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On the girth of groups acting on CAT(0) cube complexes

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We obtain a sufficient condition for lattices in the automorphism group of a finite-dimensional CAT(0) cube complex to have infinite girth. As a corollary, we get a version of the girth alternative for groups acting geometrically on CAT(0) cube complexes: any such group is either {locally finite}-by-{virtually abelian} or it has infinite girth. We produce counterexamples to show that the alternative fails in the general class of groups acting cocompactly on finite-dimensional CAT(0) cube complexes using examples of nonvirtually solvable groups which satisfy a law.

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1 Introduction

The *girth* of a finite graph is defined as the length of its shortest nontrivial cycle. If the graph has no cycle then its girth is defined to be infinite. Given a finitely generated group Γ with a fixed generating set S, it makes sense to talk about the girth of its Cayley graph, denoted by $\mathrm{Girth}(\mathrm{Cay}(\Gamma, S))$. Then, one can define the *girth* of the group Γ as the supremum of the girth of the Cayley graphs over all finite generating sets S of Γ, i.e.,

$$\mathrm{Girth}(\Gamma) = \sup_{\Gamma=\langle S\rangle, |S|<\infty} \big\{\mathrm{Girth}(\mathrm{Cay}(\Gamma, S))\big\}.$$

Any free group has infinite girth because one of its Cayley graphs is a tree and trees have infinite girth. Additionally, the fundamental group of a surface of genus g, $\pi_1(\Sigma_g)$, has infinite girth for $g \geq 2$. This is because any such surface group can be viewed as a proper HNN extension of the free group $\mathbb{F}_{2g-1}$. Infinite girth then follows from [4, Proposition 1.4]. Also, if $\mathrm{Girth}(\Gamma) = \infty$ and $\Gamma \ncong \mathbb{Z}$, then $\mathrm{Girth}(\Gamma \times H) = \infty$ for any group H [2, Proposition 1.1]. This fact gives us lots of examples of groups with infinite girth.

In contrast, examples of groups with finite girth include finite groups, noncyclic nilpotent groups, and solvable groups. In fact, any finitely generated group satisfying a nontrivial law (except for $\mathbb{Z}$) has finite girth [29]. Additionally, if a group has a finite index subgroup with finite girth, the entire group must also

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have finite girth [29]. Consequently, virtually solvable groups have finite girth. This article explores the converse of this fact: Does having finite girth imply that a group is virtually solvable?

The answer to the above question is negative in general. For example, there exists a group that satisfies a nontrivial law (and hence has finite girth) but which is not virtually solvable [14]. However, we can refine our investigation by asking under what conditions the answer may be affirmative. We say that a class of groups satisfies the girth alternative if the answer to this question is positive for all groups within that class. Introduced by Ahkmedov [2], the alternative has the following statement.

Girth alternative *For a given class $\mathcal{C}$ of finitely generated groups, $\mathcal{C}$ is said to satisfy the girth alternative if for any group $\Gamma \in \mathcal{C}$, either Γ is virtually solvable or* $\operatorname{Girth}(\Gamma) = \infty$.

The above alternative is similar in spirit to the Tits alternative. In 1972, Jacques Tits introduced and proved the Tits alternative for linear groups [30]. The alternative can be stated as follows.

Tits alternative *For a given class $\mathcal{C}$ of finitely generated groups, $\mathcal{C}$ is said to satisfy the Tits alternative if for any group $\Gamma \in \mathcal{C}$, either Γ is virtually solvable or Γ contains a nonabelian free subgroup $\mathbb{F}_2$.*

Some classes of finitely generated groups satisfying the Tits alternative are linear groups [30], Gromov hyperbolic groups [20], mapping class groups of compact surfaces [8; 22], $\mathrm{Out}(\mathbb{F}_n)$ [6; 7], groups acting geometrically on finite-dimensional CAT(0) cube complexes [13; 28], graph products of certain groups [5], etc. On the other hand, the class of subgroups of the group of orientation preserving piecewise linear homeomorphisms of the unit interval, $\mathrm{PL}_+(I)$, does not satisfy the Tits alternative. This is because Thompson's group F does not contain a copy of $\mathbb{F}_2$ [10] and it is not virtually solvable [9].

In [2], Akhmedov observed that for certain classes of groups for which the Tits alternative holds, the property of having infinite girth coincides with the property of containing a nonabelian free group. Consequently, these classes of groups also satisfy the girth alternative. A comprehensive theory of the girth alternative for different classes of groups has been developed by many authors in the past two decades. In [1; 2; 3], Akhmedov introduced and proved the girth alternative for hyperbolic, linear, one-relator groups, and subgroups of $\mathrm{PL}_+(I)$. In [31], Yamagata proves the girth alternative for convergence groups and irreducible subgroups of the mapping class groups. Independently in [25], Nakamura proves the girth alternative for all subgroups of mapping class groups and also for certain subgroups of $\mathrm{Out}(\mathbb{F}_n)$. Recently, Akhmedov and the third author of this article have studied the girth alternative phenomenon for HNN extensions and amalgamated free products [4].

Remark 1 It is not true, in general, that if a class of groups satisfies the Tits alternative, then it satisfies the girth alternative [1, Theorem 4.1]. Conversely, Akhmedov [3] proved that the class of subgroups of $\mathrm{PL}_+(I)$ satisfies the girth alternative. However, the Tits alternative fails in $\mathrm{PL}_+(I)$ because of the presence of the Thompson's group F, which is neither virtually solvable nor does it contain a copy of $\mathbb{F}_2$.

In this paper, we focus on groups acting on CAT(0) cube complexes. Different versions of the Tits alternative for groups acting on CAT(0) cube complexes are known from the work of Sageev and Wise [28]

and Caprace and Sageev [13]. Inspired by the work of Caprace and Sageev, we prove a version of the girth alternative for lattices in the group of automorphisms of a finite-dimensional CAT(0) cube complex.

Theorem A *Let $X = X_1 \times \cdots \times X_n$ be a product of n irreducible, unbounded, and locally compact* CAT(0) *cube complexes such that* $\mathrm{Aut}(X_i)$ *acts cocompactly and essentially on X_i for all i.*

Then for any (possibly nonuniform) lattice $\Gamma \leq \mathrm{Aut}(X)$ acting properly on X, either Γ is {locally finite}-by-{virtually abelian} or $\mathrm{Girth}(\Gamma) = \infty$.

Note that if a group has an upper bound on the order of its finite subgroups, then any locally finite subgroup of that group has finite order. Therefore, the following is an immediate consequence of Theorem A.

Corollary B *Suppose X satisfies the hypothesis of Theorem A. Suppose $\Gamma \leq \mathrm{Aut}(X)$ is a lattice (possibly nonuniform) acting properly on X and Γ has a bound on the order of its finite subgroups.*

Then either Γ is virtually abelian or $\mathrm{Girth}(\Gamma) = \infty$. *In particular, Γ satisfies the girth alternative.*

We obtain Theorem A as a consequence of the proof of the following more general theorem that gives sufficient conditions for a lattice to have infinite girth.

Theorem C *Suppose X satisfies the hypothesis of Theorem A. Let Γ be a (possibly nonuniform) lattice in* $\mathrm{Aut}(X)$. *If Γ virtually does not fix any point in $X \cup \partial_\infty X$, then* $\mathrm{Girth}(\Gamma) = \infty$.

Example 2 It follows from the proof of Theorem C that if Γ virtually fixes a point in $X \cup \partial_\infty X$, then each irreducible factor of X has a $\mathrm{Aut}(X)$-invariant geodesic line. In other words, each factor is $\mathbb{R}$-like, which roughly means each factor coarsely resembles the real line. Therefore, if $X = T_1 \times \cdots \times T_n$ is a product of trees, none of which are $\mathbb{R}$-like, and $\mathrm{Aut}(T_i)$ acts essentially and cocompactly on each factor T_i, then Theorem C implies that lattices in $\mathrm{Aut}(X)$ have infinite girth. For example, a group acting properly and cocompactly on a product of biregular trees, none of which are homeomorphic to the real line, has infinite girth.

In contrast to the above theorems, it is also interesting to investigate groups which do not satisfy the girth alternative. As mentioned before, this essentially boils down to finding groups satisfying a nontrivial law, but which are not virtually solvable. Cornulier and Mann [14] produce such examples of groups. More precisely, they construct finitely generated residually p-groups which satisfy a law, but are not virtually solvable. In fact, their examples show that both the girth alternative and the Tits alternative fail for the class of residually finite groups. In this paper, we provide examples of groups acting cocompactly on finite-dimensional CAT(0) cube complexes that do not satisfy the girth alternative as stated in Corollary B (see Section 4).

Sketch of proof for Theorem C Our proof relies on three main ingredients. The first is Nakamura's theorem [25] (see Theorem 6), which provides a sufficient condition for a group to have infinite girth. Specifically, the condition requires the existence of two elements that generate a free subgroup. To ensure that such elements always exist under the hypotheses of Theorem C, we use the second ingredient: a

result of Caprace and Sageev [13] (see Theorem 7), which reduces the task of finding these elements to finding mutually disjoint halfspaces in the cube complex X.

However, infinite girth does not follow merely from the existence of a nonabelian free subgroup. It depends on more subtle properties of the generators of that free subgroup, as outlined in Nakamura's theorem. To address these additional requirements, we must carefully select the "correct" generators. This brings us to our third ingredient, Lemma 8, which guarantees the existence of an abundance of mutually disjoint halfspaces in X under certain conditions. By combining the second and third ingredients, we obtain infinitely many candidates for generators of different free subgroups.

The main objective in proving Theorem C is to identify the correct generators from this pool of candidates — those that satisfy the necessary properties for applying Nakamura's theorem. This allows us to conclude that the group has infinite girth.

Overview The paper is organized as follows. In Section 2, we discuss some basic concepts pertaining to the theory of CAT(0) cube complexes. Readers who are familiar with CAT(0) cube complexes may skip this section. We prove Theorems C and A in Section 3 where they appear as Theorems 12 and 13, respectively. In Section 4, we give an example which shows Corollary B does not hold if both the assumptions properness and having a bound on the order of finite subgroups are omitted.

2 Preliminaries

2.1 Cube complexes

Let $I = [0, 1]$. We define an *n-cube*, or simply *cube*, as the product I^n (with $I^0 = \{0\}$). A *face* of I^n is the restriction of some number (possibly zero) of its coordinates to 0 or 1, and a *midcube* of I^n is the restriction of precisely one of its coordinates to $\frac{1}{2}$. Cleary, midcubes of I^n can be realized as $(n-1)$-cubes and we will often treat them as such. We will refer to 0-cubes, 1-cubes, and 2-cubes as vertices, edges, and squares, respectively.

A *cube complex* is a cell complex where the cells are n-cubes, for various n, and the attaching maps are Euclidean isometries in faces. We say the dimension of a cube complex X is n, for some $n \in \mathbb{N}$, if X contains n-cubes but not $(n+1)$-cubes, and that X is *finite-dimensional* if there exists such an n. X is *locally finite* if every vertex of X meets only finitely many cubes. In the setting of cube complexes, being locally finite is equivalent to being locally compact.

Two cube complexes X and Y are *isomorphic* if there is a bijection ϕ between their vertex sets such that, if c is an n-cube in X then $\phi(c)$ is an n-cube in Y. In particular, an isomorphism of cube complexes maps adjacent vertices to adjacent vertices.

A *subcomplex* Y of a cube complex X is any collection of cubes in X inheriting the same attaching maps. We will denote subcomplexes by $Y \leq X$. We call X *irreducible* if there do not exist cube complexes X_1, X_2 with $X = X_1 \times X_2$. Note that, if $X = X_1 \times X_2$, then X contains subcomplexes isomorphic to X_1 and X_2.

A cube complex X is a CAT(0) space if extending the Euclidean metric on individual cubes to the path length metric on X yields a CAT(0) space, in the traditional sense. That is, triangles are at least as thin as their comparison triangles in Euclidean space. Due to Gromov [20], there is a combinatorial requirement that, if satisfied by X, will guarantee that X is locally CAT(0). A finite-dimensional cube complex X is *locally CAT*(0) if the link of every vertex in X is a simplicial flag complex. If, in addition, X is simply connected then being locally CAT(0) implies that X is CAT(0). The equivalence between being locally CAT(0) and (locally) satisfying the classical definition of CAT(0) was extended to infinite-dimensional CAT(0) cube complexes in [24].

For any finite-dimensional CAT(0) cube complex X, Proposition 2.6 from [13] guarantees that the decomposition $X = X_1 \times \cdots \times X_n$ into irreducible factors is unique up to a permutation of its factors.

Proposition 3 *A finite-dimensional* CAT(0) *cube complex X admits a canonical decomposition*

$$X = X_1 \times X_2 \times \cdots \times X_p$$

into a product of irreducible cube complexes X_i. Every automorphism of X preserves that decomposition, up to a permutation of possibly isomorphic factors. In particular, the image of canonical embedding

$$\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_2) \times \cdots \times \mathrm{Aut}(X_p) \hookrightarrow \mathrm{Aut}(X)$$

has finite index in $\mathrm{Aut}(X)$.

When it is necessary to treat X as a metric space, we will use d to denote the CAT(0) metric. With this metric, X is a uniquely geodesic space.

2.2 Hyperplanes and halfspaces

For a cube complex X, we say two edges $e_1, e_2 \in X$ are *related* if $e_1 \cap e_2 = \varnothing$ and there is a square in X containing e_1 and e_2.

Informally, e_1 and e_2 are related if they are opposite edges of a square. A *hyperplane* in X, denoted by $\hat{\mathfrak{h}}$, is the transitive closure of this relation on a single edge. The collection of all hyperplanes in a cube complex X is denoted by $\widehat{\mathcal{H}}_X$. For the *geometric realization of a hyperplane*, first observe that any nonempty collection of related edges in an n-cube c uniquely determine a midcube of c. Midcubes in X therefore inherit a relation from the edges. One then sees that two midcubes M and N are related whenever their intersection is a common face. The geometric realization of $\hat{\mathfrak{h}}$ is the transitive closure of this relation on a single midcube. In this instance, midcubes M and N inherit attaching maps from their corresponding cubes.

For the remainder of this paper, we use the same symbol $\hat{\mathfrak{h}}$ to denote both a hyperplane and its geometric realization.

Properties of hyperplanes Let X be a CAT(0) cube complex and $\hat{\mathfrak{h}} \in \widehat{\mathcal{H}}_X$.

(1) $\hat{\mathfrak{h}}$ is a CAT(0) cube complex in its own right.

(2) $\hat{\mathfrak{h}}$ separates X into two components.

Recent proofs of these properties can be found in [16]. We denote the closures of the two components of $X \setminus \hat{\mathfrak{h}}$ by $\mathfrak{h}$ and $\mathfrak{h}^*$, which we also refer to as *halfspaces*. The collection of halfspaces of X is denoted by $\mathcal{H}_X$. Choosing which component to call $\mathfrak{h}$ for all $\hat{\mathfrak{h}} \in \widehat{\mathcal{H}}_X$ defines a *labeling* of $\mathcal{H}_X$. Two hyperplanes $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are *transverse*, denoted $\hat{\mathfrak{h}}_1 \pitchfork \hat{\mathfrak{h}}_2$, if there is a square in X containing edges e_1 and e_2 and a labeling of $\mathcal{H}_X$ with $e_1 \in \mathfrak{h}_1$, $e_2 \in \mathfrak{h}_2$, and $e_1 \cap e_2 \neq \varnothing$. Note that $\hat{\mathfrak{h}}_1 \pitchfork \hat{\mathfrak{h}}_2$ if and only if their geometric realizations intersect. If $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are not transverse, they are *nested*. We call a halfspace $\mathfrak{h}$ *deep* if the distance from $\hat{\mathfrak{h}}$ to points in $\mathfrak{h}$ is unbounded. Otherwise, $\mathfrak{h}$ is *shallow*. Halfspaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are *nested* if either $\mathfrak{h}_1 \subset \mathfrak{h}_2$ or $\mathfrak{h}_2 \subset \mathfrak{h}_1$. Note that if hyperplanes $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are nested then they each have a halfspace which together form a nested pair. We say $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are *separated* if there is a third hyperplane containing $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ in complimentary halfspaces, and they are *strongly separated* if they are separated and there does not exist a hyperplane transverse to both $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$. If $\hat{\mathfrak{h}}_1$ and $\hat{\mathfrak{h}}_2$ are strongly separated, we will refer to either of the halfspaces of $\hat{\mathfrak{h}}_1$ as strongly separated from either of the halfspaces of $\hat{\mathfrak{h}}_2$.

2.3 Boundaries and regular points

The collection of halfspaces of a CAT(0) cube complex X forms a poset with partial order given by set-inclusion, and comes naturally equipped with a complimentary map given by the involution $\mathfrak{h} \to \mathfrak{h}^*$. Let $\mathcal{H}_X^\circ$ denote the collection of ultrafilters of $\mathcal{H}_X$, that is, $\alpha \in \mathcal{H}_X^\circ$ satisfies the following two conditions:

(1) **Choice** For every pair $\mathfrak{h}, \mathfrak{h}^* \in \mathcal{H}_X$, exactly one of them is in α.

(2) **Consistency** If $\mathfrak{h}_1 \in \alpha$ and $\mathfrak{h}_2 \in \mathcal{H}_X$ with $\mathfrak{h}_1 \subset \mathfrak{h}_2$, then $\mathfrak{h}_2 \in \alpha$.

Given an ultrafilter $\alpha \in \mathcal{H}_X^\circ$, any subset $\{\mathfrak{h}_i\} \subseteq \alpha$ is guaranteed to satisfy the so called *partial choice condition*: for every pair $\mathfrak{h}, \mathfrak{h}^*$, at most one of them is in α.

A nested sequence of halfspaces $\{\mathfrak{h}_i\}$ is called *descending* (or a *descending chain*) if $\mathfrak{h}_{i+1} \subseteq \mathfrak{h}_i$ for all i. A descending sequence of halfspaces is *terminating* if $\mathfrak{h}_j = \mathfrak{h}_N$, for some N and all $j > N$. Otherwise, $\{\mathfrak{h}_i\}$ is *nonterminating*. Given $\sigma \in \mathcal{H}_X^\circ$, we say σ satisfies the *descending chain condition* (*DCC*) if every descending sequence in σ is terminating. Note that descending sequences necessarily satisfy the partial choice condition.

Established by Roller in [27], there is a bijective correspondence between the vertex set of X and ultrafilters in $\mathcal{H}_X^\circ$ satisfying the DCC given by $\theta : X^0 \to \mathcal{H}^\circ$, where $\theta(v) = \{\mathfrak{h} \in \mathcal{H}_X \mid v \in \mathfrak{h}\}$. Given a vertex $v \in X$, choosing the halfspace containing v to call $\mathfrak{h}$ for all hyperplanes $\hat{\mathfrak{h}}$ gives a labeling of $\mathcal{H}_X$, which we refer to as being determined by v.

Now, let $\mathbf{2} := \{0, 1\}$. Then $\mathbf{2}^{\widehat{\mathcal{H}}}$ is compact in the Tychonoff topology and the choice condition induces an inclusion $\mathcal{H}_X^\circ \hookrightarrow \mathbf{2}^{\widehat{\mathcal{H}}}$ in which $\mathcal{H}_X^\circ$ embeds as a closed subset which is therefore compact. We call the image of $\mathcal{H}_X^\circ$ the *Roller compactification* and denote it by $\overline{X}$. To realize the image of X^0 in $\mathbf{2}^{\widehat{\mathcal{H}}}$ concretely, start by defining $\phi_{\hat{\mathfrak{h}}} : X^0 \to \mathbf{2}$ by

$$\phi_{\hat{\mathfrak{h}}}(v) = \begin{cases} 1 & \text{if } x \in \mathfrak{h}, \\ 0 & \text{if } x \in \mathfrak{h}^*. \end{cases}$$

Then, the map with coordinate functions $\{\phi_{\hat{\mathfrak{h}}}\}_{\hat{\mathfrak{h}}\in\widehat{\mathcal{H}}}$ gives an embedding $X^0 \hookrightarrow \overline{X}$ as a dense open subset. See [26; 27] for details. Note that the Roller compactification behaves well with respect to products. Indeed, if $X = X_1 \times \cdots \times X_n$ then $\overline{X} = \overline{X}_1 \times \cdots \times \overline{X}_n$.

The *Roller boundary* of X, denoted by $\partial_R X$, is the remainder space $\overline{X} \setminus X$. It is a fact that $\partial_R X$ agrees with the collection of ultrafilters in $\mathcal{H}_X^\circ$ that do not satisfy the DCC. Note that an element $\xi \in \partial_R X$ necessarily contains a nonterminating descending sequence. Of paramount interest to us is the case that ξ contains a nonterminating descending sequence of strongly separated halfspaces, $\{\mathfrak{h}_i\}$. We call such ξ a *regular point*. This notion was originally defined in [17] (and independently in [23]). Lemma 10 below and [18, Proposition 5.13] imply that, if ξ is a regular point containing a nonterminating descending sequence of strongly separated halfspaces, $\{\mathfrak{h}_i\}$, and if ξ' is a regular point with $\{\mathfrak{h}_i\} \subseteq \xi'$ then $\xi' = \xi$. Due to this fact, we will often refer to a regular point, ξ, simply as a nonterminating descending sequence of strongly separated halfspaces, $\{\mathfrak{h}_i\} \subseteq \xi$.

The *visual boundary* of a CAT(0) cube complex X is the collection of equivalence classes of (CAT(0)) geodesic rays, where two such rays are equivalent if they remain within a bounded Hausdorff distance of each other. We denote the visual boundary by $\partial_\infty X$.

2.4 Groups acting on cube complexes

An element $g \in \mathrm{Isom}(X)$ is an *automorphism* if it acts as a cubical isomorphism of X. We let $\mathrm{Aut}(X)$ denote the group of automorphisms of X. A subgroup of Γ in $\mathrm{Aut}(X)$ is said to act *cocompactly* if there exists a compact subcomplex $K \leq X$ whose translates cover X, i.e., $X = \Gamma \cdot K$. Under our assumption that X is locally compact, this is equivalent to requiring that X/Γ be compact. Γ acts *essentially* if given any halfspace $\mathfrak{h}$ in X, we have $d(\Gamma \cdot \hat{\mathfrak{h}}, \hat{\mathfrak{h}})$ is unbounded. Note that, for Γ to act essentially, the halfspaces of X must be deep.

For X a CAT(0) cube complex, upon passing to its barycentric subdivision, elements $g \in \mathrm{Aut}(X)$ come in two types [21]:

(1) **Elliptic** g fixes a vertex in X.

(2) **Hyperbolic** g is not elliptic and preserves a geodesic line.

A geodesic line preserved by g is called an *axis* for g. A *rank one isometry* is a hyperbolic automorphism g, none of whose axes bound a flat halfplane, and a *contracting isometry* is a rank one isometry with axis l such that the diameter of the orthogonal projection to l of any ball disjoint from l is bounded above. An element in $\mathrm{Aut}(X)$ is *regular* if it acts as a contracting isometry on each irreducible factor of X.

Contracting isometries can be detected by looking at their action on strongly separated hyperplanes. An element $g \in \mathrm{Aut}(X)$ is said to *double skewer* the pair of hyperplanes $\{\hat{\mathfrak{h}}', \hat{\mathfrak{h}}''\}$ if there is a nested pair of halfspaces $\mathfrak{h}' \subset \mathfrak{h}''$ of X such that $\mathfrak{h}'' \subsetneq g\mathfrak{h}'$. The following appears in [13, Section 6, Lemma 6.2].

Lemma 4 (double skewering lemma) *Let X be a finite-dimensional* CAT(0) *cube complex and $g \in \mathrm{Aut}(X)$ an element which double skewers a pair of strongly separated hyperplanes in X. Then g is a contracting isometry.*

Suppose $g \in \mathrm{Aut}(X)$ double skewers $(\hat{\mathfrak{h}}', \hat{\mathfrak{h}}'')$. Then for some choice of halfspace $\mathfrak{h}'$ of the hyperplane $\hat{\mathfrak{h}}'$, $\{g^i \mathfrak{h}'\}$ and $\{g^{-i} \mathfrak{h}'^*\}$ give us two nonterminating descending sequence of halfspaces. Moreover, if $\hat{\mathfrak{h}}'$ and $\hat{\mathfrak{h}}''$ are strongly separated hyperplanes, then both $\{g^i \mathfrak{h}'\}$ and $\{g^{-i} \mathfrak{h}'^*\}$ are nonterminating descending sequences of strongly separated halfspaces, and hence correspond to two regular points in $\partial_R(X)$. In this case g can be thought as acting along an axis whose end points are $\{g^i \mathfrak{h}'\}$ and $\{g^{-i} \mathfrak{h}'^*\}$. We refer to $\{g^i \mathfrak{h}'\}$ and $\{g^{-i} \mathfrak{h}'^*\}$ as *poles* of g.

3 Girth alternative for lattices

One crucial step towards proving the girth alternative is to find a sufficient condition for infinite girth. Since free groups have infinite girth, one might expect having free subgroups is enough to guarantee infinite girth. A classical method for constructing a nonabelian free subgroup is based on the ping-pong argument, a technique first introduced by Tits [30]. We now present a formulation of Tits' theorem as stated in [25, Proposition 8].

Proposition 5 (free subgroup criterion) *Let Γ be a group acting on a set X. Suppose there exist elements $\sigma, \tau \in \Gamma$, subsets $U_\sigma, U_\tau \subset X$, and a point $x \in X$, such that*

(1) $x \notin U_\sigma \cup U_\tau$,

(2) $\sigma^k(\{x\} \cup U_\tau) \subset U_\sigma$ *for all* $k \in \mathbb{Z} - \{0\}$, *and*

(3) $\tau^k(\{x\} \cup U_\sigma) \subset U_\tau$ *for all* $k \in \mathbb{Z} - \{0\}$.

Then $\langle \sigma, \tau \rangle$ is nonabelian free subgroup of Γ.

However, mere containment of a copy of $\mathbb{F}_2$ does not guarantee infinite girth. In fact, in [1, Section 4], Akhmedov demonstrated the existence of groups with finite girth that contain free nonabelian subgroups. Nonetheless, it has been observed that certain generalizations of Tits' ping-pong technique can be applied to obtain infinite girth for specific classes of groups for which the Tits alternative holds [2, Theorems 2.6 and 3.1]. Nakamura reformulated and expanded upon Akhmedov's work, providing the following criterion for infinite girth [25, Proposition 2].

Theorem 6 (criteria for infinite girth) *Let Γ be a group acting on a set X with a finite generating set $S = \{\gamma_1, \dots, \gamma_n\}$. Suppose there exist elements $\sigma, \tau \in \Gamma$, subsets $U_\sigma, U_\tau \subset X$, and a point $x \in X$ such that*

$$x \notin (U_\sigma \cup U_\tau) \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^{n} \gamma_i^{\varepsilon}(U_\sigma \cup U_\tau), \tag{1}$$

$$\sigma^k\left(\{x\} \cup U_\tau \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^{n} \gamma_i^{\varepsilon}(U_\tau)\right) \subset U_\sigma \quad \text{for all } k \in \mathbb{Z} - \{0\}, \tag{2}$$

$$\tau^k\left(\{x\} \cup U_\sigma \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{i=1}^{n} \gamma_i^{\varepsilon}(U_\sigma)\right) \subset U_\tau \quad \text{for all } k \in \mathbb{Z} - \{0\}. \tag{3}$$

Then Γ is a noncyclic group with $\mathrm{Girth}(\Gamma) = \infty$.

Clearly, the elements σ, τ and subsets U_σ, U_τ in Theorem 6 satisfy the conditions in Proposition 5, and therefore $\langle \sigma, \tau \rangle \leq \Gamma$ must be a nonabelian free subgroup. In general, properties (2) and (3) can be interpreted as analogous to the north-south dynamics exhibited by the action of a loxodromic element in a hyperbolic group on its boundary. In the realm of irreducible CAT(0) cube complexes, the role of such isometries are played by contracting isometries.

In general, a finite-dimensional CAT(0) cube complex is a product of finitely many irreducible factors by Proposition 3. Therefore, to show that a group acting on a finite-dimensional CAT(0) cube complex has infinite girth, we must produce group elements which act as contracting isometries in each irreducible factor. In other words, we must produce regular elements.

The following consequence of [13, Theorem C] appeared in [23, Theorem 10] which says that regular elements always exist under certain conditions.

Theorem 7 (regular elements in lattices) *Let $X = X_1 \times \cdots \times X_n$ be a product of irreducible, unbounded, locally compact* CAT(0) *cube complexes such that* $\mathrm{Aut}(X_i)$ *acts cocompactly and essentially on each irreducible factor X_i. Suppose that $\Gamma \leq \mathrm{Aut}(X)$ is a lattice (possibly nonuniform). For each i, let $\hat{\mathfrak{h}}_i$ and $\hat{\mathfrak{d}}_i$ be two disjoint hyperplanes in X_i. Then there exists a regular element $g \in \Gamma$ which simultaneously double skewers $\hat{\mathfrak{h}}_i$ and $\hat{\mathfrak{d}}_i$ in each factor X_i.*

The above theorem reduces the problem of finding a regular element in $\Gamma \leq \mathrm{Aut}(X)$ to finding a pair of strongly separated hyperplanes in each irreducible factor of X. For our purpose, we will need two such regular elements whose axes in each irreducible factor avoid certain points at infinity. In particular, we want their axes to not intersect at infinity in each irreducible factor. To make this precise, we need the notion of facing hyperplanes.

A set of n hyperplanes are said to be *facing* if they have an associated set of n halfspaces that are mutually disjoint. According to Theorem 7, if we can find a facing quadruple of hyperplanes in each irreducible factor X_i, each containing two pairs of strongly separated hyperplanes, then the group elements that double skewer the strongly separated pairs in each X_i will have the desired property. Our next lemma guarantees that, under certain conditions, such a collection of facing hyperplanes exists.

Warning on notation For the rest of the paper, the notation $\{a_i, b_i\}$ will mean the set consisting of two elements a_i and b_i, whereas $\{a_i\}$ will stand for a sequence $\{a_i\}_{i=1}^{\infty}$. We will sometimes write sets with braces and at other times without braces, and we reserve parentheses for ordered sets.

Lemma 8 (abundance of facing hyperplanes) *Let X be an irreducible finite-dimensional* CAT(0) *cube complex, where* $\mathrm{Aut}(X)$ *acts essentially and without fixed points at infinity. Let $\{\hat{\mathfrak{a}}, \hat{\mathfrak{b}}, \hat{\mathfrak{c}}\}$ be a facing triple in X with $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{b}}$ strongly separated. Then for any n, there exists a collection of $2n$ facing hyperplanes $\{\hat{\mathfrak{a}}_1 = \hat{\mathfrak{a}}, \hat{\mathfrak{b}}_1 = \hat{\mathfrak{b}}, \hat{\mathfrak{a}}_2, \hat{\mathfrak{b}}_2, \ldots, \hat{\mathfrak{a}}_n, \hat{\mathfrak{b}}_n\}$ such that each pair $\{\hat{\mathfrak{a}}_i, \hat{\mathfrak{b}}_i\}$ is strongly separated for all i.*

An automorphism $g \in \mathrm{Aut}(X)$ is said to *flip* a halfspace $\mathfrak{h}$, if $g \cdot \mathfrak{h} \subsetneq \mathfrak{h}^*$. The main idea of the proof of the above lemma is to hit one pair of hyperplanes from the triple by an automorphism that flips a halfspace corresponding to the third hyperplane, and then keep repeating this process on the resulting

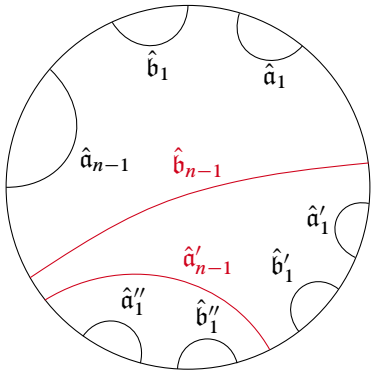

Figure 1: The above picture shows how to increase the number of facing hyperplanes. The input is four facing hyperplanes $\{\hat{\mathfrak{a}}_1, \hat{\mathfrak{b}}_1, \hat{\mathfrak{a}}_{n-1}, \hat{\mathfrak{b}}_{n-1}\}$ and the output is six facing hyperplanes $\{\hat{\mathfrak{a}}_1, \hat{\mathfrak{b}}_1, \hat{\mathfrak{a}}'_1, \hat{\mathfrak{b}}'_1, \hat{\mathfrak{a}}''_1, \hat{\mathfrak{b}}''_1\}$.

hyperplanes. The following theorem from [13, Theorem 4.1] says that any halfspace of X can be flipped under certain conditions.

Theorem 9 (flipping lemma) *Let X be finite-dimensional* CAT(0) *cube complex and $\Gamma \leq \mathrm{Aut}(X)$ be a group acting essentially without a fixed point at infinity. Then for any halfspace $\mathfrak{h}$, there is some $\gamma \in \Gamma$ such that $\gamma \cdot \mathfrak{h} \subsetneq \mathfrak{h}^*$.*

Proof of Lemma 8 It is convenient to prove the lemma in terms of halfspaces. Suppose $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ is the facing triple of halfspaces. Let $\mathfrak{a}_1 := \mathfrak{a}$ and $\mathfrak{b}_1 := \mathfrak{b}$. We start by applying an automorphism to the pair $(\mathfrak{a}_1, \mathfrak{b}_1)$ that flips the halfspace $\mathfrak{c}^*$ to get the pair $(\mathfrak{a}_2, \mathfrak{b}_2)$. Clearly the $\{\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{a}_2, \mathfrak{b}_2\}$ are facing halfspaces and the pair $\{\hat{\mathfrak{a}}_2, \hat{\mathfrak{b}}_2\}$ is strongly separated. By induction, suppose we have constructed the $2(n-1)$ facing halfspaces $\{\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{a}_2, \mathfrak{b}_2, \dots, \mathfrak{a}_{n-1}, \mathfrak{b}_{n-1}\}$ such that $\hat{\mathfrak{a}}_i$ and $\hat{\mathfrak{b}}_i$ are strongly separated for each i, where $n \geq 2$. To construct $2n$ facing hyperplanes, we apply an automorphism to the triple $(\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{a}_{n-1})$ that flips $\mathfrak{b}^*_{n-1}$ to get $(\mathfrak{a}'_1, \mathfrak{b}'_1, \mathfrak{a}'_{n-1})$ and then apply another automorphism to the pair $(\mathfrak{a}'_1, \mathfrak{b}'_1)$ that flips $\mathfrak{a}'^*_{n-1}$ to get another pair $(\mathfrak{a}''_1, \mathfrak{b}''_1)$ (see Figure 1). By construction, $\{\mathfrak{a}'_1, \mathfrak{b}'_1\}$ and $\{\mathfrak{a}''_1, \mathfrak{b}''_1\}$ are both pairs of strongly separated hyperplanes. Finally, we claim that the collection $\{\mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_{n-2}, \mathfrak{b}_{n-2}, \mathfrak{a}'_1, \mathfrak{b}'_1, \mathfrak{a}''_1, \mathfrak{b}''_1\}$ consists of facing halfspaces (Figure 1).

First, we observe that halfspaces in the collection $\{\mathfrak{a}'_1, \mathfrak{b}'_1, \mathfrak{a}''_2, \mathfrak{b}''_2\}$ are mutually disjoint. This is because $\mathfrak{a}'_1$ and $\mathfrak{b}'_1$ are contained in $\mathfrak{a}'^*_{n-1}$, and the pair $(\mathfrak{a}''_1, \mathfrak{b}''_1)$ is obtained by applying an automorphism to the pair $(\mathfrak{a}'_1, \mathfrak{b}'_1)$ that flips $\mathfrak{a}'^*_{n-1}$. It follows that both $\mathfrak{a}''_1$ and $\mathfrak{b}''_1$ are contained inside $\mathfrak{a}'_{n-1}$. The claim follows.

Next we show that both $\mathfrak{a}'_1$ and $\mathfrak{b}'_1$ are disjoint from the collection $\{\mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_{n-2}, \mathfrak{b}_{n-2}\}$. By construction, $(\mathfrak{a}'_1, \mathfrak{b}'_1)$ is obtained by applying an automorphism to the pair $(\mathfrak{a}_1, \mathfrak{b}_1)$ that flips $\mathfrak{b}^*_{n-1}$. Therefore, both $\mathfrak{a}'_1$ and $\mathfrak{b}'_1$ are contained inside $\mathfrak{b}_{n-1}$. Since $\mathfrak{b}_{n-1}$ is disjoint from any set from the collection $\{\mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_{n-2}, \mathfrak{b}_{n-2}\}$, so are both $\mathfrak{a}'_1$ and $\mathfrak{b}'_1$.

Similarly, we can show both $\mathfrak{a}''_1$ and $\mathfrak{b}''_1$ are disjoint from the collection $\{\mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_{n-2}, \mathfrak{b}_{n-2}\}$. First, note that $\mathfrak{a}'_{n-1} \subset \mathfrak{b}_{n-1}$ by construction. Also, $\mathfrak{a}''_1 \subset \mathfrak{a}'_{n-1}$ and $\mathfrak{b}''_1 \subset \mathfrak{a}'_{n-1}$. Consequently, both $\mathfrak{a}''_1$ and $\mathfrak{b}''_1$ are contained inside $\mathfrak{b}_{n-1}$ which is disjoint from any set from the collection $\{\mathfrak{a}_1, \mathfrak{b}_1, \dots, \mathfrak{a}_{n-2}, \mathfrak{b}_{n-2}\}$. □

As mentioned before, we need to find two regular elements whose axis in each irreducible factor do not intersect each other at infinity. More precisely, we want the poles of these two elements to be "disjoint" from each other in each factor. Recall that poles are defined in terms of descending sequences of halfspaces. We say, two descending sequences of halfspaces $\{P_n\}$ and $\{Q_n\}$ are *disjoint* if $P_m \cap Q_m = \varnothing$ for some m. Otherwise, we say that $\{P_n\}$ and $\{Q_n\}$ *intersect* each other. The next lemma shows that a descending sequence of strongly separated halfspaces cannot intersect two disjoint descending sequence of halfspaces simultaneously. The statement and its proof are similar to, but somewhat different from [18, Lemma 5.11]. Therefore, we include them for the convenience of the reader.

Lemma 10 (disjoint descending sequence of halfspaces) *Let $\{\mathfrak{a}_n\}$ and $\{\mathfrak{b}_n\}$ be two disjoint descending sequence of halfspaces in a finite-dimensional* CAT(0) *cube complex X. Let $\{\mathfrak{c}_n\}$ be another descending sequence of strongly separated halfspaces in X that intersects $\{\mathfrak{a}_n\}$. Then $\{\mathfrak{c}_n\}$ and $\{\mathfrak{b}_n\}$ are disjoint.*

Proof We need to show that $\mathfrak{b}_n \cap \mathfrak{c}_n = \varnothing$ for sufficiently large n. By hypothesis, there exist N such that $\mathfrak{a}_N \cap \mathfrak{b}_N = \varnothing$. Since both $\{\mathfrak{a}_n\}$ and $\{\mathfrak{b}_n\}$ are descending sequence of halfspaces, it follows that $\mathfrak{a}_i \cap \mathfrak{b}_i = \varnothing$ for all $i \geq N$. Since $\{\mathfrak{c}_n\}$ is descending and intersects $\{\mathfrak{a}_n\}$, there exists M such that $\mathfrak{a}_N \cap \mathfrak{c}_M \neq \varnothing$. So, either $\mathfrak{c}_M \subset \mathfrak{a}_N$, or $\mathfrak{c}_M^* \subset \mathfrak{a}_N$ or $\hat{\mathfrak{c}}_M \pitchfork \hat{\mathfrak{a}}_N$.

Next, we will show how to rule out the second and third cases by taking M large enough. To rule out the second case, we observe that $\mathfrak{c}_M^* \subset \mathfrak{c}_{M+k}^* \subset \mathfrak{a}_N$ for all $k \geq 0$. However, since X is finite-dimensional, there can be at most finitely many hyperplanes between any two halfspaces. So, for large enough M, the second case does not occur. To rule out the third case, suppose $\hat{\mathfrak{c}}_M \pitchfork \hat{\mathfrak{a}}_N$ for some M. Since $\mathfrak{c}_M$ and $\mathfrak{c}_{M+1}$ are strongly separated, $\mathfrak{a}_N$ cannot be transverse to $\mathfrak{c}_{M+1}$. So, either $\mathfrak{a}_N \cap \mathfrak{c}_{M+1} = \varnothing$ or $\mathfrak{c}_{M+1} \subset \mathfrak{a}_N$. By hypothesis $\mathfrak{a}_N \cap \mathfrak{c}_{M+1} \neq \varnothing$. Therefore, we have $\mathfrak{c}_{M+1} \subset \mathfrak{a}_N$.

So we are left with the first possibility, which is that $\mathfrak{c}_M \subset \mathfrak{a}_N$ for some M. Since $\mathfrak{a}_N \cap \mathfrak{b}_N = \varnothing$, it follows that $\mathfrak{c}_M \cap \mathfrak{b}_N = \varnothing$. Since the sequences of halfspaces are descending, we get $\{\mathfrak{c}_n\}$ is disjoint from $\{\mathfrak{b}_n\}$. □

We will denote by $\mathbb{R}$ the real line. We say that a CAT(0) cube complex X is $\mathbb{R}$-*like* if there is an $\mathrm{Aut}(X)$-invariant geodesic line in X. The final ingredient to our main theorem is the following lemma from [13, Lemma 7.1], which characterizes cube complexes with invariant Euclidean flats.

Lemma 11 *Let X be a finite-dimensional* CAT(0) *cube complex such that* $\mathrm{Aut}(X)$ *acts essentially. Then* $\mathrm{Aut}(X)$ *stabilizes some n-dimensional flat $\mathbb{R}^n \subset X$ if and only if X decomposes as a product $X = X_1 \times X_2 \times \cdots \times X_n$ of subcomplexes, each of which is essential and $\mathbb{R}$-like.*

We are now ready to prove our main theorem, which appears as Theorem C in the introduction.

Theorem 12 (girth alternative for lattices, version 1) *Let $X = X_1 \times \cdots \times X_n$ be a product of n irreducible, unbounded, locally compact* CAT(0) *cube complexes such that* $\mathrm{Aut}(X_i)$ *acts cocompactly and essentially on X_i for all i.*

Then for any (possibly nonuniform) lattice $\Gamma \leq \mathrm{Aut}(X)$, either Γ virtually fixes some point in $X \cup \partial_\infty X$ or $\mathrm{Girth}(\Gamma) = \infty$.

Proof Suppose $\Gamma \leq \mathrm{Aut}(X)$ does not fix a point in $X \cup \partial_\infty X$. We want to show that $\mathrm{Girth}(\Gamma) = \infty$.

Since Γ does not virtually fix points at infinity, $\mathrm{Aut}(X)$ does not have fixed points at infinity. Furthermore, since $\mathrm{Aut}(X)$ acts essentially on X, we can invoke [13, Proposition 5.1] to obtain a pair of strongly separated hyperplanes $\{\hat{\mathfrak{h}}_i, \hat{\mathfrak{d}}_i\}$ in each X_i.

We now apply Theorem 7 to obtain an element $\sigma \in \Gamma$ that double skewers $\{\hat{\mathfrak{h}}_i, \hat{\mathfrak{d}}_i\}$ in each X_i. Let $\mathfrak{h}_i$ be the halfspace corresponding to the hyperplane $\hat{\mathfrak{h}}_i$ such that $\sigma^{j+1}\mathfrak{h}_i \subset \sigma^j \mathfrak{h}_i$ for any $j \in \mathbb{N}$. Since $\{\hat{\mathfrak{h}}_i, \hat{\mathfrak{d}}_i\}$ is a pair of strongly separated hyperplanes, it follows that $\{\sigma^n \mathfrak{h}_i\}$ and $\{\sigma^{-n}\mathfrak{h}_i^*\}$ are two disjoint descending sequences of strongly separated halfspaces. In other words, they are the poles of σ in X_i (see the preliminaries).

Next, we choose an arbitrary generating set $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of Γ. We will consider the translate of poles of σ in each X_i under this generating set. More precisely, we consider the following list of descending sequences of halfspaces:

$$(\dagger) \qquad \{\sigma^n \mathfrak{h}_i\}, \{\sigma^{-n}\mathfrak{h}_i^*\}, \{\gamma_j \sigma^n \mathfrak{h}_i\}, \{\gamma_j \sigma^{-n}\mathfrak{h}_i^*\}, \{\gamma_j^{-1}\sigma^n \mathfrak{h}_i\}, \{\gamma_j^{-1}\sigma^{-n}\mathfrak{h}_i^*\},$$

where $1 \leq i \leq n$ and $1 \leq j \leq k$. Since isometries preserve strong separation, all the sequences in the above list consist of strongly separated halfspaces; in other words, they all represent regular points.

(Also note that, for a fixed i, all the halfspaces in the above list may not live in X_i, because γ_j can switch isomorphic factors. However, they all live in some X_i.)

Now, we are going to produce another regular element $\tau \in \Gamma$ whose poles in each X_i are disjoint from the list (†). In particular, its poles will be disjoint from that of σ. This is where we are going to use the fact that Γ does not fix a point in $X \cup \partial_\infty X$, ensuring that each X_i is large, in a certain sense. Specifically, we will see that each X_i has a facing triple.

First, recall that each X_i is finite-dimensional and locally compact, and $\mathrm{Aut}(X_i)$ acts essentially and cocompactly on X_i. Therefore, by [13, Theorem 7.2], either $\mathrm{Aut}(X_i)$ stabilizes some Euclidean flat or there is a facing triple of hyperplanes in X_i, for each i.

Suppose $\mathrm{Aut}(X_i)$ stabilizes some Euclidean flat $F_i \subset X_i$. Since X_i is irreducible, it follows from Lemma 11 that F_i is $\mathbb{R}$-like. Thus, $\mathrm{Aut}(X_i)$ has an index two subgroup which fixes a point at infinity, contradicting our assumption on Γ. Therefore, we have a facing triple of hyperplanes in each X_i. Moreover, we can choose this facing triple so that there is a pair of hyperplanes in the triple that are strongly separated in the following way.

First choose a pair $\{\hat{\mathfrak{h}}, \hat{\mathfrak{d}}\}$ of hyperplanes from the initial facing triple and suppose $\mathfrak{h} \cap \mathfrak{d} = \varnothing$. Since $\mathrm{Aut}(X)$ acts essentially without a fixed point at infinity, we can apply Theorem 9 to obtain an element $g \in \mathrm{Aut}(X)$ that double skewers the pair $\{\hat{\mathfrak{h}}, \hat{\mathfrak{d}}\}$. It follows that each pair of hyperplanes in the $\langle g \rangle$-orbit of $\hat{\mathfrak{h}}$ is strongly separated. Now replacing the pair $\{\hat{\mathfrak{h}}, \hat{\mathfrak{d}}\}$ by $\{g\hat{\mathfrak{h}}, g^{-1}\hat{\mathfrak{h}}\}$ in the initial facing triple of hyperplanes we obtain the desired facing triple.

By Lemma 8, there exist infinitely many pairs of strongly separated hyperplanes in each X_i that are facing each other. Given a pair of strongly separated hyperplanes taken from each X_i, we can choose an element from Γ that double skewers each pair in the corresponding X_i due to Theorem 7. Since all the

pairs are facing each other, the poles of these group elements give us infinitely many mutually disjoint descending sequences of strongly separated hyperplanes in X_i. Of these infinitely many descending sequences of halfspaces in X_i, all but finitely many are disjoint from the list (†) by Lemma 10 because the list (†) contains a finite number of descending sequences of strongly separated halfspaces. In particular, there exists a pair of disjoint halfspaces $\{\mathfrak{h}_i', \mathfrak{d}_i'\}$ in each X_i, such that for large enough N, both $\mathfrak{h}_i'$ and $\mathfrak{d}_i'$ are disjoint from the collection

$$\{\sigma^N\mathfrak{h}_i, \sigma^{-N}\mathfrak{h}_i, \gamma_j\sigma^N\mathfrak{h}_i^*, \gamma_j\sigma^{-N}\mathfrak{h}_i^*, \gamma_j^{-1}\sigma^N\mathfrak{h}_i, \gamma_j^{-1}\sigma^{-N}\mathfrak{h}_i^* \mid 1 \le i \le n, 1 \le j \le k\}.$$

Let $\tau \in \Gamma$ be an element that double skewers $\{\hat{\mathfrak{h}}_i', \hat{\mathfrak{d}}_i'\}$ in each X_i. Without loss of generality, let us assume that $\{\tau^n\mathfrak{h}_i'\}$ and $\{\tau^{-n}\mathfrak{h}_i'^*\}$ are the poles of τ in X_i.

We let $Y := \bigsqcup_{i=1}^n X_i$. Note that Γ acts on Y. We set

$$U_{\sigma'} := \bigcup_i (\sigma^N\mathfrak{h}_i \cup \sigma^{-N}\mathfrak{h}_i^*) \quad \text{and} \quad U_{\tau'} := \bigcup_i (\tau^N\mathfrak{h}_i' \cup \tau^{-N}\mathfrak{h}_i'^*).$$

Note that, the complement of $\sigma^N\mathfrak{h}_i \cup \sigma^{-N}\mathfrak{h}_i^*$ in X_i is sent to $\sigma^N\mathfrak{h}_i \cup \sigma^{-N}\mathfrak{h}_i^*$ under a high enough power of $\sigma^{\pm 1}$. Taking union over i, we obtain that the complement of U_σ in Y is sent to U_σ under a high enough power of $\sigma^{\pm 1}$. Similarly, the complement of U_τ in Y is sent to U_τ under a high enough power of $\tau^{\pm 1}$. We set $\sigma' = \sigma^M$ and $\tau' = \tau^M$, where we choose M sufficiently large that, for all $i \in \mathbb{Z} - \{0\}$,

$$(\sigma')^i(Y - U_{\sigma'}) \subset U_{\sigma'} \quad \text{and} \quad (\tau')^i(Y - U_{\tau'}) \subset U_{\tau'}. \tag{4}$$

Next, we observe that, by construction, $U_{\sigma'}$ and its translates under the generating set and their inverses do not intersect $U_{\tau'}$. In other words,

$$U_{\sigma'} \cup \bigcup_{\epsilon=\pm 1} \bigcup_{j=1}^k \gamma_j^\epsilon(U_{\sigma'}) \subset Y - U_{\tau'}. \tag{5}$$

Since $\gamma_j(U_{\sigma'}) \cap U_{\tau'} = \varnothing$ if and only if $\gamma_j^{-1}(U_{\tau'}) \cap U_{\sigma'} = \varnothing$ for any γ_j, it follows from (5) that $U_{\tau'}$ and its translates under the generating set and their inverses do not intersect $U_{\sigma'}$. In other words,

$$U_{\tau'} \cup \bigcup_{\epsilon=\pm 1} \bigcup_{j=1}^k \gamma_j^\epsilon(U_{\tau'}) \subset Y - U_{\sigma'}. \tag{6}$$

Furthermore, we can assume (by taking N large enough) that the set

$$Y - U_{\sigma'} \cup U_{\tau'} \cup \bigcup_{\varepsilon=\pm 1} \bigcup_{j=1}^k \gamma_j^\varepsilon(U_{\sigma'} \cup U_{\tau'})$$

is nonempty. For convenience, let us call this set B.

We claim that σ', τ', $U_{\sigma'}$, $U_{\tau'}$, and any point $x \in B$ satisfies the three properties of Theorem 6.

The first property is immediate by our choice of x. To check the second property, first note that $x \in Y - U_{\sigma'}$ and (6) together imply

$$\{x\} \cup U_{\tau'} \cup \bigcup_{\epsilon=\pm 1} \bigcup_{j=1}^{k} \gamma_j^{\epsilon}(U_{\tau'}) \subset Y - U_{\sigma'}.$$

Therefore, (4) gives us

$$(\sigma')^i(\{x\} \cup U_{\tau'} \cup \bigcup_{\epsilon=\pm 1} \bigcup_{j=1}^{k} \gamma_j^{\epsilon}(U_{\tau'})) \subset U_{\sigma'} \quad \text{for all} \quad i \in \mathbb{Z} - \{0\},$$

which is the desired second property.

Similarly, $x \in Y - U_{\tau'}$, together with (5) and (4) gives us

$$(\tau')^i(\{x\} \cup U_{\sigma'} \cup \bigcup_{\epsilon=\pm 1} \bigcup_{j=1}^{k} \gamma_j^{\epsilon}(U_{\sigma'})) \subset U_{\tau'},$$

which is the desired third property.

Therefore, $\mathrm{Girth}(\Gamma) = \infty$ by Theorem 6. □

We are now ready to prove Theorem A. The statement and its proof are inspired by [13, Corollary G]. We start by recalling the statement.

Theorem 13 (girth alternative for lattices, version 2) *Let $X = X_1 \times \cdots \times X_n$ be a product of n irreducible, unbounded, and locally compact* CAT(0) *cube complexes such that* $\mathrm{Aut}(X_i)$ *acts cocompactly and essentially on X_i, for all i.*

Then for any (possibly nonuniform) lattice $\Gamma \leq \mathrm{Aut}(X)$ acting properly on X, either Γ is {locally finite}-by-{virtually abelian} or $\mathrm{Girth}(\Gamma) = \infty$.

The proof of the above will rely on the proof of Theorem 12 and a result of [12]. In addition, we will need the following lemmas to equivariantly triangulate certain subsets in X.

Lemma 14 *Suppose G acts by automorphism on a locally finite cube complex X. Let $Y \subset X$ be a G-invariant subset and G does not fix Y point-wise. Then there exists $p \in Y$ such that*

$$k := \inf\{d(p, g \cdot p) \mid g \in G, p \neq g \cdot p\} > 0.$$

Proof By the assumption, we can choose $p \in Y$ such that $g \cdot p \neq p$ for some $g \in G$. We thus have two cases to consider.

Suppose p is a vertex in X. Since G acts by automorphisms, translates of p can only be other vertices. It follows that k is at least the length of an edge. Hence, $k \geq 1$.

If p is not a vertex, then p is in the interior of some cube C in X. The infimum of distances between p and all the translates of p that are inside a different cube is clearly positive. Since there are only finitely many automorphisms of C, there are only finitely many nontrivial translates of p that stay inside C. Therefore, the minimum distance from p to any of its translates inside C is also positive. □

Lemma 15 *Suppose G acts on $\mathbb{R}$ by isometries such that for some p,*

$$k := \inf\{d(p, g \cdot p) \mid g \in G, p \neq g \cdot p\} > 0.$$

Then there exists a finite index subgroup H of G and an H-equivariant triangulation of $\mathbb{R}$.

Proof Let H be the index two subgroup of G consisting of only the orientation preserving isometries. In particular, H acts by translation on $\mathbb{R}$ and hence is a subgroup of $\mathbb{R}$. Either H is trivial or $k > 0$. In either case

$$k' := \inf\{d(p, h \cdot p) \mid h \in H, p \neq h \cdot p\} > 0.$$

It follows that H is generated by the isometry that translates by k' amount. Therefore, placing the vertices at integer multiples of k', we obtain a H-equivariant triangulation of $\mathbb{R}$. □

Combining the above two lemmas, we get the following.

Lemma 16 *Suppose G acts by automorphisms on a locally finite $\mathbb{R}$-like cube complex X, and let $L \subset X$ be an $\mathrm{Aut}(X)$-invariant bi-infinite geodesic in X. Then there exists a finite index subgroup H of G and a H-equivariant triangulation of L.*

Proof If G fixes L pointwise, then any triangulation of L is G-equivariant. Otherwise, we can apply Lemma 14 with $Y := L$ to conclude that k as in Lemma 14 is positive. Therefore we can apply Lemma 15 to L which yields a finite index subgroup H of G and an H-equivariant triangulation of L. □

Proof of Theorem 13 It follows from [12, Theorem 1.7] that Γ is {locally finite}-by-{virtually abelian} if and only if Γ is amenable. Thus it is enough to show that if Γ is not amenable then $\mathrm{Girth}(\Gamma) = \infty$. Suppose Γ is nonamenable. We claim there is an irreducible factor of X that is not $\mathbb{R}$-like.

If each irreducible factor of X is $\mathbb{R}$-like, then we can take the $\mathrm{Aut}(X_i)$-invariant geodesic in each factor and take their product to get an invariant n-dimensional Euclidean flat. Let Γ' be the finite index subgroup of Γ that stabilizes each factor. In particular, Γ' stabilizes the bi-infinite geodesic lines in each factor. Applying Lemma 16 to the action of Γ' on each $\mathbb{R}$-like factor, we get a finite index subgroup of Γ' which acts by automorphisms on some triangulation of the bi-infinite geodesic line in that $\mathbb{R}$-like factor. Taking the intersection of all these finite index subgroups for each factor, we get another finite index subgroup $\Gamma'' \leq \Gamma'$ that acts by automorphisms on each triangulated geodesic line. Γ'' therefore acts by automorphisms on a Euclidean flat Y, where Y is the product of triangulated geodesic lines. By [19, Lemma 2.8], the set of bounded components in the Roller boundary of Y is finite and nonempty. Each such bounded component has a cubical structure that is respected by the induced action of Γ''. Furthermore, the action of Γ'' stabilizes the union of these bounded components. In particular, Γ'' stabilizes some finite set of vertices in the Roller boundary. Consequently, a finite index subgroup of Γ'' fixes a point in the Roller boundary of Y. By [11, Theorem A.5], Γ'' is amenable. Since $\Gamma'' \leq \Gamma$ is a finite index subgroup, Γ is amenable. This is a contradiction.

Hence, at least one irreducible factor of X is not $\mathbb{R}$-like, and thus contains a facing triple by [13, Theorem 7.2]. Now, we can discard all the $\mathbb{R}$-like factors from X to get a space Y such that each

irreducible factor of Y has a facing triple. We can then run the same argument as in Theorem 12 to conclude that the girth of Γ is infinite. □

Remark 17 In our girth alternative theorems, we require the group to be a lattice in $\mathrm{Aut}(X)$ whereas the analogous Tits alternative theorems of Caprace and Sageev [13, Theorem F] (also see [28, Theorem 1.1]) do not require the group to be a lattice. The main reason for the lattice assumption in our case is that we need regular elements that double skewer a given pair of hyperplanes in each irreducible factor to obtain infinite girth. Note that Theorem 7 gives us such elements provided the group is a lattice in $\mathrm{Aut}(X)$, whereas, in the proof of the Tits alternative, one needs to find a free subgroup. In this case, we only need two group elements that double skewer distinct pairs of hyperplanes in some irreducible factor (see [13, Theorem F] for details). Interestingly, authors in [18, Theorem 1.5] proved that regular elements exist whenever the group acts essentially and without fixed points in $X \cup \partial_\infty X$. In particular, the group does not need to be a lattice in order to contain regular elements. This suggests that the lattice assumption may be dropped from Theorem 12. However, we do not know how to use the probabilistic methods of [18] to produce an abundance of regular elements.

Remark 18 Another approach to eliminating the lattice assumption is as follows. By Proposition 3, Γ has a finite index subgroup Γ' that stabilizes each irreducible factor. Using this fact, we can apply the same argument as in Theorem 12, but now restricting our focus to the Γ'-action on X_1. This allows us to conclude that either Γ virtually fixes a point, or that $\mathrm{Girth}(\Gamma') = \infty$. Notably, this reasoning does not require Γ to be a lattice, because we can disregard the other factors and only need σ and τ to act as contracting isometries on X_1. The existence of such elements is guaranteed even without the lattice assumption, as shown in [13, Theorem F].

Thus, another way to remove the lattice assumption in Theorem 12 would be to demonstrate that $\mathrm{Girth}(\Gamma') = \infty$ implies $\mathrm{Girth}(\Gamma) = \infty$. While this seems plausible, it remains an open question whether the existence of a finite index subgroup with infinite girth necessarily ensures that the group itself has infinite girth.

4 Curious examples

As promised in the introduction, we construct spaces which are $\mathbb{R}$-like with particular group actions showing that Corollary B does not hold if both the assumptions properness and having a bound on the order of finite subgroups are omitted.

For a cube complex X, we call a pair of vertices $v, v^* \in X$ *diametrically opposed* if there exists a labeling of the halfspaces of X such that $\theta(v) = \{\mathfrak{h} \mid \hat{\mathfrak{h}} \in \widehat{\mathscr{H}}\}$ and $\theta(v^*) = \{\mathfrak{h}^* \mid \hat{\mathfrak{h}} \in \widehat{\mathscr{H}}\}$. Note that a CAT(0) cube complex X contains diametrically opposed vertices if and only if there is a geodesic path between vertices that crosses every hyperplane in X.

If X and Y are CAT(0) cube complexes each containing a pair of diametrically opposed vertices, then their product $X \times Y$ will contain a pair of diametrically opposed vertices. Indeed, if $\{\mathfrak{h}_i\}$ and $\{\mathfrak{h}_i^*\}$

define vertices $x, x^* \in X$, and $\{\mathfrak{k}_i\}$ and $\{\mathfrak{k}_i^*\}$ define vertices $y, y^* \in Y$, then (x, y) and (x^*, y^*) give diametrically opposed vertices in $X \times Y$.

Definition 19 (cube complex $\mathscr{L}[X]$) Begin with a CAT(0) cube complex X which contains diametrically opposed vertices v and v^*. For all $i \in \mathbb{Z}$, let X_i be an isomorphic copy of X, where v_i and v_i^* denote the images of v and v^*, respectively. The cube complex $\mathscr{L}[X]$ is then formed by identifying v_i^* with v_{i+1} for all $i \in \mathbb{Z}$. We think of $\mathscr{L}[X]$ informally as a "line of X's".

The space $\mathscr{L}[X]$ is clearly a cube complex as the map attaching distinct copies of X is an isometry of v. That $\mathscr{L}[X]$ is locally CAT(0) is also clear, since the link of a vertex labeled v (or v^*) in $\mathscr{L}[X]$ is isomorphic to the disjoint union of two copies of the link of v in X. Lastly, $\mathscr{L}[X]$ is simply connected by construction.

There is a natural action of $\mathbb{Z}$ on $\mathscr{L}[X]$ whereby each copy X is shifted to an adjacent copy. With $\mathrm{Aut}_{v,v^*}(X) \leq \mathrm{Aut}(X)$ denoting the automorpisms fixing both v and v^*, $\Sigma_X = \oplus \mathrm{Aut}_{v,v^*}(X)$ acts on $\mathscr{L}[X]$ by allowing only finitely many coordinate groups to act nontrivially at a time. Clearly $\mathbb{Z} \cap \Sigma_X$ is the identity element. We are interested in the semi-direct product $\mathbb{Z} \ltimes \Sigma_X \cong \mathrm{Aut}_{v,v^*}(X) \wr \mathbb{Z}$, where multiplication is defined as $(\sigma_1, \tau^m)(\sigma_2, \tau^n) = (\sigma_1 \tau^m \sigma_2 \tau^{-m}, \tau^{m+n})$ and $(\sigma, \tau^n)^{-1} = (\tau^{-n}\sigma^{-1}\tau^n, \tau^{-n})$. Note that $\mathbb{Z} \ltimes \Sigma_X$ is generated by the group $\mathrm{Aut}_{v,v^*}(X)$ and a single shift.

We define a group action of $\mathbb{Z} \ltimes \Sigma_X$ on $\mathscr{L}[X]$ by

$$(\sigma, \tau^n) \cdot (x) = \sigma \tau^n \cdot (x).$$

This action is faithful since (σ, τ^n) acts trivially if and only if both σ and τ are the identity element in Σ_X and $\mathbb{Z}$, respectively. Since $\mathbb{Z} \ltimes \Sigma_X$ acts faithfully on $\mathscr{L}[X]$, we can think of $\mathbb{Z} \ltimes \Sigma_X$ as a subgroup of $\mathrm{Aut}(\mathscr{L}[X])$.

We highlight two additional properties of $\mathbb{Z} \ltimes \Sigma_X$. It acts cocompactly on $\mathscr{L}[X]$ as any copy of X gives a compact set whose translates cover $\mathscr{L}[X]$. And, since any copy of X can be mapped to any other copy of X, the action of $\mathbb{Z} \ltimes \Sigma_X$ on $\mathscr{L}[X]$ is essential. Next, we will show that $\mathbb{Z} \ltimes \Sigma_X$ satisfies a law and thus, for nontrivial Σ_X, has finite girth.

Recall that a group Γ *satisfies a law* if there is a word $w(x_1, x_2, \ldots, x_n)$ on n letters such that $w(\gamma_1, \gamma_2, \ldots, \gamma_n) = 1$ in Γ for any $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$.

Lemma 20 *For a compact* CAT(0) *cube complex X which has diametrically opposed vertices, the group $\mathbb{Z} \ltimes \Sigma_X$ satisfies a law.*

Proof Suppose $|\mathrm{Aut}(X)| = n$, and let $x = (\sigma_1, \tau^k)$ and $y = (\sigma_2, \tau^l)$ be elements in $\mathbb{Z} \ltimes \Sigma_X$. Note that the commutator $[x, y]$ acting on $\mathscr{L}[X]$ maps each copy of X to itself. Clearly then, $[x, y]^n$ acts trivially. Since the action is faithful, $[x, y]^n$ must then be the identity. □

Although a sketch of the proof for the following result can be found elsewhere in the literature [15, Proposition 1], we provide a proof for completeness.

Lemma 21 *If a group H is not solvable, then $\bigoplus_{\mathbb{Z}} H$ is not virtually solvable.*

Proof Let $G \leq \bigoplus_{\mathbb{Z}} H$ be a finite index subgroup. We claim that, for some i, the projection map $\pi_i : G \to H$ to the i-th factor is surjective. If not, then for each i there is a nontrivial element a_i, in the i-th factor, with $a_i \notin \pi_i(G)$. Consider the elements $b_i = (\ldots, e, a_i, e, \ldots)$ in the direct sum, where a_i appears in the i-th position. It follows that $\pi_i(b_i b_j^{-1}) = a_i$ for $i \neq j$, implying that $b_i b_j^{-1} \notin G$ for all such i, j. Then, $b_i G = b_j G$ if and only if $b_i = b_j$. Since b_i is nontrivial for all $i \in \mathbb{Z}$, G has countably many distinct cosets and hence cannot be a finite index subgroup. Thus the map $\pi_i : G \to H$ is a surjection for some i, and we conclude that a finite index subgroup G of $\bigoplus_{\mathbb{Z}} H$ necessarily surjects onto H. Since H is non solvable, G is non solvable. □

Now, consider the subgroup $\Sigma_X \leq \mathbb{Z} \ltimes \Sigma_X$. By Lemma 21, if $\mathrm{Aut}_{v,v^*}(X)$ is not solvable then Σ_X is not virtually solvable, and hence neither is $\mathbb{Z} \ltimes \Sigma_X$. This fact combines with Lemma 20 to give the following.

Example 22 If $\mathrm{Aut}_{v,v^*}(X)$ is nonsolvable then $\mathbb{Z} \ltimes \Sigma_X$ provides an example of a group acting essentially and cocompactly on a finite-dimensional CAT(0) cube complex that is not virtually solvable yet satisfies a law. Consequently, this example shows that the girth alternative, as stated in Corollary B, fails for the class of groups acting cocompactly on finite-dimensional CAT(0) cube complexes.

As a special instance, for $X = I^n$, $\mathrm{Aut}_{v,v^*}(X) \cong S_n$. Thus, for $n \geq 5$, $\mathbb{Z} \ltimes \Sigma_X \cong S_n \wr \mathbb{Z}$ provides a more concrete example.

5 Questions and remarks

Question 23 *Can we drop the lattice assumption in Theorem 12 and instead assume that Γ acts essentially on X?*

As suggested in Remark 17, an affirmative answer to the above question depends on an affirmative answer to the following question: Can we drop the "lattice" assumption in Theorem 7 and instead assume only that Γ acts essentially on X?

Question 24 *Remark 18 motivates us to ask the following question. Suppose $X = X_1 \times X_2 \times \cdots \times X_n$ where X_i are irreducible and $\Gamma \leq \mathrm{Aut}(X)$. Let Γ' be the finite index subgroup of Γ that stabilizes each irreducible factor X_i. If $\mathrm{Girth}(\Gamma')$ is infinite, must $\mathrm{Girth}(\Gamma)$ be infinite?*

Question 25 *Can we generalize any of the statements of Theorems A through C to groups acting on nonlocally compact cube complexes?*

Question 25 is motivated by the analogous Tits alternatives of [13], where local compactness was not required.

Question 26 *Are the alternatives in Theorem 12 mutually exclusive? In other words, under the hypothesis of Theorem 12, if Γ virtually fixes some point in $X \cup \partial_\infty X$, does that imply $\mathrm{Girth}(\Gamma) < \infty$?*

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