AGT

Algebraic & Geometric Topology

Volume 26 (2026)

Reversible maps in groups of interval exchange transformations

NANCY GUELMAN AND ISABELLE LIOUSSE

msp

msp

Algebraic & Geometric Topology **26**:7 (2026) 2497–2518
DOI: 10.2140/agt.2026.26.2497
Published: September 1, 2026

Reversible maps in groups of interval exchange transformations

NANCY GUELMAN AND ISABELLE LIOUSSE

An element f of a group G is reversible if it is conjugated in G to its own inverse. It is plain that two elements f and h satisfying the relation $hfh^{-1} = f^{-1}$ generate a, possibly nonfaithful, $\mathrm{BS}(1,-1)$-action, where $\mathrm{BS}(1,-1)$ is the Baumslag–Solitar group $\langle a, b \mid bab^{-1} = a^{-1}\rangle$ also referred to as the Klein bottle group.

Our main result is that any reversible interval exchange transformation (IET) is reversible by a finite order IET. Its proof uses a dynamical decomposition for $\mathrm{BS}(1,-1)$-actions by IET which involves some specific IET-subgroups for which we give a precise description of the reversible elements. We also indicate consequences on the IET-realizations of a group containing reversible maps.

1 Introduction

O'Farrell and Short, motivated by the importance of reversibility in dynamical systems and group theory, wrote a quite complete survey (see [12]) exposing mostly recent works on the problem of classifying reversible elements in a large number of groups. Namely, an element f of a group G is called *reversible* if there exists $b \in G$ such that $bfb^{-1} = f^{-1}$. We say that the element b *reverses* f and f is said to be *b-reversible.*

An element f is called *strongly reversible* if there exists an involution $b \in G$ such that $bfb = f^{-1}$.

In [12], the following questions are raised: Given a group G, are all reversible elements of G strongly reversible? Let $\mathrm{Inv}(G)$ be the subgroup of G generated by its involutions. Is the minimal number of involutions that are needed for writing an element of $\mathrm{Inv}(G)$ uniformly bounded?

As we are interested in groups of interval exchange transformations, we set:

Definition 1.1 An *interval exchange transformation* (*IET*) is a bijective map $f : [0,1) \to [0,1)$ defined by a finite partition of the unit interval into half-open subintervals $\mathcal{I}_i$ and a reordering of these intervals by translations.

If the partition can be chosen with cardinal r, we say that f is an *r-IET* and we define the *associated permutation* $\pi = \pi(f) \in \mathcal{S}_r$, the r-th symmetric group, by $f(\mathcal{I}_i) = J_{\pi(i)}$, where the J_j's are the ordered images of the $\mathcal{I}_i$'s.

We denote by $\mathcal{G}$ the group of all IETs.

More generally, an *affine interval exchange transformation* (*AIET*) is a bijection on $[0,1)$ defined by a finite partition of the unit interval into half-open subintervals such that the restriction to each of these intervals is an orientation-preserving affine map. An AIET that is a homeomorphism of $[0,1)$ is called a *PL-homeomorphism*.

MSC2020: primary 37E05, 57S30; secondary 37B05, 37C85.
© 2026 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

In [6], the reversible circle homeomorphisms are studied; in particular they always have fixed points or periodic points of period 2. For IET, new dynamical phenomena appear: In [8], we exhibited a reversible periodic-point free IET (see also Section 2.1.2), this map has two minimal components (see Definition 2.9) that are exchanged by the reversing map. Moreover, reversible IETs can be minimal: the example given by Arnoux and Yoccoz (see [3]) of a minimal IET with zero SAF-invariant was constructed as a product of two involutions, so by Properties 2.1, it is strongly reversible.

Actually, [8] exhibited free $\mathrm{BS}(1,-1)$-actions in $\mathcal{G}$. We recall that an action is *free* if the only element acting with fixed points is the trivial element; in particular free actions are faithful. This implies the existence of elements in $\mathcal{G}$ that are reversible by infinite order elements. However, our main result is:

Theorem 1 *Any reversible f in $\mathcal{G}$ is reversible by a finite order element of order 2^n, with $n \in \mathbb{N}$.*

The proof of Theorem 1 uses the following results. The first one consists of a dynamical decomposition for $\mathrm{BS}(1,-1)$-actions.

Theorem 2 *Let $\tilde{f}, \tilde{h} \in \mathcal{G}$ be such that $\tilde{f}$ is $\tilde{h}$-reversible. Then there exists $\varphi \in \mathcal{G}$ such that $f = \varphi \circ \tilde{f} \circ \varphi^{-1}$ is reversible by $h = \varphi \circ \tilde{h} \circ \varphi^{-1}$ and the unit interval I decomposes as a finite union of pairwise disjoint $\langle f, h\rangle$-invariant half-open intervals, $\mathcal{I}_i$, such that either*

(1) *$f_{|\mathcal{I}_i}$ has finite order or*

(2) *the $\langle f, h\rangle$-action on $\mathcal{I}_i$ is a free $\mathrm{BS}(1,-1)$-action or*

(3) *$h_{|\mathcal{I}_i}$ has finite even order or*

(4) *$\mathcal{I}_i$ is an union of an even number of minimal f-components (see Definition 2.9).*

The second tool concerns finite order IETs. Note that, according to the Arnoux–Keane–Mayer decomposition theorem to be stated later in Section 2.2, an IET whose orbits are all finite (referred to as periodic IET) has finite order.

Proposition 1.2 *Any periodic IET is strongly reversible.*

It was showed in [7, Section 2.3] that periodic IETs are reversible. Here we adapt the proof in order to get the strong reversibility.

The third tool is a result of [8] and it establishes that free actions of $\mathrm{BS}(1,-1)$ by IET are "conjugated" to actions in some specific subgroups G_n given by:

Definition 1.3 Let n be a positive integer and $\mathbb{S}_n = \left[0, \frac{1}{n}\right]/0 = \frac{1}{n}$ be the circle of length $\frac{1}{n}$.

- Let G_n denote the set of all IETs that preserve the partition $\left[0,1\right) = \left[0, \frac{1}{n}\right) \cup \left[\frac{1}{n}, \frac{2}{n}\right) \cup \cdots \cup \left[\frac{n-1}{n}, 1\right)$ and whose restrictions to intervals $I_i = \left(\frac{i-1}{n}, \frac{i}{n}\right)$ have a unique discontinuity point.

Note that the set G_1 consists of circular rotations seen as IETs over 2 intervals and examples in G_2 are given in Section 2.1.2.

Let $g \in G_n$.

- We define σ_g to be the element of the symmetric group $\mathcal{S}_n$ given by $\sigma_g(i) = j$ if $g(I_i) = I_j$.

It follows that $g_{|I_i}(x) = R_{\alpha_i, I_i}(x) + (\sigma_g(i) - i)/n$ where $R_{\alpha_i, I_i}(x)$ is the IET of support I_i that preserves I_i and is conjugate to the rotation on $\mathbb{S}_n$ of angle $\alpha_i \in \mathbb{S}_n$ seen as an IET of $\left[0, \frac{1}{n}\right)$.

- We define $\alpha_g = (\alpha_1, \dots, \alpha_n) = (\alpha_g(1), \dots, \alpha_g(n)) \in {\mathbb{S}_n}^n$ and we write $g = (\alpha_g, \sigma_g)$.

It is plain that G_n is a group and its group law $\circ$ is given by

$$\alpha_{f\circ g} = \sigma_g(\alpha_f) + \alpha_g,$$
$$\sigma_{f\circ g} = \sigma_f \circ \sigma_g,$$

where σ_g acts on the vector α_f by permuting its coordinates (see Definition 5.1).

The next statement is [8, Corollary 4].

Proposition 1.4 *Let $\langle f, h\rangle$ be a free $\mathrm{BS}(1,-1)$-action by IET. Then there exists a map $\varphi = P \circ E$, where P is a PL-homeomorphism and $E \in \mathcal{G}$, that conjugates $\langle f, h\rangle$ to a free $\mathrm{BS}(1,-1)$-action by elements of some G_n.*

In addition, P^{-1} is affine on the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right)$ for any $k \in \{1, \dots, n\}$ and E is any IET that sends the f-components to intervals.

This motivates the study of reversible maps in the groups G_n which is the purpose of the fourth result involved in the proof of Theorem 1.

Theorem 3 *Any IET reversible in G_n is strongly reversible in G_n.*

As a consequence of Theorem 1 and Proposition 1.2, we have:

Corollary 4 *Any reversible element of $\mathcal{G}$ can be written as the product of 4 involutions.*

Tools we develop in [8] and here have consequences on the realization in $\mathcal{G}$ of groups containing reversible maps. In particular, we will consider the group

$$C_2 = \langle a, b, c \mid aba^{-1} = b^{-1}, bcb^{-1} = c^{-1}\rangle$$

that was studied in [10] and suggested by Andrés Navas. There, Leroux proves that C_2 does not admit a free action by orientation-preserving homeomorphisms of the plane.

Moreover, this group arises as an amalgamation of two $\mathrm{BS}(1,-1)$-groups and a variation, obtained by gluing $\mathrm{BS}(1,n)$-groups, has been used in [4] to study obstructions to regularization for 1-dimensional actions.

In addition, as the subgroup $\langle a, c\rangle$ of C_2 is isomorphic to $\mathbb{F}_2$ the nonabelian free group of rank two (see [10]), the existence of faithful actions of C_2 by IET is related to Katok's question: does $\mathcal{G}$ contain a subgroup isomorphic to $\mathbb{F}_2$?

Here, as a consequence of a stronger result (Proposition 8.3), we prove:

Corollary 5 *The group $C_2 = \langle a, b, c \mid aba^{-1} = b^{-1}, bcb^{-1} = c^{-1}\rangle$ does not admit a faithful minimal action by IET.*

In a forthcoming work, we will discuss features of the minimal reversible IETs as well as the connection between reversible IETs and the kernel of the SAF-invariant. However, we can already note that Theorem 3 and item 4 of Corollary 7 imply that G_n does not contain minimal reversible elements.

2 Preliminaries

2.1 Generalities

2.1.1 Basic properties on reversibility We let the reader check the following basic facts.

Properties 2.1 *Let G be a group.*

- *Involutions in G are strongly reversible and any product of two involutions $\sigma_1\sigma_2$ is strongly reversible by σ_1 and σ_2.*
- *An element f of G is strongly reversible if and only if $f = \sigma_1\sigma_2$ with $\sigma_1^2 = \sigma_2^2 = \mathrm{Id}$.*
- *If h_1 and h_2 reverse the same f in G then $h_2^{-1}h_1$ commutes with f.*
- *If f is h-reversible then h^{2p} commutes with f and h^{2p+1} reverses f for all $p \in \mathbb{Z}$. In particular, h^{-1} reverses f.*
- *If f is h-reversible then f^n is h-reversible for all $n \in \mathbb{Z}$.*
- *If f is h-reversible then f is hf^t and $f^t h$-reversible for all $t \in \mathbb{Z}$.*

As a direct consequence of the relation $bab^{-1} = a^{-1}$, we have:

Claim 2.2 *Every element of $\mathrm{BS}(1,-1)$ can be written as $b^s a^t$ with s and t integers.*

2.1.2 IET-realizations of BS(1, −1) Using the description of the law in G_n, one easily check that the element B of G_2 given by $\alpha_B = (\beta, \beta)$ and $\sigma_B = (1,2)$ reverses $A \in G_2$ given by $\alpha_A = (\alpha, -\alpha)$ and $\sigma_A = \mathrm{Id}$ (see Figure 1). This defines a morphism $\rho : \mathrm{BS}(1,-1) \to \mathcal{G}$, by setting $\rho(a) = A$ and $\rho(b) = B$. We claim that ρ is injective provided that α and β are irrational, rationally independent real numbers.

Indeed, by Claim 2.2, it suffices to prove that $\rho(b^s a^t) = \rho(b^{s'} a^{t'})$ implies that $s = s'$ and $t = t'$. This follows from the fact that $\alpha_{B^s A^t} = (t\alpha + s\beta, -t\alpha + s\beta)$.

Moreover, this provides a simple proof of the uniqueness in item (1) of the following lemma.

2.1.3 Basic algebra of $\mathrm{BS}(1,-1) = \langle a, b \mid bab^{-1} = a^{-1}\rangle$

Lemma 2.3 (1) *Every element of $\mathrm{BS}(1,-1)$ has a unique normal form $b^s a^t$ with s and t integers. In particular, a and b have infinite order.*

(2) *If s is odd then $b^s a^t$ reverses a, and if s is even then $b^s a^t$ commutes with a.*

(3) *The index-2 subgroup $\langle a, b^2\rangle$ is isomorphic to $\mathbb{Z}^2$.*

(4) *The index-2 subgroup $\langle a^2, b\rangle$ is isomorphic to $\mathrm{BS}(1,-1)$.*

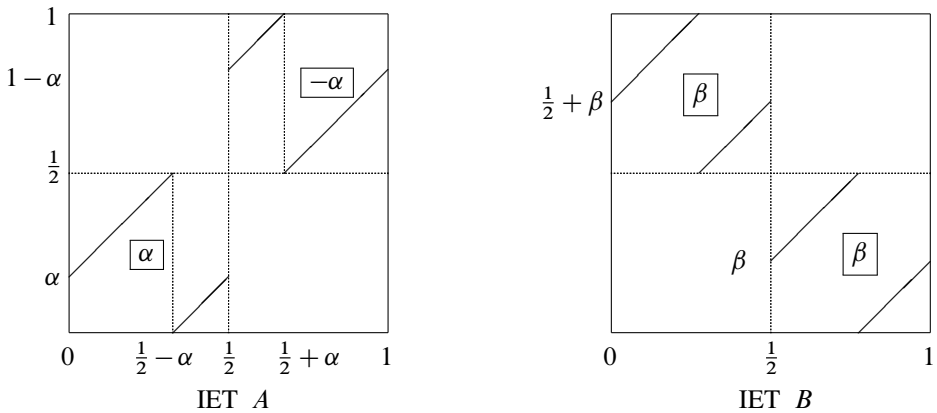

Figure 1: IET-realizations of $\mathrm{BS}(1,-1)$.

Proof The existence in item (1) is Claim 2.2 and the uniqueness is provided, as already mentioned, by the IET-realization given in Section 2.1.2.

Item (2) follows directly from Properties 2.1 and items (3) and (4) are direct consequences of the two previous items. □

2.1.4 Actions of BS(1, −1)

Lemma 2.4 *Let ρ be an action of $\mathrm{BS}(1,-1)$ by bijections on a space X.*

(1) *Any element of $\rho(\mathrm{BS}(1,-1))$ can be written as $\rho(b)^s\rho(a)^t$ with s and t integers. It commutes with $\rho(a)$ when s is even, and it reverses $\rho(a)$ when s is odd.*

(2) *ρ is not faithful if and only if either $\rho(a)$ has finite order or $\rho(b)$ has finite order.*

Proof Item (1) follows directly from Lemma 2.3(1)–(2).

For item (2), let us define $f=\rho(a)$ and $h=\rho(b)$. According to item (1), if ρ is not faithful then there exist two integers s and t such that $(s,t)\neq(0,0)$ and $h^s f^t=\mathrm{Id}$.

- If $t=0$ then $s\neq 0$ and $h^s=\mathrm{Id}$.
- If $t\neq 0$ then the map $f^s=h^{-t}$ commutes with h. But, under our assumptions, f^t is also h-reversible so f^t is a nontrivial involution. □

2.1.5 Extended centralizer and periodic points

Definition 2.5 Let G be a group.

- The *centralizer* of $f\in G$ is
$$C(f)=\{h\in G : h\circ f=f\circ h\}.$$
- The *extended centralizer* of $f\in G$ is
$$\mathrm{CR}(f)=\{h\in G : h\circ f\circ h^{-1}=f^{\pm 1}\}.$$

Note that $\mathrm{CR}(f)$ is a subgroup of G in which the centralizer $C(f)$ of f has index 1 or 2.

Definition 2.6 Let $p \in \mathbb{N}^*$. We define $\mathrm{Per}_p(f)$ to be the set of f-periodic points of period p.

Lemma 2.7 *Let f and $h \in \mathrm{CR}(f)$ be bijections of a set X. Then:*

(1) *For any $x \in X$, $h(\mathcal{O}_f(x)) = \mathcal{O}_f(h(x))$.*

(2) *For any $p \in \mathbb{N}^*$, the set $\mathrm{Per}_p(f)$ is $\langle f, h\rangle$-invariant.*

(3) *For any $s \in \mathbb{Z}$, the set $\mathrm{Per}_p(h^{2s})$ is $\langle f, h\rangle$-invariant.*

(4) *For any $g \in \langle f, h\rangle$ and any $p \in \mathbb{N}^*$, the set $\mathrm{Per}_p(g)$ is h^2-invariant.*

Proof (1) Let $x \in X$ and $n \in \mathbb{N}$. By Properties 2.1, one has $h(f^n(x)) = f^{\pm n}(h(x))$, which implies $h(\mathcal{O}_f(x)) \subset \mathcal{O}_f(h(x))$. Changing h by $h^{-1} \in \mathrm{CR}(f)$ and x by $h(x)$, we get $h(\mathcal{O}_f(x)) = \mathcal{O}_f(h(x))$.

(2) Let $p \in \mathbb{N}^*$. Then $x \in \mathrm{Per}_p(f)$ means that $\#\mathcal{O}_f(x) = p$. It is obvious that $\mathrm{Per}_p(f)$ is f-invariant. By the first item, we get that $\#\mathcal{O}_f(h(x)) = p$, and thus $h(x) \in \mathrm{Per}_p(f)$. We conclude that $\mathrm{Per}_p(f)$ is also h-invariant.

(3) Let $s \in \mathbb{N}$. By Properties 2.1, f and h commute with h^{2s}, so we have $f, h \in \mathrm{CR}(f)$. Hence, the previous item implies that $\mathrm{Per}_p(h^{2s})$ is invariant by f and h.

(4) Let $g \in \langle f, h\rangle$. By Properties 2.1, $h^2 \in C(g)$, and thus the set $\mathrm{Per}_p(g)$ is h^2-invariant. □

2.2 Specific dynamical properties for IETs

In this subsection, we collect some notions and dynamical observations relying on maps being IETs.

The *break point set* of an IET f is obtained by adding the initial point $\{0\}$ to the discontinuity set of f; it is denoted by $\mathrm{BP}(f)$. The set consisting of the f-orbits of points in $\mathrm{BP}(f)$ is denoted by $\mathrm{BP}_\infty(f)$.

We let the reader check the following:

Properties 2.8 *Let f and g be elements of $\mathcal{G}$.*

(a) $\mathrm{BP}(f \circ g) \subseteq \mathrm{BP}(g) \cup g^{-1}(\mathrm{BP}(f))$.

(b) $\mathrm{BP}(f^{-1}) = f(\mathrm{BP}(f))$.

(c) $\mathrm{BP}(f^n) \subseteq \mathrm{BP}(f) \cup f^{-1}(\mathrm{BP}(f)) \cup \cdots \cup f^{-n+1}(\mathrm{BP}(f))$.

Definition 2.9 A subset V of $[0, 1)$ is said *of type $\mathcal{M}$* (or *f-type $\mathcal{M}$*) if it is a nonempty finite union of intervals each of the form $[b, c)$, with b, c in $\mathrm{BP}_\infty(f)$. A type $\mathcal{M}$ and f-invariant set that is minimal for the inclusion among type $\mathcal{M}$ and f-invariant subsets of $[0, 1)$ is called an *f-component*. An f-component M is *minimal* if for any $x \in M$, the orbit $\mathcal{O}_f(x)$ is dense in M and for convenience M will also be denoted by $M_f(x)$.

The well-known decomposition into minimal and periodic components of an IET was first studied for measured surface flows by Mayer in 1943 [11] and restated for IETs by Arnoux [1] and Keane [9].

The Arnoux–Keane–Mayer decomposition theorem claims that given $f \in \mathcal{G}$, the unit interval can be decomposed as $[0, 1) = P_1 \cup \cdots \cup P_l \cup M_1 \cup \cdots \cup M_m$, where:

- P_i is a periodic f-component, meaning that P_i is the f-orbit of an interval $[b,c)$ with b, c in $\mathrm{BP}_\infty(f)$ and all iterates f^k of f are continuous on $[b,c)$. In particular points in P_i are periodic of the same period.
- M_j is a minimal f-component.

Remark 2.10 Note that f-periodic points of same period p may belong to distinct components. However, the set $\mathrm{Per}_p(f)$ is a finite union of periodic components and it is a type $\mathcal{M}$ and f-invariant subset. Moreover, the decomposition of $\mathrm{Per}(f)$ into periodic components is not invariant by conjugacy.

Lemma 2.11 *Let f and h be two IETs. If $h \in \mathrm{CR}(f)$ then h permutes the minimal components of f.*

Proof Let $x \in [0,1)$ be a non-f-periodic point. Then $\mathcal{O}_f(x)$ is dense in the minimal f-component $M_f(x)$. By Lemma 2.7, $h(x)$ is not f-periodic and $\mathcal{O}_f(h(x)) = h(\mathcal{O}_f(x))$. Taking closures, we get that $M_f(h(x)) = h(M_f(x))$ even if h is not continuous.

Indeed, the dynamics of the group G generated by f and h can be lifted, by a Denjoy blow-up, to that of a homeomorphisms group of $X = [0,1) \sqcup \mathrm{BP}^-_\infty(G)$, where $\mathrm{BP}^-_\infty(G)$ is an abstract copy of $\mathrm{BP}_\infty(G)$. To any $x \in \mathrm{BP}_\infty(G)$ correspond two points of X: $x^+ \in [0,1)$ and $x^- \in \mathrm{BP}^-_\infty(G)$. The topology on X is given by the total order defined on X by taking the usual order on $\mathrm{BP}^-_\infty(G)$ and $[0,1)$ and setting that for $x^- \in \mathrm{BP}^-_\infty(G)$ and $y \in [0,1)$: $x^- < y \iff x^+ \leq y$. For more details see, for example, Section 8.1 and [2, I-(1); 5, Section 3.2; 9]. □

Lemma 2.12 *Let f, h be two IETs such that h reverses f. Let p be an odd integer. Then the set $\mathrm{Per}_p(h)$ is contained in $\mathrm{Per}(f)$.*

Proof Let p be an odd integer and $x \in \mathrm{Per}_p(h)$. Since h is an IET there is an interval $I_x = [x,c) \subset \mathrm{Per}_p(h)$.

We now argue by contradiction supposing that $x \notin \mathrm{Per}(f)$, therefore $\mathcal{O}_f(x)$ is locally dense, in particular there exists a subsequence p_n such that $f^{p_n}(x) \to x_+$ (i.e., $f^{p_n}(x) \to x$ and $f^{p_n}(x) > x$). Hence, for n sufficiently large we have $f^{p_n}(x) \in I_x \subset \mathrm{Per}_p(h)$.

In addition, by reversibility, one has $h^p(f^{p_n}(x)) = f^{-p_n}(h^p(x)) = f^{-p_n}(x)$.

Thus $f^{p_n}(x) = f^{-p_n}(x)$, meaning that x is f-periodic. □

3 Dynamical decomposition of BS(1, −1)-actions. Proof of Theorem 2

Proposition 3.1 *If f is reversible in $\mathcal{G}$ by h then the unit interval I decomposes into $\langle f,h\rangle$-invariant f-type $\mathcal{M}$ subsets, that is, $I = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_l$, and either*:

(1) *There exists an integer p_i such that any point of M_i is f-periodic of period p_i.*

In the following items, each M_i is the union of finitely many minimal f-components and h acts transitively on these components.

(2) *The action of $\langle f,h\rangle$ on M_i is a free $\mathrm{BS}(1,-1)$-action.*

(3) *The action of $\langle f,h\rangle$ on M_i is a nonfaithful $\mathrm{BS}(1,-1)$-action and $h_{|M_i}$ has finite even order p_i.*

(4) *There exists an even integer p_i such that M_i is a union of p_i minimal f-components.*

Proof According to the Arnoux–Keane–Mayer decomposition theorem, I decomposes as a finite union of minimal or periodic f-components. Let N be an f-component.

If N is periodic with associated period p, we define $M = \mathrm{Per}_p(f)$.

If N is minimal, according to Lemma 2.11, for any integer j the set $h^j(N)$ is a minimal f-component. By the finiteness of the number of f-components, there exists a least positive integer s such that $h^s(N) = N$. We define $M = \bigsqcup_{j=0}^{s-1} h^j(N)$ and then $\mathrm{Per}(f_{|M}) = \varnothing$.

Case 1 (N is periodic) By Lemma 2.7 and Remark 2.10, the set M is invariant by f and h, and of type $\mathcal{M}$, therefore it satisfies item (1).

Case 2 (N is minimal and the action of $\langle f, h\rangle$ on M is free) This is the situation described in item (2).

Case 3 (N is minimal and the action of $\langle f, h\rangle$ on M is not faithful) Lemma 2.4(2) implies that the map $h_{|M}$ has finite order p. We claim that p is even.

Indeed, by contradiction, suppose that $h_{|M}$ is of odd order p. By Lemma 2.12, points that are h-periodic of period p belong to $\mathrm{Per}(f)$, thus $M \subset \mathrm{Per}(f)$. This contradicts the fact that $\mathrm{Per}(f_{|M}) = \varnothing$. So $h_{|M}$ has even order and we are in the situation described by item (3).

Case 4 (N is minimal and the action of $\langle f, h\rangle$ on M is faithful but not free) The failure of freeness on M means that there exist $x \in M$ and $g \in \langle f, h\rangle$ such that $g(x) = x$ and $g_{|M} \neq \mathrm{Id}$.

Since $\mathrm{Fix}(g_{|M}) \neq \varnothing$ and $\mathrm{Per}(f_{|M}) = \varnothing$, Lemma 2.12 implies that $g_{|M}$ does not reverse $f_{|M}$. Hence $g_{|M}$ commutes with $f_{|M}$ and the set $\mathrm{Fix}(g_{|M})$ is f-invariant and it contains x. Therefore we have $M_f(x) \subset \mathrm{Fix}(g_{|M}) \subset \mathrm{Fix}(g)$.

By Lemma 2.7(4), the set $\mathrm{Fix}(g)$ is also h^2-invariant and then $\mathrm{Fix}(g)$ contains $\bigcup_{t=0}^{s-1} h^{2t}(M_f(x))$; we recall that s satisfies $h^s(N) = N$.

Now, suppose that $s = 2p-1$ is odd. Then $h^{2p}(M_f(x)) = h(M_f(x))$ and

$$\begin{aligned}\bigcup_{t=0}^{s-1} h^{2t}(M_f(x)) &= \bigcup_{t=0}^{p-1} h^{2t}(M_f(x)) \cup \bigcup_{t=p}^{s-1} h^{2t}(M_f(x)) \\ &= M_f(x) \cup h^2(M_f(x)) \cup \cdots \cup h^{s-1}(M_f(x)) \cup h(M_f(x)) \cup h^3(M_f(x)) \\ &\qquad \cup \cdots \cup h^{s-2}(M_f(x)) \\ &= M.\end{aligned}$$

This implies that $g_{|M} = \mathrm{Id}$, a contradiction.

Thus s is even and we are in the situation described in item (4). □

We turn now to the proof of Theorem 2. Let $\langle \tilde{f}, \tilde{h}\rangle$ generate a $\mathrm{BS}(1,-1)$-action by IET. Our proof starts with the following observations:

- If M is an $\tilde{f}$-type $\mathcal{M}$ subset then there exists an IET φ with break points in $\mathrm{BP}_\infty(\tilde{f})$ such that $\varphi(M)$ is a half-open interval and it is an f-type $\mathcal{M}$ subset, where $f = \varphi \circ \tilde{f} \circ \varphi^{-1}$.

Roughly speaking, φ is an IET that sends the connected components of M into consecutive intervals, so $\varphi(M)$ is a half-open interval.

Moreover, φ can be constructed in such a way that its break point set is contained in $\mathrm{BP}_\infty(\tilde{f})$. Therefore, according to Properties 2.8, the endpoints of $\varphi(M)$ belong to $\mathrm{BP}_\infty(f)$, so $\varphi(M)$ is an f-type $\mathcal{M}$ subset.

- In addition, if M is $\langle \tilde{f}, \tilde{h}\rangle$-invariant then $\varphi(M)$ is $\langle f, h\rangle$-invariant.

We are now in position to complete the proof of Theorem 2: Applying Proposition 3.1 to $\tilde{f}$ and $\tilde{h}$, we get a decomposition in suitable pairwise disjoint $\tilde{f}$-type $\mathcal{M}$ subsets M_i. The previous observations imply the existence of IETs φ_i such that $\varphi_i(M_i) = \mathcal{I}_i$ are half-open intervals and it is easy to check that the map φ defined by $\varphi_{|M_i} = \varphi_{i\,|M_i}$ is the desired map.

4 Periodic IET are strongly reversible

Let f be a periodic IET with associated permutation π. By the Arnoux–Keane–Mayer decomposition theorem, the interval $[0, 1)$ can be written as the union of finitely many periodic f-components M_i, $i = 1, \dots, n$, of period p_i. In particular, given $i \in \{1, \dots, n\}$, one has $M_i = \bigsqcup_{k=1}^{p_i} J_k$, where $J_k = f^{k-1}(J_1)$ are half-open intervals and f is continuous on J_k.

After a possible conjugation of f by an IET, we can suppose that the J_k's are ordered consecutive intervals so that the M_i's are intervals and $\pi_i := \pi(f|_{M_i}) = (1, 2, \dots, p_i)$.

We consider the IET h, that is defined on each M_i by h is continuous on J_k and $h(J_k) = J_{\tau(k)}$, where $\tau \in \mathcal{S}_{p_i}$ satisfies $\tau^2 = \mathrm{Id}$ and $\tau\pi_i\tau^{-1}(k) = \pi^{-1}$ (such a permutation exists by Proposition 3.4 of [12]). One has that $h^2 = \mathrm{Id}$ and $h \circ f \circ h^{-1}$ is continuous on J_k and $h \circ f \circ h^{-1}(J_k) = J_{\tau^{-1}\pi_i\tau(k)} = J_{\pi_i^{-1}(k)}$. Therefore $h \circ f \circ h^{-1} = f^{-1}$ meaning that f is strongly reversible in $\mathcal{G}$.

5 A necessary and sufficient condition for strong reversibility in G_n

In this section, we describe reversible maps in G_n. We first characterize strongly reversible maps, then we show that reversible elements are strongly reversible.

Let's recall that an element of G_n is $g = (\alpha_g, \sigma_g)$ with $\sigma_g \in \mathcal{S}_n$ and $\alpha_g \in {\mathbb{S}_n}^n$ where $\mathbb{S}_n = \frac{[0,1/n]}{0=1/n}$. Therefore any equality involving α has to be understood modulo $1/n$.

For $\delta \in \mathcal{S}_n$, it is well known that δ can be decomposed as a product of cycles with disjoint supports and we include fixed points as 1-cycles.

We recall that the support of a p-cycle (i.e., cycle of length p) σ is the set $\mathrm{supp}_\sigma = \{j : \sigma(j) \neq j\}$ when $p \neq 1$ and we widen this for 1-cycles by setting $\mathrm{supp}_\sigma = \mathrm{Fix}(\sigma)$. For a p-cycle σ and $u \in \mathrm{supp}_\sigma$, we write $\sigma = (u, \sigma(u), \dots, \sigma^{p-1}(u))$.

We will prove the following:

Theorem 6 *Let $f = (\alpha_f, \sigma_f) \in G_n$. The map f is strongly reversible in G_n if and only if there exists an involution $\tau \in \mathcal{S}_n$ that reverses σ_f such that for any cycle σ in the decomposition of σ_f into cycles of disjoint supports,*

$$(\dagger) \qquad \sum_{j \in \mathrm{supp}_\sigma} \alpha_f(j) + \sum_{j \in \tau(\mathrm{supp}_\sigma)} \alpha_f(j) = 0.$$

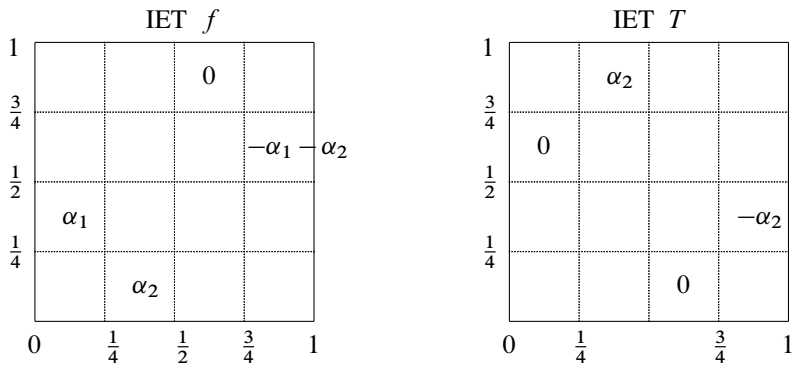

Figure 2

Remark The details in the proof of Theorem 6 are somewhat laborious, however the criterion (†) makes it easy to exhibit reversible IETs that do not present such blatant symmetries as in Section 2.1.2: for the maps described by the Figure 2, f is reversible by the involution T.

Moreover, Theorem 6 is the main ingredient for proving Theorem 3 and it also has extra consequences:

Corollary 7 (1) *A strongly reversible element of G_n whose associated permutation is a cycle has finite order at most $2n$.*

(2) *There exist finite order elements of G_n that are not strongly reversible in G_n.*

(3) *There exist strongly reversible elements in G_n that are not of finite order.*

(4) *Minimal strongly reversible elements do not exist in G_n.*

5.1 Necessary preliminaries

5.1.1 Action of the symmetric group $\mathcal{S}_n$ on vectors The symmetric group $\mathcal{S}_n$ acts linearly on $\mathbb{R}^n$ by permuting the coordinates and we denote by M_σ its associated matrix with respect to the canonical basis of $\mathbb{R}^n$. This leads to an action on the quotient space $\mathbb{S}^n_n$. More precisely:

Definition 5.1 Given $\sigma \in \mathcal{S}_n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{S}^n_n$, we define

$$\sigma(\alpha) = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}).$$

Claim *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{S}^n_n$ and $\sigma, \tau \in \mathcal{S}_n$. Then $(\sigma\tau)(\alpha) = \tau(\sigma(\alpha))$. In other words, $M_{\sigma\tau} = M_\tau M_\sigma$.*

Proof By definition $\sigma(\alpha) = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) = (\beta_1, \dots, \beta_n)$ and $\tau(\sigma(\alpha)) = (\beta_{\tau(1)}, \dots, \beta_{\tau(n)})$ where $\beta_i := \alpha_{\sigma(i)}$. In particular, $\beta_{\tau(j)} = \alpha_{\sigma(\tau(j))}$ and we finally get

$$\tau(\sigma(\alpha)) = (\alpha_{\sigma(\tau(1))}, \dots, \alpha_{\sigma(\tau(n))}) = (\sigma\tau)(\alpha). \qquad \square$$

5.1.2 Strong reversibility in $\mathcal{S}_n$ Our next lemma describes some properties about strong reversibility in $\mathcal{S}_n$.

Lemma 5.2 *Let $\delta, \tau \in \mathcal{S}_n$ such that δ is τ-reversible. Let σ be a cycle of length p in the decomposition of δ as a product of cycles with disjoint supports. Then*:

- *The set $\tau(\mathrm{supp}_\sigma)$ is the support of a cycle of length p in this decomposition of δ. In particular either $\tau(\mathrm{supp}_\sigma) = \mathrm{supp}_\sigma$ or $\tau(\mathrm{supp}_\sigma) \cap \mathrm{supp}_\sigma = \varnothing$.*
- *If $\sigma = (1, \dots, p)$ then τ satisfies that $\tau(k) = \tau(1) + 1 - k \bmod p$ for any $k \in \{1, \dots, p\}$.*
- *If $\tau(\mathrm{supp}_\sigma) = \mathrm{supp}_\sigma$ then there exists u in supp_σ such that $\tau(u) \in \{u, \sigma(u)\}$.*

Proof • The first item is a direct consequence Lemma 2.7(1)–(2), since supp_σ is the orbit of any of its points.

• For the second item, let $k \in \{2, \dots, p\}$. As $\sigma(k-1) = k$ and σ is τ-reversible, one has that $\tau\sigma(k-1) = \sigma^{-1}\tau(k-1) \bmod p$, that is, $\tau(k) = \tau(k-1) - 1 \bmod p$. By iteration, we get that $\tau(k) = \tau(k-j) - j \bmod p$ for any $j \in \{1, \dots, k-1\}$. The required formula is thus obtained when taking $j = k-1$.

• For the third item, let $v \in \mathrm{supp}_\sigma$. As $\tau(\mathrm{supp}_\sigma) = \mathrm{supp}_\sigma$, there exists an integer t such that $\tau(v) = \sigma^t(v)$. Therefore, by reversibility, for any $k \in \{1, \dots, p\}$,

$$\tau(\sigma^k(v)) = \sigma^{-k}(\tau(v)) = \sigma^{-k+t}(v) = \sigma^{-2k+t}(\sigma^k(v)).$$

Finally, taking $u = \sigma^k(v)$ with $k = \left[\frac{t}{2}\right]$ the integer part of $\frac{t}{2}$, one gets $\tau(u) \in \{u, \sigma(u)\}$. □

5.1.3 Necessary and sufficient conditions for strong reversibility In the next properties, we recall the group law of G_n and we give some useful straightforward consequences.

Properties 5.3 *Let $f = (\alpha_f, \sigma_f)$, $g = (\alpha_g, \sigma_g)$ and $T = (\alpha_T, \sigma_T)$ in G_n.*

(1) *$\sigma_{f\circ g} = \sigma_f \circ \sigma_g$ and $\alpha_{f\circ g} = \alpha_g + \sigma_g(\alpha_f)$.*

(2) *$\sigma_{f^{-1}} = \sigma_f^{-1}$ and $\alpha_{f^{-1}} = -\sigma_f^{-1}(\alpha_f)$.*

(3) *T is an involution if and only if $\sigma_T^2 = \mathrm{Id}$ and $\alpha_T = -\sigma_T(\alpha_T)$ or equivalently if and only if $\sigma_T^2 = \mathrm{Id}$ and $\alpha_T(\sigma_T(i)) = -\alpha_T(i)$ for $1 \le i \le n$.*

(4) *T reverses f if and only if $\sigma_T\sigma_f\sigma_T^{-1} = \sigma_f^{-1}$ and*

$$\alpha_{T^{-1}} + \sigma_{T^{-1}}(\alpha_f) + \sigma_{T^{-1}}(\sigma_f(\alpha_T)) = -\sigma_f^{-1}(\alpha_f).$$

5.2 Key proposition

The next proposition provides a criterion for proving that an element of G_n is strongly reversible.

Proposition 5.4 *Let $f = (\alpha_f, \sigma_f)$ and $T = (\alpha_T, \sigma_T)$ be two elements of G_n such that σ_T is an involution that reverses σ_f. Then f is strongly reversible by T if and only if*

(1) *$\alpha_T(\sigma_T(k)) = -\alpha_T(k)$ for $k \in \{1, \dots, n\}$, and*

(2) *$\alpha_T(\sigma_f(k)) = \alpha_T(k) - \alpha_f(k) - \alpha_f(\sigma_T\sigma_f(k))$ for $k \in \{1, \dots, n\}$.*

We first prove the following:

Lemma 5.5 *Let $f=(\alpha_f,\sigma_f)$ and $T=(\alpha_T,\sigma_T)$ in G_n. Then T is an involution that reverses f if and only if σ_T is an involution reversing σ_f that satisfies*

(1) $\alpha_T \in \ker(I+M_{\sigma_T})$, *and*

(2) $\alpha_T \in \ker(I+M_{\sigma_T\sigma_f})+\alpha_f$, *where I stands for the $n\times n$ identity matrix.*

Proof For simplicity, for $g\in G_n$ the matrix M_{σ_g} is denoted by M_g.

Suppose that T is an involution that reverses f.

According to Properties 5.3(3), σ_T is an involution that reverses σ_f and satisfies $\alpha_T+M_T(\alpha_T)=0$; thus item (1) is proved.

It remains to prove item (2). As T is an involution, Properties 5.3(4) leads to

$$\alpha_T+\sigma_T(\alpha_f)+\sigma_T(\sigma_f(\alpha_T))=-\sigma_f^{-1}(\alpha_f).$$

Using matrix notation, this implies that

$$(I+M_TM_f)(\alpha_T)=-(M_T+M_f^{-1})(\alpha_f).$$

Left multiplying both sides by M_T, we obtain

$$(M_T+M_f)(\alpha_T)=-(I+M_TM_f^{-1})(\alpha_f).$$

On one hand, since $M_T(\alpha_T)=-\alpha_T$, we have that

$$(M_T+M_f)(\alpha_T)=(M_T+M_f)M_TM_T(\alpha_T)=(I+M_fM_T)(-\alpha_T)=(I+M_{T\circ f})(-\alpha_T).$$

On the other hand, since $f^{-1}\circ T=T\circ f$, we get that

$$(I+M_TM_f^{-1})(\alpha_f)=(I+M_{f^{-1}\circ T})(\alpha_f)=(I+M_{T\circ f})(\alpha_f).$$

Combining these, we conclude that

$$(I+M_{T\circ f})(\alpha_T)=(I+M_{T\circ f})(\alpha_f).$$

Therefore

$$\alpha_T-\alpha_f\in\ker(I+M_{T\circ f}).$$

Note that under the assumptions that $\alpha_T\in\ker(I+M_{\sigma_T})$ and σ_T is an involution that reverses σ_f, all formulas below are equivalent since the matrix M_T is invertible. □

Proof of Proposition 5.4 The conditions provided in the items (1) of Lemma 5.5 and Proposition 5.4 respectively are clearly equivalent. Under these conditions (1), the conditions (2) are also equivalent.

Indeed,

$$\begin{aligned}
\alpha_T-\alpha_f \in \ker(I+M_{\sigma_T\sigma_f}) &\iff \big(\alpha_T-\alpha_f\big)+(\sigma_T\sigma_f)\big(\alpha_T-\alpha_f\big)=0\\
&\iff \forall k\in\{1,\dots,n\},\ \big(\alpha_T-\alpha_f\big)(k)+\big(\alpha_T-\alpha_f\big)\big((\sigma_T\sigma_f)(k)\big)=0\\
&\iff \forall k\in\{1,\dots,n\},\ \alpha_T(k)-\alpha_f(k)+\alpha_T(\sigma_T\sigma_f(k))-\alpha_f\big(\sigma_T\sigma_f(k)\big)=0\\
&\overset{(1)}{\iff} \forall k\in\{1,\dots,n\},\ \alpha_T(k)-\alpha_f(k)-\alpha_T(\sigma_f(k))-\alpha_f\big(\sigma_T\sigma_f(k)\big)=0.\ \square
\end{aligned}$$

By a straightforward induction argument, condition (2) of Proposition 5.4 leads to:

Consequence *Let $f=(\alpha_f,\sigma_f)$, $\sigma_f=(1,\dots,p)$ and $T=(\alpha_T,\sigma_T=\tau)$ be two elements of G_n. If f is strongly reversible by T then for any $2\le k\le p$,*

$$\alpha_T(k)=\alpha_T(\sigma_f^k(1))=\alpha_T(1)-\sum_{j=1}^{k-1}\alpha_f(j)-\sum_{j=2}^{k}\alpha_f(\tau(j)).$$

5.3 Proof of Theorem 6

There is no loss of generality in assuming that either $n=2p$ and σ_f is a product of two p-cycles of disjoint supports that are exchanged by τ, namely σ and $\tau\sigma^{-1}\tau$, or $n=p$ and $\sigma_f=\sigma$ is a p-cycle whose support is τ-invariant.

1 ((†) is a necessary condition for strong reversibility) Suppose that $f\in G_n$ is reversible by some involution T with $\sigma_T=\tau$.

Summing for $j\in\mathrm{supp}_\sigma$ the equalities (2) of Proposition 5.4, we get that

$$\sum_{j\in\mathrm{supp}_\sigma}\alpha_T(\sigma(j))=\sum_{j\in\mathrm{supp}_\sigma}\alpha_T(j)-\sum_{j\in\mathrm{supp}_\sigma}\alpha_f(j)-\sum_{j\in\mathrm{supp}_\sigma}\alpha_f(\tau\sigma(j))$$

and since $\sum_{j\in\mathrm{supp}_\sigma}\alpha_T(\sigma(j))=\sum_{j\in\mathrm{supp}_\sigma}\alpha_T(j)$, we get

$$\sum_{j\in\mathrm{supp}_\sigma}\alpha_f(j)+\sum_{j\in\mathrm{supp}\sigma}\alpha_f(\tau\sigma(j))=0.$$

Finally, this implies (†) (noting that $\tau\sigma_f=\sigma_f^{-1}\tau$ and changing j for $\ell=\tau(j)\in\tau(\mathrm{supp}_\sigma)$).

2 (under (†), we can find an involution T that reverses f)

Case A (supp_σ and $\tau(\mathrm{supp}_\sigma)$ are disjoint) There is no loss of generality in assuming that $\sigma=(1,\dots,p)$ and then the second cycle $\tau\sigma^{-1}\tau$ has support $\tau(\mathrm{supp}_\sigma)=(p+1,\dots,2p)$.

Let us fix arbitrarily $\alpha_T(1)$. We define a map T by $\sigma_T=\tau$ and, for any $k\in\{1,\dots,n\}$,

$$\begin{aligned}
\alpha_T(k)&=\alpha_T(1)-\sum_{j=1}^{k-1}\alpha_f(j)-\sum_{j=2}^{k}\alpha_f(\tau(j)) \quad \text{if } 2\le k\le p,\\
\alpha_T(k)&=-\alpha_T(\tau(k)) \quad \text{if } p+1\le k\le n=2p.
\end{aligned}$$

Supposing that (†) holds, that is, $\sum_{j=1}^{p}\alpha_f(j)+\sum_{j=p+1}^{n}\alpha_f(j)=0$, we check that the map T defined as before is an involution that reverses f. For that, it is enough to prove that T satisfies conditions (1) and (2) of Proposition 5.4.

- **Condition (1)** $\alpha_T(\tau(k))=-\alpha_T(k)$ is satisfied by definition.
- **Condition (2)** We have to check that $\alpha_T(\sigma_f(k))=\alpha_T(k)-\alpha_f(k)-\alpha_f(\tau\sigma_f(k))$ for $k\in\{1,\dots n=2p\}$, that is,

$$\begin{cases}\alpha_T(k+1)=\alpha_T(k)-\alpha_f(k)-\alpha_f(\tau(k+1)) & \text{if } k\in\{1,\dots,p-1\},\\ \alpha_T(1)=\alpha_T(p)-\alpha_f(p)-\alpha_f(\tau(1)) & \text{if } k=p,\\ \alpha_T(\tau\sigma^{-1}\tau(k))=\alpha_T(k)-\alpha_f(k)-\alpha_f(\sigma^{-1}\tau(k)) & \text{if } k\in\{p+1,\dots,n=2p\}.\end{cases}$$

– For $k\in\{1,\dots,p-1\}$, by definition, one has

$$\begin{aligned}\alpha_T(k+1)&=\alpha_T(1)-\sum_{j=1}^{k}\alpha_f(j)-\sum_{j=2}^{k+1}\alpha_f(\tau(j))\\ &=\alpha_T(1)-\sum_{j=1}^{k-1}\alpha_f(j)-\sum_{j=2}^{k}\alpha_f(\tau(j))-\alpha_f(k)-\alpha_f(\tau(k+1))\\ &=\alpha_T(k)-\alpha_f(k)-\alpha_f(\tau(k+1)).\end{aligned}$$

– For $k=p$, by definition $\alpha_T(p)=\alpha_T(1)-\sum_{j=1}^{p-1}\alpha_f(j)-\sum_{j=2}^{p}\alpha_f(\tau(j))$ so

$$\alpha_T(p)=\alpha_T(1)-\sum_{j=1}^{p}\alpha_f(j)-\sum_{j=1}^{p}\alpha_f(\tau(j))+\alpha_f(p)+\alpha_f(\tau(1)).$$

By (†) we get

$$\alpha_T(p)=\alpha_T(1)+\alpha_f(p)+\alpha_f(\tau(1)).$$

This leads to the required condition (2).

– Let $k\in\{p+1,\dots,n\}$, and $k'=\sigma^{-1}\tau(k)\in\{1,\dots,p\}$. By the previous two items we have

$$\alpha_T(\sigma(k'))=\alpha_T(k')-\alpha_f(k')-\alpha_f(\tau\sigma(k')).$$

Then

$$\begin{aligned}\alpha_T\big(\sigma(\sigma^{-1}\tau(k))\big)&=\alpha_T(\sigma^{-1}\tau(k))-\alpha_f(\sigma^{-1}\tau(k))-\alpha_f\big(\tau\sigma(\sigma^{-1}\tau(k))\big),\\ \alpha_T(\tau(k))&=\alpha_T(\sigma^{-1}\tau(k))-\alpha_f(\sigma^{-1}\tau(k))-\alpha_f(k).\end{aligned}$$

By condition (1) it holds that

$$-\alpha_T(k)=-\alpha_T(\tau\sigma^{-1}\tau(k))-\alpha_f(\sigma^{-1}\tau(k))-\alpha_f(k).$$

This implies condition (2) and completes the proof in case A.

Case B (supp_σ is τ-invariant) There is no loss of generality in assuming that $\sigma = (1, \dots, p)$ and $\tau(1) \in \{1, \sigma(1)\}$ by Lemma 5.2.

We first note that conditions (1) and (2) of Proposition 5.4 imply that an involution T that reverses f has to satisfy

- $\alpha_T(1) = -\alpha_T(1)$ if $\tau(1) = 1$, or
- $2\alpha_T(1) = 2\alpha_f(1)$ if $\tau(1) = \sigma(1)$.

We set $\alpha_T(1) = 0$ if $\tau(1) = 1$ and $\alpha_T(1) = \alpha_f(1)$ if $\tau(1) = \sigma(1)$.
We consider the map T defined by $\sigma_T = \tau$ and, for any $k \in \{1, \dots, n\}$,

$$\alpha_T(k) = \alpha_T(1) - \sum_{j=1}^{k-1} \alpha_f(j) - \sum_{j=2}^{k} \alpha_f(\tau(j)).$$

Supposing that (†) holds, that is, $2\sum_{j\in\mathrm{supp}_\sigma} \alpha_f(j) = 0$, we will prove that T is an involution that reverses f, by checking that T satisfies conditions (1) and (2) of Proposition 5.4.

- **Condition (1)** We have to check that $\alpha_T(\tau(k)) = -\alpha_T(k)$ and the proof depends on $\tau(1)$.
  - If $\tau(1) = 1$ then $\tau(k) = p + 2 - k$ for $k = 2, \dots, p$ (by Lemma 5.2, $\tau(k) = \tau(1) + 1 - k \bmod p$). By definition of α_T we have, on one side,

$$(*) \qquad \alpha_T(k) = \alpha_T(1) - \sum_{j=1}^{k-1} \alpha_f(j) - \sum_{j=2}^{k} \alpha_f(\tau(j)) = \alpha_T(1) - \sum_{j=1}^{k-1} \alpha_f(j) - \sum_{j=2}^{k} \alpha_f(p+2-j)$$
$$= \alpha_T(1) - \sum_{j=1}^{k-1} \alpha_f(j) - \sum_{p+2-k}^{p} \alpha_f(j).$$

  On the other side,

$$(**) \quad \alpha_T(\tau(k)) = \alpha_T(1) - \sum_{j=1}^{\tau(k)-1} \alpha_f(j) - \sum_{j=2}^{\tau(k)} \alpha_f(\tau(j)) = \alpha_T(1) - \sum_{j=1}^{p+1-k} \alpha_f(j) - \sum_{j=2}^{p+2-k} \alpha_f(p+2-j)$$
$$= \alpha_T(1) - \sum_{j=1}^{p+1-k} \alpha_f(j) - \sum_{j=k}^{p} \alpha_f(j).$$

  Summing (∗) and (∗∗), we get

$$\alpha_T(k) + \alpha_T(\tau(k)) = \alpha_T(1) - \sum_{j=1}^{k-1} \alpha_f(j) - \sum_{p+2-k}^{p} \alpha_f(j) + \alpha_T(1) - \sum_{j=1}^{p+1-k} \alpha_f(j) - \sum_{j=k}^{p} \alpha_f(j)$$
$$= 2\alpha_T(1) - 2\sum_{j=1}^{p} \alpha_f(j) = 0,$$

  by the choice of $\alpha_T(1)$ and (†).
  - If $\tau(1) = \sigma(1) = 2$ then by Lemma 5.2, $\tau(k) = p + 3 - k$ for $k = 3, \dots, p$ and the proof is analogous to the previous one.

• **Condition (2)** We have to check that $\alpha_T(\sigma_f(k)) = \alpha_T(k) - \alpha_f(k) - \alpha_f(\tau\sigma_f(k))$ for $k \in \{1,\dots,p\}$. By definition,

$$\alpha_T(k) = \alpha_T(1) - \sum_{j=1}^{k-1} \alpha_f(j) - \sum_{j=2}^{k} \alpha_f(\tau(j)),$$

so for $k \in \{1,\dots,p-1\}$, we have

$$\begin{aligned}\alpha_T(k+1) &= \alpha_T(1) - \sum_{j=1}^{k} \alpha_f(j) - \sum_{j=2}^{k+1} \alpha_f(\tau(j)) \\ &= \alpha_T(1) - \sum_{j=1}^{k-1} \alpha_f(j) - \sum_{j=2}^{k} \alpha_f(\tau(j)) - \alpha_f(k) - \alpha_f(\tau(k+1)).\end{aligned}$$

This proves condition (2) for $k \in \{1,\dots,p-1\}$.

As in case A, the condition (†) is used for treating the case $k = p$ that is left to the readers.

5.4 Proof of Corollary 7 of Theorem 6

(1) Suppose that f is a strongly reversible element with σ_f a n-cycle.

By Properties 5.3 (1), $\alpha_{f^p} = \sum_{j=0}^{p-1} \sigma_f{}^j(\alpha_f)$. Therefore $2\alpha_{f^n}(i) = 2\sum_{j=0}^{n-1}\big(\sigma_f{}^j(\alpha_f)\big)(i) = 0$ for all $i \in \{1,\dots,n\}$ by Theorem 6.

Finally, $\sigma_{f^n} = \mathrm{Id}$ and $2\alpha_{f^n}(i) = 0$ for all $i \in \{1,\dots,n\}$. Hence, f has finite order at most $2n$.

(2) The map $f = (\alpha_1, \mathrm{Id}) \in G_1$ with $\alpha_1 = \frac{1}{q}$ with $q > 2$ has finite order q but is not strongly reversible in G_1 by the previous item.

(3) Let's go back to the examples given in Section 2.1.2 and choose $\alpha_A = (\alpha, -\alpha)$ with $\alpha \notin \mathbb{Q}$ and $\alpha_B = (0,0)$. It is clear that A has infinite order and that B is an involution which reverses A.

(4) It is plain that the permutation σ_f associated to a minimal element $f \in G_n$ is a n-cycle. Then, according to item (1), minimal strongly reversible elements of G_n do not exist.

6 In G_n, reversibility implies strong reversibility. Proof of Theorem 3

In this section $f, T \in G_n$ and $f = (\alpha_f, \sigma_f)$ is reversible by $T = (\alpha_T, \sigma_T)$. Our goal is to show that f is strongly reversible. The proof will be divided into cases depending on the parity of the order of σ_T.

6.1 Proof of Theorem 3 when σ_T is of odd order

Proposition 6.1 *Let $f, T \in G_n$. If f is reversible by T and there exists s odd such that*[1] $\sigma_T^s \in \langle\sigma_f\rangle$, *then*

(1) *σ_f is an involution,*

(2) *f is strongly reversible and*

(3) *f has finite order 2 or 4.*

[1] This is weaker than requiring σ_T has odd order s because, in this case, $\sigma_T^s = \mathrm{Id} \in \langle\sigma_f\rangle$.

Proof First we reduce to the case $\sigma_T = \mathrm{Id}$. Let $f \in G_n$ be reversible by $T \in G_n$ such that $\sigma_T^s = \sigma_f^t$ for some integers t and s odd. By Lemma 2.4(1), the map $T_1 = T^s f^{-t}$ reverses f and by its definition $\sigma_{T_1} = \mathrm{Id}$.

Next, let $f \in G_n$ reversible by T with $\sigma_T = \mathrm{Id}$.

(1) As $\sigma_T = \mathrm{Id}$, one has $\sigma_f = (\sigma_f)^{-1}$.

(2) The necessary and sufficient condition of Properties 5.3(4) can be written as

$$\alpha_{T^{-1}} + \alpha_f + \sigma_f(\alpha_T) + \sigma_f(\alpha_f) = 0.$$

Composing by σ_f, one gets

$$\sigma_f(\alpha_{T^{-1}}) + \sigma_f(\alpha_f) + \alpha_T + \alpha_f = 0.$$

As $\sigma_T = \mathrm{Id}$ one has $\alpha_{T^{-1}} = -\alpha_T$. Summing the previous equalities one gets

$$2(\alpha_f + \sigma_f(\alpha_f)) = 0. \tag{A}$$

As σ_f is an involution and $\sigma_T = \mathrm{Id}$, formula (A) leads to the necessary and sufficient condition (†) for strong reversibility given by Theorem 6 when $\tau = \mathrm{Id}$.

(3) Item (1) of Properties 5.3 implies that $\alpha_{f^2} = \alpha_f + \sigma_f(\alpha_f)$ and then

$$\alpha_{f^4} = 2\big(\alpha_f + \sigma_f(\alpha_f)\big).$$

In conclusion, one has $\alpha_{f^4} = 0$ by (A) and $\sigma_f^4 = \mathrm{Id}$, so $f^4 = \mathrm{Id}$. □

6.2 Proof of Theorem 3 when σ_T is of even order

Lemma 6.2 *Let $f, T \in G_n$.*

Suppose that f is reversible by T, $\sigma_T = \tau$ is of even order, and the action of $\langle \sigma_f, \tau \rangle$ on $\{1, \dots, n\}$ is transitive.

Let $u \in \{1, \dots, n\}$ and σ be the cycle in the decomposition of σ_f as a product of cycles of disjoint supports such that $u \in \mathrm{supp}_\sigma$. Then, either:

(1) *There exists an odd integer s such that $\tau^s(u) \in \mathcal{O}_\sigma(u)$.*

(2) *For any $j \in \{1, \dots, n\}$ there exists an integer s such that $\tau^s(j) \in \mathcal{O}_\sigma(u)$. Moreover, if s' satisfies $\tau^{s'}(j) \in \mathcal{O}_\sigma(u)$, then $s - s'$ is even. In particular, the map given by $\epsilon(j) = (-1)^s$, where s is an integer such that $\tau^s(j) \in \mathcal{O}_\sigma(u)$, is well defined.*

Proof Suppose that there is no odd integer s such that $\tau^s(u) \in \mathcal{O}_\sigma(u)$.

By the transitivity assumption and Lemma 2.4, for any $j \in \{1, \dots, n\}$ there exist integers s and t such that $j = \tau^{-s}\sigma^t(u)$; thus $\tau^s(j) = \sigma^t(u) \in \mathcal{O}_\sigma(u)$.

Moreover, suppose by contradiction that there exist other integers s' and t' such that $\tau^{s'}(j) = \sigma^{t'}(u)$ (i.e., $j = \tau^{-s'}\sigma^{t'}(u)$) and $s - s'$ is odd.

Consequently, $\tau^{s-s'}(u) = \sigma^{t+t'}(u) \in \mathcal{O}_\sigma(u)$ which contradicts our initial assumption. □

We turn now on to the proof of Theorem 3.

Without loss of generality, we can suppose that the action of $\langle \sigma_f, \tau\rangle$ on $\{1,\dots,n\}$ is transitive, because the set $\{1,\dots,n\}$ decomposes in finitely many $\langle \sigma_f, \tau\rangle$-orbits on which the restrictions of the action are transitive.

Fix $u \in \{1,\dots,n\}$ and let σ be the cycle in the decomposition of σ_f as a product of cycles of disjoint supports such that $u \in \mathrm{supp}_\sigma$.

Case 1 (there is no odd integer s such that $\tau^s(u) = \sigma^t(u)$) Item (2) of Lemma 6.2 allows us to define T_0 as $T_0(x) = T^{\epsilon(j)}(x)$ for $x \in I_j$.

More precisely, writing $j = \tau^{-s}\sigma^t(u)$ we have

$$T_0(x) = \begin{cases} T(x) & \text{if } x \in I_{\tau^{-s}\sigma^t(u)} \text{ and } s \text{ is even,} \\ T^{-1}(x) & \text{if } x \in I_{\tau^{-s}\sigma^t(u)} \text{ and } s \text{ is odd.} \end{cases}$$

One can easily check that T_0 is an involution that reverses f.

Case 2 (there exists an odd integer s such that $\tau^s(u) = \sigma^t(u)$) Let $T_1 = T^s f^t$.

By Lemma 2.4(1), the map T_1 reverses f and $\sigma_{T_1}(u) = \tau^s\sigma^t(u) = \sigma^{-t}\tau^s(u) = u$. Therefore, the orbit $\langle \sigma, \sigma_{T_1}\rangle(u)$ coincides with $\mathcal{O}_\sigma(u)$, and for any k, we have $\sigma_{T_1}(\sigma^k(u)) = \sigma^{-k}(\sigma_{T_1}(u)) = \sigma^{-k}(u)$ and $\sigma_{T_1}(\sigma^{-k}(u)) = \sigma^k(u)$.

Summarizing, σ_{T_1} is an involution on $\langle \sigma, \sigma_{T_1}\rangle(u) = \mathcal{O}_\sigma(u)$ that reverses σ.

Hence, we can without loss of generality consider that $\sigma = (1,\dots,n)$ and, according to Lemma 5.2, that $\sigma_{T_1}(j) = N - j$ for some $N = 1,\dots,n+1$. Letting $x \in I_j$, one has

$$T_1 \circ f(x) = x + \alpha_f(j) + \alpha_{T_1}(j+1) \quad \text{and} \quad f^{-1} \circ T_1(x) = x + \alpha_{T_1}(j) - \alpha_f(N-1-j).$$

Therefore, as T_1 reverses f,

$$\alpha_f(j) + \alpha_{T_1}(j+1) = \alpha_{T_1}(j) - \alpha_f(N-1-j) \quad \text{for any } j = 1,\dots,n.$$

Summing from $j = 1$ to n, we get $2\sum_{j=1}^n \alpha_f(j) = 0$.

We conclude by noting that this is exactly the necessary and sufficient condition (†) for strong reversibility in the case of a σ_{T_1}-invariant σ_f-cycle.

7 Proof of Theorem 1 and Corollary 4

7.1 Proof of Theorem 1

Let $f, h \in \mathcal{G}$ be such that f is h-reversible. According to Theorem 2 and after a possible conjugation in $\mathcal{G}$, the unit interval I decomposes as a union of finitely many pairwise disjoint $\langle f, h\rangle$-invariant half-open intervals $\mathcal{I}_i$ such that either

(1) $f_{|\mathcal{I}_i}$ has finite order, or

(2) the $\langle f, h\rangle$-action on $\mathcal{I}_i$ is a free BS$(1,-1)$-action, or

(3) $h_{|\mathcal{I}_i}$ has finite even order, or

(4) $\mathcal{I}_i$ is a union of even number of minimal f-components.

Now, in any of these four cases, we prove that $f_{|\mathcal{I}_i}$ is reversible by a finite order element.

(1) We conclude that $f_{|\mathcal{I}_i}$ is strongly reversible by Proposition 1.2.

(2) After a possible conjugation by an IET of support in $\mathcal{I}_i$, we can suppose that the f-components in $\mathcal{I}_i$, which are minimal, are intervals and without loss of generality, we can suppose that h acts transitively on these intervals, so that they have the same length.

By Proposition 1.4, there exists a PL-homeomorphism P such that $f_{|\mathcal{I}_i}$ is conjugated to a reversible element $F \in G_n$ by P. By Theorem 3, F is reversible in G_n by an involution T. It remains to prove that a priori affine interval exchange transformation PTP^{-1} that is an involution that reverses $f_{|\mathcal{I}_i}$ belongs to $\mathcal{G}$.

Indeed, the map P^{-1} is affine on $\left[\frac{k-1}{n}, \frac{k}{n}\right)$ for any $k \in \{1, \dots, n\}$, and by transitivity of the action of h on the f-components, P has constant slope.

We conclude that f is strongly reversible by $PTP^{-1} \in \mathcal{G}$.

(3) Let 2ℓ be the order of $h_{|\mathcal{I}_i}$. Either ℓ is odd and h^ℓ is an involution reversing f or ℓ is even and h has order a multiple of 4.

(4) According to the proof of Proposition 3.1, $\mathcal{I}_i$ can be written as

$$\mathcal{I}_i = \bigcup_{j=0}^{p_i - 1} h^j(N),$$

where N is an f-minimal component and p_i is even. The map h_0, defined by

$$h_0(x) = \begin{cases} h(x) & \text{if } x \in h^j(N) \text{ and } j \text{ is even,} \\ h^{-1}(x) & \text{if } x \in h^j(N) \text{ and } j \text{ is odd,} \end{cases}$$

is an involution that reverses $f_{|\mathcal{I}_i}$.

To conclude, let $\{h_i\}$ be the collection of the finite order elements exhibited in the previous items, we define a map H on I by $H_{|\mathcal{I}_i} = h_i$. It is plain that H is a finite order element that reverses f. Moreover, if f is not an involution then the order p of H is even (otherwise $H^p = \mathrm{Id}$ would reverse f). Hence, $p = q2^n$ with q odd and $n \in \mathbb{N}^*$; therefore the map H^q reverses f and has order 2^n.

7.2 Proof of Corollary 4

Let f be reversible in $\mathcal{G}$. According to Theorem 1, there exists a periodic IET h of even order $2n$ that reverses f.

We first prove that $h^{-1}f$ is also a periodic IET of order $2n$. Indeed

$$(h^{-1}f)^2 = h^{-1}fh^{-1}f = h^{-2}hfh^{-1}f = h^{-2}f^{-1}f = h^{-2};$$

hence

$$(h^{-1}f)^{2n} = h^{-2n} = \mathrm{Id}.$$

Therefore $f = hh^{-1}f$ is the product of two periodic IETs. Finally, by Proposition 1.2 and Properties 2.1, we get that f is the product of at most 4 involutions.

8 The group C_2 does not admit faithful minimal actions by IET

8.1 Components of finitely generated subgroups of IET

Let G be a finitely generated subgroup of $\mathcal{G}$.

Definition 8.1 We say that V is a *G-type $\mathcal{M}$* subset if V is a nonempty finite union of intervals of the form $[b, c)$ with b, c in $\mathrm{BP}(G) = \bigcup_{g\in G} \mathrm{BP}(g)$, and *$G$-components* are defined to be minimal among G-type $\mathcal{M}$ and G-invariant sets.

Lemma 8.2 *Let M be a G-component and $g \in G$. If $x \in M$ is not g-periodic then $M_g(x) \subset M$. In particular, if g is periodic-point free on M then M decomposes as a union of minimal g-components.*

Proof We consider the Denjoy blow-up map, used in the proof of Lemma 2.11, namely

$$\pi : X = [0, 1) \sqcup \mathrm{BP}^-_\infty(G) \to [0, 1),$$

that is closed, continuous and whose restriction to $X \setminus (\mathrm{BP}_\infty(G) \cup \mathrm{BP}^-_\infty(G))$ is a homeomorphism onto its image. Any $g \in G$ extends to $\hat{g} \in \mathrm{Homeo}(X)$ defined by $\hat{g}(x) = g(x)$ if $x \in [0, 1)$ and $\hat{g}(x^-) = \lim_{y\to x^- y<x^-} \hat{g}(y)$.

It is clear that if $M = \bigsqcup_{i=1}^r [u_i, v_i)$ then M lifts to the set $M_X = \bigsqcup_{i=1}^r [\![u_i^+, v_i^-]\!]$ that is a closed and G-invariant subset of X, where $[\![\cdot, \cdot]\!]$ are intervals for the total order defined on X.

Let $x \in M$. Then $M_g(x) \subset \pi(\overline{\mathcal{O}_{\hat{g}}(x)}) \subset \pi(M_X)$. By contradiction, if $M_g(x) \not\subset M$ then there exists $v_i \in M_g(x)$. Therefore, there is a nontrivial interval $[v_i, v_i + \varepsilon)$ contained in $M_g(x)$ and then in M_X, a contradiction. □

8.2 Proof of Corollary 5

In order to prove Corollary 5, we establish a more precise statement summarized in:

Proposition 8.3 *Consider a faithful IET-action of the group $C_2 = \langle a, b, c \mid aba^{-1} = b^{-1}, bcb^{-1} = c^{-1}\rangle$, and by abuse of notation, we see $C_2 = \langle a, b, c\rangle$ as a subgroup of $\mathcal{G}$.*

Let M be a C_2-component. Then either

(1) *b has finite order on M or*

(2) *there exist nonzero integers k and m such that $[a^k, c^m] = \mathrm{Id}$ on M.*

Proof Let M be a C_2-component.

(1) If $\mathrm{Per}(b^2) \cap M \neq \varnothing$, then by items (2) and (3) of Lemma 2.7, $\mathrm{Per}(b^2)$ is invariant by a, b and c. Therefore $\mathrm{Per}(b^2) \cap M = M$, and this means that b has finite order on M.

(2) If $\mathrm{Per}(b^2) \cap M = \varnothing$ then b is periodic-point free on M.

As $\mathrm{Per}(b) \cap M = \varnothing$, Lemma 8.2 implies that the set M decomposes in finitely many minimal b-components. Moreover, since a reverses b there exists $\ell \in \mathbb{N}$, $\ell \neq 0$ such that a^ℓ preserves any b-component by Lemma 2.11.

On the other hand, as M is c-invariant, it is a finite union of c-components.

- If M_c is a periodic c-component contained in M, then for any integer j, we have $b^j(M_c) \subset \mathrm{Per}(c)$ by Lemma 2.7, and thus $M' = \bigcup_{j\in\mathbb{Z}} b^j(M_c) \subset \mathrm{Per}(c)$.

Hence, there exists a positive integer m such that $c^m = \mathrm{Id}$ on M'. Noting that $a^\ell(M') = M'$ because M' is a union of minimal b-components by Lemma 8.2, we thus get that $[a^\ell, c^m] = \mathrm{Id}$ on M'.

- If M_c is a minimal c-component contained in M, then there exists an integer r such that $b^r(M_c) = M_c$ by Lemma 2.11. The set $M' = \bigcup_{j=1}^{r} b^j(M_c)$ is thus a $\langle b, c\rangle$-component and it is a^ℓ-invariant as a union of minimal b-components by Lemma 8.2.

– If the $\langle b, c\rangle$-action on M' is free, according to Proposition 1.4 and after a possible conjugation by an IET E, there exists a PL-homeomorphism P such that the $\langle b, c\rangle$-action on M' is P-conjugate to a subgroup of some G_n.

We are now in position to apply [8, Proposition 3.12] that we recall.

Proposition [8, 3.12] *Let f be a periodic-point free IET that is conjugated to an element $F \in G_n$ by a PL-homeomorphism P such that $P^{-1}_{|I_i}$ is an affine map. Then the map P conjugates the centralizer of f in $\mathcal{G}$ to a subgroup of G_n.*

As M' is $\langle a^\ell, b, c\rangle$-invariant and $a^{2\ell}$ commutes with b, periodic-point free, on M', the map $a^{2\ell}_{|M'}$ is also P-conjugate to an element of G_n.

Consequently, the $\langle a^{2\ell}, b, c\rangle$-action on M' is P-conjugated to a subgroup of some G_n. In particular, it is virtually abelian and then there exist integers k' and m such that $[a^{k'}, c^m] = \mathrm{Id}$ on M'.

– If the $\langle b, c\rangle$-action on M' is not free, by Lemma 2.3, there exist $g = b^s c^t \in C_2 \setminus \{\mathrm{Id}\}$ and $x \in M'$ such that $g(x) = x$.

According to Lemma 2.12, if g reverses c then $\mathrm{Fix}(g) = \mathrm{Per}_1(g) \subset \mathrm{Fix}(c) = \varnothing$, a contradiction. Thus g commutes with c.

Therefore, by Lemma 2.7(2), the set $\mathrm{Fix}(g)$ is c-invariant. In addition, Lemma 2.7(4) implies that $\mathrm{Fix}(g)$ is also b^2-invariant.

As the same reasoning applies to $b(\mathrm{Fix}(g)) = \mathrm{Fix}(bgb^{-1})$, we can conclude that $\mathrm{Fix}(g) \cup b(\mathrm{Fix}(g))$ is invariant by c and b.

Finally, as M' is a $\langle b, c\rangle$-component, we have $M' \subset \mathrm{Fix}(g) \cup b(\mathrm{Fix}(g))$.

By definition of $g = b^s c^t$, it holds that $c^t = b^{-s}$ on $\mathrm{Fix}(g)$.

We claim that $c^t = b^s$ on $b(\mathrm{Fix}(g))$. Indeed, let $y \in b(\mathrm{Fix}(g))$. As $\mathrm{Fix}(g)$ is b^2- and c-invariant, $c^{-t}b(y) \in \mathrm{Fix}(g)$, and thus

$$c^t(c^{-t}b(y)) = b^{-s}(c^{-t}b(y)).$$

This leads to $b^{s+1}(y) = c^{-t}b(y) = bc^t(y)$, and thus $b^s(y) = c^t(y)$.

As $a^{2\ell}$ commutes with b, this establishes that $[a^{2\ell}, c^t] = \mathrm{Id}$ on M'.

It is easy to see that M decomposes as the union of finitely many sets of the form $M_i' = \bigcup_{j\in\mathbb{Z}} b^j(M_c^i)$ with M_c^i a periodic c-component or $M_i' = \bigcup_{j=1}^{r_i} b^j(M_c^i)$ with M_c^i a minimal c-component.

By the previous arguments, there exist integers k_i and m_i such that $[a^{k_i}, c^{m_i}] = \mathrm{Id}$ on M_i' for any i. This ends the proof of Proposition 8.3 by setting $k = \prod k_i$ and $m = \prod m_i$. □

Corollary 5 is a consequence of the fact that a minimal action has a unique component, so by Proposition 8.3, it cannot be faithful.

Acknowledgements

We thank Juan Alonso and Andrés Navas for helpful discussions. We are grateful to the referees for their comments. We acknowledge support from the MathAmSud Project GDG 18-MATH-08, the Labex CEMPI (ANR-11-LABX-0007-01), the University of Lille (BQR) and the I.F.U.M.I.

References

[1] **P Arnoux**, *Échanges d'intervalles et flots sur les surfaces*, from "Ergodic theory" (Les Plans-sur-Bex, 1980), Monogr. Enseign. Math. 29, Univ. Genève (1981) 5–38 MR

[2] **P Arnoux**, *Un invariant pour les échanges d'intervalles et les flots sur les surfaces*, PhD thesis, Université de Reims (1981)

[3] **P Arnoux**, **J-C Yoccoz**, *Construction de difféomorphismes pseudo-Anosov*, C. R. Acad. Sci. Paris Sér. I Math. 292:1 (1981) 75–78 MR

[4] **C Bonatti**, **I Monteverde**, **A Navas**, **C Rivas**, *Rigidity for C^1 actions on the interval arising from hyperbolicity, I: Solvable groups*, Math. Z. 286:3-4 (2017) 919–949 MR

[5] **Y Cornulier**, *Realizations of groups of piecewise continuous transformations of the circle*, J. Mod. Dyn. 16 (2020) 59–80 MR

[6] **N Gill**, **A G O'Farrell**, **I Short**, *Reversibility in the group of homeomorphisms of the circle*, Bull. Lond. Math. Soc. 41:5 (2009) 885–897 MR

[7] **N Guelman**, **I Liousse**, *Uniform perfectness for interval exchange transformations with or without flips*, Ann. Inst. Fourier (Grenoble) 72:4 (2022) 1477–1501 MR

[8] **N Guelman**, **I Liousse**, *Interval exchange transformations groups: free actions and dynamics of virtually abelian groups*, Transform. Groups 30:4 (2025) 1785–1807 MR

[9] **M Keane**, *Interval exchange transformations*, Math. Z. 141 (1975) 25–31 MR

[10] **F Le Roux**, *Free planar actions of the Klein bottle group*, Geom. Topol. 15:3 (2011) 1545–1567 MR

[11] **A Mayer**, *Trajectories on the closed orientable surfaces*, Rec. Math. [Mat. Sbornik] N.S. 12/54 (1943) 71–84 MR In Russian

[12] **A G O'Farrell**, **I Short**, *Reversibility in dynamics and group theory*, London Mathematical Society Lecture Note Series 416, Cambridge Univ. Press (2015) MR

NANCY GUELMAN nguelman@fing.edu.uy
Instituto de Matemática y Estadistica Rafael Laguardia, Universidad de la República, Montevideo, Uruguay

ISABELLE LIOUSSE isabelle.liousse@univ-lille.fr
Université de Lille 1, CNRS, UMR 8524 - Laboratoire Paul Painlevé, Lille, France

Received: June 19, 2024

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Vesna Stojanoska
vesna@illinois.edu
University of Illinois at Urbana-Champaign

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Daniel Isaksen	Wayne State University isaksen@math.wayne.edu
Steven Boyer	Université du Québec à Montréal boyer.steven@uqam.ca	Thomas Koberda	University of Virginia thomas.koberda@virginia.edu
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Markus Land	JGU Mainz mland@uni-mainz.de
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Octav Cornea	Université de Montreal cornea@dms.umontreal.ca	Norihiko Minami	OCAMI (Osaka Central Adv. Math. Inst.) norihikominami@gmail.com
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Kristen Hendricks	Rutgers University kristen.hendricks@rutgers.edu	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Lior Yanovski	Hebrew University of Jerusalem lior.yanovski@gmail.com
Kathryn Hess	École Polytechnique Féd. de Lausanne kathryn.hess@epfl.ch		

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2026 is US $795/year for the electronic version, and $1170/year (+$80, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

© 2026 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 26 Issue 7 (pages 2353–2708) 2026

Coxeter quotients of the automorphism group of a Coxeter group 2353
OLGA VARGHESE

An elementary proof of Novikov's theorem 2363
SAMUEL RANZ and LAURAN TOUSSAINT

Magnitude homology of geodesic spaces 2377
KIYONORI GOMI

(Non)vanishing results for extensions between simple outer functors on free groups 2421
LOUIS HAINAUT

Whitehead filtrations for computations in topological Hochschild homology 2443
LOGAN HYSLOP

Around subgroups of Artin groups: derived subgroups and acylindrical hyperbolicity in the even FC-case 2475
JONE LOPEZ DE GAMIZ ZEARRA and CONCHITA MARTÍNEZ-PÉREZ

Reversible maps in groups of interval exchange transformations 2497
NANCY GUELMAN and ISABELLE LIOUSSE

On stratifications and poset-stratified spaces 2519
LUKAS WAAS, JON WOOLF and SHOJI YOKURA

On the girth of groups acting on CAT(0) cube complexes 2539
ARKA BANERJEE, DANIEL L GULBRANDSEN, PRATYUSH MISHRA and PRAYAGDEEP PARIJA

Commutator-stable representations of hyperbolic groups 2559
ULYSSE REMFORT-AURAT

Spaces of nonresultant systems of bounded multiplicity with real coefficients 2589
ANDRZEJ KOZLOWSKI and KOHHEI YAMAGUCHI

Crossed modules and cohomology of algebras over an operad 2635
JOHAN LERAY, SALIM RIVIÈRE and FRIEDRICH WAGEMANN

On lens spaces bounding smooth 4-manifolds with $b_2 = 1$ 2659
WOOHYEOK JO, JONGIL PARK and KYUNGBAE PARK

On uniqueness of coarse median structures 2689
ELIA FIORAVANTI and ALESSANDRO SISTO

A note on a nontrivial connected component of the space of symplectic structures 2705
LUYA WANG