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Commutator-stable representations of hyperbolic groups

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Let Γ be a hyperbolic group and G be the isometry group of a Gromov-hyperbolic and geodesic metric space. We study the action of the outer automorphism group $\mathrm{Out}(\Gamma)$ on the set $\mathcal{X}(\Gamma, G)$ of conjugacy classes of representations of Γ into G. For some $\mathrm{Aut}(\Gamma)$-invariant subset $A \subset \Gamma$, we study the properties of A-stable representations which are generalizations of primitive stable representations defined by Minsky. Their conjugacy classes form $\mathrm{Out}(\Gamma)$-invariant subsets of $\mathcal{X}(\Gamma, G)$ which contains the set of conjugacy classes of quasi-isometrically embedded representations. We establish a sufficient condition for the induced action to be properly discontinuous and use it when A is the set of commutators of Γ. Working with the derived subgroup $[\Gamma, \Gamma]$, we also prove that $[\Gamma, \Gamma]$-stability is equivalent to the $[\Gamma, \Gamma]$-well-displacing property. Focusing on representations in $\mathrm{PSL}_2(\mathbb{C})$, we find new characterizations of convex cocompact subgroups of $\mathrm{PSL}_2(\mathbb{C})$.

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1 Introduction

Let Γ be a hyperbolic group and $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ be the group of orientation-preserving isometries of the 3-dimensional hyperbolic space. The group $\mathrm{Out}(\Gamma)$ of outer automorphisms of Γ naturally acts by precomposition on the set $\mathcal{X}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ of conjugacy classes of representations of Γ into G. We are interested in the dynamics of this action on invariant subsets. For example, it is well-known (see, e.g., [6]) that the set of conjugacy classes of convex cocompact representations is an open subset, invariant by the action of $\mathrm{Out}(\Gamma)$, on which the action is properly discontinuous. If ρ is a representation, an associated orbit map is a ρ-equivariant map between a Cayley graph $C(\Gamma)$ for Γ and $\mathbb{H}^3$. One way to prove the previous statement is to use the fact that (conjugacy classes of) convex cocompact representations are

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exactly those whose orbit maps are quasi-isometric embeddings, i.e., send all geodesics of $C(\Gamma)$ to uniform quasigeodesics of $\mathbb{H}^3$.

However, the dynamics of the action of $\mathrm{Out}(\Gamma)$ on the complement set of conjugacy classes of convex cocompact representations is mostly unknown and depends on the choice of the hyperbolic group Γ:

- On the one hand, when $\Gamma \cong \pi_1(S)$ is the fundamental group of a closed, orientable surface S with negative Euler characteristic, Goldman (see [13]) conjectured that the action of $\mathrm{Out}(\pi_1(S))$ on the complement set of conjugacy classes of convex cocompact representations is ergodic. The answer is known to be (almost) true when looking at the dynamics on the conjugacy classes of representations of the fundamental group of a closed surface of genus 2 in $\mathrm{PSL}_2(\mathbb{R})$ (see [20]).

- On the other hand, when $\Gamma \cong F_k$ is a free group of rank $k \geqslant 2$, Minsky [22] defined primitive-stable representations and proved that their conjugacy classes form an open $\mathrm{Out}(F_k)$-invariant subset of $\mathcal{X}(F_k, \mathrm{PSL}_2(\mathbb{C}))$ which is strictly larger than the set of conjugacy classes of convex cocompact representations, and on which $\mathrm{Out}(F_k)$ acts properly discontinuously.

In the following we consider a broader framework. Until the end of the introduction, let Γ denote a nonelementary hyperbolic group, let $G = \mathrm{Isom}(X)$ denote the isometry group of a Gromov-hyperbolic and geodesic metric space X. In particular, all rank-one symmetric spaces will be under the consideration: the real hyperbolic n-space ($\mathrm{Isom}(X) = \mathrm{SO}_0(n,1)$), the complex hyperbolic n-space ($\mathrm{Isom}(X) = \mathrm{PU}(n,1)$), the quaternionic hyperbolic n-space ($\mathrm{Isom}(X) = \mathrm{Sp}(n,1)$) and the hyperbolic plane over Cayley numbers ($\mathrm{Isom}(X) = F_{4(-20)}$). There are also interesting cases where X is not proper, such as the infinite-dimensional hyperbolic space $\mathbb{H}^\infty$, $\mathbb{R}$-trees or curve complexes associated to finite type surfaces.

In this context, we study the set of (conjugacy classes of) quasi-isometrically embedded representations, i.e., those whose orbit maps are quasi-isometric embeddings. A *domain of discontinuity* denotes an open $\mathrm{Out}(\Gamma)$-invariant subset of $\mathcal{X}(\Gamma, G)$ on which the action is properly discontinuous. The set $\mathrm{QIE}(\Gamma, G)$ of conjugacy classes of quasi-isometrically embedded representations is a domain of discontinuity. A recurring question is whether this domain of discontinuity is maximal.

Relying on the work of Minsky [22] and Lee [17], we generalize the definition of a primitive-stable representation. Starting with an $\mathrm{Aut}(\Gamma)$-invariant subset $A \subset \Gamma$, one can define the set of *A-stable representations*. Roughly speaking, an *A-stable representation* from Γ to G is a representation whose orbit map sends all geodesics corresponding to infinite-order elements of A in $C(\Gamma)$ to uniform quasigeodesics of X (see Section 4 for a more precise definition). We denote by $S_A(\Gamma, G)$ the set of conjugacy classes of A-stable representations.

Proposition 1.1 *For any* $\mathrm{Aut}(\Gamma)$*-invariant subset* $A \subset \Gamma$*, the subset* $S_A(\Gamma, G)$ *is an open* $\mathrm{Out}(\Gamma)$*-invariant subset of* $\mathcal{X}(\Gamma, G)$ *which contains* $\mathrm{QIE}(\Gamma, G)$*.*

The two types of A-stable representations which are usually studied are:

- ***Primitive-stable representations* of a free group F_k** They are defined by considering the $\mathrm{Aut}(F_k)$-invariant subset of primitive elements (i.e., elements which can be members of a free basis of F_k).

• ***Simple-stable representations* of $\pi_1(S)$, the fundamental group of a closed surface S** Elements of $\pi_1(S)$ represent free homotopy classes of loops on the surface S. A nontrivial class is called simple if there is a representative with no self-intersection. The subset of $\pi_1(S)$ consisting of simple elements of $\pi_1(S)$ is invariant under the action of $\mathrm{Aut}(\pi_1(S))$ and allows us to define simple-stable representations.

The subset of A-stable representations have mostly been studied when G is a semisimple Lie group of rank 1, to show that the set of (conjugacy classes of) convex cocompact representations does not form a maximal domain of discontinuity for the action of $\mathrm{Out}(\Gamma)$ on $\mathcal{X}(\Gamma, G)$. In addition to the example of Minsky's primitive-stable representations, Lee proved that the set of (conjugacy classes of) simple-stable representations of the fundamental group of a closed nonorientable surface with negative Euler characteristic in $\mathrm{PSL}_2(\mathbb{C})$ is a domain of discontinuity, containing strictly the set of (conjugacy classes of) convex cocompact representations (see [18]).

Wang developed a notion similar to A-stability but more suited for representations of hyperbolic groups Γ in higher-rank semisimple Lie groups G (see [31]). In this context, the class of convex cocompact representations is too rigid and we consider instead the class of Anosov representations which forms a domain of discontinuity for the action of $\mathrm{Out}(\Gamma)$ on $\mathcal{X}(\Gamma, G)$. Tholozan and Wang (see [28]) specialized in representations of the fundamental group of a closed orientable surface S with negative Euler characteristic in $\mathrm{PSL}_{2d}(\mathbb{C})$ for $d \geqslant 2$. They proved a theorem similar to Lee's result, namely that the set of (conjugacy classes of) simple-Anosov representations form a domain of discontinuity strictly containing the set of (conjugacy classes of) Anosov representations.

In contrast, we still do not know if the set of simple-stable representations of $\pi_1(S)$ in $\mathrm{PSL}_2(\mathbb{C})$ contains strictly or not the set of convex cocompact representations. If not, it would yield a weaker characterization of convex cocompact representations instead of a bigger domain of discontinuity.

In this article, we start by studying A-stable representations in the most general way possible. If g is an isometry of a metric space X with distance function d, the *translation length* of g is $l(g) = \inf_{x \in X} d(x, g(x))$. We denote by $l(\gamma)$ the translation length of $\gamma \in \Gamma$ for its isometric action on $C(\Gamma)$.

Definition (testable subset) A subset $A \subset \Gamma$ is *testable* if, for all sequences $(\varphi_n)_{n \in \mathbb{N}}$ of elements of $\mathrm{Aut}(\Gamma)$ which project onto sequences of distinct elements of $\mathrm{Out}(\Gamma)$, there is $a \in A$ such that $\limsup_{n \to \infty} l(\varphi_n(a)) = \infty$.

This notion of testability appears in [6] and provides a sufficient condition on the subset $A \subset \Gamma$ for the action of $\mathrm{Out}(\Gamma)$ on the set $S_A(\Gamma, G)$ to be properly discontinuous.

Theorem 1.2 *Let $A \subset \Gamma$ be an $\mathrm{Aut}(\Gamma)$-invariant subset of Γ. If A is testable, then the subset $S_A(\Gamma, G)$ is a domain of discontinuity of $\mathcal{X}(\Gamma, G)$ which contains $\mathrm{QIE}(\Gamma, G)$.*

Using isometric actions on $\mathbb{R}$-trees, we prove that the subset $\mathcal{C}$ of commutators of a hyperbolic group is testable and deduce:

Theorem 1.3 *The set $S_{\mathcal{C}}(\Gamma, G)$ of conjugacy classes of commutator-stable representations is a domain of discontinuity of $\mathcal{X}(\Gamma, G)$ for the action of $\mathrm{Out}(\Gamma)$.*

We also investigate the relation between A-stable and A-well-displacing representations.

If $A \subset \Gamma$ is $\mathrm{Aut}(\Gamma)$-invariant, a representation ρ of Γ in an isometry group $\mathrm{Isom}(X)$ is said to be *A-well-displacing* if the translation lengths of element of A for their action on $C(\Gamma)$ is comparable with the translation length of their images, i.e., there are $J \geqslant 1$ and $B \geqslant 0$ such that

$$\frac{1}{J} l(a) - B \leqslant l(\rho(a)) \leqslant J l(a) + B \quad \text{for all } a \in A.$$

It can be shown that an A-stable representation is always A-well-displacing. The reverse implication is known to be true for $A = \Gamma$ (see [10]) and for the primitive-stable representations of a free group of rank 2 (see [16; 25; 26]). In this paper, we prove the reverse implication for the case when A is the derived subgroup $[\Gamma, \Gamma]$ of Γ.

Theorem 1.4 *A representation $\rho \in \mathcal{R}(\Gamma, G)$ is $[\Gamma, \Gamma]$-stable if and only if it is $[\Gamma, \Gamma]$-well-displacing.*

This result does not directly come from the results in [10] because a subgroup of a hyperbolic group need not be hyperbolic. Indeed, the commutator subgroup of a free group of rank 2 is not finitely generated, hence cannot be hyperbolic. More interestingly, the authors of [3; 14] proved the existence of finitely presented subgroups of hyperbolic groups which are not hyperbolic.

There are also differences between primitive-stable and commutator-stable representations. In particular, contrary to Minsky's main theorem in [22], we prove the following theorem.

Theorem 1.5 *Let Y be a metric space and $\mathrm{Isom}(Y)$ denote its isometry group. If a representation $\rho \in \mathcal{R}(\Gamma, \mathrm{Isom}(Y))$ is commutator-well-displacing, then ρ has discrete image and finite kernel.*

We emphasize that there is no assumption on the metric space Y in the previous theorem. Consequently, it provides a criterion for checking whether a representation is discrete and faithful even for representations of hyperbolic groups into higher-rank semisimple Lie groups.

Specializing to the case $G = \mathrm{PSL}_2(\mathbb{C})$, we find the following new characterizations of convex cocompact representations:

Theorem 1.6 *Let Γ be a hyperbolic group and $\rho \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ be a representation. Let $\mathcal{C}$ denote the subset of commutators of Γ and $\mathcal{D} = [\Gamma, \Gamma]$ denote the derived subgroup of Γ. The following are equivalent:*

- *The representation ρ is convex cocompact.*
- *The representation ρ is $\mathcal{C}$-stable.*
- *The representation ρ is $\mathcal{D}$-well-displacing.*

As a by-product of our work, we answer in the affirmative a question asked by Canary [6, appendix], namely that the set of generators and products of distinct generators of a hyperbolic group is always testable.

Theorem 1.7 *For all hyperbolic groups Γ and for all finite generating sets S the set*

$$\mathcal{B}_S = S \cup \{ss' \mid s, s' \in S,\ s \neq s'\}$$

is testable.

The plan of this paper is as follows. In Section 2, we recall some preliminary facts and theorems about hyperbolic spaces and their isometry group. In Section 3, we present the set $\mathcal{X}(\Gamma, G)$ of conjugacy classes of representations of a group Γ in an isometry group G and describe the natural action of $\mathrm{Out}(\Gamma)$ on $\mathcal{X}(\Gamma, G)$. Moreover, we define some classical invariant subsets of $\mathcal{X}(\Gamma, G)$. In Section 4, starting with an $\mathrm{Aut}(\Gamma)$-invariant subset of Γ we define A-stable representations. We prove Proposition 1.1 as well as Theorem 1.2. In Section 5, we recall some theory on $\mathbb{R}$-trees and prove Theorem 1.7. In Section 6, we use previous material to recover some results about quasi-isometrically embedded and primitive-stable representations. In Section 7, we consider commutator-stable and derived-stable representations. We prove Theorem 1.3 with the material from the previous sections. Then using convergence group theory, we prove Theorem 1.5. Finally, we mix results from [10; 25] to prove Theorem 1.4. In Section 8, we focus on representations in $\mathrm{PSL}_2(\mathbb{C})$. We first generalize a theorem of Sullivan and then prove Theorem 1.6.

2 Hyperbolic metric spaces

In this section, we give definitions and some properties of Gromov-hyperbolic metric spaces and hyperbolic groups. We mainly follow [5, Section III.H.3].

2.1 Gromov-hyperbolic metric spaces and their boundaries

Let (X, d) be a metric space and $x, y, o \in X$. The *Gromov product* of x and y based at o is denoted by $\langle x, y\rangle_o$. It is defined by

$$\langle x, y\rangle_o = \tfrac{1}{2}(d(x, o) + d(y, o) - d(x, y)).$$

We say that X is *Gromov-hyperbolic* if there is $\delta > 0$ such that for all o, x, y, z in X, we have

$$\langle x, z\rangle_o \geqslant \min\{\langle x, y\rangle_o, \langle y, z\rangle_o\} - \delta.$$

We call such a δ a *hyperbolicity constant* for X.

Assume that the space X is Gromov-hyperbolic and geodesic and let $o \in X$ is a basepoint. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X *converges at infinity* if $\langle x_n, x_m\rangle_o \to \infty$ as $n, m \to \infty$. Two such sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are said to be *equivalent* if $\langle x_n, y_m\rangle_o \to \infty$ as $n, m \to \infty$. These definitions are independent of the choice of the basepoint $o \in X$.

The set of equivalence classes of sequences converging at infinity is called the *Gromov boundary* of X and denoted by ∂X. The Gromov product extends to ∂X. Let $a, b \in \partial X$. We define

$$\langle a, b\rangle_o = \sup \liminf_{n,m\to\infty} \langle x_n, y_m\rangle_o,$$

where the supremum is taken over all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_m)_{m\in\mathbb{N}}$ in X whose equivalence class is a and b, respectively.

A neighborhood system for $a \in \partial X$ is given by the sets

$$\{b \in \partial X \mid \langle a, b\rangle_o \geqslant R\} \quad \text{for } R \geqslant 0.$$

The topological spaces ∂X and $\overline{X} = X \cup \partial X$ are Hausdorff (see, e.g., [30, Sections 4 and 5] for the most general case). If X is also proper, i.e., closed ball are compact, then ∂X and $\overline{X}$ are compact. In that case, we call $\overline{X}$ the *Gromov compactification* of X.

2.2 Isometries

Let $G = \mathrm{Isom}(X)$ be the isometry group of a Gromov-hyperbolic and geodesic metric space X.

Let $g \in G$ and $o \in X$. We define two invariants of conjugacy classes of elements of G which will be helpful to get a classification of elements of G. The *translation length of* g is denoted by $l(g)$. It is defined by

$$l(g) = \inf_{x \in X} d(x, g(x)).$$

The *stable norm of* g is denoted by $N(g)$. It is defined by

$$N(g) = \lim_{n \to +\infty} \frac{d(o, g^n(o))}{n}.$$

The stable norm is well-defined by a subadditivity argument and that the limit does not depend on the choice of the basepoint $o \in X$. The next proposition shows that, in our context, these two invariants only differ by an additive constant. Its proof can be found in [7, Chapter 10, Proposition 6.4]:

Proposition 2.1 *If X is a δ-hyperbolic and geodesic metric space and $g \in \mathrm{Isom}(X)$ is an isometry of X,*

$$N(g) \leqslant l(g) \leqslant N(g) + 16\delta.$$

An isometry of X extends to a homeomorphism of $\overline{X}$. Elements of G fall into one of the following three categories:

- We say that g is *elliptic* if $N(g) = 0$ and g has a bounded orbit.
- We say that g is *parabolic* if $N(g) = 0$ and g is not elliptic. In that case, the extended action of g has a unique fixed point in ∂X.
- We say that g is *hyperbolic* if $N(g) > 0$. In that case, the extended action of g has exactly two distinct fixed points in ∂X.

2.3 Hyperbolic groups

Let Γ be a finitely generated group and S be a finite set of generators. The *Cayley graph* of Γ with respect to S is denoted by $C_S(\Gamma)$. It is the graph whose vertices are labeled by elements of Γ, and there is an edge between two vertices labeled by $\gamma, \gamma' \in \Gamma$ if there is $s \in S$ such that $\gamma' = \gamma s$ or $\gamma' = \gamma s^{-1}$. The metric on $C_S(\Gamma)$ obtained by setting each edge to be of length 1 yields a metric on Γ called the *word metric* and denoted by d_S. The *word length of* $\gamma \in \Gamma$ is $|\gamma|_S = d_S(e, \gamma)$ where e is the neutral element of Γ. The translation length of $\gamma \in \Gamma$ for its action on $C_S(\Gamma)$ is denoted by $l_S(\gamma)$.

A group Γ is *hyperbolic* if there is a finite generating set S such that $(C_S(\Gamma), d_S)$ is a Gromov-hyperbolic metric space. A hyperbolic group is *nonelementary* if it contains a free subgroup of rank 2.

The group Γ acts naturally by isometries on $(C_S(\Gamma), d_S)$ by left multiplication on the vertices. If Γ is hyperbolic, every element $\gamma \in \Gamma$ of infinite order acts as a hyperbolic isometry, and every element $\gamma \in \Gamma$ of finite order acts as an elliptic isometry. The following proposition will be helpful (see, for example, [12, Corollary, p. 224]).

Proposition 2.2 *Let Γ be a hyperbolic group and H be a subgroup of Γ. The subgroup H is infinite if and only if it contains an infinite-order element.*

We also need the following definition.

Definition 2.3 Let (X, d) and (Y, d') be two metric spaces.

- A map $f : X \to Y$ is a *quasi-isometric embedding* if there are constants $K \geqslant 1$ and $C \geqslant 0$ such that for all $x, x' \in X$,
$$\frac{1}{K} d(x, x') - C \leqslant d'(f(x), f(x')) \leqslant K d(x, x') + C.$$
- A map $f : X \to Y$ is a *quasi-isometry* if there are constants $K \geqslant 1$ and $C \geqslant 0$ such that f is a (K, C) quasi-isometric embedding and there is a (K, C) quasi-isometric embedding $g : Y \to X$ such that for all $x \in X$, $y \in Y$,
$$d(g \circ f(x), x) \leqslant C \quad \text{and} \quad d'(f \circ g(y), y) \leqslant C.$$
- A *quasigeodesic* of X is the image of a quasi-isometric embedding $f : \mathbb{Z} \to X$.

If we want to insist on constants, we will say that a map (resp. a path) is a (K, C) quasi-isometric embedding (resp. a (K, C) quasigeodesic).

When we refer to quasigeodesics in $C_S(\Gamma)$, we assume it is parametrized by arclength.

If $\gamma \in \Gamma$ has infinite order, a (K, C) *quasiaxis for* γ is a (K, C) quasigeodesic of $C_S(\Gamma)$, invariant under the action of γ and such that γ acts on it by translation of length $l_S(\gamma)$.

Suppose Γ is a hyperbolic group. The endpoints of a quasiaxis for γ are the distinct fixed points in $\partial C_S(\Gamma)$ for the extended action of γ on $\overline{C_S(\Gamma)}$. We now detail Lee's construction (see [17, Lemma II.6]) to prove the existence of quasiaxes of infinite-order elements of Γ with some uniform constants.

Let $\gamma \in \Gamma$ be of infinite order and $w_\gamma \in C_S(\Gamma)$ be such that $d_S(w_\gamma, \gamma \cdot w_\gamma) = l_S(\gamma)$. Let l_γ equal $\bigcup_{n \in \mathbb{Z}} \gamma^n \cdot [w_\gamma, \gamma \cdot w_\gamma]$ where $[w_\gamma, \gamma \cdot w_\gamma]$ denotes any geodesic segment connecting w_γ to $\gamma \cdot w_\gamma$ in $C_S(\Gamma)$.

Lemma 2.4 *If Γ is a hyperbolic group, there are constants $K' \geqslant 1$ and $C' \geqslant 0$, such that for all $\gamma \in \Gamma$ of infinite order, the path l_γ defined as above is a (K', C') quasiaxis for the action of γ on $C_S(\Gamma)$.*

Remark 2.5 For all $\gamma \in \Gamma$,
$$l_S(\gamma) = \inf_{g \in \Gamma} d_S(g, \gamma g) = \min_{g \in \Gamma} d_S(e, g^{-1}\gamma g) = \min_{g \in \Gamma} |g\gamma g^{-1}|_S.$$

Hence, if γ has minimal word length in its conjugacy class, then $\bigcup_{n \in \mathbb{Z}} [\gamma^n, \gamma^{n+1}]$ is a (K', C') quasiaxis for γ with constants K' and C' independent of γ. Moreover, if $g, h \in \Gamma$ and l_g is a (K', C') quasiaxis for g, then $h \cdot l_g$ is a (K', C') quasiaxis for hgh^{-1}.

3 Space of representations

Let X be a metric space and let $G = \mathrm{Isom}(X)$ be its isometry group endowed with the compact-open topology. Let Γ be a finitely generated group and let S be a finite set of generators.

3.1 Group action on the space of representations

The *space of representations* $\mathcal{R}(\Gamma, G)$ is the set of homomorphisms $\rho : \Gamma \to G$. We first endow $G^{|S|}$ with the product topology and then identify $\mathcal{R}(\Gamma, G)$ with a subset of $G^{|S|}$, which allows us to endow $\mathcal{R}(\Gamma, G)$ with the subspace topology. This topology does not depend on the choice of a finite generating set S.

The automorphism group of G and that of Γ act on $\mathcal{R}(\Gamma, G)$ in the following way: for any $\rho \in \mathcal{R}(\Gamma, G)$, $\psi \in \mathrm{Aut}(G)$ and $\varphi \in \mathrm{Aut}(\Gamma)$ we have,

$$\psi \cdot \rho = \psi \circ \rho \quad \text{and} \quad \varphi \cdot \rho = \rho \circ \varphi^{-1}.$$

For any group G, we denote by $\mathrm{Inn}(G)$ the normal subgroup of inner automorphisms of G and by $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ the group of outer automorphisms of G. In addition, we denote by $\mathcal{X}(\Gamma, G) = \mathcal{R}(\Gamma, G)/\mathrm{Inn}(G)$ the quotient space made of conjugacy classes of representations, and endow it with the quotient topology. We mention that this topology may not be Hausdorff (may not even be $T1$), however, this fact will not add any difficulty in what follows. The group $\mathrm{Inn}(\Gamma)$ acts trivially on $\mathcal{X}(\Gamma, G)$, therefore there is an induced action of $\mathrm{Out}(\Gamma)$ on $\mathcal{X}(\Gamma, G)$.

We denote by $\mathrm{AH}(\Gamma, G)$ the set of conjugacy classes of representations whose image is discrete in G and whose kernel is finite. If Γ is torsion-free, for any $[\rho] \in \mathcal{X}(\Gamma, G)$, it is in $\mathrm{AH}(\Gamma, G)$ if and only if it has a representative which is injective and has discrete image in G. The subset $\mathrm{AH}(\Gamma, G)$ is invariant under the action of $\mathrm{Out}(\Gamma)$.

3.2 Orbit maps and quasi-isometrically embedded representations

Let $\rho \in \mathcal{R}(\Gamma, G)$ and $x \in X$ be a basepoint. An *orbit map* $\tau_{\rho,x} : \Gamma \to X$ is a ρ-equivariant map defined by $\tau_{\rho,x}(\gamma) = \rho(\gamma)(x)$ for any $\gamma \in \Gamma$.

Recall that the finite set S of generators of Γ yields a distance function d_S on Γ. We say that $\rho \in \mathcal{R}(\Gamma, G)$ is a quasi-isometrically embedded representation if there is an orbit map which is a quasi-isometric embedding. This definition depends neither on the conjugacy class of the representation, nor on the choice of the base point $x \in X$, nor on the choice of a finite set of generators for Γ. We denote by $\mathrm{QIE}(\Gamma, G)$ the set of conjugacy classes of quasi-isometrically embedded representations. The subset $\mathrm{QIE}(\Gamma, G)$ is invariant under the action of $\mathrm{Out}(\Gamma)$ and contained in $\mathrm{AH}(\Gamma, G)$.

Remark 3.1 Let $\rho \in \mathcal{R}(\Gamma, G)$ and $\tau_{\rho,x}$ be an orbit map. The map $\tau_{\rho,x}$ is κ-Lipschitz where κ equals $\max_{s \in S}\{d(x, \tau_{\rho,x}(s))\}$.

We now describe the work of the authors of [10] who gave an equivalent definition when Γ is a hyperbolic group. For $\gamma \in \Gamma$, we denote by $l_S(\gamma)$ the translation length of γ for its action on $C_S(\Gamma)$ and

by $l(\rho(\gamma))$ the translation length of $\rho(\gamma)$ for its action on X. We say that $\rho \in \mathcal{R}(\Gamma, G)$ is *well-displacing* if there are $J \geqslant 1$ and $B \geqslant 0$ such that

$$\frac{1}{J} l_S(\gamma) - B \leqslant l(\rho(\gamma)) \leqslant J l_S(\gamma) + B \quad \text{for all } \gamma \in \Gamma.$$

The following theorem is proved in [10, Sections 3 and 4].

Theorem 3.2 *Let Γ be a hyperbolic group. A representation $\rho \in \mathcal{R}(\Gamma, G)$ is quasi-isometrically embedded if and only if ρ is well-displacing.*

We now study a special case when X is a Gromov-hyperbolic, geodesic and proper metric space. We still denote by $G = \mathrm{Isom}(X)$ the isometry group of X. Say that $X' \subset X$ is *quasiconvex* if there is $\eta > 0$ such that any two points of X' can be connected by a geodesic segment of X contained in an η-neighborhood of X'. A subgroup H of G is *quasiconvex* if it acts properly discontinuously on X and if there is a point $x \in X$ such that the orbit $H(x)$ is quasiconvex. A subgroup H of G is *convex cocompact* if it acts properly discontinuously on X and if there is a convex and H-invariant subset of X on which the action of H is cocompact.

The following proposition is proved in [2, Proposition 1.8.2; 6, Section 4].

Proposition 3.3 *Let X be a Gromov-hyperbolic, geodesic and proper metric space and $G = \mathrm{Isom}(X)$ denotes its isometry group. Let Γ be a finitely generated group and $\rho \in \mathcal{R}(\Gamma, G)$. The following statements are equivalent*:

- *ρ has an orbit map which is a quasi-isometric embedding.*
- *$\rho(\Gamma)$ is a quasiconvex subgroup of G.*
- *$\rho(\Gamma)$ is a convex cocompact subgroup of G.*

Moreover, if there exists such a representation, then Γ is a hyperbolic group.

A similar statement also holds when $X = \mathbb{H}^\infty$ is the infinite-dimensional hyperbolic space (see [32, Theorem 1.1] for more details).

4 A-stable representations

In this section, Γ denotes a hyperbolic group, S is a finite set of generators for Γ, $G = \mathrm{Isom}(X)$ is the isometry group of a geodesic and Gromov-hyperbolic metric space X and $x \in X$ is a basepoint.

For any $\mathrm{Aut}(\Gamma)$-invariant subset $A \subset \Gamma$, we define an $\mathrm{Out}(\Gamma)$-invariant open subset $S_A(\Gamma, G) \subset \mathcal{X}(\Gamma, G)$ called the set of A-stable representations. We give a sufficient condition on $A \subset \Gamma$ for the action of $\mathrm{Out}(\Gamma)$ on $S_A(\Gamma, G)$ to be properly discontinuous. We end this section by giving a geometric characterization, due to Lee [17], of discrete, faithful and A-stable representations when $G = \mathrm{PSL}_2(\mathbb{C})$.

4.1 Definition of A-stable representation

If $\gamma \in \Gamma$ has infinite order, γ_- and γ_+ will denote its fixed points in $\partial C_S(\Gamma)$ for the extended action of γ on $\overline{C_S(\Gamma)}$. Moreover we denote by $L_S(\gamma)$ the set of geodesics of $C_S(\Gamma)$ connecting γ_- to γ_+. If $A \subset \Gamma$, we name A_∞ the subset consisting of infinite-order elements of A and $L_S(A) = \bigcup_{a \in A_\infty} L_S(a)$. When we discuss quasigeodesics in $\tau_{\rho,x}(C_S(\Gamma)$, we assume that the parametrization is the composition of the parametrization by arclength in $C_S(\Gamma)$ and $\tau_{\rho,x}$.

Definition 4.1 Let $A \subset \Gamma$ be a subset invariant under automorphisms of Γ. A representation $\rho \in \mathcal{R}(\Gamma, G)$ is *A-stable* if there are constants $K \geqslant 1$ and $C \geqslant 0$ such that for all $l \in L_S(A)$, $\tau_{\rho,x}(l)$ is a (K, C) quasigeodesic.

Being A-stable depends neither on the conjugacy class of the representation, nor on the choice of the basepoint $x \in X$, nor on the choice of a finite set of generators for Γ. We write $S_A(\Gamma, G)$ the set of conjugacy classes of A-stable representations.

If we want to insist on constants, we will say that a representation $\rho \in \mathcal{R}(\Gamma, G)$ is a (K, C) A-stable representation. A conjugacy class of a representation $[\rho] \in \mathcal{X}(\Gamma, G)$ is said to be a (K, C) A-stable if it has a representative which is a (K, C) A-stable representation.

4.2 An open Out(Γ)-invariant subset

We now aim at proving the following proposition which is a strengthening of Proposition 1.1.

Proposition 4.2 *The set $S_A(\Gamma, G)$ is an* Out(Γ)*-invariant subset of $\mathcal{X}(\Gamma, G)$. Moreover, if δ is a hyperbolicity constant for X, for all constants $K_0 \geqslant 1$, $C_0 \geqslant 0$, there are constants $K \geqslant 1$, $C \geqslant 0$ depending only on K_0, C_0 and δ with the following property: every conjugacy class $[\rho_0]$ which is (K_0, C_0) A-stable has an open neighborhood U_0 of $[\rho_0]$ such that every $[\rho] \in U_0$ is (K, C) A-stable.*

The proof relies on the local-to-global principle for quasigeodesics in a Gromov-hyperbolic space.

Definition 4.3 Let $K \geqslant 1$, $C \geqslant 0$ and $M \geqslant 0$. A (K, C, M) *local quasigeodesic* is the image of $f : \mathbb{Z} \to X$ such that f restricted to any segment of length at most M is a (K, C) quasi-isometric embedding.

The next lemma is proved in [7, Chapter 3, Theorem 1.4] and gives us a sufficient condition for a family of paths in a Gromov-hyperbolic metric space to be uniform quasigeodesics.

Lemma 4.4 *Let X be a δ-hyperbolic and geodesic metric space. For all $K' \geqslant 1$ and $C' \geqslant 0$ there are $K \geqslant 1$, $C \geqslant 0$ and $M \geqslant 0$ depending only on δ, K' and C' such that any (K', C', M) local quasigeodesic of X is a (K, C) quasigeodesic.*

Proof of Proposition 4.2 We start by proving that $S_A(\Gamma, G)$ is Out(Γ)-invariant. Let $\rho \in \mathcal{R}(\Gamma, G)$ be an A-stable representation, i.e., $\tau_{\rho,x}$ maps all geodesics of $L_S(A)$ to uniform quasigeodesics of X. Let $\varphi \in \mathrm{Aut}(\Gamma)$ be an automorphism of Γ. The map

$$\varphi_* : (C_{\varphi(S)}(\Gamma), d_{\varphi(S)}) \to (C_S(\Gamma), d_S), \quad \gamma \mapsto \varphi^{-1}(\gamma),$$

is an isometry. Hence, if $l \in L_{\varphi(S)}(A)$ is a geodesic of $C_{\varphi(S)}(\Gamma)$ whose extremities are the fixed points of $a \in A$, then $\varphi_*(l)$ is a geodesic of $C_S(\Gamma)$ which is in $L_S(A)$ since A is $\mathrm{Aut}(\Gamma)$-invariant.

Because ρ is A-stable, the map $\tau_{\rho,x} \circ \varphi_* : C_{\varphi(S)}(\Gamma) \to X$, which is an orbit map for $\rho \circ \varphi^{-1}$, sends every geodesic in $L_{\varphi(S)}(A)$ to uniform quasigeodesic of X.

Since, A-stability does not depend on the choice of the orbit map, we proved that $\varphi \cdot \rho$ is A-stable.

Let us now prove the second part. Let $[\rho_0] \in S_A(\Gamma, G)$ be the conjugacy class of a (K_0, C_0) A-stable representation ρ_0. According to Lemma 4.4, there are constants K, C and M depending only on δ, K_0 and C_0 such that every $(K_0, 2C_0, M)$ local quasigeodesic of X is a (K, C) quasigeodesic.

Consider the open neighborhood U_0 of $[\rho_0]$ in $\mathcal{X}(\Gamma, G)$ such that for any $[\rho] \in U_0$, there exists a representative ρ satisfying

$$d(\tau_{\rho,x}(\gamma), \tau_{\rho_0,x}(\gamma)) \leqslant C_0 \quad \text{for all } \gamma \in \overline{B_S(e, M)},$$

where $\overline{B_S(e, M)}$ is the closed ball of radius M around the identity element in the Cayley graph $C_S(\Gamma)$.

We will show that every element of U_0 is (K, C) A-stable by verifying that for every geodesic $l \in L_S(A)$, the map $\tau_{\rho,x}(l)$ is a $(K_0, 2C_0, M)$ local quasigeodesic in X.

Let $l = \{l_n\}_{n\in\mathbb{Z}}$ be a geodesic of $L_S(A)$, where each l_n is a vertex in $C_S(\Gamma)$. Let $n, m \in \mathbb{Z}$ be integers such that $|n-m| \leqslant M$. Using the equivariance of orbit maps and the fact that ρ_0 is A-stable, we have

$$\begin{aligned} d(\tau_{\rho,x}(l_n), \tau_{\rho,x}(l_m)) &= d(\tau_{\rho,x}(e), \tau_{\rho,x}(l_n^{-1} l_m)) \\ &\leqslant d(x, \tau_{\rho_0,x}(l_n^{-1} l_m)) + d(\tau_{\rho_0,x}(l_n^{-1} l_m), \tau_{\rho,x}(l_n^{-1} l_m)) \\ &\leqslant d(\tau_{\rho_0,x}(l_n), \tau_{\rho_0,x}(l_m)) + C_0 \\ &\leqslant K_0 |n-m| + 2C_0. \end{aligned}$$

Moreover, by symmetry

$$d(\tau_{\rho_0,x}(l_n), \tau_{\rho_0,x}(l_m)) \leqslant d(\tau_{\rho,x}(l_n), \tau_{\rho,x}(l_m)) + C_0.$$

We deduce that

$$\begin{aligned} d(\tau_{\rho,x}(l_n), \tau_{\rho,x}(l_m)) &\geqslant d(\tau_{\rho_0,x}(l_n), \tau_{\rho_0,x}(l_m)) - C_0 \\ &\geqslant \frac{|n-m|}{K_0} - 2C_0. \end{aligned}$$

Therefore, $\tau_{\rho,x}$ maps every geodesic $l \in L_S(A)$ to $(K_0, 2C_0, M)$ local quasigeodesics of X. By Lemma 4.4, these are (K, C) quasigeodesics of X, where K and C depend only on δ, K_0, and C_0, not on the specific $[\rho] \in U_0$ or $l \in L_S(A)$.

We conclude that every conjugacy class of representation in U_0 is (K, C) A-stable. □

4.3 A-well-displacing representations

We now describe the relation between A-stability and another property that we call the A-well-displacing property.

Definition 4.5 A representation $\rho \in \mathcal{R}(\Gamma, G)$ is *A-well-displacing* if there are constants $J \geqslant 1$ and $B \geqslant 0$ such that

$$\frac{1}{J} l_S(a) - B \leqslant l(\rho(a)) \leqslant J l_S(a) + B \quad \text{for all } a \in A.$$

If we want to insist on constants, we will say $\rho \in \mathcal{R}(\Gamma, G)$ is a (J, B) A-well-displacing representation.

Since there are only finitely many conjugacy classes of finite-order element in a hyperbolic group (see, for instance, [12, Corollaire, p. 218]), it is sufficient to show that the previous relation holds for all $a \in A_\infty$.

Proposition 4.6 *Given two constants $K \geqslant 1$ and $C \geqslant 0$, there are constants $J \geqslant 1$ and $B \geqslant 0$ such that every (K, C) A-stable representation is (J, B) A-well-displacing.*

To prove this proposition, we will apply the following lemma. While originally established for $X = \mathbb{H}^3$ in [17, Lemma II.9], the proof relies solely on the Gromov hyperbolicity of $C_S(\Gamma)$ and requires no specific assumptions about X.

Lemma 4.7 *Let $\rho \in \mathcal{R}(\Gamma, G)$ be a representation, let $x \in X$ be a basepoint, let $K, K' \geqslant 1$ and $C, C' \geqslant 0$ be some constants and let l be a geodesic of $C_S(\Gamma)$. There are constants $K'' \geqslant 1$ and $C'' \geqslant 0$ such that, if $\tau_{\rho,x}(l)$ is a (K, C) quasigeodesic of X, then for any (K', C') quasigeodesic l' of $C_S(\Gamma)$ with the same endpoints as l, the map $\tau_{\rho,x}(l')$ is a (K'', C'') quasigeodesic of X.*

Proof of Proposition 4.6 Let $\rho \in \mathcal{R}(\Gamma, G)$ be a (K, C) A-stable representation and $x \in X$ be a base point. According, to Lemma 2.4, there are uniform constants K' and C' such that every $a \in A_\infty$ has a (K', C') quasiaxis for its action on $C_S(\Gamma)$ denoted by l_a. Now, Lemma 4.7 gives the existence of uniform constants K'' and C'' such that for every $a \in A_\infty$, $\tau_{\rho,x}(l_a)$ is a (K'', C'') quasigeodesic of X. Let $a \in A_\infty$ be of minimal word length in its conjugacy class. Hence for all integers $n \in \mathbb{Z}$ we have $a^n \in l_a$, therefore

$$\frac{1}{K''} |a^n|_S - C'' \leqslant d(x, \rho(a^n)(x)) \leqslant K'' |a^n|_S + C''.$$

Dividing the previous inequality by n and taking the limit yield

$$\frac{1}{K''} N_S(a) \leqslant N(\rho(a)) \leqslant K'' N_S(a),$$

where $N_S(a)$ (resp. $N(\rho(a))$) denote the stable norm of a for its action on $C_S(\Gamma)$ (resp. of $\rho(a)$ for its action on X). Since translation length and stable norm differ by an additive constant according to Proposition 2.1, we actually proved the existence of uniform constants $J \geqslant 1$ and $B \geqslant 0$ such that

$$\frac{1}{J} l_S(a) - B \leqslant l(\rho(a)) \leqslant J l_S(a) + B \quad \text{for all } a \in A_\infty. \qquad \square$$

Combining Propositions 4.2 and 4.6, we now show that:

Corollary 4.8 *Let R be a compact subset of $S_A(\Gamma, G)$. There are uniform constants $J_R \geqslant 1$ and $B_R \geqslant 0$ such that every element in R is (J_R, B_R) A-well-displacing.*

Proof Proposition 4.2 implies that for every representation $[\rho] \in R$, there are $U_{[\rho]}$ an open neighborhood of $[\rho]$ in $\mathcal{X}(\Gamma, G)$, $K_{[\rho]} \geqslant 1$ and $C_{[\rho]} \geqslant 0$ such that every representation in $U_{[\rho]}$ is $(K_{[\rho]}, C_{[\rho]})$ A-stable. Proposition 4.6 now gives us $J_{[\rho]}$ and $B_{[\rho]}$ such that for every $\sigma \in U_{[\rho]}$ and all $a \in A$,

$$\frac{1}{J_{[\rho]}} l_S(a) - B_{[\rho]} \leqslant l(\sigma(a)) \leqslant J_{[\rho]} l_S(a) + B_{[\rho]} \quad \text{for all } \sigma \in U_{[\rho]} \text{ and all } a \in A.$$

Since R is compact, we can consider a finite subcover of $(U_{[\rho]})_{[\rho]\in R}$: there are $[\rho_1], \dots, [\rho_k] \in R$ such that $R \subset \bigcup_{i=1}^{k} U_{[\rho_i]}$. We take $J_R = \max\{J_{[\rho_1]}, \dots, J_{[\rho_k]}\}$ and $B_R = \max\{B_{[\rho_1]}, \dots, B_{[\rho_k]}\}$. □

4.4 Dynamics of Out(Γ) on $S_A(\Gamma, G)$

Let H be a group acting continuously on a topological space Y and $Y' \subset Y$ be invariant by the action of H. Recall that H acts *properly discontinuously* on Y' if for every compact set $R \subset Y'$, the set $\{h \in H \mid h(R) \cap R \neq \varnothing\}$ is finite. We say that Y' is a *domain of discontinuity* if Y' is open, H-invariant, and that the action of H on Y' is properly discontinuous.

Definition 4.9 A subset $A \subset \Gamma$ is *testable* if, for all sequences $(\varphi_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{Aut}(\Gamma)$ which project onto sequences of distinct elements of $\mathrm{Out}(\Gamma)$, there is $a \in A$ such that

$$\limsup_{n\to\infty} l_S(\varphi_n(a)) = \infty.$$

Note that this property does not depend on the choice of the set S of generators.

Theorem 1.2 *Let $A \subset \Gamma$ be an $\mathrm{Aut}(\Gamma)$-invariant subset of Γ. If A is testable, then the subset $S_A(\Gamma, G)$ is a domain of discontinuity of $\mathcal{X}(\Gamma, G)$ which contains $\mathrm{QIE}(\Gamma, G)$.*

Proof By contradiction, we suppose that there are a compact subset R of $S_A(\Gamma, G)$ and a sequence $([\varphi_n])_{n\in\mathbb{N}}$ of distinct elements of $\mathrm{Out}(\Gamma)$ such that $[\varphi_n](R) \cap R \neq \varnothing$ for infinitely many $n \in \mathbb{N}$. Without loss of generality, suppose that $[\varphi_n](R) \cap R \neq \varnothing$ for all $n \in \mathbb{N}$. There is a sequence $([\rho_n])_{n\in\mathbb{N}}$ of elements of R such that $[\varphi_n] \cdot [\rho_n] = [\rho_n \circ \varphi_n^{-1}] \in R$ for all $n \in \mathbb{N}$. We write $[\rho'_n] = [\rho_n \circ \varphi_n^{-1}]$ this other sequence of elements of R.

Since A is testable, there is $a \in A$ such that $\limsup_{n\to\infty} l_S(\varphi_n^{-1}(a)) = \infty$. Let J_R and B_R be the constants given by Corollary 4.8. As both $[\rho_n]$ and $[\rho'_n]$ belong to R, we have

$$\frac{1}{J_R} l_S(\varphi_n^{-1}(a)) - B_R \leqslant l\big(\rho_n(\varphi_n^{-1}(a))\big) = l(\rho'_n(a)) \leqslant J_R l_S(a) + B_R \quad \text{for all } n \in \mathbb{N}.$$

We get a contradiction since $\limsup_{n\to\infty} l_S(\varphi_n^{-1}(a)) = \infty$. □

4.5 Geometric interpretation when $G = \mathrm{PSL}_2(\mathbb{C})$

This section ends with a geometric interpretation, due to Lee [17], of discrete and faithful A-stable representations when Γ is torsion-free and $G = \mathrm{PSL}_2(\mathbb{C})$ is identified with the group of orientation-preserving isometries of the 3-dimensional hyperbolic space $\mathbb{H}^3$. This result will be used in Section 8 and may help some readers to be more familiar with those representations.

Any discrete torsion-free subgroup H of $\mathrm{PSL}_2(\mathbb{C})$ acts freely and properly discontinuously on $\mathbb{H}^3$ and hence yields a 3-dimensional hyperbolic manifold $\mathbb{H}^3/H$ with fundamental group isomorphic to H. A subgroup H of $\mathrm{PSL}_2(\mathbb{C})$ is convex cocompact if and only if there is a convex and compact submanifold N of $\mathbb{H}^3/H$ such that the inclusion $N \hookrightarrow \mathbb{H}^3/H$ is a homotopy equivalence.

Let Γ be a torsion-free hyperbolic group and $A \subset \Gamma$ be an $\mathrm{Aut}(\Gamma)$-invariant subset of Γ.

Let $\rho \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ be a representation and $a \in A$ be an element of A. If $\rho(a)$ is a hyperbolic isometry of $\mathbb{H}^3$, then we denote by $L_\rho(a)$ the geodesic connecting $\rho(a)^-$ to $\rho(a)^+$ in $\mathbb{H}^3$.

Remark 4.10 When specializing to $\mathrm{PSL}_2(\mathbb{C})$, the nature of an element as an isometry of $\mathbb{H}^3$ is determined by its trace (which is well-defined up to sign). An element $[M] \in \mathrm{PSL}_2(\mathbb{C})$ has two distinct fixed points on the boundary if and only if its trace is not a real number whose absolute value is less than or equal to 2. Among such elements, some authors distinguish between cases, calling $[M]$ hyperbolic when M has real trace and loxodromic otherwise. In the following, we adhere to our previous definition and call all those elements hyperbolic.

The following equivalence is due to Lee [17, Lemma III.10]:

Proposition 4.11 *Let $\rho \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ be a discrete and faithful representation. The representation ρ is A-stable if and only if*

(1) *$\rho(a)$ is hyperbolic for all $a \in A \setminus \{e\}$,*

(2) *there is a $\rho(\Gamma)$-invariant subset $\Omega \subset \mathbb{H}^3$ such that $\Omega/\rho(\Gamma)$ is compact and $\bigcup_{a \in A\setminus\{e\}} L_\rho(a)$ is contained in Ω.*

This proposition leads us to consider the following definition:

Definition 4.12 We say that a representation $\rho \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ is *A-convex cocompact* if

(1) $\rho(a)$ is hyperbolic for all $a \in A \setminus \{e\}$,

(2) there is a $\rho(\Gamma)$-invariant subset $\Omega \subset \mathbb{H}^3$ such that $\Omega/\rho(\Gamma)$ is compact and $\bigcup_{a \in A\setminus\{e\}} L_\rho(a)$ is contained in Ω.

5 Actions on $\mathbb{R}$-trees and testable subsets

In this section, Γ will denote a hyperbolic group such that its outer automorphism group $\mathrm{Out}(\Gamma)$ is infinite and S will denote a finite set of generators for Γ.

Before giving examples of A-stable representations, we focus on certain subsets of Γ and ask for their testability (see Definition 4.9). In order to do so, we recall some theory about isometric actions on $\mathbb{R}$-trees. Then we make use of a theorem of Paulin [24] that asserts that if $\mathrm{Out}(\Gamma)$ is infinite, then the group Γ acts isometrically on an $\mathbb{R}$-tree without global fixed points. We take advantage of this approach to prove Theorem 1.7 and to show that the commutator subset of Γ is testable.

5.1 $\mathbb{R}$-trees

We refer the reader to [8; 23] for a more detailed introduction on $\mathbb{R}$-trees.

Definition 5.1 Let (X, d) be a metric space and let $x, y \in X$. An *arc* from x to y is the image of a topological embedding $\alpha : [a, b] \to X$ of a closed interval $[a, b]$ such that $\alpha(a) = x$ and $\alpha(b) = y$.

Definition 5.2 An $\mathbb{R}$-*tree* is a nonempty metric space (T, d) such that for every $x, y \in T$, there is a unique arc connecting x to y and this arc is a geodesic segment. We say that $T' \subset T$ is a *subtree* if $T' \neq \varnothing$ and T' is connected.

Note that $\mathbb{R}$-trees are geodesic, 0-hyperbolic metric spaces but need not to be proper. Let $G = \mathrm{Isom}(T)$ be the isometry group of an $\mathbb{R}$-tree T. We denote by $l_T(g) = l(g)$ the translation length of $g \in G$. The *characteristic set of* g is defined as the set

$$C_g = \{x \in T \mid d(x, g(x)) = l_T(g)\}.$$

If $l_T(g) = 0$, then g is elliptic and C_g is the set of fixed points of g. If $l_T(g) > 0$, then g is hyperbolic, C_g is isometric to $\mathbb{R}$ and is called the *axis* of g. Furthermore, g acts on C_g by translation of length $l_T(g)$.

Proofs of the previous and following statements can be found in [8, Section 1].

Proposition 5.3 *For all $g, h \in G$,*

- *C_g is a closed subtree of T, invariant by the action of g,*
- *$C_{g^{-1}} = C_g$,*
- *$C_{ghg^{-1}} = g(C_h)$.*

We now give two propositions which are proved in [8, Proposition 1.2; 23, Proposition 1.8], respectively.

Proposition 5.4 *If T_1, T_2, T_3 are three closed subtrees of an $\mathbb{R}$-tree T such that $T_i \cap T_j$ is not empty for all $1 \leqslant i, j \leqslant 3$, then $T_1 \cap T_2 \cap T_3$ is not empty.*

Remark 5.5 By induction, if $T_1, \ldots, T_k$ are k closed subtrees of an $\mathbb{R}$-tree T such that $T_i \cap T_j$ is not empty for all $1 \leqslant i, j \leqslant k$, then $T_1 \cap \cdots \cap T_k$ is not empty.

Proposition 5.6 *If g, h are two elliptic isometries of an $\mathbb{R}$-tree (T, d), then $l_T(gh) = 2d(C_g, C_h)$. In particular, if g, h and gh are elliptic, then $C_g \cap C_h \neq \varnothing$.*

5.2 Isometric actions on $\mathbb{R}$-trees

We now follow the work of Bestvina [1].

A Γ-*space* is a pair (X, ρ) where X is a metric space and $\rho \in \mathcal{R}(\Gamma, \mathrm{Isom}(X))$ is an isometric action of Γ on X. A Γ-*tree* is a Γ-space (T, ρ) where T is an $\mathbb{R}$-tree. If there is no ambiguity, we denote the translation length of $\rho(\gamma)$ by $l_T(\gamma)$. A Γ-tree (T, ρ) is *trivial* if there is a global fixed point in T. Let $\mathcal{B}_S = S \cup \{ss' \mid s, s' \in S,\ s \neq s'\}$ be the set of generators and product of distinct generators.

Proposition 5.7 *Let Γ be a finitely generated group and (T,ρ) be a Γ-tree. The following are equivalent*:

- *(T,ρ) is trivial.*
- *$l_T(\gamma)=0$ for all $\gamma\in\Gamma$.*
- *$l_T(\gamma)=0$ for all $\gamma\in\mathcal{B}_S$.*

Proof If (T,ρ) is trivial, every element of Γ fixes a point, hence is elliptic and $l_T(\gamma)=0$ for all $\gamma\in\Gamma$. It is clear that the second condition implies the third one. Suppose now that $l_T(\gamma)=0$ for all $\gamma\in\mathcal{B}_S$. Let $s,s'\in S$ be such that $s\neq s'$. Since s, s' and ss' are elliptic, the set of fixed points C_s and $C_{s'}$ have nonempty intersection according to Proposition 5.6. Proposition 5.4 now implies that the intersection $\bigcap_{s\in S}C_s$ is nonempty: therefore there is a point of T fixed by all generators of Γ. This point is therefore fixed by all elements of Γ and (T,ρ) is a trivial Γ-tree. □

An arbitrary group is *small* if it does not contain a free subgroup of rank 2. We say that a Γ-tree (T,ρ) is *small* if the pointwise stabilizer of any nondegenerate segment is small.

5.3 Sequences of automorphisms and convergence of based Γ-spaces

A *pseudometric* on Γ is a symmetric application $d:\Gamma\times\Gamma\to[0,\infty[$, vanishing on the diagonal and satisfying the triangular inequality. Let us consider the space $\mathcal{D}(\Gamma)$ of all pseudometrics on Γ endowed with the compact-open topology. The group Γ acts diagonally on $\Gamma\times\Gamma$ by left multiplication. We denote by $\mathcal{ED}(\Gamma)\subset\mathcal{D}(\Gamma)$ the subset of pseudometrics on Γ which are invariant under this action. The group $\mathbb{R}_+^*$ acts freely on $\mathcal{ED}(\Gamma)$ by scaling, we denote by $\mathcal{PED}(\Gamma)$ the quotient space of projectivized Γ-equivariant pseudometrics endowed with quotient topology.

A *based Γ-space* is a triple (X,x,ρ) where (X,ρ) is a Γ-space, $\rho\in\mathcal{R}(\Gamma,\mathrm{Isom}(X))$ is an isometric action of Γ on X and $x\in X$ is a basepoint which is not fixed by all elements of $\rho(\Gamma)$.

A based Γ-space (X,x,ρ) induces a Γ-equivariant pseudometric $d_{(X,x,\rho)}$ on Γ by setting

$$d_{(X,x,\rho)}(\gamma,\gamma')=d_X(\rho(\gamma)(x),\rho(\gamma')(x)) \quad \text{for all } \gamma,\gamma'\in\Gamma.$$

Let $((X_n,x_n,\rho_n))_{n\in\mathbb{N}}$ be a sequence of based Γ-spaces. We say that the sequence $((X_n,x_n,\rho_n))_{n\in\mathbb{N}}$ *converges* to the based Γ-space (X,x,ρ) if $([d_{(X_n,x_n,\rho_n)}])_{n\in\mathbb{N}}$ converges to $[d_{(X,x,\rho)}]$ in $\mathcal{PED}(\Gamma)$.

We consider the special case when $X=C_S(\Gamma)$. From now on, $(\varphi_n)_{n\in\mathbb{N}}$ will denote a sequence of elements of $\mathrm{Aut}(\Gamma)$ that projects onto a sequence of distinct elements of $\mathrm{Out}(\Gamma)$.

For all $n\in\mathbb{N}$, we introduce the application

$$f_n:C_S(\Gamma)\to\mathbb{N},\quad x\mapsto\max_{s\in S}d_S(x,\varphi_n(s)x).$$

Let $\lambda_n=\min\{f_n(x)\mid x\in C_S(\Gamma)\}$ and choose arbitrarily $x_n\in C_S(\Gamma)$ that realize this minimum. Note that x_n need not be a vertex of $C_S(\Gamma)$.

We also denote by $\rho_0\in\mathcal{R}\big(\Gamma,\mathrm{Isom}(C_S(\Gamma))\big)$ the natural action of Γ on $C_S(\Gamma)$, and we write $\rho_n=\rho_0\circ\varphi_n\in\mathcal{R}\big(\Gamma,\mathrm{Isom}(C_S(\Gamma))\big)$.

The sequences $(x_n)_{n\in\mathbb{N}} \in C_S(\Gamma)^{\mathbb{N}}$, $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}} \in \mathcal{R}\big(\Gamma, \mathrm{Isom}(C_S(\Gamma))\big)^{\mathbb{N}}$ will be called *sequences induced by* $(\varphi_n)_{n\in\mathbb{N}}$.

Proposition 5.8 *The sequence of integers $(\lambda_n)_{n\in\mathbb{N}}$ induced by $(\varphi_n)_{n\in\mathbb{N}}$ is not bounded.*

Proof The proof goes by contradiction: Suppose that $(\lambda_n)_{n\in\mathbb{N}}$ is bounded by M. Then for any vertex y_n closest to x_n, we would have $d_S(e, y_n^{-1}\varphi_n(s)y_n) = d_S(y_n, \varphi_n(s)y_n) \leqslant M+2$ for all $s \in S$ and $n \in \mathbb{N}$. Since there are only finitely many vertices in the ball of radius $M+2$ about e, this bound would imply the existence of integers $n, m \in \mathbb{N}$, $n \neq m$, such that $y_n^{-1}\varphi_n(s)y_n = y_m^{-1}\varphi_m(s)y_m$ for all $s \in S$. Thereby, φ_n and φ_m would be equal in $\mathrm{Out}(\Gamma)$; this is a contradiction. □

The next theorem is a special case of [1, Theorem 3.9] when considering $C_S(\Gamma)$. The fact that the limit based Γ-tree is small is a consequence of [1, Proposition 3.10].

Theorem 5.9 *Let $(x_n)_{n\in\mathbb{N}}$, $(\rho_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ be the sequences induced by $(\varphi_n)_{n\in\mathbb{N}}$. The sequence $\big((C_S(\Gamma), x_n, \rho_n)\big)_{n\in\mathbb{N}}$ of based Γ-spaces converges (up to subsequence) to a small and nontrivial based Γ-tree (T, x, ρ). More precisely,*

$$d_{(T,x,\rho)}(\cdot,\cdot) = \lim_{n\to\infty} \frac{d_{(C_S(\Gamma),x_n,\rho_n)}}{\lambda_n}(\cdot,\cdot).$$

Moreover,

$$\forall \gamma \in \Gamma, \quad l_T(\rho(\gamma)) = \lim_{n\to\infty} \frac{l_S(\varphi_n(\gamma))}{\lambda_n}.$$

The based Γ-tree (T, x, ρ) given by this theorem will be called a *based Γ-tree induced by* $(\varphi_n)_{n\in\mathbb{N}}$.

Corollary 5.10 *Let (T, x, ρ) be a based Γ-tree induced by $(\varphi_n)_{n\in\mathbb{N}}$. If $\gamma \in \Gamma$ is such that $\big(l_S(\varphi_n(\gamma))\big)_{n\in\mathbb{N}}$ is bounded, then $\rho(\gamma)$ is elliptic.*

Proof Recall that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of integers induced by $(\varphi_n)_{n\in\mathbb{N}}$ is not bounded according to Proposition 5.8. The last equality of Theorem 5.9 yields that $l_T(\rho(\gamma)) = 0$, hence $\rho(\gamma)$ is an elliptic isometry of T. □

5.4 Application: testable subsets

For any $\phi \in \mathrm{Out}(\Gamma)$, $\varphi \in \mathrm{Aut}(\Gamma)$ a representative of ϕ and $\gamma \in \Gamma$, since the number $l_S(\varphi(\gamma))$ does not depend on the choice of the representative φ, we will abuse the notation and allow ourselves to write $l_S(\phi(\gamma))$ instead of $l_S(\varphi(\gamma))$.

5.4.1 $\mathcal{B}_S$ is testable Recall that $\mathcal{B}_S = S \cup \{ss' \mid s, s' \in S,\ s \neq s'\}$ denotes the set of generators and product of distinct generators. Richard Canary [6, appendix] asked the following question: is it true that for all hyperbolic groups Γ and all $M \geqslant 0$,

$$\{\phi \in \mathrm{Out}(\Gamma) \mid l_S(\phi(b)) \leqslant M \text{ for all } b \in \mathcal{B}_S\}$$

is finite?

The next proposition gives an affirmative answer to this question, and will be used in Section 6 to prove the proper discontinuity of the action of $\mathrm{Out}(\Gamma)$ on the set of conjugacy classes of classical A-stable representations.

Theorem 1.7 *For all hyperbolic groups Γ and for all finite generating sets S the set*

$$\mathcal{B}_S = S \cup \{ss' \mid s, s' \in S,\ s \neq s'\}$$

is testable.

Proof The proof goes by contradiction: Suppose that there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ of distinct elements of $\mathrm{Out}(\Gamma)$ such that for all $b \in \mathcal{B}_S$, $\big(l_S(\phi_n(b))\big)_{n\in\mathbb{N}}$ is bounded. Denote by $(\varphi_n)_{n\in\mathbb{N}}$ any sequence of elements of $\mathrm{Aut}(\Gamma)$ which projects onto $(\phi_n)_{n\in\mathbb{N}}$ and let (T, x, ρ) be a based Γ-tree induced by $(\varphi_n)_{n\in\mathbb{N}}$. Hence $\rho(b)$ would be elliptic for all $b \in \mathcal{B}_S$ according to Corollary 5.10. Proposition 5.7 now implies that (T, ρ) is trivial, contradicting Theorem 5.9. □

5.4.2 The commutator subset is testable In this section, we show that the subset $\mathcal{C} = \{[g, h] \mid g, h \in \Gamma\}$ of commutators of Γ is testable.

A Γ-tree (T, ρ) is *minimal* if there is no $\rho(\Gamma)$-invariant subtree T' properly contained in T. If (T, ρ) is nontrivial, there is a unique minimal invariant subtree T': the union of all axes of hyperbolic elements of $\rho(\Gamma)$ (see [23, Proposition 2.4] for a proof).

We say that a Γ-tree (T, ρ) is *irreducible* if (T, ρ) is nontrivial and $\rho(\Gamma)$ does not fix any point of ∂T.

Culler and Morgan [8, Lemma 5.1 and Corollary 2.3] proved the following:

Lemma 5.11 *If Γ is a nonelementary hyperbolic group and (T, ρ) is a nontrivial, minimal and small Γ-tree, then (T, ρ) is irreducible.*

Lemma 5.12 *If (T, ρ) is an irreducible Γ-tree, then there are $g, h \in \Gamma$ such that $l_T([g, h]) > 0$.*

We deduce:

Proposition 5.13 *If Γ is a nonelementary hyperbolic group, then the subset $\mathcal{C}$ of commutators of Γ is testable.*

Proof Once again, the proof goes by contradiction: Suppose there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ of distinct elements of $\mathrm{Out}(\Gamma)$ and such that $\big(l_S(\phi_n(w))\big)_{n\in\mathbb{N}}$ is bounded for all $w \in \mathcal{C}$. Denote by $(\varphi_n)_{n\in\mathbb{N}}$ any sequence of elements of $\mathrm{Aut}(\Gamma)$ which projects onto $(\phi_n)_{n\in\mathbb{N}}$ and let (T, x, ρ) be a based Γ-tree induced by $(\varphi_n)_{n\in\mathbb{N}}$. (T, ρ) is a nontrivial and small Γ-tree according to Theorem 5.9. Up to restricting to T', where T' denotes the union of all axes of hyperbolic elements of $\rho(\Gamma)$, we can suppose that the action is minimal. Hence (T', ρ) is irreducible according to Lemma 5.11 and Lemma 5.12 now states that there is $w \in \mathcal{C}$ such that $\rho(w)$ is hyperbolic. Corollary 5.10 leads us to a contradiction. □

6 First examples

Let Γ be a hyperbolic group, let S be a finite set of generators of Γ and let $G = \mathrm{Isom}(X)$ be the isometry group of a Gromov-hyperbolic and geodesic metric space X.

If $A \subset \Gamma$ is an $\mathrm{Aut}(\Gamma)$-invariant subset, we denote by $D_A(\Gamma, G)$ the set of conjugacy classes of A-well-displacing representations.

6.1 Quasi-isometrically embedded representations

We start with a trivial case, when $A = \Gamma$:

Proposition 6.1 *A representation ρ in $\mathcal{R}(\Gamma, G)$ is quasi-isometrically embedded if and only if it is Γ-stable.*

Proof The first implication is trivial. Conversely, if ρ is a Γ-stable representation, Proposition 4.6 implies that ρ must be Γ-well-displacing, hence quasi-isometrically embedded according to Theorem 3.2. □

We deduce:

Proposition 6.2 *The set $\mathrm{QIE}(\Gamma, G)$ is a domain of discontinuity for the action of $\mathrm{Out}(\Gamma)$ on $\mathcal{X}(\Gamma, G)$.*

Proof The group Γ is testable since it contains the subset $\mathcal{B}_S = S \cup \{ss' \mid s, s' \in S,\ s \neq s'\}$ which is testable according to Theorem 1.7. The proof follows from Theorem 1.2. □

We have the situation

$$\mathrm{QIE}(\Gamma, G) = S_\Gamma(\Gamma, G) = D_\Gamma(\Gamma, G) \subset \mathrm{AH}(\Gamma, G).$$

6.2 Primitive-stable representations of a free group

Let $\Gamma = F_k$ be a free group of rank $k \geqslant 2$, S be a minimal set of generators and

$$\mathcal{P} = \{\varphi(s) \mid \varphi \in \mathrm{Aut}(F_k),\ s \in S\}$$

be the set of *primitive* elements of F_k. The subset $\mathcal{P}$ is clearly invariant by automorphisms of F_k. Moreover, $\mathcal{P}$ contains the testable subset $\mathcal{B}_S$, therefore $\mathcal{P}$ is testable. We deduce from Theorem 1.2:

Proposition 6.3 *The set $S_{\mathcal{P}}(F_k, G)$ is a domain of discontinuity for the action of $\mathrm{Out}(F_k)$ on $\mathcal{X}(F_k, G)$.*

Moreover, for $G = \mathrm{PSL}_2(\mathbb{C})$, Minsky [22, Theorem 1.1] proved:

Theorem 6.4 *The set $S_{\mathcal{P}}(F_k, \mathrm{PSL}_2(\mathbb{C}))$ is a domain of discontinuity of $\mathcal{X}(F_k, \mathrm{PSL}_2(\mathbb{C}))$ for the action of $\mathrm{Out}(F_k)$ which is strictly larger than the set $\mathrm{QIE}(F_k, \mathrm{PSL}_2(\mathbb{C}))$ of quasi-isometrically embedded (i.e., convex cocompact) representations. Moreover, there are primitive-stable representations which are not discrete and faithful.*

Say that a geodesic and Gromov-hyperbolic metric space is a *visibility space* if every two distinct points in ∂X are the endpoints of some geodesic in X.

We know that primitive stable representations are $\mathcal{P}$-well-displacing according to Proposition 4.6. The converse is also true when $k = 2$ and X is a visibility space (see [16; 26] when $G = \mathrm{PSL}_2(\mathbb{C})$ and [25, Theorem 1.3] for the following general result):

Proposition 6.5 *Let F_2 be the free group of rank 2 and G be the isometry group of a Gromov-hyperbolic, geodesic and visibility metric space. A representation $\rho \in \mathcal{R}(F_2, G)$ is $\mathcal{P}$-stable if and only if ρ is $\mathcal{P}$-well-displacing*

We recap the inclusions between different $\mathrm{Out}(F_k)$-invariant subsets of $\mathcal{X}(F_k, \mathrm{PSL}_2(\mathbb{C}))$ and of $\mathcal{X}(F_2, G)$ for G as above:

$$S_{\mathcal{P}}(F_2, G) = D_{\mathcal{P}}(F_2, G), \quad S_{\mathcal{P}}(F_k, \mathrm{PSL}_2(\mathbb{C})) \not\subset \mathrm{AH}(F_k, \mathrm{PSL}_2(\mathbb{C})).$$

7 New examples

Let $G = \mathrm{Isom}(X)$ be the isometry group of a Gromov-hyperbolic and geodesic metric space X. Let Γ be a nonelementary hyperbolic group, let S be a finite set of generators of Γ.

In this section, we focus on special cases of A-stable representations: when A is either the subset of commutators or the derived subgroup of Γ.

7.1 Commutator-stable representations

Let $\mathcal{C} = \{[g, h] \mid g, h \in \Gamma\}$ denote the subset of commutators of Γ. The subset $\mathcal{C}$ is clearly invariant under automorphisms of Γ; hence we can consider the set of *commutator-stable representations*. We first show that primitive-stable and commutator-stable representations share common properties.

Theorem 1.3 *The set $S_{\mathcal{C}}(\Gamma, G)$ of conjugacy classes of commutator-stable representations is a domain of discontinuity of $\mathcal{X}(\Gamma, G)$ for the action of $\mathrm{Out}(\Gamma)$.*

Proof The proof is a direct combination of Proposition 5.13 and Theorem 1.2. □

However, the following theorem shows that unlike primitive-stable representations, a commutator-stable representation must be discrete and have finite kernel.

Theorem 1.5 *Let Y be a metric space and $\mathrm{Isom}(Y)$ denote its isometry group. If a representation $\rho \in \mathcal{R}(\Gamma, \mathrm{Isom}(Y))$ is commutator-well-displacing, then ρ has discrete image and finite kernel.*

We emphasize that there is no assumption on the metric space Y. Indeed, the proof of Theorem 1.5 only uses the Gromov-hyperbolicity of the Cayley graph $C_S(\Gamma)$. More precisely, we use the fact that the isometric action of Γ on $C_S(\Gamma)$ extends in a convergence action, as defined by Gehring and Martin [11], on the Gromov compactification $\overline{C_S(\Gamma)}$.

We first state results for G the isometry group of a Gromov-hyperbolic, geodesic and proper metric space X and use them when $X = C_S(\Gamma)$ in the end. Recall that an isometry $g \in G$ extends to a homeomorphism of the Gromov compactification $\overline{X}$, which will again be denoted by g.

Definition 7.1 Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of G and $a, b \in \partial X$. We say that $(g_n)_{n\in\mathbb{N}}$ is a *convergence sequence with base* (a, b) if the restriction of $(g_n)_{n\in\mathbb{N}}$ to $\overline{X} \setminus \{a\}$ converges to b uniformly on compact subsets.

A proof of the next theorem can be found in [29, Theorem 3A]:

Theorem 7.2 *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of G. One of the two following statements is true*:

- *The sequence $(g_n)_{n\in\mathbb{N}}$ is relatively compact.*
- *Up to a subsequence, there are $a, b \in \partial X$ such that $(g_n)_{n\in\mathbb{N}}$ is a convergence sequence with base (a, b).*

One can check that:

Proposition 7.3 *Let $a, b \in \partial X$ and $(g_n)_{n\in\mathbb{N}}$ be a convergence sequence with base (a, b)*:

- *The sequence $(g_n^{-1})_{n\in\mathbb{N}}$ is a convergence sequence with base (b, a).*
- *If $h : X \to X$ is a homeomorphism, then $(h \circ g_n \circ h^{-1})_{n\in\mathbb{N}}$ is a convergence sequence with base $(h(a), h(b))$.*
- *If $c, d \in \partial X$, $b \neq c$ and $(h_n)_{n\in\mathbb{N}}$ is a convergence sequence with base (c, d), then $(h_n g_n)_{n\in\mathbb{N}}$ is a convergence sequence with base (a, d).*

We will need the following lemmas (see [29, Corollary 2E] for a proof of the first one):

Lemma 7.4 *Let $a, b \in \partial X$ such that $a \neq b$. If $(g_n)_{n\in\mathbb{N}}$ is a convergence sequence with base (a, b), then g_n is hyperbolic for n large enough and*

$$g_n^- \to a \quad \text{and} \quad g_n^+ \to b,$$

where g_n^- and g_n^+ denote the distinct fixed points in ∂X of g_n.

We now take $X = C_S(\Gamma)$ which is a Gromov-hyperbolic, geodesic and proper metric space. Until the end of this section, we will deliberately confuse the elements of Γ with their images in $\mathrm{Isom}(C_S(\Gamma))$ so as not to complicate the notation.

Lemma 7.5 *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of elements of Γ which is a convergence sequence with base (a, b) where $a, b \in \partial C_S(\Gamma)$ and $a \neq b$. The sequence $(l_S(\gamma_n))_{n\in\mathbb{N}}$ is not bounded.*

Proof Let $(\gamma_n)_{n\in\mathbb{N}}$ be a convergence sequence with base (a, b) such that $a \neq b$ and l be a geodesic of $C_S(\Gamma)$ connecting a to b. According to Lemma 7.4, we can assume up to a subsequence, that γ_n is hyperbolic for all $n \in \mathbb{N}$. Lemma 2.4 gives us uniform constants $K' \geqslant 1, C' \geqslant 0$ such that γ_n has a (K', C') quasiaxis l_n connecting γ_n^- to γ_n^+. By contradiction, let us suppose that $(l_S(\gamma_n))_{n\in\mathbb{N}}$ is bounded. Let $x \in l$ and for all $n \in \mathbb{N}$, choose $x_n \in l_n$ to be a closest point projection on l_n, i.e., such that $d_S(x, x_n) = d_S(x, l_n)$. It follows that

$$\forall n \in \mathbb{N}, \quad d_S(x, \gamma_n(x)) \leqslant 2d_S(x, x_n) + d_S(x_n, \gamma_n(x_n)).$$

Combining Lemma 7.4 and the Morse lemma, we get that $(d_S(x, x_n))_{n\in\mathbb{N}}$ is a bounded sequence. Hence, the assumption that $\big(d_S(x_n, \gamma_n(x_n))\big)_{n\in\mathbb{N}} = (l_S(\gamma_n))_{n\in\mathbb{N}}$ is also bounded would contradict the fact that $(\gamma_n)_{n\in\mathbb{N}}$ is a convergence sequence. □

We introduce a final lemma:

Lemma 7.6 *Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements of Γ. There is some element $h\in\Gamma$ such that $(l_S([\gamma_n,h]))_{n\in\mathbb{N}}$ is not bounded.*

Proof Since Γ acts properly discontinuously on $C_S(\Gamma)$, the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is not relatively compact in $\mathrm{Isom}(C_S(\Gamma))$, hence according to Theorem 7.2, there are $\gamma^-,\gamma^+\in\partial C_S(\Gamma)$ such that up to subsequence, $(\gamma_n)_{n\in\mathbb{N}}$ is a convergence sequence with base (γ^-,γ^+). Let $h\in\Gamma$ with infinite order be such that $\{h^-,h^+\}$ is disjoint from $\{\gamma^-,\gamma^+\}$. According to Proposition 7.3, the sequence $(h\gamma_n^{-1}h^{-1})_{n\in\mathbb{N}}$ is a convergence sequence with base $(h(\gamma^+),h(\gamma^-))$, hence $([\gamma_n,h])_{n\in\mathbb{N}}$ is a convergence sequence with base $(h(\gamma^+),\gamma^+)$ where $\gamma^+\neq h(\gamma^+)$. We conclude using Lemma 7.5. □

We are now able to prove Theorem 1.5.

Proof of Theorem 1.5 We show the contraposition. Suppose that a representation $\rho\in\mathcal{R}(\Gamma,\mathrm{Isom}(Y))$ has not a finite kernel. According to Proposition 2.2, this implies that there is an infinite-order element $\gamma\in\Gamma$ such that $\rho(\gamma)=\mathrm{Id}_Y$. This implies that $\rho([\gamma^n,h])=\mathrm{Id}_Y$ for all $n\in\mathbb{N}$ and all $h\in\Gamma$. According to Lemma 7.6, there is $h\in\Gamma$ such that $(l_S([\gamma^n,h]))_{n\in\mathbb{N}}$ is not bounded whereas $l(\rho([\gamma^n,h]))=0$ for all $n\in\mathbb{N}$.

Suppose now that a representation $\rho\in\mathcal{R}(\Gamma,\mathrm{Isom}(Y))$ is not discrete, i.e, there is a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of elements of Γ such that $(\rho(\gamma_n))_{n\in\mathbb{N}}$ is a sequence of distinct elements of $\mathrm{Isom}(Y)$ and $\rho(\gamma_n)\to\mathrm{Id}_Y$. This implies that $\rho([\gamma_n,h])\to\mathrm{Id}_Y$ for all $h\in\Gamma$. Once more, according to Lemma 7.6, there is $h\in\Gamma$ such that $(l_S([\gamma_n,h]))_{n\in\mathbb{N}}$ is not bounded whereas $l(\rho([\gamma_n,h]))\to 0$.

In both cases, ρ cannot be $\mathcal{C}$-well-displacing. □

Going back to our assumption that G is the isometry group of a geodesic and Gromov-hyperbolic metric space, we have the following inclusions of $\mathrm{Out}(\Gamma)$-invariant subsets of $\mathcal{X}(\Gamma,G)$:

$$\mathrm{QIE}(\Gamma,G)\subset S_{\mathcal{C}}(\Gamma,G)\subset D_{\mathcal{C}}(\Gamma,G)\subset \mathrm{AH}(\Gamma,G).$$

7.2 Derived-stable representations

In this section, $G=\mathrm{Isom}(X)$ is the isometry group of a Gromov-hyperbolic and geodesic metric space X. Recall that Γ is a nonelementary hyperbolic group and let $\mathcal{D}=[\Gamma,\Gamma]$ denote the derived subgroup of Γ, i.e., the subgroup generated by all commutators of Γ. The subgroup $\mathcal{D}$ is also clearly invariant under automorphisms of Γ; hence we can consider the set of *derived-stable representations*.

Proposition 7.7 *The set $S_{\mathcal{D}}(\Gamma,G)$ of conjugacy classes of derived-stable representations is a domain of discontinuity of $\mathcal{X}(\Gamma,G)$ for the action of* $\mathrm{Out}(\Gamma)$.

Proof Identical to Theorem 1.3. □

The main objective of this section is to prove the following theorem.

Theorem 1.4 *A representation $\rho\in\mathcal{R}(\Gamma,G)$ is $[\Gamma,\Gamma]$-stable if and only if it is $[\Gamma,\Gamma]$-well-displacing.*

We first follow [10] to prove that if ρ is a $\mathcal{D}$-well-displacing representation, then the distance between elements of $\mathcal{D}$ in a Cayley graph are comparable to the distance between their images by an orbit map.

Definition 7.8 Let S be a finite set of generators for Γ, δ_S be a hyperbolicity constant of $C_S(\Gamma)$ and $u, v \in \Gamma$. We say that (u, v) is a ping-pong pair if

(1) $\min(|u|_S, |v|_S) \geqslant 100\delta_S$,

(2) $\langle u^{\pm 1}, v^{\pm 1}\rangle_e \leqslant \frac{1}{2}\min(|u|_S, |v|_S) - 20\delta_S$,

(3) $\langle u, u^{-1}\rangle_e \leqslant \frac{1}{2}|u|_S - 20\delta_S$ and $\langle v, v^{-1}\rangle_e \leqslant \frac{1}{2}|v|_S - 20\delta_S$,

where $|\cdot|_S = d_S(e, \cdot)$ denotes the word length with respect to S, and $\langle \cdot, \cdot \rangle_e$ denotes the Gromov product based at e.

A ping-pong pair generates a free subgroup of rank two (see [9] for a proof) and reciprocally, one can always find a ping-pong pair in a free subgroup of rank two. Recall that if $\gamma \in \Gamma$ has infinite order, then γ^- and γ^+ denote its distinct fixed points in $\partial C_S(\Gamma)$.

We need the two following lemmas from [10, Section 4].

Lemma 7.9 *If $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma$ are two elements of infinite order and $\{\gamma_1^-, \gamma_1^+\} \cap \{\gamma_2^-, \gamma_2^+\} = \varnothing$, then there is $N \in \mathbb{N}$ such that $(\gamma_1^N, \gamma_2\gamma_1^{-N}\gamma_2^{-1})$ is a ping-pong pair.*

Lemma 7.10 *If $u, v \in \Gamma$ is a ping-pong pair, then there is some $\alpha \geqslant 0$ such that*

$$|\gamma|_S \leqslant 3\max\{l_S(\gamma), l_S(u\gamma), l_S(v\gamma)\} + \alpha \quad \text{for all } \gamma \in \Gamma.$$

We deduce the next proposition:

Proposition 7.11 *Let H be a subgroup of Γ which contains a free subgroup of rank two. There are $\alpha \geqslant 0$ and $h_1, h_2 \in H$ such that*

$$|\gamma|_S \leqslant 3\max\{l_S(\gamma), l_S(h_1\gamma), l_S(h_2\gamma)\} + \alpha \quad \text{for all } \gamma \in \Gamma.$$

Proof Let $b_1, b_2 \in H$ be two elements of H generating a free subgroup of rank 2. We have that $\{b_1^-, b_1^+\} \cap \{b_2^-, b_2^+\} = \varnothing$, hence Lemma 7.9 gives an $N \in \mathbb{N}$ such that $(b_1^N, b_2 b_1^{-N} b_2^{-1})$ is a ping-pong pair. Take $h_1 = b_1^N$, $h_2 = b_2 b_1^{-N} b_2^{-1}$ and use Lemma 7.10. □

Since Γ is supposed to be nonelementary, every infinite normal subgroup is also nonelementary; see, e.g., [15, Theorem 12.2, Item (5)]. Following the proof of [10, Lemma 4.0.4], we deduce the following.

Proposition 7.12 *Let H be an infinite normal subgroup of Γ, let $\rho \in D_H(\Gamma, G)$ be an H-well-displacing representation and $x \in X$.*

There are two constants $K \geqslant 1$ and $C \geqslant 0$ such that for all $h \in H$,

$$\frac{1}{K}|h|_S - C \leqslant d(x, \rho(h)(x)) \leqslant K|h|_S + C.$$

Proof Consider an H-well-displacing representation $\rho \in \mathcal{R}(\Gamma, G)$: there are $J \geqslant 1$ and $B \geqslant 0$ such that

$$\forall h \in H, \quad \frac{1}{J} l_S(h) - B \leqslant l(\rho(h)) \leqslant J l_S(h) + B.$$

Specifically, for all $x \in X$ and for all $h \in H$, we have $d(x, \rho(h)((x)) \geqslant \frac{1}{J} l_S(h) - B$.

Since H contains a free group of rank 2, according to Proposition 7.11, there are elements $h_1, h_2 \in H$ and a constant $\alpha \geqslant 0$ such that

$$\forall h \in H, \quad \tfrac{1}{3}|h|_S - \tfrac{1}{3}\alpha \leqslant \max\{l_S(h), l_S(h_1 h), l_S(h_2 h)\}.$$

For all $x \in X$ and for all $h \in H$, we have

$$\begin{aligned} d(x, \rho(h)(x)) &\geqslant \max\{d(x, \rho(h)(x)), d(x, \rho(h_1 h)(x)), d(x, \rho(h_2 h)(x))\} \\ &\qquad - \max\{d(x, \rho(h_1)(x)), d(x, \rho(h_2)(x))\} \\ &\geqslant \max\{l(\rho(h)), l(\rho(h_1 h)), l(\rho(h_2 h))\} - \max\{d(x, \rho(h_1)(x)), d(x, \rho(h_2)(x))\} \\ &\geqslant \frac{1}{J}\max\{l_S(h), l_S(h_1 h), l_S(h_2 h)\} - B - \max\{d(x, \rho(h_1)(x)), d(x, \rho(h_2)(x))\} \\ &\geqslant \frac{1}{3J}|h|_S - \frac{\alpha}{3J} - B - \max\{d(x, \rho(h_1)(x)), d(x, \rho(h_2)(x))\}. \end{aligned}$$

Since $\tau_{\rho,x}$ is Lipschitz continuous (see Remark 3.1), we actually proved the existence of uniform constants $K \geqslant 1$ and $C \geqslant 0$ such that for all $h \in H$,

$$\frac{1}{K} d_S(e, h) - C \leqslant d(x, \rho(h)(x)) \leqslant K d_S(e, h) + C. \qquad \square$$

We now use material from [25, Section 5].

Let $\epsilon > 0$, $\gamma \in \Gamma$, $\rho \in \mathcal{R}(\Gamma, G)$ and $\tau_{\rho,x}$ an associated orbit map. We say that γ is an *ϵ-quasiloop for $\tau_{\rho,x}$* if

$$d(x, \rho(\gamma)(x)) \leqslant \epsilon |\gamma|_S.$$

Note that if γ is an ϵ-quasiloop for $\tau_{\rho,x}$, then γ^{-1} is also an ϵ-quasiloop for $\tau_{\rho,x}$. Recall that $\tau_{\rho,x}$ is κ-Lipschitz where κ is defined in Remark 3.1.

The following lemma is proved in [25, Lemma 5.14]. For the reader's convenience, we detail its proof.

Lemma 7.13 *Let $\epsilon > 0$ and $s \in S$ be a generator of Γ. If $w \in \Gamma$ is an ϵ-quasiloop such that $|w|_S \geqslant \frac{\kappa}{\epsilon} + 2$, then ws is a 2ϵ-quasiloop.*

Proof Let $w \in \Gamma$ be an ϵ-quasiloop for $\tau_{\rho,x}$ and $s \in S$ be a generator.

If $|ws|_S = |w|_S - 1$, then

$$\begin{aligned} d(x, \rho(ws)(x)) &\leqslant d(x, \rho(w)(x)) + d(\rho(w)(x), \rho(ws)(x)) \leqslant d(x, \rho(w)(x)) + d(x, \rho(s)(x)) \\ &\leqslant \epsilon|w|_S + \kappa \leqslant \epsilon|w|_S + (|w|_S - 2)\epsilon \leqslant 2\epsilon|w|_S - 2\epsilon = 2\epsilon|ws|_S. \end{aligned}$$

If $|ws|_S = |w|_S + 1$, the same calculations yield

$$d(x, \rho(ws)(x)) \leqslant 2\epsilon|w|_S - 2\epsilon \leqslant 2\epsilon|w|_S + 2\epsilon = 2\epsilon|ws|_S. \qquad \square$$

We also need the following proposition that we prove in Appendix B.

Proposition 7.14 *There are two infinite-order elements $\gamma_1, \gamma_2 \in \Gamma$ and a constant $\beta \geqslant 0$ with the following property: for all $w \in \Gamma$, there is $i \in \{1, 2\}$ with $|w|_S \leqslant |w\gamma_i w^{-1}|_S + \beta$.*

We now prove Theorem 1.4 by showing that a representation which is not $\mathcal{D}$-stable cannot be $\mathcal{D}$-well-displacing.

Proof of Theorem 1.4 First, recall that a $\mathcal{D}$-stable representation is $\mathcal{D}$-well-displacing according to Proposition 4.6.

Let $\rho \in \mathcal{R}(\Gamma, G)$ be a representation which is not $\mathcal{D}$-stable. If γ is an infinite-order element of Γ, let l_γ denote any geodesic of $C_S(\Gamma)$ with endpoints γ_- and γ_+. Since $\tau_{\rho,x}$ is κ-Lipschitz, our hypothesis is equivalent to the existence of a sequence $(h_n)_{n\in\mathbb{N}}$ of elements of infinite-order elements of $\mathcal{D}$ and two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of vertices of $C_S(\Gamma)$ on l_{h_n} such that for all $n \in \mathbb{N}$,

$$d(\rho(u_n)(x), \rho(v_n)(x)) \leqslant \frac{1}{n} d_S(u_n, v_n) - 1.$$

Let $(w_n)_{n\in\mathbb{N}}$ be the sequence of elements of Γ defined by $w_n = u_n^{-1} v_n$. The previous inequality yields $|w_n|_S = d_S(u_n, v_n) \geqslant n$ and $d(x, \rho(w_n)(x)) \leqslant \frac{1}{n}|w_n|_S$.

Hence for all $n \in \mathbb{N}$, w_n is an $\frac{1}{n}$-quasiloop for $\tau_{\rho,x}$.

Let $\gamma_1, \gamma_2 \in \Gamma$ and $\beta > 0$ be given by Proposition 7.14. Define $(i_n)_{n\in\mathbb{N}}$ to be a sequence taking its values in $\{1, 2\}$ such that for all $n \in \mathbb{N}$, $|w_n|_S < |w_n \gamma_{i_n} w_n^{-1}|_S + \beta$.

Let $\epsilon > 0$, define $M = \max\{|\gamma_1|_S, |\gamma_2|_S\}$ and $\epsilon' = 2^M \epsilon$.

According to Lemma 7.13, there is $n_\epsilon \in \mathbb{N}$ such that for all $n \geqslant n_\epsilon$, $w_n\gamma_{i_n}$ and $w_n^{-1}\gamma_{i_n}^{-1}$ are ϵ'-quasiloops for $\tau_{\rho,x}$. This implies

$$\begin{aligned}
d(x, \rho(w_n\gamma_{i_n}w_n^{-1}\gamma_{i_n}^{-1})(x)) &\leqslant d(x, \rho(w_n\gamma_{i_n})(x)) + d(\rho(w_n\gamma_{i_n})(x), \rho(w_n\gamma_{i_n}w_n^{-1}\gamma_{i_n}^{-1})(x)) \\
&\leqslant d(x, \rho(w_n\gamma_{i_n})(x)) + d(x, \rho(w_n^{-1}\gamma_{i_n}^{-1})(x)) \leqslant \epsilon'|w_n\gamma_{i_n}|_S + \epsilon'|w_n^{-1}\gamma_{i_n}^{-1}|_S \\
&\leqslant 2\epsilon'|w_n|_S + 2\epsilon'|\gamma_{i_n}|_S \leqslant 2\epsilon'|w_n|_S + 2\epsilon' M \leqslant 2\epsilon'(|w_n\gamma_{i_n}w_n^{-1}|_S + \beta) + 2\epsilon' M \\
&\leqslant 2\epsilon'|w_n\gamma_{i_n}w_n^{-1}|_S + 2\epsilon'(M+\beta) \leqslant 2\epsilon'|w_n\gamma_{i_n}w_n^{-1}\gamma_{i_n}^{-1}|_S + 2\epsilon'(2M+\beta).
\end{aligned}$$

Since ϵ (and hence ϵ') can be taken arbitrarily small, the last inequality proves that there can be no $K \geqslant 1$ and $C \geqslant 0$ such that for all h in $\mathcal{D}$,

$$\frac{1}{K} d_S(e, h) - C \leqslant d(x, \rho(h)(x)).$$

This means that ρ cannot be $\mathcal{D}$-well-displacing according to Proposition 7.12. □

We have the following inclusions of $\mathrm{Out}(\Gamma)$-invariant subsets of $\mathcal{X}(\Gamma, G)$:

$$\mathrm{QIE}(\Gamma, G) \subset S_{\mathcal{D}}(\Gamma, G) = D_{\mathcal{D}}(\Gamma, G) \subset \mathrm{AH}(\Gamma, G).$$

8 Representations in $\mathrm{PSL}_2(\mathbb{C})$

We now aim to prove Theorem 1.6 which uses previous material to give some new characterizations of convex cocompact representations in $\mathrm{PSL}_2(\mathbb{C})$.

When $G = \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$, Sullivan (see [27, Theorem A, Section 7]) proved that the set $\mathrm{CC}(\Gamma, G)$ of conjugacy classes of convex cocompact (i.e., quasiconvex) representations of a torsion-free

hyperbolic group is precisely the interior of the set $\mathrm{AH}(\Gamma, G)$ of conjugacy classes of discrete and faithful representations. Matsuzaki (see [21, Theorem 1]) generalized Sullivan's theorem when Γ has torsion:

Theorem 8.1 *Let Γ be a hyperbolic group. The set $\mathrm{CC}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ is precisely the interior of $\mathrm{AH}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ in $\mathcal{X}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$.*

We are now ready to prove:

Theorem 1.6 *Let Γ be a hyperbolic group and $\rho \in \mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ be a representation. Let $\mathcal{C}$ denote the subset of commutators of Γ and $\mathcal{D} = [\Gamma, \Gamma]$ denote the derived subgroup of Γ. The following are equivalent:*

- *The representation ρ is convex cocompact.*
- *The representation ρ is $\mathcal{C}$-stable.*
- *The representation ρ is $\mathcal{D}$-well-displacing.*

Proof The first equivalence comes from Theorem 8.1 and the fact that $S_{\mathcal{C}}(\Gamma, G)$ is an open set which naturally contains $\mathrm{CC}(\Gamma, G)$ and is contained in $\mathrm{AH}(\Gamma, G)$ according to Theorem 1.5. For the equivalence between convex cocompact and $\mathcal{D}$-well-displacing representations, recall that a convex cocompact representation is Γ-well-displacing according to Theorem 3.2, hence $\mathcal{D}$-well-displacing and conversely, a $\mathcal{D}$-well-displacing representation is $\mathcal{D}$-stable according to Theorem 1.4, hence $\mathcal{C}$-stable, hence convex cocompact. □

Recalling the geometric interpretation of A-stable representations due to Lee and Definition 4.12 we can deduce the following corollary:

Corollary 8.2 *Let Γ be a torsion-free nonelementary hyperbolic group and let $\mathcal{C}$ be the subset of commutators of Γ. A discrete and faithful representation in $\mathcal{R}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ is convex cocompact if and only if it is $\mathcal{C}$-convex cocompact.*

Appendix A A finite testable subset for $[F_k, F_k]$

Let F_k denote the free group of rank $k \geqslant 3$. We already know that its derived subgroup $[F_k, F_k]$ is testable (it is a consequence of Proposition 5.13). In this appendix, we show that the derived subgroup $[F_k, F_k]$ actually contains a finite testable subset.

Let $S = \{x_1, \dots, x_k\}$ be a finite and minimal set of generators and $\mathcal{B}_S = \{x_i, x_i x_j \mid x_i, x_j \in S,\ i \neq j\}$ be the subset of elements of S and product of distinct elements of S. We define the following finite subsets of the derived subgroup $[F_k, F_k]$:

- $\mathcal{W}_{1,S} = \mathcal{W}_1 = \{[b, b'] \mid b, b' \in \mathcal{B}_S\}$,
- $\mathcal{W}_{2,S} = \mathcal{W}_2 = \{[b, w] \mid b \in \mathcal{B}_S,\ w \in \mathcal{W}_1\}$,
- $\mathcal{W}_{3,S} = \mathcal{W}_3 = \{ww' \mid w, w' \in \mathcal{W}_1\}$,
- $\mathcal{W}_S = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$.

Proposition A.1 *The subset $\mathcal{W}_S$ is testable.*

In order to prove this proposition, we will need the following lemma.

Lemma A.2 *Let T be an $\mathbb{R}$-tree and $g, h \in \mathrm{Isom}(T)$ such that g is hyperbolic and h is elliptic. If $l_T([g,h]) = 0$, then $C_g \cap C_h$ is a geodesic segment of length at least $l_T(g)$.*

Proof We write $A_g = C_g$ for the axis of g on which g acts by translation and we recall that C_h is the set of fixed points of h. The commutator $[g,h] = ghg^{-1}h^{-1}$ is the product of two elliptic elements, namely ghg^{-1} and h^{-1} with set of fixed points $g(C_h)$ and C_h, respectively, according to Proposition 5.3.

The proof goes by contraposition: If $A_g \cap C_h = \varnothing$, then $g(C_h)$ and C_h are disjoint (see [8, 1.7]). If $A_g \cap C_h \neq \varnothing$ and $A_g \cap C_h$ is a geodesic segment of length strictly less than $l_T(g)$, then the fact that g acts by translation of length $l_T(g)$ on A_g yields that $A_g \cap g(C_h) \neq \varnothing$ and $A_g \cap C_h \cap g(C_h) = \varnothing$. Proposition 5.4 now implies that $C_h \cap g(C_h) = \varnothing$.

In both cases, Proposition 5.6 implies that $l_T(ghg^{-1}h^{-1}) > 0$. □

Proof of Proposition A.1 Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of $\mathrm{Aut}(F_k)$ that projects on a sequence of distinct elements of $\mathrm{Out}(F_k)$. Let (T, x, ρ) be a based F_k-tree induced by $(\varphi_n)_{n\in\mathbb{N}}$ in Theorem 5.9.

By contradiction, suppose that $\big(l_T(\varphi_n(w))\big)_{n\in\mathbb{N}}$ is bounded for all $w \in \mathcal{W}_S$. Corollary 5.10 implies that $l_T(w) = l(\rho(w)) = 0$ for all $w \in \mathcal{W}_S$. Since (T, ρ) is not a trivial F_k-tree, there is $b \in \mathcal{B}_S$ such that $l_T(b) > 0$ according to Proposition 5.7. We denote by $A_b = C_b$ the axis of b for its action on T, we recall that b acts on A_b by translation of length $l_T(b)$.

Let $w_1, w_2, w_3 \in \mathcal{W}_1$ be 3 distinct elements such that the subgroup $< w_i, w_j >$ generated by w_i and w_j is isomorphic to a free group of rank 2 for all $i \neq j$. We denote by $C_i = C_{w_i}$ the set of fixed points of w_i for its action on T.

Since $l_T(w_i) = l_T(w_j) = l_T(w_i w_j) = 0$ for all $i \neq j$ by hypothesis, Proposition 5.6 yields $C_i \cap C_j \neq \varnothing$ for all $i \neq j$. Moreover, $l_T([b, w_i]) = 0$, hence Lemma A.2 implies that $A_b \cap C_i$ is a geodesic segment of T of length at least $l_T(b)$ for all i. Proposition 5.4 now implies that $A_b \cap C_i \cap C_j \neq \varnothing$ for all $i \neq j$.

If $A_b \cap C_1 \cap C_2$ is a geodesic segment of positive length, we get a contradiction since (T, ρ) is a small F_k-tree and the subgroup $< w_1, w_2 >$ is free of rank 2. If $A_b \cap C_1 \cap C_2$ is just a point, then $A_b \cap C_3 \cap C_2$ or $A_b \cap C_3 \cap C_1$ must be a geodesic segment of positive length, contradicting once more the hypothesis that (T, ρ) is a small F_k-tree.

We conclude that for all sequences $(\phi_n)_{n\in\mathbb{N}}$ of distinct elements of $\mathrm{Out}(F_k)$, there is $w \in \mathcal{W}_S$ such that $\limsup_n l_S(\phi_n(w)) = \infty$. □

Appendix B A proof of Proposition 7.14

In this section, Γ denotes a hyperbolic group, S is a finite set of generators and $\delta > 0$ is a hyperbolicity constant for $C_S(\Gamma)$. Before proving Proposition 7.14, we state definitions and lemmas for X a δ-hyperbolic, geodesic and proper metric space and apply it to $X = C_S(\Gamma)$ at the end.

Let l be a geodesic of X and $x \in X$. Say that $p \in l$ is a *closest point projection of x on l* if $d(x,p) = d(x,l)$, i.e., for all $q \in l$, $d(x,p) \leqslant d(x,q)$. While such projections need not be unique, all closest point projections of x onto l lie within a 6δ neighborhood of each other (see [19, Proposition 3.3]).

The following lemma bounds the distance from a basepoint on l to a closest point projection in terms of the Gromov product (see, e.g., [4, Lemma 2.2]).

Lemma B.1 *There exists a constant $K_1 \geqslant 0$ depending only on δ with the following property. Let l be a geodesic of X with endpoints l_- and l_+ in ∂X, and let $o \in l$ be a basepoint on l. For all $x \in X$, if p is a closest point projection of x on l, then there is $l_\pm \in \{l_-, l_+\}$ such that*

$$d(o,p) \leqslant \langle x, l_\pm \rangle_o + K_1.$$

We first deduce:

Lemma B.2 *Let l be a geodesic of X with endpoints l_- and l_+ in ∂X. Let $o \in l$ be a basepoint on l. For all open neighborhoods U_- (resp. U_+) of l_- (resp. l_+) in $\overline{X}$, there is $R \geqslant 0$ such that closest point projections on l of elements of $X \setminus (U_- \cup U_+)$ are at distance at most R from o.*

Proof Let $x \in X \setminus (U_- \cup U_+)$. By definition of the topology on ∂X, there is $R' > 0$ such that $\langle l_-, x \rangle_o \leqslant R'$ and $\langle l_+, x \rangle_o \leqslant R'$. Denote by $p \in l$ be a closest point projection of x on l. Using Lemma B.1, we know that there is $l_\pm \in \{l_-, l_+\}$ such that

$$d(o,p) \leqslant \langle l_\pm, x \rangle_o + K_1 \leqslant R' + K_1.$$

We conclude by taking $R = R' + K_1$. □

As a consequence, one can prove:

Lemma B.3 *Let l_1 and l_2 be two geodesics of X with four distinct endpoints $l_1^+, l_1^-, l_2^+, l_2^-$ in ∂X. Let $o \in X$ be a basepoint in X. There is $R > 0$ such that for all $x \in X$, there is $i \in \{1,2\}$ such that closest point projections of x onto l_i are at distance at most R from o.*

Proof Since the Gromov compactification $\overline{X}$ is Hausdorff, we can find $U_1^+, U_1^-, U_2^+, U_2^-$ four disjoints associated neighborhoods in $\overline{X}$ of $l_1^+, l_1^-, l_2^+, l_2^-$, respectively. Let $x \in X$ and $i \in \{1,2\}$ such that $x \in X \setminus (U_i^+ \cup U_i^-)$. Denote by x_i (resp. o_i) a closest point projection of x (resp. o) on l_i. According to Lemma B.2, there is R_i such that $d(x_i, o_i) \leqslant R_i$. This implies that $d(x_i, o) \leqslant R_i + d(o_i, o)$. We conclude by taking $R = \max\{R_1 + d(o_1,o), R_2 + d(o_2,o)\}$. □

We will need this last classical lemma which highlights the "tree-like" geometry of Gromov-hyperbolic spaces (see, e.g., [19, Proposition 3.4] for a proof).

Lemma B.4 *There exist two constants $K_2, K_3 \geqslant 0$ depending only on δ with the following property. Let l be a geodesic of X and $x, y \in X$. Denote by p_x (resp. p_y) a closest point projection of x (resp. y) on l. If $d(p_x, p_y) \geqslant K_2$, then*

$$d(x,y) \geqslant d(x,p_x) + d(p_x,p_y) + d(p_y,y) - K_3.$$

Proposition 7.14 *There are two infinite-order elements $\gamma_1, \gamma_2 \in \Gamma$ and a constant $\beta \geqslant 0$ with the following property: for all $w \in \Gamma$, there is $i \in \{1,2\}$ with $|w|_S \leqslant |w\gamma_i w^{-1}|_S + \beta$.*

Proof Let γ be an infinite-order element of Γ and l be a geodesic of $C_S(\Gamma)$ joining γ^+ and γ^-. For $w \in \Gamma$, denote by p_w (resp. $p_{\gamma w}$) a closest point projection of w (resp. γw) on l. Notice that γp_w is a closest point projection of γw on $\gamma(l)$, which may be a geodesic in $C_S(\Gamma)$ different from l, but with the same endpoints as l. The Morse lemma implies the existence of a constant $K_4 \geqslant 0$ depending only on δ such that $d_S(\gamma p_w, p_{\gamma w}) \leqslant K_4$.

Let K_2 and K_3 be the constants defined as in Lemma B.4. Choose γ_1 and γ_2 two infinite-order elements of Γ with distinct fixed points and such that $l_S(\gamma_i) \geqslant K_2 + K_4$ for $i \in \{1,2\}$. Denote by l_i a geodesic of $C_S(\Gamma)$ joining γ_i^+ and γ_i^-.

According to Lemma B.3, there is a constant $R > 0$ such that for all $w \in \Gamma$, there is $i \in \{1,2\}$ such that closest point projections of w^{-1} on l_i lie at distance at most R from e.

We want to compare

$$|w\gamma_i w^{-1}|_S = d_S(e, w\gamma_i w^{-1}) = d_S(w^{-1}, \gamma_i w^{-1}) \quad \text{and} \quad |w|_S = d_S(e,w) = d_S(e, w^{-1}).$$

In order to lighten the notation, we now replace w by w^{-1} and write γ and l instead of γ_i and l_i. Recall that p_w (resp. $p_{\gamma w}$) denotes a closest point projection of w (resp. γw) on l and that $d_S(e, p_w) \leqslant R$.

First, note that $d_S(p_w, p_{\gamma w}) \geqslant d_S(p_w, \gamma p_w) - K_4 \geqslant l_S(\gamma) - K_4 \geqslant K_2$. Therefore, we can apply Lemma B.4 and we get

$$\begin{aligned} d_S(w, \gamma w) &\geqslant d_S(w, p_w) + d_S(p_w, p_{\gamma w}) + d_S(p_{\gamma w}, \gamma w) - K_3 \\ &\geqslant d_S(w,e) - d_S(e, p_w) + d_S(\gamma w, \gamma) - d_S(\gamma, p_{\gamma w}) - K_3 \\ &\geqslant d_S(w,e) - R + d_S(w,e) - d_S(\gamma, \gamma p_w) - K_3 - K_4 \geqslant 2d_S(w,e) - 2R - K_3 - K_4 \\ &\geqslant d_S(w,e) - 2R - K_3 - K_4. \end{aligned}$$

This completes the proof by taking $\beta = 2R + K_3 + K_4$. □

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