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A note on a nontrivial connected component of the space of symplectic structures

LUYA WANG

This note provides the first example of a nontrivial connected component of the space of symplectic structures standard at infinity in dimension four.

1 Introduction

Given a symplectic manifold (X, ω), one may consider the space of symplectic structures representing the same cohomology class, denoted by $\mathcal{S}_{[\omega]}$. Two symplectic forms are *isotopic* if there exists a path of cohomologous symplectic forms connecting them. In the case when X is open, we consider only the symplectic structures that are fixed at infinity and relative cohomology. In [9, Section 9.4; 10, Section 13.1; 11], the following question was asked.

Uniqueness question *Is the space $\mathcal{S}_{[\omega]}$ connected?*

This question was answered in the negative in dimension eight and six in [8] and [9, Theorem 9.7.4] for closed manifolds. In dimension four, the above uniqueness question is still open for closed manifolds. We give a negative answer to the open case:

Theorem 1 *There exist infinitely many symplectic forms on $S^1 \times D^3$ standard at infinity, such that they are not pairwise isotopic through symplectic forms standard at infinity.*

In particular, the space $\mathcal{S}_0$ is not connected for $S^1 \times D^3$. The proof is essentially based on either [1] or [13] which studies the compactly supported diffeomorphisms nonisotopic to the identity, and a result of [4] on the compactly supported symplectic mapping class group of $S^1 \times D^3$, itself building on [6].

2 Proof of Theorem 1

Extending work of Gromov [6] on the group of compactly supported symplectomorphisms $\mathrm{Symp}_c(D^4, \omega_0)$, Evans showed the following. The standard symplectic structure on $S^1 \times D^3 \cong \mathbb{C}^* \times \mathbb{C}$ is given by the product symplectic form.

Theorem 2.1 [4, Theorem 6] *The compactly supported symplectomorphism group $\mathrm{Symp}_c(S^1 \times D^3, \omega_{\mathrm{std}})$ is weakly contractible.*

Our result is then an immediate corollary of the works of Budney and Gabai [1] and Watanabe [13]. Among other results, Budney, Gabai and Watanabe proved the following.

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Theorem 2.2 [1, Theorem 3.15; 13, Corollary 1.5] *There exist infinitely many elements of*

$$\pi_0(\mathrm{Diff}_c(S^1 \times D^3)).$$

Proof of Theorem 1 Let $f \in \mathrm{Diff}_c(S^1 \times D^3)$ be a nontrivial element from Theorem 2.2 and $\omega_1 := f^*\omega_{\mathrm{std}}$. Since $H^2(S^1 \times D^3, S^1 \times S^2; \mathbb{R}) = 0$, ω_1 and ω_{std} have the same cohomology class. We show that ω_1 is not isotopic to ω_{std}. The pairwise version works exactly the same. Suppose for the contradiction that ω_1 is isotopic to ω_{std}. By Moser isotopy, e.g., [10, Theorem 3.2.4 and Exercise 3.2.6], we have an isotopy ϕ_t, such that $\phi_0 = \mathrm{id}$ and $\phi_1^*\omega_1 = \omega_{\mathrm{std}}$. Therefore, $\phi_1^* f^*\omega_{\mathrm{std}} = \omega_{\mathrm{std}}$. So $f \circ \phi_1$ is a symplectomorphism. Then, by Theorem 2.1, we know that $f \circ \phi_1$ is isotopic to the identity. This is a contradiction. □

The above arguments can be readily applied to study higher homotopy groups of the space of symplectic structures as well.

Lemma 2.3 *Given $k \geq 0$, the k-th homotopy group of the space of symplectic structures is nontrivial for any (X, ω) such that $H^2(X, \partial X; \mathbb{R}) = 0$, $\pi_k(\mathrm{Diff}_c(X)) \neq \{*\}$ and $\pi_k(\mathrm{Symp}_c(X, \omega)) = \{*\}$.*

Proof The proof is similar to that of Theorem 1. Consider a nontrivial k-parameter family of compactly supported diffeomorphisms $f_{r_1,\dots,r_k}$, where $r_i \in [0, 1]$, given by $\pi_k(\mathrm{Diff}_c(X)) \neq \{*\}$. Suppose for the sake of contradiction that $f^*_{r_1,\dots,r_k}\omega_{\mathrm{std}}$ is isotopic to ω_{std} via symplectic forms that are fixed at infinity. Then by Moser isotopy, we have a k-parameter family of compactly supported isotopies $\phi_{t;r_1,\dots,r_k}$ such that $\phi^*_{1;r_1,\dots,r_k} f^*_{r_1,\dots,t_k}\omega_{\mathrm{std}} = \omega_{\mathrm{std}}$ and $\phi_{0;r_1,\dots,r_k} = \mathrm{id}$. Again, now $f_{r_1,\dots,t_k} \circ \phi_{1;r_1,\dots,r_k}$ is a k-parameter family of compactly supported symplectomorphisms, which by $\pi_k(\mathrm{Symp}_c(X, \omega)) = \{*\}$, must be isotopic to the identity. This contradicts the assumption on $f_{r_1,\dots,r_k}$. □

Corollary 2.4 [12, Section 1.1(3)] *The space of symplectic structures on D^4 standard at infinity is not contractible.*

Proof By Gromov [6] showing weak contractibility of $\mathrm{Symp}_c(D^4, \omega_{\mathrm{std}})$, together with the constructions in [12, Theorem 1.1] on nontriviality of certain higher homotopy groups of $\mathrm{Diff}_c(D^4)$, we have the conclusion by applying Lemma 2.3. □

Remark 2.5 As remarked in [2; 7; 9; 11; 12], given a symplectic form ω, one can consider the orbits of diffeomorphisms acting on the space $\mathcal{S}_{[\omega]}$. By Moser isotopy, we have that given any two forms in the same connected component of $\mathcal{S}_{[\omega]}$, there is an isotopy connecting them, i.e., the action is transitive on each connected component. Denote the connected component of ω by $\mathcal{S}_\omega$. Now, one needs to restrict to the identity smooth components (or restricting to symplectic forms that are in the orbit under $\mathrm{Diff}(X)$ of ω) for $\mathrm{Symp}(X, \omega)$ and $\mathrm{Diff}(X)$ in order to obtain a fibration

$$\mathrm{Symp}(X, \omega) \cap \mathrm{Diff}_0(X) \to \mathrm{Diff}_0(X) \to \mathcal{S}_\omega.$$

To utilize the exotic diffeomorphisms in [1; 12; 13], we therefore prefer to adopt the direct arguments as in the proof of Theorem 1 and Lemma 2.3.

3 Additional observations and questions

As evidenced in the proof of Theorem 1, the question of finding nontrivial connected components of the space $\mathcal{S}_{[\omega]}$ is deeply connected to finding a diffeomorphism that is not isotopic to the identity. We refer to them as exotic symplectic forms and exotic diffeomorphisms.

The following can be considered as a symplectic version of the Smale conjecture, asked by Eliashberg in [2, Problem 8] and [7, Problem 4.141(D)].

Question 3.1 *Is the space $\mathcal{S}_0$ of all standard-at-infinity symplectic structures on $\mathbb{R}^4$ connected?*

Question 3.2 (π_0 version of 4-dimensional Smale conjecture) *Is the space $\mathrm{Diff}_c(\mathbb{R}^4)$ of all compactly supported diffeomorphisms on $\mathbb{R}^4$ connected?*

Lemma 3.3 *Question 3.1 is equivalent to Question 3.2.*

Proof Suppose that we have a symplectic structure ω_{exot} on $\mathbb{R}^4$ that is standard at infinity and not isotopic to ω_{std}, the standard symplectic form on $\mathbb{R}^4$. By [6, Section 0.3.C.], we know that there exists a compactly supported diffeomorphism g such that $g^*\omega_{\text{exot}} = \omega_{\text{std}}$. However, by assumption there is no compactly supported diffeomorphism isotopic to the identity pulling ω_{exot} back to ω_{std}. Therefore, g is not isotopic to the identity. On the other hand, suppose that $\pi_0(\mathrm{Diff}_c(\mathbb{R}^4)) \neq \{*\}$. Then we can use the same argument as that in the proof of Theorem 1, together with Gromov's work on $\pi_0(\mathrm{Diff}_c(\mathbb{R}^4, \omega_{\text{std}})) = \{*\}$ as discussed above, to obtain a nontrivial connected component of $\mathcal{S}_0$ standard-at-infinity on $\mathbb{R}^4$. □

Therefore, a natural question to ask is whether the forms obtained by attaching the standard Weinstein 2-handles along the Legendrian knots smoothly isotopic to $S^1 \times \{*\} \subset S^1 \times S^2$ are isotopic or not. Clearly the symplectic form obtained from attaching the standard Weinstein 2-handle on $(S^1 \times D^3, \omega_{\text{std}})$ is the standard symplectic form on D^4.

Question 3.4 *Are the forms on D^4 obtained by attaching the standard Weinstein 2-handles on $(S^1 \times D^3, \omega_i)$ given in Theorem 1 isotopic to the standard symplectic form?*

A negative answer to Question 3.4 would give a negative answer to Question 3.1 as well as Question 3.2. One could hope to construct a isotopy between the symplectic forms obtained by carving out a family of 2-handles on D^4 via composing the Moser isotopy with an isotopy given by [3]. The difficulty is to make sure that this family of 2-handles does not interact with the exotic diffeomorphisms, or interacts in a controlled way.

Remark 3.5 Since the appearance of this note, a negative answer has been claimed for Question 3.2 in [5]. Nevertheless, it would be interesting to find a symplectic proof of Question 3.1 directly.

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