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# On the Deligne–Beilinson cohomology sheaves

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We prove that the Deligne–Beilinson cohomology sheaves  $\mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})$  are torsion-free as a consequence of the Bloch–Kato conjectures as proven by Rost and Voevodsky. This implies that  $H^0(X, \mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})) = 0$  if  $X$  is unirational. For a surface  $X$  with  $p_g = 0$  we show that the Albanese kernel, identified with  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$ , can be characterized using the integral part of the sheaves associated to the Hodge filtration.

## Introduction

For a compact algebraic manifold  $X$  over  $\mathbb{C}$ , the Deligne cohomology  $H^*(X, \mathbb{Z}(\cdot)_{\mathcal{D}})$  is defined by taking the hypercohomology of the truncated de Rham complex augmented over  $\mathbb{Z}$ . The extension of such a cohomology theory to arbitrary algebraic complex varieties is usually called Deligne–Beilinson cohomology (for example, see [Gillet 1984; Esnault and Viehweg 1988] for definitions, properties and details). Since Deligne–Beilinson cohomology yields a Poincaré duality theory with supports, the associated Zariski sheaves  $\mathcal{H}^*(\mathbb{Z}(\cdot)_{\mathcal{D}})$  have groups of global sections which are birational invariants of smooth complete varieties (see [Barbieri-Viale 1994; 1997]). The motivation for this paper is to start an investigation of these invariants.

We can show that the Deligne–Beilinson cohomology sheaves  $\mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})$  are *torsion-free* (see Theorem 2.5) as a consequence of the Bloch–Kato isomorphisms, proven by Rost and Voevodsky (for example, see [Haesemeyer and Weibel 2014; Voevodsky 2011; Weibel 2008]). In particular,  $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$  is torsion-free thanks to the Merkurjev–Suslin theorem [1983] on  $K_2$ . Thus, the corresponding invariants *vanish* for unirational varieties (see Corollary 2.8). Note that also the singular cohomology sheaves  $\mathcal{H}^q(\mathbb{Z})$  are torsion-free. Indeed, a conditional proof (depending on the validity of the Bloch–Kato conjectures) of these properties has been known for a long time (see Remark 2.13).

Furthermore, if only  $H^2(X, \mathcal{O}_X) = 0$ , we can show that the group of global sections of  $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$  is *exactly* the kernel of the cycle map  $CH^2(X) \rightarrow H^4(X, \mathbb{Z}(2)_{\mathcal{D}})$  in Deligne cohomology, which contains the kernel of the Abel–Jacobi map (see

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**Proposition 3.3** and **Remark 3.6**). This fact generalizes the result of H. Esnault [1990a, Theorem 2.5] for 0-cycles in the case of codimension-2 cycles to  $X$  of arbitrary dimension, and it is obtained by a different proof. Concerning the discrete part  $\mathcal{F}_{\mathbb{Z}}^{2,2}$  of the Deligne–Beilinson cohomology sheaf  $\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})$ , we can describe, for any  $X$  proper and smooth, the torsion of  $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$  in terms of “transcendental cycles” and  $H^3(X, \mathbb{Z})_{\text{tors}}$ , and we can see that it has no nonzero global sections (see **Proposition 3.3**). For surfaces with  $p_g = 0$ , we are able to compute the group of global sections of  $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$  — Bloch’s conjecture is that  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = 0$  — by means of some short exact sequences (described in **Proposition 4.3**) involving the discrete part  $\mathcal{F}_{\mathbb{Z}}^{2,2}$  and the Hodge filtration. In order to do that, we argue first with arithmetic resolutions of the Zariski sheaves associated with the presheaves of mixed Hodge structures defined by singular cohomology: the Hodge and weight filtrations do have corresponding coniveau spectral sequences, the  $E_2$  terms of which are given by the cohomology groups of the Zariski sheaves associated to such filtrations (see **Proposition 1.2**, **Theorem 1.6** and compare with the local Hodge theory of [Barbieri-Viale 2002, §3]).

**Notation.** Throughout this note  $X$  is a complex algebraic variety. Let  $H^*(X, A)$  and  $H_*(X, A)$  be the singular cohomology and Borel–Moore homology of the associated analytic space  $X_{\text{an}}$  with coefficients in  $A$ , respectively, where  $A$  could be  $\mathbb{Z}$ ,  $\mathbb{Z}/n$ ,  $\mathbb{C}$  or  $\mathbb{C}^*$ . Let  $H^*(X)$  and  $H_*(X)$  be the corresponding mixed Hodge structures, respectively (see [Deligne 1971]). Denote by  $W_i H^*(X)$  and  $W_{-i} H_*(X)$  the  $\mathbb{Q}$ -vector spaces given by the weight filtration, and by  $F^i H^*(X)$  and  $F^{-i} H_*(X)$  the complex vector spaces given by the Hodge filtration, respectively. For the ring  $\mathbb{Z}$  of integers, we will denote by  $\mathbb{Z}(r)$  the Tate twist in Hodge theory and by  $H^*(X, \mathbb{Z}(r)_{\mathcal{D}})$  the Deligne–Beilinson cohomology groups (see [Esnault and Viehweg 1988; Gillet 1984]). The Tate twist induces the twist  $A \otimes \mathbb{Z}(r)$  in the coefficients, which we shall denote by  $A(r)$  for short. Denote by  $\mathcal{H}^*(A(r))$  and  $\mathcal{H}^*(\mathbb{Z}(r)_{\mathcal{D}})$  the Zariski sheaves on a given  $X$  associated to singular cohomology and Deligne–Beilinson cohomology, respectively.

## 1. Arithmetic resolutions in mixed Hodge theory

Let  $Z \hookrightarrow X$  be a closed subscheme of the complex algebraic variety  $X$ . According to [Deligne 1974, (8.2.2) and (8.3.8)], the singular cohomology groups  $H_{\mathbb{Z}}^*(X, \mathbb{Z})$  carry a mixed Hodge structure fitting into long exact sequences

$$\cdots \rightarrow H_{\mathbb{Z}}^j(X) \rightarrow H_T^j(X) \rightarrow H_{T-Z}^j(X-Z) \rightarrow H_{\mathbb{Z}}^{j+1}(X) \rightarrow \cdots \quad (1.1)$$

for any pair  $Z \subset T$  of closed subschemes of  $X$ . As has been remarked in [Jannsen 1990], the assignment

$$Z \subseteq X \rightsquigarrow (H_{\mathbb{Z}}^*(X), H_*(Z))$$

yields a Poincaré duality theory with supports (see [Bloch and Ogus 1974]), and furthermore this theory is appropriate for algebraic cycles (in the sense of [Barbieri-Viale 1997]) with values in the abelian tensor category of mixed Hodge structures. In particular, sheafifying the presheaves of vector spaces

$$U \rightsquigarrow F^i H^j(U) \quad \text{and} \quad U \rightsquigarrow W_i H^j(U)$$

on a fixed variety  $X$ , we obtain Zariski sheaves  $\mathcal{F}^i \mathcal{H}^j$  and  $\mathcal{W}_i \mathcal{H}^j$ , respectively, filtering the sheaves  $\mathcal{H}^j(\mathbb{C})$ . We then have:

**Proposition 1.2.** *Let  $X$  be smooth. The “arithmetic resolution”*

$$0 \rightarrow \mathcal{H}^q(\mathbb{C}) \rightarrow \coprod_{x \in X^0} (i_x)_* H^q(x) \rightarrow \coprod_{x \in X^1} (i_x)_* H^{q-1}(x) \rightarrow \cdots \rightarrow \coprod_{x \in X^q} (i_x)_* \mathbb{C} \rightarrow 0$$

is a bifiltered quasi-isomorphism

$$(\mathcal{H}^q(\mathbb{C}), \mathcal{F}, \mathcal{W}) \rightsquigarrow \left( \coprod_{x \in X^\circ} (i_x)_* H^{q-\circ}(x), \coprod_{x \in X^\circ} (i_x)_* F, \coprod_{x \in X^\circ} (i_x)_* W \right)$$

yielding flasque resolutions

$$0 \rightarrow \mathrm{gr}_{\mathcal{F}}^i \mathrm{gr}_{\mathcal{W}}^j \mathcal{H}^q(\mathbb{C}) \rightarrow \coprod_{x \in X^0} (i_x)_* \mathrm{gr}_F^i \mathrm{gr}_j^W H^q(x) \rightarrow \cdots \\ \rightarrow \coprod_{x \in X^q} (i_x)_* \mathrm{gr}_F^{i-q} \mathrm{gr}_{j-2q}^W H^0(x) \rightarrow 0.$$

*Proof.* By [Deligne 1971, Théorèmes 1.2.10 and 2.3.5] the functors  $F^n$ ,  $W_n$  and  $\mathrm{gr}_F^n$  (for any  $n \in \mathbb{Z}$ ) from the category of mixed Hodge structures to that of vector spaces are exact;  $\mathrm{gr}_n^W$  is exact as a functor from mixed Hodge structures to pure  $\mathbb{Q}$ -Hodge structures. So the claimed results are obtained via the “locally homologically effaceable” property (see [Bloch and Ogus 1974, Claim, p. 191]) by construction of the arithmetic resolution (given by [Bloch and Ogus 1974, Theorem 4.2]). For example, by applying  $F^i$  to the long exact sequences (1.1), taking direct limits over pairs  $Z \subset T$  filtered by codimension and sheafifying, we obtain a flasque resolution

$$0 \rightarrow \mathcal{F}^i \mathcal{H}^q \rightarrow \coprod_{x \in X^0} (i_x)_* F^i H^q(x) \rightarrow \coprod_{x \in X^1} (i_x)_* F^{i-1} H^{q-1}(x) \rightarrow \cdots$$

of length  $q$ , where

$$F^* H^*(x) := \varinjlim_{U \subset \{x\}} F^* H^*(U).$$

By this method we obtain as well a resolution of  $\mathcal{W}_j$ ,

$$0 \rightarrow \mathcal{W}_j \mathcal{H}^q \rightarrow \coprod_{x \in X^0} (i_x)_* W_j H^q(x) \rightarrow \coprod_{x \in X^1} (i_x)_* W_{j-2} H^{q-1}(x) \rightarrow \cdots$$

These resolutions give us the claimed bifiltered quasi-isomorphism. (Note: for  $X$  of dimension  $d$ , the fundamental class  $\eta_X$  belongs to  $W_{-2d}H_{2d}(X) \cap F^{-d}H_{2d}(X)$ , so that “local purity” yields the shift by two for the weight filtration and the shift by one for the Hodge filtration). In the same way we obtain resolutions of  $\mathrm{gr}_{\mathcal{F}}^i$ ,  $\mathrm{gr}_j^{\mathcal{W}}$  and  $\mathrm{gr}_{\mathcal{F}}^i \mathrm{gr}_j^{\mathcal{W}}$ .  $\square$

We may consider the twisted Poincaré duality theory  $(F^n H^*, F^{-m} H_*)$  where the integers  $n$  and  $m$  play the role of twisting and indeed we have

$$F^{d-n} H_Z^{2d-k}(X) \cong F^{-n} H_k(Z)$$

for  $X$  smooth of dimension  $d$ . Via the arithmetic resolution of  $\mathcal{F}^i \mathcal{H}^q$ , we then have the following:

**Corollary 1.3.** *Let us assume that  $X$  is smooth, and let  $i$  be a fixed integer. We then have a “coniveau spectral sequence”*

$$E_2^{p,q} = H^p(X, \mathcal{F}^i \mathcal{H}^q) \implies F^i H^{p+q}(X), \quad (1.4)$$

where  $H^p(X, \mathcal{F}^i \mathcal{H}^q) = 0$  if  $q < i$  or  $q < p$ .

**Remark 1.5.** Concerning the Zariski sheaves  $\mathrm{gr}_{\mathcal{F}}^i \mathcal{H}^q$  and  $\mathcal{H}^q / \mathcal{F}^i$ , we indeed obtain corresponding coniveau spectral sequences as above.

Because of the maps of “Poincaré duality theories”  $F^i H^*(-) \rightarrow H^*(-, \mathbb{C})$ , we also have maps of coniveau spectral sequences; on the  $E_2$ -terms the map

$$H^p(X, \mathcal{F}^i \mathcal{H}^q) \rightarrow H^p(X, \mathcal{H}^q(\mathbb{C}))$$

is given by taking Zariski cohomology of  $\mathcal{F}^i \mathcal{H}^q \hookrightarrow \mathcal{H}^q(\mathbb{C})$ . For example, if  $i < p$ , we clearly have (by comparing the arithmetic resolutions)  $H^p(X, \mathcal{F}^i \mathcal{H}^p) \cong H^p(X, \mathcal{H}^p(\mathbb{C}))$ , and

$$H^p(X, \mathcal{H}^p(\mathbb{C})) \cong NS^p(X) \otimes \mathbb{C}$$

by [Bloch and Ogus 1974, Remark 7.6], where  $NS^p(X)$  is the group of cycles of codimension  $p$  modulo algebraic equivalence. For  $i = p$  we still have:

**Theorem 1.6.** *Let  $X$  be smooth. Then*

$$H^p(X, \mathcal{F}^p \mathcal{H}^p) \cong NS^p(X) \otimes \mathbb{C}.$$

*Proof.* By Proposition 1.2, we have

$$H^p(X, \mathcal{F}^p \mathcal{H}^p) \cong \mathrm{coker} \left( \coprod_{x \in X^{p-1}} F^1 H^1(x) \rightarrow \coprod_{x \in X^p} \mathbb{C} \right),$$

whence the canonical map  $H^p(X, \mathcal{F}^p \mathcal{H}^p) \rightarrow NS^p(X) \otimes \mathbb{C}$  is surjective. To show the injectivity, via the arithmetic resolution we see that

$$H_{Z^{p-1}}^{2p-1}(X, \mathbb{C}) \cong \text{Ker} \left( \coprod_{x \in X^{p-1}} H^1(x) \rightarrow \coprod_{x \in X^p} \mathbb{C} \right),$$

where  $H_{Z^i}^*$  denotes the direct limit of the cohomology groups with support on closed subsets of codimension  $\geq i$ ; indeed, this formula is obtained by taking the direct limit of (1.1) over pairs  $Z \subset T$  of codimension  $\geq p$  and  $\geq p-1$ , respectively, since  $H_{Z^p}^{2p-1} = 0$  and

$$H_{Z^p}^{2p}(X, \mathbb{C}) = \coprod_{x \in X^p} \mathbb{C}.$$

Furthermore,

$$F^p H_{Z^{p-1}}^{2p-1} \cong \text{Ker} \left( \coprod_{x \in X^{p-1}} F^1 H^1(x) \rightarrow \coprod_{x \in X^p} \mathbb{C} \right)$$

and

$$H_{Z^{p-1}}^{2p-1}/F^p \cong \coprod_{x \in X^{p-1}} \text{gr}_F^0 H^1(x)$$

since the arithmetic resolution of  $\mathcal{H}^p/\mathcal{F}^p$  has length  $p-1$ . Thus, we have

$$\text{Im} \left( \coprod_{x \in X^{p-1}} F^1 H^1(x) \rightarrow \coprod_{x \in X^p} \mathbb{C} \right) = \text{Im} \left( \coprod_{x \in X^{p-1}} H^1(x) \rightarrow \coprod_{x \in X^p} \mathbb{C} \right). \quad \square$$

**Remark 1.7.** By considering the sheaf  $\mathcal{H}^q(\mathbb{C})$  (which equals  $\mathcal{F}^0 \mathcal{H}^q$ ) on  $X$  filtered by the subsheaves  $\mathcal{F}^i \mathcal{H}^q$  we have, as usual, (see [Deligne 1971, (1.4.5)]) a spectral sequence

$${}_{\mathcal{F}} E_1^{r,s} = H^{r+s}(X, \text{gr}_{\mathcal{F}}^s \mathcal{H}^q) \implies H^{r+s}(X, \mathcal{H}^q(\mathbb{C}))$$

with induced “aboutissement” filtration

$$F^i H^p(X, \mathcal{H}^q) := \text{Im}(H^p(X, \mathcal{F}^i \mathcal{H}^q) \rightarrow H^p(X, \mathcal{H}^q(\mathbb{C}))).$$

The interested reader can check that this spectral sequence degenerates. This reproves Theorem 1.6, and also yields that the filtration  $F^i H^p(X, \mathcal{H}^p)$  is the Néron–Severi group  $NS^p(X) \otimes \mathbb{C}$  if  $i \leq p$  and vanishes otherwise.

**Remark 1.8.** As an immediate consequence of this Theorem 1.6, via the coniveau spectral sequence (1.4), we see the well-known fact that the image of the cycle map  $c\ell^p : NS^p(X) \otimes \mathbb{C} \rightarrow H^{2p}(X, \mathbb{C})$  is contained in  $F^p H^{2p}(X)$ .

For any  $X$  smooth and proper, we have  $F^2 H^2(X) = H^0(X, \mathcal{F}^2 \mathcal{H}^2) = H^0(X, \Omega_X^2)$  and

$$H^0(X, \mathcal{H}^2/\mathcal{F}^2) \cong H^0(X, \mathcal{H}^2(\mathbb{C}))/H^0(X, \Omega_X^2) \cong \frac{H^2(X, \mathbb{C})}{H^0(X, \Omega_X^2) \oplus NS(X) \otimes \mathbb{C}}, \quad (1.9)$$

where  $H^0(X, \mathcal{H}^2(\mathbb{C})) \cong H^0(X, \mathcal{H}^2(\mathbb{Z})) \otimes \mathbb{C}$  and

$$H^0(X, \mathcal{H}^2(\mathbb{Z})) = \text{Im}(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X))$$

is the lattice of “transcendental cycles”. The formula (1.9) can be obtained, for example, by the exact sequence (given by the coniveau spectral sequence since  $\mathcal{F}^2 \mathcal{H}^1 = 0$ )

$$0 \rightarrow H^1(X, \mathcal{H}^1(\mathbb{C})) \rightarrow H^2(X)/F^2 \rightarrow H^0(X, \mathcal{H}^2/\mathcal{F}^2) \rightarrow 0$$

since  $H^1(X, \mathcal{H}^1(\mathbb{C})) = NS(X) \otimes \mathbb{C}$ .

## 2. Deligne–Beilinson cohomology sheaves

Let  $X$  be smooth over  $\mathbb{C}$ . Let us consider the Zariski sheaf  $\mathcal{H}^*(\mathbb{Z}(r)_{\mathcal{D}})$  associated to the presheaf of Deligne–Beilinson cohomology groups  $U \rightsquigarrow H^*(U, \mathbb{Z}(r)_{\mathcal{D}})$  on  $X$ . We have canonical long exact sequences of sheaves

$$\dots \rightarrow \mathcal{H}^q(\mathbb{Z}(r)) \rightarrow \mathcal{H}^q(\mathbb{C})/\mathcal{F}^r \rightarrow \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}}) \rightarrow \mathcal{H}^{q+1}(\mathbb{Z}(r)) \rightarrow \dots, \quad (2.1)$$

$$\dots \rightarrow \mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}}) \rightarrow \mathcal{H}^q(\mathbb{Z}(r)) \oplus \mathcal{F}^r \mathcal{H}^q \rightarrow \mathcal{H}^q(\mathbb{C}) \rightarrow \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}}) \rightarrow \dots, \quad (2.2)$$

$$\dots \rightarrow \mathcal{F}^r \mathcal{H}^q \rightarrow \mathcal{H}^q(\mathbb{C}^*(r)) \rightarrow \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}}) \rightarrow \mathcal{F}^r \mathcal{H}^{q+1} \rightarrow \dots \quad (2.3)$$

on  $X$  obtained by sheafifying the usual long exact sequences coming with Deligne–Beilinson cohomology (see [Esnault and Viehweg 1988, Corollary 2.10]).

Define the “discrete part”  $\mathcal{F}_{\mathbb{Z}}^{r,q}$  (cf. [Esnault 1990a, §1]) of the Deligne–Beilinson cohomology sheaves by

$$\mathcal{F}_{\mathbb{Z}}^{r,q} := \text{Im}(\mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}}) \rightarrow \mathcal{H}^q(\mathbb{Z}(r))),$$

or, equivalently by (2.1),  $\mathcal{F}_{\mathbb{Z}}^{r,q}$  is the integral part of  $\mathcal{F}^r \mathcal{H}^q$ . Note that  $\mathcal{F}_{\mathbb{Z}}^{r,q}$  is given by the inverse image of  $\mathcal{F}^r \mathcal{H}^q$  under the canonical map  $\mathcal{H}^q(\mathbb{Z}) \rightarrow \mathcal{H}^q(\mathbb{C})$ .

We may define the “transcendental part” of the Deligne–Beilinson cohomology sheaves to be

$$\mathcal{T}_{\mathcal{D}}^{r,q} := \text{Ker}(\mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}}) \rightarrow \mathcal{H}^q(\mathbb{Z}(r))).$$

We then have the short exact sequence

$$0 \rightarrow \mathcal{H}^q(\mathbb{Z}(r))/\mathcal{F}_{\mathbb{Z}}^{r,q} \rightarrow \mathcal{H}^q(\mathbb{C})/\mathcal{F}^r \rightarrow \mathcal{T}_{\mathcal{D}}^{r,q+1} \rightarrow 0 \quad (2.4)$$

induced by (2.1). Note that if  $r = 0$  then  $\mathcal{H}^q(\mathbb{C})/\mathcal{F}^0 = 0$  and (2.1) yields the isomorphism  $\mathcal{H}^*(\mathbb{Z}(0)_{\mathcal{D}}) \cong \mathcal{H}^*(\mathbb{Z})$ , so that (2.2) splits in trivial short exact sequences.

**Theorem 2.5.** *Let  $X$  be smooth over  $\mathbb{C}$  and assume  $q \geq 0$ .*

- (i) *The sheaf  $\mathcal{H}^q(\mathbb{Z})$  is torsion-free.*
- (ii) *The sheaf  $\mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})$  is torsion-free.*

(iii) *There is a canonical isomorphism  $\mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}}) \otimes \mathbb{Z}/n \cong \mathcal{H}^q(\mathbb{Z}/n)$*

*Proof.* (i) In order to show that  $\mathcal{H}^{q+1}(\mathbb{Z})$  is torsion-free it suffices to see that  $\mathcal{H}^q(\mathbb{Z}) \rightarrow \mathcal{H}^q(\mathbb{Z}/n)$  is an epimorphism for any  $n \in \mathbb{Z}$ ; via the canonical map  $\mathcal{O}_X^* \rightarrow \mathcal{H}^1(\mathbb{Z})$  and cup product we obtain a map  $(\mathcal{O}_X^*)^{\otimes q} \rightarrow \mathcal{H}^q(\mathbb{Z})$ . The composition

$$(\mathcal{O}_X^*)^{\otimes q} \rightarrow \mathcal{H}^q(\mathbb{Z}) \rightarrow \mathcal{H}^q(\mathbb{Z}/n)$$

can be obtained as well as (cf. [Bloch and Srinivas 1983, p. 1240]) the composition

$$(\mathcal{O}_X^*)^{\otimes q} \xrightarrow{\text{sym}} \mathcal{K}_q^M \rightarrow \mathcal{H}^q(\mathbb{Z}/n),$$

where by definition of Milnor’s  $K$ -theory sheaf the symbol map  $\text{sym}$  is an epimorphism. Thus it is left to show that the Galois symbol  $\mathcal{K}_q^M \rightarrow \mathcal{H}^q(\mathbb{Z}/n)$  is an epimorphism (for the sake of exposition we are tacitly fixing an  $n$ -th root of unity, yielding a noncanonical isomorphism  $\mathcal{H}_{\text{ét}}^q(\mu_n^{\otimes r}) \cong \mathcal{H}^q(\mathbb{Z}/n)$ ). The Galois symbol map can be obtained by mapping the Gersten resolution for Milnor’s  $K$ -theory (for example, see [Kerz 2009, Theorem 7.1]) to the Bloch–Ogus arithmetic resolution of the sheaf  $\mathcal{H}^q(\mathbb{Z}/n)$ . In fact, there is a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{K}_q^M & \longrightarrow & \coprod_{\eta \in X^0} (i_{\eta})_* \mathcal{K}_q^M(k(\eta)) & \longrightarrow & \coprod_{x \in X^1} (i_x)_* \mathcal{K}_{q-1}^M(k(x)) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}^q(\mathbb{Z}/n) & \longrightarrow & \coprod_{\eta \in X^0} (i_{\eta})_* H^q(\eta) & \longrightarrow & \coprod_{x \in X^1} (i_x)_* H^{q-1}(x) \end{array}$$

where  $H^*(\text{point})$  is the Galois cohomology of  $k(\text{point})$  with  $\mathbb{Z}/n$ -coefficients. Thus, the Bloch–Kato isomorphism  $\mathcal{K}_*^M(k(\text{point}))/n \xrightarrow{\sim} H^*(\text{point})$  (see [Haesemeyer and Weibel 2014, Theorem A]) and the exactness of the Gersten complex for Milnor’s  $K$ -theory mod  $n$  (for example, see [Kerz 2009, Theorem 7.8]) yields the desired projection  $\mathcal{K}_q^M \rightarrow \mathcal{K}_q^M/n \xrightarrow{\sim} \mathcal{H}^q(\mathbb{Z}/n)$ .

(ii) & (iii) By considering the Bloch–Beilinson regulators

$$\mathcal{K}_q^M \rightarrow \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}})$$

(simply obtained by the fact that  $\mathcal{K}_1^M = \mathcal{O}_X^* \cong \mathcal{H}^1(\mathbb{Z}(1)_{\mathcal{D}})$  and using the cup product) we have that the composition

$$\mathcal{K}_q^M \rightarrow \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}}) \rightarrow \mathcal{H}^q(\mathbb{Z}(q)) \rightarrow \mathcal{H}^q(\mathbb{Z}/n)$$

is the Galois symbol (see [Esnault 1990b, §0, p. 375]). Hence the composition

$$\mathcal{K}_q^M \rightarrow \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}})/n \rightarrow \mathcal{H}^q(\mathbb{Z}/n)$$



is an epimorphism. Therefore, by comparing with (2.7) below, the proof of the theorem is finished.  $\square$

**Lemma 2.6.** *We have a short exact sequence of sheaves*

$$0 \rightarrow \mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}})/n \rightarrow \mathcal{H}^q(\mathbb{Z}/n) \rightarrow \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}} \rightarrow 0 \quad (2.7)$$

for all  $q, r \geq 0$  and  $n \in \mathbb{Z}$ .

*Proof.* The sequence (2.7) is obtained from the long exact sequence (2.1) as follows. Since the sheaf  $\mathcal{H}^q(\mathbb{Z}(r)) \cong \mathcal{H}^q(\mathbb{Z})$  for all  $r \geq 0$  is torsion-free, we have  $\mathcal{T}_{\mathcal{D}, n\text{-tors}}^{r, q+1} = \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}}$ . Using (2.4), since the sheaf  $\mathcal{H}^q(\mathbb{C})/\mathcal{F}^r$  is uniquely divisible, we have that

$$\mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}} = (\mathcal{H}^q(\mathbb{Z}(r))/\mathcal{F}_{\mathbb{Z}}^{r, q}) \otimes \mathbb{Z}/n.$$

Thus we get a short exact sequence

$$0 \rightarrow \mathcal{F}_{\mathbb{Z}}^{r, q}/n \rightarrow \mathcal{H}^q(\mathbb{Z}/n) \rightarrow \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}} \rightarrow 0$$

by tensoring with  $\mathbb{Z}/n$  the canonical one induced by the subsheaf  $\mathcal{F}_{\mathbb{Z}}^{r, q} \hookrightarrow \mathcal{H}^q(\mathbb{Z}(r))$ . Since  $\mathcal{T}_{\mathcal{D}}^{r, q}$  is divisible, we are done.  $\square$

By a standard argument (see [Barbieri-Viale 1994, §2; 1997]) we have:

**Corollary 2.8.** *Suppose that  $X$  is a smooth unirational complete variety. Then*

$$H^0(X, \mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})) = H^0(X, \mathcal{H}^q(\mathbb{Z})) = 0.$$

**Remark 2.9.** In particular, from Theorem 2.5(i) we get the short exact sequence

$$0 \rightarrow \mathcal{H}^q(\mathbb{Z}) \rightarrow \mathcal{H}^q(\mathbb{C}) \rightarrow \mathcal{H}^q(\mathbb{C}^*) \rightarrow 0. \quad (2.10)$$

Moreover, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \mathcal{H}^q(\mathbb{Z}(q))/\mathcal{F}_{\mathbb{Z}}^{q, q} & \longrightarrow & \mathcal{H}^q(\mathbb{C})/\mathcal{F}^q & \longrightarrow & \mathcal{T}_{\mathcal{D}}^{q, q+1} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \mathcal{H}^q(\mathbb{Z}(q)) & \longrightarrow & \mathcal{H}^q(\mathbb{C}) & \longrightarrow & \mathcal{H}^q(\mathbb{C}^*(q)) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \mathcal{F}_{\mathbb{Z}}^{q, q} & \longrightarrow & \mathcal{F}^q & \longrightarrow & \mathcal{F}^q/\mathcal{F}_{\mathbb{Z}}^{q, q} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array} \quad (2.11)$$

where the middle row is given by (2.10) and the top one is just given by (2.4). Finally, we have (see [Esnault 1990a, (1.3) $\alpha$ ]) the short exact sequence

$$0 \rightarrow \mathcal{H}^{q-1}(\mathbb{C}^*(q)) \rightarrow \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}}) \rightarrow \mathcal{F}_{\mathbb{Z}}^{q,q} \rightarrow 0 \quad (2.12)$$

given by (2.3) or (2.1), taking account of (2.10).

**Remark 2.13.** The argument in the proof of [Theorem 2.5](#), assuming the validity of Bloch–Kato conjecture, was given in the previous version of this paper, available at [arXiv:alg-geom/9412006v1](https://arxiv.org/abs/alg-geom/9412006v1). Indeed, O. Gabber announced (at the end of 1992) the (universal) exactness of the Gersten complex of Milnor’s  $K$ -groups, and some discussions with B. Kahn directed my attention to Gabber’s announcement. Actually, [Theorem 2.5\(i\)](#) was first considered in [Bloch and Srinivas 1983, p. 1240] for  $q = 3$ , it was conjectured in [Barbieri-Viale 1994, §7] in general, and a proof also appears in [Colliot-Thélène and Voisin 2012, Théorème 3.1].

### 3. Coniveau versus Hodge filtrations

Recall the existence of arithmetic resolutions of the sheaves  $\mathcal{H}^*(\mathbb{Z}(\cdot)_{\mathcal{D}})$ , the coniveau spectral sequence

$${}_{\mathcal{D}}E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathbb{Z}(\cdot)_{\mathcal{D}})) \implies H^{p+q}(X, \mathbb{Z}(\cdot)_{\mathcal{D}}), \quad (3.1)$$

and the formula  $H^p(X, \mathcal{H}^p(\mathbb{Z}(p)_{\mathcal{D}})) \cong CH^p(X)$  (see [Gillet 1984]). By the spectral sequence (3.1), we have a long exact sequence

$$0 \rightarrow H^1(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \rightarrow H^3(X, \mathbb{Z}(2)_{\mathcal{D}}) \xrightarrow{\rho} H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \xrightarrow{\delta} CH^2(X). \quad (3.2)$$

The mapping  $\delta$  is just a differential between  ${}_{\mathcal{D}}E_2$ -terms of the coniveau spectral sequence (3.1); we still have

$$\text{Im } \delta = \text{Ker}(CH^2(X) \xrightarrow{c\ell} H^4(X, \mathbb{Z}(2)_{\mathcal{D}})),$$

where  $c\ell$  is the cycle class map in Deligne–Beilinson cohomology.

**Proposition 3.3.** *Let  $X$  be proper and smooth. Then*

$$H^0(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) = 0,$$

*the group  $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$  is infinitely divisible, and*

$$H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})_{\text{tors}} \cong H^0(X, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(2))).$$

*If  $H^2(X, \mathcal{O}_X) = 0$ , then*

$$H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \cong \text{Ker}(CH^2(X) \xrightarrow{c\ell} H^4(X, \mathbb{Z}(2)_{\mathcal{D}})),$$

*i.e.,  $\rho = 0$  in (3.2), and  $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) \cong H^3(X, \mathbb{Z})_{\text{tors}}$ .*

*Proof.* By the canonical map of ‘‘Poincaré duality theories’’

$$H^{\sharp-1}(-, \mathbb{C}^*(\cdot)) \rightarrow H^{\sharp}(-, \mathbb{Z}(\cdot)_{\mathcal{D}}),$$

we obtain a map of coniveau spectral sequences and the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) & \longrightarrow & H^3(X, \mathbb{Z}(2)_{\mathcal{D}}) & \xrightarrow{\rho} & H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \\ & & \uparrow \iota & & \uparrow & & \uparrow \\ 0 & \longrightarrow & NS(X) \otimes \mathbb{C}^*(2) & \longrightarrow & H^2(X, \mathbb{C}^*(2)) & \longrightarrow & H^0(X, \mathcal{H}^2(\mathbb{C}^*(2))) \longrightarrow 0 \end{array} \quad (3.4)$$

In fact, we have

$$H^1(X, \mathcal{H}^1(\mathbb{C}^*(2))) \cong NS(X) \otimes \mathbb{C}^*(2),$$

and the exactness on the right of the bottom exact sequence is provided by the vanishing  $H^2(X, \mathcal{H}^1(\mathbb{C}^*(2))) = 0$ . The left-most map  $\iota$  is induced by the corresponding map in the long exact sequence

$$\begin{aligned} \dots & \xrightarrow{\alpha} H^0(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) \rightarrow H^1(X, \mathcal{H}^1(\mathbb{C}^*(2))) \\ & \rightarrow H^1(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \rightarrow H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) \xrightarrow{\beta} \dots \end{aligned} \quad (3.5)$$

obtained from the short exact sequence of sheaves (2.12) for  $q = 2$ .

It is then easy to see that  $H^0(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$  and  $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$  are, respectively, the kernel and the cokernel of  $\iota$ . In fact,  $\beta = 0$  in (3.5) because  $H^2(X, \mathcal{H}^1(\mathbb{C}^*(2))) = 0$ . To see that  $\alpha = 0$  in (3.5), note that, by the coniveau spectral sequences,

$$H^0(X, \mathcal{H}^1(\mathbb{C}^*(2))) \cong H^1(X, \mathbb{C}^*(2)) \quad \text{and} \quad H^0(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \cong H^2(X, \mathbb{Z}(2)_{\mathcal{D}})$$

since  $\mathbb{C}^*(2) \cong \mathcal{H}^1(\mathbb{Z}(2)_{\mathcal{D}})$  is Zariski constant; now, since  $X$  is proper,

$$H^1(X, \mathbb{C}^*(2)) \cong H^2(X, \mathbb{Z}(2)_{\mathcal{D}}),$$

that is,

$$F^2 H^1 = 0 \quad \text{and} \quad F^2 H^2 \hookrightarrow H^2(X, \mathbb{C}^*(2))$$

in the global version of (2.3), so that  $\alpha = 0$  in (3.5), as claimed. Furthermore, we have

$$H^3(X, \mathbb{Z}(2)_{\mathcal{D}}) \cong H^2(X, \mathbb{C}^*(2))/F^2 H^2, \quad \text{since} \quad F^2 H^3 \hookrightarrow H^3(X, \mathbb{C}^*(2)).$$

We conclude that  $H^0(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$  vanishes because  $F^2 H^2 \cap NS(X) \otimes \mathbb{C}^*(2) = 0$ . Moreover,  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$  is torsion-free (by Theorem 2.5) so that  $\text{Im } \rho$  is torsion-free and  $H^3(X, \mathbb{Z}(2)_{\mathcal{D}}) \otimes \mathbb{Q}/\mathbb{Z} = 0$  indeed; therefore, tensoring the top row of (3.4) with  $\mathbb{Q}/\mathbb{Z}$ , we get

$$H^1(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \otimes \mathbb{Q}/\mathbb{Z} = 0.$$

Then the vanishing  $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) \otimes \mathbb{Q}/\mathbb{Z} = 0$  follows from the description of the cokernel of  $\iota$ . Further, by taking the torsion subgroups in the diagram (3.4), we obtain the assertion about the torsion of  $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$ . If  $H^2(X, \mathcal{O}_X) = 0$  then  $NS(X) \cong H^2(X, \mathbb{Z})$ . We then have (by the bottom row of the diagram (3.4) above) that  $H^0(X, \mathcal{H}^2(\mathbb{C}^*(2))) \cong H^3(X, \mathbb{Z})_{\text{tors}}$ , and its image in  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$  is equal to the image of  $\rho$ , whence the image of  $\rho$  is zero since it is torsion-free. Since  $F^2H^2 = 0$  by a final diagram chase, we obtain the last claim.  $\square$

**Remark 3.6.**  $H^0(X, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}))$  is actually an extension of  $H^0(X, \mathcal{H}^2(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z}$  by  $H^3(X, \mathbb{Z})_{\text{tors}}$  because  $H^0(X, \mathcal{H}^3(\mathbb{Z}))$  is torsion-free. Recall the commutative diagram (see [Barbieri-Viale 1994, 6.1; Esnault and Viehweg 1988, §7])

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A^2(X) & \longrightarrow & CH^2(X) & \longrightarrow & NS^2(X) \longrightarrow 0 \\
 & & \downarrow & & \text{cl} \downarrow & & \downarrow \\
 0 & \longrightarrow & J^2(X) & \longrightarrow & H^4(X, \mathbb{Z}(2)_{\mathcal{D}}) & \longrightarrow & H_{\mathbb{Z}}^{2,2} \longrightarrow 0
 \end{array} \tag{3.7}$$

where  $J^2(X)$  is the intermediate Jacobian,  $A^2(X) \subset CH^2(X)$  is the subgroup of cycles which are algebraically equivalent to zero,  $H_{\mathbb{Z}}^{2,2} \subset H^4(X, \mathbb{Z}(2))$  are integral Hodge cycles and the composition  $NS^2(X) \rightarrow H_{\mathbb{Z}}^{2,2} \subset H^4(X, \mathbb{Z}(2))$  is the classical cycle class map in singular cohomology. Recall that we also have an exact sequence

$$H^0(X, \mathcal{H}^3(\mathbb{Z})) \rightarrow NS^2(X) \rightarrow H^4(X, \mathbb{Z}(2)).$$

The vanishing  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = H^0(X, \mathcal{H}^3(\mathbb{Z})) = 0$ , e.g., if  $X$  is unirational by Corollary 2.8, would imply the finite generation of  $NS^2(X)$  and the representability of  $A^2(X)$ .

In order to detect elements in the mysterious group  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$  of global sections, we dispose of the image of  $H^0(X, \mathcal{H}^2/\mathcal{F}^2)$  (see (1.9)) which is the same (see the diagram (2.11)) as the image of

$$H^0(X, \mathcal{H}^2(\mathbb{Z})) \otimes \mathbb{C}/\mathbb{Q}(2) = H^0(X, \mathcal{H}^2(\mathbb{C}^*(2))) \otimes \mathbb{Q}.$$

Unfortunately these images cannot be the entire group, in general. Indeed, whenever the map

$$H^0(X, \mathcal{H}^2(\mathbb{C}^*(2))) \rightarrow H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$$

is surjective then  $\rho$  is surjective in (3.2) (because of (3.4)), whence the cycle map is injective, which is not the case in general (indeed, for any surface with  $p_g \neq 0$  the cycle map is not injective, by [Mumford 1968]).

#### 4. Surfaces with $p_g = 0$

In the following we let  $X$  denote a complex algebraic surface which is smooth and complete. Let  $A_0(X)$  be the subgroup of  $CH^2(X)$  of cycles of degree zero. Let

$$\phi : A_0(X) \rightarrow J^2(X)$$

be induced by the canonical mapping to the Albanese variety. It is well known (see [Gillet 1984, Theorem 2 and Corollary]) that  $c\ell|_{A_0(X)} = \phi$ , where  $c\ell$  is the cycle map in Deligne cohomology, i.e., in the diagram (3.7) we have that  $NS^2(X) \cong H^4(X, \mathbb{Z}(2))$  and  $A^2(X) \cong A_0(X)$  under our assumptions. Actually, it is known that (see [Barbieri-Viale and Srinivas 1995]) on such a surface  $X$  the sheaf  $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$  is *flasque* and

$$H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \cong \varinjlim_{U \subset X} \frac{H^2(U, \mathbb{C})}{F^2 H^2(U) + H^2(U, \mathbb{Z}(2))},$$

where the limit is taken over the nonempty Zariski open subsets of  $X$ . We then have

$$H^1(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = 0. \quad (4.1)$$

Moreover, the sheaf  $\mathcal{H}^4(\mathbb{Z}(2)_{\mathcal{D}})$  vanishes on a surface (as it is easy to see via the exact sequence (2.1)). Thus, via the spectral sequence (3.1), the vanishing (4.1) corresponds to the vanishing of the cokernel of the cycle map

$$CH^2(X) \xrightarrow{c\ell} H^4(X, \mathbb{Z}(2)_{\mathcal{D}}),$$

which is equivalent (using the diagram (3.7)) to the well-known surjectivity of  $\phi$ . Finally, we have that  $H^2(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \cong H^5(X, \mathbb{Z}(2)_{\mathcal{D}}) = 0$ . In conclusion, the only possibly nonzero terms in the spectral sequence (3.1) are: those giving the exact sequence (3.2),  $H^0(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \cong H^2(X, \mathbb{Z}(2)_{\mathcal{D}})$  and  $H^0(X, \mathcal{H}^1(\mathbb{Z}(2)_{\mathcal{D}})) = \mathbb{C}^*$ . Since  $\text{Ker } \phi = \text{Ker } c\ell = \text{Im } \delta$  in (3.2), we obtain:

**Lemma 4.2.** *The group  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$  is uniquely divisible for any surface  $X$  which is smooth and complete.*

*Proof.* Note that  $A^2(X)$  is always divisible (for example, see [Bloch and Ogus 1974, Lemma 7.10]) and  $A_0(X)_{\text{tors}} \cong J^2(X)_{\text{tors}}$  by [Rojtman 1980], so  $\text{Ker } \phi \otimes \mathbb{Q}/\mathbb{Z} = 0$ . Now (3.2) yields  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \otimes \mathbb{Q}/\mathbb{Z} = 0$  since  $\text{Im } \delta = \text{Ker } \phi$  and  $\text{Ker } \delta = \text{Im } \rho$  both vanish when tensored with  $\mathbb{Q}/\mathbb{Z}$ . Using Theorem 2.5 we are done.  $\square$

We know (see [Mumford 1968]) that  $A_0(X) \cong J^2(X)$  implies that  $p_g = 0$ . Conversely, Bloch's conjecture is that if  $p_g = 0$  then  $A_0(X) \cong J^2(X)$ . Therefore, by Proposition 3.3, we obtain that  $p_g = 0$  if and only if  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = 0$ , assuming the validity of Bloch's conjecture (see [Barbieri-Viale and Srinivas 1995];

Rosenschon 1999; Esnault 1990a; Gillet 1984]). A first characterization of the uniquely divisible group  $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$  is given by the following:

**Proposition 4.3.** *Let  $X$  be a smooth complete surface with  $p_g = 0$ . We then have the following canonical short exact sequences*

$$0 \rightarrow H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \rightarrow H^1(X, \mathcal{F}^2/\mathcal{F}_{\mathbb{Z}}^{2,2}) \rightarrow H^3(X, \mathbb{C}^*(2)) \rightarrow 0, \quad (4.4)$$

$$0 \rightarrow H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \rightarrow H^1(X, \mathcal{H}^2(\mathbb{Z}(2))/\mathcal{F}_{\mathbb{Z}}^{2,2}) \rightarrow H^3/F^2 \rightarrow 0, \quad (4.5)$$

where

$$0 \rightarrow F^2 H^3 \rightarrow H^1(X, \mathcal{F}^2/\mathcal{F}_{\mathbb{Z}}^{2,2}) \rightarrow A_0(X) \rightarrow 0, \quad (4.6)$$

$$0 \rightarrow H^3(X, \mathbb{Z})/\text{tors} \rightarrow H^1(X, \mathcal{H}^2(\mathbb{Z}(2))/\mathcal{F}_{\mathbb{Z}}^{2,2}) \rightarrow A_0(X) \rightarrow 0 \quad (4.7)$$

are also exact.

*Proof.* All these exact sequences are obtained by considering the exact diagram of cohomology groups associated with the diagram of sheaves (2.11) (where  $\mathcal{T}_{\mathcal{D}}^{2,3} = \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$  on a surface) taking account of Theorems 2.5 and 1.6, Proposition 3.3 and the coniveau spectral sequence (1.4). For example, the sequence (4.4) is obtained by taking the long exact sequence of cohomology groups associated with the right-most column of (2.11), the fact that  $H^1(X, \mathcal{H}^2(\mathbb{C}^*(2))) \cong H^3(X, \mathbb{C}^*(2))$  on a surface and the formula (4.1). For (4.5) one has to use the top row of (2.11), the formulas (1.9) and (4.1), and the fact that  $H^3/F^2 \cong H^1(X, \mathcal{H}^2/\mathcal{F}^2)$ . The left-most column of (2.11) yields (4.7), since

$$H^2(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) \cong H^2(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \cong CH^2(X)$$

by (2.12) (see [Esnault 1990a, Theorem 1.3]) and the map of sheaves  $\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}}) \rightarrow \mathcal{H}^2(\mathbb{Z}(2))$  induces the degree map on  $H^2$ . For (4.6) one has to argue with the commutative square in the left bottom corner of (2.11) and the isomorphism

$$H^2(X, \mathcal{F}^2) \cong H^2(X, \mathcal{H}^2(\mathbb{C})) \cong \mathbb{C};$$

remember that  $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) = H^3(X, \mathbb{Z})_{\text{tors}}$ , whence it goes to zero in  $F^2 H^3 \cong H^1(X, \mathcal{F}^2 \mathcal{H}^2)$ .  $\square$

**Remark 4.8.** Because of Proposition 4.3, Bloch’s conjecture is equivalent to showing that the canonical injections of sheaves

$$\mathcal{F}^2/\mathcal{F}_{\mathbb{Z}}^{2,2} \hookrightarrow \mathcal{H}^2(\mathbb{C}^*(2)) \quad \text{and} \quad \mathcal{H}^2(\mathbb{Z}(2))/\mathcal{F}_{\mathbb{Z}}^{2,2} \hookrightarrow \mathcal{H}^2/\mathcal{F}^2$$

remain injections on  $H^1$ . It would be very nice to know of any reasonable description of the Zariski cohomology classes of these subsheaves.

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