AGT

Algebraic & Geometric Topology

Volume 26 (2026)

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Algebraic & Geometric Topology **26**:7 (2026) 2589–2633
DOI: 10.2140/agt.2026.26.2589
Published: September 1, 2026

Spaces of nonresultant systems of bounded multiplicity with real coefficients

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For each pair (m,n) of positive integers with $(m,n)\neq(1,1)$ and an arbitrary field $\mathbb{F}$ with algebraic closure $\overline{\mathbb{F}}$, let $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ denote the space of m-tuples $(f_1(z),\dots,f_m(z))\in\mathbb{F}[z]^m$ of monic polynomials with coefficients in $\mathbb{F}$ of the same degree d such that the polynomials $\{f_k(z)\}_{k=1}^m$ have no common root in $\overline{\mathbb{F}}$ of multiplicity $\geq n$. These spaces $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ were first defined and studied by B. Farb and J. Wolfson as generalizations of spaces first studied by Arnold, Vassiliev and Segal and others in several different contexts. In a previous paper we determined explicitly the homotopy type of this space when $\mathbb{F}=\mathbb{C}$. Here we investigate this space when $\mathbb{F}=\mathbb{R}$.

1 Introduction

Motivation We build on the works of V. I. Arnold [1], V. Vassiliev [22], G. Segal [17], and B. Farb and J. Wolfson [4].

Arnold considered the space $\mathrm{SP}_n^d(\mathbb{C})$ of complex-coefficient monic polynomials of degree d without roots of multiplicity $\geq n$, which plays a role in the theory of singularities. For example, if $n=2$, this space is the same as the space of complex monic polynomials of degree d without repeated roots, and this is homotopy equivalent to the Eilenberg– Mac Lane space $K(\mathrm{Br}(d),1)$, where $\mathrm{Br}(d)$ denotes the Artin braid group of d strings. Arnold [1] computed the homology of these spaces and established their homology stability. His results were much extended and generalized by Vassiliev [22].

Analogous results were discovered by Segal [17] in a different context inspired by control theory. Segal considered the space $\mathrm{Hol}_d^*(S^2,\mathbb{C}\mathrm{P}^{m-1})$ of basepoint-preserving holomorphic maps of degree d from the Riemann sphere S^2 to the $(m-1)$-dimensional complex projective space $\mathbb{C}\mathrm{P}^{m-1}$, and its inclusion into the space $\mathrm{Map}_d^*(S^2,\mathbb{C}\mathrm{P}^{m-1})=\Omega_d^2\mathbb{C}\mathrm{P}^{m-1}$ of corresponding basepoint-preserving continuous maps. Intuitive considerations based on Morse theory suggest that the homotopy type of the first space should approximate that of the second space more and more closely as the degree d increases. Segal proved that this is true by observing that the space $\mathrm{Hol}_d^*(S^2,\mathbb{C}\mathrm{P}^{m-1})$ can be identified with the space of m-tuples $(f_1(z),\dots,f_m(z))\in\mathbb{C}[z]^m$ of monic polynomials of the same degree d without common roots. He defined a stabilization map $\mathrm{Hol}_d^*(S^2,\mathbb{C}\mathrm{P}^{m-1})\to\mathrm{Hol}_{d+1}^*(S^2,\mathbb{C}\mathrm{P}^{m-1})$ and proved that the induced maps on homotopy groups are isomorphisms up to some dimension increasing with d. Using a different technique he also proved that there is a homotopy equivalence $S:\varinjlim\mathrm{Hol}_d^*(S^2,\mathbb{C}\mathrm{P}^{m-1})\xrightarrow{\simeq}\Omega_0^2\mathbb{C}\mathrm{P}^{m-1}$

MSC2020: primary 55P15; secondary 55P35, 55R80.
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defined by a "scanning of particles", and that this equivalence is homotopic to the inclusion of the space of all holomorphic maps into the space of all continuous maps.

Inspired by the classical theory of resultants and the algebraic nature of Arnold and Segal's arguments, Farb and Wolfson [4; 5] defined algebraic varieties $\mathrm{Poly}_n^{d,m}(\mathbb{F})$, which generalize the spaces considered by Arnold and Segal as above. These varieties $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ are defined as follows.

Definition 1.1 For a field $\mathbb{F}$ with its algebraic closure $\bar{\mathbb{F}}$, let $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ denote the space of m-tuples $(f_1(z),\dots,f_m(z)) \in \mathbb{F}[z]^m$ of monic $\mathbb{F}$-coefficient polynomials of the same degree d with no common root in $\overline{\mathbb{F}}$ of multiplicity $\geq n$.

Note that the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$ is an affine variety defined by systems of polynomial equations with integer coefficients by the classical theory of resultants. Thus this variety can be defined over $\mathbb{Z}$ and over any field $\mathbb{F}$.

Farb and Wolfson computed various algebraic and geometric invariants of these varieties (such as the number of points for a finite field $\mathbb{F}_q$, étale cohomology, etc.) and found that the varieties $\mathrm{Poly}_n^{dn,m}(\mathbb{F})$ and $\mathrm{Poly}_1^{d,mn}(\mathbb{F})$ are identical for the case $\mathbb{F} = \mathbb{F}_q$. They conjectured that these varieties are algebraically isomorphic. Although this conjecture was disproved in [19], analogous results (e.g., (2-15) and (2-21)) hold in the homotopy category. So this analogy between the algebraic and the topological situation remains intriguing.

From the topological point of view, there has been a lot of work on the homotopy type of $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ for $\mathbb{F} = \mathbb{C}$. For example, if $\mathbb{F} = \mathbb{C}$, $\mathrm{Poly}_n^{d,1}(\mathbb{C}) = \mathrm{SP}_n^d(\mathbb{C})$ and $\mathrm{Poly}_1^{d,m}(\mathbb{C})$ can be identified with the space $\mathrm{Hol}_d^*(S^2, \mathbb{C}\mathrm{P}^{m-1})$. The space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$ has been studied by Vassiliev [22], Segal [17], M. Guest and the present authors [7], and by the present authors [11].

Here we study the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$. Note that the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ is already known for $mn = 2$. Indeed, if $(m,n) = (2,1)$, the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ was determined in [17, Propositions 7.1 and 7.2]. The case $(m,n) = (1,2)$ can be easily determined by using [16].[1] Note that the case $n = 1$ (with $m \geq 3$) was also studied in [9].

To explain our main result, we need the following definition:

Definition 1.2 Let $f : X \to Y$ be a based map between based spaces X and Y. Then the map f is called *a homotopy equivalence through dimension N* (resp. *a homology equivalence through dimension N*) if the induced homomorphism

$$f_* : \pi_k(X) \to \pi_k(Y) \quad (\text{resp. } f_* : H_k(X;\mathbb{Z}) \to H_k(Y;\mathbb{Z}))$$

is an isomorphism for any integer $k \leq N$.

Our main goal is to investigate the homotopy type of the space $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ when $mn \geq 3$. In particular, we prove that an Atiyah–Jones–Segal type result holds for this space and determine its stable homotopy type explicitly. Then our main results can be summarized as follows:

[1] Since this does not appear to be stated anywhere, we consider this case in the Appendix; see also Remark 1.4(ii).

Theorem 1.3 (Theorems 2.7 and 2.11, and Corollary 2.12) *Let $m, n, d \geq 1$ be positive integers such that $(m,n) \neq (1,1)$ with $d \geq n$,[2] and let $D(d;m,n)$ denote the positive integer defined by*

$$D(d;m,n) = (mn-2)(\lfloor d/n \rfloor + 1) - 1, \tag{1-1}$$

where $\lfloor x \rfloor$ denotes the integer part of a real number x.

(i) *The natural map (defined by (2-6))*

$$i^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to (\Omega^2_d \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} \simeq \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}$$

is a homotopy equivalence through dimension $D(d;m,n)$ if $mn \geq 4$, and a homology equivalence through dimension $D(d;m,n)$ if $mn = 3$.

(ii) *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\begin{aligned}\mathrm{Poly}^{d,m}_n(\mathbb{R}) &\simeq_s \mathrm{Poly}^{\lfloor d/2 \rfloor, m}_n(\mathbb{C}) \vee B^{d,m}_n \vee \mathrm{Q}^{d,m}_n(\mathbb{R}) \\ &\simeq_s \Big(\bigvee_{i=1}^{\lfloor d/n \rfloor} S^{(mn-2)i} \Big) \vee \Big(\bigvee_{i \geq 0, j \geq 1, i+2j \leq \lfloor d/n \rfloor} \Sigma^{(mn-2)(i+2j)} D_j \Big),\end{aligned}$$

where $B^{d,m}_n$ and D_j denote the spaces defined by (2-20) and (4-6), respectively.

Remark 1.4 (i) If $mn \geq 4$ then the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ is simply connected. So in this case homotopy stability follows from homology stability (see Lemma 6.3 and Theorem 1.3). On the other hand, when $mn = 3$ with $d \geq n$, this space is not simply connected and $\pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) = \mathbb{Z}$ (see Lemma 6.3). If $mn = 3$ with $d \geq n$, then, since homology stability holds and the map $i^{d,m}_{n,\mathbb{R}}$ induces an isomorphism on fundamental groups (Theorem 1.3 and Corollary 8.1), it is reasonable to expect that the corresponding homotopy stability also holds. This is proved in [14] for $(m,n) = (3,1)$. [3]

(ii) Next consider the case $mn = 2$. If $(m,n) = (2,1)$, Segal [17, Proposition 1.4] already proved that the space $\mathrm{Poly}^{d,2}_1(\mathbb{R})$ has $d+1$ path components and that homotopy stability holds for each component. If $(m,n) = (1,2)$, the space $\mathrm{Poly}^{d,1}_2(\mathbb{R})$ has $(\lfloor d/2 \rfloor + 1)$ path components, and homology stability holds for each component, but homotopy stability does not hold (this will be proved in Theorem A.3). □

The organization of this paper is as follows. In Section 2 we recall several definitions, notations, and known results. After that we give the main results (Theorems 2.7 and 2.11). In Section 3 we investigate the homotopy type of the spaces $(\Omega^2_d \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$ and $\mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+)$. In Section 4 we construct the Vassiliev spectral sequence converging to the homology of $\mathrm{Poly}^{d,m}_n(\mathbb{R})$, and compute its E^1-terms. In Section 5, we construct pairing maps which induce loop sums and stabilization maps for the spaces $\mathrm{Poly}^{d,m}_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathrm{Q}^{d,m}_n(\mathbb{R})$, and use them to prove Theorem 5.6. In Section 6 we prove the homology stability theorem for the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ (Theorem 6.2). In Section 7 we consider configuration space models for $\mathrm{Poly}^{d,m}_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\mathrm{Q}^{d,m}_n(\mathbb{R})$, define corresponding stabilization maps and use

[2] If $d < n$, the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ is contractible by (2-12), so we assume that $d \geq n$.

[3] The remaining case $(m,n) = (1,3)$ is still open.

them to prove the stable theorems (Theorems 7.5 and 7.9). In Section 8 we study the homotopy type of $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ and give the proofs of our main results (Theorems 2.7 and 2.11) and their corollaries (Corollaries 2.9, 2.10, 2.13 and 2.14). Finally in the Appendix we deal with the case $(m,n)=(1,2)$ for the sake of completeness.

2 Basic definitions and main results

Before describing our main results precisely, we shall recall several definitions and facts needed to give the precise statements of the main results.

Basic definitions and notations For connected spaces X and Y, let $\mathrm{Map}(X,Y)$ (resp. $\mathrm{Map}^*(X,Y)$) denote the space consisting of all continuous maps (resp. basepoint-preserving continuous maps) from X to Y with the compact–open topology. For each element $D\in\pi_0(\mathrm{Map}^*(X,Y))$, let $\mathrm{Map}^*_D(X,Y)$ denote the path component of $\mathrm{Map}^*(X,Y)$ which corresponds to D. When X and Y are complex manifolds, let $\mathrm{Hol}^*_D(X,Y)\subset\mathrm{Map}^*_D(X,Y)$ denote the subspace of all based holomorphic maps from X to Y.

Let $\mathbb{R}\mathrm{P}^N$ (resp. $\mathbb{C}\mathrm{P}^N$) denote the N-dimensional real projective space (resp. N-dimensional complex projective space). Note that the space $\mathrm{Map}^*(S^1,\mathbb{R}\mathrm{P}^N)=\Omega\mathbb{R}\mathrm{P}^N$ has two path components $\Omega_\epsilon\mathbb{R}\mathrm{P}^N$ ($\epsilon\in\{0,1\}$) for $N\geq 2$, where the space $\Omega_0\mathbb{R}\mathrm{P}^N$ is the path component of null homotopic maps and $\Omega_1\mathbb{R}\mathrm{P}^N$ is the path component which contains the natural inclusion of the bottom cell S^1 into $\mathbb{R}\mathrm{P}^N$. Similarly, for each integer $d\in\mathbb{Z}=\pi_0(\mathrm{Map}^*(S^2,\mathbb{C}\mathrm{P}^N))$, let $\Omega^2_d\mathbb{C}\mathrm{P}^N=\mathrm{Map}^*_d(S^2,\mathbb{C}\mathrm{P}^N)$ denote the path component of $\Omega^2\mathbb{C}\mathrm{P}^N$ of basepoint-preserving maps from S^2 to $\mathbb{C}\mathrm{P}^N$ of degree d.

Definition 2.1 Let $\mathbb{N}$ be the set of all positive integers. From now on, let $d\in\mathbb{N}$, let $(m,n)\in\mathbb{N}^2$ be a pair of positive integers such that $(m,n)\neq(1,1)$, and let $\mathbb{F}$ be a field with its algebraic closure $\overline{\mathbb{F}}$.

(i) Let $\mathrm{P}_d(\mathbb{F})$ denote the space of all $\mathbb{F}$-coefficient monic polynomials

$$f(z)=z^d+a_1z^{d-1}+\cdots+a_{d-1}z+a_d\in\mathbb{F}[z]$$

of degree d. Note that there is a natural homeomorphism $\mathrm{P}_d(\mathbb{F})\cong\mathbb{F}^d$ given by

$$f(z)=z^d+\sum_{k=1}^{d}a_kz^{d-k}\mapsto(a_1,\dots,a_d). \tag{2-1}$$

(ii) For each m-tuple $D=(d_1,\dots,d_m)\in\mathbb{N}^m$ of positive integers, we denote by $\mathrm{Poly}_n^{D,m}(\mathbb{F})=\mathrm{Poly}_n^{(d_1,\dots,d_m),m}(\mathbb{F})$ the space of all m-tuples $(f_1(z),\dots,f_m(z))\in\mathrm{P}_{d_1}(\mathbb{F})\times\mathrm{P}_{d_2}(\mathbb{F})\times\cdots\times\mathrm{P}_{d_m}(\mathbb{F})$ of monic polynomials such that the polynomials $\{f_j(z)\}_{j=1}^m$ have no common root in $\overline{\mathbb{F}}$ of multiplicity $\geq n$. We call the space $\mathrm{Poly}_n^{D,m}(\mathbb{F})$ *the space of nonresultant systems of bounded multiplicity n with coefficients in $\mathbb{F}$*.[4]

[4]Recall that the classical resultant of a system of polynomials vanishes if and only if they have a common solution in an algebraically closed field containing the coefficients. Systems which have no common roots are called "nonresultant". This is the intuition behind our choice of the term "nonresultant system of bounded multiplicity."

In particular, when $D_m = (d, d, \dots, d) \in \mathbb{N}^m$ (m times), we write

$$\mathrm{Poly}_n^{d,m}(\mathbb{F}) = \mathrm{Poly}_n^{D_m,m}(\mathbb{F}) = \mathrm{Poly}_n^{(d,d,\dots,d),m}(\mathbb{F}). \tag{2-2}$$

Definition 2.2 From now on, let us suppose that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(i) Let $\mathrm{Q}_n^{d,m}(\mathbb{K})$ denote the space of all m-tuples $(f_1(z), \dots, f_m(z)) \in \mathrm{P}_d(\mathbb{K})^m$ of $\mathbb{K}$-coefficient monic polynomials of the same degree d such that the polynomials $\{f_j(z)\}_{j=1}^m$ have no common *real* root of multiplicity $\geq n$ (but may have complex common roots of any multiplicity).

Note that there are the following two inclusions:

$$\mathrm{Poly}_n^{d,m}(\mathbb{C}) \xleftarrow[\supset]{\iota_{n,\mathbb{C}}^{d,m}} \mathrm{Poly}_n^{d,m}(\mathbb{R}) \xrightarrow[\subset]{\iota_{n,\mathbb{R}}^{d,m}} \mathrm{Q}_n^{d,m}(\mathbb{R}). \tag{2-3}$$

(ii) For a monic polynomial $f(z) \in \mathrm{P}_d(\mathbb{K})$, we define the n-tuple $F_n(f) = F_n(f)(z) \in \mathrm{P}_d(\mathbb{K})^n$ of the monic polynomials of the same degree d by

$$F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \dots, f(z) + f^{(n-1)}(z)). \tag{2-4}$$

Note that $f(z) \in \mathrm{P}_d(\mathbb{K})$ is not divisible by $(z-\alpha)^n$ for some $\alpha \in \mathbb{K}$ if and only if $F_n(f)(\alpha) \neq \mathbf{0}_n$, where we set $\mathbf{0}_n = (0, 0, \dots, 0) \in \mathbb{K}^n$.

(iii) When $\mathbb{K} = \mathbb{C}$, by identifying S^2 with $\mathbb{C} \cup \infty$ we define *a natural map*

$$i_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{C}) \to \Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1} \tag{2-5}$$

by

$$i_{n,\mathbb{C}}^{d,m}(\mathrm{f})(\alpha) = \begin{cases} [F_n(f_1)(\alpha) : F_n(f_2)(\alpha) : \cdots : F_n(f_m)(\alpha)] & \text{if } \alpha \in \mathbb{C}, \\ [1 : 1 : \cdots : 1] & \text{if } \alpha = \infty, \end{cases}$$

for $\mathrm{f} = (f_1(z), \dots, f_m(z)) \in \mathrm{Poly}_n^{d,m}(\mathbb{C})$ and $\alpha \in \mathbb{C} \cup \infty = S^2$, where we choose the points ∞ and $* = [1 : 1 : \cdots : 1]$ as the basepoints of S^2 and $\mathbb{C}\mathrm{P}^{mn-1}$, respectively.

Definition 2.3 Let $\mathbb{Z}_2 = \{\pm 1\}$ denote the (multiplicative) cyclic group of order 2. From now on, we will regard the two spaces $S^2 = \mathbb{C} \cup \infty$ and $\mathbb{C}\mathrm{P}^{mn-1}$ as $\mathbb{Z}_2$-spaces with actions induced by the complex conjugation on $\mathbb{C}$.

(i) Let $(\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}$ denote the space consisting of all $\mathbb{Z}_2$-equivariant based maps $f : (S^2, \infty) \to (\mathbb{C}\mathrm{P}^{mn-1}, *)$.

(ii) Since $\mathrm{Poly}_n^{d,m}(\mathbb{R}) \subset \mathrm{Poly}_n^{d,m}(\mathbb{C})$ and $i_{n,\mathbb{C}}^{d,m}(\mathrm{Poly}_n^{d,m}(\mathbb{R})) \subset (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}$, we also define *the natural map*

$$i_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} \tag{2-6}$$

by the restriction

$$i_{n,\mathbb{R}}^{d,m} = i_{n,\mathbb{C}}^{d,m} | \mathrm{Poly}_n^{d,m}(\mathbb{R}) : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}.$$

(iii) When $mn \geq 3$, by identifying S^1 with $\mathbb{R} \cup \infty$ we define *a natural map*

$$i_n^{d,m} : \mathrm{Q}_n^{d,m}(\mathbb{R}) \to \Omega_{[d]_2}\mathbb{R}\mathrm{P}^{mn-1} \simeq \Omega S^{mn-1} \tag{2-7}$$

by

$$i_n^{d,m}(\mathrm{f})(\alpha) = \begin{cases} [F_n(f_1)(\alpha) : F_n(f_2)(\alpha) : \cdots : F_n(f_m)(\alpha)] & \text{if } \alpha \in \mathbb{R}, \\ [1 : 1 : \cdots : 1] & \text{if } \alpha = \infty, \end{cases}$$

for $\mathrm{f} = (f_1(z), \ldots, f_m(z)) \in \mathrm{Q}_n^{d,m}(\mathbb{R})$ and $\alpha \in \mathbb{R} \cup \infty = S^1$, where $[d]_2 \in \{0, 1\}$ is the integer d mod 2 and we choose the points ∞ and $* = [1 : 1 : \cdots : 1]$ as the basepoints of $S^1 = \mathbb{R} \cup \infty$ and $\mathbb{R}\mathrm{P}^{mn-1}$, respectively.

(iv) For a positive integer $n \geq 3$, we define *the jet embedding*

$$j_n^d : \mathrm{Poly}_n^{d,1}(\mathbb{R}) \to \mathrm{Poly}_1^{d,n}(\mathbb{R}) \tag{2-8}$$

by

$$j_n^d(f(z)) = (f(z), f(z) + f'(z), f(z) + f''(z), \ldots, f(z) + f^{(n-1)}(z))$$

for $f(z) \in \mathrm{Poly}_n^{d,1}(\mathbb{R})$.

Definition 2.4 (i) Let G be a group and $f : X \to Y$ be a G-equivariant basepoint-preserving map between G-spaces X and Y. Then the map f is called *a G-equivariant homotopy equivalence through dimension N* (resp. a *G-equivariant homology equivalence through dimension N*) if the restriction map

$$f^H = f|X^H : X^H \to Y^H$$

is a homotopy equivalence through dimension N (resp. a homology equivalence through dimension N) for any subgroup $H \subset G$, where W^H denotes the H-fixed subspace of a G-space W given by

$$W^H = \{x \in W : h \cdot x = x \text{ for any } h \in H\}. \tag{2-9}$$

(ii) Recall from [8] that there is a homotopy equivalence for $N \geq 2$ obtained by using the reduced product:

$$\Omega S^{N+1} \simeq S^N \cup e^{2N} \cup e^{3N} \cup \cdots \cup e^{(k-1)N} \cup e^{kN} \cup e^{(k+1)N} \cup \cdots. \tag{2-10}$$

We denote by $J_k(S^N)$ the kN-skeleton of ΩS^{N+1}, i.e.,

$$J_k(S^N) = S^N \cup e^{2N} \cup e^{3N} \cup \cdots \cup e^{(k-1)N} \cup e^{kN}, \tag{2-11}$$

which is usually called *the k-th stage James filtration* of ΩS^{N+1}.

Some known results Note that there are homeomorphisms

$$\mathrm{Poly}_n^{d,m}(\mathbb{K}) \cong \mathbb{K}^{dm} \quad (\mathbb{K} = \mathbb{R}, \mathbb{C}) \qquad \text{and} \qquad \mathrm{Q}_n^{d,m}(\mathbb{R}) \cong \mathbb{R}^{dm} \quad \text{if } d < n. \tag{2-12}$$

Thus, these spaces are contractible if $d < n$. From now on, we always assume that d is a positive integer such that

$$d \geq n. \tag{2-13}$$

Now recall the following two known results.

Theorem 2.5 [11] *Let $m,n\geq 1$ be positive integers such that $mn\geq 3$.*

(i) *The natural map*

$$i^{d,m}_{n,\mathbb{C}}:\mathrm{Poly}^{d,m}_n(\mathbb{C})\to\Omega^2_d\mathbb{C}\mathrm{P}^{mn-1}\simeq\Omega^2S^{2mn-1}$$

is a homotopy equivalence through dimension $D(d;m,n;\mathbb{C})$, where $\lfloor x\rfloor$ denotes the integer part of a real number x and the positive integer $D(d;m,n;\mathbb{C})$ is defined by

$$D(d;m,n;\mathbb{C})=(2mn-3)(\lfloor d/n\rfloor+1)-1. \tag{2-14}$$

(ii) *There is a homotopy equivalence*

$$\mathrm{Poly}^{d,m}_n(\mathbb{C})\simeq\mathrm{Poly}^{\lfloor d/n\rfloor,mn}_1(\mathbb{C}). \tag{2-15}$$

(iii) *There is a stable homotopy equivalence*

$$\mathrm{Poly}^{d,m}_n(\mathbb{C})\simeq_s\bigvee_{j=1}^{\lfloor d/n\rfloor}\Sigma^{2(mn-2)j}D_j, \tag{2-16}$$

where Σ^j and $D_j=D_j(S^1)=F(\mathbb{C},j)_+\wedge_{S_j}(S^1)^{\wedge j}$ denote the j-fold reduced suspension and the space defined by (4-6), respectively. □

Theorem 2.6 [13; 22; 23] (i) *The natural map*

$$i^{d,m}_n:\mathrm{Q}^{d,m}_n(\mathbb{R})\to\Omega_{[d]_2}\mathbb{R}\mathrm{P}^{mn-1}\simeq\Omega S^{mn-1}$$

is a homotopy equivalence through dimension $D(d;m,n)$ if $mn\geq 4$ and a homology equivalence through dimension $D(d;m,n)$ if $mn=3$,[5] *where the positive integer $D(d;m,n)$ is defined by*

$$D(d;m,n)=(mn-2)(\lfloor d/n\rfloor+1)-1\quad(\text{as in (1-1)}). \tag{2-17}$$

(ii) *If $mn\geq 4$, there is a homotopy equivalence*

$$\mathrm{Q}^{d,m}_n(\mathbb{R})\simeq J_{\lfloor d/n\rfloor}(S^{mn-2}). \tag{2-18}$$

□

The main results The main purpose of this paper is to investigate the homotopy type of the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ and to prove for it an Atiyah–Jones–Segal type theorem.

First, we consider the unstable homotopy type of the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$. For this purpose, the key observation will be the homotopy equivalence (see Lemma 3.3 below)

$$(\Omega^2_d\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}\simeq\Omega^2S^{2N+1}\times\Omega S^N\qquad\text{for } N\geq 2. \tag{2-19}$$

We will use it to prove the following result, closely related to Theorems 2.5 and 2.6.

[5] If $mn\geq 4$, the assertion (i) follows from [13]. The assertion (i) follows from [20] and [22, Theorem 3 (page 88)] if $(m,n)=(1,3)$, and it follows from [23] if $(m,n)=(3,1)$.

Theorem 2.7 *The natural map*

$$i^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to (\Omega^2_d \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} \simeq \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}$$

is a homotopy equivalence through dimension $D(d;m,n)$ if $mn \geq 4$ and it is a homology equivalence through dimension $D(d;m,n)$ if $mn = 3$.

Since $\mathrm{Poly}^{d,m}_n(\mathbb{C})^{\mathbb{Z}_2} = \mathrm{Poly}^{d,m}_n(\mathbb{R})$ and $(i^{d,m}_{n,\mathbb{C}})^{\mathbb{Z}_2} = i^{d,m}_{n,\mathbb{R}}$, by using Theorems 2.5 and 2.7, we also obtain the following:

Corollary 2.8 *The natural map*

$$i^{d,m}_{n,\mathbb{C}} : \mathrm{Poly}^{d,m}_n(\mathbb{C}) \to \Omega^2_d \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1}$$

is a $\mathbb{Z}_2$-equivariant homotopy equivalence through dimension $D(d;m,n)$ if $mn \geq 4$, and a $\mathbb{Z}_2$-equivariant homology equivalence through dimension $D(d;m,n)$ if $mn = 3$. □

Corollary 2.9 *Let $mn \geq 3$, let $I^{d,m}_n : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}$ denote the map defined by (8-5), and let q_k denote the projection to the k-th factor, for $k = 1, 2$, given by (8-2).*

(i) *The map $q_1 \circ I^{d,m}_n : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \Omega^2 S^{2mn-1}$ induces an epimorphism on the homology group $H_k(\,;\mathbb{Z})$ for any $k \leq D(\lfloor d/2 \rfloor; m, n; \mathbb{C})$.*

(ii) *The map $q_2 \circ I^{d,m}_n : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \Omega S^{mn-1}$ induces an epimorphism on the homology group $H_k(\,;\mathbb{Z})$ for any $k \leq D(d;m,n)$.*

Corollary 2.10 *The jet embedding (defined by (2-8))*

$$j^d_n : \mathrm{Poly}^{d,1}_n(\mathbb{R}) \to \mathrm{Poly}^{d,n}_1(\mathbb{R})$$

is a homotopy equivalence through dimension $(n-2)(\lfloor d/n \rfloor + 1) - 1$ if $n \geq 4$, and a homology equivalence through dimension $\lfloor d/3 \rfloor$ if $n = 3$.

Next, we consider the stable homotopy type of the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$.

Theorem 2.11 *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\mathrm{Poly}^{d,m}_n(\mathbb{R}) \simeq_s \mathrm{Poly}^{\lfloor d/2 \rfloor, m}_n(\mathbb{C}) \vee B^{d,m}_n \vee \mathrm{Q}^{d,m}_n(\mathbb{R}),$$

where the space $B^{d,m}_n$ is defined by

$$B^{d,m}_n = \bigvee_{i,j \geq 1, i+2j \leqq \lfloor d/n \rfloor} \Sigma^{(mn-2)(i+2j)} D_j. \tag{2-20}$$

By using the stable homotopy equivalences (2-16), (4-26) and the equality $\lfloor \lfloor d/n \rfloor / 2 \rfloor = \lfloor \lfloor d/2 \rfloor / n \rfloor$ (by (iii) of Lemma 3.9), we obtain the following result.

Corollary 2.12 *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\mathrm{Poly}^{d,m}_n(\mathbb{R}) \simeq_s \left(\bigvee_{i=1}^{\lfloor d/n \rfloor} S^{(mn-2)i} \right) \vee \left(\bigvee_{i \geq 0, j \geq 1, i+2j \leq \lfloor d/n \rfloor} \Sigma^{(mn-2)(i+2j)} D_j \right).$$ □

In particular, we obtain the following result, analogous to (2-15).

Corollary 2.13 *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\mathrm{Poly}_n^{d,m}(\mathbb{R}) \simeq_s \mathrm{Poly}_1^{\lfloor d/n \rfloor, mn}(\mathbb{R}). \tag{2-21}$$

By using Theorems 2.11 and 5.6, we also have the following result.

Corollary 2.14 *Let $mn \geq 3$.*

(i) *The inclusion map $\iota_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \xrightarrow{\subset} \mathrm{Q}_n^{d,m}(\mathbb{R})$ induces a split epimorphism on the homology group $H_*(\,;\mathbb{Z})$.*

(ii) *The inclusion map $\iota_{n,\mathrm{H}_+}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+) \xrightarrow{\subset} \mathrm{Poly}_n^{d,m}(\mathbb{R})$ induces a split monomorphism on the homology group $H_*(\,;\mathbb{Z})$.*[6]

3 The spaces $(\Omega_d^2\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$ and $\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+)$

In this section, we shall investigate the homotopy type of the spaces $(\Omega_d^2\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$ and $\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+)$.

We first consider the space $(\Omega_d^2\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$. By identifying S^2 with $\mathbb{C}\cup\infty$, the space S^2 acquires a $\mathbb{Z}_2$-action induced by complex conjugation on $\mathbb{C}$. The space $\mathbb{C}\mathrm{P}^N$ also has a natural $\mathbb{Z}_2$-action induced by complex conjugation. Thus, we can consider the space $(\Omega_d^2\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$ of based $\mathbb{Z}_2$-equivariant maps from S^2 to $\mathbb{C}\mathrm{P}^N$.

Definition 3.1 Let (X, A) and (Y, B) be pairs of connected based spaces.

(i) We denote by $\mathrm{Map}^*(X, A; Y, B)$ the subspace of $\mathrm{Map}^*(X, Y)$ consisting of all based maps $f \in \mathrm{Map}^*(X, Y)$ such that $f(A) \subset B$.

(ii) Let $\mathrm{Map}_0^*(X, Y)$ denote the path component of $\mathrm{Map}^*(X, Y)$ containing nullhomotopic maps. Then for a based map $g \in \mathrm{Map}_0^*(A, Y)$, let $F(X, A; Y; g) \subset \mathrm{Map}_0^*(X, Y)$ denote the subspace

$$F(X, A; Y; g) = \{f \in \mathrm{Map}_0^*(X, Y) : f|A = g\}. \tag{3-1}$$

(iii) The pair (X, A) is called an NDR-pair if the inclusion $i_A : A \xrightarrow{\subset} X$ is a closed cofibration. In this situation, we have a cofibration sequence

$$A \xrightarrow{i_A} X \xrightarrow{q_A} X/A, \tag{3-2}$$

where $q_A : X \to X/A$ denotes the natural pinching map.

Lemma 3.2 *If (X, A) is an NDR-pair, for each pair of spaces (Y, B),*

$$\mathrm{Map}^*(X/A, Y) \xrightarrow{q_A^\#} \mathrm{Map}^*(X, A; Y, B) \xrightarrow{r_A} \mathrm{Map}^*(A, B) \tag{3-3}$$

[6] Here H_+ denotes the upper half-plane in the complex plane $\mathbb{C}$. The map $\iota_{n,\mathrm{H}_+}^{d,m}$ and the space $\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+)$ will be defined in Definition 3.6.

is a fibration sequence, where the two maps $q_A^\#$ and r_A are defined by $q_A^\#(f) = f \circ q_A$ and $r_A(g) = g \circ i_A = g|A$ for $(f, g) \in \mathrm{Map}^*(X/A, Y) \times \mathrm{Map}^*(X, A; Y, B)$.

Proof By (3-2) we obtain a fibration sequence

$$\mathrm{Map}^*(X/A, Y) \xrightarrow{q_A^\#} \mathrm{Map}^*(X, Y) \xrightarrow{re_A} \mathrm{Map}^*(A, Y). \tag{3-4}$$

Let $i_B : B \to Y$ be an inclusion map and let $i_{B\#} : \mathrm{Map}^*(A, B) \to \mathrm{Map}^*(A, Y)$ denote the map given by $i_{B\#}(h) = i_B \circ h$ for $h \in \mathrm{Map}^*(A, B)$.

Now consider the fibration induced from (3-4) by the map $i_{B\#}$

$$\begin{array}{ccccc} \mathrm{Map}^*(X/A, Y) & \longrightarrow & E & \xrightarrow{r_A} & \mathrm{Map}^*(A, B) \\ \| & & \downarrow & & \downarrow{\scriptstyle i_{B\#}} \\ \mathrm{Map}^*(X/A, Y) & \xrightarrow{q_A^\#} & \mathrm{Map}^*(X, Y) & \xrightarrow{re_A} & \mathrm{Map}^*(A, Y) \end{array} \tag{3-5}$$

where

$$\begin{aligned} E &= \{(f, g) \in \mathrm{Map}^*(X, Y) \times \mathrm{Map}^*(A, B) : re_A(f) = i_{B\#}(g)\} \\ &= \{(f, g) \in \mathrm{Map}^*(X, Y) \times \mathrm{Map}^*(A, B) : f|A = g\} = \mathrm{Map}^*(X, A; Y, B). \end{aligned}$$

Thus, we have obtained the fibration sequence (3-3). □

Lemma 3.3 *If $N \geq 2$, there is a homotopy equivalence*

$$(\Omega_d^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2} \simeq \Omega^2 S^{2N+1} \times \Omega S^N. \tag{3-6}$$

Proof For any fixed map $g \in \Omega_d^2 \mathbb{C}\mathrm{P}^N$, let $A_g : \Omega_0^2 \mathbb{C}\mathrm{P}^N \to \Omega_d^2 \mathbb{C}\mathrm{P}^N$ be the map defined by $A_g(f) = f \cdot g$ for $f \in \Omega_0^2 \mathbb{C}\mathrm{P}^N$.[7] One can easily prove that it is a homotopy equivalence. If $g \in (\Omega_0^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$, $A_g((\Omega_0^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}) \subset (\Omega_d^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$ and the restriction defines the map $(\Omega_0^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2} \to (\Omega_d^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$. Since the space $(\Omega_0^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$ is path connected, one can show that this restriction gives a homotopy equivalence

$$(\Omega_0^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2} \xrightarrow{\simeq} (\Omega_d^2 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}. \tag{3-7}$$

Hence, it suffices to prove the assertion (3-6) for the case $d = 0$. Let $j : \mathbb{R}\mathrm{P}^N \to \mathbb{C}\mathrm{P}^N$ and $i_N : S^N \to S^{2N+1}$ denote the natural inclusions. We denote by ξ_N the restriction fibration $r : \mathrm{Map}^*(D^2, \mathbb{C}\mathrm{P}^N) \to \Omega\mathbb{C}\mathrm{P}^N$ given by $r(f) = f|S^1$.

By applying (3-3) to the case $(X, A; Y, B) = (D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N)$, we obtain the fibration sequence

$$\Omega^2 \mathbb{C}\mathrm{P}^N \to \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N) \xrightarrow{r_{S^1}} \Omega_0 \mathbb{R}\mathrm{P}^N. \tag{3-8}$$

It follows from the proof of Lemma 3.2 that (3-8) is the pullback of the fibration ξ_N by the map $\Omega j : \Omega_0 \mathbb{R}\mathrm{P}^N \to \Omega\mathbb{C}\mathrm{P}^N$. Thus, the space $\mathrm{Map}_0^* = \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N)$ can be identified with

[7] The multiplication is induced by the loop product.

the space

$$\mathrm{Map}_0^* = \{(f,\omega) \in \mathrm{Map}^*(D^2, \mathbb{C}\mathrm{P}^N) \times \Omega_0 \mathbb{R}\mathrm{P}^N : r(f) = (\Omega j)(\omega)\}. \tag{3-9}$$

Let $\gamma_N : S^{2N+1} \to \mathbb{C}\mathrm{P}^N$ and $\gamma_{N,\mathbb{R}} : S^N \to \mathbb{R}\mathrm{P}^N$ denote the Hopf fibration and the usual double covering, respectively. Consider the commutative diagram

$$\begin{array}{ccc} \Omega S^N & \xrightarrow{\Omega i_N} & \Omega S^{2N+1} \\ {\scriptstyle \Omega\gamma_{N,\mathbb{R}}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \Omega\gamma_N}\downarrow \\ \Omega_0 \mathbb{R}\mathrm{P}^N & \xrightarrow{\Omega j} & \Omega \mathbb{C}\mathrm{P}^N \end{array}$$

Since the map i_N is nullhomotopic, the map Ωj is nullhomotopic. Since the space $\mathrm{Map}^*(D^2, \mathbb{C}\mathrm{P}^N)$ is contractible, by (3-9) we have the homotopy equivalence

$$\begin{aligned} \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N) &= \{(f,\omega) \in \mathrm{Map}^*(D^2, \mathbb{C}\mathrm{P}^N) \times \Omega_0 \mathbb{R}\mathrm{P}^N : r(f) = (\Omega j)(\omega)\} \\ &\simeq \{(f,\omega) \in \mathrm{Map}^*(D^2, \mathbb{C}\mathrm{P}^N) \times \Omega_0 \mathbb{R}\mathrm{P}^N : r(f) = *\} \\ &\simeq \Omega^2 \mathbb{C}\mathrm{P}^N \times \Omega_0 \mathbb{R}\mathrm{P}^N \simeq \Omega^2 S^{2N+1} \times \Omega S^N \times \mathbb{Z}. \end{aligned}$$

Thus, we have obtained a homotopy equivalence

$$\mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N) \simeq \Omega^2 S^{2N+1} \times \Omega S^N \times \mathbb{Z}. \tag{3-10}$$

Now we regard S^2 as the union $S^2 = D^2_+ \cup D^2_-$, where we let D^2_+ and D^2_- denote the northern and southern hemispheres given by

$$\begin{aligned} D^2_+ &= \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 0\}, \\ D^2_- &= \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \leq 0\}. \end{aligned}$$

We identify the intersection $D^2_+ \cap D^2_- = \{(x,y,0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ with the unit circle S^1. Then each $\mathbb{Z}_2$-equivariant map $f : S^2 \to \mathbb{C}\mathrm{P}^N$ can be identified with the map $(D^2_+, S^1) \to (\mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N)$.

If we identify D^2 with D^2_+, the space $(\Omega^2_0 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2}$ is homotopy equivalent to a path component of $\mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N)$; see [14, (4.12) and Lemma 4.5]. Thus there is a homotopy equivalence

$$(\Omega^2_0 \mathbb{C}\mathrm{P}^N)^{\mathbb{Z}_2} \simeq \Omega^2 S^{2N+1} \times \Omega S^N. \tag{3-11}$$

This completes the proof. □

Lemma 3.4 *If $N \geq 2$, there is a homotopy equivalence*

$$F(D^2, S^1; \mathbb{C}\mathrm{P}^N; g) \simeq \Omega^2 S^{2N+1} \quad \textit{for any } g \in \Omega \mathbb{C}\mathrm{P}^N. \tag{3-12}$$

Proof Let $r : \mathrm{Map}^*(D^2, \mathbb{C}\mathrm{P}^N) \to \Omega \mathbb{C}\mathrm{P}^N$ be the restriction fibration given by $r(f) = f|S^1$ as in the proof of Lemma 3.3. Since $\mathrm{Map}^*(D^2, \mathbb{C}\mathrm{P}^N)$ is contractible, each fiber of r is homotopy equivalent to the space $\Omega^2_0 \mathbb{C}\mathrm{P}^N \simeq \Omega^2 S^{2N+1}$. Thus, there is a homotopy equivalence $F(D^2, S^1; \mathbb{C}\mathrm{P}^N; g) = r^{-1}(g) \simeq \Omega^2 S^{2N+1}$. □

Lemma 3.5 *If $N \geq 2$, there is a homotopy commutative diagram*

$$\begin{array}{ccc} \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, *) & \xrightarrow[\simeq]{\iota_{\mathbb{C}}} & \Omega^2 S^{2N+1} \\ \downarrow \hat{j} & & \uparrow q_1 \\ \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N) & \xrightarrow[\simeq]{\iota_{\mathbb{C}\mathbb{R}}} & \Omega^2 S^{2N+1} \times \Omega S^N \\ \downarrow r_{S^1} & & \downarrow q_2 \\ \Omega_0 \mathbb{R}\mathrm{P}^N & \xrightarrow[\simeq]{\iota_{\mathbb{R}}} & \Omega S^N \end{array} \tag{3-13}$$

where $\hat{j} : \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, *) \subsetneq \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N)$ denotes the natural inclusion, $r_{S^1} : \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^N, \mathbb{R}\mathrm{P}^N) \to \Omega_0 \mathbb{R}\mathrm{P}^N$ is the restriction map given by $r_{S^1}(f) = f|S^1$, $\iota_{\mathbb{C}\mathbb{R}}$ and $\iota_{\mathbb{C}}$ are homotopy equivalences given by (3-10) and (3-12), and q_1 and q_2 are the projections to the first and second factors, respectively.

Proof The assertion easily follows from the proofs of Lemmas 3.3 and 3.4. □

Next, we define the space $\mathrm{Poly}_n^{d,m}(\mathbb{R}; \mathrm{H}_+)$.

Definition 3.6 Let $\mathrm{H}_+ = \{\alpha \in \mathbb{C} : \mathrm{Im}\,\alpha > 0\}$ denote the upper half-plane in the complex plane $\mathbb{C}$.

(i) For each even positive integer $d = 2d_0 \in \mathbb{N}$, let $\mathrm{P}_d(\mathbb{R}; \mathrm{H}_+)$ denote the space of monic polynomials $f(z) \in \mathbb{R}[z]$ of degree d such that it has just d_0 roots in H_+. Thus, if $f(z) \in \mathrm{P}_d(\mathbb{R}; \mathrm{H}_+)$, it has the form

$$f(z) = \prod_{k=1}^{d_0} (z - \alpha_k)(z - \overline{\alpha_k}) \quad \text{for some } \alpha_k \in \mathrm{H}_+ \quad (1 \leq k \leq d_0). \tag{3-14}$$

(ii) For each $d \in \mathbb{N}$, we define the space $\mathrm{Poly}_n^{d,m}(\mathbb{R}; \mathrm{H}_+)$ as follows.

(ii-0) When $d \in \mathbb{N}$ is an even integer, the space $\mathrm{Poly}_n^{d,m}(\mathbb{R}; \mathrm{H}_+)$ is

$$\mathrm{Poly}_n^{d,m}(\mathbb{R}; \mathrm{H}_+) = \mathrm{Poly}_n^{d,m}(\mathbb{R}) \cap (\mathrm{P}_d(\mathbb{R}; \mathrm{H}_+))^m. \tag{3-15}$$

(ii-1) When $d \in \mathbb{N}$ is an odd integer, the space $\mathrm{Poly}_n^{d,m}(\mathbb{R}; \mathrm{H}_+)$ is the subspace of $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ consisting of all elements of the form

$$((z-1) f_1(z), (z-2) f_2(z), \ldots, (z-m) f_m(z)), \tag{3-16}$$

where $(f_1(z), \ldots, f_m(z)) \in \mathrm{Poly}_n^{d-1,m}(\mathbb{R}; \mathrm{H}_+)$.

(iii) It is easy to see that there is a homeomorphism

$$\mathrm{Poly}_n^{2d_0+1,m}(\mathbb{R}; \mathrm{H}_+) \cong \mathrm{Poly}_n^{2d_0,m}(\mathbb{R}; \mathrm{H}_+) \quad \text{for any } d_0 \in \mathbb{N}, \tag{3-17}$$

and that there is an inclusion $\mathrm{Poly}_n^{d,m}(\mathbb{R}; \mathrm{H}_+) \subset \mathrm{Poly}_n^{d,m}(\mathbb{R})$ for any $d \in \mathbb{N}$. We denote this natural inclusion map by

$$\iota_{n,\mathrm{H}_+}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}; \mathrm{H}_+) \subsetneq \mathrm{Poly}_n^{d,m}(\mathbb{R}). \tag{3-18}$$

(iv) By making the identification $S^2 = \mathrm{H}_+ \cup \infty$, we obtain a map

$$i^{d,m}_{n,\mathrm{H}_+} : \mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+) \to \Omega^2_d \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1} \quad \text{given by} \tag{3-19}$$

$$i^{d,m}_{n,\mathrm{H}_+}(f)(\alpha) = \begin{cases} [F_n(f_1)(\alpha) : F_n(f_2)(\alpha) : \cdots : F_n(f_m)(\alpha)] & \text{if } \alpha \in \mathrm{H}_+, \\ [1:1:\cdots:1] & \text{if } \alpha = \infty, \end{cases}$$

for $f = (f_1(z), \dots, f_m(z)) \in \mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+)$ and $\alpha \in S^2 = \mathrm{H}_+ \cup \infty$.

The next lemma is a simple but crucial observation.

Lemma 3.7 (i) *If $d \in \mathbb{N}$ is an even positive integer, there is a homeomorphism* $\mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+) \cong \mathrm{Poly}^{\lfloor d/2 \rfloor, m}_n(\mathbb{C})$.

(ii) *There is a homeomorphism* $\mathrm{Poly}^{\lfloor d/2 \rfloor, m}_n(\mathbb{C}) \cong \mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+)$ *for any $d \geq 2$.*

Proof (i) Let $d = 2d_0$ and let $\psi : \mathbb{C} \xrightarrow{\cong} \mathrm{H}_+$ be any fixed homeomorphism. Then we have the homeomorphism $\psi_d : \mathrm{P}_{d_0}(\mathbb{C}) \xrightarrow{\cong} \mathrm{P}_d(\mathbb{R};\mathrm{H}_+)$ given by

$$\psi_d\Big(\prod_{k=1}^{d_0}(z-\alpha_k)\Big) = \prod_{k=1}^{d_0}(z-\psi(\alpha_k))(z-\overline{\psi(\alpha_k)})$$

for $\alpha_k \in \mathbb{C}$. This naturally extends to the desired homeomorphism

$$\mathrm{Poly}^{d_0,m}_n(\mathbb{C}) \xrightarrow{\cong} \mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+) \tag{3-20}$$

given by $(f_1(z), \dots, f_m(z)) \mapsto (\psi_d(f_1(z)), \dots, \psi_d(f_m(z)))$.

(ii) The assertion (ii) easily follows from (3-17) and the assertion (i). □

Theorem 3.8 *If $mn \geq 3$, the natural map*

$$i^{d,m}_{n,\mathrm{H}_+} : \mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+) \to \Omega^2_d \mathbb{C}\mathrm{P}^{mn-1} \simeq \Omega^2 S^{2mn-1}$$

is a homotopy equivalence through dimension $D(\lfloor d/2 \rfloor; m, n; \mathbb{C})$.

Proof Since there is a homeomorphism $\mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+) \cong \mathrm{Poly}^{\lfloor d/2 \rfloor, m}_n(\mathbb{C})$ (by Lemma 3.7), the proof of [11, Theorem 1.8] works verbatim by replacing $\mathbb{C}$ by H_+ in the case when d is even. The case of odd d can be easily reduced to the even one by (3-17). □

Lemma 3.9 *If $d \in \mathbb{N}$ and $d_0 = \lfloor d/2 \rfloor$, the following assertions hold*:

(i) $\lfloor d/n \rfloor = 2\lfloor d_0/n \rfloor$ or $\lfloor d/n \rfloor = 2\lfloor d_0/n \rfloor + 1$.

(ii) $\lfloor d/(2n) \rfloor = \lfloor d_0/n \rfloor$.

(iii) $\lfloor \lfloor d/n \rfloor /2 \rfloor = \lfloor d_0/n \rfloor$, and $D(d;m,n) < D(\lfloor d/2 \rfloor; m, n; \mathbb{C})$.

(iv) *Define a finite subset $\mathcal{F}^{d,m}_n \subset \mathbb{N}^2$ by*

$$\mathcal{F}^{d,m}_n = \{(i,j) \in \mathbb{N}^2 : i + 2j \leq \lfloor d/n \rfloor\}. \tag{3-21}$$

Then if $(i,j)\in\mathscr{F}_n^{d,m}$, $1\le i<\lfloor d/n\rfloor$ *and* $1\le j\le\lfloor d_0/n\rfloor$, *and*

$$\mathscr{F}_n^{d,m}\subset\{(i,j)\in\mathbb{N}^2 : 1\le i<\lfloor d/n\rfloor, 1\le j\le\lfloor d_0/n\rfloor, i+2j\le\lfloor d/n\rfloor\}. \tag{3-22}$$

Proof (i) Let us write $q=\lfloor d_0/n\rfloor$. Then we can also write

$$d=2d_0+\epsilon_0 \text{ with } \epsilon_0\in\{0,1\},\quad d_0=nq+\epsilon \text{ with } 0\le\epsilon\le n-1.$$

Since

$$\begin{cases}0\le 2\epsilon+\epsilon_0\le 2(n-1)+1=2n-1<2n, \text{ and}\\ d=2d_0+\epsilon_0=2(nq+\epsilon)+\epsilon_0=2nq+(2\epsilon+\epsilon_0),\end{cases}$$
$$2nq\le d<(2q+2)n.\quad \text{So } 2q\le\lfloor d/n\rfloor<2q+2.$$

Hence, $\lfloor d/n\rfloor=2q=2\lfloor d_0/n\rfloor$ or $\lfloor d/n\rfloor=2q+1=2\lfloor d_0/n\rfloor+1$.

(ii) Since $2nq\le d<(2q+2)n=2n(q+1)$ by (i), $q\le d/(2n)<q+1$. Thus, we also have $\lfloor d/(2n)\rfloor=q=\lfloor d_0/n\rfloor$.

(iii) By using (i), we have

$$\lfloor d/n\rfloor/2=q=\lfloor d_0/n\rfloor\quad\text{or}\quad\lfloor d/n\rfloor/2=q+\tfrac{1}{2}=\lfloor d_0/n\rfloor+\tfrac{1}{2}.$$

Thus, $\lfloor\lfloor d/n\rfloor/2\rfloor=q=\lfloor d_0/n\rfloor$. Moreover, since $\lfloor d/n\rfloor\leqq 2q+1$ (by (i)),

$$\begin{aligned}\delta&=D(d;m,n)-D(\lfloor d/2\rfloor;m,n;\mathbb{C})\\&=\{(mn-2)(\lfloor d/n\rfloor+1)-1\}-\{(2mn-3)(q+1)-1\}\\&\leqq(mn-2)(2q+2)-(2mn-3)(q+1)\leqq-(q+1)<0.\end{aligned}$$

(iv) Suppose that $(i,j)\in\mathscr{F}_n^{d,m}$. The first assertion is immediate, so it remains to prove the second. Since $i\ge 1$, we have $1\le 2j<\lfloor d/n\rfloor$.

Hence, by using (iii), we have $1\le j\le\lfloor\lfloor d/n\rfloor/2\rfloor=\lfloor d_0/n\rfloor$. □

4 The Vassiliev spectral sequence

In this section we construct a Vassiliev-type spectral sequence converging to the homology of $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ by means of *nondegenerate* simplicial resolutions of discriminants, and compute its E^1-terms.

First, we summarize the basic facts of the theory of nondegenerate simplicial resolutions [22; 21] (see also [15]) and the spectral sequences associated with them.

Definition 4.1 For a finite set $\boldsymbol{v}=\{v_1,\dots,v_l\}\subset\mathbb{R}^N$, let $\sigma(\boldsymbol{v})$ denote the convex hull spanned by $\boldsymbol{v}$. Suppose that $h:X\to Y$ is a surjective map such that $h^{-1}(y)$ is a finite set for any $y\in Y$, and let $i:X\to\mathbb{R}^N$ be an embedding.

(i) Then let $\mathscr{X}^\Delta$ and $h^\Delta:\mathscr{X}^\Delta\to Y$ denote the space and the map defined by

$$\mathscr{X}^\Delta=\{(y,\boldsymbol{u})\in Y\times\mathbb{R}^N : \boldsymbol{u}\in\sigma(i(h^{-1}(y)))\}\subset Y\times\mathbb{R}^N,\quad h^\Delta(y,\boldsymbol{u})=y. \tag{4-1}$$

The pair $(\mathscr{X}^\Delta, h^\Delta)$ is called *the simplicial resolution of* (h,i). In particular, $(\mathscr{X}^\Delta, h^\Delta)$ is called *a nondegenerate simplicial resolution* if for each $y \in Y$ any k points of $i(h^{-1}(y))$ span a $(k-1)$-dimensional simplex of $\mathbb{R}^N$.

(ii) For each $k \geq 0$, let $\mathscr{X}^\Delta_k \subset \mathscr{X}^\Delta$ be the subspace given by

$$\mathscr{X}^\Delta_k = \{(y,\boldsymbol{u}) \in \mathscr{X}^\Delta : \boldsymbol{u} \in \sigma(\boldsymbol{v}),\ \boldsymbol{v} = \{v_1,\dots,v_l\} \subset i(h^{-1}(y)),\ l \leq k\}. \tag{4-2}$$

We make the identification $X = \mathscr{X}^\Delta_1$ by identifying $x \in X$ with $(h(x), i(x)) \in \mathscr{X}^\Delta_1$, and we note that there is an increasing filtration

$$\varnothing = \mathscr{X}^\Delta_0 \subset X = \mathscr{X}^\Delta_1 \subset \mathscr{X}^\Delta_2 \subset \cdots \subset \mathscr{X}^\Delta_k \subset \mathscr{X}^\Delta_{k+1} \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathscr{X}^\Delta_k = \mathscr{X}^\Delta.$$

Since the map h^Δ is proper, it extends to the map $h^\Delta_+ : \mathscr{X}^\Delta_+ \to Y_+$ between one-point compactifications, where X_+ denotes the one-point compactification of a locally compact space X.

Theorem 4.2 ([22; 21]; see also [10; 15]) *Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, let $i : X \to \mathbb{R}^N$ be an embedding, and let $(\mathscr{X}^\Delta, h^\Delta)$ denote the simplicial resolution of (h,i).*

(i) *If X and Y are semialgebraic spaces and the two maps h, i are semialgebraic maps, then $h^\Delta_+ : \mathscr{X}^\Delta_+ \xrightarrow{\simeq} Y_+$ is a homology equivalence.*

(ii) *If there is an embedding $j : X \to \mathbb{R}^M$ such that its associated simplicial resolution $(\widetilde{\mathscr{X}}^\Delta, \tilde{h}^\Delta)$ is nondegenerate, the space $\widetilde{\mathscr{X}}^\Delta$ is uniquely determined up to homeomorphism and there is a filtration preserving homotopy equivalence $q^\Delta : \widetilde{\mathscr{X}}^\Delta \xrightarrow{\simeq} \mathscr{X}^\Delta$ such that $q^\Delta | X = \mathrm{id}_X$.*

Proof See the proof of [13, Theorem 2.2] for the details. □

Remark 4.3 It is known that the map h^Δ_+ is a homotopy equivalence [21, page 156]. However, this stronger assertion is not needed in this paper. □

Now recall several basic definitions and notations.

Definition 4.4 (i) For a connected space X, let $F(X,d)$ denote *the ordered configuration space* of distinct d points of X defined by

$$F(X,d) = \{(x_1,\dots,x_d) \in X^d : x_i \neq x_j \text{ if } i \neq j\}. \tag{4-3}$$

(ii) Let S_d denote the symmetric group of d-letters. Then the group S_d acts on $F(X,d)$ by the coordinate permutation and let $C_d(X)$ denote *the unordered configuration space of d-distinct points* of X defined by the orbit space

$$C_d(X) = F(X,d)/S_d. \tag{4-4}$$

(iii) For a connected space X, let $D_j(X)$ denote *the equivariant half-smash product of X* defined by

$$D_j(X) = F(X,j)_+ \wedge_{S_j} X^{\wedge j}, \tag{4-5}$$

where we set $F(X,j)_+ = F(X,j) \cup \{*\}$ (disjoint union), $X^{\wedge j} = X \wedge X \wedge \cdots \wedge X$ (j times) and the j-th symmetric group S_j acts on $X^{\wedge j}$ by the coordinate permutation. In particular, for $X = S^1$, we set

$$D_j = D_j(S^1) = F(\mathbb{C}, j)_+ \wedge_{S_j} (S^1)^{\wedge j}. \tag{4-6}$$

Let $mn \geq 3$; we shall construct the Vassiliev-type spectral sequence.

Definition 4.5 (i) Let $\Sigma_n^{d,m}$ denote *the discriminant* of $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ in $\mathrm{P}_d(\mathbb{R})^m$ given by the complement

$$\Sigma_n^{d,m} = \mathrm{P}_d(\mathbb{R})^m \setminus \mathrm{Poly}_n^{d,m}(\mathbb{R}). \tag{4-7}$$

(ii) For each m-tuple $(f_1(z), \ldots, f_m(z)) \in \mathrm{P}_d(\mathbb{R})^m$ let $F_n^m(f_1, \ldots, f_m)$ denote the mn-tuple of $\mathbb{R}$-coefficient monic polynomials of the same degree d defined by

$$F_n^m(f_1, \ldots, f_m) = F_n^m(f_1, \ldots, f_m)(z) = (F_n(f_1), \ldots, F_n(f_m)). \tag{4-8}$$

Since $F_n^m(f_1, \ldots, f_m)$ is an mn-tuple of $\mathbb{R}$-coefficient polynomials,

$$F_n^m(f_1, \ldots, f_m)(\alpha) = 0 \iff F_n^m(f_1, \ldots, f_m)(\overline{\alpha}) = 0 \quad \text{for } \alpha \in \mathrm{H}_+.$$

Thus, the space $\Sigma_n^{d,m}$ is given by

$$\Sigma_n^{d,m} = \{f \in \mathrm{P}_d(\mathbb{R})^m : F_n^m(f)(x) = \mathbf{0} \text{ for some } x \in \overline{\mathrm{H}}_+\}, \tag{4-9}$$

where $\mathbf{0} \in \mathbb{C}^{mn}$, $F_n^m(f) = F_n^m(f_1, \ldots, f_m)$ for $f = (f_1(z), \ldots, f_m(z)) \in \mathrm{P}_d(\mathbb{R})^m$ and $\overline{\mathrm{H}}_+$ denotes the space defined by

$$\overline{\mathrm{H}}_+ = \mathrm{H}_+ \cup \mathbb{R} = \{\alpha \in \mathbb{C} : \mathrm{Im}\,\alpha \geq 0\}. \tag{4-10}$$

(iii) Let $Z_n^{d,m} \subset \Sigma_n^{d,m} \times \mathbb{C}$ denote *the tautological normalization* of $\Sigma_n^{d,m}$ given by

$$Z_n^{d,m} = \{((f_1(z), \ldots, f_m(z)), x) \in \Sigma_n^{d,m} \times \overline{\mathrm{H}}_+ : F_n^m(f_1, \ldots, f_m)(x) = \mathbf{0}\}.$$

Projection on the first factor gives the surjective map

$$\pi_n^{d,m} : Z_n^{d,m} \to \Sigma_n^{d,m}. \tag{4-11}$$

(iv) Let $\varphi_{\mathbb{R}} : \mathrm{P}_d(\mathbb{R})^{nm} \cong \mathbb{R}^{dmn}$ be any fixed homeomorphism. We identify $\mathbb{C}$ with $\mathbb{R}^2$ and define the embedding $i_{\mathbb{R}} : Z_n^{d,m} \to \mathbb{R}^{dmn} \times \mathbb{C} = \mathbb{R}^{dmn+2}$ by

$$i_{\mathbb{R}}((f_1, \ldots, f_m), x) = (\varphi_{\mathbb{R}}(F_n(f_1), \ldots, F_n(f_m)), x) \tag{4-12}$$

for $((f_1, \ldots, f_m), x) \in Z_n^{d,m}$.

(v) Let $(\mathscr{X}^d, \pi^\Delta : \mathscr{X}^d \to \Sigma_n^{d,m})$ be a nondegenerate simplicial resolution of $(\pi_n^{d,m}, i_{\mathbb{R}})$. Then it is easy to see that there is a natural increasing filtration

$$\varnothing = \mathscr{X}_0^d \subset \mathscr{X}_1^d \subset \mathscr{X}_2^d \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathscr{X}_k^d = \mathscr{X}^d$$

such that $\mathscr{X}_k^d = \mathscr{X}^d$ if $k > \lfloor d/n \rfloor$. □

By Theorem 4.2, the map $\pi_+^\Delta : \mathscr{X}_+^d \xrightarrow{\simeq} \Sigma_{n+}^{d,m}$ is a homology equivalence. The filtration on $\mathscr{X}_+^d$ gives rise to a spectral sequence

$$\{E_t^{k,s}(d), d_t : E_t^{k,s}(d) \to E_t^{k+t,s+1-t}(d)\} \Rightarrow H_c^{k+s}(\Sigma_n^{d,m}; \mathbb{Z}), \tag{4-13}$$

where $H_c^k(X;\mathbb{Z})$ denotes the cohomology group with compact supports of a locally compact space X given by $H_c^k(X;\mathbb{Z}) = \widetilde{H}^k(X_+;\mathbb{Z})$ and

$$E_1^{k,s}(d) = \widetilde{H}^{k+s}(\mathscr{X}_{k+}^d/\mathscr{X}_{k-1+}^d;\mathbb{Z}) = H_c^{k+s}(\mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d;\mathbb{Z}) \tag{4-14}$$

(since $\mathscr{X}_{k+}^d/\mathscr{X}_{k-1+}^d \cong (\mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d)_+$).

Since there is a homeomorphism $\mathrm{P}_d(\mathbb{R})^m \cong \mathbb{R}^{dm}$, by Alexander duality there is a natural isomorphism

$$\widetilde{H}_k(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{Z}) \cong H_c^{dm-k-1}(\Sigma_n^{d,m};\mathbb{Z}) \quad \text{for any } k. \tag{4-15}$$

By reindexing we obtain *the Vassiliev-type spectral sequence*

$$\{E_{k,s}^t(d),\ d^t : E_{k,s}^t(d) \to E_{k+t,s+t-1}^t(d)\} \Rightarrow \widetilde{H}_{s-k}(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{Z}), \tag{4-16}$$

where we set

$$E_{k,s}^1(d) = H_c^{dm+k-s-1}(\mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d;\mathbb{Z}). \tag{4-17}$$

Definition 4.6 Let $\mathrm{H}_+ = \{\alpha \in \mathbb{C} : \mathrm{Im}\,\alpha > 0\}$ denote the upper half-plane, and let $(d,k) \in \mathbb{N}^2$ be a pair of positive integers such that $1 \le k \le \lfloor d/n \rfloor$.

(i) Let $\Sigma_n^{d,m}(k) \subset \Sigma_n^{d,m}$ be the subspace consisting of all m-tuples $(f_1(z),\dots,f_m(z)) \in \Sigma_n^{d,m}$ such that the polynomials $\{f_t(z)\}_{t=1}^m$ have exactly k common roots of multiplicity $\ge n$ in $\overline{\mathrm{H}}_+$.

(ii) Let $(f_1(z),\dots,f_m(z)) \in \Sigma_n^{d,m}(k)$. Since each of these polynomials has real coefficients, these common roots of multiplicity $\ge n$ can be uniquely represented as a set

$$\tilde{c} = \{x_1,\dots,x_i;\alpha_1,\overline{\alpha}_1,\dots,\alpha_j,\overline{\alpha}_j\} \quad (i+j=k \text{ and } i,j \ge 0) \tag{4-18}$$

of $k+j$ complex numbers which satisfy the following two conditions:

$$x_s \in \mathbb{R} \text{ for each } 1 \le s \le i \text{ and } x_s \ne x_l \text{ if } s \ne l. \tag{((4-18).1)}$$

$$\alpha_t \in \mathrm{H}_+ \text{ for each } 1 \le t \le j \text{ and } \alpha_t \ne \alpha_l \text{ if } t \ne l. \tag{((4-18).2)}$$

Define a subspace $\Sigma_n^{d,m}(i,j) \subset \Sigma_n^{d,m}(k)$ as the space of all m-tuples $(f_1(z),\dots,f_m(z)) \in \Sigma_n^{d,m}(k)$ whose common roots of multiplicity $\ge n$ satisfy ((4-18).1) and ((4-18).2).

(iii) Note that $(\pi^\Delta)^{-1}(\Sigma_n^{d,m}(k)) = \mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d$. For each pair $(i,j) \in (\mathbb{Z}_{\ge 0})^2$ with $i+j=k$, define a subspace $\mathscr{X}_k^d(i,j) \subset \mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d$ by

$$\mathscr{X}_k^d(i,j) = (\pi^\Delta)^{-1}(\Sigma_n^{d,m}(i,j)). \tag{4-19}$$

It is easy to see that

$$\mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d = \coprod_{i+j=k} \mathscr{X}_k^d(i,j) \quad \text{(disjoint union)}, \tag{4-20}$$

where $\{\mathscr{X}_k^d(i,j) : i \geq 0,\ j \geq 0,\ i+j=k\}$ are the path components of $\mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d$.

Lemma 4.7 *If $mn \geq 3$, $1 \leq k \leq \lfloor d/n \rfloor$ and $(i,j) \in (\mathbb{Z}_{\geq 0})^2$ with $i+j=k$, the space $\mathscr{X}_k^d(i,j)$ is homeomorphic to the total space of a real affine bundle $\xi_k^d(i,j)$ over $C_i(\mathbb{R}) \times C_j(\mathrm{H}_+)$ with real rank $l_{d,k}(i,j) = m(d-nk-nj)+k-1$.*

Proof The argument is exactly analogous to the one used in the proof of [13, Lemma 3.2] or [11, Lemma 3.3]. Namely, an element of $\mathscr{X}_k^d(i,j)$ is represented by the $(m+1)$-tuple $(f_1(z), \dots, f_m(z), \boldsymbol{u})$, where $(f_1(z), \dots, f_m(z))$ is an m-tuple of monic polynomials of the same degree d in $\Sigma_n^{d,m}$ and $\boldsymbol{u}$ is an element of the interior of the span of the images of k distinct points

$$c = (\{x_1, \dots, x_i\}, \{\alpha_1, \dots, \alpha_j\}) \in C_i(\mathbb{R}) \times C_j(\mathrm{H}_+)$$

under a suitable embedding i_k.[8] Note that the following $k+j$ points

$$\tilde{c} = \{x_1, \dots, x_i, \alpha_1, \overline{\alpha}_1, \alpha_2, \overline{\alpha}_2, \dots, \alpha_j, \overline{\alpha}_j\}$$

are common roots of $\{f_s(z)\}_{s=1}^m$ of multiplicity n.

By the definition of the nondegenerate simplicial resolution and [13, Lemma 2.5], the k distinct points c are uniquely determined by $\boldsymbol{u}$. Thus, there is the projection map

$$\pi_{k;i,j}^d : \mathscr{X}_k^d(i,j) \to C_i(\mathbb{R}) \times C_j(\mathrm{H}_+) \tag{4-21}$$

defined by $((f_1, \dots, f_m), \boldsymbol{u}) \mapsto (\{x_1, \dots, x_i\}, \{\alpha_1, \dots, \alpha_j\})$.

Now suppose that $1 \leq k \leq \lfloor d/n \rfloor$, $i,j \geq 0$ with $i+j=k$. Let $c = (\{x_1, \dots, x_i\}, \{\alpha_1, \dots, \alpha_j\}) \in C_i(\mathbb{R}) \times C_j(\mathrm{H}_+)$ be any fixed element. Consider the fiber $(\pi_{k;i,j}^d)^{-1}(c)$. It is easy to see that the condition that a polynomial $f_s(z) \in \mathrm{P}_d(\mathbb{R})$ is divisible by $\prod_{u=1}^i (z-x_u)^n$ is equivalent to the condition

$$f_s^{(t)}(x_u) = 0 \quad \text{for all } 0 \leq t < n,\ 1 \leq u \leq i. \tag{4-22}$$

In general, for each $0 \leq t < n$ and $1 \leq u \leq i$, the condition $f_s^{(t)}(x_u) = 0$ gives one linear condition on the coefficients of $f_s(z)$, and this determines an affine hyperplane in $\mathrm{P}_d(\mathbb{R}) \cong \mathbb{R}^d$. For example, if $f_s(z) = z^d + \sum_{l=1}^d a_l z^{d-l}$, then $f_s(x_u) = 0$ for all $1 \leq u \leq i$ if and only if

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{d-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{d-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & x_i & x_i^2 & x_i^3 & \cdots & x_i^{d-1} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} x_1^d \\ x_2^d \\ \vdots \\ x_i^d \end{bmatrix}.$$

[8] This means that the embedding i_k satisfies the condition [13, (2.3)].

Similarly, $f_s'(x_u)=0$ for all $1\le u\le i$ if and only if

$$\begin{bmatrix} 0 & 1 & 2x_1 & 3x_1^2 & \cdots & (d-1)x_1^{d-2} \\ 0 & 1 & 2x_2 & 3x_2^2 & \cdots & (d-1)x_2^{d-2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 2x_i & 3x_i^2 & \cdots & (d-1)x_i^{d-2} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} dx_1^{d-1} \\ dx_2^{d-1} \\ \vdots \\ dx_i^{d-1} \end{bmatrix}$$

and $f_s''(x_u)=0$ for all $1\le u\le i$ if and only if

$$\begin{bmatrix} 0 & 0 & 2 & 6x_1 & \cdots & (d-1)(d-2)x_1^{d-3} \\ 0 & 0 & 2 & 6x_2 & \cdots & (d-1)(d-2)x_2^{d-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 2 & 6x_i & \cdots & (d-1)(d-2)x_i^{d-3} \end{bmatrix} \cdot \begin{bmatrix} a_d \\ a_{d-1} \\ \vdots \\ a_1 \end{bmatrix} = - \begin{bmatrix} d(d-1)x_1^{d-2} \\ d(d-1)x_2^{d-1} \\ \vdots \\ d(d-1)x_i^{d-2} \end{bmatrix}$$

and so on. Since $1\le i\le k\le \lfloor d/n\rfloor$ and $\{x_u\}_{u=1}^i\in C_i(\mathbb{R})$, it follows from the properties of Vandermonde matrices and Gaussian elimination as in the proof of [13, Lemma 3.2] that the condition (4-22) is equivalent to exactly ni affinely independent conditions on the coefficients of $f_s(z)$. Hence, the space of m-tuples $(f_1(z),\dots,f_m(z))\in \mathrm{P}_d(\mathbb{R})^m$ which satisfy the condition (4-22) for each $1\le s\le m$ is the intersection of mni real affine hyperplanes in general position, and has real codimension mni in $\mathrm{P}_d(\mathbb{R})^m$.

Arguing in exactly the same manner, we see that the condition that each polynomial $f_s(z)$ is divisible by $\prod_{t=1}^j (z-\alpha_t)^n$ for each $1\le s\le m$, gives a subspace of *complex* codimension mnj in $\mathrm{P}_d(\mathbb{C})^m$. Since $f_s(z)$ is a real-coefficient polynomial, $z=\alpha_t$ is a root of $f_s(z)$ of multiplicity n if and only if the same holds for $z=\overline{\alpha}_t$. Thus, the fiber $(\pi^d_{k;i,j})^{-1}(c)$ is homeomorphic to the product of an open $(k-1)$-simplex with the real affine space of dimension $dm-(mni+2mnj)=dm-mn(k+j)=m(d-nk-nj)$. We can check that local triviality holds. Hence, we see that $\mathscr{X}^d_k(i,j)$ is a real affine bundle over $C_i(\mathbb{R})\times C_j(\mathrm{H}_+)$ of rank $l_{d,k}(i,j)=m(d-nk-nj)+k-1$. □

Lemma 4.8 *If $1\le k\le\lfloor d/n\rfloor$ and $mn\ge 3$, there is a natural isomorphism*

$$E^1_{k,s}(d)\cong\left(\bigoplus_{j=1}^{k}\widetilde{H}_{s-(mn-1)k}(\Sigma^{(mn-2)j}D_j;\mathbb{Z})\right)\oplus\widetilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z}).$$

Proof First, consider the case $j=0$. Then $1\le k=i\le\lfloor d/n\rfloor$. Since $C_k(\mathbb{R})\cong\mathbb{R}^k$, the affine bundle $\xi^d_k(k,0)$ is trivial. Hence, there is a homeomorphism $\mathscr{X}^d_k(k,0)_+\cong(\mathbb{R}^k\times\mathbb{R}^{l_{d,k}(k,0)})_+=S^{dm-(mn-1)k+k-1}$. Therefore, there is an isomorphism

$$\begin{aligned} H_c^{dm+k-s-1}(\mathscr{X}^d_k(k,0);\mathbb{Z})&\cong\widetilde{H}^{dm+k-s-1}(S^{dm-(mn-1)k+k-1};\mathbb{Z})\\ &\cong\widetilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z}).\end{aligned}$$

Next consider the case $j\ge 1$. Since

$$\begin{cases} dm+k-s-1-l_{d,k}(i,j)=mn(k+j)-s,\\ 2j-\{mn(k+j)-s-i\}=s-(mn-1)k-(mn-1)j,\end{cases}$$

by the Thom isomorphism and Poincaré duality, there are isomorphisms

$$\begin{aligned} H_c^{dm+k-s-1}(\mathscr{X}_k^d(i,j);\mathbb{Z}) &\cong H_c^{dm+k-s-1-l_{d,k}(i,j)}(C_i(\mathbb{R})\times C_j(\mathrm{H}_+);\pm\mathbb{Z}) \\ &= H_c^{mn(k+j)-s}(C_i(\mathbb{R})\times C_j(\mathrm{H}_+);\pm\mathbb{Z}) \cong H_c^{mn(k+j)-s-i}(C_j(\mathrm{H}_+);\pm\mathbb{Z}) \\ &\cong \widetilde{H}_{s-(mn-1)k-(mn-1)j}(C_j(\mathrm{H}_+);\pm\mathbb{Z}) \\ &\cong \widetilde{H}_{s-(mn-1)k-(mn-1)j}(C_j(\mathbb{C});\pm\mathbb{Z}). \end{aligned}$$

Hence, by (4-20) we have the isomorphisms

$$\begin{aligned} E^1_{k,s}(d) = H_c^{dm+k-s-1}(\mathscr{X}_k^d \setminus \mathscr{X}_{k-1}^d;\mathbb{Z}) &= \bigoplus_{i+j=k} H_c^{dm+k-s-1}(\mathscr{X}_k^d(i,j);\mathbb{Z}) \\ &\cong \Bigg(\bigoplus_{j=1}^{k} \widetilde{H}_{s-(mn-1)k-(mn-1)j}(C_j(\mathbb{C});\pm\mathbb{Z})\Bigg) \oplus \widetilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z}). \end{aligned}$$

It follows from [3] that $D_j = D_j(S^1)$ is the Thom space of the following j-dimensional vector bundle over $C_j(\mathbb{C})$:

$$F(\mathbb{C},j)\times_{S_j}\mathbb{R}^j \to F(\mathbb{C},j)\times_{S_j}\{*\} = F(\mathbb{C},j)/S_j = C_j(\mathbb{C}). \tag{4-23}$$

Thus, by the Thom isomorphism theorem, there is an isomorphism

$$\widetilde{H}_{*+j}(D_j;\mathbb{Z}) \cong \widetilde{H}_*(C_j(\mathbb{C});\pm\mathbb{Z}). \tag{4-24}$$

Hence, we have the isomorphisms

$$\begin{aligned} E^1_{k,s}(d) &\cong \Bigg(\bigoplus_{j=1}^{k} \widetilde{H}_{s-(mn-1)k-(mn-1)j}(C_j(\mathbb{C});\pm\mathbb{Z})\Bigg) \oplus \widetilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z}) \\ &\cong \Bigg(\bigoplus_{j=1}^{k} \widetilde{H}_{s-(mn-1)k-(mn-2)j}(D_j;\mathbb{Z})\Bigg) \oplus \widetilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z}) \\ &\cong \Bigg(\bigoplus_{j=1}^{k} \widetilde{H}_{s-(mn-1)k}(\Sigma^{(mn-2)j}D_j;\mathbb{Z})\Bigg) \oplus \widetilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z}), \end{aligned}$$

and this completes the proof. □

Corollary 4.9 *If $mn \geq 3$, there is a natural isomorphism*

$$E^1_{k,s}(d) = \begin{cases} A_{k,s} \oplus \widetilde{H}_{s-(mn-1)k}(S^0;\mathbb{Z}) & \text{if } 1 \leq k \leq \lfloor d/n \rfloor \text{ and } s \geq (mn-1)k, \\ 0 & \text{otherwise,} \end{cases}$$

where $A_{k,s}$ denotes the abelian group defined by

$$A_{k,s} = \bigoplus_{j=1}^{k} \widetilde{H}_{s-(mn-1)k}(\Sigma^{(mn-2)j}D_j;\mathbb{Z}). \tag{4-25}$$

Proof The assertion easily follows from Lemma 4.8. □

The following results will be needed in the proof of Theorem 2.11.

Lemma 4.10 *If $mn \geq 3$, there is a stable homotopy equivalence*

$$\mathrm{Q}_n^{d,m}(\mathbb{R}) \simeq_s \bigvee_{k=1}^{\lfloor d/n \rfloor} S^{k(mn-2)}. \tag{4-26}$$

Proof First consider $mn \geq 4$. By (2-18) there is a homotopy equivalence $\mathrm{Q}_n^{d,m}(\mathbb{R}) \simeq J_{\lfloor d/n \rfloor}(S^{mn-2})$. From [8] it follows that there is a homotopy equivalence $\Sigma \mathrm{Q}_n^{d,m}(\mathbb{R}) \simeq \bigvee_{k=1}^{\lfloor d/n \rfloor} S^{k(mn-2)+1}$, and we obtain the stable homotopy equivalence (4-26).

Next, consider the case $mn = 3$, i.e., $(m,n) = (3,1)$ or $(1,3)$.

If $(m,n) = (3,1)$, the assertion easily follows from [23, Theorems A and B]. If $(m,n) = (1,3)$, then by using Theorem 2.6, [20] and [22, Theorem 3 (page 88)], we obtain the following result:

(∗∗) The natural map $i_3^{d,1} : \mathrm{Q}_3^{d,1}(\mathbb{R}) \to \Omega S^2$ is a homology equivalence through dimension $D(d;1,3) = \lfloor d/3 \rfloor$, and $H_i(\mathrm{Q}_3^{d,1}(\mathbb{R}); \mathbb{Z}) = 0$ for any $i > \lfloor d/3 \rfloor$.

Let us consider the stable map given by the composite of stable maps

$$\mathrm{Q}_3^{d,1}(\mathbb{R}) \xrightarrow{i_3^{d,1}} \Omega S^2 \xrightarrow{\simeq_s} \bigvee_{i=1}^{\infty} S^i \xrightarrow{q} \bigvee_{i=1}^{\lfloor d/3 \rfloor} S^i,$$

where q is the pinching map. It is easy to see that this map induces an isomorphism on the homology groups $H_*(\,;\mathbb{Z})$ and hence gives a stable homotopy equivalence $\mathrm{Q}_3^{d,1}(\mathbb{R}) \simeq_s \bigvee_{i=1}^{\lfloor d/3 \rfloor} S^i$. □

Definition 4.11 Let $\mathrm{P}_n^{d,m}$ denote the space given by

$$\mathrm{P}_n^{d,m} = \mathrm{Poly}_n^{\lfloor d/2 \rfloor, m}(\mathbb{C}) \vee B_n^{d,m} \vee \mathrm{Q}_n^{d,m}(\mathbb{R}), \tag{4-27}$$

where $B_n^{d,m} = \bigvee_{i,j \geq 1, i+2j \leq \lfloor d/n \rfloor} \Sigma^{(mn-2)(i+2j)} D_j$ as in (2-20).

Lemma 4.12 *If $mn \geq 3$, there is an isomorphism*

$$E_s^1 := \bigoplus_{k \in \mathbb{Z}} E_{k,k+s}^1(d) \cong H_s(\mathrm{P}_n^{d,m}; \mathbb{Z}) \quad \text{for any } s \geq 1.$$

Proof Since, in general, the total number of roots of multiplicity n is $k + j = i + 2j$, we only need to consider the case

$$i \geq 0, \quad j \geq 0, \quad i + 2j \leq \lfloor d/n \rfloor. \tag{4-28}$$

Suppose that $s \geq 1$. Since $k = i + j$, by (3-21), we have

$$\bigoplus_{k\in\mathbb{Z}} A_{k,k+s} = \bigoplus_{i\geq 0, j\geq 1, i+2j\leq \lfloor d/n\rfloor} \tilde{H}_{s-(mn-2)(i+j)}(\Sigma^{(mn-2)j} D_j; \mathbb{Z})$$
$$\cong \bigoplus_{(i,j)\in \mathcal{G}_n^{d,m}} \tilde{H}_s(\Sigma^{(mn-2)(i+2j)} D_j; \mathbb{Z}),$$

where we set $\mathcal{G}_n^{d,m} = \{(i, j) : i \geq 0, j \geq 1, i + 2j \leq \lfloor d/n\rfloor\}$.

Thus, by Corollary 4.9 and Lemma 4.10,

$$E_s^1 = \bigoplus_{k\in\mathbb{Z}} E^1_{k,k+s}(d) = \bigoplus_{k=1}^{\lfloor d/n\rfloor} (A_{k,s} \oplus \tilde{H}_{s-(mn-2)k}(S^0; \mathbb{Z}))$$
$$\cong \Bigg(\bigoplus_{(i,j)\in \mathcal{G}_n^{d,m}} \tilde{H}_s(\Sigma^{(mn-2)(i+2j)} D_j; \mathbb{Z}) \Bigg) \oplus \Bigg(\bigoplus_{k=1}^{\lfloor d/n\rfloor} \tilde{H}_{s-(mn-2)k}(S^0; \mathbb{Z}) \Bigg)$$
$$\cong \Bigg(\bigoplus_{(i,j)\in \mathcal{G}_n^{d,m}} \tilde{H}_s(\Sigma^{(mn-2)(i+2j)} D_j; \mathbb{Z}) \Bigg) \oplus \tilde{H}_s\Big(\bigvee_{k=1}^{\lfloor d/n\rfloor} S^{(mn-2)k}; \mathbb{Z}\Big)$$
$$\cong \Bigg(\bigoplus_{(i,j)\in \mathcal{G}_n^{d,m}} \tilde{H}_s(\Sigma^{(mn-2)(i+2j)} D_j; \mathbb{Z}) \Bigg) \oplus \tilde{H}_s(\mathrm{Q}_n^{d,m}(\mathbb{R}); \mathbb{Z})$$
$$\cong \tilde{H}_s\Bigg(\Bigg(\bigvee_{(i,j)\in \mathcal{G}_n^{d,m}} \Sigma^{(mn-2)(i+2j)} D_j \Bigg) \vee \mathrm{Q}_n^{d,m}(\mathbb{R}); \mathbb{Z}\Bigg).$$

Let $d_0 = \lfloor d/2\rfloor$ and let $C_n^{d,m} = \bigvee_{(i,j)\in \mathcal{G}_n^{d,m}} \Sigma^{(mn-2)(i+2j)} D_j$.

Then by Lemma 3.9 and (2-16),

$$C_n^{d,m} = \bigvee_{i\geq 0, j\geq 1, i+2j\leq \lfloor d/n\rfloor} \Sigma^{(mn-2)(i+2j)} D_j$$
$$= \Bigg(\bigvee_{i,j\geq 1, i+2j\leq \lfloor d/n\rfloor} \Sigma^{(mn-2)(i+2j)} D_j \Bigg) \vee \Bigg(\bigvee_{2j\leq \lfloor d/n\rfloor,\, j\geq 1} \Sigma^{2(mn-2)j} D_j \Bigg)$$
$$= B_n^{d,m} \vee \Bigg(\bigvee_{j=1}^{\lfloor d_0/n\rfloor} \Sigma^{2(mn-2)j} D_j \Bigg) \simeq_s B_n^{d,m} \vee \mathrm{Poly}_n^{d_0,m}(\mathbb{C}) = B_n^{d,m} \vee \mathrm{Poly}_n^{\lfloor d/2\rfloor,m}(\mathbb{C}).$$

Hence, for $s \geq 1$, there are isomorphisms

$$E_s^1 = \bigoplus_{k\in\mathbb{Z}} E^1_{k,k+s}(d) \cong H_s(C_n^{d,m} \vee \mathrm{Q}_n^{d,m}(\mathbb{R}); \mathbb{Z})$$
$$= H_s(B_n^{d,m} \vee \mathrm{Poly}_n^{\lfloor d/2\rfloor,m}(\mathbb{C}) \vee \mathrm{Q}_n^{d,m}(\mathbb{R}); \mathbb{Z}) = H_s(\mathrm{P}_n^{d,m}; \mathbb{Z})$$

and the assertion follows. □

5 Pairing maps and stabilization maps

In this section we construct four pairing maps which induce loop sums and stabilization maps on the spaces $\mathrm{Poly}_n^{d,m}(\mathbb{K})$ and $\mathrm{Q}_n^{d,m}(\mathbb{R})$, and use them to prove Theorem 5.6.

Definition 5.1 Let $\varphi : \mathbb{C} \xrightarrow{\cong} (0,\infty)\times\mathbb{R}$ and $\psi : \mathbb{C} \xrightarrow{\cong} (-\infty,0)\times\mathbb{R}$ be any fixed homeomorphisms satisfying the following two conditions:

$$\begin{cases} \varphi(\mathbb{R}) = (0,\infty)\times\{0\}, & \varphi(H_+) = (0,\infty)\times(0,\infty), \\ \psi(\mathbb{R}) = (-\infty,0)\times\{0\}, & \psi(\mathrm{H}_+) = (-\infty,0)\times(0,\infty). \end{cases} \tag{5.1.1}$$

$$\varphi(\overline{\alpha}) = \overline{\varphi(\alpha)} \text{ and } \psi(\overline{\alpha}) = \overline{\psi(\alpha)} \text{ for any } \alpha\in\mathbb{C}. \tag{5.1.2}$$

For each monic polynomial $f(z) = \prod_{k=1}^{d}(z-x_k) \in \mathrm{P}_d(\mathbb{C})$, let $\varphi(f)$ and $\psi(f)$ denote the monic polynomials of the same degree d given by

$$\widetilde{\varphi}(f) = \prod_{k=1}^{d}(z-\varphi(x_k)) \quad \text{and} \quad \widetilde{\psi}(f) = \prod_{k=1}^{d}(z-\psi(x_k)). \tag{5-1}$$

Remark 5.2 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. It is easy to see that the following hold:

(i) If $f(z)\in \mathrm{P}_d(\mathbb{R})$, then we have that $\widetilde{\varphi}(f)\in \mathrm{P}_d(\mathbb{R})$ and $\widetilde{\psi}(f)\in \mathrm{P}_d(\mathbb{R})$.

(ii) If $(f,g) = ((f_1(z),\dots,f_m(z)),(g_1(z),\dots,g_m(z))) \in \mathrm{Poly}_n^{d_1,m}(\mathbb{K})\times\mathrm{Poly}_n^{d_2,m}(\mathbb{K})$, then we have that $(\widetilde{\varphi}(f_1)\widetilde{\psi}(g_1),\dots,\widetilde{\varphi}(f_m)\widetilde{\psi}(g_m)) \in \mathrm{Poly}_n^{d_1+d_2,m}(\mathbb{K})$.

(iii) If $(f,g) = ((f_1(z),\dots,f_m(z)),(g_1(z),\dots,g_m(z))) \in \mathrm{Q}_n^{d_1,m}(\mathbb{K})\times\mathrm{Q}_n^{d_2,m}(\mathbb{K})$, then we have that $(\widetilde{\varphi}(f_1)\widetilde{\psi}(g_1),\dots,\widetilde{\varphi}(f_m)\widetilde{\psi}(g_m)) \in \mathrm{Q}_n^{d_1+d_2,m}(\mathbb{R})$. □

Using an idea from [2, Definition 4.9] and Remark 5.2 we will now define four pairing maps which induce loop sums.

Definition 5.3 (i) Define the pairing map

$$\mu^{\mathbb{C}}_{d_1,d_2} : \mathrm{Poly}_n^{d_1,m}(\mathbb{C})\times\mathrm{Poly}_n^{d_2,m}(\mathbb{C}) \to \mathrm{Poly}_n^{d_1+d_2,m}(\mathbb{C}) \quad \text{by the juxtaposition} \tag{5-2}$$

$$\mu^{\mathbb{C}}_{d_1,d_2}(f,g) = (\widetilde{\varphi}(f_1)\widetilde{\psi}(g_1),\dots,\widetilde{\varphi}(f_m)\widetilde{\psi}(g_m))$$

for $(f,g)\in \mathrm{Poly}_n^{d_1,m}(\mathbb{C})\times\mathrm{Poly}_n^{d_2,m}(\mathbb{C})$, where we write

$$(f,g) = ((f_1(z),\dots,f_m(z)),(g_1(z),\dots,g_m(z))).$$

(ii) It is easy to see that

$$\mu^{\mathbb{C}}_{d_1,d_2}(\mathrm{Poly}_n^{d_1,m}(\mathbb{R})\times\mathrm{Poly}_n^{d_2,m}(\mathbb{R})) \subset \mathrm{Poly}_n^{d_1+d_2,m}(\mathbb{R}). \tag{5-3}$$

Hence, one can define the pairing map

$$\mu^{\mathbb{R}}_{d_1,d_2} : \mathrm{Poly}_n^{d_1,m}(\mathbb{R})\times\mathrm{Poly}_n^{d_2,m}(\mathbb{R}) \to \mathrm{Poly}_n^{d_1+d_2,m}(\mathbb{R}) \tag{5-4}$$

as the restriction $\mu^{\mathbb{R}}_{d_1,d_2} = \mu^{\mathbb{C}}_{d_1,d_2}|\mathrm{Poly}_n^{d_1,m}(\mathbb{R})\times\mathrm{Poly}_n^{d_2,m}(\mathbb{R})$.

(iii) Since the relation

$$\mu^{\mathbb{R}}_{d_1,d_2}(\mathrm{Poly}^{d_1,m}_n(\mathbb{R};\mathrm{H}_+)\times \mathrm{Poly}^{d_2,m}_n(\mathbb{R};\mathrm{H}_+))\subset \mathrm{Poly}^{d_1+d_2,m}_n(\mathbb{R};\mathrm{H}_+), \tag{5-5}$$

also holds, define the pairing map

$$\mu^{\mathrm{H}_+}_{d_1,d_2}:\mathrm{Poly}^{d_1,m}_n(\mathbb{R};\mathrm{H}_+)\times \mathrm{Poly}^{d_2,m}_n(\mathbb{R};\mathrm{H}_+)\to \mathrm{Poly}^{d_1+d_2,m}_n(\mathbb{R};\mathrm{H}_+) \tag{5-6}$$

as the restriction $\mu^{\mathrm{H}_+}_{d_1,d_2}=\mu^{\mathbb{R}}_{d_1,d_2}|\mathrm{Poly}^{d_1,m}_n(\mathbb{R};\mathrm{H}_+)\times \mathrm{Poly}^{d_2,m}_n(\mathbb{R};\mathrm{H}_+)$.

(iv) Similarly, we define the pairing map by the juxtaposition

$$\mu_{d_1,d_2}:\mathrm{Q}^{d_1,m}_n(\mathbb{R})\times \mathrm{Q}^{d_2,m}_n(\mathbb{R})\to \mathrm{Q}^{d_1+d_2,m}_n(\mathbb{R}) \quad \text{by} \tag{5-7}$$

$$\mu_{d_1,d_2}(f,g)=(\widetilde{\varphi}(f_1)\widetilde{\psi}(g_1),\dots,\widetilde{\varphi}(f_m)\widetilde{\psi}(g_m))$$

for $(f,g)=((f_1(z),\dots,f_m(z)),(g_1(z),\dots,g_m(z)))\in \mathrm{Q}^{d_1,m}_n(\mathbb{R})\times \mathrm{Q}^{d_2,m}_n(\mathbb{R})$.

Next, recall the definitions of stabilization maps.

Definition 5.4 Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. For each integer $d\geq 1$, let $\{x_{d,i}:1\leq i\leq m\}\subset(d,d+1)$ be any fixed real numbers such that $x_{d,i}\neq x_{d,k}$ if $i\neq k$, and let $\varphi_d:\mathbb{C}\overset{\cong}{\to}\mathbb{C}_d=\{\alpha\in\mathbb{C}:\mathrm{Re}(\alpha)<d\}$ be any homeomorphism satisfying the following condition:

((5-7).1) $\varphi_d(\mathbb{R})=(-\infty,d)\times\{0\}$ and $\varphi_d(\overline{\alpha})=\overline{\varphi_d(\alpha)}$ for any $\alpha\in\mathbb{C}$, where we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way.

(i) Define the stabilization map $s^{d,m}_{n,\mathbb{K}}:\mathrm{Poly}^{d,m}_n(\mathbb{K})\to\mathrm{Poly}^{d+1,m}_n(\mathbb{K})$ by

$$s^{d,m}_{n,\mathbb{K}}(f_1(z),\dots,f_m(z))=((z-x_{d,1})\widetilde{\varphi}_d(f_1),\dots,(z-x_{d,m})\widetilde{\varphi}_d(f_m)) \tag{5-8}$$

for $(f_1(z),\dots,f_m(z))\in\mathrm{Poly}^{d,m}_n(\mathbb{K})$, where we set $\widetilde{\varphi}_d(f)=\prod^d_{k=1}(z-\varphi_d(x_k))$ if $f=f(z)=\prod^d_{k=1}(z-x_k)\in\mathrm{P}_d(\mathbb{K})$.

Note that the map $s^{d,m}_{n,\mathbb{K}}$ depends on the choice of points $\{x_{d,k}\}^m_{k=1}$ and the homeomorphism φ_d, but its homotopy type does not, as in [12, Definition 3.11].

(ii) Let $s^{d,m}_n:\mathrm{Q}^{d,m}_n(\mathbb{R})\to\mathrm{Q}^{d+1,m}_n(\mathbb{R})$ be the stabilization map given by

$$s^{d,m}_n(f_1(z),\dots,f_m(z))=((z-x_{d,1})\widetilde{\varphi}_d(f_1),\dots,(z-x_{d,m})\widetilde{\varphi}_d(f_m)) \tag{5-9}$$

for $(f_1(z),\dots,f_m(z))\in\mathrm{Q}^{d,m}_n(\mathbb{R})$.

(iii) Since one can easily see that $s^{d,m}_{n,\mathbb{R}}(\mathrm{Poly}^{d,m}_{n,\mathrm{H}_+})\subset\mathrm{Poly}^{d+1,m}_{n,\mathrm{H}_+}$, we define the stabilization map $s^{d,m}_{n,\mathrm{H}_+}:\mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+)\to\mathrm{Poly}^{d+1,m}_n(\mathbb{R};\mathrm{H}_+)$ by the restriction

$$s^{d,m}_{n,\mathrm{H}_+}=s^{d,m}_{n,\mathbb{R}}|\mathrm{Poly}^{d,m}_n(\mathbb{R};\mathrm{H}_+). \tag{5-10}$$

Remark 5.5 By the definitions of (5-8), (5-9) and (5-10) the following diagram is commutative:

$$\begin{array}{ccccc}
\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+) & \xrightarrow[\subset]{\iota_{n,\mathrm{H}_+}^{d,m}} & \mathrm{Poly}_n^{d,m}(\mathbb{R}) & \xrightarrow[\subset]{\iota_{n,\mathbb{R}}^{d,m}} & \mathrm{Q}_n^{d,m}(\mathbb{R}) \\
\downarrow{\scriptstyle s_{n,\mathrm{H}_+}^{d,m}} & & \downarrow{\scriptstyle s_{n,\mathbb{R}}^{d,m}} & & \downarrow{\scriptstyle s_n^{d,m}} \\
\mathrm{Poly}_n^{d+1,m}(\mathbb{R};\mathrm{H}_+) & \xrightarrow[\subset]{\iota_{n,\mathrm{H}_+}^{d+1,m}} & \mathrm{Poly}_n^{d+1,m}(\mathbb{R}) & \xrightarrow[\subset]{\iota_{n,\mathbb{R}}^{d+1,m}} & \mathrm{Q}_n^{d+1,m}(\mathbb{R})
\end{array} \tag{5-11}$$

Moreover, by using the method invented by C. Boyer and B. Mann [2, Definition 4.9] we obtain two homotopy commutative diagrams

$$\begin{array}{ccc}
\mathrm{Poly}_n^{d_1,m}(\mathbb{R};\mathrm{H}_+)\times\mathrm{Poly}_n^{d_2,m}(\mathbb{R};\mathrm{H}_+) & \xrightarrow{\mu_{d_1,d_2}^{\mathrm{H}_+}} & \mathrm{Poly}_n^{d_1+d_2,m}(\mathbb{R};\mathrm{H}_+) \\
\downarrow{\scriptstyle \iota_{n,\mathrm{H}_+}^{d_1,m}\times\iota_{n,\mathrm{H}_+}^{d_2,m}}\cap & & \downarrow{\scriptstyle \iota_{n,\mathrm{H}_+}^{d_1+d_2,m}}\cap \\
\mathrm{Poly}_n^{d_1,m}(\mathbb{R})\times\mathrm{Poly}_n^{d_2,m}(\mathbb{R}) & \xrightarrow{\mu_{d_1,d_2}^{\mathbb{R}}} & \mathrm{Poly}_n^{d_1+d_2,m}(\mathbb{R}) \\
\downarrow{\scriptstyle \iota_{n,\mathbb{R}}^{d_1,m}\times\iota_{n,\mathbb{R}}^{d_2,m}}\cap & & \downarrow{\scriptstyle \iota_{n,\mathbb{R}}^{d_1+d_2,m}}\cap \\
\mathrm{Q}_n^{d_1,m}(\mathbb{R})\times\mathrm{Q}_n^{d_2,m}(\mathbb{R}) & \xrightarrow{\mu_{d_1,d_2}} & \mathrm{Q}_n^{d_1+d_2,m}(\mathbb{R}) \\
\downarrow{\scriptstyle i_n^{d_1,m}\times i_n^{d_2,m}} & & \downarrow{\scriptstyle i_n^{d_1+d_2,m}} \\
\Omega S^{mn-1}\times\Omega S^{mn-1} & \xrightarrow{l_S} & \Omega S^{mn-1}
\end{array} \tag{5-12}$$

where l_S denotes the loop product on the loop space ΩS^{mn-1}. □

Now we are ready to give a proof of the following result.

Theorem 5.6 *If $mn \geq 3$, the inclusion map*

$$\iota_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \overset{\subseteq}{\hookrightarrow} \mathrm{Q}_n^{d,m}(\mathbb{R})$$

induces a split epimorphism on the homology group $H_*(\,;\mathbb{Z})$.

Proof Recall that there is an isomorphism $H_*(\Omega S^{mn-1};\mathbb{Z}) = \mathbb{Z}[\iota_{mn-2}]$ for the generator $\iota_{mn-2} \in H_{mn-2}(\Omega S^{mn-1};\mathbb{Z}) \cong \mathbb{Z}$. It follows from (2-18) that there is a generator $\iota_Q \in H_{mn-2}(\mathrm{Q}_n^{d,m}(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$ which satisfies the equality

$$(i_n^{d,m})_*(\iota_Q) = \iota_{mn-2}, \tag{5-13}$$

where $n_0 = \lfloor d/n \rfloor$ and $H_*(\mathrm{Q}_n^{d,m}(\mathbb{R});\mathbb{Z}) = \mathbb{Z}[\iota_Q]/((\iota_Q)^{n_0+1})$.

Note that if a polynomial $f(z) \in \mathbb{R}[z]$ has a complex root $\alpha \in \mathbb{C}\setminus\mathbb{R}$, its conjugate $\overline{\alpha}$ is also a root of $f(z)$. Hence

$$\mathrm{Poly}_n^{d,m}(\mathbb{R}) = \mathrm{Q}_n^{d,m}(\mathbb{R}) \quad \text{if } d < 2n. \tag{5-14}$$

Thus, the inclusion map $\iota^{d,m}_{n,\mathbb{R}}$ is the identity map if $d < 2n$.

Since there is a homotopy equivalence $\mathrm{Poly}^{n,m}_n(\mathbb{R}) \cong \mathbb{R}^{mn} \setminus \mathbb{R} \simeq S^{mn-2}$, there is a generator $\iota_P \in H_{mn-2}(\mathrm{Poly}^{n,m}_n(\mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}$ such that

$$(\iota^{n,m}_{n,\mathbb{R}})_*(\iota_P) = \iota_Q. \tag{5-15}$$

Thus, for any $1 \le k \le d_0 = \lfloor d/n \rfloor$, by using the diagrams (5-12) and (5-11) we see that $(\iota_Q)^k = ((\iota^{n,m}_{n,\mathbb{R}})_*(\iota_P))^k = (\iota^{kn,m}_{n,\mathbb{R}})_*((\iota_P)^k)$. However, since $kn \le d$,

$$(\iota_Q)^k = (\iota^{kn,m}_{n,\mathbb{R}})_*((\iota_P)^k) \in (\iota^{d,m}_{n,\mathbb{R}})_*(H_{k(mn-2)}(\mathrm{Poly}^{d,m}_n(\mathbb{R}); \mathbb{Z})).$$

Hence, the map $\iota^{d,m}_{n,\mathbb{R}}$ induces a split epimorphism on the homology group $H_k(\,;\mathbb{Z})$ for any k. □

6 The homology stability theorem

In this section we consider the homology stability of the stabilization maps $s^{d,m}_{n,\mathbb{R}}$ and prove the homology stability theorem (Theorem 6.2).

Consider the stabilization map $s^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{R})$ given by (5-8). Note that the map $s^{d,m}_{n,\mathbb{R}}$ clearly extends to a map

$$\mathrm{P}_d(\mathbb{R})^m \to \mathrm{P}_{d+1}(\mathbb{R})^m, \tag{6-1}$$

and its restriction gives a stabilization map $\tilde{s}^{d,m}_{n,\mathbb{R}} : \Sigma^{d,m}_n \to \Sigma^{d+1,m}_n$ between discriminants. It is easy to see that it also extends to an open embedding

$$\tilde{s}^{d,m}_{n,\mathbb{R}} : \Sigma^{d,m}_n \times \mathbb{R}^m \to \Sigma^{d+1,m}_n. \tag{6-2}$$

Since one-point compactification is contravariant for open embeddings, it induces the map

$$(\tilde{s}^{d,m}_{n,\mathbb{R}})_+ : (\Sigma^{d+1,m}_n)_+ \to (\Sigma^{d,m}_n \times \mathbb{R}^m)_+ = \Sigma^{d,m}_{n+} \wedge S^m \tag{6-3}$$

between one-point compactifications. Thus

$$\begin{array}{ccc} \widetilde{H}_k(\mathrm{Poly}^{d,m}_n(\mathbb{R}); \mathbb{Z}) & \xrightarrow{(s^{d,m}_{n,\mathbb{R}})_*} & \widetilde{H}_k(\mathrm{Poly}^{d+1,m}_n(\mathbb{R}); \mathbb{Z}) \\ {\scriptstyle Al}\downarrow\cong & & {\scriptstyle Al}\downarrow\cong \\ H^{dm-k-1}_c(\Sigma^{d,m}_n; \mathbb{Z}) & \xrightarrow{(\tilde{s}^{d,m}_{n,\mathbb{R}+})^*} & H^{(d+1)m-k-1}_c(\Sigma^{d+1,m}_n; \mathbb{Z}) \end{array} \tag{6-4}$$

is commutative, where Al denotes the Alexander duality isomorphism and the homomorphism $(\tilde{s}^{d,m}_{n,\mathbb{R}+})^*$ denotes the composite of the suspension isomorphism with the homomorphism $\tilde{s}^{d,m*}_{n,\mathbb{R}+}$,

$$H^{dm-k-1}_c(\Sigma^{d,m}_n; \mathbb{Z}) \cong H^{(d+1)m-k-1}_c(\Sigma^{d,m}_n \times \mathbb{R}^m; \mathbb{Z}) \xrightarrow{\tilde{s}^{d,m*}_{n,\mathbb{R}+}} H^{(d+1)m-k-1}_c(\Sigma^{d+1,m}_n; \mathbb{Z}).$$

The map $\tilde{s}^{d,m}_{n,\mathbb{R}}$ naturally extends to a filtration-preserving open map

$$\hat{s}^{d,m}_{n,\mathbb{R}} : \mathscr{X}^d \times \mathbb{R}^m \to \mathscr{X}^{d+1}, \tag{6-5}$$

and this extends to a filtration-preserving map $(\hat{s}^{d,m}_{n,\mathbb{R}})_+ : \mathscr{X}^{d+1}_+ \to \mathscr{X}^d_+ \wedge S^m$. This induces a homomorphism of spectral sequences

$$\{\theta^t_{k,s} : E^t_{k,s}(d) \to E^t_{k,s}(d+1)\}. \tag{6-6}$$

It is easy to see that $\hat{s}^{d,m}_{n,\mathbb{R}}(\mathscr{X}^d_k(i,j) \times \mathbb{R}^m) \subset \mathscr{X}^{d+1}_k(i,j)$ if $1 \le k \le \lfloor d/n \rfloor$ and $i + j = k$. Hence, if $i + j = k$ and $1 \le k \le \lfloor d/n \rfloor$, one can define a map

$$\hat{s}^{d,m}_{n;i,j} : \mathscr{X}^d_k(i,j) \times \mathbb{R}^m \to \mathscr{X}^{d+1}_k(i,j) \tag{6-7}$$

as the restriction $\hat{s}^{d,m}_{n;i,j} = \hat{s}^{d,m}_{n,\mathbb{R}} | \mathscr{X}^d_k(i,j)$. Since $\hat{s}^{d,m}_{n;i,j}$ is an open embedding, it induces a map

$$\hat{s}^{d,m}_{n;i,j+} : \mathscr{X}^{d+1}_k(i,j)_+ \to \mathscr{X}^d_k(i,j)_+ \wedge S^m \tag{6-8}$$

such that

$$\theta^1_{k,s} = \sum_{i+j=k} (\hat{s}^{d,m}_{n;i,j+})^* : E^1_{k,s}(d) \to E^1_{k,s}(d+1) \qquad \text{if } 1 \le k \le \left\lfloor \frac{d}{n} \right\rfloor, \tag{6-9}$$

where

$$(\hat{s}^{d,m}_{n;i,j+})^* : H_c^{dm+k-s-1}(\mathscr{X}^d_k(i,j); \mathbb{Z}) \to H_c^{(d+1)m+k-s-1}(\mathscr{X}^{d+1}_k(i,j); \mathbb{Z}),$$

and

$$\begin{cases} E^1_{k,s}(d) = \bigoplus_{i+j=k} H_c^{dm+k-s-1}(\mathscr{X}^d_k(i,j); \mathbb{Z}), \\ E^1_{k,s}(d+1) = \bigoplus_{i+j=k} H_c^{(d+1)m+k-s-1}(\mathscr{X}^{d+1}_k(i,j); \mathbb{Z}). \end{cases}$$

Lemma 6.1 *If $1 \le k \le \lfloor d/n \rfloor$, then $\theta^1_{k,s} : E^1_{k,s}(d) \cong E^1_{k,s}(d+1)$ is an isomorphism for any s.*

Proof Suppose that $1 \le k \le \lfloor d/n \rfloor$ with $i + j = k$. By (6-9) it suffices to show that $(\hat{s}^{d,m}_{n;i,j+})^*$ is an isomorphism. Note that the projection $\pi^d_{k;i,j}$ (defined in (4-21)) also naturally extends to a map $\hat{\pi}^d_{k;i,j} : \mathscr{X}^d_k(i,j) \times \mathbb{R}^m \to C_i(\mathbb{R}) \times \mathbb{C}_j(\mathrm{H}_+)$ and it is easy to check that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{X}^d_k(i,j) \times \mathbb{R}^m & \xrightarrow{\hat{\pi}^d_{k;i,j}} & C_i(\mathbb{R}) \times \mathbb{C}_j(\mathrm{H}_+) \\ \Big\downarrow{\scriptstyle \hat{s}^{d,m}_{n;i,j}} & & \Big\| \\ \mathscr{X}^{d+1}_k(i,j) & \xrightarrow{\pi^{d+1}_{k;i,j}} & C_i(\mathbb{R}) \times C_j(\mathrm{H}_+) \end{array}$$

It follows from the naturality of the Thom isomorphism that the homomorphism $(\hat{s}^{d,m}_{n;i,j+})^*$ is indeed an isomorphism if $1 \le k \le \lfloor d/n \rfloor$. □

Theorem 6.2 *If $mn \geq 3$, the stabilization map*

$$s^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{R})$$

is a homology equivalence for $\lfloor d/n \rfloor = \lfloor (d+1)/n \rfloor$, and a homology equivalence through dimension $D(d;m,n)$ for $\lfloor d/n \rfloor < \lfloor (d+1)/n \rfloor$, where $D(d;m,n)$ denotes the positive integer given by (2-17).

Proof First, let $\lfloor d/n \rfloor = \lfloor (d+1)/n \rfloor$. In this case, by using Corollary 4.9 and Lemma 6.1 it is easy to show that $\theta^1_{k,s} : E^1_{k,s}(d) \xrightarrow{\cong} E^1_{k,s}(d+1)$ is an isomorphism for any (k,s). Hence, $\theta^\infty_{k,s}$ is an isomorphism for any (k,s). Since $\theta^t_{k,s}$ is induced from $\hat{s}^{d,m}_{n,\mathbb{R}}$, it follows from (6-4) that the map $s^{d,m}_{n,\mathbb{R}}$ is a homology equivalence.

Next assume that $\lfloor d/n \rfloor < \lfloor (d+1)/n \rfloor$, i.e., $\lfloor (d+1)/n \rfloor = \lfloor d/n \rfloor + 1$, and let $\epsilon \in \{0,1\}$. In this case, by considering the differential $d^t : E^t_{k,s}(d+\epsilon) \to E^t_{k+t,s+t-1}(d+\epsilon)$, Lemma 6.1 and Corollary 4.9, one can show that $\theta^t_{k,s} : E^t_{k,s}(d) \to E^t_{k,s}(d+1)$ is an isomorphism for any (k,s) and any $t \geq 1$ as long as $s - t \leq D(d;m,n)$. Hence, if $s - t \leq D(d;m,n)$, $\theta^\infty_{k,s}$ is always an isomorphism and so the map $s^{d,m}_{n,\mathbb{R}}$ is a homology equivalence through dimension $D(d;m,n)$. □

The following two results will be needed for the proof of Corollary 7.11.

Lemma 6.3 *The space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ is simply connected if $mn \geq 4$, and there is an isomorphism $\pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) \cong \mathbb{Z}$ if $mn = 3$ and $d \geq n$.*

Proof We use the description of the fundamental group in terms of braids as in [13, Lemma 3.5]. We represent elements of the group as strings of m-different colors, which can move continuously, with only the following case not allowed to occur:

$(*)$ All strings of multiplicity $\geq n$ of m-different colors pass through a single point.

Since $mn \geq 3$, this representation shows that $\pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R}))$ is an abelian group. Hence, there is an isomorphism $\pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) \cong H_1(\mathrm{Poly}^{d,m}_n(\mathbb{R});\mathbb{Z})$. Now consider the spectral sequence (4-16). Then, by Corollary 4.9 , we see that

$$E^1_{k,k+1}(d) = \begin{cases} \mathbb{Z} & \text{if } k = 1 \text{ and } mn = 3, \\ 0 & \text{otherwise.} \end{cases}$$

First, consider the case $mn \geq 4$. We see that $H_1(\mathrm{Poly}^{d,m}_n(\mathbb{R});\mathbb{Z}) = 0$, and the space $\mathrm{Poly}^{d,m}_n(|\mathbb{R})$ is simply connected.

Next, consider the case $mn = 3$. Since $E^1_{2,2}(d) = 0$ by Corollary 4.9, by considering the differential $d^t : E^t_{k,s}(d) \to E^t_{k+t,s+t+1}(d)$, we see that $\bigoplus_{k\in\mathbb{Z}} E^1_{k,k+1}(d) = E^1_{1,2}(d) = \mathbb{Z} = E^\infty_{1,2}(d)$. Hence, if $mn = 3$,

$$\pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) \cong H_1(\mathrm{Poly}^{d,m}_n(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}.$$ □

Corollary 6.4 *If $mn \geq 4$, the stabilization map*

$$\mathrm{s}^{d,m}_{n,\mathbb{R}} : \mathrm{Poly}^{d,m}_n(\mathbb{R}) \to \mathrm{Poly}^{d+1,m}_n(\mathbb{R})$$

is a homotopy equivalence if $\lfloor d/n \rfloor = \lfloor (d+1)/n \rfloor$ and a homotopy equivalence through dimension $D(d;m,n)$ otherwise.

Proof This follows from Theorem 6.2 and Lemma 6.3. □

Definition 6.5 Let $D = (d_1, \dots, d_m) \in \mathbb{N}^m$ be an m-tuple of positive integers. Let $x_d \in (d, d+1)$ be any fixed real number and let $\varphi_d : \mathbb{C} \xrightarrow{\cong} \mathbb{C}_d = \{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) < d\}$ be any fixed homeomorphism satisfying the condition ((5-7).1).

Then for each $1 \le i \le m$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let

$$s^{D,i;m}_{n,\mathbb{K}} : \mathrm{Poly}^{(d_1,\dots,d_m),m}_n(\mathbb{K}) \to \mathrm{Poly}^{(d_1,\dots,d_{i-1},d_i+1,d_{i+1},\dots,d_m),m}_n(\mathbb{K}) \tag{6-10}$$

denote the stabilization map defined by

$$s^{D,i;m}_{n,\mathbb{K}}(f) = (\widetilde{\varphi}_d(f_1), \dots, \widetilde{\varphi}_d(f_{i-1}), (z - x_d)\widetilde{\varphi}_d(f_i), \widetilde{\varphi}_d(f_{i+1}), \dots, \widetilde{\varphi}_d(f_m))$$

for $f = (f_1(z), \dots, f_m(z)) \in \mathrm{Poly}^{(d_1,\dots,d_m),m}_n(\mathbb{K})$.

Theorem 6.6 *Let $1 \le i \le m$, $mn \ge 3$ and $D = (d_1, \dots, d_m) \in \mathbb{N}^m$ be an m-tuple of positive integers.*

(i) *The stabilization map*

$$s^{D,i;m}_{n,\mathbb{R}} : \mathrm{Poly}^{(d_1,\dots,d_m),m}_n(\mathbb{R}) \to \mathrm{Poly}^{(d_1,\dots,d_{i-1},d_i+1,d_{i+1},\dots,d_m),m}_n(\mathbb{R})$$

is a homology equivalence if $\lfloor d_i/n \rfloor = \lfloor (d_i+1)/n \rfloor$ and a homology equivalence through dimension $D(d_i;m,n)$ otherwise.

(ii) *The stabilization map*

$$s^{D,i;m}_{n,\mathbb{C}} : \mathrm{Poly}^{(d_1,\dots,d_m),m}_n(\mathbb{C}) \to \mathrm{Poly}^{(d_1,\dots,d_{i-1},d_i+1,d_{i+1},\dots,d_m),m}_n(\mathbb{C})$$

is a homotopy equivalence if $\lfloor d_i/n \rfloor = \lfloor (d_i+1)/n \rfloor$ and a homotopy equivalence through dimension $D(d_i;m,n;\mathbb{C})$ otherwise.

Proof The assertion (i) can be proved in exactly the same way as Theorem 6.2 and the assertion (ii) can be proved analogously to [11, Theorem 1.8]. □

7 Configuration spaces and scanning maps

In this section we define the "horizontal scanning maps" and then use them to prove our stable results (Theorems 7.5 and 7.9).

Definition 7.1 For a space X, let $\mathrm{SP}^d(X)$ denote the d-th *symmetric product* defined by the quotient space

$$\mathrm{SP}^d(X) = X^d/S_d, \tag{7-1}$$

where the symmetric group S_d of d letters acts on X^d by the permutation of coordinates. Since $F(X,d)$ is an S_d-invariant subspace of X^d and $C_d(X) = F(X,d)/S_d$, there is a natural inclusion $C_d(X) \subset \mathrm{SP}^d(X)$.

Note that an element $\alpha \in \mathrm{SP}^d(X)$ may be identified with the formal linear combination

$$\alpha = \sum_{i=1}^{k} d_i x_i, \qquad \text{where } d_i \geq 1,\ \{x_i\}_{i=1}^{k} \in C_k(X) \text{ and } \sum_{i=1}^{k} d_i = d. \tag{7-2}$$

We shall refer to α as *a configuration* (or 0-*cycle*) having *a multiplicity* d_i at the point x_i.

Remark 7.2 (i) If we use the notation (7-2), the space $\mathrm{P}_d(\mathbb{C})$ can be easily identified with the space $\mathrm{SP}^d(\mathbb{C})$ by the homeomorphism $\Phi_d : \mathrm{P}_d(\mathbb{C}) \xrightarrow{\cong} \mathrm{SP}^d(\mathbb{C})$ defined by

$$\Phi_d\Big(\prod_{i=1}^{k}(z-\alpha_i)^{d_i}\Big) = \sum_{i=1}^{k} d_i\alpha_i. \tag{7-3}$$

(ii) Since there is a natural inclusion $\mathrm{P}_d(\mathbb{R}) \subset \mathrm{P}_d(\mathbb{C})$, one can define a subspace $\mathrm{SP}^d_{\mathbb{R}} \subset \mathrm{SP}^d(\mathbb{C})$ as the image

$$\mathrm{SP}^d_{\mathbb{R}} = \Phi_d(\mathrm{P}_d(\mathbb{R})). \tag{7-4}$$

Then the restriction gives a homeomorphism

$$\Phi_d|\mathrm{P}_d(\mathbb{R}) : \mathrm{P}_d(\mathbb{R}) \xrightarrow{\cong} \mathrm{SP}^d_{\mathbb{R}}. \tag{7-5}$$

Note that any element $\alpha \in \mathrm{SP}^d_{\mathbb{R}}$ can be represented as the formal sum

$$\alpha = \sum_{k=1}^{s} d_k x_k + \sum_{j=1}^{t} e_j(y_j + \overline{y_j}) \quad (d_k, e_j \in \mathbb{N},\ x_k \in \mathbb{R},\ y_j \in \mathrm{H}_+), \tag{7-6}$$

where $x_k \neq x_l$ if $k \neq l$, $y_j \neq y_i$ if $j \neq i$ and $\sum_{k=1}^{s} d_k + 2\sum_{j=1}^{t} e_j = d$.

Definition 7.3 Let X be a space and $D = (d_1, \ldots, d_m) \in (\mathbb{Z}_{\geq 0})^m$ be an m-tuple of nonnegative integers.

(i) Let $\mathrm{SP}^0(X) = \{\varnothing\}$ and let $\mathrm{SP}^D(X)$ denote the space given by

$$\mathrm{SP}^D(X) = \mathrm{SP}^{(d_1,\ldots,d_m)}(X) = \mathrm{SP}^{d_1}(X) \times \mathrm{SP}^{d_2}(X) \times \cdots \times \mathrm{SP}^{d_m}(X). \tag{7-7}$$

We define the space $\mathrm{Pol}^{D,m}_n(X) = \mathrm{Pol}^{(d_1,\ldots,d_m),m}_n(X) \subset \mathrm{SP}^D(X)$ by

$$\mathrm{Pol}^{D,m}_n(X) = \mathrm{Pol}^{(d_1,\ldots,d_m),m}_n(X) = \{(\xi_1, \ldots, \xi_m) \in \mathrm{SP}^D(X) : (*)_n\}, \tag{7-8}$$

where the condition $(*)_n$ is given by

$(*)_n$ $\bigcap_{i=1}^{m} \xi_i$ does not contain any element of multiplicity $\geq n$.

(ii) When $X \subset \mathbb{C}$, let $\mathrm{SP}^D_{\mathbb{R}}(X)$ denote the space defined by

$$\mathrm{SP}^D_{\mathbb{R}}(X) = (\mathrm{SP}^{d_1}_{\mathbb{R}} \times \mathrm{SP}^{d_2}_{\mathbb{R}} \times \cdots \times \mathrm{SP}^{d_m}_{\mathbb{R}}) \cap \mathrm{SP}^D(X). \tag{7-9}$$

We define the space $\mathcal{Q}^{D,m}_n(X) = \mathcal{Q}^{(d_1,\ldots,d_m),m}_n(X) \subset \mathrm{SP}^D_{\mathbb{R}}(X)$ by

$$\mathcal{Q}^{D,m}_n(X) = \mathcal{Q}^{(d_1,\ldots,d_m),m}_n(X) = \{(\xi_1, \ldots, \xi_m) \in \mathrm{SP}^D_{\mathbb{R}}(X) : (*)^{\mathbb{R}}_n\}, \tag{7-10}$$

where the condition $(*)_n^{\mathbb{R}}$ is given by

$(*)_n^{\mathbb{R}}$ $\big(\bigcap_{i=1}^m \xi_i\big) \cap \mathbb{R}$ does not contain any element of multiplicity $\geq n$.

In particular, when $D_m = (d, \dots, d) \in \mathbb{N}^m$ (m times), we write

$$\mathrm{Pol}_n^{d,m}(X) = \mathrm{Pol}_n^{D_m,m}(X) \quad \text{and} \quad \mathcal{Q}_n^{d,m}(X) = \mathcal{Q}_n^{D_m,m}(X). \tag{7-11}$$

(iii) If $A \subset X$ is a closed subspace, we define the space $\mathrm{Pol}_n^{D,m}(X, A)$ by

$$\mathrm{Pol}_n^{D,m}(X, A) = \mathrm{Pol}_n^{(d_1,\dots,d_m),m}(X, A) = \mathrm{Pol}_n^{D,m}(X)/\sim, \tag{7-12}$$

where the equivalence relation "$\sim$" is defined by

$$(\xi_1, \dots, \xi_m) \sim (\eta_1, \dots, \eta_m) \quad \text{if } \xi_i \cap (X \setminus A) = \eta_i \cap (X \setminus A)$$

for all $1 \leq i \leq m$. Therefore, points in A are ignored. When $A \neq \varnothing$, there is a natural inclusion

$$\mathrm{Pol}_n^{(d_1,\dots,d_i,\dots,d_m),m}(X, A) \subset \mathrm{Pol}_n^{(d_1,\dots,d_i+1,\dots,d_m),m}(X, A)$$

by adding a point of A into the i-th part. Let $\mathrm{Pol}_n^{(0,\dots,0),m}(X, A) = \{\varnothing\}$ and define the space $\mathrm{Pol}_n^m(X, A)$ by the union

$$\mathrm{Pol}_n^m(X, A) = \bigcup_{(d_1,,\dots,d_m)\in(\mathbb{Z}_{\geq 0})^m} \mathrm{Pol}_n^{(d_1,\dots,d_m),m}(X, A). \tag{7-13}$$

(iv) If $X \subset \mathbb{C}$ and $A \subset X$ is a closed subspace, we also define the space $\mathcal{Q}_n^{D,m}(X, A)$ by

$$\mathcal{Q}_n^{D,m}(X, A) = \mathcal{Q}_n^{(d_1,\dots,d_m),m}(X, A) = \mathcal{Q}_n^{D,m}(X)/\sim, \tag{7-14}$$

where the equivalence relation "$\sim$" is defined by

$$(\xi_1, \dots, \xi_m) \sim (\eta_1, \dots, \eta_m) \quad \text{if } \xi_i \cap (X \setminus A) \cap \mathbb{R} = \eta_i \cap (X \setminus A) \cap \mathbb{R}$$

for all $1 \leq i \leq m$. Again, when $A \neq \varnothing$, there is a natural inclusion

$$\mathcal{Q}_n^{(d_1,\dots,d_i,\dots,d_m),m}(X, A) \subset \mathcal{Q}_n^{(d_1,\dots,d_i+1,\dots,d_m),m}(X, A)$$

by adding a point of A into the i-th part. Let $\mathcal{Q}_n^{(0,\dots,0),m}(X, A) = \{\varnothing\}$, and define the space $\mathcal{Q}_n^m(X, A)$ by the union

$$\mathcal{Q}_n^m(X, A) = \bigcup_{(d_1,\dots,d_m)\in(\mathbb{Z}_{\geq 0})^m} \mathcal{Q}_n^{(d_1,\dots,d_m),m}(X, A). \tag{7-15}$$

(v) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and define the stabilized spaces $\mathrm{Poly}_n^{\infty,m}(\mathbb{K})$ and $\mathrm{Q}_n^{\infty,m}(\mathbb{R})$ by the colimits

$$\mathrm{Poly}_n^{\infty,m}(\mathbb{K}) = \lim_{d\to\infty} \mathrm{Poly}_n^{d,m}(\mathbb{K}), \quad \mathrm{Q}_n^{\infty,m}(\mathbb{R}) = \lim_{d\to\infty} \mathrm{Q}_n^{d,m}(\mathbb{R}) \tag{7-16}$$

taken over the stabilization maps $\{s_{n,\mathbb{K}}^{d,m}\}$ and $\{s_n^{d,m}\}$ given by (5-8) and (5-9), respectively.

We will need two kinds of horizontal scanning maps. First, we define the scanning map for a configuration space of particles. From now on, we make the identification $\mathbb{C} = \mathbb{R}^2$ by identifying $\mathbb{C} \ni x + \sqrt{-1}y$ with $(x, y) \in \mathbb{R}^2$ in the usual way.

Definition 7.4 For a rectangle X in $\mathbb{C} = \mathbb{R}^2$, let σX denote the union of the sides of X which are parallel to the y-axis, and for a subspace $Z \subset \mathbb{C} = \mathbb{R}^2$, let $\overline{Z}$ denote the closure of Z.[9]

For example, if $X = [a, b] \times [c, d] = \{(t, s) \in \mathbb{R}^2 : a \le t \le b, c \le s \le d\}$, then $\sigma X = \{a, b\} \times [c, d] = \{(t, s) \in \mathbb{R}^2 : t \in \{a, b\}, c \le s \le d\}$.

From now on, let I denote the interval $I = [-1, 1]$ and let $\epsilon > 0$ be a fixed positive real number. For each $x \in \mathbb{R}$, let $V(x)$ be the set defined by

$$V(x) = \{w \in \mathbb{C} : |\mathrm{Re}(w) - x| < \epsilon, |\mathrm{Im}(w)| < 1\} = (x - \epsilon, x + \epsilon) \times (-1, 1), \tag{7-17}$$

and let us identify $I \times I = I^2$ with the closed unit rectangle

$$\{t + s\sqrt{-1} \in \mathbb{C} : -1 \le t,\ s \le 1\} \subset \mathbb{C}.$$

(i) First, we define the *horizontal scanning map*

$$sc_n^{d,m} : \mathrm{Q}_n^{d,m}(\mathbb{R}) \to \Omega \mathbb{Q}_n^m(I^2, \partial I \times I) = \Omega \mathbb{Q}_n^m(I^2, \sigma I^2) \tag{7-18}$$

of the space $\mathrm{Q}_n^{d,m}(\mathbb{R})$ as follows.

For each m-tuple $\alpha = (\xi_1, \dots, \xi_m) \in \mathbb{Q}_n^{d,m}(\mathbb{R})$ of configurations, let $sc_n^{d,m}(\alpha) : \mathbb{R} \to \mathbb{Q}_n^{d,m}(I^2, \partial I \times I) = \mathbb{Q}_n^{d,m}(I^2, \sigma I^2)$ denote the map given by

$$\mathbb{R} \ni x \mapsto (\xi_1 \cap \overline{V}(x), \dots, \xi_m \cap \overline{V}(x)) \in \mathbb{Q}_n^m(\overline{V}(x), \sigma\overline{V}(x)) \cong \mathbb{Q}_n^m(I^2, \sigma I^2),$$

where we use the canonical identification $(\overline{V}(x), \sigma\overline{V}(x)) \cong (I^2, \sigma I^2)$.

Since $\lim_{x \to \pm\infty} sc_n^{d,m}(\alpha)(x) = (\varnothing, \dots, \varnothing)$, by setting $sc_n^{d,m}(\alpha)(\infty) = (\varnothing, \dots, \varnothing)$ we obtain a based loop $sc_n^{d,m}(\alpha) \in \Omega \mathbb{Q}_n^m(I^2, \sigma I^2)$, where we identify S^1 with $\mathbb{R} \cup \infty$ and we choose the empty configuration $(\varnothing, \dots, \varnothing)$ as the basepoint of $\mathrm{Q}_n^m(I^2, \sigma I^2)$.

Finally, making the identifications $\mathrm{Q}_n^{d,m}(\mathbb{R}) \cong \mathbb{Q}_n^{d,m}(\mathbb{R}) = \mathbb{Q}_n^{D_m;m}(\mathbb{R})$, we obtain a map $sc_n^{d,m} : \mathrm{Q}_n^{d,m}(\mathbb{R}) \to \Omega \mathbb{Q}_n^m(I^2, \sigma I^2)$.

Since $sc_n^{d+1,m} \circ s_n^{d,m} \simeq sc_n^{d,m}$ (up to homotopy), by setting $S = \lim_{d \to \infty} sc_n^{d,m}$, we also obtain *the stable horizontal scanning map*

$$S : \mathrm{Q}_n^{\infty,m}(\mathbb{R}) \to \Omega \mathbb{Q}_n^m(I^2, \partial I \times I) = \Omega \mathbb{Q}_n^m(I^2, \sigma I^2), \tag{7-19}$$

where $\mathrm{Q}_n^{\infty,m}(\mathbb{R})$ denotes the stabilized space given by (7-16).

(ii) Next, we define the *horizontal scanning map*

$$sc_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{C}) \to \Omega\, \mathrm{Pol}_n^m(I^2, \partial I \times I) = \Omega\, \mathrm{Pol}_n^m(I^2, \sigma I^2) \tag{7-20}$$

of the space $\mathrm{Poly}_n^{d,m}(\mathbb{C})$ as follows.

[9] In general, a rectangle need not necessarily have edges parallel to x or y-axis. However, we only consider such rectangles.

For each m-tuple $\alpha = (\xi_1, \dots, \xi_m) \in \mathrm{Pol}_n^{d,m}(\mathbb{C})$ of configurations, let

$$sc_{n,\mathbb{C}}^{d,m}(\alpha) : \mathbb{R} \to \mathrm{Pol}_n^{d,m}(I^2, \partial I \times I) = \mathrm{Pol}_n^{d,m}(I^2, \sigma I^2)$$

denote the map given by

$$\mathbb{R} \ni x \mapsto (\xi_1 \cap \overline{V}(x), \dots, \xi_m \cap \overline{V}(x)) \in \mathrm{Pol}_n^m(\overline{V}(x), \sigma\overline{V}(x)) \cong \mathrm{Pol}_n^m(I^2, \sigma I^2),$$

where we use the canonical identification $(\overline{V}(x), \sigma\overline{V}(x)) \cong (I^2, \sigma I^2)$.

Since $\lim_{x\to\pm\infty} sc_{n,\mathbb{C}}^{d,m}(\alpha)(x) = (\varnothing, \dots, \varnothing)$, by setting $sc_{n,\mathbb{C}}^{d,m}(\alpha)(\infty) = (\varnothing, \dots, \varnothing)$ we obtain a based loop $sc_{n,\mathbb{C}}^{d,m}(\alpha) \in \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2)$, where we identify S^1 with $\mathbb{R} \cup \infty$ and choose the empty configuration $(\varnothing, \dots, \varnothing)$ as the basepoint of $\mathrm{Pol}_n^m(I^2, \sigma I^2)$.

Finally, by making the identifications $\mathrm{Poly}_n^{d,m}(\mathbb{C}) \cong \mathrm{Pol}_n^{d,m}(\mathbb{C}) = \mathrm{Pol}_n^{D_m,m}(\mathbb{C})$, we obtain a map $sc_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{C}) \to \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2)$.

Since $sc_{n,\mathbb{C}}^{d+1,m} \circ s_{n,\mathbb{C}}^{d,m} \simeq sc_{n,\mathbb{C}}^{d,m}$ (up to homotopy), by setting $S_{\mathbb{C}} = \lim_{d\to\infty} sc_{n,\mathbb{C}}^{d,m}$ we obtain *the stable horizontal scanning map*

$$S_{\mathbb{C}} : \mathrm{Poly}_n^{\infty,m}(\mathbb{C}) \to \Omega\mathrm{Pol}_n^m(I^2, \sigma I^2), \tag{7-21}$$

where $\mathrm{Poly}_n^{\infty,m}(\mathbb{C})$ denotes the stabilized space given by (7-16). Although the scanning map itself depends on the choice of ϵ, its homotopy class does not.

(iii) Let $\mathbb{Z}_2 = \{\pm 1\}$ denote the multiplicative cyclic group of order 2. Complex conjugation in $\mathbb{C}$ naturally induces a $\mathbb{Z}_2$-action on $\mathrm{Poly}_n^{D;m}(\mathbb{C})$. It is easy to see that its fixed-point set is $\mathrm{Poly}_n^{d,m}(\mathbb{C})^{\mathbb{Z}_2} = \mathrm{Poly}_n^{d,m}(\mathbb{R})$. Since the stabilization maps $\{s_{n,\mathbb{C}}^{d,m}\}_{d\geq 1}$ are $\mathbb{Z}_2$-equivariant, $\mathrm{Poly}_n^{\infty,m}(\mathbb{C})^{\mathbb{Z}_2} = \mathrm{Poly}_n^{\infty,m}(\mathbb{R})$. Moreover, since the scanning maps $\{sc_{n,\mathbb{C}}^{d,m}\}_{d\geq 1}$ are also $\mathbb{Z}_2$-equivariant, by setting $S_{\mathbb{R}} = S_{\mathbb{C}}|\mathrm{Poly}_n^{\infty,m}(\mathbb{R})$ we also obtain the *stable horizontal scanning map*

$$S_{\mathbb{R}} : \mathrm{Poly}_n^{\infty,m}(\mathbb{R}) \to \Omega\mathrm{Pol}_n^m(I^2, \sigma I^2)^{\mathbb{Z}_2}. \tag{7-22}$$

(iv) We define the map

$$r_I : \mathbb{Q}_n^m(I^2, \sigma I^2) \to \mathbb{Q}_n^m(I, \partial I) \tag{7-23}$$

by the restriction $r_I(\xi_1, \dots, \xi_m) = (\xi_1 \cap \mathbb{R}, \dots, \xi_m \cap \mathbb{R})$ for $(\xi_1, \dots, \xi_m) \in \mathbb{Q}_n^m(I^2, \sigma I^2)$.

Theorem 7.5 [17; 6] *If $mn \geq 3$, the stable horizontal scanning maps*

$$\begin{cases} S : \mathrm{Q}_n^{\infty,m}(\mathbb{R}) \xrightarrow{\simeq} \Omega\mathbb{Q}_n^m(I^2, \sigma I^2), \\ S_{\mathbb{C}} : \mathrm{Poly}_n^{\infty,m}(\mathbb{C}) \xrightarrow{\simeq} \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2), \\ S_{\mathbb{R}} : \mathrm{Poly}_n^{\infty,m}(\mathbb{R}) \xrightarrow{\simeq} \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2)^{\mathbb{Z}_2}, \end{cases}$$

are homotopy equivalences.

Proof The assertions can be proved by using the idea indicated in [17, Proposition 3.2 and Lemma 3.4] (and also in [6, Proposition 2]). However, since the argument in [17] is very sketchy, we provided a

detailed proof for the map S in [13, Theorem 5.6].[10] The proof for the map $S_{\mathbb{C}}$ carries over word for word to the present case, if we replace the condition "$(*)_n^{\mathbb{R}}$" by the condition "$(*)_n$". A very similar argument also works in the case of the map $S_{\mathbb{R}}$. □

Corollary 7.6 *If $mn \geq 3$, the map $S_{\mathbb{C}} : \mathrm{Poly}_n^{\infty,m}(\mathbb{C}) \stackrel{\simeq}{\Rightarrow} \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2)$ is a $\mathbb{Z}_2$-equivariant homotopy equivalence.*

Proof Consider the $\mathbb{Z}_2$-action on the space $\mathrm{Poly}_n^{\infty,m}(\mathbb{C})$ induced from the conjugation on $\mathbb{C}$. If we consider S^1 as a $\mathbb{Z}_2$-space with trivial $\mathbb{Z}_2$-action, we see that $(\Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2))^{\mathbb{Z}_2} = \Omega\mathbb{Q}_n^m(I^2, \sigma I^2)^{\mathbb{Z}_2}$. Since $(S_{\mathbb{C}})^{\mathbb{Z}_2} = S_{\mathbb{R}}$, the assertion easily follows from Theorem 7.5. □

Lemma 7.7 *The map $r_I : \mathbb{Q}_n^m(I^2, \sigma I^2) \stackrel{\simeq}{\Rightarrow} \mathbb{Q}_n^m(I, \partial I)$ is a deformation retraction.*

Proof We identify I^2 with $\{a + b\sqrt{-1} \in \mathbb{C} : -1 \leq a, b \leq 1\} \subset \mathbb{C}$ as before. Let $\Pi \subset I^2$ denote the subspace defined by $\Pi = \{a + b\sqrt{-1} \in I^2 : b \in \{0, \pm\frac{1}{2}\}\}$. For $b \in \mathbb{R}$, let $\epsilon(b) = \frac{b}{|b|}$ if $b \neq 0$ and $\epsilon(0) = 0$. Now consider the homotopy $\varphi : I^2 \times [0,1] \to I^2$ given by $\varphi(\alpha, t) = a + \{(1-t)b + \frac{\epsilon(b)t}{2}\}\sqrt{-1}$ for $\alpha = a + b\sqrt{-1} \in I^2$ $(a, b \in \mathbb{R})$. By means of this homotopy, one can define a deformation retraction $R : \mathbb{Q}_n^m(I^2, \sigma I^2) \stackrel{\simeq}{\Rightarrow} \mathbb{Q}_n^m\big(\Pi, \partial I \times \{0, \pm\frac{1}{2}\}\big)$.

Next, by using the homotopy given by $f_t(a + b\sqrt{-1}) = ta + (1-t) + b\sqrt{-1}$ if $b = \pm\frac{1}{2}$ and $f_t(a + b\sqrt{-1}) = a$ if $b = 0$, one can also define a deformation retraction $\varphi : \mathbb{Q}_n^m\big(\Pi, \partial I \times \{0, \pm\frac{1}{2}\}\big) \stackrel{\simeq}{\Rightarrow} \mathbb{Q}_n^m(I, \partial I)$. Since $r_I = \varphi \circ R$, it is also a deformation retraction. □

Proposition 7.8 (i) *If $mn \geq 3$, there is a homotopy equivalence*

$$\Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2) \stackrel{\simeq}{\Rightarrow} \Omega^2 S^{2mn-1}.$$

(ii) *If $mn \geq 3$, there is a homotopy equivalence*

$$\Omega\mathbb{Q}_n^m(I, \partial I) \stackrel{\simeq}{\Rightarrow} \Omega S^{mn-1}.$$

Proof (i) It follows from [11, Theorem 1.8] that there is a homotopy equivalence $\mathrm{Poly}_n^{\infty,m}(\mathbb{C}) \simeq \Omega^2 S^{2mn-1}$. Since the map $S_{\mathbb{C}}$ is a homotopy equivalence by Theorem 7.5, we obtain the homotopy equivalence $\Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2) \simeq \Omega^2 S^{2mn-1}$.

(ii) Consider the map $\widehat{S} : \mathrm{Q}_n^{\infty,m}(\mathbb{R}) \to \Omega\mathbb{Q}_n^m(I, \partial I)$ defined as the composite of maps $\widehat{S} = (\Omega r_I) \circ S$. It follows from Theorem 7.5 and Lemma 7.7 that $\widehat{S}$ is a homotopy equivalence. Moreover, it follows from Theorem 2.6 that there is a map $i_n^{\infty,m} = \lim_{d\to\infty} i_n^{d,m} : \mathrm{Q}_n^{\infty,m}(\mathbb{R}) \stackrel{\simeq}{\Rightarrow} \Omega S^{mn-1}$ which is a homotopy equivalence for $mn \geq 4$ and is a homology equivalence for $mn = 3$. Now consider the composite map

$$\Omega\mathbb{Q}_n^m(I, \partial I) \stackrel{\widetilde{S}}{\underset{\simeq}{\Rightarrow}} \mathrm{Q}_n^{\infty,m}(\mathbb{R}) \xrightarrow[\simeq]{i_n^{\infty,m}} \Omega S^{mn-1},$$

where $\widetilde{S}$ denotes a homotopy inverse of $\widehat{S}$. This map is the desired homotopy equivalence if $mn \geq 4$ and homology equivalence if $mn = 3$. Since two spaces $\Omega\mathbb{Q}_n^m(I, \partial I)$ and ΩS^{mn-1} are loop spaces, they are simple. Thus, this map is indeed a homotopy equivalence even if $mn = 3$. □

[10] Although the proof of [13, Theorem 5.6] was given under the condition $mn \geq 4$, the same proof works for the case $mn = 3$.

Theorem 7.9 *If $mn \geq 3$, there is a homotopy equivalence*

$$\mathrm{Poly}_n^{\infty,m}(\mathbb{R}) \xrightarrow{\simeq} \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}.$$

Proof Since $S_{\mathbb{R}}$ is a homotopy equivalence, it follows from Proposition 7.8 that it suffices to prove that there is a homotopy equivalence

$$\Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2)^{\mathbb{Z}_2} \simeq \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2) \times \Omega \mathcal{Q}_n^m(I, \partial I). \tag{7-24}$$

Consider the map $\hat{r}_I : \mathrm{Pol}_n^m(I^2, \sigma I^2)^{\mathbb{Z}_2} \to \mathcal{Q}_n^m(I, \partial I)$ given by the restriction

$$\hat{r}_I(\xi_1, \dots, \xi_m) = (\xi_1 \cap \mathbb{R}, \dots, \xi_m \cap \mathbb{R}).$$

Note that $\hat{r}_I$ is a quasifibration with fiber

$$\begin{aligned} F &= \mathrm{Pol}_n^m(I \times ([-1,0) \cup (0,1]), \partial I \times ([-1,0) \cup (0,1]))^{\mathbb{Z}_2} \\ &\cong \mathrm{Pol}_n^m(I \times (0,1], \partial I \times (0,1]) = \bigcup_{0<\epsilon<1} \mathrm{Pol}_n^m(I \times [\epsilon,1], \partial I \times [\epsilon,1]). \end{aligned}$$

Since the space $\mathrm{P}(\epsilon) = \mathrm{Pol}_n^m(I \times [\epsilon,1], \partial I \times [\epsilon,1])$ is homotopy equivalent to the space $\mathrm{P}\big(\frac{1}{2}\big) = \mathrm{Pol}_n^m\big(I \times \big[\frac{1}{2},1\big], \partial I \times \big[\frac{1}{2},1\big]\big)$ for any $0 < \epsilon < 1$ by radial expansion, there is a homotopy equivalence

$$F \simeq \mathrm{P}\big(\tfrac{1}{2}\big) = \mathrm{Pol}_n^m\big(I \times \big[\tfrac{1}{2},1\big], \partial I \times \big[\tfrac{1}{2},1\big]\big) \cong \mathrm{Pol}_n^m(I^2, \sigma I^2).$$

Thus, we obtain a fibration sequence (up to homotopy)

$$\mathrm{Pol}_n^m(I^2, \sigma I^2) \xrightarrow{s_I} \mathrm{Pol}_n^m(I^2, \sigma I^2)^{\mathbb{Z}_2} \xrightarrow{\hat{r}_I} \mathcal{Q}_n^m(I, \partial I). \tag{7-25}$$

Let $i : (I, \partial I) \xhookrightarrow{\subset} (I^2, \sigma I^2)$ denote the natural inclusion map given by $i(x) = (x, 0)$ for $x \in I$. This inclusion naturally extends to the map $i_\# : \mathcal{Q}_n^m(I, \partial I) \to \mathrm{Pol}_n^m(I^2, \sigma I^2)$. Since $\hat{r}_I \circ i_\# = \mathrm{id}_{\mathcal{Q}_n^m(I,\partial I)}$, the loop sum map

$$\Omega s_I + \Omega i_\# : \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2) \times \Omega \mathcal{Q}_n^m(I, \partial I) \xrightarrow{\simeq} \Omega\,\mathrm{Pol}_n^m(I^2, \sigma I^2)^{\mathbb{Z}_2}$$

is a homotopy equivalence. □

Corollary 7.10 *If $mn \geq 3$, there is a homotopy equivalence*

$$j_n^{\infty,m} : \mathrm{Poly}_n^{\infty,m}(\mathbb{R}) \xrightarrow{\simeq} (\Omega^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}.$$

Proof The assertion follows from Theorem 7.9 and (3-6). □

Corollary 7.11 *There is a map*

$$f_n^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} \simeq \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}$$

which is a homotopy equivalence through dimension $D(d; m, n)$ if $mn \geq 4$, and a homology equivalence through dimension $D(d; m, n)$ if $mn = 3$.

Proof Let $\hat{\iota}_n^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to \mathrm{Poly}_n^{\infty,m}(\mathbb{R})$ denote the natural map, and let us consider the composite of maps

$$f_n^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \xrightarrow{\hat{\iota}_n^{d,m}} \mathrm{Poly}_n^{\infty,m}(\mathbb{R}) \xrightarrow[\simeq]{j_n^{\infty,m}} (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}.$$

Then by using Theorem 6.2, Corollaries 6.4 and 7.10, we see that the map $f_n^{d,m}$ is a homotopy equivalence through dimension $D(d;m,n)$ if $mn \geq 4$ and it is a homology equivalence through dimension $D(d;m,n)$ if $mn = 3$. □

8 The homotopy type of $\mathrm{Poly}_n^{d,m}(\mathbb{R})$

In this section we give proofs of our main results (Theorems 2.7 and 2.11) and their corollaries (Corollaries 2.9, 2.10, 2.13 and 2.14).

Before giving the proof of Theorem 2.7, we make some general comments about it. It seems plausible that one could give a proof of Theorem 2.7 by a method similar to that used in the proof of [11, Theorem 1.8]. However, this approach would require a study of $\mathbb{Z}_2$-equivariant homotopy of spaces of $\mathbb{Z}_2$-equivariant maps, which seems difficult. For this reason we decided to use an indirect approach, combining the corresponding results given in [11, Theorem 1.8; 13, Theorem 1.8].

Proof of Theorem 2.7 Assume that $mn \geq 3$. From now on, we use the same notations as in Lemma 3.5. We denote by

$$r_{S^1} : \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^{mn-1}, \mathbb{R}\mathrm{P}^{mn-1}) \to \Omega_0 \mathbb{R}\mathrm{P}^{mn-1} \tag{8-1}$$

the restriction map given by $r_{S^1}(f) = f|S^1$, and let

$$\Omega^2 S^{2mn-1} \xleftarrow{q_1} \Omega^2 S^{2mn-1} \times \Omega S^{mn-1} \xrightarrow{q_2} \Omega S^{mn-1} \tag{8-2}$$

denote the projections onto the first and the second factor, respectively. Then if we write

$$\mathrm{Map}_0^* = \mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^{mn-1}, \mathbb{R}\mathrm{P}^{mn-1}), \tag{8-3}$$

it follows from diagram (3-13) and the definitions of the three natural maps $i_{n,\mathrm{H}_+}^{d,m}$, $i_{n,\mathbb{R}}^{d,m}$ and $i_n^{d,m}$ that the following diagram is homotopy commutative:

$$\begin{array}{ccccc}
\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+) & \xrightarrow[\subset]{\iota_{n,\mathrm{H}_+}^{d,m}} & \mathrm{Poly}_n^{d,m}(\mathbb{R}) & \xrightarrow[\subset]{\iota_n^{d,m}} & \mathrm{Q}_n^{d,m}(\mathbb{R}) \\
\downarrow{\scriptstyle i_{n,\mathrm{H}+}^{d,m}} & & \downarrow{\scriptstyle i_{n,\mathbb{R}}^{d,m}} & & \downarrow{\scriptstyle i_n^{d,m}} \\
\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1} & & (\Omega_d^2 \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2} & & \Omega_{[d]_2} \mathbb{R}\mathrm{P}^{mn-1} \\
\downarrow{\scriptstyle \iota'_{\mathbb{C}}}{\scriptstyle \simeq} & & \downarrow{\scriptstyle \iota'}{\scriptstyle \simeq} & & \downarrow{\scriptstyle \iota'_{\mathbb{R}}}{\scriptstyle \simeq} \\
\mathrm{Map}_0^*(D^2, S^1; \mathbb{C}\mathrm{P}^{mn-1}, *) & \xrightarrow[\subset]{\hat{j}} & \mathrm{Map}_0^* & \xrightarrow{r_{S^1}} & \Omega_0 \mathbb{R}\mathrm{P}^{mn-1} \\
\downarrow{\scriptstyle \iota_{\mathbb{C}}}{\scriptstyle \simeq} & & \downarrow{\scriptstyle \iota_{\mathbb{C}\mathbb{R}}}{\scriptstyle \simeq} & & \downarrow{\scriptstyle \iota_{\mathbb{R}}}{\scriptstyle \simeq} \\
\Omega^2 S^{2mn-1} & \xleftarrow{q_1} & \Omega^2 S^{2mn-1} \times \Omega S^{mn-1} & \xrightarrow{q_2} & \Omega S^{mn-1}
\end{array} \tag{8-4}$$

Now consider the following three maps given by

$$\text{(8-5)}\qquad \begin{cases} I_n^{d,m} = \iota_{\mathbb{C}\mathbb{R}} \circ \iota' \circ i_{n,\mathbb{R}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to \Omega^2 S^{2mn-1} \times \Omega S^{mn-1}, \\ I_{n,\mathrm{H}_+}^{d,m} = \iota_{\mathbb{C}} \circ \iota'_{\mathbb{C}} \circ i_{n,\mathrm{H}_+}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+) \to \Omega^2 S^{2mn-1}, \\ I_{n,\mathbb{R}}^{d,m} = \iota_{\mathbb{R}} \circ \iota'_{\mathbb{R}} \circ i_n^{d,m} : \mathrm{Q}_n^{d,m}(\mathbb{R}) \to \Omega S^{mn-1}. \end{cases}$$

Let $\mathbb{F}$ be any fixed field. It follows from Theorems 3.8 and 2.6 that the two maps $I_{n,\mathrm{H}_+}^{d,m}$ and $I_{n,\mathbb{R}}^{d,m}$ are homology equivalences through dimension $D(\lfloor d/2 \rfloor; m, n; \mathbb{C})$ and $D(d; m, n)$, respectively.

Since $D(d; m, n) < D(\lfloor d/2 \rfloor; m, n; \mathbb{C})$ (by Lemma 3.9), by using the diagram (8-4), we see that the induced homomorphism

$$\text{(8-6)}\qquad (I_n^{d,m})_* : H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R}); \mathbb{F}) \to H_s(\Omega^2 S^{2mn-1} \times \Omega S^{mn-1}; \mathbb{F})$$

is an epimorphism for any $s \le D(d; m, n)$. However, since

$$\dim_{\mathbb{F}} H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R}); \mathbb{F}) = \dim_{\mathbb{F}} H_s(\Omega^2 S^{2mn-1} \times \Omega S^{mn-1}; \mathbb{F}) < \infty$$

for any $s \le D(d; m, n)$ (by Corollary 7.11), we notice that the homomorphism $(I_n^{d,m})_*$ (given by (8-6)) is an isomorphism for any $s \le D(d; m, n)$.

Then, by putting $\mathbb{F} = \mathbb{Z}/p$ (p: prime) or $\mathbb{Q}$, it follows from the universal coefficient Theorem that the map $I_n^{d,m}$ induces an isomorphism on the homology group $H_s(\,;\mathbb{Z})$ for any $s \le D(d; m, n)$. Thus, we see that the map $i_{n,\mathbb{R}}^{d,m}$ is a homology equivalence through dimension $D(d; m, n)$.

Next, assume that $mn \ge 4$. Then by Lemma 6.3, we see that two spaces $\mathrm{Poly}_n^{d,m}(\mathbb{R})$ and $\Omega^2 S^{2mn-1} \times \Omega S^{mn-1}$ are simply connected. Hence, we obtain that the map $i_{n,\mathbb{R}}^{d,m}$ is a homotopy equivalence through dimension $D(d; m, n)$. This completes the proof of Theorem 2.7. □

Proof of Corollary 2.9 The assertion follows easily from the proof of Theorem 2.7, so we omit the details. □

Proof of Corollary 2.10 Consider the commutative diagram

$$\text{(8-7)}\qquad \begin{array}{ccccc} \mathrm{Poly}_n^{d,1}(\mathbb{R}) & \xrightarrow{i_{n,\mathbb{R}}^{d,1}} & (\Omega_d^2 \mathbb{C}\mathrm{P}^{n-1})^{\mathbb{Z}_2} & \xrightarrow{\simeq} & \Omega^2 S^{2n-1} \times \Omega S^{n-1} \\ {\scriptstyle j_n^d}\downarrow & & \| & & \| \\ \mathrm{Poly}_1^{d,n}(\mathbb{R}) & \xrightarrow{i_{1,\mathbb{R}}^{d,n}} & (\Omega_d^2 \mathbb{C}\mathrm{P}^{n-1})^{\mathbb{Z}_2} & \xrightarrow{\simeq} & \Omega^2 S^{2n-1} \times \Omega S^{n-1} \end{array}$$

First, consider the case $n \ge 4$. It follows from Theorem 2.7 that the maps $i_{n,\mathbb{R}}^{d,1}$ and $i_{1,\mathbb{R}}^{d,n}$ are homotopy equivalences through dimension $D(d; 1, n)$ and $D(d; n, 1)$, respectively. Since $D(d; 1, n) < D(d; n, 1)$, by using the diagram (8-7) we see that the map j_n^d is a homotopy equivalence through dimension $D(d; 1, n) = (n-2)(\lfloor d/n \rfloor + 1) - 1$.

Next, if $n = 3$, by Theorem 2.7 we see that the maps $i_{3,\mathbb{R}}^{d,1}$ and $i_{1,\mathbb{R}}^{d,3}$ are homology equivalences through dimension $D(d; 1, 3) = \lfloor d/3 \rfloor$ and $D(d; 3, 1) = d$, respectively. Thus, we see that the map j_3^d is a homology equivalence through dimension $\lfloor d/3 \rfloor$. □

Corollary 8.1 (i) *If $mn = 3$ and $d \geq n$, two maps $i^{d,m}_{n,\mathbb{R}}$ and $s^{d,m}_{n,\mathbb{R}}$ induce isomorphisms*

$$\begin{cases} (i^{d,m}_{n,\mathbb{R}})_* : \pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) \xrightarrow{\cong} \pi_1((\Omega^2_d \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2}) \cong \pi_1(\Omega^2 S^5 \times \Omega S^2) \cong \mathbb{Z}, \\ (s^{d,m}_{n,\mathbb{R}})_* : \pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) \xrightarrow{\cong} \pi_1(\mathrm{Poly}^{d+1,m}_n(\mathbb{R})) \cong \mathbb{Z}. \end{cases}$$

(ii) *If $d \geq 3$, the map $j^d_3 : \mathrm{Poly}^{d,1}_3(\mathbb{R}) \to \mathrm{Poly}^{d,3}_1(\mathbb{R})$ induces an isomorphism*

$$(j^d_3)_* : \pi_1(\mathrm{Poly}^{d,1}_3(\mathbb{R})) \xrightarrow{\cong} \pi_1(\mathrm{Poly}^{d,3}_1(\mathbb{R})) \cong \mathbb{Z}.$$

Proof (i) Let $mn = 3$ and consider the commutative diagram

$$\begin{array}{ccccc} \pi_1(\mathrm{Poly}^{d+1,m}_n(\mathbb{R})) & \xleftarrow{(s^{d,m}_{n,\mathbb{R}})_*} & \pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) & \xrightarrow{(i^{d,m}_{n,\mathbb{R}})_*} & \pi_1((\Omega^2_d \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2}) \\ \downarrow{\scriptstyle h_1}\cong & & \downarrow{\scriptstyle h_2}\cong & & \downarrow{\scriptstyle h_3}\cong \\ H_1(\mathrm{Poly}^{d+1,m}_n(\mathbb{R});\mathbb{Z}) & \xleftarrow[\cong]{(s^{d,m}_{n,\mathbb{R}})_\#} & H_1(\mathrm{Poly}^{d,m}_n(\mathbb{R});\mathbb{Z}) & \xrightarrow[\cong]{(i^{d,m}_{n,\mathbb{R}})_\#} & H_1((\Omega^2_d \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2};\mathbb{Z}) \end{array}$$

where h_k $(k = 1, 2, 3)$ are corresponding Hurewicz homomorphisms.

Since $\pi_1(\mathrm{Poly}^{d,m}_n(\mathbb{R})) \cong \pi_1(\mathrm{Poly}^{d+1,m}_n(\mathbb{R})) \cong \pi_1((\Omega^2_d \mathbb{C}\mathrm{P}^{mn-1})^{\mathbb{Z}_2}) \cong \mathbb{Z}$ by Lemma 6.3, by the Hurewicz theorem each h_k is an isomorphism. It follows from Theorems 2.7 and 6.2 that $(i^{d,m}_{n,\mathbb{R}})_\#$ and $(s^{d,m}_{n,\mathbb{R}})_\#$ are isomorphisms, and the assertion (i) easily follows.

(ii) Consider the commutative diagram

$$\begin{array}{ccccc} \pi_1(\mathrm{Poly}^{d,1}_3(\mathbb{R})) & \xrightarrow[\cong]{(i^{d,1}_{3,\mathbb{R}})_*} & \pi_1((\Omega^2_d \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2}) & \xrightarrow[\cong]{} & \pi_1(\Omega^2 S^5 \times \Omega S^2) = \mathbb{Z} \\ \downarrow{\scriptstyle (j^d_3)_*} & & \| & & \| \\ \pi_1(\mathrm{Poly}^{d,3}_1(\mathbb{R})) & \xrightarrow[\cong]{(i^{d,1}_{3,\mathbb{R}})_*} & \pi_1((\Omega^2_d \mathbb{C}\mathrm{P}^2)^{\mathbb{Z}_2}) & \xrightarrow[\cong]{} & \pi_1(\Omega^2 S^5 \times \Omega S^2) = \mathbb{Z} \end{array}$$

Since $(i^{d,1}_{3,\mathbb{R}})_*$ and $(i^{d,3}_{1,\mathbb{R}})_*$ are isomorphisms (by (i)), the homomorphism $(j^d_3)_*$ is also an isomorphism. □

Next, consider the stable homotopy type of $\mathrm{Poly}^{d,m}_n(\mathbb{R})$. For this purpose, recall the following result:

Lemma 8.2 *For any field $\mathbb{F}$ and any $s \geq 1$,*

$$\dim_{\mathbb{F}} H_s(\mathrm{Poly}^{d,m}_n(\mathbb{R});\mathbb{F}) \leq \dim_{\mathbb{F}} H_s(\mathrm{P}^{d,m}_n;\mathbb{F}) < \infty.$$

Proof Consider the Vassiliev-type spectral sequence

$$\{E^t_{k,s}(d), d^t : E^t_{k,s}(d) \to E^t_{k+t,s+t-1}(d)\} \Rightarrow H_{s-k}(\mathrm{Poly}^{d,m}_n(\mathbb{R});\mathbb{F}).$$

Since $E^1_s = \bigoplus_{k\in\mathbb{Z}} E^1_{k,k+s}(d) \cong H_s(\mathrm{P}^{d,m}_n;\mathbb{F})$ for any $s \geq 1$ by Lemma 4.12, the assertion follows. □

Now we can give the proof of Theorem 2.11.

Proof of Theorem 2.11 From now on, let $d_0 = \lfloor d/2 \rfloor$, and we write

$$\mathrm{Poly}_n^{d_0,m} = \mathrm{Poly}_n^{d_0,m}(\mathbb{C}), \quad \mathrm{Q}_n^{d,m} = \mathrm{Q}_n^{d,m}(\mathbb{R}). \tag{8-8}$$

Let $\mathcal{F}_n^{d,m} = \{(i,j) \in \mathbb{N}^2 : i + 2j \le \lfloor d/n \rfloor\}$ as in (3-21), and recall the spaces $B_n^{d,m}$ and $\mathrm{P}_n^{d,m}(\mathbb{R})$ given by (see (2-20) and (4-27))

$$B_n^{d,m} = \bigvee_{(i,j)\in\mathcal{F}_n^{d,m}} S^{(mn-2)i} \wedge \Sigma^{2(mn-2)j} D_j, \quad \mathrm{P}_n^{d,m} = \mathrm{Poly}_n^{d_0,m} \vee B_n^{d,m} \vee \mathrm{Q}_n^{d,m}.$$

It follows from (2-16) and (4-26) that there are stable homotopy equivalences

$$\theta_{\mathbb{C}} : \bigvee_{j=1}^{\lfloor d_0/n \rfloor} \Sigma^{2(mn-2)j} D_j \xrightarrow{\simeq_s} \mathrm{Poly}_n^{d_0,m}, \quad \theta_{\mathbb{R}} : \bigvee_{i=1}^{\lfloor d/n \rfloor} S^{(mn-2)i} \xrightarrow{\simeq_s} \mathrm{Q}_n^{d,m}. \tag{8-9}$$

Hence, the space $\mathrm{P}_n^{d,m}$ is stably homotopy equivalent to the space

$$\Big(\bigvee_{j=1}^{\lfloor d_0/n \rfloor} \Sigma^{2(mn-2)j} D_j\Big) \vee \Big(\bigvee_{(i,j)\in\mathcal{F}_n^{d,m}} (S^{(mn-2)i} \wedge \Sigma^{2(mn-2)j} D_j)\Big) \vee \Big(\bigvee_{i=1}^{\lfloor d/n \rfloor} S^{(mn-2)i}\Big).$$

It follows from [8] and the Snaith splitting [18] that there are two stable homotopy equivalences

$$\Omega^2 S^{2mn-1} \simeq_s A_n^m(1) := \bigvee_{j=1}^{\infty} \Sigma^{2(mn-2)j} D_j, \quad \Omega S^{mn-1} \simeq_s A_n^m(3) := \bigvee_{i=1}^{\infty} S^{(mn-2)i}. \tag{8-10}$$

Thus, there is a stable homotopy equivalence

$$\Omega^2 S^{2mn-1} \times \Omega S^{mn-1} \simeq_s \bigvee_{k=1}^{3} A_n^m(k) = A_n^m(1) \vee A_n^m(2) \vee A_n^m(3),$$

where the space $A_n^m(2)$ is defined by

$$A_n^m(2) = A_n^m(1) \wedge A_n^m(3) = \bigvee_{i,j\ge 1} S^{(mn-2)i} \wedge \Sigma^{2(mn-2)j} D_j. \tag{8-11}$$

Let q_1' and q_3' denote the natural projections to the first and the third factors given by

$$A_n^m(1) \xleftarrow{q_1'} A_n^m(1) \vee A_n^m(2) \vee A_n^m(3) \xrightarrow{q_3'} A_n^m(3). \tag{8-12}$$

Similarly, let $p_{\mathbb{C}}$ and $p_{\mathbb{R}}$ also denote the natural projections to the first and the third factors given by

$$\mathrm{Poly}_n^{\lfloor d_0 \rfloor,m} \xleftarrow{p_{\mathbb{C}}} \mathrm{P}_n^{d,m} = \mathrm{Poly}_n^{\lfloor d_0 \rfloor,m} \vee B_n^{d,m} \vee \mathrm{Q}_n^{d,m} \xrightarrow{p_{\mathbb{R}}} \mathrm{Q}_n^{d,m}. \tag{8-13}$$

Let p_1, p_2 and p_3 denote the corresponding natural pinching maps given by

$$p_1 : A_n^m(1) = \bigvee_{j=1}^{\infty} \Sigma^{2(mn-2)j} D_j \to \mathrm{P}(1) := \bigvee_{j=1}^{\lfloor d_0/n \rfloor} \Sigma^{2(mn-2)j} D_j,$$

$$p_2 : A_n^m(2) = \bigvee_{i,j\geq 1} S^{(mn-2)i} \wedge \Sigma^{2(mn-2)j} D_j \to \bigvee_{(i,j)\in \mathscr{F}_n^{d,m}} S^{(mn-2)i} \wedge \Sigma^{2(mn-2)j} D_j,$$

$$p_3 : A_n^m(3) = \bigvee_{i=1}^{\infty} S^{(mn-2)i} \to \mathrm{P}(3) := \bigvee_{i=1}^{\lfloor d/n \rfloor} S^{(mn-2)i}.$$

By the diagram (8-4) and (8-5), we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+) & \xrightarrow[\subset]{\iota_{n,\mathrm{H}_+}^{d,m}} & \mathrm{Poly}_n^{d,m}(\mathbb{R}) & \xrightarrow[\subset]{\iota_n^{d,m}} & \mathrm{Q}_n^{d,m}(\mathbb{R}) \\
\downarrow{\scriptstyle I_{n,\mathrm{H}_+}^{d,m}} & & \downarrow{\scriptstyle I_n^{d,m}} & & \downarrow{\scriptstyle I_{n,\mathbb{R}}^{d,m}} \\
\Omega^2 S^{2mn-1} & \xleftarrow{q_1} & \Omega^2 S^{2mn-1} \times \Omega S^{mn-1} & \xrightarrow{q_2} & \Omega S^{mn-1} \\
\downarrow{\scriptstyle \pi'}\simeq_s & & \downarrow{\scriptstyle \pi}\simeq_s & & \downarrow{\scriptstyle \pi''}\simeq_s \\
A_n^m(1) & \xleftarrow{q_1'} & A_1^m(1) \vee A_n^m(2) \vee A_n^m(3) & \xrightarrow{q_3'} & A_n^m(3) \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1\vee p_2\vee p_3} & & \downarrow{\scriptstyle p_3} \\
\mathrm{P}(1) & \xleftarrow{p_1'} & \mathrm{P}(1) \vee B_n^{d,m} \vee \mathrm{P}(3) & \xrightarrow{p_3'} & \mathrm{P}(3) \\
\downarrow{\scriptstyle \theta_{\mathbb{C}}}\simeq_s & & \downarrow{\scriptstyle \theta_{\mathbb{C}}\vee \mathrm{id}\vee\theta_{\mathbb{R}}}\simeq_s & & \downarrow{\scriptstyle \theta_{\mathbb{R}}}\simeq_s \\
\mathrm{Poly}_n^{d_0,m} & \xleftarrow{p_{\mathbb{C}}} & \mathrm{P}_n^{d,m} = \mathrm{Poly}_n^{d_0,m} \vee B_n^{d,m} \vee \mathrm{Q}_n^{d,m} & \xrightarrow{p_{\mathbb{R}}} & \mathrm{Q}_n^{d,m}
\end{array} \tag{8-14}$$

Now consider the three maps

$$\begin{cases} J_{n,\mathbb{C}}^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+) \to \mathrm{Poly}_n^{d_0,m}, \\ J_{n,\mathbb{R}}^{d,m} : \mathrm{Q}_n^{d,m} \to \mathrm{Q}_n^{d,m}, \\ J_n^{d,m} : \mathrm{Poly}_n^{d,m}(\mathbb{R}) \to \mathrm{P}_n^{d,m} = \mathrm{Poly}_n^{d_0,m} \vee B_n^{d,m} \vee \mathrm{Q}_n^{d,m}, \end{cases} \tag{8-15}$$

defined by

$$\begin{cases} J_{n,\mathbb{C}}^{d,m} = \theta_{\mathbb{C}} \circ p_1 \circ \pi' \circ I_{n,\mathrm{H}_+}^{d,m}, \\ J_{n,\mathbb{R}}^{d,m} = \theta_{\mathbb{R}} \circ p_3 \circ \pi'' \circ I_{n,\mathbb{R}}^{d,m}, \\ J_n^{d,m} = (\theta_{\mathbb{C}} \vee \mathrm{id} \vee \theta_{\mathbb{R}}) \circ (p_1 \vee p_2 \vee p_3) \circ \pi \circ I_n^{d,m}. \end{cases} \tag{8-16}$$

It suffices to prove that the map $J_n^{d,m}$ is a stable homotopy equivalence. It is easy to see that

$$J_{n,\mathbb{R}}^{d,m} = \mathrm{id} : \mathrm{Q}_n^{d,m}(\mathbb{R}) \to \mathrm{Q}_n^{d,m}(\mathbb{R}) \quad \text{(up to homotopy)}. \tag{8-17}$$

Now let $\mathbb{F}$ be any fixed field, and consider the homomorphism

$$(J_{n,\mathbb{C}}^{d,m})_* : H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+);\mathbb{F}) \to H_s(\mathrm{Poly}_n^{d_0,m};\mathbb{F}) \quad \text{for } s \geq 1.$$

We will need the following lemma.

Lemma 8.3 *The induced homomorphism*

$$(J_n^{d,m})_* : H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{F}) \to H_s(\mathrm{P}_m^{d,m};\mathbb{F}) \tag{8-18}$$

is an epimorphism for any $s \geq 1$ and for any field $\mathbb{F}$.

We postpone the proof of Lemma 8.3, and first complete the proof of Theorem 2.11 by using this lemma. By Lemma 8.3 we have

$$\dim_{\mathbb{F}} H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{F}) \geq \dim_{\mathbb{F}} H_s(\mathrm{P}_n^{d,m};\mathbb{F}) \quad \text{for any } s \geq 1. \tag{8-19}$$

Combining this with Lemma 8.2 we obtain the equality:

$$\dim_{\mathbb{F}} H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{F}) = \dim_{\mathbb{F}} H_s(\mathrm{P}_n^{d,m};\mathbb{F}) < \infty \quad \text{for any } s \geq 1. \tag{8-20}$$

Since $(J_n^{d,m})_*$ is an epimorphism, (8-20) implies that it is, in fact, an isomorphism for any $s \geq 1$. By putting $\mathbb{F} = \mathbb{Z}/p$ (p: prime) or $\mathbb{F} = \mathbb{Q}$, and using the universal coefficients theorem, we conclude that

$$(J_n^{d,m})_* : H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{Z}) \xrightarrow{\cong} H_s(\mathrm{P}_n^{d,m};\mathbb{Z})$$

is an isomorphism for every $s \geq 1$. Hence, the map $J_n^{d,m}$ is a stable homotopy equivalence. This completes the proof of Theorem 2.11. □

Proof of Lemma 8.3 It follows from Corollary 2.9, Lemma 3.7 and (8-17) that the maps $J_{n,\mathbb{C}}^{d,m}$ and $J_{n,\mathbb{R}}^{d,m}$ induce epimorphisms on homology groups $H_*(\,;\mathbb{F})$. Moreover, since $\mathrm{P}_n^{d,m} = \mathrm{Poly}_n^{d_0,m}(\mathbb{C}) \vee B_n^{d,m} \vee \mathrm{Q}_n^{d,m}(\mathbb{R})$ (by (4-27)), to prove Lemma 8.3, it suffices to prove the following assertion:

(†) If $x \neq 0 \in H_s(B_n^{d,m};\mathbb{F})$, there is an element $y \in H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{F})$ such that $(J_n^{d,m})_*(y) = x$.

Since $B_n^{d,m} = \bigvee_{(i,j)\in\mathcal{F}_n^{d,m}} \Sigma^{(mn-2)(i+2j)} D_j$, we may assume, without loss of generality, that $x \in H_s(\Sigma^{(mn-2)(i+2j)} D_j;\mathbb{F}) \cong H_{s-(mn-2)i}(\Sigma^{2(mn-2)j} D_j;\mathbb{F})$ for some $(i,j) \in \mathcal{F}_n^{d,m}$. By Lemma 3.9, $1 \leq j \leq \lfloor d_0/n \rfloor$.

On the other hand, it follows from (2-16) and Lemma 3.7 that there is a stable homotopy equivalence

$$\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+) \simeq \mathrm{Poly}_n^{d_0,m}(\mathbb{C}) \simeq_s \bigvee_{k=1}^{\lfloor d_0/n \rfloor} \Sigma^{2(mn-2)k} D_k.$$

So there exists an element $y_1 \in H_{s-(mn-2)i}(\mathrm{Poly}_n^{d,m}(\mathbb{R};\mathrm{H}_+);\mathbb{F})$ satisfying $(J_{n,\mathbb{C}}^{d,m})_*(\sigma^{(mn-2)i}(y_1)) = x$, where $\sigma^{(mn-2)i}$ denotes the $(mn-2)i$-fold suspension isomorphism. Then by using the commutative diagram (8-14), we see that

$$x = (J_n^{d,m})_*(\iota_{n,\mathrm{H}_+}^{d,m})_*(\sigma^{(mn-2)i}(y_1)).$$

Hence, if we put $y = (\iota_{n,\mathrm{H}_+}^{d,m})_*(\sigma^{(mn-2)i}(y_1)) \in H_s(\mathrm{Poly}_n^{d,m}(\mathbb{R});\mathbb{F})$, the assertion (†) is satisfied. This completes the proof of Lemma 8.3. □

The next assertion easily follows from (8-20).

Corollary 8.4 *If $mn \geq 3$, the Vassiliev spectral sequence*

$$\{E^t_{k,s}(d),\ d^t : E^t_{k,s}(d) \to E^t_{k+t,s+t-1}(d)\} \Rightarrow H_{s-k}(\mathrm{Poly}^{d,m}_n(\mathbb{R}); \mathbb{Z}).$$

collapses at E^1-terms, i.e., $E^1_{**}(d) = E^\infty_{**}(d)$. □

Next we give the proofs of Corollaries 2.13 and 2.14.

Proof of Corollary 2.13 It follows from Theorem 2.11 that there is a stable homotopy equivalence

$$\mathrm{Poly}^{\lfloor d/n\rfloor,mn}_1(\mathbb{R}) \simeq_s \left(\bigvee_{i=1}^{\lfloor d/n\rfloor} S^{(mn-2)i}\right) \vee \left(\bigvee_{i\geq 0, j\geq 1, i+2j\leq \lfloor d/n\rfloor} \Sigma^{(mn-2)(i+2j)} D_j\right).$$

Thus, by using Corollary 2.12 we easily obtain the stable homotopy equivalence (2-21). □

Proof of Corollary 2.14 The assertion (i) follows from Theorem 5.6 and it remains to show (ii). Recall from Lemma 3.7 that there is a homeomorphism $\mathrm{Poly}^{d,m}_n(\mathbb{R}; \mathrm{H}_+) \cong \mathrm{Poly}^{\lfloor d/2\rfloor,m}_n(\mathbb{C})$. Consider the homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Poly}^{d,m}_n(\mathbb{R}; \mathrm{H}_+) & \xrightarrow[\subset]{\iota^{d,m}_{n,\mathrm{H}_+}} & \mathrm{Poly}^{d,m}_n(\mathbb{R}) \\ \| & & \downarrow \simeq_s \\ \mathrm{Poly}^{d,m}_n(\mathbb{R}; \mathrm{H}_+) \cong \mathrm{Poly}^{\lfloor d/2\rfloor,m}_n(\mathbb{C}) & \xrightarrow[\subset]{j_1} & \mathrm{Poly}^{\lfloor d/2\rfloor,m}_n(\mathbb{C}) \vee B^{d,m}_n \vee \mathrm{Q}^{d,m}_n(\mathbb{R}) \end{array}$$

where j_1 denotes the inclusion to the first factor. Then (ii) easily follows from Theorem 2.11. □

Remark 8.5 Let $mn \geq 3$. Then it follows from (2-16), (4-26) and Theorem 2.11 that there are stable homotopy equivalences

$$\begin{aligned} \mathrm{Poly}^{\lfloor d/2\rfloor,m}_n(\mathbb{C}) \times \mathrm{Q}^{d,m}_n(\mathbb{R}) &\simeq_s \mathrm{Poly}^{\lfloor d/2\rfloor,m}_n(\mathbb{C}) \vee A^{d,m}_n \vee \mathrm{Q}^{d,m}_n(\mathbb{R}), \\ \mathrm{Poly}^{d,m}_n(\mathbb{R}) &\simeq_s \mathrm{Poly}^{\lfloor d/2\rfloor,m}_n(\mathbb{C}) \vee B^{d,m}_n \vee \mathrm{Q}^{d,m}_n(\mathbb{R}), \end{aligned}$$

where $A^{d,m}_n$ denotes the space defined by

$$A^{d,m}_n = \bigvee_{1\leq i\leq \lfloor d/n\rfloor, 1\leq j\leq \lfloor d_0/n\rfloor} \Sigma^{(mn-2)(i+2j)} D_j. \tag{8-21}$$

Since $B^{d,m}_n = \bigvee_{(i,j)\in \mathcal{F}^{d,m}_n} \Sigma^{(mn-2)(i+2j)} D_j \subset A^{d,m}_n$ (by Lemma 3.9), the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ can be regarded as the subspace of $\mathrm{Poly}^{\lfloor d/2\rfloor,m}_n(\mathbb{C}) \times \mathrm{Q}^{d,m}_n(\mathbb{R})$ in the stable homotopy category. □

Appendix: The case $(m, n) = (1, 2)$

In this section we consider the homotopy type of the space $\mathrm{Poly}^{d,m}_n(\mathbb{R})$ for the case $(m, n) = (1, 2)$. In fact, the homology stability follows directly from Segal's seminal article [17]. However, as it does not appear to be stated anywhere, we provide a detailed proof below.

Let $f(z) \in \mathrm{Poly}_2^{d,1}(\mathbb{R})$. Then $f(z) \in \mathbb{R}[z]$ is a monic polynomial of degree d without multiple roots. If $\alpha \in \mathbb{C} \setminus \mathbb{R}$ is a complex root of $f(z)$, its conjugate $\overline{\alpha}$ is also a root of $f(z)$. Thus it can be represented as

$$f(z) = \Big(\prod_{i=1}^{d-2j} (z - x_i) \Big) \Big(\prod_{k=1}^{j} (z - \alpha_k)(z - \overline{\alpha}_k) \Big) \tag{A-1}$$

for some $(\{x_i\}_{i=1}^{d-2j}, \{\alpha_k\}_{k=1}^{j}) \in C_{d-2j}(\mathbb{R}) \times C_j(\mathrm{H}_+)$.

Definition A.1 For each nonnegative integer $0 \le j \le \lfloor d/2 \rfloor$, let $\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R})$ denote the subspace of $\mathrm{Poly}_2^{d,1}(\mathbb{R})$ consisting of all monic polynomials $f(z) \in \mathrm{Poly}_2^{d,1}(\mathbb{R})$ which have only $2j$ nonreal roots.

It is easy to see that each polynomial $f(z) \in \mathrm{Poly}_{2,j}^{d,1}(\mathbb{R})$ can be represented in the form (A-1). □

Remark A.2 (i) It is also easy to show that there is a homeomorphism

$$\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) \cong C_{d-2j}(\mathbb{R}) \times C_j(\mathrm{H}_+). \tag{A-2}$$

Let $\varphi : \mathrm{H}_+ \xrightarrow{\cong} \mathbb{C}$ be any homeomorphism (which we now fix), and let

$$\overline{\varphi} : C_j(\mathrm{H}_+) \xrightarrow{\cong} C_j(\mathbb{C}) \tag{A-3}$$

denote the homeomorphism given by $\overline{\varphi}(\{\alpha_k\}_{k=1}^{j}) = \{\varphi(\alpha_k)\}_{k=1}^{j}$. Since there is a homeomorphism $C_{d-2j}(\mathbb{R}) \cong \mathbb{R}^{d-2j}$ and a homotopy equivalence $C_j(\mathbb{C}) \simeq K(\mathrm{Br}(j), 1)$, there is a homotopy equivalence

$$\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) \xrightarrow{\simeq} C_j(\mathbb{C}) \simeq K(\mathrm{Br}(j), 1). \tag{A-4}$$

(ii) A map $f : X \to Y$ is called *a homology equivalence up to dimension N* if the induced homomorphism $f_* : H_k(X; \mathbb{Z}) \to H_k(Y; \mathbb{Z})$ is an isomorphism for any $k < N$ and an epimorphism for $k = N$. □

Theorem A.3 (i) *The space* $\mathrm{Poly}_2^{d,1}(\mathbb{R})$ *consists of* $(\lfloor d/2 \rfloor + 1)$ *connected components* $\{\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) : 0 \le j \le \lfloor d/2 \rfloor\}$, *and there is a homotopy equivalence*

$$\mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) \simeq K(\mathrm{Br}(j), 1) \quad \textit{for each } 0 \le j \le \lfloor d/2 \rfloor.$$

(ii) *There is a natural map* $i_{2,j}^{d,1} : \mathrm{Poly}_{2,j}^{d,1}(\mathbb{R}) \to \Omega_j^2 \mathbb{C}\mathrm{P}^1 \simeq \Omega_j^2 S^2 \simeq \Omega^2 S^3$ *which is a homology equivalence up to dimension* $\lfloor j/2 \rfloor$ *if* $j \ge 3$, *and a homotopy equivalence through dimension* 1 *if* $j = 2$.

Proof Since (i) is obvious, we only deal with (ii). We make the usual identification of S^2 with $\mathbb{C} \cup \infty$ and assume that $j \ge 2$. Define the map $i_{2,j} : C_j(\mathbb{C}) \to \Omega_j^2 \mathbb{C}\mathrm{P}^1$ by

$$i_{2,j}(\{a_k\}_{k=1}^{j})(\alpha) = \begin{cases} [f(\alpha) : f(\alpha) + f'(\alpha)] = [F_2(f)(\alpha)] & \text{if } \alpha \in \mathbb{C}, \\ [1:1] & \text{if } \alpha = \infty, \end{cases} \tag{A-5}$$

for $\alpha \in S^2 = \mathbb{C} \cup \infty$ and $f(z) = \prod_{k=1}^{j}(z - a_k)$. Note that

$$\frac{f(z) + f'(z)}{f(z)} = 1 + \frac{f'(z)}{f(z)} = 1 + \sum_{k=1}^{j} \frac{1}{z - a_k}. \tag{A-6}$$

It then follows from [17, page 42] that the map $i_{2,j}$ is a homology equivalence up to dimension $\lfloor j/2 \rfloor$. Now let $2 \leq j \leq \lfloor d/2 \rfloor$, and consider the map

$$i^d_{2,j} : \mathrm{Poly}^{d,1}_{2,j}(\mathbb{R}) \to \Omega^2_j \mathbb{C}\mathrm{P}^1 \cong \Omega^2_j S^2 \simeq \Omega^2 S^3 \tag{A-7}$$

given by the composite of maps

$$\mathrm{Poly}^{d,1}_{2,j}(\mathbb{R}) \xrightarrow{\simeq} C_j(\mathbb{C}) \xrightarrow{i_{2,j}} \Omega^2_j \mathbb{C}\mathrm{P}^1 \simeq \Omega^2_j S^2 \simeq \Omega^2 S^3. \tag{A-8}$$

Since $i_{2,j}$ is a homology equivalence up to dimension $\lfloor j/2 \rfloor$, the map $i^{d,1}_{2,j}$ is also a homology equivalence up to dimension $\lfloor j/2 \rfloor$.

Finally consider the case $j = 2$. Since $\mathrm{Br}(2) \cong \mathbb{Z}$, there is a homotopy equivalence $\mathrm{Poly}^{d,1}_{2,2}(\mathbb{R}) \simeq S^1$. Since the map $i^{d,1}_{2,2}$ induces an epimorphism

$$(i^{d,1}_{2,2})_* : \mathbb{Z} \cong H_1(\mathrm{Poly}^{d,1}_{2,2}(\mathbb{R}); \mathbb{Z}) \to H_1(\Omega^2_j S^2; \mathbb{Z}) \cong \mathbb{Z},$$

it is indeed an isomorphism. Moreover, note that there is an isomorphism

$$\pi_1(\mathrm{Poly}^{d,1}_{2,2}(\mathbb{R})) \cong \pi_1(S^1) \cong \mathbb{Z} \cong \pi_1(\Omega^2_j S^2).$$

Next, by the Hurewicz theorem we know that the map $i^{d,1}_{2,2}$ induces an isomorphism on the fundamental group $\pi_1(\,)$. □

Remark A.4 (i) It follows from (A-6) that the map $i_{2,j}$ is equivalent to *the electric field map* $E_c : \mathbb{C} \cup \infty \to \mathbb{C} \cup \infty$, described in [16, page 213].

Moreover, we can easily see that $i^{d,1}_{2,j} = i^{d,2}_{1,\mathbb{R}} | \mathrm{Poly}^{d,2}_{1,j}(\mathbb{R})$ (up to homotopy).

(ii) Since

$$\pi_1(\mathrm{Poly}^{d,1}_{2,j}(\mathbb{R})) = \mathrm{Br}(j) \ncong \mathbb{Z} = \pi_1(\Omega^2 S^3) \quad \text{for } j \geq 3,$$

the homotopy stability does not hold for the map $i^{d,2}_{1,j}$ when $j \geq 3$. □

Acknowledgements

The authors would like to take this opportunity to thank Professor Martin Guest for his many valuable insights and suggestions, especially concerning scanning maps. Yamaguchi was supported by JSPS KAKENHI grant JP22K03283. This work was also supported by the Research Institute for Mathematical Sciences, a joint usage/research center located in Kyoto University.

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Received: December 3, 2024 Revised: June 26, 2025

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

AGT peer review and production are managed by EditFlow® from MSP.

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