathrm{gp}}.$$

Since Q is degreewise the disjoint union of finite copies of $(i_\sharp i^* P)^{\mathrm{ex}}$ and $(i_\sharp i^* P)^{\mathrm{ex}} \amalg (i_\sharp i^* P)^{\mathrm{ex}}$ with the switching involution as an $(i_\sharp i^* P)^{\mathrm{ex}}$-set, Lemma 2.7 yields a natural equivalence of $\mathbb{Z}/2$-spectra

$$\mathbb{S}[Q] \wedge_{\mathbb{S}[(i_\sharp i^* P)^{\mathrm{ex}}]} \mathbb{S}[P] \simeq \mathbb{S}[P \times \mathrm{N}^{\sigma} P^{\mathrm{gp}}]. \tag{2-5}$$

Let us show that the map $Q \to P$ given by

$$(x, g_0, \dots, g_q) \mapsto x$$

in simplicial degree q is a $\mathbb{Z}/2$-homotopy equivalence. Its underlying map of simplicial sets is the projection $P \times \mathrm{E}P^{\mathrm{gp}} \to P$, which is a homotopy equivalence. The $\mathbb{Z}/2$-fixed point of the Segal subdivision $\mathrm{sd}_\sigma Q$ is in simplicial degree q the set

$$\{(x, g_0, \dots, g_q, w(x) - w(g_q), \dots, w(x) - w(g_0)) : x \in P^{\mathbb{Z}/2}, g_0, \dots, g_q \in P^{\mathrm{gp}}\}.$$

From this description, we see that the induced map

$$(\mathrm{sd}_\sigma Q)^{\mathbb{Z}/2} \to (\mathrm{sd}_\sigma P)^{\mathbb{Z}/2}$$

can be identified with the projection $P \times \mathrm{E}P^{\mathrm{gp}} \to P$. Hence the map $Q \to P$ is a $\mathbb{Z}/2$-homotopy equivalence. Combine this with Proposition 2.11 and (2-5) to obtain the desired equivalence. □

Remark 2.14 For a commutative monoid P with involution, we can regard $\mathbb{S}[\mathrm{B}^{\mathrm{di}} P]$ and $\mathbb{S}[\mathrm{B}^{\mathrm{drep}} P]$ as objects of $\mathrm{NAlg}^{\mathbb{Z}/2}$ by Propositions 2.8 and 2.13.

Proposition 2.15 *Let P be a commutative monoid with involution. Then there are natural equivalences of $\mathbb{Z}/2$-spectra*

$$i^*\mathbb{S}[\mathrm{B}^{\mathrm{di}}P] \simeq \mathbb{S}[\mathrm{B}^{\mathrm{cy}}i^*P] \quad \textit{and} \quad i^*\mathbb{S}[\mathrm{B}^{\mathrm{drep}}P] \simeq \mathbb{S}[\mathrm{B}^{\mathrm{rep}}i^*P].$$

Proof Observe that we obtain $\mathrm{B}^{\mathrm{cy}}i^*P$ (resp. $\mathrm{B}^{\mathrm{rep}}i^*P$) by forgetting the involution structure on $\mathrm{B}^{\mathrm{di}}P$ (resp. $\mathrm{B}^{\mathrm{drep}}P$). □

3 THR of log rings with involutions

For $R \in \mathrm{NAlg}^{\mathbb{Z}/2}$, the *real topological Hochschild homology of R* is defined to be

$$\mathrm{THR}(R) := R \wedge_{N^{\mathbb{Z}/2}i^*R} R,$$

where the maps $N^{\mathbb{Z}/2}i^*R \to R$ in the formulation are the counit of the adjunction pair $(N^{\mathbb{Z}/2}, i^*)$. If A is a commutative ring with involution, then the equivariant Eilenberg–Mac Lane spectrum $\mathrm{H}A$ can be realized as a commutative orthogonal $\mathbb{Z}/2$-ring spectrum by [31, Example 11.12], so we have $\mathrm{H}A \in \mathrm{NAlg}^{\mathbb{Z}/2}$ together with [19, Remark A.2.3]. We set $\mathrm{THR}(A) := \mathrm{THR}(\mathrm{H}A)$. For a commutative monoid P with involution, there is a natural equivalence of $\mathbb{Z}/2$-spectra

$$\mathrm{THR}(\mathbb{S}[P]) \simeq \mathbb{S}[\mathrm{B}^{\mathrm{di}}P]; \tag{3-1}$$

see [18] and also [10, Proposition 5.9]. One can also show this using Proposition 2.8 and the natural equivalence of $\mathbb{Z}/2$-spectra $N^{\mathbb{Z}/2}i^*\mathbb{S}[P] \simeq \mathbb{S}[i_\sharp i^*P]$.

Recall from [25, Definition III.1.2.3] that a *log ring*[1] (A, P) is a commutative ring A equipped with a homomorphism $P \to A$ of commutative monoids, where the monoid operation on A is the multiplication. A *homomorphism of log rings* $(A, P) \to (B, Q)$ is a pair of homomorphisms $A \to B$ and $P \to Q$ such that the square

$$\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & & \downarrow \\ Q & \longrightarrow & B \end{array}$$

commutes. We regard a commutative ring A as a log ring $(A, \{1\})$.

Rognes [29, Definition 8.11, Remark 8.12] defined the *topological Hochschild homology of a log ring* (A, P) as the coproduct, in CAlg,

$$\mathrm{THH}(A, P) := \mathrm{THH}(A) \wedge_{\mathbb{S}[\mathrm{B}^{\mathrm{cy}}P]} \mathbb{S}[\mathrm{B}^{\mathrm{rep}}P], \tag{3-2}$$

where the map $\mathbb{S}[\mathrm{B}^{\mathrm{cy}}P] \simeq \mathrm{THH}(\mathbb{S}[P]) \to \mathrm{THH}(A)$ in the formulation is obtained by applying THH to the induced map $\mathbb{S}[P] \to \mathrm{H}A$.

[1] A log ring is often called a *prelog ring* in the literature; see, e.g., [29, Definition 2.1].

Definition 3.1 The *real topological Hochschild homology of a log ring* (A, P) *with involution* is the coproduct, in $\mathrm{NAlg}^{\mathbb{Z}/2}$,

$$\mathrm{THR}(A, P) := \mathrm{THR}(A) \wedge_{\mathbb{S}[\mathrm{B}^{\mathrm{di}} P]} \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P], \tag{3-3}$$

where the map $\mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \simeq \mathrm{THR}(\mathbb{S}[P]) \to \mathrm{THR}(A)$ in this formulation is obtained by applying THR to the induced map $\mathbb{S}[P] \to \mathrm{H}A$.

We obviously have $\mathrm{THR}(A, \{1\}) \simeq \mathrm{THR}(A)$ for every commutative ring A.

Definition 3.2 Let $P \to M$ be a homomorphism of commutative monoids with involutions. For notational convenience, we set

$$\mathrm{THR}(\mathbb{S}[M], P) := \mathbb{S}[\mathrm{B}^{\mathrm{di}} M] \wedge_{\mathbb{S}[\mathrm{B}^{\mathrm{di}} P]} \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P].$$

Proposition 3.3 *Let $Q \to M$ be a homomorphism of commutative monoids with involutions. Then there is a natural equivalence of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(A[M], P \oplus Q) \simeq \mathrm{THR}(A, P) \wedge \mathrm{THR}(\mathbb{S}[M], Q).$$

Proof By [19, Proposition 4.2.4] and Proposition 2.10, there are natural isomorphisms of dihedral sets

$$\mathrm{N}^{\mathrm{di}}(P \oplus Q) \simeq \mathrm{N}^{\mathrm{di}} P \times \mathrm{N}^{\mathrm{di}} Q \text{ and } \mathrm{N}^{\mathrm{drep}}(P \oplus Q) \simeq \mathrm{N}^{\mathrm{drep}} P \times \mathrm{N}^{\mathrm{drep}} Q.$$

Apply $\mathbb{S}[-]$ to these, and use (3-3) to obtain the desired equivalence. □

Let S^σ be S^1 with the involution given by $e^{i\theta} \in S^1 \mapsto e^{-i\theta}$.

Example 3.4 By [19, (4.12)], we have an equivalence of $\mathbb{Z}/2$-spectra

$$\mathrm{THR}(\mathbb{S}[\mathbb{N}], \mathbb{N}) \simeq \bigoplus_{d=0}^{\infty} \mathbb{S}[S^\sigma].$$

Furthermore, [19, Propositions 4.2.11 and 4.2.12] implies that the induced map

$$\mathrm{THR}(\mathbb{S}[\mathbb{N}]) \to \mathrm{THR}(\mathbb{S}[\mathbb{N}], \mathbb{N})$$

can be written as the componentwise induced map

$$\mathbb{S} \oplus \bigoplus_{d=1}^{\infty} \mathbb{S}[S^\sigma] \to \bigoplus_{d=0}^{\infty} \mathbb{S}[S^\sigma].$$

In the remaining part of this section, we investigate how THH and THR interact with the functors i^* and i_*.

Proposition 3.5 *Let (A, P) be a log ring with involution. Then there is a natural equivalence of spectra*

$$i^*\mathrm{THR}(A, P) \simeq \mathrm{THH}(i^* A, i^* P).$$

Proof We have a natural equivalence $i^*\mathrm{THR}(A) \simeq \mathrm{THH}(i^*A)$ by [19, Proposition 3.4.7]. Since the functor $i^* : \mathrm{NAlg}^{\mathbb{Z}/2} \to \mathrm{CAlg}$ preserves colimits, we have a natural equivalence

$$i^*\mathrm{THR}(A, P) \simeq i^*\mathrm{THR}(A) \wedge_{i^*\mathbb{S}[\mathrm{B}^{\mathrm{di}} P]} i^*\mathbb{S}[\mathrm{B}^{\mathrm{drep}} P].$$

Together with Proposition 2.15, we obtain the desired equivalence. □

For a commutative monoid P, let $\underline{P}$ denote the commutative monoid P with the trivial involution.

Construction 3.6 Let (A, P) be a log ring. We have the composite map of spectra

$$i^*\mathrm{THR}(i_*A, \underline{P}) \xrightarrow{\simeq} \mathrm{THH}(A \oplus A, P) \to \mathrm{THH}(A, P),$$

where log structure homomorphism $\underline{P} \to i_*A$ sends $p \in P$ to $(\alpha(p), \alpha(p))$, where $\alpha : P \to A$ is the log structure homomorphism. and the first arrow is obtained by Proposition 3.5, and the second arrow is induced by the summation homomorphism $A \oplus A \to A$. By adjunction, we obtain a map of $\mathbb{Z}/2$-spectra

$$\mathrm{THR}(i_*A, \underline{P}) \to i_*\mathrm{THH}(A, P). \tag{3-4}$$

Proposition 3.7 *Let (A, P) be a log ring. Then (3-4) is an equivalence.*

Proof Lemma 2.4 and Proposition 2.15 yield natural equivalences of $\mathbb{Z}/2$-spectra

$$\begin{aligned} i_*\mathrm{THH}(A, P) &\simeq i_*(\mathrm{THH}(A) \wedge_{i^*\mathbb{S}[\mathrm{B}^{\mathrm{di}} \underline{P}]} i^*\mathbb{S}[\mathrm{B}^{\mathrm{drep}} \underline{P}]) \\ &\simeq i_*\mathrm{THH}(A) \wedge_{\mathbb{S}[\mathrm{B}^{\mathrm{di}} \underline{P}]} \mathbb{S}[\mathrm{B}^{\mathrm{drep}} \underline{P}]. \end{aligned}$$

By [19, Propositions 2.1.4, 2.3.3], we have a natural equivalence of $\mathbb{Z}/2$-spectra

$$i_*\mathrm{THH}(A) \simeq \mathrm{THR}(i_*A).$$

Combine these with the definition of $\mathrm{THR}(i_*A, \underline{P})$ to conclude. □

Proposition 3.8 *The functor* THR *from the category of log rings with involutions to $\mathbb{Z}/2$-spectra preserves filtered colimits.*

Proof One can directly check that the endofunctors $P \mapsto i_\sharp i^* P, (i_\sharp i^* P)^{\mathrm{ex}}$ on the category of commutative monoids with involutions preserve colimits. By Propositions 2.8 and 2.13, the functors $P \mapsto \mathbb{S}[\mathrm{B}^{\mathrm{di}} P], \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P]$ from the category of commutative monoids with involutions to $\mathrm{NAlg}_{\mathbb{Z}/2}$ preserve colimits.

On the other hand, the functor $A \mapsto \mathrm{THR}(A)$ from the category of commutative rings with involutions to $\mathrm{NAlg}_{\mathbb{Z}/2}$ preserves filtered colimits since the Eilenberg–Mac Lane functor preserves filtered colimits, $N^{\mathbb{Z}/2}$ and i^* preserve colimits, and $\wedge$ is the coproduct. It follows that the functor $(A, P) \mapsto \mathrm{THR}(A, P)$ from the category of commutative rings with involutions to $\mathrm{NAlg}_{\mathbb{Z}/2}$ preserves filtered colimits. The forgetful functor $\mathrm{NAlg}_{\mathbb{Z}/2} \to \mathrm{Sp}_{\mathbb{Z}/2}$ preserves filtered colimits as observed in [19, §1], which finishes the proof. □

4 THR of log schemes with involutions

So far, we have discussed THR of log rings (A, P) with involutions. The purpose of this section is to define $\mathrm{THR}(X)$ for every separated log scheme X and to show that a canonical map

$$\mathrm{THR}(A, P) \to \mathrm{THR}(\mathrm{Spec}(A, P))$$

is an equivalence of $\mathbb{Z}/2$-spectra.

Let us briefly review basic notation and terminology in log geometry. We refer to Ogus's book [25] for the details. For a commutative monoid P, let P^* denote its submonoid of units. We set $\bar{P} := P/P^*$. We say that P is *integral* if the induced homomorphism $P \to P^{\mathrm{gp}}$ is injective.

- For a log scheme X, let $\underline{X}$ be its underlying scheme, and let $\mathcal{M}_X$ be its structure sheaf of monoids.
- A morphism of log schemes $f : Y \to X$ is *strict* if the induced morphism $Y \to X \times_{\underline{X}} \underline{Y}$ is an isomorphism.
- For a log ring (A, P), let $\mathrm{Spec}(A, P)$ denote its associated log scheme in the sense of Definition III.1.2.3 in [25].
- For a commutative monoid P, we set $\mathbb{A}_P := \mathrm{Spec}(\mathbb{Z}[P], P)$, whose structure homomorphism $P \to \mathbb{Z}[P]$ sends $p \in P$ to p.
- A *chart* P *of a log scheme* X is a commutative monoid P together with a strict morphism $X \to \mathbb{A}_P$ of log schemes.
- A log scheme X is *integral* if $\mathcal{M}_X$ is a sheaf of integral monoids.
- A log scheme X is *fine saturated* (or simply *fs*) if X admits strict étale locally a chart P such that P is a fine saturated monoid.

Let G be a finite group. For a point x of a G-scheme X, the *scheme-theoretic stabilizer of* X *at* x is

$$G_x := \ker\bigl(\{g \in G : gx = x\} \to \mathrm{Aut}(k(x))\bigr).$$

A morphism of G-schemes $Y \to X$ is an *isovariant étale cover* if the induced homomorphism $G_y \to G_{f(x)}$ is an isomorphism for every point $y \in Y$ and f is étale and surjective after forgetting the G-action.

For a commutative ring A with involution, an A*-algebra with involution* is a commutative ring B with involution equipped with a $\mathbb{Z}/2$-equivariant map $A \to B$.

Proposition 4.1 *Let* (A, P) *be a log ring with involution. Then the presheaf* $\mathrm{THR}(-, P)$ *on the opposite category of* A*-algebras with involution is an isovariant étale hypersheaf.*

Proof Let $B \to C$ be an isovariant étale homomorphism of A-algebras with involutions. By Theorem 3.2.3 of [19], there are natural equivalences of $\mathbb{Z}/2$-spectra

$$\mathrm{THR}(B, P) \wedge_{\mathrm{H}B} \mathrm{H}C \simeq \mathrm{THR}(B, P) \wedge_{\mathrm{H}\iota(B^{\mathbb{Z}/2})} \mathrm{H}\iota(C^{\mathbb{Z}/2}) \simeq \mathrm{THR}(C, P),$$

where ιM denotes the commutative monoid with the trivial involution for a commutative monoid M. Argue as in the proof of [19, Theorem 3.4.3] to show the claim. □

Lemma 4.2 *Let $\theta : P \to Q$ be a homomorphism of integral monoids. If $\bar{\theta} : \bar{P} \to \bar{Q}$ is an isomorphism, then the induced homomorphism of monoids*

$$\eta : P \oplus_{P^*} Q^* \to Q \tag{4-1}$$

is an isomorphism.

Proof Since $\bar{\theta}$ is an isomorphism, η is surjective. To show that η is injective, assume $\eta(p, v) = \eta(p', v')$ with $p, p' \in P$ and $v, v' \in Q^*$. This implies $\theta(p) + v = \theta(p') + v'$. Since $\bar{\theta}$ is an isomorphism, there exists $u \in P^*$ satisfying $p = p' + u$. Together with the assumption that Q is integral, we have $v' = v + \theta(u)$. Use [25, Proposition I.1.1.5(3)] to see that $(p, v) = (p', v')$ in $P \oplus_{P^*} Q^*$. Hence η is an isomorphism. □

Lemma 4.3 *Let $\theta : P \to Q$ be a homomorphism of integral monoids with involution. If $\bar{\theta} : \bar{P} \to \bar{Q}$ is surjective, then the induced map*

$$\mathrm{THR}(\mathbb{S}[Q], P) \to \mathrm{THR}(\mathbb{S}[Q], Q)$$

is an equivalence of $\mathbb{Z}/2$-spectra.

Proof Due to Propositions 2.8 and 2.13, it suffices to show that the induced map

$$(\mathbb{S}[Q] \wedge_{\mathbb{S}[i_\sharp i^* Q]} \mathbb{S}[Q]) \wedge_{(\mathbb{S}[P] \wedge_{\mathbb{S}[i_\sharp i^* P]} \mathbb{S}[P])} (\mathbb{S}[P] \wedge_{\mathbb{S}[(i_\sharp i^* P)^{\mathrm{ex}}]} \mathbb{S}[P]) \to \mathbb{S}[Q] \wedge_{\mathbb{S}[(i_\sharp i^* Q)^{\mathrm{ex}}]} \mathbb{S}[Q]$$

is an equivalence of $\mathbb{Z}/2$-spectra. The left-hand side is equivalent to

$$\mathbb{S}[Q] \wedge_{(\mathbb{S}[i_\sharp i^* Q] \wedge_{\mathbb{S}[i_\sharp i^* P]} \mathbb{S}[(i_\sharp i^* P)^{\mathrm{ex}}])} \mathbb{S}[Q].$$

Hence it suffices to show that the induced map

$$\mathbb{S}[i_\sharp i^* Q] \wedge_{\mathbb{S}[i_\sharp i^* P]} \mathbb{S}[(i_\sharp i^* P)^{\mathrm{ex}}] \to \mathbb{S}[(i_\sharp i^* Q)^{\mathrm{ex}}]$$

is an equivalence of $\mathbb{Z}/2$-spectra. Since P is integral, $u + x = x$ implies $u = 0$ for $u \in P^*$ and $x \in P$. It follows that $i^* P$ is a free $i^* P^*$-set, so the induced map

$$\mathbb{S}[i_\sharp i^* Q^*] \wedge_{\mathbb{S}[i_\sharp i^* P^*]} \mathbb{S}[i_\sharp i^* P] \to \mathbb{S}[i_\sharp i^* Q]$$

is an equivalence of $\mathbb{Z}/2$-spectra. Hence it suffices to show that the induced map

$$\mathbb{S}[i_\sharp i^* Q^*] \wedge_{\mathbb{S}[i_\sharp i^* P^*]} \mathbb{S}[(i_\sharp i^* P)^{\mathrm{ex}}] \to \mathbb{S}[(i_\sharp i^* Q)^{\mathrm{ex}}] \tag{4-2}$$

is an equivalence of $\mathbb{Z}/2$-spectra.

Consider the localization P_F with respect to the face $F := \theta^{-1}(Q^*)$ of P. By [25, Theorem I.4.5.7, Proposition I.4.6.3(3), Remark I.4.6.6], $i^* P_F$ is a filtered colimit of free $i^* P$-sets. This implies that $\mathbb{S}[i_\sharp i^* P_F]$ is a filtered colimit of free $\mathbb{S}[i_\sharp i^* P]$-modules. It follows that the induced map

$$\mathbb{S}[i_\sharp i^* P_F] \wedge_{\mathbb{S}[i_\sharp i^* P]} \mathbb{S}[(i_\sharp i^* P)^{\mathrm{ex}}] \to \mathbb{S}[(i_\sharp i^* P_F)^{\mathrm{ex}}]$$

is an equivalence. Replace $P \to Q$ with $P_F \to Q$ to reduce to the case when $\bar{\theta}$ is an isomorphism. Then we have $Q \simeq P \oplus_{P^*} Q^*$ by Lemma 4.2.

We have the induced cocartesian square

$$\begin{array}{ccc} i^*P^* \oplus i^*P^* & \xrightarrow{\alpha} & i^*P \oplus i^*P^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ i^*Q^* \oplus i^*Q^* & \longrightarrow & i^*Q \oplus i^*Q^{\mathrm{gp}} \end{array} \tag{4-3}$$

where α sends (p, p') to $(p+p', p')$. Observe that $i^*P \oplus i^*P^{\mathrm{gp}}$ is a free $i^*P^* \oplus i^*P^*$-set. Apply $\mathbb{S}[-]$ to this square to see that the induced map

$$\mathbb{S}[i^*Q \oplus i^*Q] \wedge_{\mathbb{S}[i^*P \oplus i^*P]} \mathbb{S}[i^*P \oplus i^*P^{\mathrm{gp}}] \to \mathbb{S}[i^*Q \oplus i^*Q^{\mathrm{gp}}]$$

is an equivalence of spectra, i.e., (4-2) becomes an equivalence after applying i^*. Hence it remains to show that (4-2) becomes an equivalence after applying $\Phi^{\mathbb{Z}/2}$.

We claim that θ is exact, i.e., the induced square

$$\begin{array}{ccc} P & \xrightarrow{\eta} & P^{\mathrm{gp}} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ Q & \xrightarrow{\eta'} & Q^{\mathrm{gp}} \end{array}$$

is cartesian. Assume that $\eta'(q) = \theta(p')$ for some $q \in Q$ and $p' \in P^{\mathrm{gp}}$. Using [25, Proposition I.1.1.5(3)] for $Q \simeq P \oplus_{P^*} Q^*$, we can write $q = (p, v)$ for some $p \in P$ and $v \in Q^*$. Since the group completion functor is a left adjoint, we have an isomorphism $Q^{\mathrm{gp}} \simeq P^{\mathrm{gp}} \oplus_{P^*} Q^*$. Using [25, Proposition I.1.1.5(3)] for this, we have $(\eta(p), v) = (p', 0)$ in Q'^{gp}, and hence we have $v = \theta(u)$ for some $u \in P^*$. This implies that $q = \theta(p+u)$ and $p' = \eta(p+u)$, so θ is exact.

Observe that the $\mathbb{Z}/2$-fixed point monoids of $i_\sharp i^*P^*$ and $i_\sharp i^*Q^*$ are isomorphic to i^*P^* and i^*Q^*. Let w denote the involutions on P and Q, and let M and N be the $\mathbb{Z}/2$-fixed point monoids of $(i_\sharp i^*P)^{\mathrm{ex}}$ and $(i_\sharp i^*Q)^{\mathrm{ex}}$. Observe that N is a submonoid of $Q \oplus Q^{\mathrm{gp}}$ consisting of (q, y) such that $\eta'(q) = y + w(y)$. We have a similar description for M too. Using $Q^{\mathrm{gp}} \simeq P^{\mathrm{gp}} \oplus_{P^*} Q^*$, we can find $x \in P^{\mathrm{gp}}$ and $v \in Q^*$ such that $y = \theta(x) + v$. Since θ is exact, there exists a unique $p \in P$ such that $\eta(p) = x + w(x)$ and $\theta(p) = q - v - w(v)$. This implies that the induced homomorphism $M \oplus_{i^*P^*} i^*Q^* \to N$ is surjective.

On the other hand, if $(p, x) \in M$ and $v \in Q^*$ satisfies $(\theta(p), \theta(x)) = (v + w(v), v)$ in N, then we have $p \in \theta^{-1}(Q^*)$. By [25, Proposition I.1.1.5(3)] for $Q \cong P \oplus_{P^*} Q^*$, we have $\theta^{-1}(Q^*) = P^*$, so we have $p \in P^*$. This implies that the induced homomorphism $M \oplus_{i^*P^*} i^*Q^* \to N$ is injective and hence an isomorphism. Since M is a free i^*P^*-set, we deduce that the induced map

$$\mathbb{S}[i^*Q] \wedge_{\mathbb{S}[i^*P]} \mathbb{S}[M] \to \mathbb{S}[N]$$

is an equivalence of spectra, i.e., (4-2) becomes an equivalence after applying $\Phi^{\mathbb{Z}/2}$. □

A morphism of log schemes with involutions $f : Y \to X$ is a *strict isovariant étale cover* if f is strict and its underlying morphism of schemes with involutions $\underline{f} : \underline{Y} \to \underline{X}$ is an isovariant étale cover. The *strict isovariant étale topology* is the topology generated by strict isovariant étale covers. Let sisoét be the shorthand for this topology.

Proposition 4.4 *Let X be a separated integral log scheme with involution. Then there exists a strict isovariant étale covering $\{U_i \to X\}_{i\in I}$ such that each $\underline{U_i}$ is an affine scheme with involution.*

Proof Let w be the involution on X. Choose an open neighborhood U_x of x in X such that $\underline{U_x}$ is an affine scheme. If $x = w(x)$, then the underlying scheme of $U_x \cap w(U_x)$ is an affine scheme with involution since X is separated. If $x \neq w(x)$, then the underlying scheme of $U_x \amalg w(U_x)$ is an affine scheme with involution. We finish the proof using [14, Corollary 2.19]. □

Let lSch denote the category of separated integral log schemes. Following Definition 2.5, we obtain the category $\mathrm{lSch}_{\mathbb{Z}/2}$, and we have the forgetful functor $i^* : \mathrm{lSch}_{\mathbb{Z}/2} \to \mathrm{lSch}$ with a left adjoint $i_\sharp$.

Definition 4.5 Let X be a separated integral log scheme with involution. Consider the presheaf $\mathrm{THR}|_X$ of $\mathbb{Z}/2$-spectra given by

$$\mathrm{THR}|_X(U) := \mathrm{THR}(\Gamma(U, \mathcal{O}_U), \Gamma(U, \mathcal{M}_U))$$

for every strict isovariant étale morphism $U \to X$. The *real topological Hochschild homology of X* is

$$\mathrm{THR}(X) := (L_{\mathrm{sisoét}} \mathrm{THR}|_X)(X) \in \mathrm{Sp}^{\mathbb{Z}/2},$$

where $L_{\mathrm{sisoét}} : \mathrm{PSh}(\mathrm{lSch}_{\mathbb{Z}/2}, \mathrm{Sp}^{\mathbb{Z}/2}) \to \mathrm{Sh}_{\mathrm{sisoét}}(\mathrm{lSch}_{\mathbb{Z}/2}, \mathrm{Sp}^{\mathbb{Z}/2})$ denotes the strict isovariant étale sheafification functor; see [21, Lemma 1.3.4.3] for sheafification functors.

Observe that THR is a strict isovariant étale sheaf by definition.

Theorem 4.6 *Let (A, P) be an integral log ring with involution. Then the induced map of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(A, P) \to \mathrm{THR}(\mathrm{Spec}(A, P))$$

is an equivalence.

Proof Consider the presheaf $\mathrm{THR}|_{(A,P)}$ of $\mathbb{Z}/2$-spectra given by

$$\mathrm{THR}|_{(A,P)}(B) := \mathrm{THR}(B, P)$$

for every A-algebra B with involution such that $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is isovariant étale. We have the induced map of presheaves

$$\mathrm{THR}|_{(A,P)} \to \mathrm{THR}|_{\mathrm{Spec}(A,P)}.$$

Observe that the left-hand side is an isovariant étale sheaf by Proposition 4.1. It suffices to show that this map of presheaves becomes an equivalence after sheafification since taking the global sections produces the desired equivalence.

For this, it suffices to show that the map of stalks is an equivalence. For every sheaf $\mathcal{F}$ and a point x of X, the stalk $\mathcal{F}_x$ is given by the filtered colimit $\mathrm{colim}_{U \ni x} \mathcal{F}(U)$. Hence by Propositions 3.8 and 4.4, we reduce to the case when A is a local ring with involution. In this case, we need to show that the induced map

$$\mathrm{THR}(A,P) \to \mathrm{THR}(A,P^a)$$

is an equivalence, where P^a is the logification of P, which is given by $P^a := P \oplus_{\theta^{-1}(A^*)} A^*$ if $\theta : P \to A$ is the structure map. This is a consequence of Lemma 4.3 for $P \to P^a$. □

In the next results, we explain how THR is related with THH under the functors i^* and i_*. We review the definition of THH of schemes in [7, Definition 8.3.7] as follows. We consider the presheaf of spectra $\mathrm{THH}|_X$ on $X_{\text{ét}}$ given by

$$\mathrm{THH}|_X(U) := \mathrm{THH}(\Gamma(U,\mathcal{O}_U),\Gamma(U,\mathcal{M}_U))$$

for $U \in X_{\text{ét}}$. The topological Hochschild homology of X is

$$\mathrm{THH}(X) := (L_{\text{ét}}\mathrm{THH}|_X)(X) \in \mathrm{Sp},$$

where $L_{\text{ét}}$ denotes the étale sheafification functor. The induced map

$$\mathrm{THH}(A,P) \simeq \mathrm{THH}(\mathrm{Spec}(A,P))$$

is an equivalence of spectra for every integral log ring (A,P) as Theorem 4.6.

Proposition 4.7 *Let X be a separated integral log scheme with involution. Then there is a natural equivalence of spectra*

$$\mathrm{THH}(i^*X) \simeq i^*\mathrm{THR}(X).$$

Proof By Proposition 4.4, we reduce to the case when $X = \mathrm{Spec}(A,P)$ for some log ring (A,P) with involution. Proposition 3.5 finishes the proof. □

Proposition 4.8 *Let X be a separated integral log scheme. Then there is a natural equivalence of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(i_\sharp X) \simeq i_*\mathrm{THH}(X).$$

Proof As above, we reduce to the case when $X = \mathrm{Spec}(A,P)$ for some log ring (A,P). In this case, there is a natural isomorphism

$$i_\sharp \,\mathrm{Spec}(A,P) \simeq \mathrm{Spec}(i_*A,\underline{P}).$$

Proposition 3.7 finishes the proof. □

Corollary 4.9 *Let X be a separated integral log scheme with involution. Then there is a natural equivalence of spectra*

$$\mathrm{THR}(i_\sharp X)^{\mathbb{Z}/2} \simeq \mathrm{THH}(X).$$

Proof Combine Proposition 4.8 with [19, Proposition A.2.7(2)] to conclude. □

5 TCR of log schemes with involutions

Throughout this section, p is a prime number. This section is relying on [27; 28] due to Quigley and Shah, which we review as follows. Keep in mind that our C_p and $\mathbb{Z}/2$ correspond to their μ_p and C_2. See [26, §3] for another review.

A $\mathbb{Z}/2$-∞-*category* (resp. $\mathbb{Z}/2$-∞-*space*) is a presheaf of ∞-categories (resp. spaces) on the orbit category $\mathcal{O}_{\mathbb{Z}/2}$. Recall that $\mathcal{O}_{\mathbb{Z}/2}$ can be described as the diagram

$$w \circlearrowright (\mathbb{Z}/2)/e \xrightarrow{\text{res}} (\mathbb{Z}/2)/(\mathbb{Z}/2).$$

Observe that we can naturally view a $\mathbb{Z}/2$-space as a $\mathbb{Z}/2$-category. A $\mathbb{Z}/2$-functor of $\mathbb{Z}/2$-categories is a morphism of presheaves. For $\mathbb{Z}/2$-categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathrm{Fun}_{\mathbb{Z}/2}(\mathcal{C},\mathcal{D})$ be the ∞-category of $\mathbb{Z}/2$-functors $\mathcal{C}\to\mathcal{D}$.

Consider the $\mathbb{Z}/2$-∞-categories $\underline{\mathrm{Sp}}^{\mathbb{Z}/2}$ and $\underline{\mathrm{NAlg}}^{\mathbb{Z}/2}$ in [26, Example 3.2]. We have the equations $\underline{\mathrm{Sp}}^{\mathbb{Z}/2}((\mathbb{Z}/2)/e)=\mathrm{Sp}$, $\underline{\mathrm{Sp}}^{\mathbb{Z}/2}((\mathbb{Z}/2)/(\mathbb{Z}/2))=\mathrm{Sp}^{\mathbb{Z}/2}$, and $\underline{\mathrm{Sp}}^{\mathbb{Z}/2}(\mathrm{res})=i^*$. We also have a similar description for $\underline{\mathrm{NAlg}}^{\mathbb{Z}/2}$. Let BS^σ and BC_p^σ be the classifying $\mathbb{Z}/2$-spaces of S^σ and C_p^σ, where C_p^σ denotes the $\mathbb{Z}/2$-subspace of S^σ whose underlying space is C_p. We have the ∞-categories

$$(\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} := \mathrm{Fun}_{\mathbb{Z}/2}(BS^\sigma,\underline{\mathrm{Sp}}^{\mathbb{Z}/2}),$$
$$(\mathrm{Sp}^{\mathbb{Z}/2})^{BC_p^\sigma} := \mathrm{Fun}_{\mathbb{Z}/2}(BC_p^\sigma,\underline{\mathrm{Sp}}^{\mathbb{Z}/2}).$$

We similarly have the ∞-categories $(\mathrm{NAlg}^{\mathbb{Z}/2})^{BS^\sigma}$ and $(\mathrm{NAlg}^{\mathbb{Z}/2})^{BC_p^\sigma}$.

For equivariant homotopy theory, we refer to [32, §3] for the model-categorical approach (the group G can be a compact Lie group) and [2, §9] for the ∞-categorical approach (G is only a profinite group). For a compact Lie group G, let Sp^G be the ∞-category of G-spectra, which is the underlying ∞-category of orthogonal G-spectra. According to [28, Theorem A, Remark 1.8], we have the forgetful functors

$$j^*:\mathrm{Sp}^{O(2)}\to(\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma},$$
$$j^*:\mathrm{Sp}^{D_{2p}}\to(\mathrm{Sp}^{\mathbb{Z}/2})^{BC_p^\sigma},$$

which admit right adjoints j_*. We have the fixed point functors

$$(-)^{C_p}:\mathrm{Sp}^{O(2)}\to\mathrm{Sp}^{O(2)/C_p}\simeq\mathrm{Sp}^{O(2)},$$
$$(-)^{C_p}:\mathrm{Sp}^{D_{2p}}\to\mathrm{Sp}^{\mathbb{Z}/2}.$$

We have the norm functor

$$N_{\mathbb{Z}/2}^{D_{2p}}:\mathrm{Sp}^{\mathbb{Z}/2}\to\mathrm{Sp}^{D_{2p}}.$$

We have the geometric fixed point functors

$$\Phi^{C_p}:\mathrm{Sp}^{O(2)}\to\mathrm{Sp}^{O(2)/C_p}\simeq\mathrm{Sp}^{O(2)},$$
$$\Phi^{C_p}:\mathrm{Sp}^{D_{2p}}\to\mathrm{Sp}^{\mathbb{Z}/2}.$$

Recall from [28, Definition 1.6, Remark 1.8] that the *parametrized Tate constructions* are

$$(-)^{tC_p^\sigma} := j^*\Phi^{C_p} j_* : (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \to (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma},$$
$$(-)^{tC_p^\sigma} := \Phi^{C_p} j_* : (\mathrm{Sp}^{\mathbb{Z}/2})^{BC_p^\sigma} \to \mathrm{Sp}^{\mathbb{Z}/2}.$$

For $X \in \mathrm{Sp}^{\mathbb{Z}/2}$, we often use the notation $X^{\otimes C_p^\sigma} := j^* N_{\mathbb{Z}/2}^{D_{2p}} X$. We have the parametrized Tate diagonal given by the composite natural map

$$\Delta : X \xrightarrow{\simeq} \Phi^{C_p} N_{\mathbb{Z}/2}^{D_{2p}} X \to \Phi^{C_p} j_* j^* N_{\mathbb{Z}/2}^{D_{2p}} X = (X^{\otimes C_p^\sigma})^{tC_p^\sigma}.$$

A *real cyclotomic spectrum* X is an object of $(\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma}$ equipped with maps

$$\varphi_p : X \to X^{tC_p^\sigma}$$

in $(\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma}$ for all primes p. In [27, Definition 1.20], the ∞-category of real cyclotomic spectra is defined to be the lax equalizer

$$\mathbb{R}\mathrm{CycSp} := \mathrm{LEq}\Big((\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \underset{\prod_p (-)^{tC_p^\sigma}}{\overset{\mathrm{id}}{\rightrightarrows}} \textstyle\prod_p (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma}\Big).$$

See [27, Remark 1.21] for the $\mathbb{Z}/2$-∞-category of real cyclotomic spectra $\underline{\mathbb{R}\mathrm{CycSp}}$. We refer to [27, Remark 4.3] for the functor

$$\mathrm{TCR} : \mathbb{R}\mathrm{CycSp} \to \mathrm{Sp}^{\mathbb{Z}/2}.$$

By [26, Proposition 3.7], the forgetful functor

$$q^* : (\mathrm{NAlg}^{\mathbb{Z}/2})^{BS^\sigma} \to \mathrm{NAlg}^{\mathbb{Z}/2}$$

admits a left adjoint $q_\otimes$. For $A \in \mathrm{NAlg}^{\mathbb{Z}/2}$, following [27, §5] (see also [26, Definition 3.11 and Proposition 3.12], we set

$$\mathrm{THR}(A) := q_\otimes A.$$

Recall from [27, §5] the following facts:

(1) There exists a unique map

$$\varphi_p : \mathrm{THR}(A) \to \mathrm{THR}(A)^{tC_p^\sigma}$$

such that the induced square

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{THR}(A) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\varphi_p} \\ (A^{\otimes C_p^\sigma})^{tC_p^\sigma} & \longrightarrow & \mathrm{THR}(A)^{tC_p^\sigma} \end{array} \tag{5-1}$$

commutes, where the lower horizontal map is induced by the inclusion $C_p \to S^1$.

(2) φ_p is a map in $\mathrm{NAlg}^{\mathbb{Z}/2}$.

(3) As a consequence of (2), $\mathrm{THR}(A)$ is an object of $\mathrm{NAlg}(\mathbb{R}\mathrm{CycSp})$, where $\mathrm{NAlg}(\mathbb{R}\mathrm{CycSp})$ denotes the ∞-category of normed algebras in $\underline{\mathbb{R}\mathrm{CycSp}}$ defined as in [2, Definition 9.14]. We write down the definition as follows. For $X \in \mathrm{FinGpd}$, let Fin_X denote the category of finite coverings of X in FinGpd. We have a functor

$$\underline{\mathbb{R}\mathrm{CycSp}}^{\otimes} : \mathrm{Span}(\mathrm{Fin}_{B(\mathbb{Z}/2)}) \to \mathrm{Cat}_\infty$$

in [27, Point 3 after Remark 4.3]. A *normed algebra in* $\underline{\mathbb{R}\mathrm{CycSp}}$ is a section of $\underline{\mathbb{R}\mathrm{CycSp}}^{\otimes}$ over $\mathrm{Span}(\mathrm{Fin}_{B(\mathbb{Z}/2)})$ that is cocartesian over the backward morphisms in $\mathrm{Span}(\mathrm{Fin}_{B(\mathbb{Z}/2)})$. Recall that a backward morphism in a span ∞-category is a morphism of the form $X \leftarrow Y \xrightarrow{\mathrm{id}} Y$; see [2, Appendix C].

Proposition 5.1 *Let P be a commutative monoid with involution. Then for every prime p, there exists a natural equivalence of $\mathbb{Z}/2$-spaces*

$$(\mathrm{B}^{\mathrm{di}} P)^{C_p} \simeq \mathrm{B}^{\mathrm{di}} P. \tag{5-2}$$

Proof Let sd_p be the p-fold subdivision functor in [8]. For every integer $q \geq 0$, the set of q-simplices in $(\mathrm{sd}_p \mathrm{N}^{\mathrm{di}} P)^{C_p}$ is the set

$$\{(x_0, \dots, x_q, \dots, x_0, \dots, x_q) : x_0, \dots, x_q \in P\},$$

which is isomorphic to the set of q-simplices in $\mathrm{N}^{\mathrm{di}} P$. This isomorphism is also compatible with the involutions, so we obtain the desired equivalence. □

Proposition 5.2 *Let P be a commutative monoid with involution. Then for every prime p, there exists a natural equivalence of $\mathbb{Z}/2$-spaces*

$$(\mathrm{B}^{\mathrm{drep}} P)^{C_p} \simeq \mathrm{B}^{\mathrm{drep}} P. \tag{5-3}$$

Proof The isomorphism $\mathrm{N}^{\mathrm{di}} P^{\mathrm{gp}} \cong (\mathrm{sd}_p \mathrm{N}^{\mathrm{di}} P^{\mathrm{gp}})^{C_p}$ obtained by the proof of Proposition 5.1 can be restricted to an isomorphism $\mathrm{N}^{\mathrm{drep}} P \cong (\mathrm{sd}_p \mathrm{N}^{\mathrm{drep}} P)^{C_p}$. This produces the desired equivalence. □

Proposition 5.3 *Let P be a commutative monoid with involution. Then the Frobenius $\varphi_p : \mathrm{THR}(\mathbb{S}[P]) \to \mathrm{THR}(\mathbb{S}[P])^{tC_p^\sigma}$ is equivalent to the composite*

$$\mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \xrightarrow{\simeq} \mathbb{S}[(\mathrm{B}^{\mathrm{di}} P)^{C_p}] \xrightarrow{\simeq} \Phi^{C_p} \mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \to \mathbb{S}[\mathrm{B}^{\mathrm{di}} P]^{tC_p^\sigma}, \tag{5-4}$$

where the first map is obtained by (5-2).

Proof Consider the commutative square

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{B}^{\mathrm{di}} P \\ \simeq\big\uparrow & & \big\uparrow\simeq \\ (P^{\oplus C_p})^{C_p} & \longrightarrow & (\mathrm{B}^{\mathrm{di}} P)^{C_p} \end{array}$$

where the right vertical arrow is obtained by (5-2), and the lower horizontal arrow is induced by the inclusion $C_p \to S^1$. After taking $\mathbb{S}[-]$, we obtain the commutative square

$$\begin{array}{ccc} \mathbb{S}[P] & \longrightarrow & \mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \\ \simeq\uparrow & & \uparrow\simeq \\ \Phi^{C_p} N_{\mathbb{Z}/2}^{D_{2p}} \mathbb{S}[P] & \longrightarrow & \Phi^{C_p} \mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \end{array}$$

Using the natural transformation $\Phi^{C_p} \to \Phi^{C_p} j_* j^*$, we obtain the commutative square

$$\begin{array}{ccc} \mathbb{S}[P] & \longrightarrow & \mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \\ \downarrow & & \downarrow \\ (\mathbb{S}[P]^{\otimes C_p^\sigma})^{tC_p^\sigma} & \longrightarrow & \mathbb{S}[\mathrm{B}^{\mathrm{di}} P]^{tC_p^\sigma} \end{array}$$

Compare this with (5-1) to conclude. □

Definition 5.4 Let P be a commutative monoid with involution. We have the Frobenius $\varphi : \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P] \to \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P]^{tC_p^\sigma}$ given by the composite

$$\mathbb{S}[\mathrm{B}^{\mathrm{drep}} P] \xrightarrow{\simeq} \mathbb{S}[(\mathrm{B}^{\mathrm{drep}} P)^{C_p}] \xrightarrow{\simeq} \Phi^{C_p} \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P] \to \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P]^{tC_p^\sigma}, \tag{5-5}$$

where the first map is obtained by (5-3). Since the canonical map $\mathrm{B}^{\mathrm{di}} P \to \mathrm{B}^{\mathrm{drep}} P$ is a map of commutative monoids in the category of topological $O(2)$-spaces, we obtain a natural map

$$\mathbb{S}[\mathrm{B}^{\mathrm{di}} P] \to \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P] \tag{5-6}$$

in $(\mathrm{NAlg}^{\mathbb{Z}/2})^{BS^\sigma}$. Compare (5-4) and (5-5) to see that (5-6) is compatible with φ_p. Hence we can promote (5-6) to a map in $\mathrm{NAlg}(\mathbb{R}\mathrm{CycSp})$.

Let (A, P) be a log ring with involution. We take the coproduct

$$\mathrm{THR}(A, P) := \mathrm{THR}(A) \wedge_{\mathbb{S}[\mathrm{B}^{\mathrm{di}} P]} \mathbb{S}[\mathrm{B}^{\mathrm{drep}} P]$$

in $\mathrm{NAlg}(\mathbb{R}\mathrm{CycSp})$. The *real topological cyclic homology of* (A, P) is

$$\mathrm{TCR}(A, P) := \mathrm{TCR}(\mathrm{THR}(A, P)).$$

Proposition 5.5 *The forgetful functor*

$$\mathbb{R}\mathrm{CycSp} \to \mathrm{Sp}^{\mathbb{Z}/2}$$

is conservative, exact, symmetric monoidal, and preserves colimits and finite limits.

Proof We refer to [27, Remark 4.3]. □

Definition 5.6 A morphism of log schemes with involutions is a *strict equivariant Nisnevich cover* if it is strict and its underlying morphism of schemes with involutions is an equivariant Nisnevich cover in the sense of Voevodsky [9, §3.1]. See also [14, §2].

The *strict equivariant Nisnevich topology* is the topology generated by strict equivariant Nisnevich covers. Let seNis be the shorthand for this topology. Observe that the strict equivariant Nisnevich topology is coarser than the strict isovariant étale topology.

Proposition 5.7 *Let (A, P) be a finite-dimensional noetherian log ring with involution. Then the presheaf* $\mathrm{THR}(-, P)$ *on the opposite category of A-algebras of finite type with involution is an equivariant Nisnevich sheaf of real cyclotomic spectra.*

Proof For an equivariant Nisnevich distinguished square Q (see [14, §2.1]) consisting of A-algebras of finite type with involution, we need to show that $\mathrm{THR}(Q, P)$ is a cocartesian square of real cyclotomic spectra. By Proposition 5.5, it suffices to show that $\mathrm{THR}(Q, P)$ is a cocartesian square of $\mathbb{Z}/2$-spectra. This is a consequence of Proposition 4.1. □

Remark 5.8 We do not know whether $\mathrm{THR}(-, P)$ in Proposition 5.7 is an isovariant étale sheaf of real cyclotomic spectra. An affirmative answer would remove the finite-dimensional noetherian assumption in Theorem 5.12 below.

Proposition 5.9 *Let X be a finite-dimensional noetherian separated log scheme with involution. Then there exists an equivariant Nisnevich covering $\{U_i \to X\}_{i \in I}$ such that each $\underline{U_i}$ is an affine scheme with involution.*

Proof This is a consequence of [14, Lemma 2.20]. □

Proposition 5.10 *The functor* THR *from the category of log rings with involutions to $\mathbb{Z}/2$-spectra preserves filtered colimits.*

Proof The is a consequence of Propositions 3.8 and 5.5. □

Definition 5.11 Let X be a finite-dimensional noetherian separated integral log scheme with involution. Consider the presheaf $\mathrm{THR}|_X$ of real cyclotomic spectra given by

$$\mathrm{THR}|_X(U) := \mathrm{THR}(\Gamma(U, \mathcal{O}_U), \Gamma(U, \mathcal{M}_U))$$

for every strict isovariant étale morphism $U \to X$. Consider

$$\mathrm{THR}(X) := (L_{\mathrm{seNis}} \mathrm{THR}|_X)(X) \in \mathbb{R}\mathrm{CycSp},$$

where L_{seNis} denotes the strict equivariant Nisnevich sheafification functor.

Observe that THR is a strict equivariant Nisnevich sheaf of real cyclotomic spectra by definition. Furthermore, if we forget the real cyclotomic structure on $\mathrm{THR}(X)$, then we recover $\mathrm{THR}(X)$ in Definition 4.5 by Theorem 5.12 below.

The *real topological cyclic homology of X* is

$$\mathrm{TCR}(X) := \mathrm{TCR}(\mathrm{THR}(X)).$$

Theorem 5.12 *Let (A, P) be a finite-dimensional noetherian integral log ring with involution. Then the induced map*

$$\mathrm{THR}(A, P) \to \mathrm{THR}(\mathrm{Spec}(A, P))$$

is an equivalence of real cyclotomic spectra. In particular, the induced map

$$\mathrm{TCR}(A, P) \to \mathrm{TCR}(\mathrm{Spec}(A, P))$$

is an equivalence of $\mathbb{Z}/2$-spectra.

Proof Argue as in the proof of Theorem 4.6, but use instead Propositions 5.7, 5.9, and 5.10 to reduce to the case when A is a local ring. In this case, we need to show that the induced map

$$\mathrm{THR}(A, P) \to \mathrm{THR}(A, P^a)$$

is an equivalence of real cyclotomic spectra, where $P^a := P \oplus_{\theta^{-1}(A^*)} A^*$ if $\theta : P \to A$ is the structure map. This is a consequence of Theorem 4.6 and Proposition 5.5. □

The purpose of the remaining part of this section is to show Proposition 5.30, which is needed for Proposition 6.17. For this, we will check that various squares commute.

The map of $\mathbb{Z}/2$-spaces $q : * \to BS^\sigma$ induces the forgetful functor

$$q^* : (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \to \mathrm{Sp}^{\mathbb{Z}/2},$$

which is conservative by [26, Proposition 3.8] and admits a left adjoint $q_\sharp$ and right adjoint q_* by [26, Proposition 3.7].

Proposition 5.13 *There are natural equivalences*

(5-7) $$q^* q_\sharp \simeq \Sigma^\infty S^\sigma_+ \wedge (-),$$

(5-8) $$q^* q_* \simeq \Sigma^{-\sigma} \Sigma^\infty S^\sigma_+ \wedge (-).$$

Proof For (5-7), consider the cartesian square

$$\begin{array}{ccc} S^\sigma & \xrightarrow{r} & * \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} \\ * & \xrightarrow{q} & BS^\sigma \end{array}$$

where $r : S^\sigma \to *$ is the unique map. By [28, Lemma 4.3], we have a natural equivalence $q^* q_\sharp \simeq r_\sharp r^*$. To obtain (5-7), observe that the projection formula [28, Lemma 5.44] yields a natural equivalence $r_\sharp r^* \simeq \Sigma^\infty S^\sigma_+ \wedge (-)$.

We have $\mathrm{fib}(\mathrm{id} \to q^* q_\sharp) \simeq \Sigma^\sigma$. By taking right adjoints, we have $\mathrm{cofib}(q^* q_* \to \mathrm{id}) \simeq \Sigma^{-\sigma}$, which yields (5-8). □

Remark 5.14 We similarly have the conservative forgetful functor

$$q^* : \mathrm{Sp}^{BS^1} \to \mathrm{Sp}$$

with a left adjoint $q_\sharp$ and right adjoint q_*. There are also natural equivalences

$$q^* q_\sharp \simeq \Sigma^\infty S^1_+ \wedge (-),$$
$$q^* q_* \simeq \Sigma^{-1}\Sigma^\infty S^1_+ \wedge (-).$$

Proposition 5.15 *There are induced commutative squares*

$$\begin{array}{ccc} \mathrm{Sp}^{\mathbb{Z}/2} & \xrightarrow{q_\sharp} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathrm{Sp} & \xrightarrow{q_\sharp} & \mathrm{Sp}^{BS^1} \end{array} \qquad \begin{array}{ccc} \mathrm{Sp}^{\mathbb{Z}/2} & \xrightarrow{q_*} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathrm{Sp} & \xrightarrow{q_*} & \mathrm{Sp}^{BS^1} \end{array}$$

Proof We focus on the first square since the proofs are similar. Since $q^* : \mathrm{Sp}^{BS^1} \to \mathrm{Sp}$ is conservative, it suffices to show that the composite natural transformation

$$q^* q_\sharp i^* \to q^* i^* q_\sharp \xrightarrow{\simeq} i^* q^* q_\sharp$$

is an isomorphism. Using Proposition 5.13 and Remark 5.14, this natural transformation can be identified with the natural transformation

$$\Sigma^\infty S^1_+ \wedge i^* \to i^*(\Sigma^\infty S^\sigma_+ \wedge (-)).$$

This is an equivalence since $i^*\Sigma^\infty S^\sigma \simeq \Sigma^\infty S^1$. □

Let $v : BS^\sigma \to B(S^\sigma/C_p^\sigma) \simeq BS^\sigma$ be the functor induced by the quotient map $S^\sigma \to S^\sigma/C_p^\sigma$. We have the induced functor

$$v^* : (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \to (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma},$$

whose left adjoint is $(-)_{hC_p^\sigma}$ and right adjoint is $(-)^{hC_p^\sigma}$, which correspond to $(-)_{h_{C_2}S^1}$ and $(-)^{h_{C_2}S^1}$ used in [28, Example 5.57]. Similarly, we have the induced functor

$$v^* : \mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{BS^1},$$

whose left adjoint is $(-)_{hC_p}$ and right adjoint is $(-)^{hC_p}$, which are used in [24, Theorem 1.3].

Proposition 5.16 *There are induced commutative squares*

$$\begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{v^*} & \mathrm{Sp}^{BS^1} \\ {\scriptstyle i_\sharp}\downarrow & & \downarrow{\scriptstyle i_\sharp} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{v^*} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array} \qquad \begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{v^*} & \mathrm{Sp}^{BS^1} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{v^*} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array}$$

Proof We focus on the first square since the proofs are similar. We have the induced commutative diagram

$$\begin{array}{ccccc} i_\sharp q^* v^* & \longrightarrow & q^* i_\sharp v^* & \longrightarrow & q^* v^* i_\sharp \\ \simeq \downarrow & & & & \downarrow \simeq \\ i_\sharp q^* & & \longrightarrow & & q^* i_\sharp \end{array}$$

The vertical arrows are equivalences since the composite $* \xrightarrow{q} BS^\sigma \xrightarrow{v} BS^\sigma$ agrees with q and the same claim holds for BS^1 too. The lower horizontal and upper right horizontal arrows are equivalences by Proposition 5.15. Hence the upper left horizontal arrow is an equivalence. To conclude, use the fact that $q^* : \mathrm{Sp}^{BS^1} \to \mathrm{Sp}$ is conservative. □

Let $u : BS^\sigma \to *$ be the unique map. We have the induced functor $u^* : \mathrm{Sp}^{\mathbb{Z}/2} \to (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma}$ whose left adjoint is $(-)_{hS^\sigma}$ and right adjoint is $(-)^{hS^1}$. Similarly, we have the induced functor $u^* : \mathrm{Sp} \to \mathrm{Sp}^{BS^1}$ whose left adjoint is $(-)_{hS^1}$ and right adjoint is $(-)^{hS^1}$.

Proposition 5.17 *There are induced commutative squares*

$$\begin{array}{ccc} \mathrm{Sp} & \xrightarrow{u^*} & \mathrm{Sp}^{BS^1} \\ i_\sharp \downarrow & & \downarrow i_\sharp \\ \mathrm{Sp}^{\mathbb{Z}/2} & \xrightarrow{u^*} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array} \qquad \begin{array}{ccc} \mathrm{Sp} & \xrightarrow{u^*} & \mathrm{Sp}^{BS^1} \\ i_* \downarrow & & \downarrow i_* \\ \mathrm{Sp}^{\mathbb{Z}/2} & \xrightarrow{u^*} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array}$$

Proof Argue as in Proposition 5.16. □

Proposition 5.18 *There are induced commutative squares*

$$\begin{array}{ccc} (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)^{tC_p^\sigma}} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \\ i^* \downarrow & & \downarrow i^* \\ \mathrm{Sp}^{BS^1} & \xrightarrow{(-)^{tC_p}} & \mathrm{Sp}^{BS^1} \end{array} \qquad \begin{array}{ccc} (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)^{tS^\sigma}} & \mathrm{Sp}^{\mathbb{Z}/2} \\ i^* \downarrow & & \downarrow i^* \\ \mathrm{Sp}^{BS^1} & \xrightarrow{(-)^{tS^1}} & \mathrm{Sp} \end{array}$$

Proof Consider the appropriate adjoint squares of the squares in Proposition 5.16, and compare the cofiber sequences $(-)_{hC_p} \to (-)^{hC_p} \to (-)^{tC_p}$ and $(-)_{hC_p^\sigma} \to (-)^{hC_p^\sigma} \to (-)^{tC_p^\sigma}$ to obtain the left square. Argue similarly for the right square, but use Proposition 5.17 instead. □

Construction 5.19 The motivic purity transformation [1] can be adapted to the equivariant homotopy theory as follows. Let $f : X \to S$ be a finite covering in FinGpd. Consider the induced diagram

$$\begin{array}{ccccc} X & \xrightarrow{a} & X \times_S X & \xrightarrow{p_2} & X \\ & & p_1 \downarrow & & \downarrow f \\ & & X & \xrightarrow{f} & S \end{array}$$

where a is the diagonal morphism, and p_1 (resp. p_2) is the first (resp. second) projection. We set $\Sigma_f := p_{2\sharp}a_*$. We have the natural transformation

$$f_\sharp \xrightarrow{\mathfrak{p}_f} f_*\Sigma_f : \mathrm{SH}(X) \to \mathrm{SH}(S)$$

given by the composite

$$f_\sharp \xrightarrow{\simeq} f_\sharp p_{1*}a_* \to f_* p_{2\sharp}a_*.$$

Recall that $i : * \to B(\mathbb{Z}/2)$ denotes the unique morphism.

Proposition 5.20 *The functor Σ_i is equivalent to the identity functor, and the natural transformation $\mathfrak{p}_i : i_\sharp \to i_*\Sigma_i$ is an equivalence.*

Proof Let $r : \mathbb{Z}/2 \to *$ be the unique map of finite groupoids, and let $a : * \to \mathbb{Z}/2$ be the inclusion to the first point. The functor $a_* : \mathrm{Sp} \to \mathrm{Sp} \times \mathrm{Sp}$ is $(\mathrm{id}, 0)$, and the functor $r_\sharp : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ is the direct sum functor. Hence $\Sigma_i := r_\sharp a_*$ is equivalent to the identity functor.

The pair of functors $(i^*, \Phi^{\mathbb{Z}/2})$ is conservative. Since $\Phi^{\mathbb{Z}/2} i_\sharp \simeq \Phi^{\mathbb{Z}/2} i_* \simeq 0$ by Proposition A.2.7(3), (5) in [19], it suffices to show that $i^*\mathfrak{p}_i$ is an equivalence. Using Proposition 5.15, it suffices to show that $\mathfrak{p}_r$ is an equivalence. This can be shown directly using the following description in [19, Proposition A.1.5]: for a finite set X with n elements, $\mathrm{SH}(X)$ is equivalent to the product of n copies of Sp. □

Construction 5.21 Consider the composite functor

$$\Sigma_i : \mathrm{Sp}^{BS^1} \xrightarrow{a_*} \mathrm{Sp}^{BS^1} \times \mathrm{Sp}^{BS^1} \xrightarrow{r_\sharp} \mathrm{Sp}^{BS^1}, \tag{5-9}$$

where $a_* := (\mathrm{id}, 0)$, and $r_\sharp$ is the direct sum functor. As in Construction 5.19, we have the natural transformation

$$i_\sharp \xrightarrow{\mathfrak{p}_f} i_*\Sigma_i : (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \to \mathrm{Sp}^{BS^1}. \tag{5-10}$$

Proposition 5.22 *The functor (5-9) is equivalent to the identity functor, and the natural transformation (5-10) is an equivalence.*

Proof The descriptions of a_* and $r_\sharp$ show that (5-9) is equivalent to the identity functor. Since $q^* : \mathrm{Sp}^{BS^1} \to \mathrm{Sp}$ is conservative, the second claim is reduced to Proposition 5.20 by Proposition 5.15. □

Proposition 5.23 *There are induced commutative squares*

$$\begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{(-)_{hC_p}} & \mathrm{Sp}^{BS^1} \\ {\scriptstyle i_\sharp}\downarrow & & \downarrow{\scriptstyle i_\sharp} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)_{hC_p^\sigma}} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array} \qquad \begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{(-)^{hC_p}} & \mathrm{Sp}^{BS^1} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)^{hC_p^\sigma}} & \mathrm{Sp}^{\mathbb{Z}/2} \end{array}$$

$$\begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{(-)_{hS^1}} & \mathrm{Sp} \\ {\scriptstyle i_\sharp}\downarrow & & \downarrow{\scriptstyle i_\sharp} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)_{hS^\sigma}} & \mathrm{Sp}^{\mathbb{Z}/2} \end{array} \qquad \begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{(-)^{hS^1}} & \mathrm{Sp} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)^{hS^\sigma}} & \mathrm{Sp}^{\mathbb{Z}/2} \end{array}$$

Proof The right squares commute since their left adjoint squares commute. Together with Propositions 5.20 and 5.22, we see that the left squares commute too. □

Proposition 5.24 *There are induced commutative squares*

$$\begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{(-)^{tC_p}} & \mathrm{Sp}^{BS^1} \\ {\scriptstyle i_\sharp}\downarrow & & \downarrow{\scriptstyle i_\sharp} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)^{tC_p^\sigma}} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array} \qquad \begin{array}{ccc} \mathrm{Sp}^{BS^1} & \xrightarrow{(-)^{tC_p}} & \mathrm{Sp}^{BS^1} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \xrightarrow{(-)^{tC_p^\sigma}} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array}$$

Proof Argue as in Proposition 5.18, but use Proposition 5.23 instead. □

Proposition 5.25 *There is an adjunction*

$$i^* : \mathbb{R}\mathrm{CycSp} \rightleftarrows \mathrm{CycSp} : i_*$$

satisfying the following properties:

(1) *For $Y \in \mathbb{R}\mathrm{CycSp}$, the Frobenius $i^*Y \to (i^*Y)^{tC_p}$ is identified with $i^*\varphi$ if $\varphi : Y \to Y^{tC_p^\sigma}$ is the Frobenius.*

(2) *For $X \in \mathrm{CycSp}$, the Frobenius $i_*X \to (i_*X)^{tC_p^\sigma}$ is identified with $i_*\varphi$ if $\varphi : X \to X^{tC_p}$ is the Frobenius.*

(3) *i^* is symmetric monoidal.*

Proof The functor $i^* : \mathbb{R}\mathrm{CycSp} \to \mathrm{CycSp}$ is obtained by taking lax equalizers to the rows of the diagram

$$\begin{array}{ccc} (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \overset{\mathrm{id}}{\underset{\prod_p (-)^{tC_p^\sigma}}{\rightrightarrows}} & \prod_p (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathrm{Sp}^{BS^1} & \overset{\mathrm{id}}{\underset{\prod_p (-)^{tC_p}}{\rightrightarrows}} & \prod_p \mathrm{Sp}^{BS^1} \end{array} \tag{5-11}$$

whose two squares commute by Proposition 5.18. The two squares in the induced diagram

$$\begin{array}{ccc} \mathrm{Sp}^{BS^1} & \overset{\mathrm{id}}{\underset{\prod_p (-)^{tC_p}}{\rightrightarrows}} & \prod_p \mathrm{Sp}^{BS^1} \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} & \overset{\mathrm{id}}{\underset{\prod_p (-)^{tC_p^\sigma}}{\rightrightarrows}} & \prod_p (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array} \tag{5-12}$$

commute by Proposition 5.24. Hence we see that a right adjoint i_* is obtained by taking lax equalizers to the rows of (5-12) since we can check the counit-unit identities pointwise. This implies the claims (1) and (2).

Recall from [24, Construction IV.2.1(ii)] the following fact: Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be symmetric monoidal ∞-categories. If $F : \mathcal{C} \to \mathcal{D}$ (resp. $G : \mathcal{C} \to \mathcal{D}$) is a symmetric (resp. lax symmetric) monoidal functor, then having a symmetric monoidal functor $\mathcal{E} \to \mathrm{LEq}(F, G)$ is equivalent to having a symmetric monoidal functor $H : \mathcal{E} \to \mathcal{F}$ and a lax symmetric monoidal transformation $F \circ H \to G \circ H$.

The canonical functor $\mathbb{R}\mathrm{CycSp} \to (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma}$ is symmetric monoidal by the above paragraph, and one can show that the forgetful functor $i^*: (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \to \mathrm{Sp}^{BS^1}$ is symmetric monoidal using the descriptions of the symmetric monoidal structures on Sp^{BS^1} and $(\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma}$ obtained by [24, Construction IV.2.1(1); 28, Example 5.13]. Hence the composite functor $\mathbb{R}\mathrm{CycSp} \to \mathrm{Sp}^{BS^1}$ is symmetric monoidal too. Furthermore, we have the lax symmetric natural transformation of the two composite functors

$$\mathbb{R}\mathrm{CycSp} \longrightarrow (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \xrightarrow{i^*} \mathrm{Sp}^{\mathbb{Z}/2} \underset{\prod_p (-)^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} \prod_p \mathrm{Sp}^{BS^1}$$

using (5-11) and the above paragraph. We deduce the claim (3) using the above paragraph again. □

Proposition 5.26 *For $X \in \mathrm{CycSp}$ and $Y \in \mathbb{R}\mathrm{CycSp}$, there are natural equivalences*

$$i^*\mathrm{TCR}(Y) \simeq \mathrm{TC}(i^*Y),$$
$$i_*\mathrm{TC}(X) \simeq \mathrm{TCR}(i_*X).$$

Proof Using the descriptions of $i^*Y \to (i^*Y)^{tC_p}$ and $i_*X \to (i_*X)^{tC_p^\sigma}$ in Proposition 5.25, the claim follows from [24, Corollary 1.5; 27, Theorem C(3)], and Propositions 5.17 and 5.23. □

Proposition 5.27 *Let (A, P) be a log ring with involution. Then there is a natural equivalence of cyclotomic spectra*

$$i^*\mathrm{THR}(A, P) \simeq \mathrm{THH}(i^*A, i^*P)$$

and hence an equivalence of spectra

$$i^*\mathrm{TCR}(A, P) \simeq \mathrm{TCR}(i^*A, i^*P).$$

Proof We have a natural equivalence of spectra $i^*\mathrm{THR}(A) \simeq \mathrm{THH}(i^*A)$ by [19, Proposition 3.4.7]. Apply i^* to (5-1), and compare this with the square in [24, p. 342] to show that $i^*\mathrm{THR}(A) \xrightarrow{i^*\varphi_p} i^*(\mathrm{THR}(A)^{tC_p^\sigma})$ can be identified with $\mathrm{THH}(A) \xrightarrow{\varphi_p} \mathrm{THH}(A)^{tC_p}$. We also need Proposition 5.18 here. Together with Proposition 5.25(2), we have an equivalence of cyclotomic spectra $i^*\mathrm{THR}(A) \simeq \mathrm{THH}(A)$.

On the other hand, if we apply i^* to (5-4) and (5-5), then we get φ_p for $\mathbb{S}[\mathrm{B}^{\mathrm{cy}}P]$ and $\mathbb{S}[\mathrm{B}^{\mathrm{rep}}P]$ by Proposition 2.15 by Proposition 5.18. Together with Proposition 5.25(2), we have equivalences of cyclotomic spectra $i^*\mathbb{S}[\mathrm{B}^{\mathrm{di}}P] \simeq \mathbb{S}[\mathrm{B}^{\mathrm{cy}}P]$ and $i^*\mathbb{S}[\mathrm{B}^{\mathrm{drep}}P] \simeq \mathbb{S}[\mathrm{B}^{\mathrm{rep}}P]$. Proposition 5.25(3) finishes the proof. □

Proposition 5.28 *There is a commutative square*

$$\begin{array}{ccc} \mathrm{CycSp} & \xrightarrow{i_*} & \mathbb{R}\mathrm{CycSp} \\ \downarrow & & \downarrow \\ \mathrm{Sp} & \xrightarrow{i_*} & \mathrm{Sp}^{\mathbb{Z}/2} \end{array}$$

where the vertical arrows are the forgetful functors.

Proof By Proposition 5.16, it suffices to show that there is a commutative square

$$\begin{array}{ccc} \mathrm{CycSp} & \xrightarrow{i_*} & \mathbb{R}\mathrm{CycSp} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{BS^1} & \xrightarrow{i_*} & (\mathrm{Sp}^{\mathbb{Z}/2})^{BS^\sigma} \end{array}$$

where the vertical arrows are the forgetful functors. This can be obtained using (5-12). □

Proposition 5.29 *Let (A, P) be a log ring. Then there is a natural equivalence of real cyclotomic spectra*

$$\mathrm{THR}(i_*A, \underline{P}) \simeq i_*\mathrm{THH}(A, P).$$

Proof Argue as in Construction 3.6 but use Proposition 5.27 instead to construct a natural map of real cyclotomic spectra $\mathrm{THR}(i_*A, \underline{P}) \to i_*\mathrm{THR}(A, P)$. By Proposition 5.5, it suffices to show that this becomes an equivalence of $\mathbb{Z}/2$-spectra after applying the forgetful functor $\mathbb{R}\mathrm{CycSp} \to \mathrm{Sp}^{\mathbb{Z}/2}$. This is a consequence of Propositions 3.7 and 5.28. □

Proposition 5.30 *Let X be a finite-dimensional noetherian separated integral log scheme. Then there is a natural equivalence of real cyclotomic spectra*

$$\mathrm{THR}(i_\sharp X) \simeq i_*\mathrm{THH}(X)$$

and hence an equivalence of $\mathbb{Z}/2$-spectra

$$\mathrm{TCR}(i_\sharp X) \simeq i_*\mathrm{TC}(X)$$

and an equivalence of spectra

$$\mathrm{TCR}(i_\sharp X)^{\mathbb{Z}/2} \simeq \mathrm{TC}(X).$$

Proof Argue as in Proposition 4.8, but use Propositions 5.9 and 5.29 to obtain the first equivalence. For the remaining equivalences, use Proposition 5.26 and [19, Proposition A.2.7(2)]. □

We will use the notation

$$\mathrm{THR}^{\mathbb{Z}/2}(-) := \mathrm{THR}(-)^{\mathbb{Z}/2}, \quad \mathrm{TCR}^{\mathbb{Z}/2}(-) := \mathrm{TCR}(-)^{\mathbb{Z}/2}.$$

6 Motivic representability

Throughout this section, we fix a finite-dimensional noetherian separated scheme S (with the trivial log structure and involution). The purpose of this section is to represent $\mathrm{THH}^{\mathbb{Z}/2}$ and $\mathrm{TCR}^{\mathbb{Z}/2}$ in $\mathrm{logSH}(S)$ and THR and TCR in a $\mathbb{Z}/2$-equivariant analogue of $\mathrm{logSH}(S)$.

We begin with recalling the ingredients for defining $\mathrm{logSH}(S)$. Let SmlSm/S denote the category of fs log schemes Y of finite type over S such that $Y \to S$ is log smooth and $\underline{Y} \to S$ is smooth. A morphism in SmlSm/S is a *strict Nisnevich cover* if it is strict and its underlying morphism of schemes is a Nisnevich cover. We have the ∞-category of strict Nisnevich sheaves of spectra $\mathrm{Sh}_{\mathrm{sNis}}(\mathrm{SmlSm}/S, \mathrm{Sp})$.

For a smooth scheme X of finite type over S with a strict normal crossing divisor D, let (X, D) denote the fs log scheme with the underlying scheme X and the compactifying log structure associated with the open immersion $X - D \to X$. By [6, Lemma A.5.10], every object of SmlSm/S arises as this form. We often regard $\mathbb{P}^{n-1}$ as a closed complement of $\mathbb{A}^n$ in $\mathbb{P}^n$, and we form $(\mathbb{P}^n, \mathbb{P}^{n-1})$. We set $\square := (\mathbb{P}^1, \infty)$.

Now, the *∞-category of logarithmic motivic S^1-spectra* is defined to be the localization

$$\mathrm{logSH}_{S^1}(S) := (\mathbb{P}^\bullet, \mathbb{P}^{\bullet-1})^{-1}\mathrm{Sh}_{\mathrm{sNis}}(\mathrm{SmlSm}/S, \mathrm{Sp}),$$

where $(\mathbb{P}^\bullet, \mathbb{P}^{\bullet-1})^{-1}$ denotes the class of projections $(\mathbb{P}^n, \mathbb{P}^{n-1}) \times X \to X$ for all $X \in \mathrm{SmlSm}/S$ and integers $n \geq 1$. This is one of the various models of $\mathrm{logSH}_{S^1}(S)$ in [7, §3.4]. The *∞-category of logarithmic motivic spectra* is defined to be the ∞-category of $\mathbb{P}^1$-spectra in $\mathrm{logSH}_{S^1}(S)$, that is,

$$\mathrm{logSH}(S) := \mathrm{Sp}_{\mathbb{P}^1}(\mathrm{logSH}_{S^1}(S)).$$

Remark 6.1 In this section, we often compute $\mathrm{THR}(X \times Y)$ in terms of $\mathrm{THR}(X)$ for noetherian separated fs log schemes X and Y such that there is a Zariski covering $\{\mathrm{Spec}(\mathbb{Z}[M_i], P_i) \to Y\}_{i \in I}$ for some homomorphisms of monoids $P_i \to M_i$. In view of Proposition 3.3, the base X is irrelevant for the computation. We will often argue as if everything takes place on the spectral scheme $\mathrm{Spec}(\mathbb{S})$ instead of X for notational convenience.

Recall that S^σ is S^1 with the involution given by $e^{i\theta} \in S^1 \mapsto e^{-i\theta}$. We have the functor

$$\Sigma^\sigma := \Sigma^\infty S^\sigma \wedge (-) : \mathrm{Sp}^{\mathbb{Z}/2} \to \mathrm{Sp}^{\mathbb{Z}/2}.$$

Let us recall the computation of THR for the projective spaces with the trivial involutions as follows.

Proposition 6.2 *Let X be a noetherian separated fs log scheme. Then there exists a natural equivalence of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(X \times \mathbb{P}^n) \simeq \begin{cases} \mathrm{THR}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_*\mathrm{THH}(X) & \text{if } n \text{ is even,} \\ \mathrm{THR}(X) \oplus \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_*\mathrm{THH}(X) \oplus \Sigma^{n(\sigma-1)}\mathrm{THR}(X) & \text{if } n \text{ is odd.} \end{cases}$$

Proof Following Remark 6.1, we can argue as if everything takes place on $\mathbb{S}$ instead of X. Then the claim is due to [19, Theorem 5.2.6]. □

Our next goal is to show that $\mathrm{THR}^{\mathbb{Z}/2}$ and $\mathrm{TCR}^{\mathbb{Z}/2}$ are representable in $\mathrm{logSH}_{S^1}(S)$. For this, we need the notion of cubes [22, Definition 6.1.1.2], which we recall as follows.

For a set I, let $\mathbf{P}(I)$ denote the set of subsets of I. We impose the partially ordered set structure on $\mathbf{P}(I)$ with respect to the inclusion, and we regard $\mathbf{P}(I)$ as the associated category. For an ∞-category $\mathcal{C}$, an *I-cube in $\mathcal{C}$* is a functor

$$Q : \mathbf{P}(I) \to \mathcal{C}.$$

If the cardinal of I is an integer n, then an I-cube is called an *n-cube*. The *total cofiber of Q* is

$$\mathrm{tcofib}(Q) := \mathrm{cofib}(\mathrm{colim}(Q|_{\mathbf{P}(I)-\{I\}}) \to Q(I)),$$

whenever the colimit and cofiber exist. The *total fiber of* Q is

$$\mathrm{tfib}(Q) := \mathrm{fib}(Q(\varnothing) \to \lim(Q|_{\mathbf{P}(I)-\{\varnothing\}})),$$

whenever the limit and fiber exist.

Theorem 6.3 *Let* X *be a noetherian separated fs log scheme with involution. Then the induced map*

$$\mathrm{THR}(X) \to \mathrm{THR}(X \times (\mathbb{P}^n, \mathbb{P}^{n-1}))$$

is an equivalence of real cyclotomic spectra, where we impose the trivial involution on $(\mathbb{P}^n, \mathbb{P}^{n-1})$.

Proof By Proposition 5.5, it suffices to show that the map is an equivalence of $\mathbb{Z}/2$-spectra.

Following Remark 6.1, we will argue for $\mathbb{S}$ instead of X. Let $\underline{U_0}, \dots, \underline{U_n}$ be the standard cover of $\mathbb{P}^n_{\mathbb{S}}$, and we set $U_i := (\mathbb{P}^n_{\mathbb{S}}, \mathbb{P}^{n-1}_{\mathbb{S}}) \times_{\mathbb{P}^n_{\mathbb{S}}} \underline{U_i}$. For every nonempty subset $I \subset [n]$, we set $U(I) := U_I := \bigcap_{i \in I} U_i$. We also set $U_\varnothing := \mathbb{S}$. We regard U as an $(n+1)$-cube. We need to show $\mathrm{tcofib}(\mathrm{THR}(U)) \simeq 0$.

For notational convenience, we consider the commutative monoids

$$\begin{aligned} P_j &:= \{(x_1, \dots, x_n) \in \mathbb{Z}^n : x_j \geq 0\} \quad \text{for } j = 1, \dots, n, \\ P_0 &:= \{(x_1, \dots, x_n) \in \mathbb{Z}^n : x_1 + \dots + x_n \leq 0\}, \\ F_0 &:= \{(x_1, \dots, x_n) \in \mathbb{Z}^n : x_1 + \dots + x_n = 0\}. \end{aligned}$$

For every subset $J \subset [n]$, we set $P_J := \bigcap_{j \in J} P_j$. By convention, we have $P_\varnothing := \mathbb{Z}^n$. There is a canonical equivalence

$$\mathrm{THR}(U_I) \simeq \begin{cases} \mathrm{THR}(\mathbb{S}[P_{[n]-I}]) & \text{if } 0 \in I, \\ \mathrm{THR}(\mathbb{S}[P_{[n]-I}], P_{[n]-I} \cap F_0) & \text{if } 0 \notin I. \end{cases}$$

Each P_J is isomorphic to products of finite copies of $\mathbb{Z}$ and $\mathbb{N}$. The computations of $\mathrm{B}^{\mathrm{di}}\mathbb{N}$ and $\mathrm{B}^{\mathrm{di}}\mathbb{Z}$ in [19, (4.12), Propositions 4.2.11, 4.2.12] and the fact B^{di} preserves finite products [19, Proposition 4.2.4] yield a natural equivalence of $\mathbb{Z}/2$-spectra

$$\mathrm{THR}(U_I) \simeq \mathbb{S}[\Phi(I)]$$

with

$$\begin{aligned} \Phi(I) &:= \coprod_{(x_1,\dots,x_n) \in \mathbb{Z}^n} \Phi(I; (x_1, \dots, x_n)), \\ \Phi(I; (x_1, \dots, x_n)) &:= \begin{cases} \Phi_1(I; x_1) \times \dots \times \Phi_n(I; x_n) & \text{if } (x_1, \dots, x_n) \in P_{[n]-I}, \\ \varnothing & \text{otherwise}, \end{cases} \\ \Phi_i(I; x_i) &:= \begin{cases} * & \text{if } x_i = 0 \text{ and } i \in I, \\ S^\sigma & \text{otherwise}. \end{cases} \end{aligned}$$

Since there is a natural decomposition

$$\mathrm{tcofib}(\Phi) \simeq \coprod_{x \in \mathbb{Z}^n} \mathrm{tcofib}(\Phi(-; x)),$$

it suffices to show $\mathrm{tcofib}(\Phi(-; x)) \simeq 0$ for every $x \in \mathbb{Z}^n$.

If $x \neq (0, \dots, 0)$, then there exists $i \in [n]$ such that $x \notin P_i$. The induced map $\Phi(I; x) \to \Phi(I \cup \{i\}; x)$ is a $\mathbb{Z}/2$-homeomorphism whenever $i \notin I$. Together with the categorical result [7, Proposition A.6.5], we have $\mathrm{tcofib}(\Phi(-; x)) \simeq 0$.

If $x = (0, \dots, 0)$, then the induced map $\Phi(I; x) \to \Phi(I \cup \{0\}; x)$ is a $\mathbb{Z}/2$-homeomorphism whenever $0 \notin I$. We have $\mathrm{tcofib}(\Phi(-; x)) \simeq 0$ similarly. □

The THH and TC parts of the following result are proved in [7, Theorem 8.4.4].

Theorem 6.4 *Let S be a noetherian separated scheme. Then the sheaves*

$$\mathrm{THH}, \mathrm{THR}^{\mathbb{Z}/2}, \mathrm{TC}, \mathrm{TCR}^{\mathbb{Z}/2} \in \mathrm{Sh}_{\mathrm{sNis}}(\mathrm{SmlSm}/S, \mathrm{Sp})$$

are in $\mathrm{logSH}_{S^1}(S)$.

Proof Immediate from Theorem 6.3. □

Since THR is not a sheaf of spectra but a sheaf of $\mathbb{Z}/2$-spectra, we cannot directly apply the results in [6; 7] to THR. Instead, we will often use the fact that the pair $(i^*, (-)^{\mathbb{Z}/2})$ is conservative to reduce to the case of THH and $\mathrm{THR}^{\mathbb{Z}/2}$.

For any vector bundle $\mathcal{E} \to X$ with $X \in \mathrm{SmlSm}/S$, the *real topological Hochschild homology of the Thom space* $\mathrm{Th}(\mathcal{E})$ is defined to be

$$\mathrm{THR}(\mathrm{Th}(\mathcal{E})) := \mathrm{fib}(\mathrm{THR}(\mathcal{E}) \to \mathrm{THR}(\mathrm{Bl}_Z \mathcal{E}, E)), \tag{6-1}$$

where Z is the zero section of $\mathcal{E}$, and E is the exceptional divisor.

Proposition 6.5 *With the above notation, there are natural equivalences of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(\mathrm{Th}(\mathcal{E})) \simeq \mathrm{fib}\big(\mathrm{THR}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O})) \to \mathrm{THR}(\mathbb{P}(\mathcal{E}))\big),$$
$$\mathrm{TCR}(\mathrm{Th}(\mathcal{E})) \simeq \mathrm{fib}\big(\mathrm{TCR}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O})) \to \mathrm{TCR}(\mathbb{P}(\mathcal{E}))\big).$$

Proof We focus on the case of THR since the proofs are similar.

Let Y and Y' be the blow-up of $\mathcal{E}$ and $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})$ along the zero section, and let E and E' be the exceptional divisors. By the proof of [6, Proposition 7.4.5], there exists a commutative diagram

$$\begin{array}{ccccc}
(Y, E) & \longrightarrow & (Y', E') & \longleftarrow & \mathbb{P}(\mathcal{E}) \\
\downarrow & & \downarrow & & \\
\mathcal{E} & \longrightarrow & \mathbb{P}(\mathcal{E} \oplus \mathcal{O}) & &
\end{array}$$

such that the induced maps

$$\mathrm{fib}\big(\mathcal{F}(\mathbb{P}(\mathcal{E} \oplus \mathcal{O})) \to \mathcal{F}(Y', E')\big) \to \mathrm{fib}(\mathcal{F}(\mathcal{E}) \to \mathcal{F}(Y, E)),$$
$$\mathcal{F}(Y', E') \to \mathcal{F}(\mathbb{P}(\mathcal{E}))$$

are equivalences for $\mathcal{F} \in \mathrm{logSH}_{S^1}(S)$. In particular, these are equivalences for THH and $\mathrm{THR}^{\mathbb{Z}/2}$ by Theorem 6.4. From this, we obtain the desired equivalence. □

Lemma 6.6 *Let $\mathcal{E}_1 \to X_1$ and $\mathcal{E}_2 \to X_2$ be vector bundles, where $X_1, X_2 \in \mathrm{SmlSm}/S$. We regard $T := \mathcal{E}_1 \times_S \mathcal{E}_2$ as a vector bundle over $X_1 \times_S X_2$. We set*

$$T_1 := (\mathrm{Bl}_{Z_1}\mathcal{E}_1, E_1) \times_S \mathcal{E}_2, \quad T_2 := \mathcal{E}_1 \times_S (\mathrm{Bl}_{Z_2}\mathcal{E}_2, E_2), \quad T_{12} := T_1 \times_S T_2,$$

where Z_1 and Z_2 are the zero sections of $\mathcal{E}_1$ and $\mathcal{E}_2$. Then there are natural equivalences of $\mathbb{Z}/2$-spectra

$$\mathrm{THR}(\mathrm{Th}(T)) \simeq \mathrm{tfib}(\mathrm{THR}(C_3)), \quad \mathrm{TCR}(\mathrm{Th}(T)) \simeq \mathrm{tfib}(\mathrm{TCR}(C_3)),$$

where C_3 is the cartesian square

$$\begin{array}{ccc} T_{12} & \longrightarrow & T_1 \\ \downarrow & & \downarrow \\ T_2 & \longrightarrow & T \end{array}$$

Proof Again, we focus on THR. Consider the fs log schemes $T_I \in \mathrm{SmlSm}/S$ for every subset I of $\{1,2,4\}$ in [6, Construction 7.4.14], and consider the induced squares

$$C_1 := \begin{array}{ccc} T_4 & \longrightarrow & T_4 \\ \downarrow & & \downarrow \\ T_4 & \longrightarrow & T \end{array} \qquad C_2 := \begin{array}{ccc} T_{124} & \longrightarrow & T_{14} \\ \downarrow & & \downarrow \\ T_{24} & \longrightarrow & T \end{array}$$

We have the induced maps of squares $C_1 \leftarrow C_2 \to C_3$. By Theorem 6.4 and the proof of Proposition 7.4.15 in [6], the induced maps

$$\mathrm{tfib}(\mathcal{F}(C_1)) \to \mathrm{tfib}(\mathcal{F}(C_2)) \leftarrow \mathrm{tfib}(\mathcal{F}(C_3))$$

are equivalences for $\mathcal{F} := \mathrm{THH}, \mathrm{THR}^{\mathbb{Z}/2}$. This implies that we have a natural equivalence

$$\mathrm{tfib}(\mathrm{THR}(C_1)) \simeq \mathrm{tfib}(\mathrm{THR}(C_3)),$$

which yields the desired equivalence. □

Proposition 6.7 *For $X \in \mathrm{SmlSm}/S$ and integer $n \geq 0$, there are natural equivalences of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(\mathrm{Th}(X \times \mathbb{A}^n)) \simeq \Sigma^{n(\sigma-1)}\mathrm{THR}(X), \quad \mathrm{TCR}(\mathrm{Th}(X \times \mathbb{A}^n)) \simeq \Sigma^{n(\sigma-1)}\mathrm{TCR}(X),$$

where we regard $X \times \mathbb{A}^n$ as the rank-n trivial bundle over X.

Proof Again, we focus on THR. We proceed by induction on n. The claim is trivial if $n = 0$. Assume $n > 0$. We set $X_1 := X$, $\mathcal{E}_1 := X \times \mathbb{A}^{n-1}$, $X_2 := S$, and $\mathcal{E}_2 := S \times \mathbb{A}^1$. The square C_3 in Lemma 6.6 becomes

$$\begin{array}{ccc} (\mathrm{Bl}_X(X \times \mathbb{A}^{n-1}), E_{n-1}) \times (\mathbb{A}^1, 0) & \longrightarrow & (\mathrm{Bl}_X(X \times \mathbb{A}^{n-1}), E_{n-1}) \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^{n-1} \times (\mathbb{A}^1, 0) & \longrightarrow & X \times \mathbb{A}^{n-1} \times \mathbb{A}^1 \end{array}$$

where E_{n-1} is the exceptional divisor. By [19, Proposition 5.2.4], Proposition 3.3, and Lemma 6.6, we have a natural equivalence

$$\mathrm{THR}(\mathrm{Th}(X \times \mathbb{A}^n)) \simeq \mathrm{THR}(\mathrm{Th}(X \times \mathbb{A}^{n-1})) \wedge \mathrm{fib}(\mathrm{THR}(\mathbb{S}[\mathbb{N}]) \to \mathrm{THR}(\mathbb{S}[\mathbb{N}], \mathbb{N})).$$

Combine this with Example 3.4 and use the induction hypothesis to conclude. □

Next, we are concerned about the Gysin cofiber sequence in logarithmic motivic homotopy theory. For this, we recall the deformation to the normal cone construction in the logarithmic setting as follows. Assume that $X \in \mathrm{SmlSm}/S$ has the form $(\underline{X}, Z_1 + \cdots + Z_n)$, and let Z be a strict closed subscheme of X such that $\underline{Z}$ is strict normal crossing with $Z_1 + \cdots + Z_n$ in the sense of [6, Definition 7.2.1]. The *blow-up of X along Z* is

$$\mathrm{Bl}_Z X := (\mathrm{Bl}_{\underline{Z}} \underline{X}, \widetilde{Z}_1 + \cdots + \widetilde{Z}_n),$$

where $\widetilde{Z}_i$ is the strict transform of Z_i for $1 \le i \le n$. The *normal bundle of Z in X* is defined to be

$$\mathrm{N}_Z X := \mathrm{N}_{\underline{Z}} \underline{X} \times_{\underline{X}} X,$$

where $\mathrm{N}_{\underline{Z}} \underline{X}$ denotes the normal bundle of $\underline{Z}$ in $\underline{X}$. The *deformation space associated with $Z \to X$* is defined to be

$$\mathrm{D}_Z X := \mathrm{Bl}_{Z \times \{0\}}(X \times \square) - \mathrm{Bl}_{Z \times \{0\}}(X \times \{0\}).$$

Theorem 6.8 *With the above notation, there exist natural cofiber sequences of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(\mathrm{Th}(\mathrm{N}_Z X)) \to \mathrm{THR}(X) \to \mathrm{THR}(\mathrm{Bl}_Z X, E),$$
$$\mathrm{TCR}(\mathrm{Th}(\mathrm{N}_Z X)) \to \mathrm{TCR}(X) \to \mathrm{TCR}(\mathrm{Bl}_Z X, E),$$

where E is the exceptional divisor.

Proof Again, we focus on THR. We have the induced maps

$$\mathrm{fib}(\mathrm{THR}(X) \to \mathrm{THR}(\mathrm{Bl}_Z X, E)) \leftarrow \mathrm{fib}\big(\mathrm{THR}(\mathrm{D}_Z X) \to \mathrm{THR}(\mathrm{Bl}_{Z \times \square}(\mathrm{D}_Z X), E^D)\big) \to \mathrm{fib}\big(\mathrm{THR}(\mathrm{N}_Z X) \to \mathrm{THR}(\mathrm{Bl}_Z(\mathrm{N}_Z X), E^N)\big),$$

where E, E^D, and E^N are the exceptional divisors. It suffices to show that the corresponding maps for THH and $\mathrm{THR}^{\mathbb{Z}/2}$ are equivalences of spectra since the pair of functors $(i^*, (-)^{\mathbb{Z}/2})$ is conservative. This is a consequence of [6, Theorem 7.5.4] (see also [7, Theorem 3.2.14]) and Theorem 6.4. □

Recall from [7, Definition 8.5.3] that we have the log motivic spectra

$$\mathbf{THH} := (\mathrm{THH}, \mathrm{THH}, \dots),$$
$$\mathbf{TC} := (\mathrm{TC}, \mathrm{TC}, \dots),$$

whose bonding maps $\mathrm{THH} \to \Omega_{\mathbb{P}^1} \mathrm{THH}$ and $\mathrm{TC} \to \Omega_{\mathbb{P}^1} \mathrm{TC}$ are obtained by the projective bundle formula.

Example 6.9 Let X be a scheme, and let $\mathcal{E} \to X$ be a rank-n vector bundle. There exists a Thom equivalence

$$\mathrm{THH}(\mathrm{Th}(\mathcal{E})) \simeq \mathrm{THH}(X);$$

see [7, Proposition 8.6.9]. This is a consequence of the fact that **THH** is an orientable logarithmic motivic spectrum [7, Definition 7.1.3, Theorem 8.6.7]. Unlike this, in general, we do not have an equivalence

$$\mathrm{THR}(\mathrm{Th}(\mathcal{E})) \simeq \Sigma^{n(\sigma-1)}\mathrm{THR}(X),$$

i.e., $\mathrm{THR}(\mathrm{Th}(\mathcal{E})) \not\simeq \mathrm{THR}(\mathrm{Th}(X \times \mathbb{A}^n))$ by Proposition 6.7.

For example, if $X := \mathbb{P}^n_{\mathbb{S}}$ and $Z := \mathbb{P}^{n-1}_{\mathbb{S}}$ with even $n \geq 2$, then Proposition 6.2 and Theorems 6.3 and 6.8 yield

$$\mathrm{THR}(\mathrm{Th}(\mathrm{N}_Z X)) \simeq \bigoplus_{j=1}^{\lfloor n/2 \rfloor} i_*\mathbb{S}.$$

This is not equivalent to

$$\Sigma^{\sigma-1}\mathrm{THR}(Z) \simeq \Sigma^{\sigma-1} \bigoplus_{j=1}^{\lfloor (n-1)/2 \rfloor} i_*\mathbb{S} \oplus \Sigma^{n(\sigma-1)}$$

that is obtained by Proposition 6.2. We have a similar conclusion if n is odd too.

We also obtain the following descent result with respect to blow-ups along smooth centers (with the trivial involutions).

Theorem 6.10 *Let $Z \to X$ be a closed immersion of smooth schemes over S. Then the induced squares of $\mathbb{Z}/2$-spectra*

$$\begin{array}{ccc} \mathrm{THR}(X) & \longrightarrow & \mathrm{THR}(Z) \\ \downarrow & & \downarrow \\ \mathrm{THR}(Z \times_X \mathrm{Bl}_Z X) & \longrightarrow & \mathrm{THR}(\mathrm{Bl}_Z X) \end{array} \qquad \begin{array}{ccc} \mathrm{TCR}(X) & \longrightarrow & \mathrm{TCR}(Z) \\ \downarrow & & \downarrow \\ \mathrm{TCR}(Z \times_X \mathrm{Bl}_Z X) & \longrightarrow & \mathrm{TCR}(\mathrm{Bl}_Z X) \end{array}$$

are cartesian.

Proof Again, we focus on THR. The corresponding squares of spectra for THH and $\mathrm{THR}^{\mathbb{Z}/2}$ are cartesian by [6, Theorem 7.3.3] (see also [7, Theorem 3.3.5]) and Theorem 6.4. This implies the claim since the pair of functors $(i^*, (-)^{\mathbb{Z}/2})$ is conservative. □

Let $\mathbb{P}^\sigma$ be the scheme $\mathbb{P}^1$ with the involution given by $[x:y] \mapsto [y:x]$. Let 1 be the base points of $\mathbb{P}^1$ and $\mathbb{P}^\sigma$, and then we form the endofunctors $\Omega_{\mathbb{P}^1}$, $\Omega_{\mathbb{P}^\sigma}$, and $\Omega_{\mathbb{P}^1 \wedge \mathbb{P}^\sigma} \simeq \Omega_{\mathbb{P}^\sigma}\Omega_{\mathbb{P}^1}$ on $\mathrm{PSh}((\mathrm{SmlSm}/S)_{\mathbb{Z}/2}, \mathrm{Sp})$ and $\mathrm{PSh}((\mathrm{SmlSm}/S)_{\mathbb{Z}/2}, \mathrm{Sp}^{\mathbb{Z}/2})$.

Proposition 6.11 *Let X be a noetherian separated fs log scheme. Then there are natural equivalences of $\mathbb{Z}/2$-spectra*

$$\mathrm{THR}(X) \simeq \Omega_{\mathbb{P}^1 \wedge \mathbb{P}^\sigma}\mathrm{THR}(X), \quad \mathrm{TCR}(X) \simeq \Omega_{\mathbb{P}^1 \wedge \mathbb{P}^\sigma}\mathrm{TCR}(X).$$

Proof Again, we focus on THR. Following Remark 6.1, we reduce to [19, Proposition 5.1.5]. □

Definition 6.12 The ∞-*category of prelogarithmic motivic* $\mathbb{Z}/2$-*spectra over* S is defined to be

$$\mathrm{prelogSH}^{\mathbb{Z}/2}(S) := \mathrm{Sp}_{\mathbb{P}^1\wedge\mathbb{P}^\sigma}\big((\mathbb{P}^\bullet,\mathbb{P}^{\bullet-1})^{-1}\mathrm{Sh}_{\mathrm{seNis}}((\mathrm{SmlSm}/S)_{\mathbb{Z}/2},\mathrm{Sp})\big).$$

Remark 6.13 Let $\mathbb{A}^w$ denote the affine line $\mathbb{A}^1$ with the involution w given by $x \mapsto -x$. We have the $\mathbb{A}^1$-homotopy

$$\mathbb{A}^1 \times \mathbb{A}^w \to \mathbb{A}^w \tag{6-2}$$

given by $(x,y)\mapsto xy$. Due to this, the projections $X\times\mathbb{A}^w \to X$ are automatically inverted if we invert the projections $X\times\mathbb{A}^1\to X$ for all $X\in(\mathrm{Sm}/S)_{\mathbb{Z}/2}$.

However, this phenomenon is not generalized to the logarithmic setting since there exists no morphism of fs log schemes

$$\square\times\square^w\to\square^w$$

extending (6-2), where $\square^w$ denotes the fs log scheme $\square$ with the involution $[x,y]\mapsto[-x,y]$. This indicates that inverting the projections $X\times\square^w\to X$ for all $X\in(\mathrm{SmlSm}/S)_{\mathbb{Z}/2}$ is at least required for a better behaved ∞-category of logarithmic motivic $\mathbb{Z}/2$-spectra.

Definition 6.14 The *prelogarithmic motivic* $\mathbb{Z}/2$-*fixed point functor* is defined to be the functor

$$(-)^{\mathbb{Z}/2} : \mathrm{prelogSH}^{\mathbb{Z}/2}(S)\to\mathrm{logSH}(S)$$

sending a $\mathbb{P}^1\wedge\mathbb{P}^\sigma$-spectrum $(\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2,\dots)$ to the $\mathbb{P}^1$-spectrum

$$(\mathcal{F}_0,\Omega_{\mathbb{P}^\sigma}\mathcal{F}_1,\Omega_{\mathbb{P}^{2\sigma}}\mathcal{F}_2,\dots)$$

restricting to the objects of the category SmlSm/S with the trivial involutions, where the bonding map $\Omega_{\mathbb{P}^{n\sigma}}\mathcal{F}_n\to\Omega_{\mathbb{P}^1\wedge\mathbb{P}^{(n+1)\sigma}}\mathcal{F}_{n+1}$ is induced by the bounding map $\mathcal{F}_n\to\Omega_{\mathbb{P}^1\wedge\mathbb{P}^\sigma}\mathcal{F}_{n+1}$.

Definition 6.15 The *motivic real topological Hochschild* $\mathbb{Z}/2$-*spectrum* and *motivic real topological cyclic* $\mathbb{Z}/2$-*spectrum* are the $\mathbb{P}^1\wedge\mathbb{P}^\sigma$-spectra

$$\mathbf{THR} := (\mathrm{THR}^{\mathbb{Z}/2},\mathrm{THR}^{\mathbb{Z}/2},\dots)\in\mathrm{prelogSH}^{\mathbb{Z}/2}(S),$$
$$\mathbf{TCR} := (\mathrm{TCR}^{\mathbb{Z}/2},\mathrm{TCR}^{\mathbb{Z}/2},\dots)\in\mathrm{prelogSH}^{\mathbb{Z}/2}(S),$$

whose bonding maps are given by Proposition 6.11. We have the induced motivic spectra

$$\mathbf{THR}^{\mathbb{Z}/2},\mathbf{TCR}^{\mathbb{Z}/2}\in\mathrm{logSH}(S).$$

Definition 6.16 For a strict Nisnevich sheaf of spectra $\mathcal{F}$ on $(\mathrm{SmlSm}/S)_{\mathbb{Z}/2}$, let $i^*\mathcal{F}$ be the strict Nisnevich sheaf of spectra on SmlSm/S given by

$$i^*\mathcal{F}(X) := \mathcal{F}(i_\sharp X).$$

Observe that for every integer $n \geq 1$, $i^*\mathcal{F}$ is $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariant if $\mathcal{F}$ is $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariant. We have the "forgetful" functor

$$i^* : \mathrm{prelogSH}^{\mathbb{Z}/2}(S) \to \mathrm{logSH}(S)$$

sending a $\mathbb{P}^1 \wedge \mathbb{P}^\sigma$-spectrum $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots)$ to the $\mathbb{P}^1$-spectrum

$$(i^*\mathcal{F}_0, \Omega_{\mathbb{P}^\sigma} i^*\mathcal{F}_1, \Omega_{\mathbb{P}^{2\sigma}} i^*\mathcal{F}_2, \dots)$$

with the induced bonding maps.

Proposition 6.17 *There are equivalences, in* $\mathrm{logSH}(S)$,

$$i^*\mathbf{THR} \simeq \mathbf{THH},$$
$$i^*\mathbf{TCR} \simeq \mathbf{TC}.$$

Proof This is a consequence of Corollary 4.9 and Proposition 5.30. □

Remark 6.18 In this section, we have considered mainly THR and TCR, but we can similarly show the analogous results for $\mathrm{TCR}^- := \mathrm{THR}^{hS^\sigma}$, TCR_{hS^σ}, and $\mathrm{TPR} := \mathrm{THR}^{tS^\sigma}$.

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