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An elementary proof of Novikov's theorem

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Novikov's theorem states that, given a taut (codimension-one) foliation on a closed 3-manifold M, the fundamental group of any leaf injects into the fundamental group of M. We use foliated branched covers to give an elementary proof of this result.

1 Introduction

Codimension-one foliations[1] are extremely flexible and, apart from the Euler characteristic, do not restrict the topology of the manifolds on which they live [5; 11; 20]. Therefore, it is interesting to consider special classes of foliations. One such class consists of taut foliations, and some of their topological properties are described by Novikov's theorem [15]. In this note, we provide a new proof of this classical result.

Let us start by recalling the main definition:

Definition 1 A codimension-one foliation $\mathcal{F}$ on a closed, orientable, 3-manifold M is *taut* if any of the following equivalent conditions hold:

(i) There exists a transverse loop intersecting every leaf of $\mathcal{F}$.

(ii) For any leaf L of $\mathcal{F}$ there exists a closed transversal intersecting L.

(iii) There exists a closed form $\omega \in \Omega^2(M)$ whose restriction $\omega|_{\mathcal{F}}$ is nondegenerate.

(iv) There exists a Riemannian metric on M for which the leaves of $\mathcal{F}$ are minimal surfaces.

For a detailed discussion of this definition we refer the reader to [1; 3]. Taut foliations closely interact with the topology of the ambient manifold, as the following theorem shows.

Theorem [15] *Let $\mathcal{F}$ be a taut foliation on a closed, orientable 3-manifold M. Then for any leaf L the inclusion $\iota : L \to M$ induces an injection*

$$\iota_* : \pi_1(L) \to \pi_1(M).$$

Novikov's original proof [15], although elementary, is rather delicate. The most recent proofs rely on the theory of minimal surfaces [8; 16; 18; 19]; see [1] for an excellent exposition.

In this note we reprove this result using another characterization of taut foliations. In short, given a foliated manifold $(M, \mathcal{F})$, a foliated branched cover is a smooth map $\pi : (M, \mathcal{F}) \to \mathbb{CP}^1$ whose restriction to any leaf of $\mathcal{F}$ defines a branched cover; see Definition 4 for a more detailed description. The regular fibers of π are closed transversals to the leaves of $\mathcal{F}$. Consequently, any foliation admitting a foliated

[1] Unless explicitly stated otherwise, we will always assume our foliations to be smooth, coorientable, and codimension-one.

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branched cover is taut in the sense of Definition 1. The converse can be proven in several ways. An analytical proof is given in [10, Corollary 1.3] while the main theorem of [2] provides a combinatorial proof.

Theorem [2] *A foliation $\mathcal{F}$ on a closed 3-manifold is taut if and only if it admits a foliated branched cover.*

In light of this characterization, Novikov's theorem is a consequence of the following, which is our main result:

Main Theorem *Let $\mathcal{F}$ be a foliation on a closed, orientable 3-manifold M which admits a foliated branched cover. Then, for any leaf L the inclusion $\iota : L \to M$ induces an injection*

$$\iota_* : \pi_1(L) \to \pi_1(M).$$

The proof consists of two steps. First, we show that a foliation admitting a foliated branched cover does not have any vanishing cycles; see Proposition 6. We then follow a standard argument to show that for such foliations the fundamental group of a leaf injects into the fundamental group of the ambient space; see Section 5.

Organization of the paper

The first four sections contain the required preliminaries. In Section 2 we recall Ascoli's theorem. Section 3 contains the basics on foliated branched covers, including Proposition 6 which is the main ingredient for the proof of Main Theorem. Section 4 contains the definition of Morse maps into foliations and some properties of their characteristic foliations. The proof of the main theorem, is then given in Section 5.

2 Ascoli's theorem

We recall here the well-known Ascoli's theorem in the setting of jet bundles. Its main use will be to show that foliated branched covers do not admit vanishing cycles; see Definition 5 and Proposition 6.

Consider manifolds X and Y, and suppose Y is endowed with a metric d. Consider a subset $S \subset C^\infty(X, Y)$ of the space of smooth maps.

- The set S is said to be *equicontinuous* at $x_0 \in X$ if, given $\varepsilon > 0$, there exists a neighborhood U of x_0 such that
$$d(f(x), f(x_0)) < \varepsilon$$
for all $f \in S$ and $x \in U$. We say S is equicontinuous if it is equicontinuous at every $x \in X$.
- The set S is said to be *pointwise precompact* if for each $x \in X$ the set
$$S_x := \{f(x) \mid f \in S\} \subset Y$$
has compact closure.

Next, recall that the r-th order jet bundle $\pi : J^r(X,Y) \to X$ consists of all r-jets of maps from X to Y. Its space of sections is denoted by $\Gamma(J^r(X,Y))$. A section $s \in \Gamma(J^r(X,Y))$ is called holonomic if $s = j^r f$ for some $f : X \to Y$, i.e., if s is the r-th order jet of a map f. See [6, Chapter 2] for a detailed introduction to the theory of jet bundles. Given a subset $S \subset C^\infty(X,Y)$, we denote by

$$J^k S := \{j^k f \mid f \in S\} \subset \Gamma(J^k(X,Y)) \tag{1}$$

the family of k-th order jets of maps in S.

Theorem 2 (Ascoli's theorem) *Suppose we are given a subset $S \subset C^\infty(X,Y)$ such that $J^k S$ is equicontinuous and pointwise precompact. Then, any sequence in $J^k S$ contains a subsequence converging (in the C^0-norm on $\Gamma(J^k(X,Y))$) to a holonomic section.*

Proof Since J^k is equicontinuous and pointwise precompact, Ascoli's theorem implies it has compact closure. For a proof of this result we refer the reader to [14, Theorem 47.1]. Furthermore, the space of holonomic sections is closed with respect to the C^0-norm; see, for example, [17, Theorem 7.17]. It follows that any sequence in $J^k S$ has a subsequence converging (in the C^0-norm) to a holonomic section. □

A map $f : X \to (Y, \mathcal{F})$ from a manifold X into a foliated manifold $(Y, \mathcal{F})$ is called a leafwise map if its image is contained in a single leaf of $\mathcal{F}$. Ascoli's theorem yields the following sufficient conditions for the convergence of a sequence of leafwise maps:

Lemma 3 *Consider a compact manifold Y (possibly with nonempty boundary) endowed with an auxiliary metric $d : Y \times Y \to \mathbb{R}$, and a foliation $\mathcal{F}$. Furthermore, let $\pi : Y \to B$ be a surjective submersion from Y onto a compact manifold B, whose fibers are transverse to $\mathcal{F}$. Suppose we are given a compact, connected manifold X and a family of leafwise maps $g_n : X \to (Y, \mathcal{F})$, $n \in \mathbb{N}$, satisfying:*

(i) *All the g_n lift the same map $f : X \to B$, i.e.,*

$$\pi \circ g_n = f \quad \text{for all } n \in \mathbb{N}.$$

(ii) *There exist $x_0 \in X$ and $y_0 \in Y$ such that*

$$\lim_{n\to\infty} g_n(x_0) = y_0.$$

Then, there exists a unique leafwise map $g : X \to (Y, \mathcal{F})$ lifting f, and such that the g_n converge to g in the C^∞-topology.

Proof We define the set $S := \{g_n \mid n \in \mathbb{N}\} \subset C^\infty(X,Y)$, and using the notation of (1), we consider

$$J^k S = \{j^k g_n \mid n \in \mathbb{N}\} \subset \Gamma(J^k(X,Y)) \tag{2}$$

for $k \geq 0$.

The first step of the proof is to show that, for any $k \geq 0$, the set $J^k S$ is equicontinuous and pointwise precompact. By Theorem 2 this implies that, for each $k \geq 0$, there exists a subsequence (of the

sequence $(g_n)_{n\in\mathbb{N}}$) converging in the C^k-norm on $C^\infty(X,Y)$. Hence, using a standard diagonalization argument, we find a subsequence converging in the C^∞-norm to a leafwise map $g : X \to (Y,\mathcal{F})$ lifting f. The second step in the proof consists of showing that this limit map is unique.

Step 1 (J^kS is equicontinuous and pointwise precompact) In order to show that the sets J^kS from (2) are equicontinuous and pointwise precompact, it suffices to show that for each $k \geq 0$ the set

$$T_k := \{j^k g_n(x) \mid x \in X, n \in \mathbb{N}\} \subset J^k(X,Y)$$

is contained in a compact set. Indeed, since $J^k(X,Y)$ is Hausdorff, this implies that

$$J^kS_x := \{j^k g_n(x) \mid n \in \mathbb{N}\} \subset T_k$$

has compact closure for any $x \in X$. Hence, J^kS is pointwise precompact. To see that J^kS is equicontinuous we choose some auxiliary metrics d_X and d_Y on X and Y, respectively. This induces metrics on $J^k(X,Y)$ for each $k \geq 0$, which by slight abuse of notation we all denote by d. If we identify $j^{k+1}g_n$ with the derivative of $j^k g_n : X \to J^k(X,Y)$, then it follows that for any fixed $x \in X$ we have

$$d(j^k g_n(x'), j^k g_n(x)) \leq \max_{x'\in X} \|j^{k+1} g_n(x')\|\, d_X(x',x) \leq C d_X(x',x) \quad \text{for all } x' \in X$$

for some constant $C > 0$ independent of n which exists by compactness of T_k. This implies J^kS is equicontinuous at $x \in X$. Since x was arbitrary we conclude that J^kS is equicontinuous.

Thus, in light of the previous observations, we want to prove that T_k is contained in a compact set. Given a map $h : B \to Y$ and point $b \in B$, we will write $[h]_b \in C^\infty_b(B,Y)$ to denote the germ of the map h at the point b. We also use the notation $\mathcal{O}\mathrm{p}(y)$ to denote an arbitrarily small but unspecified open neighborhood of a point $y \in Y$.

Consider the identity map id_B on B. Since the leaves of $\mathcal{F}$ are transverse to the fibers of π, it follows that for any $y \in Y$ there is a unique germ of map $[h_y]_{\pi(y)} \in C^\infty_{\pi(y)}(B,Y)$ such that

$$h_y : \mathcal{O}\mathrm{p}(\pi(y)) \to \mathcal{O}\mathrm{p}(y)$$

lifts id_B and maps into the leaf of $\mathcal{F}$ containing y. As such, we obtain a set

$$K := \{j^k h_y(\pi(y)) \mid y \in Y\} \subset J^k(B,Y),$$

which is parametrized by the compact space Y, and hence compact.

Recall that the pullback map induced by $f : X \to B$,

$$f^* : X \times_B J^k(B,Y) \to J^k(X,Y), \quad \big(x, j^k h(f(x))\big) \mapsto j^k(h \circ f)(x),$$

is continuous. Since X and K are compact, it follows that $f^*(X \times_B K) \subset J^k(X,Y)$ is also compact. It remains to observe that since each of the g_n lifts f, we have

$$j^k g_n(x) = j^k(h_{g_n(x)} \circ f)(x) = f^*\big(x, j^k h_{g_n(x)}(f(x))\big) \in f^*(X \times_B K),$$

so that T_k is contained in the compact set $f^*(X \times_B K)$, as claimed.

Step 2 (the limit is unique) As explained at the beginning of the proof, since $J^k S$ from (2) satisfies the conditions of Theorem 2, we obtain a subsequence $(g_{n_i})_{i\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ converging in the C^∞-norm to a leafwise map $g : X \to (Y, \mathcal{F})$ lifting f. It remains to show that this limit map is unique. To this end, suppose we are given two such subsequences n_i and $\tilde{n}_i$, with corresponding limit maps g and $\tilde{g}$, respectively. We show that the set

$$A := \{x \in X \mid g(x) = \tilde{g}(x)\}$$

is open, closed and nonempty. By connectedness of X this implies that g and $\tilde{g}$ agree on the whole of X, thus proving uniqueness of the limit.

To prove that A is nonempty we use the hypothesis that $\lim_{n\to\infty} g_n(x_0) = y_0$. This implies that $\lim_{i\to\infty} g_{n_i}(x_0) = y_0 = \lim_{i\to\infty} g_{\tilde{n}_i}(x_0)$, so that $x_0 \in A$. To see that A is closed, consider a sequence of points $(x_j)_{j\in\mathbb{N}} \in A$ such that $\lim_{j\to\infty} x_j = x$. Then

$$g(x) = \lim_{j\to\infty} g(x_j) = \lim_{j\to\infty} \tilde{g}(x_j) = \tilde{g}(x),$$

since both g and $\tilde{g}$ are continuous. Hence, $x \in A$ so that A is closed. Lastly, to see that A is open, fix some $x \in A$. Then, there exist neighborhoods $U \subset X$ around x, $V \subset B$ around $f(x)$, and $W \simeq \pi^{-1}(f(x)) \times V \subset Y$ around $\pi^{-1}(f(x))$ such that $f(U) \subset V$,

$$\mathcal{F}|_W = \bigcup_{t\in\pi^{-1}(f(x))} \{t\} \times V,$$

and the restriction $\pi|_W$ equals

$$\pi|_W : \pi^{-1}(f(x)) \times V \to V, \quad (t, v) \mapsto v.$$

In these coordinates we can write

$$g(x') = (t_0, f(x')) \quad \text{and} \quad \tilde{g}(x') = (t_1, f(x')) \qquad \text{for all } x' \in U$$

for some $t_0, t_1 \in \pi^{-1}(f(x))$. In fact, since $g(x_0) = \tilde{g}(x_0)$ it follows that $t_0 = t_1$, implying that $g|_U = \tilde{g}|_U$. We conclude that $U \subset A$ so that A is open. □

3 Foliated branched covers

Definition 4 Let M be a closed, orientable 3-manifold endowed with a codimension-one foliation $\mathcal{F}$. A smooth map

$$\pi : (M, \mathcal{F}) \to \mathbb{CP}^1$$

is a *foliated branched cover* if the critical set, $C := \mathrm{Crit}(\pi) \subset M$, is a closed, not necessarily connected, one-dimensional submanifold satisfying the following local model: around each $x \in C$ there are coordinates $(z, t) \in \mathbb{D}^2 \times \mathbb{R} \subset M$, and a holomorphic chart $\mathbb{D}^2 \subset \mathbb{CP}^1$ around $\pi(x)$ such that

(i) $\mathcal{F} = \bigcup_{t\in\mathbb{R}} \mathbb{D}^2 \times \{t\}$;

(ii) $\pi(z, t) = \gamma(t) + z^2$, where $\gamma : \mathbb{R} \to \mathbb{D}^2$ is a curve satisfying $\gamma(0) = \pi(x)$.

In other words, a smooth map $\pi : (M, \mathcal{F}) \to \mathbb{CP}^1$ which restricts to each leaf as a branched cover is a foliated branched cover. The existence of a foliated branched cover imposes strong restrictions on the topology of the foliation. In particular, as the following proposition shows, $\mathcal{F}$ does not contain any vanishing cycles. For completeness we first recall the definition of a vanishing cycle:

Definition 5 Consider a foliated manifold $(M, \mathcal{F})$ and a loop $\gamma : S^1 \to M$ whose image is contained in a leaf L of $\mathcal{F}$. We say that γ is a *vanishing cycle* if there exists a map $\gamma : \mathbb{S}^1 \times [0, 1] \to M$ such that (using the notation $\gamma_t(\theta) := \gamma(\theta, t)$)

(i) the image of $\gamma_t : \mathbb{S}^1 \to M$ is contained in a leaf L_t of $\mathcal{F}$ for all $t \in [0, 1]$,

(ii) for each fixed $\theta \in \mathbb{S}^1$, the curve $t \mapsto \gamma_t(\theta)$ is transverse to $\mathcal{F}$,

(iii) $\gamma = \gamma_0$ is not contractible in L,

(iv) γ_t is contractible in L_t for all $t \in (0, 1]$.

Note that by a C^0-small perturbation of γ in the leaf L, we may assume without loss of generality that γ is immersed. Hence, for the proof of our main theorem given in Section 5 below it suffices to prove the following:

Proposition 6 *Let $\pi : (M, \mathcal{F}) \to \mathbb{CP}^1$ be a foliated branched cover. Suppose we are given a family of immersions of the disk $g_t : D \to (M, \mathcal{F})$, $t \in (0, 1]$, such that*

- *the image of g_t is contained in a leaf L_t of $\mathcal{F}$;*
- *for every $x \in D$, the curve $g_t(x)$, $t \in (0, 1]$, is transverse to $\mathcal{F}$;*
- *as t goes to zero, the restrictions $g_t|_{\partial D} : \partial D \to M$ converge (in the C^0-topology) to an immersion $g_\partial : \partial D \to M$, whose image is contained in a leaf L.*

Then, there exists a map $g : D \to L$ extending g_∂.

Proof We start by proving the proposition under some extra assumptions. Afterwards we will show how to reduce to this special case.

Step 1 (special case) So, let us assume for the moment that there exist finitely many points $p_1, \dots, p_k$ in the interior of the disk D and a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}} \subset (0, 1]$ converging to 0 such that, for all $n \in \mathbb{N}$,

- $\pi \circ g_{t_n} = \pi \circ g_{t_0}$;
- the critical points of $\pi \circ g_{t_n}$ are $p_1, \dots, p_k$.

For any $\varepsilon > 0$ define

$$D_\varepsilon := D \setminus \bigcup_{i=1}^{k} \{x \in D \mid |x - p_i| < \varepsilon\}.$$

The assumptions above imply that D_ε gets mapped by each g_{t_n} into the complement of an open neighborhood of the critical locus of π. That is, there exists a compact set $K \subset M$ such that the image of each g_{t_n}

is contained in K, and such that $\pi|_K$ is a submersion whose fibers are transverse to $\mathcal{F}$. Together with the fact that $\partial D \subset D_\varepsilon$ and $g_{t_n}|_{\partial D}$ converges, this means that we are in the setup of Lemma 3. We conclude (using Lemma 3) that $g_{t_n}|_{D_\varepsilon}$ converges (in the C^∞-norm) to a leafwise map $D_\varepsilon \to (M, \mathcal{F})$ extending g_∂.

Since $\varepsilon > 0$ is arbitrary this defines the limit g on $D \setminus \{p_1, \dots, p_k\}$. We claim that it uniquely extends to a map $g : D \to (M, \mathcal{F})$. To see how to extend g over p_1 (the argument for extending g over the points $p_2, \dots, p_k$ is the same) consider the sequence $(g_{t_n}(p_1))_{n\in\mathbb{N}} \subset M$. By compactness of M there exists a converging subsequence $(g_{t_n'}(p_1))_{n\in\mathbb{N}}$ whose limit we denote by $q := \lim_{n\to\infty} g_{t_n'}(p_1)$. Let $U \simeq \mathbb{D}^2 \times [-1, 1]$ be a coordinate chart as in Definition 4 around q. Using that the family g_t is continuous (in both domain and parameter) and converges uniformly on D_ε, we see that there exists a small disk $D_{p_1} \subset D$ centered at p_1, such that $g_{t_n'}(D_{p_1})$ is contained in U for all n sufficiently large. The assumption that $\pi \circ g_{t_n'} = \pi \circ g_{t_0}$ then implies that (after reparametrizing the interval direction if necessary), in the coordinates defined by U, we can write

$$g_{t_n'}|_{D_{p_1}} : D_{p_1} \to I \times \mathbb{D}^2, \quad x \mapsto (t_n', \tilde{g}(x)),$$

for a uniquely defined map $\tilde{g} : D_{p_1} \to \mathbb{D}^2$ independent of t_n. In particular, the limit map g, which is defined on $D_{p_1} \setminus p_1$, equals

$$g|_{D_{p_1}\setminus p_1} : D_{p_1} \setminus p_1 \to I \times \mathbb{D}^2, \quad x \mapsto (0, \tilde{g}(x)).$$

As such it is clear that g can be extended over p_1 by setting $g(p_1) := (0, \tilde{g}(p_1))$. This finishes the proof using the extra assumptions discussed above.

Step 2 (reduction to special case) It remains to show that we can always reduce to this special case. We start by observing that if ϕ_s, $s \in [0, 1]$, is a leafwise isotopy of $(M, \mathcal{F})$, then proving the proposition for g_t and g_∂ is equivalent to proving it for $\phi_1 \circ g_t$ and $\phi_1 \circ g_\partial$. Hence, we can assume that the image of g_∂ does not contain any critical points of π. Indeed, there is a (C^∞-small) leafwise isotopy making the image of g_∂ disjoint from the critical points of π. We will slightly abuse notation and keep denoting the isotoped family by g_t and g_∂.

As we are interested in $\lim_{t\to 0} g_t$, we can restrict the family to $t \in (0, \varepsilon]$ for $\varepsilon > 0$ arbitrarily small. Since the image of g_∂ does not contain any critical points of π, we may assume the same is true for $g_t|_{\partial D}$ for $t \in (0, \varepsilon]$. Thus, after restricting the family to $t \in (0, \varepsilon]$, there exists a leafwise isotopy taking g_t to a family for which

$$\pi \circ g_t|_{\partial D} = \pi \circ g_\partial. \tag{3}$$

The family g_t defines a submersion

$$G : (0, \varepsilon] \times D \to (M, \mathcal{F}), \quad (t, x) \mapsto g_t(x), \tag{4}$$

with the property that

$$G^*\mathcal{F} = \bigcup_{t\in(0,\varepsilon]} \{t\} \times D.$$

Moreover, since the critical set of π is transverse to $\mathcal{F}$ and does not intersect the image of $g_t|_{\partial D}$, its preimage under G is transverse to the levels $\{t\} \times D$ and does not intersect the boundary. It follows that the number of critical points k of $\pi \circ g_t$ is independent of t, and finite by compactness of D.

After (parametrically) reparametrizing the domain of g_t, we can assume that the critical locus of $\pi \circ g_t$ equals $\{p_1, \dots, p_k\} \subset D$ for all t. This allows us to define curves

$$\lambda_i : (0, \varepsilon] \to M, \quad t \mapsto g_t(p_i), \qquad i = 1, \dots, k,$$

into the critical locus of π. Since the critical locus is compact there exists a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}} \subset (0, \varepsilon]$, such that the sequence $\lambda_i(t_n)$ converges for all i. We denote the limit points by

$$q_i := \lim_{n\to\infty} \lambda_i(t_n) \in M, \quad i = 1, \dots, k.$$

For $i = 1, \dots, k$, we fix local coordinates in $U_i \simeq \mathbb{D}^2 \subset D$ around p_i, $V_i \simeq \mathbb{D}^2 \times [-1, 1] \subset M$ around q_i, and $W_i \simeq \mathbb{D}^2 \subset \mathbb{CP}^1$ around $\pi(q_i)$, in which the foliation equals

$$\mathcal{F} = \bigcup_{t\in[-1,1]} \mathbb{D}^2 \times \{t\},$$

and the foliated branched cover π is given by

$$\pi|_{V_i} : V_i \to W_i, \quad (z, t) \mapsto z^2 + \gamma_i(t).$$

Let $\phi_{i,t} : W_i \to W_i$, $t \in [-1, 1]$, be a smooth family of diffeomorphisms whose support is contained in the interior of W_i, and such that

- $\phi_{i,t} = \mathrm{id}$ for $t \in \mathcal{O}\mathrm{p}(\pm 1)$,
- there exists $\delta > 0$ such that $\phi_{i,t}(\gamma_i(t)) = \gamma_i(0)$ for $t \in [-\delta, \delta]$,

where we use the notation $\mathcal{O}\mathrm{p}(\pm 1)$ to indicate an arbitrarily small open neighborhood of ± 1. We perturb π on V_i by setting

$$\widetilde{\pi}|_{V_i} : V_i \to W_i, \quad (z, t) \mapsto (\phi_{i,t} \circ \pi)(z, t).$$

By construction we have that $\widetilde{\pi}$ agrees with π on the complement of the V_i, while on $\widetilde{V}_i := \mathbb{D}^2 \times [-\delta, \delta] \subset V_i$ we have

$$\widetilde{\pi}|_{\widetilde{V}_i} : \widetilde{V}_i \to W_i, \quad (z, t) \mapsto z^2 + \gamma_i(0). \tag{5}$$

Again, abusing notation we continue to denote $\widetilde{\pi}$ and $\widetilde{V}_i$ by π and V_i, respectively.

Shrinking the parameter interval $(0, \varepsilon]$ even further if necessary, we can assume that the intersection of the image of g_{t_n} with the critical locus of π is contained in the union of the V_i. By (5) this means that

$$\pi \circ g_{t_n}|_{\{p_1,\dots,p_k\}} = \pi \circ g_\varepsilon|_{\{p_1,\dots,p_k\}}.$$

It remains to arrange that $\pi\circ g_{t_n}=\pi\circ g_\varepsilon$ on the rest of D. To this end, consider the map $G: D\times(0,\varepsilon]\to M$ from (4), and the subset

$$B := \partial D\times(0,\varepsilon]\cup\bigcup_{n\in\mathbb{N}}\{p_1,\dots,p_k\}\times\{t_n\}$$

of its domain. The fibers of the map $\pi\circ G$ are transverse to the D-factor, and at points in B are tangent to the $(0,\varepsilon]$-factor. Hence, there exists a vector field $X\in\mathfrak{X}(D\times(0,\varepsilon])$ tangent to the fibers of $\pi\circ G$ satisfying $\mathrm{d}t(X)=-1$ and $X|_B=-\partial_t$. We use its flow ϕ_t to define the maps

$$\tilde g_{t_n}: D\to M,\quad (x,t)\mapsto G(\phi_{t_n}(x,\varepsilon)),\qquad \text{for all } n\in\mathbb{N}.$$

By construction these maps satisfy $\pi\circ\tilde g_{t_n}=\pi\circ\tilde g_\varepsilon$ on the whole of D. □

4 Morse maps into foliations

In this section we recall the basic properties of Morse functions into codimension-one foliations; see, for example, [13]. A smooth function $f: M\to\mathbb{R}$ is called Morse if all of its critical points are nondegenerate. It is well known that Morse functions form an open and dense subset of $C^\infty(M)$.[2] See, for example, [7; 12].

Lemma 7 *Let $f\in C^\infty(M)$ be a smooth function which is Morse on a neighborhood of a (possibly empty) closed set $A\subset M$. Then there exists a Morse function $g\in C^\infty(M)$ arbitrarily close to f and satisfying $g|_A=f|_A$.*

Consider a cooriented codimension-one foliation $\mathcal{F}$ on N. Around any point $x\in N$ there exists an open neighborhood $U\subset N$ and a submersion $\phi: U\to\mathbb{R}$ such that

$$\mathcal{F}|_U=\ker\mathrm{d}\phi$$

as cooriented distributions. We refer to (U,ϕ) as a *submersion chart* for $\mathcal{F}$. Suppose $f: M\to(N,\mathcal{F})$ is a smooth map. We say that a point $x\in M$ is a *singularity of tangency* of f with respect to $\mathcal{F}$ if $\mathrm{d}f(T_xM)$ and $T_{f(x)}\mathcal{F}$ are not transverse as subspaces of $T_{f(x)}N$. That is, f has a singularity of tangency if and only if $\phi\circ f: f^{-1}(U)\to\mathbb{R}$ has a critical point for some (and hence any) submersion chart (U,ϕ) around $f(x)$. A singularity of tangency x of f is *nondegenerate* if $\phi_U\circ f$ has a nondegenerate critical point at x. In this case we define the *index* of x to be the index of $\phi\circ f$ at x.

Definition 8 A map $f: M\to(N,\mathcal{F})$ is called *Morse* if all of its singularities of tangency are nondegenerate. If M has nonempty boundary we require the singularities of f to be contained in the interior.

Lemma 9 *If M is compact the following hold*:

- *The set of Morse maps from M to $(N,\mathcal{F})$ is open and dense in $C^\infty(M,N)$.*
- *Suppose $f: M\to(N,\mathcal{F})$ is a smooth map which is Morse on a neighborhood of a closed subset $A\subset M$. Then there exists a Morse map g arbitrarily close to f such that $g|_A=f|_A$.*

[2] Given smooth manifolds M and N, we consider $C^\infty(M,N)$ to be equipped with the (strong) C^∞-topology. In this topology a sequence of functions f_k, $k\in\mathbb{N}$, converges to f if the derivatives of every order converge uniformly.

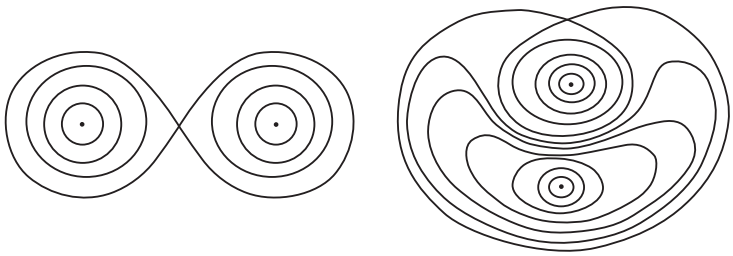

Figure 1: The two possible configurations of a saddle singularity together with its corresponding center singularities.

Proof Consider a smooth map $f : M \to (N, \mathcal{F})$. Since M is compact, its image can be covered by finitely many submersion charts (U_i, ϕ_i) for $\mathcal{F}$. The proof follows from inductively applying Lemma 7 to $\phi_i \circ f : f^{-1}(U_i) \to \mathbb{R}$. □

As for Morse functions, the singularities of $f : M \to (N, \mathcal{F})$ are isolated and satisfy a local model: Let $\mathcal{F}$ be the foliation by horizontal hyperplanes in $\mathbb{R}^{n+1}$. For $1 \leq k \leq n$, the embedding

$$f_k : \mathbb{R}^n \to (\mathbb{R}^{n+1}, \mathcal{F}), \quad (x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_n, \sum_{i=1}^{k} -x_i^2 + \sum_{i=k+1}^{n} x_i^2\right), \tag{6}$$

is Morse, and has an index-k singularity at the origin. Locally, around an index-k singularity, any Morse map is equivalent to this model.

Lemma 10 *Let M be compact and $f : M \to (N, \mathcal{F})$ a Morse map. Then*

$$\chi(M) = \sum_{p \in \mathrm{Sing(f)}} (-1)^{\mathrm{Index}(f,p)}.$$

Proof Cover the image of f by submersion charts (U_i, ϕ_i) for $\mathcal{F}$. This allows us to define the gradient of $\phi_i \circ f$ on $f^{-1}(U_i)$. Using a partition, these local vector fields can be patched together to a vector field X on M. It is easily checked that for any critical point p the index of f and the gradient X are related by

$$\mathrm{Index}(X, p) = (-1)^{\mathrm{Index}(f,p)}.$$

As such the result follows from the Poincaré–Hopf theorem. See, for example, [7]. □

We now restrict to $\dim M = 2$ and $\dim N = 3$. In this case the pullback $f^*\mathcal{F}$ defines a singular foliation by lines on M, called the *characteristic foliation* of f. To be precise, a leaf of $f^*\mathcal{F}$ is defined to be the preimage under f of a leaf of $\mathcal{F}$. On the complement of the critical points of f the characteristic foliation is regular, and the singularities of $f^*\mathcal{F}$ correspond to the critical points of f. Their local models are given by (6). The singularity corresponding to a critical point of index 0 or 2 is called a *center*, while the singularity corresponding to a critical point of index 1 is called a *saddle*; see also Figure 1.

Thus, Lemma 10 immediately implies the following:

Corollary 11 *Suppose $f : \mathbb{D}^2 \to (M^3, \mathcal{F})$ is Morse. Then, the number of centers of the characteristic foliation $f^*\mathcal{F}$ is one more than the number of saddles. In particular, it has at least one center singularity.*

5 Proof of the main theorem

In light of Proposition 6 it suffices to show that if $(M,\mathcal{F})$ does not have any vanishing cycles, then for any leaf L there is an injection $\pi_1(L)\to\pi_1(M)$. This is a classic result which can be found in many places in the literature (e.g., [1, Theorem 4.35; 3, Proposition 9.2.5]). We provide a proof here for completeness. In particular, in the following lemma we clarify the existence of an immersed capping disk, which is often implicitly assumed.

Lemma 12 *Consider a foliated manifold $(M^3,\mathcal{F})$ and $\gamma:\mathbb{S}^1\to L$ a leafwise curve which is contractible in M. Then, there exists an immersion $f:\mathbb{D}^2\to M$ satisfying*:

(i) *$f(\partial\mathbb{D}^2)\subset L$, and $f|_{\partial\mathbb{D}^2}$ is leafwise homotopic (inside L) to γ.*

(ii) *$f|_{\mathcal{O}\mathrm{p}(\partial\mathbb{D}^2)}$ is transverse to $\mathcal{F}$.*

Proof By a C^0-small leafwise homotopy we can assume γ is immersed. Slightly pushing it off, transverse to $\mathcal{F}$, we obtain an immersed cylinder transversely to $\mathcal{F}$. Since any homotopy of γ stays contractible in M, we find a smooth map $g:\mathbb{D}^2\to M$ extending the immersed cylinder. That is, $g|_{\partial\mathbb{D}^2}=\gamma$, and $g|_{\mathcal{O}\mathrm{p}(\partial\mathbb{D}^2)}$ is an immersion transverse to $\mathcal{F}$.

The Smale–Hirsch immersion theorem [9], see also [4, Theorem 8.2.1], implies that to find our desired immersion f, it suffices to find an injective bundle map $G:T\mathbb{D}^2\to TM$ covering g, and such that G agrees with $\mathrm{d}g$ at points in $\partial\mathbb{D}^2$.

The differential of g defines a bundle map $\mathrm{d}g:T\mathbb{D}^2\to g^*TM$ covering the identity. Using that the disk is contractible we fix a framing $e:=(e_1,e_2,e_3)$ of g^*TM such that $\mathcal{F}=\langle e_1,e_2\rangle$ (implying $e_3\pitchfork\mathcal{F}$), and

$$\mathrm{d}_pg(\partial_\theta)=e_2,\quad \mathrm{d}_pg(\partial_r)=e_3\qquad\text{for all } p\in\partial\mathbb{D}^2,$$

where (r,θ) denote polar coordinates on $\mathbb{D}^2$. In the above framing $\mathrm{d}g|_{\partial D^2}$ can be interpreted as family of orthonormal 2-frames

$$G_\partial:\mathbb{S}^1\to V_2(\mathbb{R}^3),\quad t\mapsto(\dot\gamma(t),e_3).$$

As such, the existence of G is equivalent to G_∂ defining the zero class in $\pi_1(V_2(\mathbb{R}^3))=\mathbb{Z}_2$, which in turn is equivalent to the rotation number of $\dot\gamma:\mathbb{S}^1\to\mathbb{R}^2$ being odd.

Let $U\subset L$ be a contractible neighborhood of $\gamma(0)$, and extend the framing e over U. Let $\sigma:\mathbb{S}^1\to U$ be an immersed curve with $\sigma(0)=\gamma(0)$, $\dot\sigma(0)=\dot\gamma(0)$, and $\mathrm{rot}(\dot\sigma)=1$. Then, γ is leafwise homotopic (though not through immersed curves) to the immersed curve $\gamma\circ\sigma$ which has rotation number $\mathrm{rot}(\gamma\circ\sigma)=\mathrm{rot}(\gamma)+1$. Thus, we can assume that γ has odd rotation number implying that an injective bundle map $G:T\mathbb{D}^2\to TM$ covering g exists. Applying the Smale–Hirsch immersion theorem relative to the boundary ∂D^2 gives the desired immersion f. □

Proof of the main theorem By assumption we are given a foliated manifold $(M^3,\mathcal{F})$ and a leafwise curve $\gamma:S^1\to L$ which is contractible in M. We will show that γ is contractible in L.

Fix an immersed capping disk $f:\mathbb{D}^2\to M$ as in Lemma 12. The induced characteristic foliation $\widetilde{\mathcal{F}}:=f^*\mathcal{F}$ is tangent to the boundary $\partial\mathbb{D}^2$. By compactness it contains finitely many center and saddle

singularities. After slightly perturbing f if necessary, we may assume that each of the leaves of $\widetilde{\mathcal{F}}$ contains at most one singularity. For each center singularity c, let $\Delta \subset \mathbb{D}^2$ be the maximal set satisfying:

- Δ contains c and is $\widetilde{\mathcal{F}}$ saturated.
- Each leaf of $\widetilde{\mathcal{F}}$ contained in Δ bounds an immersed disk in the corresponding leaf of $\mathcal{F}$.

The local model of a center singularity shows that Δ is nonempty, and it follows from local Reeb stability that Δ is open.

We claim that Δ is also closed. To see this, we will assume Δ is not closed and show this leads to a contradiction. Since Δ is topologically a disk, the boundary $\partial\Delta := \overline{\Delta} \setminus \mathring{\Delta}$ is a closed (not necessarily immersed) curve contained in a leaf of $\widetilde{\mathcal{F}}$.

Now there are several possibilities. First, assume that $\widetilde{\mathcal{F}}$ has only one singularity which is necessarily a center. In this case $\partial\Delta$ is contained in a nonsingular leaf of $\widetilde{\mathcal{F}}$ and hence itself immersed. That is, $\partial\Delta$ is a vanishing cycle giving a contradiction by Proposition 6.

Next, suppose that $\widetilde{\mathcal{F}}$ has more than one center. The previous argument shows that for each center c the corresponding curve $\partial\Delta$ is singular. Since each leaf of $\widetilde{\mathcal{F}}$ contains at most one singular point, so does $\partial\Delta$, and this singularity is a saddle. By Corollary 11 there must be at least two centers c_1 and c_2 for which the boundaries $\partial\Delta_1$ and $\partial\Delta_2$ share the same saddle, and thus intersect in a point. There are two possible configurations, depicted in Figure 1.

In the first case, the union $\partial\Delta_1 \cup \partial\Delta_2$ bounds a simply connected domain in the corresponding leaf of $\mathcal{F}$. In the second case, suppose without loss of generality that $\partial\Delta_1 \subset \Delta_2$, and denote the saddle singularity by s. Then, the closed curve $\partial\Delta_2 \setminus \partial\Delta_1 \cup s$ bounds a disk in the corresponding leaf of $\mathcal{F}$. Thus, in both cases, f can be replaced by $\tilde{f}$ for which the induced characteristic foliation has one less center singularity. Hence we can inductively reduce to the case that $\widetilde{\mathcal{F}}$ has only one center, which we considered above.

We conclude from the arguments above that Δ is open, closed and nonempty. That is, it equals the whole immersed disk $\mathbb{D}^2$. This shows that $\gamma = \partial\mathbb{D}^2$ bounds an immersed disk in the leaf L of $\mathcal{F}$. □

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