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OLGA VARGHESE

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# Coxeter quotients of the automorphism group of a Coxeter group

OLGA VARGHESE

We show that for a large class $\mathcal{W}$ of infinite Coxeter groups the following holds: Given a group $W_\Gamma$ in $\mathcal{W}$, the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto some infinite Coxeter group. In particular, the group $\mathrm{Aut}(W_\Gamma)$ does not have Kazhdan's property (T). Moreover, we show that $\mathrm{Aut}(W_\Gamma)$ is amenable if and only if $W_\Gamma$ is virtually abelian.

## 1 Introduction

One fascinating property of a group is property (T). Property (T) is a representation-theoretic gadget for locally compact groups introduced by Kazhdan [15]. This property was reformulated by Delorme and Guichardet using geometric group theory. A countable group $G$ has Kazhdan's property (T) if every action of $G$ on a real Hilbert space by isometries has a global fixed point [2, Theorem 2.12.4].

Examples of groups satisfying this property are $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$ and $\mathrm{Out}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Z})$ for $n \geq 4$; see [2; 13; 14; 18].

Given a finite simplicial graph $\Gamma$ with the vertex set $V(\Gamma)$, the edge set $E(\Gamma)$ and with an edge-labeling $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$, the Coxeter group $W_\Gamma$ associated to $\Gamma$ is the group with the presentation

$$W_\Gamma = \langle V(\Gamma) \mid v^2, (vw)^{m(\{v,w\})} \text{ for all } v \in V(\Gamma) \text{ and whenever } \{v, w\} \in E(\Gamma)\rangle.$$

Coxeter groups are well understood objects in geometric group theory, but there are many open questions concerning their automorphism groups. We conjecture:

**Conjecture 1.1** *The group* $\mathrm{Aut}(W_\Gamma)$ *has Kazhdan's property* (T) *if and only if* $W_\Gamma$ *is finite.*

In particular, we are interested in algebraic properties of $\mathrm{Aut}(W_\Gamma)$ that prohibit this group from having property (T). After presenting several results regarding the above conjecture at the virtual geometric group theory conference in Luminy, 2020, Alain Valette formulated the following conjecture:

**Conjecture 1.2** *For every infinite Coxeter group* $W_\Gamma$, *the automorphism group* $\mathrm{Aut}(W_\Gamma)$ *virtually maps onto some infinite Coxeter group.*

We want to point out that a verification of Conjecture 1.2 would imply Conjecture 1.1, since it is known that infinite Coxeter groups virtually map onto $\mathbb{Z}$, see [7], and therefore do not have Kazhdan's property (T).

Our main goal is to verify Conjecture 1.2 for a large class of Coxeter groups.

Let $\Gamma$ be a Coxeter graph. For $S \subseteq V(\Gamma)$ the subgroup generated by $S$ is called a *special parabolic subgroup* of $W_\Gamma$. We note that the subgroup $\langle S\rangle \subseteq W_\Gamma$ is canonically isomorphic to $W_\Omega$ where the

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graph $\Omega$ is the full subgraph of $\Gamma$ with $V(\Omega) = S$. A special parabolic subgroup $W_\Omega$ is called *spherical* if $W_\Omega$ is finite.

Our first result regarding Conjecture 1.2 is about Coxeter groups with finite outer automorphism group.

**Lemma 1.3** *Let $W_\Gamma$ be a Coxeter group. If $W_\Gamma$ is infinite and the outer automorphism group* $\mathrm{Out}(W_\Gamma) := \mathrm{Aut}(W_\Gamma)/\mathrm{Inn}(W_\Gamma)$ *is finite, then* $\mathrm{Aut}(W_\Gamma)$ *virtually maps onto an infinite special parabolic subgroup of $W_\Gamma$.*

Twist-rigid Coxeter groups and virtually abelian Coxeter groups have finite outer automorphism groups; see [5, Corollary 1.2; 6]. An example of a Coxeter group $W_\Gamma$ which is not of any of those types and where $\mathrm{Out}(W_\Gamma)$ is finite is given in Figure 1 on page 2357.

Let $\Gamma$ be a Coxeter graph. We denote by $\Gamma_{\mathrm{odd}} \subseteq \Gamma$ the subgraph obtained from $\Gamma$ by removing all edges with even labels. A special parabolic subgroup $W_\Omega$ is called an *odd factor* of $W_\Gamma$ if $V(\Omega)$ is a union of the vertex sets of some connected components of $\Gamma_{\mathrm{odd}}$. Hence, if all edge-labels in $\Gamma$ are even, then every special parabolic subgroup of $W_\Gamma$ is an odd factor.

**Theorem 1.4** *Let $W_\Gamma$ be a Coxeter group. If*

(1) *there exists a nonspherical odd factor $W_\Omega$ of $W_\Gamma$ such that the outer automorphism group* $\mathrm{Out}(W_\Omega)$ *is finite, or*

(2) *the Coxeter graph $\Gamma$ is disconnected,*

*then* $\mathrm{Aut}(W_\Gamma)$ *virtually surjects onto some infinite Coxeter group.*

By definition, a Coxeter group $W_\Gamma$ is called *even* if $E(\Gamma) = \varnothing$ or $m(E(\Gamma)) \subseteq 2\mathbb{N}$. As a corollary of Theorem 1.4 we obtain the following result.

**Corollary 1.5** *Let $W_\Gamma$ be an infinite Coxeter group. If $W_\Gamma$ is even, then* $\mathrm{Aut}(W_\Gamma)$ *virtually surjects onto some infinite Coxeter group.*

If a group does not have Kazhdan's property (T), it is natural to ask if this group is amenable.

**Proposition 1.6** *Let $W_\Gamma$ be a Coxeter group. The group* $\mathrm{Aut}(W_\Gamma)$ *is amenable if and only if $W_\Gamma$ is virtually abelian.*

## Structure of the paper

In Section 2 we review some of the standard facts on simplicial graphs and Coxeter groups. This section also contains the proof of Proposition 1.6. Some results concerning finiteness of the outer automorphism group of a Coxeter group can also be find in this section. Further, in this section we recall useful facts about retractions on special parabolic subgroups of a Coxeter group and define characteristic subgroups with respect to a subgroup of an automorphism group. We end this section by proving Lemma 2.12 which is the main ingredient in the proof of Theorem 1.4. Section 3 contains the proofs of Theorem 1.4 and Corollary 1.5.

# 2 Preliminaries

## 2.1 Simplicial graphs

In this section we define simplicial graphs and collect some important properties of these combinatorial objects which we will need to define the groups of our interest. We largely follow [9].

A *simplicial graph* $\Gamma$ consists of a set $V(\Gamma)$ of *vertices* and a set $E(\Gamma)$ of subsets of $V(\Gamma)$ of cardinality two which are called *edges*. A graph $\Gamma$ is said to be *finite* if $V(\Gamma)$ is a finite set. Two vertices $v, w \in V(\Gamma)$ are called *adjacent*, if $\{v, w\}$ is an edge of $\Gamma$. By an *edge-labeling* of $\Gamma$ we mean just a map $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$. If all the vertices of $\Gamma$ are pairwise adjacent, then $\Gamma$ is called *complete*. A graph $\Lambda = (X, Y)$ is a subgraph of $\Gamma$ if $X \subseteq V(\Gamma)$, $Y \subseteq E(\Gamma)$ and all vertices of the edges in $Y$ are in $X$. A (*maximal*) *clique* in $\Gamma$ is a (maximal) complete subgraph of $\Gamma$. Given a subset $S \subseteq V(\Gamma)$, the *graph generated/induced by* $S$, denoted by $\langle S \rangle$, is the graph with vertex set $S$ and edge set $\{\{v, w\} \in E(\Gamma) \mid v, w \in S\}$. A subgraph $\Lambda = (X, Y)$ of $\Gamma$ is called *full* if $\langle X \rangle = \Lambda$. A graph $\Gamma$ is called *connected* if for every two vertices $v, w \in V(\Gamma)$ there exist vertices $v_1, \ldots, v_n \in V(\Gamma)$ such that $\{v, v_1\}, \{v_i, v_{i+1}\}, \{v_n, w\} \in E(\Gamma)$ for $i = 1, \ldots, n-1$. A subgraph $\Lambda$ is a *connected component* of $\Gamma$ if $\Lambda$ is connected and is maximal with this property.

## 2.2 Coxeter groups

Let us recall the basic definitions and notation concerning Coxeter groups. For further details we refer to [8].

**Definition 2.1** Let $\Gamma$ be a finite simplicial graph with an edge-labeling $m : E(\Gamma) \to \mathbb{N}_{\geq 2}$. We call such a labeled graph a *Coxeter graph*. The Coxeter group $W_\Gamma$ is defined as

$$W_\Gamma = \langle V(\Gamma) \mid v^2, (vw)^{m(\{v,w\})} \text{ for all } v \in V(\Gamma) \text{ and whenever } \{v, w\} \in E(\Gamma) \rangle.$$

Given a Coxeter graph $\Gamma$, the next proposition shows that each subset $X \subseteq V(\Gamma)$ generates a Coxeter group. For a proof we refer to [8, Theorem 4.1.6].

**Proposition 2.2** *Let $\Gamma$ be a Coxeter graph. For each $X \subseteq V(\Gamma)$, the subgroup which is generated by $X$ is canonically isomorphic to the Coxeter group $W_\Lambda$ where $\Lambda$ is the subgraph of $\Gamma$ generated by $X$.*

The above proposition leads to the following definition.

**Definition 2.3** Let $\Gamma$ be a Coxeter graph. For a full subgraph $\Lambda \subseteq \Gamma$, the subgroup $W_\Lambda$ of $W_\Gamma$ is called *special parabolic* and any conjugate of a special parabolic subgroup is called *parabolic*. Note that sometimes we write for $W_\Lambda$ just $W_{V(\Lambda)}$.

Let $W_\Gamma$ be a Coxeter group. By definition, $W_\Gamma$ is called *$k$-spherical* if every special parabolic subgroup $W_\Lambda$ with $|V(\Lambda)| = k$ is finite. In particular, $W_\Gamma$ is 2-spherical if and only if $\Gamma$ is a complete graph.

The building blocks of a Coxeter group are irreducible special parabolic subgroups.

**Definition 2.4** A Coxeter group $W_\Gamma$ is called *reducible* if $V(\Gamma)$ can be partitioned into two nonempty disjoint subsets $V_1$ and $V_2$ such that $\{x, y\} \in E(\Gamma)$ for all $x \in V_1$, $y \in V_2$ and $m(\{x, y\}) = 2$ for all $x \in V_1$ and $y \in V_2$. In that case $W_\Gamma \cong W_{V_1} \times W_{V_2}$ [8, Proposition 4.1.7]. If this sort of decomposition is not possible, we call the Coxeter group $W_\Gamma$ *irreducible*.

We are now going to recall the definition of the centralizer of a subgroup $H$ in a group $G$. The *centralizer* of $H$ in $G$, denoted by $Z_G(H)$, is defined as

$$Z_G(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

The *center* of $G$ is the subgroup $Z(G) := Z_G(G)$.

Given a Coxeter group $W_\Gamma$ and an arbitrary subgroup $H$, it is in general not known what $Z_{W_\Gamma}(H)$ is. However, when $H$ is a special parabolic subgroup one can say a lot about the structure of the centralizer of $H$ in $W_\Gamma$; see [1; 3; 4; 19]. In particular, the center of $W_\Gamma$ can be calculated as follows.

**Lemma 2.5** *Let $W_\Gamma$ be a Coxeter group. Write $W_\Gamma \cong W_{\Gamma_1} \times \cdots \times W_{\Gamma_n}$ as a direct product of irreducible special parabolic subgroups. We define $J := \{i \in \{1, \dots, n\} \mid W_{\Gamma_i} \text{ is finite}\}$. Then $Z(W_\Gamma) \cong \prod_{j \in J} Z(W_{\Gamma_j})$.*

**Proof** The center of $W_\Gamma$ is isomorphic to $Z(W_{\Gamma_1}) \times \cdots \times Z(W_{\Gamma_n})$. Since the center of an irreducible infinite Coxeter group is trivial [8, Theorem D.2.10] we obtain $Z(W_\Gamma) \cong \prod_{j \in J} Z(W_{\Gamma_j})$. □

## 2.3 Finite outer automorphism group

In this section we are in particular interested in Coxeter groups with finite outer automorphism group. First we give a proof of Lemma 1.3.

**Proof of Lemma 1.3** We write $W_\Gamma$ as a direct product of irreducible special parabolic subgroups $W_\Gamma \cong W_{\Gamma_1} \times W_{\Gamma_2} \times \cdots \times W_{\Gamma_n}$. Since $W_\Gamma$ is infinite by assumption, there exists $j \in \{1, \dots, n\}$ such that $W_{\Gamma_j}$ is infinite. Note that the center of $W_{\Gamma_j}$ is trivial. We obtain

$$\mathrm{Inn}(W_\Gamma) \cong W_\Gamma / Z(W_\Gamma) \cong W_{\Gamma_1}/Z(W_{\Gamma_1}) \times W_{\Gamma_2}/Z(W_{\Gamma_2}) \times \cdots \times W_{\Gamma_n}/Z(W_{\Gamma_n}) \xrightarrow{\pi} W_{\Gamma_j},$$

where $\pi$ is the canonical projection onto $W_{\Gamma_j}$. By assumption the index of $\mathrm{Inn}(W_\Gamma)$ in $\mathrm{Aut}(W_\Gamma)$ is finite, hence $\mathrm{Aut}(W_\Gamma)$ maps virtually onto the special parabolic subgroup $W_{\Gamma_j}$. □

Now we prove Proposition 1.6.

**Proof of Proposition 1.6** Let $W_\Gamma$ be a Coxeter group. If $W_\Gamma$ is virtually abelian, then $\mathrm{Out}(W_\Gamma)$ is finite; see [6]. In particular, $\mathrm{Aut}(W_\Gamma)$ is virtually abelian and therefore amenable. If $W_\Gamma$ is not virtually abelian, then the free group $F_2$ is a subgroup of $W_\Gamma$ by [8, Theorem 14.1.2]. Thus $F_2 \subseteq \mathrm{Inn}(W_\Gamma)$. It is known that $F_2$ is not amenable and amenability is preserved by taking subgroups, hence $\mathrm{Aut}(W_\Gamma)$ is not amenable. □

It was shown in [5, Corollary 1.2] that the outer automorphism group of a twist-rigid Coxeter group is finite. For definition of twist-rigidity, see [5].

**Corollary 2.6** *Let $W_\Gamma$ be an infinite Coxeter group. If $W_\Gamma$ is twist-rigid, then $\mathrm{Aut}(W_\Gamma)$ satisfies Conjecture 1.2.*

Let us consider the Coxeter group $W_\Gamma$ where $\Gamma$ is isomorphic to the graph in Figure 1.

The Coxeter group $W_\Gamma$ is not twist-rigid. Our next goal is to show that the outer automorphism group of $W_\Gamma$ is finite.

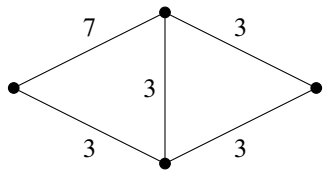


Figure 1: Coxeter graph $\Gamma$.

**Theorem 2.7** [17, Theorem 28] *Let $\Gamma$ be a Coxeter graph and $\Delta_1, \dots, \Delta_n$ be all maximal cliques of $\Gamma$. Let $C(W_\Gamma)$ be the subgroup of all $f \in \mathrm{Aut}(W_\Gamma)$ such that for $i \in \{1, \dots, n\}$ there exists an element $w_{i,f} \in W_\Gamma$ so that*

$$f(x) = w_{i,f} \cdot x \cdot {w_{i,f}}^{-1} \quad \text{for all } x \in W_{\Delta_i}.$$

*The subgroup $C(W_\Gamma)$ has finite index in $\mathrm{Aut}(W_\Gamma)$.*

A natural question is when the subgroup $C(W_\Gamma)$ is equal to the inner automorphism group of $W_\Gamma$. If $\Gamma$ is not connected, then in most cases $C(W_\Gamma)$ is much larger than $\mathrm{Inn}(W_\Gamma)$ as the following observation shows: Let $\Gamma_1, \dots, \Gamma_n$ be the connected components of $\Gamma$ and $W_{\Gamma_1}, \dots, W_{\Gamma_n}$ the corresponding special parabolic subgroups. Then $W_\Gamma = W_{\Gamma_1} * \cdots * W_{\Gamma_n}$. We assume that $n \geq 2$. Let $a$ be a noncentral element in $W_{\Gamma_1}$, more precisely $a \in W_{\Gamma_1}$ and $a \notin Z(W_{\Gamma_1})$. We define $f_{12} : W_\Gamma \to W_\Gamma$ as follows: $f_{12}(v) = v$ for all $v \in V(\Gamma) - V(\Gamma_2)$ and $f_{12}(w) = awa^{-1}$ for all $w \in V(\Gamma_2)$. The map $f_{12}$ induces naturally an automorphism of $W_\Gamma$. Moreover $f_{12}$ is contained in $C(W_\Gamma)$. Since the element $a$ does not commute with $W_{\Gamma_1}$ the automorphism $f_{12}$ is not inner.

**Theorem 2.8** *Let $\Gamma$ be a connected Coxeter graph. We denote by $\Delta_1, \dots, \Delta_n$ its maximal cliques and by $\Omega_1, \dots, \Omega_m$ the nonempty intersections of these maximal cliques where the intersection is taken over at least two maximal cliques.*

*If the centralizer of the special parabolic subgroup $W_{\Omega_i}$ is trivial for all $i = 1, \dots, m$, then the subgroup $C(W_\Gamma)$ equals $\mathrm{Inn}(W_\Gamma)$. In particular, $\mathrm{Inn}(W_\Gamma)$ has finite index in $\mathrm{Aut}(W_\Gamma)$.*

**Proof** Let $f$ be in $C(W_\Gamma)$. Our goal is to show that $f$ is an inner automorphism of $W_\Gamma$. Clearly, it is sufficient to show that there exists $a \in W_\Gamma$ such that for all $v \in V(\Gamma)$ we have $f(v) = ava^{-1}$.

By the definition of $C(W_\Gamma)$ we know that there exists $w_{1,f} \in W_\Gamma$ such that for all $x \in W_{\Delta_1}$ we have $f(x) = w_{1,f} \cdot x \cdot {w_{i,f}}^{-1}$. If $\Gamma$ itself is a clique, then we are done. Otherwise, since $\Gamma$ is connected there exists a vertex $v \in V(\Gamma) - V(\Delta_1)$ and $w \in V(\Delta_1)$ such that $\{v, w\} \in E(\Gamma)$. Further, there exists $\Delta_j$ such that $v, w \in V(\Delta_j)$. Hence, the intersection of $\Delta_1$ and $\Delta_j$ is nonempty. Again, by definition of $C(W_\Gamma)$ there exists $w_{j,f} \in W_\Gamma$ such that $f(x) = w_{j,f} \cdot x \cdot {w_{j,f}}^{-1}$ for all $x \in W_{\Delta_j}$.

For $x \in W_{\Delta_1} \cap W_{\Delta_j} = W_{\Delta_1 \cap \Delta_j}$ we have

$$f(x) = w_{1,f} \cdot x \cdot {w_{1,f}}^{-1} = w_{j,f} \cdot x \cdot {w_{j,f}}^{-1}.$$

Hence ${w_{j,f}}^{-1} w_{1,f} \in Z_{W_\Gamma}(W_{\Delta_1 \cap \Delta_j})$. By assumption, this centralizer is trivial, thus $w_{1,f} = w_{j,f}$.

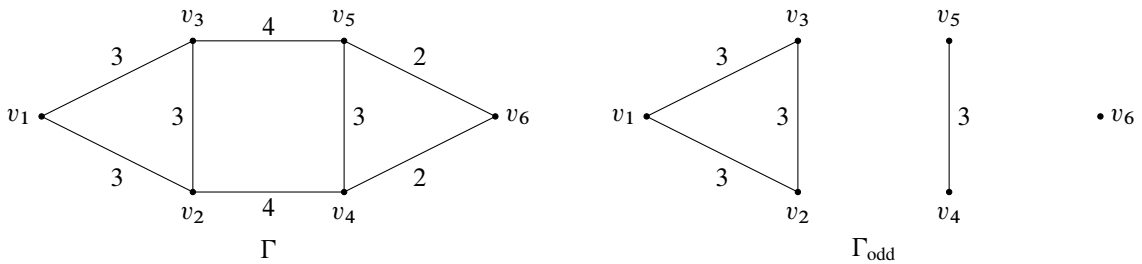


Figure 2: A Coxeter graph $\Gamma$ and its odd graph $\Gamma_{\text{odd}}$.

In particular, the above proof strategy shows that for $k \in \{1, \dots, n\}$ where $\Delta_k \cap \Delta_1 \neq \varnothing$ we have

$$f(v) = w_{1,f} \cdot v \cdot {w_{1,f}}^{-1} \quad \text{for all } v \in V(\Delta_k).$$

Let $J := \{k \in \{1, \dots, n\} \mid \Delta_k \cap \Delta_1 \neq \varnothing\}$. If $V(\Gamma) = \bigcup_{k \in J} V(\Delta_k)$, then we have proved that $f$ is a global conjugation by $w_{1,f}$. Otherwise the set $V(\Gamma) - \bigcup_{k \in J} V(\Delta_k)$ is nonempty. Since $\Gamma$ is connected, there exist $v \in V(\Gamma) - \bigcup_{k \in J} V(\Delta_k)$ and $w \in \bigcup_{k \in J} V(\Delta_k)$ such that $\{v, w\} \in E(\Gamma)$. We choose $\Delta_l$ such that $v, w \in V(\Delta_l)$. Hence there exists $k \in J$ such that $\Delta_l \cap \Delta_k \neq \varnothing$. Now we proceed as above to show that $f(x) = w_{1,f} \cdot x \cdot {w_{1,f}}^{-1}$ for all $x \in \Delta_l$.

After applying the above proof strategy finitely many times, we see that $f$ is a global conjugation by $w_{1,f}$. □

## 2.4 Retractions on special parabolic subgroups

By definition, a Coxeter group $W_\Gamma$ is called *right-angled* if $E(\Gamma) = \varnothing$ or $m(E(\Gamma)) = \{2\}$. It is well known that for a right-angled Coxeter group $W_\Gamma$ and a special parabolic subgroup $W_\Lambda$ there exists a retraction homomorphism $r : W_\Gamma \twoheadrightarrow W_\Lambda$ such that $r(v) = v$ for all $v \in V(\Lambda)$ and $r(w) = 1$ for all $w \in V(\Gamma) - V(\Lambda)$. This kind of a retraction map exists also for other Coxeter groups.

Let $\Gamma$ be a Coxeter graph. We denote by $\Gamma_{\text{odd}} \subseteq \Gamma$ the subgraph obtained from $\Gamma$ by removing all edges with even labels. We note that the abelianization of $W_\Gamma$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}^k$ where $k$ is the number of connected components of $\Gamma_{\text{odd}}$. A special parabolic subgroup $W_\Omega$ is called an *odd factor* of $W_\Gamma$ if $V(\Omega)$ is a union of the vertex sets of some connected components of $\Gamma_{\text{odd}}$.

**Proposition 2.9** [10, Proposition 2.1] *Let $W_\Gamma$ be a Coxeter group and $W_\Omega$ be a special parabolic subgroup. If $W_\Omega$ is an odd factor in $W_\Gamma$, then*

(i) *there exists a well-defined homomorphism $r : W_\Gamma \to W_\Omega$ such that $r_{|W_\Omega} = \mathrm{id}$ and $r(w) = 1$ for all $w \in V(\Gamma) - V(\Omega)$;*

(ii) *the kernel of $r$ is the normal closure of the special parabolic subgroup $W_{V(\Gamma)-V(\Omega)}$.*

Let us examine the Coxeter group $W_\Gamma$ defined by the graph $\Gamma$ in Figure 2. Applying Proposition 2.9, some retractions of $W_\Gamma$ are $W_\Gamma \twoheadrightarrow W_{\{v_1,v_2,v_3,v_6\}}$, $W_\Gamma \twoheadrightarrow W_{\{v_4,v_5\}}$ and $W_\Gamma \twoheadrightarrow W_{\{v_6\}}$.

## 2.5 Characteristic subgroups

For the proof of Theorem 1.4 the notion of characteristic subgroups with respect to a subgroup of the automorphism group will be needed. We refer to [16] for more information on these subgroups.

**Definition 2.10** Let $G$ be a group and $H \subseteq \mathrm{Aut}(G)$ be a subgroup. A normal subgroup $N \trianglelefteq G$ is called *characteristic with respect to* $H$ if for all $f \in H$ we have $f(N) = N$.

Let $N \trianglelefteq G$ be characteristic with respect to $H \subseteq \mathrm{Aut}(G)$. Then the map $H \to \mathrm{Aut}(G/N)$ given by $f \mapsto [gN \mapsto f(g)N]$ is a well-defined group homomorphism; see [16, Lemma 7.4].

**Definition 2.11** Let $W_\Gamma$ be a Coxeter group. Define *the set of special automorphisms* $\mathrm{Spe}(W_\Gamma) \subseteq \mathrm{Aut}(W_\Gamma)$ as follows: $f \in \mathrm{Spe}(W_\Gamma)$ if and only if for every $w \in W_\Gamma$ with $w^2 = 1$ there exists $a \in W_\Gamma$ such that $f(w) = awa^{-1}$.

Clearly, $\mathrm{Spe}(W_\Gamma)$ is a subgroup and $\mathrm{Inn}(W_\Gamma)$ is contained in $\mathrm{Spe}(W_\Gamma)$. We can also define the subgroup $\mathrm{Spe}(W_\Gamma)$ as follows: It was proven in [20] that the cardinality of the set of conjugacy classes of subgroups of order 2 in $W_\Gamma$ is finite. The group $\mathrm{Aut}(W_\Gamma)$ acts on these conjugacy classes in the canonical way. The kernel of this action is precisely the subgroup $\mathrm{Spe}(W_\Gamma)$. In particular, the normal subgroup $\mathrm{Spe}(W_\Gamma)$ has finite index in $\mathrm{Aut}(W_\Gamma)$.

Let $W_\Gamma$ be a Coxeter group. A special parabolic subgroup $W_\Omega$ is called a *free factor* of $W_\Gamma$ if $V(\Omega)$ is a union of the vertex sets of some connected components of $\Gamma$.

**Lemma 2.12** *Let $W_\Gamma$ be a Coxeter group and $W_\Omega$ a special parabolic subgroup. If $W_\Omega$ is an odd-factor, then the kernel of the canonical retraction map $r : W_\Gamma \twoheadrightarrow W_\Omega$ is a characteristic subgroup with respect to* $\mathrm{Spe}(W_\Gamma)$. *Thus each special automorphism $f \in \mathrm{Spe}(W_\Gamma)$ induces an automorphism $\bar{f}$ of $W_\Gamma/\ker(r) \cong W_\Omega$.*

*Furthermore, the image of the group homomorphism*

$$\Phi_\Omega : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_\Omega) \quad \text{given by } f \mapsto \bar{f}$$

*contains* $\mathrm{Inn}(W_\Omega)$. *Moreover, if $W_\Omega$ is a free factor of $W_\Gamma$, then $\Phi_\Omega$ is surjective.*

**Proof** Let $W_\Omega$ be an odd-factor of $W_\Gamma$ and $r : W_\Gamma \twoheadrightarrow W_\Omega$ be the retraction map. Let $K := \ker(r)$ and let $f$ be a special automorphism of $W_\Gamma$. By Proposition 2.9 the group $K$ is generated by the set $K' := \{xvx^{-1} \mid v \in V(\Gamma) - V(\Omega), x \in W_\Gamma\}$. Let $xvx^{-1} \in K'$. Since $f \in \mathrm{Spe}(W_\Gamma)$ and $\mathrm{ord}(xvx^{-1}) = 2$, there exists $a \in W_\Gamma$ such that

$$f(xvx^{-1}) = axvx^{-1}a^{-1} \in K'.$$

Thus $f(K) \subseteq K$. This shows that for all $g \in \mathrm{Spe}(W_\Gamma)$ we have $g(K) \subseteq K$. In particular, $f^{-1}(K) \subseteq K$ which yields $K \subseteq f(K)$ and hence $f(K) = K$. Thus, $K$ is characteristic with respect to $\mathrm{Spe}(W_\Gamma)$.

A straightforward calculation shows that the group $\mathrm{Inn}(W_\Omega)$ is contained in the image of $\Phi_\Omega$.

If $W_\Omega$ is a free factor of $W_\Gamma$, then every (special) automorphism $f$ of $W_\Omega$ has a canonical extension to $f' : W_\Gamma \to W_\Gamma$ via $f'(v) = v$ for $v \in V(\Gamma) - V(\Omega)$. Thus, if $W_\Omega$ is a free factor of $W_\Gamma$, then $\Phi_\Omega$ is surjective. □

## 3 Proof of Theorem 1.4

**Proof of Theorem 1.4** Let $W_\Gamma$ be an infinite Coxeter group.

(1) Let $W_\Omega$ be a nonspherical odd factor of $W_\Gamma$ such that the outer automorphism group $\mathrm{Out}(W_\Omega)$ is finite.

We claim $\mathrm{Spe}(W_\Gamma)$ surjects virtually onto $\mathrm{Inn}(W_\Omega)$. By Lemma 2.12 we have a group homomorphism

$$\Phi_\Omega : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_\Omega)$$

and the image of $\Phi_\Omega$ contains $\mathrm{Inn}(W_\Omega)$. Since $\mathrm{Out}(W_\Omega)$ is finite, the group $\Phi_\Omega^{-1}(\mathrm{Inn}(W_\Omega))$ has finite index in $\mathrm{Spe}(W_\Gamma)$ and surjects onto $\mathrm{Inn}(W_\Omega)$. The proof of Lemma 1.3 shows that $\mathrm{Inn}(W_\Omega)$ maps onto an infinite special parabolic subgroup of $W_\Omega$. Hence $\mathrm{Aut}(W_\Gamma)$ maps virtually onto an infinite special parabolic subgroup of $W_\Omega$.

(2) Let $\Gamma_1, \dots, \Gamma_k$ be the connected components of $\Gamma$. By assumption $\Gamma$ is disconnected, thus $k \geq 2$. We consider the special parabolic subgroup $W_{\Gamma_1} * W_{\Gamma_2}$. Clearly, this special parabolic subgroup is a free factor of $W_\Gamma$. Thus by Lemma 2.12 the map

$$\Phi : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_{\Gamma_1} * W_{\Gamma_2})$$

is surjective.

Now let $\rho : W_{\Gamma_i} * W_{\Gamma_j} \twoheadrightarrow W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}$ be the group homomorphism induced by the abelianization $W_{\Gamma_i} \twoheadrightarrow W_{\Gamma_i}^{\mathrm{ab}}$. The kernel $\ker(\rho)$ is characteristic with respect to $\mathrm{Spe}(W_{\Gamma_1} * W_{\Gamma_2})$.

Thus we obtain a group homomorphism

$$\Sigma \circ \Phi : \mathrm{Spe}(W_\Gamma) \to \mathrm{Spe}(W_{\Gamma_1} * W_{\Gamma_2}) \to \mathrm{Spe}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}).$$

The subgroup $\mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}})$ is contained in the image of $\Sigma \circ \Phi$. By [11, Corollary 3.21] the subgroup $\mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}})$ has finite index in $\mathrm{Aut}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}})$. Thus the subgroup $H := (\Sigma \circ \Phi)^{-1}(\mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}))$ has finite index in $\mathrm{Spe}(W_\Gamma)$ and we have an epimorphism

$$H \twoheadrightarrow \mathrm{Inn}(W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}}) \cong W_{\Gamma_i}^{\mathrm{ab}} * W_{\Gamma_j}^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z}^k * \mathbb{Z}/2\mathbb{Z}^l.$$

This shows that $\mathrm{Aut}(W_\Gamma)$ virtually maps onto the infinite right-angled Coxeter group $\mathbb{Z}/2\mathbb{Z}^k * \mathbb{Z}/2\mathbb{Z}^l$. □

**Proof of Corollary 1.5** Let $W_\Gamma$ be an infinite even Coxeter group. If $\Gamma$ is 2-spherical, then by [5; 12] the outer automorphism group $\mathrm{Out}(W_\Gamma)$ is finite and by Lemma 1.3 it follows that $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto some infinite Coxeter group.

Assume now that $\Gamma$ is not 2-spherical. Then there exist $v, w \in V(\Gamma)$ such that $\{v, w\} \notin E(\Gamma)$. The special parabolic subgroup $W_{\{v,w\}}$ is a nonspherical odd factor with finite outer automorphism group. Hence, by Theorem 1.4 $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto some infinite Coxeter group. □

Let us calculate an example. The Coxeter graph $\Gamma$ in Figure 3 has exactly two connected components $\Gamma_1$ and $\Gamma_2$. The abelianization of $W_{\Gamma_i}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2$. Thus $\mathrm{Aut}(W_\Gamma)$ projects virtually onto $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

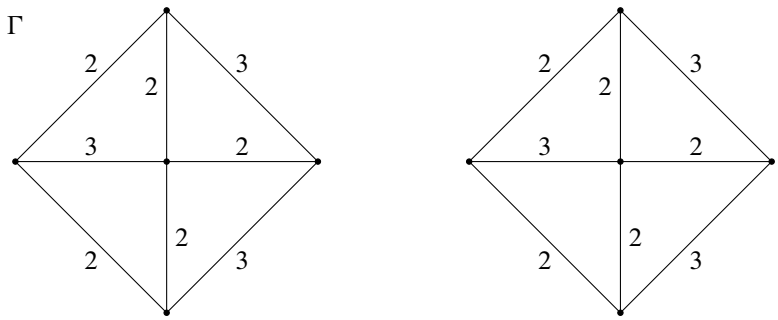


Figure 3: A Coxeter graph $\Gamma$ with exactly two connected components.

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OLGA VARGHESE olga.varghese@uni-muenster.de
*Faculty of Mathematics and Computer Science, University of Münster, Münster, Germany*

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