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Coxeter quotients of the automorphism group of a Coxeter group

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We show that for a large class $\mathcal{W}$ of infinite Coxeter groups the following holds: Given a group W_Γ in $\mathcal{W}$, the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto some infinite Coxeter group. In particular, the group $\mathrm{Aut}(W_\Gamma)$ does not have Kazhdan's property (T). Moreover, we show that $\mathrm{Aut}(W_\Gamma)$ is amenable if and only if W_Γ is virtually abelian.

1 Introduction

One fascinating property of a group is property (T). Property (T) is a representation-theoretic gadget for locally compact groups introduced by Kazhdan [15]. This property was reformulated by Delorme and Guichardet using geometric group theory. A countable group G has Kazhdan's property (T) if every action of G on a real Hilbert space by isometries has a global fixed point [2, Theorem 2.12.4].

Examples of groups satisfying this property are $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$ and $\mathrm{Out}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ for $n \geq 4$; see [2; 13; 14; 18].

Given a finite simplicial graph Γ with the vertex set $V(\Gamma)$, the edge set $E(\Gamma)$ and with an edge-labeling $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$, the Coxeter group W_Γ associated to Γ is the group with the presentation

$$W_\Gamma = \langle V(\Gamma) \mid v^2, (vw)^{m(\{v,w\})} \text{ for all } v \in V(\Gamma) \text{ and whenever } \{v, w\} \in E(\Gamma)\rangle.$$

Coxeter groups are well understood objects in geometric group theory, but there are many open questions concerning their automorphism groups. We conjecture:

Conjecture 1.1 *The group* $\mathrm{Aut}(W_\Gamma)$ *has Kazhdan's property* (T) *if and only if* W_Γ *is finite.*

In particular, we are interested in algebraic properties of $\mathrm{Aut}(W_\Gamma)$ that prohibit this group from having property (T). After presenting several results regarding the above conjecture at the virtual geometric group theory conference in Luminy, 2020, Alain Valette formulated the following conjecture:

Conjecture 1.2 *For every infinite Coxeter group* W_Γ, *the automorphism group* $\mathrm{Aut}(W_\Gamma)$ *virtually maps onto some infinite Coxeter group.*

We want to point out that a verification of Conjecture 1.2 would imply Conjecture 1.1, since it is known that infinite Coxeter groups virtually map onto $\mathbb{Z}$, see [7], and therefore do not have Kazhdan's property (T).

Our main goal is to verify Conjecture 1.2 for a large class of Coxeter groups.

Let Γ be a Coxeter graph. For $S \subseteq V(\Gamma)$ the subgroup generated by S is called a *special parabolic subgroup* of W_Γ. We note that the subgroup $\langle S\rangle \subseteq W_\Gamma$ is canonically isomorphic to W_Ω where the

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graph Ω is the full subgraph of Γ with $V(\Omega) = S$. A special parabolic subgroup W_Ω is called *spherical* if W_Ω is finite.

Our first result regarding Conjecture 1.2 is about Coxeter groups with finite outer automorphism group.

Lemma 1.3 *Let W_Γ be a Coxeter group. If W_Γ is infinite and the outer automorphism group* $\mathrm{Out}(W_\Gamma) := \mathrm{Aut}(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$ *is finite, then* $\mathrm{Aut}(W_\Gamma)$ *virtually maps onto an infinite special parabolic subgroup of W_Γ.*

Twist-rigid Coxeter groups and virtually abelian Coxeter groups have finite outer automorphism groups; see [5, Corollary 1.2; 6]. An example of a Coxeter group W_Γ which is not of any of those types and where $\mathrm{Out}(W_\Gamma)$ is finite is given in Figure 1 on page 2357.

Let Γ be a Coxeter graph. We denote by $\Gamma_{\mathrm{odd}} \subseteq \Gamma$ the subgraph obtained from Γ by removing all edges with even labels. A special parabolic subgroup W_Ω is called an *odd factor* of W_Γ if $V(\Omega)$ is a union of the vertex sets of some connected components of Γ_{odd}. Hence, if all edge-labels in Γ are even, then every special parabolic subgroup of W_Γ is an odd factor.

Theorem 1.4 *Let W_Γ be a Coxeter group. If*

(1) *there exists a nonspherical odd factor W_Ω of W_Γ such that the outer automorphism group* $\mathrm{Out}(W_\Omega)$ *is finite, or*

(2) *the Coxeter graph Γ is disconnected,*

then $\mathrm{Aut}(W_\Gamma)$ *virtually surjects onto some infinite Coxeter group.*

By definition, a Coxeter group W_Γ is called *even* if $E(\Gamma) = \varnothing$ or $m(E(\Gamma)) \subseteq 2\mathbb{N}$. As a corollary of Theorem 1.4 we obtain the following result.

Corollary 1.5 *Let W_Γ be an infinite Coxeter group. If W_Γ is even, then* $\mathrm{Aut}(W_\Gamma)$ *virtually surjects onto some infinite Coxeter group.*

If a group does not have Kazhdan's property (T), it is natural to ask if this group is amenable.

Proposition 1.6 *Let W_Γ be a Coxeter group. The group* $\mathrm{Aut}(W_\Gamma)$ *is amenable if and only if W_Γ is virtually abelian.*

Structure of the paper

In Section 2 we review some of the standard facts on simplicial graphs and Coxeter groups. This section also contains the proof of Proposition 1.6. Some results concerning finiteness of the outer automorphism group of a Coxeter group can also be find in this section. Further, in this section we recall useful facts about retractions on special parabolic subgroups of a Coxeter group and define characteristic subgroups with respect to a subgroup of an automorphism group. We end this section by proving Lemma 2.12 which is the main ingredient in the proof of Theorem 1.4. Section 3 contains the proofs of Theorem 1.4 and Corollary 1.5.

2 Preliminaries

2.1 Simplicial graphs

In this section we define simplicial graphs and collect some important properties of these combinatorial objects which we will need to define the groups of our interest. We largely follow [9].

A *simplicial graph* Γ consists of a set $V(\Gamma)$ of *vertices* and a set $E(\Gamma)$ of subsets of $V(\Gamma)$ of cardinality two which are called *edges*. A graph Γ is said to be *finite* if $V(\Gamma)$ is a finite set. Two vertices $v, w \in V(\Gamma)$ are called *adjacent*, if $\{v, w\}$ is an edge of Γ. By an *edge-labeling* of Γ we mean just a map $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$. If all the vertices of Γ are pairwise adjacent, then Γ is called *complete*. A graph $\Lambda = (X, Y)$ is a subgraph of Γ if $X \subseteq V(\Gamma)$, $Y \subseteq E(\Gamma)$ and all vertices of the edges in Y are in X. A (*maximal*) *clique* in Γ is a (maximal) complete subgraph of Γ. Given a subset $S \subseteq V(\Gamma)$, the *graph generated/induced by* S, denoted by $\langle S \rangle$, is the graph with vertex set S and edge set $\{\{v, w\} \in E(\Gamma) \mid v, w \in S\}$. A subgraph $\Lambda = (X, Y)$ of Γ is called *full* if $\langle X \rangle = \Lambda$. A graph Γ is called *connected* if for every two vertices $v, w \in V(\Gamma)$ there exist vertices $v_1, \ldots, v_n \in V(\Gamma)$ such that $\{v, v_1\}, \{v_i, v_{i+1}\}, \{v_n, w\} \in E(\Gamma)$ for $i = 1, \ldots, n-1$. A subgraph Λ is a *connected component* of Γ if Λ is connected and is maximal with this property.

2.2 Coxeter groups

Let us recall the basic definitions and notation concerning Coxeter groups. For further details we refer to [8].

Definition 2.1 Let Γ be a finite simplicial graph with an edge-labeling $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$. We call such a labeled graph a *Coxeter graph*. The Coxeter group W_Γ is defined as

$$W_\Gamma = \langle V(\Gamma) \mid v^2, (vw)^{m(\{v,w\})} \text{ for all } v \in V(\Gamma) \text{ and whenever } \{v, w\} \in E(\Gamma) \rangle.$$

Given a Coxeter graph Γ, the next proposition shows that each subset $X \subseteq V(\Gamma)$ generates a Coxeter group. For a proof we refer to [8, Theorem 4.1.6].

Proposition 2.2 *Let Γ be a Coxeter graph. For each $X \subseteq V(\Gamma)$, the subgroup which is generated by X is canonically isomorphic to the Coxeter group W_Λ where Λ is the subgraph of Γ generated by X.*

The above proposition leads to the following definition.

Definition 2.3 Let Γ be a Coxeter graph. For a full subgraph $\Lambda \subseteq \Gamma$, the subgroup W_Λ of W_Γ is called *special parabolic* and any conjugate of a special parabolic subgroup is called *parabolic*. Note that sometimes we write for W_Λ just $W_{V(\Lambda)}$.

Let W_Γ be a Coxeter group. By definition, W_Γ is called *k-spherical* if every special parabolic subgroup W_Λ with $|V(\Lambda)| = k$ is finite. In particular, W_Γ is 2-spherical if and only if Γ is a complete graph.

The building blocks of a Coxeter group are irreducible special parabolic subgroups.

Definition 2.4 A Coxeter group W_Γ is called *reducible* if $V(\Gamma)$ can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that $\{x, y\} \in E(\Gamma)$ for all $x \in V_1$, $y \in V_2$ and $m(\{x, y\}) = 2$ for all $x \in V_1$ and $y \in V_2$. In that case $W_\Gamma \cong W_{V_1} \times W_{V_2}$ [8, Proposition 4.1.7]. If this sort of decomposition is not possible, we call the Coxeter group W_Γ *irreducible*.

We are now going to recall the definition of the centralizer of a subgroup H in a group G. The *centralizer* of H in G, denoted by $Z_G(H)$, is defined as

$$Z_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

The *center* of G is the subgroup $Z(G) := Z_G(G)$.

Given a Coxeter group W_Γ and an arbitrary subgroup H, it is in general not known what $Z_{W_\Gamma}(H)$ is. However, when H is a special parabolic subgroup one can say a lot about the structure of the centralizer of H in W_Γ; see [1; 3; 4; 19]. In particular, the center of W_Γ can be calculated as follows.

Lemma 2.5 *Let W_Γ be a Coxeter group. Write $W_\Gamma \cong W_{\Gamma_1} \times \cdots \times W_{\Gamma_n}$ as a direct product of irreducible special parabolic subgroups. We define $J := \{i \in \{1, \dots, n\} \mid W_{\Gamma_i} \text{ is finite}\}$. Then $Z(W_\Gamma) \cong \prod_{j \in J} Z(W_{\Gamma_j})$.*

Proof The center of W_Γ is isomorphic to $Z(W_{\Gamma_1}) \times \cdots \times Z(W_{\Gamma_n})$. Since the center of an irreducible infinite Coxeter group is trivial [8, Theorem D.2.10] we obtain $Z(W_\Gamma) \cong \prod_{j \in J} Z(W_{\Gamma_j})$. □

2.3 Finite outer automorphism group

In this section we are in particular interested in Coxeter groups with finite outer automorphism group. First we give a proof of Lemma 1.3.

Proof of Lemma 1.3 We write W_Γ as a direct product of irreducible special parabolic subgroups $W_\Gamma \cong W_{\Gamma_1} \times W_{\Gamma_2} \times \cdots \times W_{\Gamma_n}$. Since W_Γ is infinite by assumption, there exists $j \in \{1, \dots, n\}$ such that W_{Γ_j} is infinite. Note that the center of W_{Γ_j} is trivial. We obtain

$$\mathrm{Inn}(W_\Gamma) \cong W_\Gamma / Z(W_\Gamma) \cong W_{\Gamma_1}/Z(W_{\Gamma_1}) \times W_{\Gamma_2}/Z(W_{\Gamma_2}) \times \cdots \times W_{\Gamma_n}/Z(W_{\Gamma_n}) \xrightarrow{\pi} W_{\Gamma_j},$$

where π is the canonical projection onto W_{Γ_j}. By assumption the index of $\mathrm{Inn}(W_\Gamma)$ in $\mathrm{Aut}(W_\Gamma)$ is finite, hence $\mathrm{Aut}(W_\Gamma)$ maps virtually onto the special parabolic subgroup W_{Γ_j}. □

Now we prove Proposition 1.6.

Proof of Proposition 1.6 Let W_Γ be a Coxeter group. If W_Γ is virtually abelian, then $\mathrm{Out}(W_\Gamma)$ is finite; see [6]. In particular, $\mathrm{Aut}(W_\Gamma)$ is virtually abelian and therefore amenable. If W_Γ is not virtually abelian, then the free group F_2 is a subgroup of W_Γ by [8, Theorem 14.1.2]. Thus $F_2 \subseteq \mathrm{Inn}(W_\Gamma)$. It is known that F_2 is not amenable and amenability is preserved by taking subgroups, hence $\mathrm{Aut}(W_\Gamma)$ is not amenable. □

It was shown in [5, Corollary 1.2] that the outer automorphism group of a twist-rigid Coxeter group is finite. For definition of twist-rigidity, see [5].

Corollary 2.6 *Let W_Γ be an infinite Coxeter group. If W_Γ is twist-rigid, then $\mathrm{Aut}(W_\Gamma)$ satisfies Conjecture 1.2.*

Let us consider the Coxeter group W_Γ where Γ is isomorphic to the graph in Figure 1.

The Coxeter group W_Γ is not twist-rigid. Our next goal is to show that the outer automorphism group of W_Γ is finite.

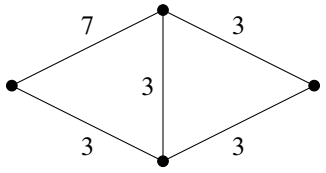

Figure 1: Coxeter graph Γ.

Theorem 2.7 [17, Theorem 28] *Let Γ be a Coxeter graph and $\Delta_1, \dots, \Delta_n$ be all maximal cliques of Γ. Let $C(W_\Gamma)$ be the subgroup of all $f \in \mathrm{Aut}(W_\Gamma)$ such that for $i \in \{1, \dots, n\}$ there exists an element $w_{i,f} \in W_\Gamma$ so that*

$$f(x) = w_{i,f} \cdot x \cdot {w_{i,f}}^{-1} \quad \text{for all } x \in W_{\Delta_i}.$$

The subgroup $C(W_\Gamma)$ has finite index in $\mathrm{Aut}(W_\Gamma)$.

A natural question is when the subgroup $C(W_\Gamma)$ is equal to the inner automorphism group of W_Γ. If Γ is not connected, then in most cases $C(W_\Gamma)$ is much larger than $\mathrm{Inn}(W_\Gamma)$ as the following observation shows: Let $\Gamma_1, \dots, \Gamma_n$ be the connected components of Γ and $W_{\Gamma_1}, \dots, W_{\Gamma_n}$ the corresponding special parabolic subgroups. Then $W_\Gamma = W_{\Gamma_1} * \cdots * W_{\Gamma_n}$. We assume that $n \geq 2$. Let a be a noncentral element in W_{Γ_1}, more precisely $a \in W_{\Gamma_1}$ and $a \notin Z(W_{\Gamma_1})$. We define $f_{12} : W_\Gamma \to W_\Gamma$ as follows: $f_{12}(v) = v$ for all $v \in V(\Gamma) - V(\Gamma_2)$ and $f_{12}(w) = awa^{-1}$ for all $w \in V(\Gamma_2)$. The map f_{12} induces naturally an automorphism of W_Γ. Moreover f_{12} is contained in $C(W_\Gamma)$. Since the element a does not commute with W_{Γ_1} the automorphism f_{12} is not inner.

Theorem 2.8 *Let Γ be a connected Coxeter graph. We denote by $\Delta_1, \dots, \Delta_n$ its maximal cliques and by $\Omega_1, \dots, \Omega_m$ the nonempty intersections of these maximal cliques where the intersection is taken over at least two maximal cliques.*

If the centralizer of the special parabolic subgroup W_{Ω_i} is trivial for all $i = 1, \dots, m$, then the subgroup $C(W_\Gamma)$ equals $\mathrm{Inn}(W_\Gamma)$. In particular, $\mathrm{Inn}(W_\Gamma)$ has finite index in $\mathrm{Aut}(W_\Gamma)$.

Proof Let f be in $C(W_\Gamma)$. Our goal is to show that f is an inner automorphism of W_Γ. Clearly, it is sufficient to show that there exists $a \in W_\Gamma$ such that for all $v \in V(\Gamma)$ we have $f(v) = ava^{-1}$.

By the definition of $C(W_\Gamma)$ we know that there exists $w_{1,f} \in W_\Gamma$ such that for all $x \in W_{\Delta_1}$ we have $f(x) = w_{1,f} \cdot x \cdot {w_{i,f}}^{-1}$. If Γ itself is a clique, then we are done. Otherwise, since Γ is connected there exists a vertex $v \in V(\Gamma) - V(\Delta_1)$ and $w \in V(\Delta_1)$ such that $\{v, w\} \in E(\Gamma)$. Further, there exists Δ_j such that $v, w \in V(\Delta_j)$. Hence, the intersection of Δ_1 and Δ_j is nonempty. Again, by definition of $C(W_\Gamma)$ there exists $w_{j,f} \in W_\Gamma$ such that $f(x) = w_{j,f} \cdot x \cdot {w_{j,f}}^{-1}$ for all $x \in W_{\Delta_j}$.

For $x \in W_{\Delta_1} \cap W_{\Delta_j} = W_{\Delta_1 \cap \Delta_j}$ we have

$$f(x) = w_{1,f} \cdot x \cdot {w_{1,f}}^{-1} = w_{j,f} \cdot x \cdot {w_{j,f}}^{-1}.$$

Hence ${w_{j,f}}^{-1} w_{1,f} \in Z_{W_\Gamma}(W_{\Delta_1 \cap \Delta_j})$. By assumption, this centralizer is trivial, thus $w_{1,f} = w_{j,f}$.

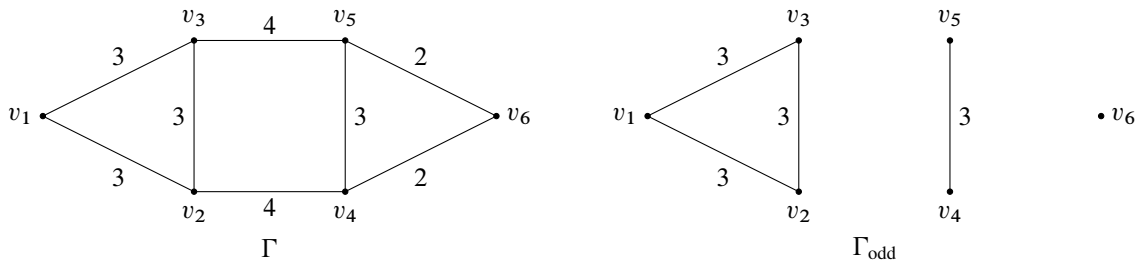

Figure 2: A Coxeter graph Γ and its odd graph Γ_{odd}.

In particular, the above proof strategy shows that for $k \in \{1, \dots, n\}$ where $\Delta_k \cap \Delta_1 \neq \varnothing$ we have

$$f(v) = w_{1,f} \cdot v \cdot {w_{1,f}}^{-1} \quad \text{for all } v \in V(\Delta_k).$$

Let $J := \{k \in \{1, \dots, n\} \mid \Delta_k \cap \Delta_1 \neq \varnothing\}$. If $V(\Gamma) = \bigcup_{k \in J} V(\Delta_k)$, then we have proved that f is a global conjugation by $w_{1,f}$. Otherwise the set $V(\Gamma) - \bigcup_{k \in J} V(\Delta_k)$ is nonempty. Since Γ is connected, there exist $v \in V(\Gamma) - \bigcup_{k \in J} V(\Delta_k)$ and $w \in \bigcup_{k \in J} V(\Delta_k)$ such that $\{v, w\} \in E(\Gamma)$. We choose Δ_l such that $v, w \in V(\Delta_l)$. Hence there exists $k \in J$ such that $\Delta_l \cap \Delta_k \neq \varnothing$. Now we proceed as above to show that $f(x) = w_{1,f} \cdot x \cdot {w_{1,f}}^{-1}$ for all $x \in \Delta_l$.

After applying the above proof strategy finitely many times, we see that f is a global conjugation by $w_{1,f}$. □

2.4 Retractions on special parabolic subgroups

By definition, a Coxeter group W_Γ is called *right-angled* if $E(\Gamma) = \varnothing$ or $m(E(\Gamma)) = \{2\}$. It is well known that for a right-angled Coxeter group W_Γ and a special parabolic subgroup W_Λ there exists a retraction homomorphism $r : W_\Gamma \twoheadrightarrow W_\Lambda$ such that $r(v) = v$ for all $v \in V(\Lambda)$ and $r(w) = 1$ for all $w \in V(\Gamma) - V(\Lambda)$. This kind of a retraction map exists also for other Coxeter groups.

Let Γ be a Coxeter graph. We denote by $\Gamma_{\text{odd}} \subseteq \Gamma$ the subgraph obtained from Γ by removing all edges with even labels. We note that the abelianization of W_Γ is isomorphic to $\mathbb{Z}/2\mathbb{Z}^k$ where k is the number of connected components of Γ_{odd}. A special parabolic subgroup W_Ω is called an *odd factor* of W_Γ if $V(\Omega)$ is a union of the vertex sets of some connected components of Γ_{odd}.

Proposition 2.9 [10, Proposition 2.1] *Let W_Γ be a Coxeter group and W_Ω be a special parabolic subgroup. If W_Ω is an odd factor in W_Γ, then*

(i) *there exists a well-defined homomorphism $r : W_\Gamma \to W_\Omega$ such that $r_{|W_\Omega} = \mathrm{id}$ and $r(w) = 1$ for all $w \in V(\Gamma) - V(\Omega)$;*

(ii) *the kernel of r is the normal closure of the special parabolic subgroup $W_{V(\Gamma)-V(\Omega)}$.*

Let us examine the Coxeter group W_Γ defined by the graph Γ in Figure 2. Applying Proposition 2.9, some retractions of W_Γ are $W_\Gamma \twoheadrightarrow W_{\{v_1,v_2,v_3,v_6\}}$, $W_\Gamma \twoheadrightarrow W_{\{v_4,v_5\}}$ and $W_\Gamma \twoheadrightarrow W_{\{v_6\}}$.

2.5 Characteristic subgroups

For the proof of Theorem 1.4 the notion of characteristic subgroups with respect to a subgroup of the automorphism group will be needed. We refer to [16] for more information on these subgroups.

Definition 2.10 Let G be a group and $H \subseteq \mathrm{Aut}(G)$ be a subgroup. A normal subgroup $N \trianglelefteq G$ is called *characteristic with respect to* H if for all $f \in H$ we have $f(N) = N$.

Let $N \trianglelefteq G$ be characteristic with respect to $H \subseteq \mathrm{Aut}(G)$. Then the map $H \to \mathrm{Aut}(G/N)$ given by $f \mapsto [gN \mapsto f(g)N]$ is a well-defined group homomorphism; see [16, Lemma 7.4].

Definition 2.11 Let W_Γ be a Coxeter group. Define *the set of special automorphisms* $\mathrm{Spe}(W_\Gamma) \subseteq \mathrm{Aut}(W_\Gamma)$ as follows: $f \in \mathrm{Spe}(W_\Gamma)$ if and only if for every $w \in W_\Gamma$ with $w^2 = 1$ there exists $a \in W_\Gamma$ such that $f(w) = awa^{-1}$.

Clearly, $\mathrm{Spe}(W_\Gamma)$ is a subgroup and $\mathrm{Inn}(W_\Gamma)$ is contained in $\mathrm{Spe}(W_\Gamma)$. We can also define the subgroup $\mathrm{Spe}(W_\Gamma)$ as follows: It was proven in [20] that the cardinality of the set of conjugacy classes of subgroups of order 2 in W_Γ is finite. The group $\mathrm{Aut}(W_\Gamma)$ acts on these conjugacy classes in the canonical way. The kernel of this action is precisely the subgroup $\mathrm{Spe}(W_\Gamma)$. In particular, the normal subgroup $\mathrm{Spe}(W_\Gamma)$ has finite index in $\mathrm{Aut}(W_\Gamma)$.

Let W_Γ be a Coxeter group. A special parabolic subgroup W_Ω is called a *free factor* of W_Γ if $V(\Omega)$ is a union of the vertex sets of some connected components of Γ.

Lemma 2.12 *Let W_Γ be a Coxeter group and W_Ω a special parabolic subgroup. If W_Ω is an odd-factor, then the kernel of the canonical retraction map $r : W_\Gamma \twoheadrightarrow W_\Omega$ is a characteristic subgroup with respect to* $\mathrm{Spe}(W_\Gamma)$. *Thus each special automorphism $f \in \mathrm{Spe}(W_\Gamma)$ induces an automorphism $\bar{f}$ of $W_\Gamma/\ker(r) \cong W_\Omega$.*

Furthermore, the image of the group homomorphism

$$\Phi_\Omega : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_\Omega) \quad \text{given by } f \mapsto \bar{f}$$

contains $\mathrm{Inn}(W_\Omega)$. *Moreover, if W_Ω is a free factor of W_Γ, then Φ_Ω is surjective.*

Proof Let W_Ω be an odd-factor of W_Γ and $r : W_\Gamma \twoheadrightarrow W_\Omega$ be the retraction map. Let $K := \ker(r)$ and let f be a special automorphism of W_Γ. By Proposition 2.9 the group K is generated by the set $K' := \{xvx^{-1} \mid v \in V(\Gamma) - V(\Omega), x \in W_\Gamma\}$. Let $xvx^{-1} \in K'$. Since $f \in \mathrm{Spe}(W_\Gamma)$ and $\mathrm{ord}(xvx^{-1}) = 2$, there exists $a \in W_\Gamma$ such that

$$f(xvx^{-1}) = axvx^{-1}a^{-1} \in K'.$$

Thus $f(K) \subseteq K$. This shows that for all $g \in \mathrm{Spe}(W_\Gamma)$ we have $g(K) \subseteq K$. In particular, $f^{-1}(K) \subseteq K$ which yields $K \subseteq f(K)$ and hence $f(K) = K$. Thus, K is characteristic with respect to $\mathrm{Spe}(W_\Gamma)$.

A straightforward calculation shows that the group $\mathrm{Inn}(W_\Omega)$ is contained in the image of Φ_Ω.

If W_Ω is a free factor of W_Γ, then every (special) automorphism f of W_Ω has a canonical extension to $f' : W_\Gamma \to W_\Gamma$ via $f'(v) = v$ for $v \in V(\Gamma) - V(\Omega)$. Thus, if W_Ω is a free factor of W_Γ, then Φ_Ω is surjective. □

3 Proof of Theorem 1.4

Proof of Theorem 1.4 Let W_Γ be an infinite Coxeter group.

(1) Let W_Ω be a nonspherical odd factor of W_Γ such that the outer automorphism group $\mathrm{Out}(W_\Omega)$ is finite.

We claim $\mathrm{Spe}(W_\Gamma)$ surjects virtually onto $\mathrm{Inn}(W_\Omega)$. By Lemma 2.12 we have a group homomorphism

$$\Phi_\Omega : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_\Omega)$$

and the image of Φ_Ω contains $\mathrm{Inn}(W_\Omega)$. Since $\mathrm{Out}(W_\Omega)$ is finite, the group $\Phi_\Omega^{-1}(\mathrm{Inn}(W_\Omega))$ has finite index in $\mathrm{Spe}(W_\Gamma)$ and surjects onto $\mathrm{Inn}(W_\Omega)$. The proof of Lemma 1.3 shows that $\mathrm{Inn}(W_\Omega)$ maps onto an infinite special parabolic subgroup of W_Ω. Hence $\mathrm{Aut}(W_\Gamma)$ maps virtually onto an infinite special parabolic subgroup of W_Ω.

(2) Let $\Gamma_1, \dots, \Gamma_k$ be the connected components of Γ. By assumption Γ is disconnected, thus $k \geq 2$. We consider the special parabolic subgroup $W_{\Gamma_1} * W_{\Gamma_2}$. Clearly, this special parabolic subgroup is a free factor of W_Γ. Thus by Lemma 2.12 the map

$$\Phi : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_{\Gamma_1} * W_{\Gamma_2})$$

is surjective.

Now let $\rho : W_{\Gamma_i} * W_{\Gamma_j} \twoheadrightarrow W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}$ be the group homomorphism induced by the abelianization $W_{\Gamma_i} \twoheadrightarrow W_{\Gamma_i}^{\mathrm{ab}}$. The kernel $\ker(\rho)$ is characteristic with respect to $\mathrm{Spe}(W_{\Gamma_1} * W_{\Gamma_2})$.

Thus we obtain a group homomorphism

$$\Sigma \circ \Phi : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_{\Gamma_1} * W_{\Gamma_2}) \to \mathrm{Spe}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}).$$

The subgroup $\mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}})$ is contained in the image of $\Sigma \circ \Phi$. By [11, Corollary 3.21] the subgroup $\mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}})$ has finite index in $\mathrm{Aut}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}})$. Thus the subgroup $H := (\Sigma \circ \Phi)^{-1}(\mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}))$ has finite index in $\mathrm{Spe}(W_\Gamma)$ and we have an epimorphism

$$H \twoheadrightarrow \mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}) \cong W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z}^k * \mathbb{Z}/2\mathbb{Z}^l.$$

This shows that $\mathrm{Aut}(W_\Gamma)$ virtually maps onto the infinite right-angled Coxeter group $\mathbb{Z}/2\mathbb{Z}^k * \mathbb{Z}/2\mathbb{Z}^l$. □

Proof of Corollary 1.5 Let W_Γ be an infinite even Coxeter group. If Γ is 2-spherical, then by [5; 12] the outer automorphism group $\mathrm{Out}(W_\Gamma)$ is finite and by Lemma 1.3 it follows that $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto some infinite Coxeter group.

Assume now that Γ is not 2-spherical. Then there exist $v, w \in V(\Gamma)$ such that $\{v, w\} \notin E(\Gamma)$. The special parabolic subgroup $W_{\{v,w\}}$ is a nonspherical odd factor with finite outer automorphism group. Hence, by Theorem 1.4 $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto some infinite Coxeter group. □

Let us calculate an example. The Coxeter graph Γ in Figure 3 has exactly two connected components Γ_1 and Γ_2. The abelianization of W_{Γ_i} is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2$. Thus $\mathrm{Aut}(W_\Gamma)$ projects virtually onto $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

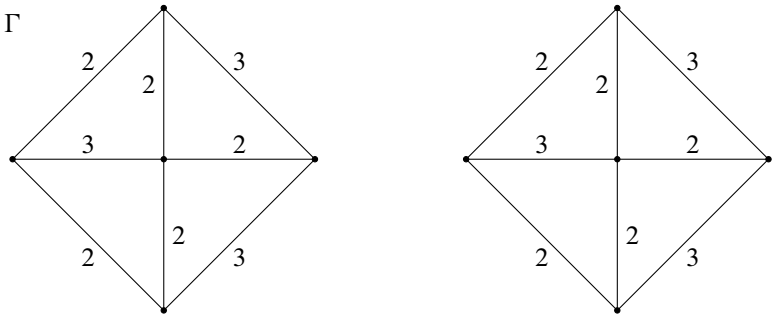

Figure 3: A Coxeter graph Γ with exactly two connected components.

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