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The joint spectral flow and localization of the indices of elliptic operators

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We introduce the notion of the joint spectral flow, which is a generalization of the spectral flow, by using Segal's model of the connective K-theory spectrum. We apply it for some localization results of indices motivated by Witten's deformation of Dirac operators, and rephrase some analytic techniques in terms of topology.

1. Introduction

We give a topological viewpoint for the index and localization phenomena for elliptic operators on certain fiber bundles, using the notion of the joint spectral flow, which is a generalization of the spectral flow introduced by Atiyah, Patodi and Singer [Atiyah et al. 1976]. The spectral flow has various generalizations: for example, the higher spectral flow given by Dai and Zhang [1998] and the noncommutative spectral flow by Leichtnam and Piazza [2003] and Wahl [2007]. However, what we introduce here is a completely different new generalization.

The spectral flow for a one-parameter family of self-adjoint operators is an integer counting the number of eigenvalues with multiplicity crossing over zero. In geometric situations, it is related to the index of some Fredholm operators, as shown in [Atiyah et al. 1976] as follows. For a one-parameter family of self-adjoint elliptic differential operators D_t of first order ($t \in S^1$) on $\Gamma(Y, E)$, where Y is a closed manifold and E is a hermitian vector bundle on Y, the first-order differential operator $d/dt + D_t$ on $\Gamma(Y \times S^1, \pi^* E)$ is also elliptic, and its index coincides with

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the spectral flow. The proof is given essentially by the index theorem for families over the closed 1-dimensional manifold S^1.

The joint spectral flow deals with an n-parameter family of n-tuples of mutually commuting self-adjoint operators and their joint spectra. We deal with continuous or smooth families of commuting Fredholm n-tuples, which are defined in Definition 2.3, and the "Dirac operators" associated with them. In the special case $n = 1$, the joint spectral flow coincides with the usual spectral flow. We also relate it with the index of some elliptic operators, as in the case of the ordinal spectral flow.

Theorem 3.19. *Let B be a closed n-dimensional* Spin^c *manifold, $Z \to M \to B$ a smooth fiber bundle over B such that the total space M is also a* Spin^c*-manifold, E a smooth complex vector bundle over M, and V an n-dimensional* Spin^c *vector bundle over B. For a bundle map $\{D_v(x)\}$ from $V \setminus \{0\}$ to the bundle of fiberwise pseudodifferential operators $\Psi^1_f(M, E)$ satisfying Condition 3.18, we have*

$$\operatorname{ind}(\pi^* \not{\mathfrak{D}}_B + D(x)) = \operatorname{jsf}\big(\{D(x)\}\big).$$

The proof also works in a similar way to the original one. The crucial theorem introduced by Segal [1977] is that the space of n-tuples of mutually commuting compact self-adjoint operators is a model for the spectrum of connective K-theory.

The joint spectral flow and its index formula imply some localization results. E. Witten [1982] reinterpreted and reproved some localization formulas for the indices of Dirac operators from the viewpoint of supersymmetry. He deformed Dirac operators by adding potential terms coming from Morse functions or Killing vectors. Recently, Fujita, Furuta and Yoshida [2010] used an infinite-dimensional analogue to localize the Riemann–Roch numbers of certain completely integrable systems and their prequantum data on their Bohr–Sommerfeld fibers. Here a fiber of a Lagrangian fiber bundle is Bohr–Sommerfeld if the restriction of the prequantum line bundle to it is trivially flat (flat with trivial monodromy). In this case the indices of Dirac operators on fiber bundles localize on some special fibers instead of points. Here we relate them with our joint spectral flow and give a topological viewpoint for this analytic way of localization. A strong point of our method is that we give a precise way to compute the multiplicity at each point on which the index localizes. As a consequence, we reprove and generalize theorems of Witten and Fujita, Furuta and Yoshida.

Corollary 4.3 [Andersen 1997; Fujita et al. 2010]. *Let (M, ω) be a symplectic manifold of dimension $2n$, $\mathbb{T}^n \to M \to B$ a Lagrangian fiber bundle, and (L, ∇^L, h) its prequantum data. Then its Riemann–Roch number* $\mathrm{RR}(M, L)$ *coincides with the number of Bohr–Sommerfeld fibers.*

Finally we consider an operator-theoretic problem.

Unfortunately, there are not many examples of geometrically important operators (for example Dirac operators) represented as Dirac operators associated with commuting Fredholm n-tuples coming from differential operators. Compared with the case where their principal symbols "decompose" as the sum of commuting n-tuples, which is the easiest case because this occurs when when their tangent bundles decompose, the case where the Dirac operators themselves decompose is much more difficult because it requires some integrability of decompositions of tangent bundles. However, the bounded operators $\not{D}(1+\not{D}^2)^{-1/2}$ associated with the Dirac operators $\not{D}$ and zeroth-order pseudodifferential operators are much easier to deal with than first-order differential operators. We glue two commuting n-tuples of pseudodifferential operators by using topological methods to show that the indices for families are complete obstructions to decomposing families of Dirac operators. Here the theories of extensions of C^*-algebras and of Cuntz's quasihomomorphisms play an important role.

Theorem 5.3. *Let $Z \to M \to B$ be a fiber bundle. We assume that there are vector bundles $V_1, \dots, V_l$ on B and $E_1, \dots, E_l$ on M such that the vertical tangent bundle $T_V M$ is isomorphic to $\pi^* V_1 \otimes E_1 \oplus \cdots \oplus \pi^* V_l \otimes E_l$. Then its fiberwise Dirac operator $\not{D}^E_f$ is n-decomposable* (in the sense of Definition 5.2) *if and only if the index for a family* $\operatorname{ind}(\not{D}^E_f)$ *is in the image of $K^n(B, B^{(n-1)}) \to K^n(B)$, or equivalently the image of $\tilde{k}^n(B) \to K^n(B)$.*

This paper is organized as follows. In Section 2, we relate Segal's description of the connective K-theory with the theory of Fredholm operators. In Section 3, we introduce the notion of the joint spectral flow and prove its index formula. In Section 4, we apply the theory and reprove or generalize some classical facts. In Section 5 we deal with the problem of decomposing Dirac operators and give an index-theoretic complete obstruction.

Conventions. We use the following notation throughout this paper:

First, any topological space is assumed to be locally compact and Hausdorff unless otherwise noted (there are exceptions, which are mentioned individually).

Second, we use some topological terms as follows. For a based space $(X, *)$, we denote by ΣX the suspension $X \times S^1/(X \times *_{S^1} \cup *_X \times S^1)$ and by ΩX the reduced loop space $\operatorname{Map}((S^1, *), (X, *))$. On the other hand, for an unbased space X we denote by ΣX and IX the direct sums $X \times (0, 1)$ and $X \times [0, 1]$, respectively. Similarly, for a C^*-algebra A we denote by ΣA and IA its suspensions $A \otimes C_0(0, 1)$ and $A \otimes C[0, 1]$. In particular, we denote by just Σ (resp. I) the topological space $(0, 1)$ or the C^*-algebra $C_0(0, 1)$ (resp. $[0, 1]$ or $C[0, 1]$).

2. Fredholm picture of the connective K-theory

In this section, we first summarize the notion of connective K-theory and its relation to operator algebras according to [Segal 1977] and [Dădărlat and Némethi 1990]. Then we connect it with a model of the K-theory spectrum that is related to the space of Fredholm operators. Finally we generalize the theory for the twisted case. This is fundamental to describing the joint spectral flow.

Let $\{H^i\}_{i\in\mathbb{Z}}$ be a generalized cohomology theory. We say $\{h_i\}_{i\in\mathbb{Z}}$ is the connective cohomology theory associated to $\{H_i\}$ if it is a generalized cohomology theory satisfying the following properties:

(1) There is a canonical natural transformation $h^i \to H^i$ that induces an isomorphism $h^i(\mathrm{pt}) \to H^i(\mathrm{pt})$ for $i \leq 0$.

(2) We have $h^i(\mathrm{pt}) = 0$ for $i > 0$.

Then (reduced) connective K-theory is the connective cohomology theory that is associated to (reduced) K-theory.

Segal [1977] gave an explicit realization of connective K-theory spectra by using operator-algebraic methods.

For a pair of compact Hausdorff spaces (X, A), we denote by $F(X, A)$ the configuration space with labels in finite-dimensional subspaces of a fixed (separable infinite-dimensional) Hilbert space. More precisely, an element of $F(X, A)$ is a pair $(S, \{V_x\}_{x\in S})$, where S is a countable subset of $X \setminus A$ whose cluster points are all in A and each V_x is a nonzero finite-dimensional subspace of a Hilbert space $\mathcal{H}$ such that V_x and V_y are orthogonal if $x \neq y$. It is a non-locally compact topological space with canonical topology satisfying the following:

(1) When two sequences $\{x_i\}$, $\{y_i\}$ converge to the same point z and V_z is the limit of $\{V_{i,x_i} \oplus V_{i,y_i}\}$, the limit of $(\{x_i, y_i\}, \{V_{i,x_i}, V_{i,y_i}\})$ is $(\{z\}, \{V_z\})$.

(2) When all cluster points of a sequence $\{x_i\}$ are in A, the limit of $(\{x_i\}, \{V_{i,x_i}\})$ is $(\varnothing, \varnothing)$.

Then the following holds for this topological space:

Proposition 2.1. *Let (X, A) be a pair of compact Hausdorff spaces. We assume that X is connected, A is path-connected, and A is a neighborhood deformation retract in X. Then the space $F(X, A)$ is homotopy-equivalent to its subspace $F_{\mathrm{fin}}(X, A) := \{(S, \{V_x\}_{x\in S}) \in F(X, A) \mid \#S < \infty\}$ and a sequence $F_{\mathrm{fin}}(A, *) \to F_{\mathrm{fin}}(X, *) \to F_{\mathrm{fin}}(X, A)$ is a quasifibration. Here morphisms are induced by continuous maps $(A, *) \to (X, *) \to (X, A)$. Hence the map $F(X, A) \to \Omega F(SX, SA)$ induces a homotopy equivalence.*

Proof. See [Segal 1977, Proposition 1.3; Dădărlat and Némethi 1990, Section 3.1]. □

This means that $\{F(S^n, *)\}_{n=1,2,\dots}$ is an Ω-spectrum, and hence homotopy classes of continuous maps to it realize some cohomology theory.

Now we introduce two other non-locally compact spaces. First, let $F_n(\mathcal{H})$ be the space of $(n+1)$-tuples $\{T_i\}_{i=0,\dots,n}$ of self-adjoint bounded operators on $\mathcal{H}$ that satisfy the following:

(1) The operator $T^2 := \sum T_i^2$ is equal to the identity.

(2) The operator T_i commutes with T_j for any i and j.

(3) The operators T_i $(i = 1, 2, \dots, n)$ and $T_0 - 1$ are compact.

Then there is a canonical one-to-one correspondence between $F_n(\mathcal{H})$ and $F(S^n, *)$. If we have an element $(S, \{V_x\})$ of $F(S^n, *)$, then we obtain an $(n+1)$-tuple $(T_0, \dots, T_n)$ by setting $T_i := \sum_{x \in S} x_i P_{V_x}$, where P_V is the orthogonal projection onto V and x_i the i-th coordinate of x in $S^n \subset \mathbb{R}^{n+1}$. Conversely, if we have an element $(T_0, \dots, T_n)$ in $F_n(\mathcal{H})$, then we obtain data of joint spectra and the eigenspaces because the T_i are simultaneously diagonalizable. Actually, this correspondence is homeomorphic.

On the other hand, if we have an element $(T_0, \dots, T_n) \in F_n(\mathcal{H})$, then there is a canonical inclusion from the spectrum of the abelian C^*-algebra $C^*(T_0, \dots, T_n)$ into the unit sphere of $\mathbb{R}^{n+1}$ according to condition (1). It gives a $*$-homomorphism $C(S^n) \to \mathbb{B}(\mathcal{H})$ sending x_i to T_i. Now, by virtue of condition (3), the image of its restriction to $C_0(S^n \setminus \{*\})$ (where $* = (1, 0, \dots, 0)$) is in the compact operator algebra $\mathbb{K} = \mathbb{K}(\mathcal{H})$. Conversely, if we have a $*$-homomorphism $\varphi : C_0(\mathbb{R}^n) \to \mathbb{K}$, then we obtain an element $(\varphi(x_0 - 1) + 1, \varphi(x_1), \dots, \varphi(x_n))$ in $F_n(\mathcal{H})$. This gives a canonical one-to-one correspondence between $F_n(\mathcal{H})$ and $\operatorname{Hom}(C_0(\mathbb{R}^n), \mathbb{K})$. This correspondence is also a homeomorphism when we equip $\operatorname{Hom}(C_0(\mathbb{R}^n), \mathbb{K})$ with the strong topology. Moreover, a continuous family of $*$-homomorphisms $\{\varphi_x\}_{x \in X}$ parametrized by a finite CW-complex X is regarded as a $*$-homomorphism $C_0(\mathbb{R}^n) \to C(X) \otimes \mathbb{K} \cong C(X, \mathbb{K})$.

Proposition 2.2 [Segal 1977; Dădărlat and Némethi 1990]. *Let X be a finite CW-complex and $n \in \mathbb{Z}_{>0}$. The three sets*

(1) $[X, F(S^n, *)]$,

(2) $[X, F_n(\mathcal{H})]$,

(3) $[C_0(\mathbb{R}^n), C(X) \otimes \mathbb{K}]$

are canonically mutually isomorphic and form the n-th reduced connective K-group $\tilde{k}^n(X)$. Here the first two are the sets of homotopy classes of continuous maps and the third is that of homotopy classes of $*$-homomorphisms.

Proof. We have already seen that these three sets are canonically isomorphic and $\{F(S^n,*)\}_{n=1,2,\dots}$ is an Ω-spectrum. The desired canonical natural transform is a canonical map Φ from $[C_0(\mathbb{R}^n), C(X)\otimes\mathbb{K}]$ to $KK(C_0(\mathbb{R}^n), C(X)\otimes\mathbb{K}) \cong K^n(X)$ that sends a homotopy class $[\varphi]$ to $[\mathcal{H}\otimes C(X), \varphi, 0]$. Hence we only have to compute $\pi_i(F(S^n,*))$. First for a general C^*-algebra A, the map $[C_0(\mathbb{R}), A] \to K_1(A)$ is an isomorphism because a $*$-homomorphism from $C_0(\mathbb{R})$ to A is determined by a unitary operator. Hence $[X, F(S^1,*)]$ is isomorphic to $K^1(X)$. In the case $i \geq n$ we have

$$\pi_i(F(S^n,*)) \cong \pi_{i-n+1}(F(S^1,*)) \cong K^1(\mathbb{R}^{i-n+1}),$$

that is, $\mathbb{Z}$ when $i-n$ is even and 0 when $i-n$ is odd. In the case $i<n$ we have $\pi_i(F(S^n,*)) \cong \pi_0(F(S^{n-i},*)) \cong 0$ because $F(S^{n-i},*)$ is connected. □

Next we relate this picture to a realization of K-theory that uses the space of Fredholm operators.

Atiyah and Singer [1969] gave a realization of the K-theory spectrum. Let $\mathbb{C}\ell_n$ be the complex Clifford algebra associated to $\mathbb{C}^n$ and its canonical inner product, $e_1,\dots,e_n$ its canonical self-adjoint generators with relations $e_ie_j + e_je_i = 2\delta_{ij}$ and $\mathcal{H}$ a Hilbert space with a $\mathbb{Z}/2$-grading and a $\mathbb{Z}/2$-graded $\mathbb{C}\ell_n$-action c. Then the (non-locally compact) space of odd bounded self-adjoint Fredholm operators T that commute with the $\mathbb{C}\ell_n$-action, and such that $c(e_1)\cdots c(e_n)T|_{\mathcal{H}^0}$ is neither positive- nor negative-definite modulo compact operators if n is odd, represents the K^{-n}-functor.

Similarly, we represent the K^n-functor for $n>0$ as a space of Fredholm operators. For an ungraded separable infinite-dimensional Hilbert space $\mathcal{H}$, let $\mathcal{H}_{\mathbb{C}\ell_n}$ be the $\mathbb{Z}/2$-graded Hilbert $\mathbb{C}\ell_n$-module $\mathcal{H}\hat{\otimes}\mathbb{C}\ell_n$. Now, for $n>0$, let $\mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})$ be the (non-locally compact) space of odd bounded self-adjoint operators in $\mathbb{B}(\mathcal{H}_{\mathbb{C}\ell_n})$ that is Fredholm, that is, invertible modulo $\mathbb{K}(\mathcal{H}_{\mathbb{C}\ell_n})$. Moreover, if n is odd, we additionally assume that $c(e_1)\cdots c(e_n)T|_{\mathcal{H}\otimes\mathbb{C}\ell_n^0}$ is neither positive- nor negative-definite. Then it represents the K^n-functor. It can be understood from the viewpoint of Kasparov's [1980b] KK-theory (or bivariant K-theory). As is well-known, the KK-theory has various formulations, and the original one of Kasparov is deeply related to the theory of Fredholm operators and their indices (see also [Blackadar 1998]). For separable $\mathbb{Z}/2$-graded C^*-algebras A and B, a cycle in $KK(A,B)$ is of the form $[E,\varphi,F]$, where E is a countably generated $\mathbb{Z}/2$-graded Hilbert B-module, φ a $*$-homomorphism from A to $\mathbb{B}(E)$ and F an odd self-adjoint "Fredholm" operator on E relative to A. More precisely, F is an operator in $\mathbb{B}(E)$ such that $[\varphi(a),F]$, $\varphi(a)(F^2-1)$ and $\varphi(a)(F-F^*)$ are in $\mathbb{K}(E)$ for any $a\in A$. A continuous family (in the norm topology) of $\mathbb{C}\ell_n$-equivariant odd Fredholm operators $F(x)$ $(x\in X)$ gives a cycle $[\mathcal{H}_{\mathbb{C}\ell_n}\hat{\otimes}C(X), 1, F]$ in $KK(\mathbb{C}, C(X)\hat{\otimes}\mathbb{C}\ell_n)$ by regarding F as an element in $\mathbb{B}(\mathcal{H}_{\mathbb{C}\ell_n}\otimes C(X))$ by pointwise multiplication. Because this KK-cycle depends only on its homotopy class, this correspondence gives a map from $[X, \mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})]$ to $KK(\mathbb{C}, C_0(X) \hat{\otimes} \mathbb{C}\ell_n)$. We can see that it is actually an isomorphism by using the equivalence relations called the operator homotopy [Kasparov 1980b]. Here we do not have to care about additions of degenerate cycles by virtue of the Kasparov stabilization theorem [Kasparov 1980a].

Now we have shown that there is some operator-theoretic description of the connective K-theory, but it is not consistent to the Fredholm picture of KK-theory and our construction of the K-theory spectrum. Next we see that these two are canonically related.

Both of the two groups $KK(C_0(\mathbb{R}^n), C(X))$ and $KK(\mathbb{C}, C(X) \hat{\otimes} \mathbb{C}\ell_n)$ are isomorphic to $K^n(X)$. The canonical isomorphism

$$KK(C_0(\mathbb{R}^n), C(X)) \to KK(\mathbb{C}, C(X) \hat{\otimes} \mathbb{C}\ell_n)$$

is given by taking the Kasparov product [1980b] with the canonical generator of $KK(\mathbb{C}, C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell_n)$ from the left. This canonical generator also has many identifications, and here we use the one in [Kasparov 1980b]. It is based on the Fredholm picture and is of the form $[C_0(\mathbb{R}^n) \hat{\otimes} \mathbb{C}\ell_n, 1, C]$, where $C := \sum c_i x_i (1 + |x|^2)^{-1/2}$. Here $c_i := c(e_i)$ is left multiplication of e_i on $\mathbb{C}\ell_n$, which is a $\mathbb{C}\ell_n$-module by right multiplication.

Now we apply it for cycles that come from $\varphi \in \mathrm{Hom}(C_0(\mathbb{R}^n), C(X) \otimes \mathbb{K})$. We then have

$$\begin{aligned}
&[C_0(\mathbb{R}^n) \hat{\otimes} \mathbb{C}\ell_n, 1, C] \otimes_{C_0(\mathbb{R}^n)} [\mathcal{H} \hat{\otimes} C(X), \varphi, 0] \\
&\qquad = \big[C_0(\mathbb{R}^n) \otimes_\varphi (\mathcal{H} \otimes C(X)) \hat{\otimes} \mathbb{C}\ell_n, 1, C \otimes_\varphi \mathrm{id}\big] \\
&\qquad = \Big[\mathcal{E}(\varphi) \hat{\otimes} \mathbb{C}\ell_n, 1, \sum c_i T_i\Big].
\end{aligned}$$

Here we denote by $\mathcal{E}(\varphi)$ the Hilbert $C(X)$-module $\{\overline{\varphi_x(C_0(\mathbb{R}^n))\mathcal{H}}\}_{x \in X}$ (more precisely, the submodule of $C(X) \otimes \mathcal{H}$ that consists of $\mathcal{H}$-valued functions on X whose evaluations at x are in $\overline{\varphi_x(C_0(\mathbb{R}^n))\mathcal{H}}$). A $*$-homomorphism $\varphi : C_0(\mathbb{R}^n) \to \mathbb{B}(\mathcal{E}(\varphi))$ uniquely extends to $\tilde{\varphi} : C_b(\mathbb{R}^n) \to \mathbb{B}(\mathcal{E}(\varphi))$ because φ is nondegenerate onto $\mathbb{B}(\mathcal{E}(\varphi))$ (see Section 5 of [Lance 1995]). We set $T_i := \tilde{\varphi}(x_i(1 + |x|^2)^{-1/2})$.

This can be regarded as the Fredholm picture of connective K-theory. However, unfortunately it is not useful for our purpose because $\mathcal{E}(\varphi)$ may not be locally trivial and hence not a bundle of Hilbert spaces in general. Nonetheless, cycles arising in geometric contexts, which are our main interest, have a better description, as follows:

Definition 2.3. • An n-tuple of bounded self-adjoint operators $(T_1, \ldots, T_n)$ on $\mathcal{H}$ is called a *bounded commuting Fredholm n-tuple* if it satisfies the following:

(1) The operator $T^2 := \sum T_i^2$ is in $1 + \mathbb{K}(\mathcal{H})$.

(2) The operator T_i commutes with T_j for any i and j.

We denote by $\mathcal{F}_n(\mathcal{H})$ the set of bounded commuting Fredholm n-tuples equipped with the norm topology.

• An n-tuple of unbounded self-adjoint operators $(D_1, \dots, D_n)$ on $\mathcal{H}$ is an *unbounded commuting Fredholm n-tuple* if it satisfies the following:

(1) The operator $D^2 := \sum D_i^2$ is densely defined, Fredholm, and has compact resolvents.

(2) The operator D_i commutes with D_j for any i and j on $\mathrm{dom}(D^2)^2$.

We denote the set of unbounded commuting Fredholm n-tuples by $\mathscr{F}_n(\mathcal{H})$. It is equipped with the strongest topology such that the map

$$(D^1, \dots, D^n) \mapsto (D^1(1 + D^2)^{-1/2}, \dots, D^n(1 + D^2)^{-1/2})$$

is continuous. This definition is an analogue of the Riesz topology on the space of self-adjoint operators.

• For a bounded (resp. unbounded) commuting Fredholm n-tuple $(T_1, \dots, T_n)$ (resp. $(D_1, \dots, D_n)$), we say that an odd self-adjoint operator $T := c_1 T_1 + \cdots + c_n T_n$ on $\mathcal{H} \hat{\otimes} \mathbb{C}\ell_n$ (resp. $D := c_1 D_1 + \cdots + c_n D_n$ with the domain $\mathrm{dom}(D^2)^{1/2}$) is the *Dirac operator* associated with $(T_1, \dots, T_n)$. For simplicity of notation, hereafter we use the same letter T (resp. D) for commuting Fredholm n-tuples and the Dirac operators associated with them.

The continuous map $(\overline{\mathbb{D}^n}, \partial\mathbb{D}^n) \to (S^n, *)$ that collapses the boundary, more precisely of the form

$$(T_1, \dots, T_n) \mapsto \big(2T^2 - 1, 2(1 - T^2)^{1/2} T_1, \dots, 2(1 - T^2)^{1/2} T_n\big),$$

which is the unique continuous extension of the composition map of the canonical isomorphism between $\mathbb{D}^n$ and $\mathbb{R}^n$ and the stereographic projection, induces a continuous map $\iota : \mathcal{F}_n(\mathcal{H}) \to F_n(\mathcal{H})$ by functional calculus and definition of the topology on $\mathscr{F}_n(\mathcal{H})$. On the other hand, for $(T_1, \dots, T_n) \in \mathcal{F}_n(\mathcal{H})$, the Dirac operator T is in $\mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})$. This correspondence gives a map from $[X, \mathcal{F}_n(\mathcal{H})]$ to $[X, \mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})] \cong KK(\mathbb{C}, C(X) \otimes \mathbb{C}\ell_n)$; the interpretation, in a geometric context, is taking the index bundle with $\mathbb{C}\ell_n$-module structure for the continuous family of Dirac operators associated with $(T_1, \dots, T_n)$. Hence we denote it by ind.

Theorem 2.4. *The following diagram commutes*:

$$\begin{tikzcd}
{[X, \mathcal{F}_n(\mathcal{H})]} \arrow[r, "\mathrm{ind}"] \arrow[d, "\iota"'] & KK(\mathbb{C}, C(X) \hat{\otimes} \mathbb{C}\ell_n) \\
{[X, F_n(\mathcal{H})]} \arrow[r, "\Phi"] & KK(C_0(\mathbb{R}^n), C(X)). \arrow[u, "\sim"]
\end{tikzcd}$$

Proof. Let $\{T(x)\}_{x\in X} := \{(T_1(x), \dots, T_n(x))\}_{x\in X}$ be a continuous family of bounded commuting Fredholm n-tuples and φ^T its image under ι. Then $\Phi \circ \iota[\{T(x)\}]$ is of the form $[\mathcal{E}(\varphi^T) \hat{\otimes} \mathbb{C}\ell_n, 1, T]$. Now we give a homotopy connecting $[\operatorname{ind} T] = \big[(\mathcal{H} \otimes C(X)) \hat{\otimes} \mathbb{C}\ell_n, 1, T(x)\big]$ and $[\mathcal{E}(\varphi^T) \hat{\otimes} \mathbb{C}\ell_n, 1, T(x)]$ directly. It is given by the Kasparov $\mathbb{C}$-$IC(X)$-bimodule

$$\big[\mathcal{E}(\varphi^T) \oplus_{\mathrm{ev}_0} (\mathcal{H}_{C(X)} \otimes I), 1, T\big],$$

where $\mathcal{E}(\varphi^T) \oplus_{\mathrm{ev}_0} (\mathcal{H}_{C(X)} \otimes I) := \big\{(x, f) \in \mathcal{E}(\varphi^T) \oplus (\mathcal{H}_{C(X)} \otimes I) \mid f(0) = x\big\}$. □

Remark 2.5. For a general locally compact CW-complex we have an analogue of K-theory with compact support. The K-group with compact support $K^n_{\mathrm{cpt}}(X)$ is defined as the kernel of the canonical morphism $K^n(X^+) \to K^n(x_0)$, where X^+ is the one-point compactification of X and $\{x_0\} = X^+ \setminus X$. It coincides with the K-group of the nonunital C^*-algebra $C_0(X)$ by definition. Similarly, we write $k^n_{\mathrm{cpt}}(X)$ for the kernel of $k^n(X^+) \to k^n(x_0)$. When X^+ has a relatively compact deformation retract of $\{x_0\}$, $\tilde{k}^n_{\mathrm{cpt}}(X)$ is isomorphic to the set of compactly supported homotopy classes of maps from X to $F(S^n, *)$ with compact support, $F_n(\mathcal{H})$, or $\mathrm{Hom}(C_0(\mathbb{R}^n), \mathbb{K}))$. Hence it is also isomorphic to $\mathrm{Hom}(C_0(\mathbb{R}^n), C_0(X) \otimes \mathbb{K})$. In terms of our Fredholm picture, a continuous family of Fredholm n-tuples on X which is *bounded below* by some $\kappa > 0$ (i.e., $D(x)^2 \geq \kappa$) outside some compact subset $K \subset X$ determines a k^n-cycle on X. For simplicity we write just $\tilde{k}(X)$ instead of $\tilde{k}_{\mathrm{cpt}}(X)$ in this paper.

Remark 2.6. The above formulation is compatible with the product of cohomology theories. We define the product of continuous families of bounded commuting Fredholm n-tuples $T(x) = (T_1(x), \dots, T_n(x))$ in $\mathrm{Map}(X, \mathcal{F}_n(\mathcal{H}))$ and m-tuples $S(x) = (S_1(x), \dots, S_m(x))$ in $\mathrm{Map}(X, \mathcal{F}_m(\mathcal{H}'))$ by

$$\begin{aligned}
T(x) \times S(x) &= (T_1(x), \dots, T_n(x)) \times (S_1(x), \dots, S_m(x)) \\
&:= \big(T_1(x) \otimes 1, \dots, T_n(x) \otimes 1, 1 \otimes S_1(x), \dots, 1 \otimes S_m(x)\big) \\
&\qquad\qquad \in \mathrm{Map}(X, \mathcal{F}_{n+m}(\mathcal{H} \otimes \mathcal{H}')).
\end{aligned}$$

Then the homotopy class of $T(x) \times S(x)$ depends only on the homotopy classes of $T(x)$ and $S(x)$. Consequently $[\{T(x)\}] \cup [\{S(x)\}] := [\{T(x) \times S(x)\}]$ gives a well-defined product $[X, \mathcal{F}_n(\mathcal{H})] \times [X, \mathcal{F}_m(\mathcal{H})] \to [X, \mathcal{F}_{n+m}(\mathcal{H})]$ that is compatible with the product of connective K-groups, which is induced from the canonical map $(S^n,*)\times(S^m,*)\to(S^n,*)\wedge(S^m,*)\cong(S^{n+m},*)$. By a similar argument we can define the product for unbounded commuting Fredholm n-tuples.

The twisted case. Next, we generalize the above theory to twisted connective K-theory. In the above argument, we have used the action of the Clifford algebra $\mathbb{C}\ell_n$ as the coefficients to construct a Dirac operator associated with a family of commuting Fredholm n-tuples. Now we regard it as the Clifford algebra bundle $\mathbb{C}\ell(\underline{\mathbb{C}^n})$ associated with the trivial bundle. We generalize the notion of commuting Fredholm n-tuples and apply the general Clifford algebra bundles $\mathbb{C}\ell(V_{\mathbb{C}})$ associated with Spin^c vector bundles V for the coefficients of the Dirac operators associated with them.

We consider the canonical actions of $\mathrm{GL}(n;\mathbb{R})$ on the spaces $F(S^n,*)$, $F_n(\mathcal{H})$ and $\mathrm{Hom}(C_0(\mathbb{R}^n),C(X)\otimes\mathbb{K})$. For example, on $F_{n+m}(\mathcal{H})$ the action is of the form

$$g\cdot(T_0,T_1,\dots,T_n):=\Big(\sum g_{1j}T_j,\dots,\sum g_{nj}T_j\Big).$$

Let V be a real vector bundle over X. We denote a fiber bundle

$$\mathrm{GL}(V)\times_{\mathrm{GL}(n;\mathbb{R})}F(S^n,*)\quad(\text{resp. }\mathrm{GL}(V)\times_{\mathrm{GL}(n;\mathbb{R})}F_n(\mathcal{H}))$$

by F_V (resp. $F_V(\mathcal{H})$). Similarly, $\mathrm{GL}(n,\mathbb{R})$ acts on the space of bounded (resp. unbounded) commuting Fredholm n-tuples $\mathcal{F}_n(\mathcal{H})$ (resp. $\mathscr{F}_n(\mathcal{H})$), and we denote by $\mathcal{F}_V(\mathcal{H})$ (resp. $\mathscr{F}_V(\mathcal{H})$) the corresponding fiber bundle.

Definition 2.7. A V-twisted family of bounded (resp. unbounded) commuting Fredholm n-tuples is a continuous section $T=T(x)$ in $\Gamma(X,\mathcal{F}_V(\mathcal{H}))$ (resp. in $\Gamma(X,\mathscr{F}_V(\mathcal{H}))$).

In the similar way as in the above argument, the space of continuous sections $\Gamma\mathbb{C}\ell(V)=\Gamma(X,\mathbb{C}\ell(V))$ is a C^*-algebra and a continuous section $T\in\Gamma(X,\mathcal{F}_V(\mathcal{H}))$ defines a Kasparov $\mathbb{C}$-$\Gamma\mathbb{C}\ell(V)$-bimodule

$$\big[\mathcal{H}\hat{\otimes}\mathbb{C}\ell(V),1,c(e_1)T_{e_1}(x)+\dots+c(e_n)T_{e_n}(x)\big],$$

which is independent of the choice of an orthonormal basis $\{e_1,\dots,e_n\}\in V_x$. Therefore we obtain a map $\pi_0(\Gamma(X,\mathcal{F}))\to KK(\mathbb{C},\Gamma\mathbb{C}\ell(V))$.

Proposition 2.8. *Let X be a finite CW-complex and V a real vector bundle. The three sets*

(1) $\Gamma(X,F_V)$,

(2) $\Gamma(X,F_V(\mathcal{H}))$,

(3) $\mathrm{Hom}_{C(X)}(C_0(V),C(X)\otimes\mathbb{K})$

are canonically mutually homeomorphic, and their connected components form the twisted reduced connective K-group associated with the principal bundle

$$\mathrm{GL}(V) \times_{\mathrm{GL}(n,\mathbb{R})} \mathcal{G}_k^{\mathrm{mod}},$$

which we denote by $\tilde{k}^V(X)$ (see Section 3 of [Atiyah and Segal 2004]). *Here* $\mathrm{Hom}_{C(X)}(C_0(V \times \mathbb{R}^k), C(X) \otimes \mathbb{K})$ *is the set of $C(X)$-homomorphisms, that is, $*$-homomorphisms that are compatible with their $C(X)$-module structures.*

Theorem 2.9. *Let X be a finite CW-complex. Then the following diagram commutes*:

$$\begin{tikzcd}
\pi_0\big(\Gamma(X, \mathcal{F}_V(\mathcal{H}))\big) \arrow[r, "\mathrm{ind}"] \arrow[d, "\iota"'] & KK(\mathbb{C}, \Gamma\mathbb{C}\ell(V)) \\
\pi_0\big(\Gamma(X, F_V(\mathcal{H}))\big) \arrow[r, "\Phi"] & \mathcal{R}KK(X; C_0(V), C(X)). \arrow[u, "\sim"]
\end{tikzcd}$$

Here $\mathcal{R}KK(X; C_0(V), C(X))$ is the representable KK-group [Kasparov 1988].

In the same way as in K-theory, the Thom isomorphism holds for twisted connective K-theory.

Proposition 2.10. *The following isomorphism holds*:

$$k^W(X) \cong k^{\pi^*V \oplus \pi^*W}(V)$$

Proof. Let F be a closed subspace of X and denote by V_F the restriction $V|_F$ of a vector bundle V. Then there is a morphism

$$\mathrm{Hom}_{C(F)}(C_0(W_F), C(F) \otimes \mathbb{K}) \to \mathrm{Hom}_{C_0(V_F)}(C_0(\pi^*(V \oplus W)_{V_F}), C_0(V_F) \otimes \mathbb{K}),$$
$$\varphi \mapsto \mathrm{id}_V \otimes \varphi,$$

which is an isomorphism if V is trivial on F, and functorial with respect to inclusions. The Mayer–Vietoris exact sequence implies the global isomorphism. □

In particular, combining with the Thom isomorphism of connective K-theory, we obtain that the twist associated with V is trivial if V has a Spin^c structure.

3. The joint spectral flow

Now we give the precise definition of the joint spectral flow by using the notions introduced in Section 2. Next, we prove an index theorem generalizing the spectral flow index theorem of Atiyah, Patodi and Singer [Atiyah et al. 1976]. Finally we generalize it to the case in which the coefficients c_i are globally twisted by a Spin^c vector bundle.

3A. *Definitions and an index theorem.* In the previous section we have seen that $F(S^n, *)$ represents connective K-theory. Now we introduce another configuration space $P(X, A)$ with labels in positive integers on X relative to A. More precisely, an element of $P(X, A)$ is a pair $(S, \{n_x\}_{x \in S})$, where S is a countable subset of $X \setminus A$ whose cluster points are all in A and each n_x is a positive integer. The topology is defined in the same way as that of $F(X, A)$. Then $P(S^n, *)$ is canonically homotopy equivalent to the infinite symmetric product of $(S^n, *)$, which is a model of the Eilenberg–MacLane space $K(\mathbb{Z}, n)$ by virtue of the Dold–Thom theorem [1958]. There is a canonical continuous map j from $F(S^n, *)$ to $P(S^n, *)$ "forgetting" data about vector spaces except their dimensions, which is given more precisely by

$$(S, \{V_x\}_{x \in S}) \longmapsto (S, \{\dim V_x\}_{x \in S}).$$

In the viewpoint of commuting Fredholm n-tuples, this map forgets their eigenspaces and keeps only their joint spectra with multiplicity. It induces a group homomorphism

$$j_* : \tilde{k}^n(X) \longrightarrow H^n(X; \mathbb{Z}).$$

Now we introduce the notion of the joint spectral flow:

Definition 3.1. Let X be an oriented closed manifold of dimension n. For a continuous family $\{T(x)\} = \{(T_0(x), \dots, T_n(x))\}_{x \in X}$ of elements in $F_n(\mathcal{H})$ parametrized by X, we say that $\langle j_*[\{T(x)\}], [X]\rangle \in \mathbb{Z}$ is its *joint spectral flow*, which we denote by $\mathrm{jsf}(\{T(x)\})$. For a continuous family of bounded or unbounded commuting Fredholm n-tuple $\{T_1, \dots, T_n\}$, we say $\mathrm{jsf}(\iota\{T(x)\})$ is its joint spectral flow, denoted simply by $\mathrm{jsf}(\{T(x)\})$.

Example 3.2 (the case $n = 1$). According to Section 7 of [Atiyah et al. 1976], the spectral flow is defined as the canonical group isomorphism $\mathrm{sf} : \pi_1(F_1(\mathcal{H})) \to \mathbb{Z}$ as follows. For a continuous map $T : S^1 \to F_1(\mathcal{H})$ such that the essential spectrum of each T_t is $\{-1, 1\}$, there is a family of continuous functions $j_i : [0, 1] \to [-1, 1]$ such that $-1 = j = 0 \le j_1 \le \cdots \le j_m = 1$ and $\sigma(T(t)) = \{j_0(t), \dots, j_m(t)\}$ for any $t \in [0, 1]$. Then we obtain the integer l such that $j_k(1) = j_{k+l}(0)$ for any k. This l is called the spectral flow. Now let $\{T(t)\}$ be a continuous family of bounded self-adjoint Fredholm operators such that $\sigma(T(t)) = \{0, (t+1)/2, 1\}$ and the eigenspace $E_{(t+1)/2}$ is of dimension 1. Then by definition its spectral flow $\mathrm{sf}(\{T(t)\})$ is equal to 1. On the other hand, we obtain $j_*(\{T(t)\}) = 1 \in H^1(S^1; \mathbb{Z})$ since the canonical inclusion $S^1 \to \mathrm{Sym}^\infty(S^1, *)$ gives a generator $1 \in H^1(S^1; \mathbb{Z}) \cong [S^1, \mathrm{Sym}^\infty(S^1, *)]$ (see [Dold and Thom 1958] or Proposition 5.2.23 of [Aguilar et al. 2002]). This means that the joint spectral flow coincides with the ordinary spectral flow in the case $X = S^1$.

Proposition 3.3. *The homomorphism j_* is a natural transform of multiplicative cohomology theories.*

Proof. According to Section 3 of [Dădărlat and Némethi 1990],

$$S : \mathrm{Hom}(C_0(\mathbb{R}^n), \mathbb{K}) \to \mathrm{Hom}(C_0(\mathbb{R}^{n+1}), C_0(\mathbb{R}) \otimes \mathbb{K}),$$
$$\varphi \mapsto \mathrm{id}_{\mathbb{R}} \otimes \varphi$$

or equivalently

$$S : F(S^n, *) \to \Omega F(S^n \times I, S^n \times \{0, 1\} \cup \{*\} \times I),$$
$$(S, \{V_x\}_{x \in S}) \mapsto \{t \mapsto ((x, t), \{V_x\}_{x \in S})\}$$

gives the structure map $F(S^n, *) \to \Omega F(S^{n+1}, *)$. By the same argument we obtain that

$$S : P(S^n, *) \to \Omega P(S^n \times I, S^n \times \{0, 1\} \cup \{*\} \times I),$$
$$(S, \{n_x\}_{x \in S}) \mapsto \{t \mapsto ((x, t), \{n_x\}_{x \in S})\}$$

gives the structure map $P(S^n, *) \to \Omega P(S^{n+1}, *)$. Now, by definition the following diagram commutes:

$$\begin{array}{ccc} F(S^n, *) & \xrightarrow{S} & \Omega F(S^{n+1}, *) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ P(S^n, *) & \xrightarrow{S} & \Omega P(S^{n+1}, *). \end{array}$$

The multiplicativity of j_* follows immediately since the multiplicative structure on $\{F(S^n, *)\}_{n=0,1,2,\dots}$ and $P(S^n, *)_{n=0,1,2,\dots}$ are induced from the map

$$(S^n, *) \times (S^m, *) \to (S^{n+m}, *)$$

coming from the wedge product. □

To prove a generalization of the spectral flow index theorem, we note the relation between the joint spectral flow and the Chern character. The Chern character is a natural transform from the K-functor to the rational cohomology functor. Here there is a generalization of the Chern character for a general cohomology theory, which was introduced by Dold [1962] and is called the Chern–Dold character.

Now we identify $k^*(X)$ with $\tilde{k}^{*+1}(SX)$ to extend j_* to a natural transform between unreduced cohomology theories $k^*(X) \to H^*(X; \mathbb{Z})$. It is compatible with the original j_* according to Proposition 3.3.

Proposition 3.4. *The n-th Chern–Dold character* $\mathrm{ch}_n : k^n(X) \otimes \mathbb{Q} \to H^n(X; \mathbb{Q})$ *coincides with j_* rationally.*

Proof. The diagram

$$\begin{tikzcd} k^n(X)\otimes\mathbb{Q} \arrow[r, "\mathrm{ch}"'] \arrow[d, "j_*"'] & H^n(X; k^*(\mathrm{pt})\otimes\mathbb{Q}) \arrow[d, "1\otimes j_*"'] \\ H^n(X;\mathbb{Q}) \arrow[r, "\sim", "\mathrm{ch}=\mathrm{id}"'] & H^n(X; H^*(\mathrm{pt})\otimes\mathbb{Q}) \end{tikzcd}$$

commutes by Proposition 3.3 and naturality of the Chern–Dold character. In fact, Dold [1962] proved that there is a one-to-one correspondence between natural transforms of multiplicative cohomology theories $h \to h'$ and graded ring homomorphisms $h(\mathrm{pt}) \to h'(\mathrm{pt})$ if $h'(\mathrm{pt})$ is a graded vector space over $\mathbb{Q}$. The Chern–Dold character is induced from the ring homomorphism $h^*(\mathrm{pt}) \to \mathbb{Q}\otimes_{\mathbb{Z}} h^*(\mathrm{pt})$. Naturality follows from uniqueness.

Now $k^*(\mathrm{pt}) \cong \mathbb{Z}[\beta]$ (β is of degree -2), $H^*(\mathrm{pt}) \cong \mathbb{Z}$ and the ring homomorphism j_* from $\mathbb{Z}[\beta]$ to $\mathbb{Z}$ is given by $1 \mapsto 1$ and $\beta \mapsto 0$. Hence $(1\otimes j_*)\circ \mathrm{ch}$ coincides with the n-th Chern–Dold character ch_n. This implies that $j_* = \mathrm{ch}_n$. $\square$

Let X be a closed Spin^c manifold, $\mathfrak{S}_{\mathbb{C}}(X)$ the associated $\mathbb{C}\ell_n$-module bundle of $\mathrm{Spin}^c(X)$ by the left multiplication on $\mathbb{C}\ell_n$ as a right $\mathbb{C}\ell_n$-module and $\mathfrak{D}_X$ the $\mathbb{C}\ell_n$-Dirac operator on $\mathfrak{S}_{\mathbb{C}}(X)$. Now $\mathfrak{S}_{\mathbb{C}}(X)$ is equipped with the canonical $\mathbb{Z}/2$-grading and $\mathfrak{D}_X$ is an odd operator. Then it gives an element of $K_n(X) \cong KK(C(X)\hat{\otimes}\mathbb{C}\ell_n, \mathbb{C})$

$$[\mathfrak{D}_X] := \big[L^2(X, \mathfrak{S}_{\mathbb{C}}(X)), m, \mathfrak{D}_X(1+\mathfrak{D}_X^2)^{-1/2}\big],$$

which is the fundamental class of K-theory. Here $m : C(X)\hat{\otimes}\mathbb{C}\ell_n \to \mathbb{B}\big(L^2(\mathfrak{S}_{\mathbb{C}}(X))\big)$ is given by Clifford multiplication.

Lemma 3.5. *Let $\{T(x)\}_{x\in X}$ be a continuous family of commuting Fredholm n-tuples. Then*

$$\langle [\operatorname{ind} T], [\mathfrak{D}_X]\rangle_n = \mathrm{jsf}\{T(x)\}.$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle_n$ is the canonical pairing between $K^n(X)$ and $K_n(X)$.

Proof. First, we prove the lemma in the case of even n. In this case we have a unique irreducible representation Δ_n of $\mathbb{C}\ell_n$ and the Dirac operator $\not{D}_X$ on $\not{S}_{\mathbb{C}}(X) := \mathrm{Spin}^c(X)\times_{\mathbb{C}\ell_n}\Delta_n$. Now Δ_n is equipped with a canonical $\mathbb{Z}/2$-grading and $\not{D}$ is an odd operator. It defines a KK-cycle

$$[\not{D}_X] := \big[L^2(X, \not{S}_{\mathbb{C}}(X)), m, \not{D}_X(1+\not{D}_X^2)^{-1/2}\big] \in KK(C(X), \mathbb{C}).$$

We denote by $[\![\operatorname{ind} T]\!]$ a KK-cycle $[\mathcal{H}\otimes\Delta_n, 1, T] \in KK(\mathbb{C}, C(X))$. Since $\mathbb{C}\ell_n \cong \Delta_n\otimes\Delta_n^*$ as $\mathbb{C}\ell_n$-$\mathbb{C}\ell_n$-bimodules, we have the equalities $[\mathfrak{D}_X] = [\not{D}_X]\otimes\Delta_n$ and

$[\operatorname{ind} T] = [\![\operatorname{ind} T]\!] \hat{\otimes} \Delta_n$ (in particular $\operatorname{ch}[\operatorname{ind} T] = \operatorname{ch}[\![\operatorname{ind} T]\!]$). Here Δ^* is a Hilbert $\mathbb{C}\ell_n$-module by the inner product $\langle x, y \rangle := x^* y$.

The pairing $\langle \cdot, \cdot \rangle_n$ is given by the Kasparov product

$$KK(\mathbb{C}, C(X) \otimes \mathbb{C}\ell_n) \otimes KK(C(X) \otimes \mathbb{C}\ell_n, \mathbb{C}) \to \mathbb{Z}.$$

Therefore

$$\begin{aligned} \langle [\operatorname{ind} T], [\mathfrak{D}] \rangle_n &= [\operatorname{ind} T] \otimes_{C(X) \otimes \mathbb{C}\ell_n} [\mathfrak{D}_X] \\ &= ([\![\operatorname{ind} T]\!] \otimes_{C(X)} [\not{D}_X]) \otimes (\Delta^* \otimes_{\mathbb{C}\ell_n} \Delta) = [\![\operatorname{ind} T]\!] \otimes_{C(X)} [\not{D}_X]. \end{aligned}$$

Now we use the Chern character for K-homology that is compatible with pairing. The Chern character of the Spin^c Dirac operator $\not{D}_X$ is given by the Todd class associated with the Spin^c structure of TX. Hence

$$\begin{aligned} \langle [\{T(x)\}], [\not{D}_X] \rangle &= \langle \operatorname{ch}([\![\operatorname{ind} T]\!]), \operatorname{ch}([\not{D}_X]) \rangle \\ &= \langle \operatorname{ch}([\operatorname{ind} T]), \operatorname{Td}(X) \cap [X] \rangle \\ &= \langle \operatorname{ch}_n([\operatorname{ind} T]), [X] \rangle = \operatorname{jsf}\{T(x)\}. \end{aligned}$$

Here the third equality holds because $\operatorname{ch}([\operatorname{ind} T])$ is in $\bigoplus_{k \geq 0} H^{n+2k}(X; \mathbb{Q}) \cong H^n(X; \mathbb{Q})$ and the zeroth Todd class $\operatorname{Td}_0(X)$ is equal to 1. The last equality holds by Proposition 3.4.

Finally, we prove the lemma in the case of odd n. We can reduce the problem to the case $n = 1$ because, for a family of self-adjoint operators $S(t)$ parametrized by S^1 whose spectral flow is 1 (hence $[\operatorname{ind} S] = 1 \in K^1(S^1) \cong \mathbb{Z}$), we have

$$\begin{aligned} \langle [\operatorname{ind} T], [\mathfrak{D}] \rangle_n &= \big\langle [\operatorname{ind} T] \cup [\operatorname{ind} S], [\mathfrak{D}_X] \otimes [\mathfrak{D}_{S^1}] \big\rangle_{n+1} \\ &= \operatorname{jsf}(\{T(x)\} \times \{S(t)\}) = \operatorname{jsf}\{T(x)\}. \end{aligned}$$

Here we use the fact that the joint spectral flow of the product family $\{T(x)\} \times \{S(t)\}$ coincides with the product $\operatorname{jsf}(\{T(x)\}) \cdot \operatorname{jsf}(\{S(t)\})$. □

Now we give an index theorem that is a generalization of the spectral flow index theorem in [Atiyah et al. 1976].

Let B be a closed n-dimensional Spin^c manifold, $Z \to M \to B$ a smooth fiber bundle over B and E a smooth complex vector bundle over M. We fix a decomposition $TM = T_V M \oplus T_H M$ of the tangent bundle, where $T_V M := \{v \in TM \mid \pi_* v = 0\}$ is the vertical tangent bundle. For a hermitian vector bundle E, we denote by $\pi^* \mathfrak{S}^E_{\mathbb{C}}(B)$ the $\mathbb{C}\ell_n$-module bundle $\pi^* \mathfrak{S}_{\mathbb{C}}(B) \otimes E$ on M. Now we define the *pull-back* of the $\mathbb{C}\ell_n$-Dirac operator $\mathfrak{D}_B$ on B twisted by E as

$$\begin{aligned} \pi^* \mathfrak{D}_B : \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B)) &\xrightarrow{\nabla} \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B) \otimes T^* M) \\ &\xrightarrow{p_{T^*_H M}} \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B) \otimes T^*_H M) \xrightarrow{h} \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B)). \end{aligned}$$

Here, h is the left Clifford action of $\mathbb{C}\ell(TB) \cong \mathbb{C}\ell(T_H M)$ on $\pi^*\mathfrak{S}^E_{\mathbb{C}}(B)$. This operator is expressed using an orthogonal basis $\{e_1, \dots, e_n\}$ of $T_{\pi(x)}B \cong T^*_{\pi(x)}B$ as

$$\pi^*\mathfrak{D}_B = \sum h(\pi^* e_i)\nabla^{\pi^*\mathfrak{S}^E_{\mathbb{C}}(B)}_{\pi^* e_i}.$$

Now it satisfies

$$\pi^*\mathfrak{D}_B(\pi^*\varphi) = \pi^*(\mathfrak{D}_B\varphi).$$

Let $\{D_1, \dots, D_n\}$ be an n-tuple of fiberwise first-order pseudodifferential operators on E, that is, a smooth family $\{D_1(x), \dots, D_n(x)\}$ of pseudodifferential operators on $\Gamma(Z_x, E|_{Z_x})$. Moreover, we assume two conditions:

Condition 3.6. (1) The operators D_i and D_j commute for any i and j.

(2) The square sum $\sum_{i=1}^n D_i^2$ is fiberwise elliptic, that is, its principal symbol is invertible on $S(T_V M)$.

Then, by taking a trivialization of the Hilbert bundle of fiberwise L^2-sections $\mathcal{L}^2_f(M, E \hat{\otimes} \mathbb{C}\ell_n) := \{L^2(Z_x, E_x \hat{\otimes} \mathbb{C}\ell_n)\}_{x \in B}$,

$$\{D(x)\} = \{(D_1(x), \dots, D_n(x))\}$$

forms a continuous family of unbounded commuting Fredholm n-tuples parametrized by B.

Indeed, according to Kuiper's theorem, any Hilbert space bundles are trivial and $[D(x)]$ is independent of the choice of a trivialization. The second assertion holds because a trivialization of Hilbert bundle $\mathcal{V}$ gives a unitary $U \in \mathrm{Hom}_{C(X)}(C(X) \otimes \mathcal{H}, \Gamma(X, \mathcal{V}))$, and hence two trivializations U and U' give a norm-continuous unitary-valued function $U^{-1}U'$, which is homotopic to the identity. Combining with a connection on $\pi^*\mathfrak{S}_{\mathbb{C}}(B)$, which is fiberwise flat, the Dirac operator $D(x) = c_1 D_1(x) + \cdots + c_n D_n(x)$ associated with $\{D(x)\}$ (here we denote by c the $\mathbb{C}\ell_n$-action on $\mathfrak{S}_{\mathbb{C}}(B)$ and write $c_i := c(e_i)$ for an orthonormal basis $\{e_i\}$) also defines a first-order pseudodifferential operator on $\pi^*\mathfrak{S}^E_{\mathbb{C}}(B)$.

Now we state our main theorem:

Theorem 3.7. *Let B, M, E, and $\{D(x)\}$ be as above. Then*

$$\mathrm{ind}_0(\pi^*\mathfrak{D}_B + D(x)) = \mathrm{jsf}\{D(x)\}.$$

Here, for an odd self-adjoint operator D on $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, we denote by $\mathrm{ind}_0 D$ *the Fredholm index of $D^0 : \mathcal{H}^0 \to \mathcal{H}^1$.*

To prove this theorem, we use a lemma about an operator inequality. In this section we denote $D(x)$ and $\pi^*\mathfrak{D}_B$ simply by D_f and D_b.

Lemma 3.8. *For any $\alpha \geq 0$ there is a constant $C > 0$ such that, for any $\xi \in \Gamma(M, \pi^*\mathfrak{S}^E_{\mathbb{C}}(B))$,*

$$\langle [D_b, D_f]\xi, \xi\rangle \geq -\alpha \| D_f \xi\|^2 - C\|\xi\|^2. \tag{3.9}$$

Proof. First, we observe that $[D_b, D_f]$ is a fiberwise first-order pseudodifferential operator as well. Let $(V, x_b^1, \dots, x_b^n)$ be a local coordinate of $x \in B$ and $(U, x_b^1, \dots, x_b^n, x_f^1, \dots, x_f^m)$ a local coordinate in $\pi^{-1}(V)$ such that the tangent vectors $\partial_{x_b^i}(p)$ are in $(T_H M)_p$ for any $p \in \pi^{-1}(x)$. We get such a coordinate by identifying a neighborhood of the zero section of $T_H M|_{\pi^{-1}(x)} \cong N\pi^{-1}(x)$ with a tubular neighborhood of $\pi^{-1}(x)$. We assume that $\pi^*\mathfrak{S}^E_{\mathbb{C}}(B)$ is trivial on U, and fix a trivialization. Then, for any fiberwise pseudodifferential operator P supported in U, the operator $[\partial_{x_b^i}, P]$ is also fiberwise pseudodifferential. Indeed, when we write down a fiberwise pseudodifferential operator P on a bounded open subset of $\mathbb{R}^{n+m} = \mathbb{R}^n_{x_b} \times \mathbb{R}^m_{x_f}$ as

$$Pu(x_b, x_f) = \int_{(y_f, \xi_f) \in \mathbb{R}^m \times \mathbb{R}^m} e^{i\langle x_f - y_f, \xi_f\rangle} a(x_b, x_f, y_f, \xi_f) u(x_b, y_f)\, dy_f\, d\xi_f,$$

we have

$$\begin{aligned}
[\partial_{x_b^i}, P]u(x_b, x_f) &= \int \partial_{x_b^i}\big(e^{i\langle x_f - y_f, \xi_f\rangle} a(x_b, x_f, y_f, \xi_f) u(x_b, y_f)\big)\, dy_f\, d\xi_f \\
&\qquad - \int e^{i\langle x_f - y_f, \xi_f\rangle} a(x_b, x_f, y_f, \xi_f)\, \partial_{x_b^i} u(x_b, y_f)\, dy_f\, d\xi_f \\
&= \int e^{i\langle x_f - y_f, \xi_f\rangle} (\partial_{x_b^i}(a(x_b, x_f, y_f, \xi_f)) u(x_b, y_f))\, dy_f\, d\xi_f.
\end{aligned}$$

Let $D_b' := \sum g^{ij} h(\partial_{x_b^i}) \nabla_{\partial_{x_b^j}}$. Since the Riemannian metric g^{ij} on $T_H M$ only depends on the local coordinate of B (i.e., is a function on B), an operator $[D_b', P] = \big[\sum g^{ij} h(\partial_{x_b^i})(\partial_{x_b^j} + \omega(\partial_{x_b^j})), P\big]$ is also fiberwise pseudodifferential.

For any $\xi \in \Gamma(U, \pi^*\mathfrak{S}^E_{\mathbb{C}}(B)|_U)$, the section $[D_b, P]\xi|_{\pi^{-1}(x)}$ depends only on the restriction of ξ and its differentials in normal direction on $\pi^{-1}(x)$. Since the Dirac operator D_b coincides with D_b' on $U_0 := U \cap \pi^{-1}(x)$ and $[P, D_b']$ is fiberwise pseudodifferential, $[D_b, P]\xi|_{\pi^{-1}(x)} = [D_b', P]\xi|_{\pi^{-1}(x)}$ does not depend on the differentials of ξ. Now, the above argument is independent of the choice of $x \in B$. As a consequence, $[D_b, P]$ is also fiberwise pseudodifferential. By using a partition of unity, we can see that $[D_b, D_f]$ is also a fiberwise pseudodifferential operator.

As a consequence, we obtain that $[D_b, D_f](1 + D_f^2)^{-1/2}$ is a zeroth order pseudodifferential operator. In particular, it is bounded. Now, for any $\lambda > 0$, we obtain the inequality

$$\begin{aligned}\langle [D_b, D_f]\xi, \xi\rangle &= \langle \lambda[D_b, D_f]\xi, \lambda^{-1}\xi\rangle \\ &\geq -\tfrac{1}{2}\lambda^2\langle [D_b, D_f]\xi, [D_b, D_f]\xi\rangle - \tfrac{1}{2}\lambda^{-2}\langle \xi, \xi\rangle \\ &\geq -\tfrac{1}{2}\lambda^2\|[D_b, D_f](1+D_f^2)^{-1/2}\|^2\langle (1+D_f^2)\xi, \xi\rangle - \tfrac{1}{2}\lambda^{-2}\langle \xi, \xi\rangle \\ &= -\tfrac{1}{2}\lambda^2\|[D_b, D_f](1+D_f^2)^{-1/2}\|^2\langle D_f\xi, D_f\xi\rangle \\ &\quad - \tfrac{1}{2}\big(\lambda^2\|[D_b, D_f](1+D_f^2)^{-1/2}\|^2 + \lambda^{-2}\big)\langle \xi, \xi\rangle,\end{aligned}$$

as is introduced in Lemma 7.5 of [Kaad and Lesch 2012]. Now, by choosing $\lambda := \sqrt{2\alpha}\|[D_b, D_f](1+D_f^2)^{-1/2}\|^{-1}$ and $C := \alpha + \lambda^{-2}/2$, we show this C satisfies the above condition. □

Now we use the Connes–Skandalis-type sufficiency condition to realize the Kasparov product of unbounded Kasparov bimodules introduced by Kucerovsky [1997].

Theorem 3.10. *Suppose that* (E_1, φ_1, D_1), (E_2, φ_2, D_2), *and* $(E_1 \hat{\otimes} E_2, \varphi_1 \hat{\otimes} 1, D)$ *are unbounded Kasparov bimodules for* (A, B), (B, C), *and* (A, C) *such that the following conditions hold*:

(1) *For all x in some dense subset of $\varphi_1(A)E_1$, the operator*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2\end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T_x^* & 0\end{pmatrix}\right]$$

is bounded on $\mathrm{dom}(D \oplus D_2)$.

(2) *The resolvent of D is compatible with* $D_1 \hat{\otimes} 1$.

(3) *For all x in the domain,* $\langle D_1 x, Dx\rangle + \langle Dx, D_1 x\rangle \geq \kappa\langle x, x\rangle$.

Here $x \in E_1$ is homogeneous and $T_x : E_2 \to E$ maps $e \mapsto x \hat{\otimes} e$. Then

$$[E_1 \hat{\otimes} E_1, \varphi_1 \hat{\otimes} 1, D] \in KK(A, C)$$

represents the Kasparov product of $[E_1, \varphi_1, D_1] \in KK(A, C)$ *and* $[E_2, \varphi_2, D_2] \in KK(B, C)$.

Here the resolvent of D is said to be *compatible* with D' if there is a dense submodule $\mathcal{W} \subset E_1 \hat{\otimes} E_2$ such that $D'(i\mu + D)^{-1}(i\mu' + D')^{-1}$ is defined on $\mathcal{W}$ for any $\mu, \mu' \in \mathbb{R} \setminus \{0\}$. It holds, for example, in the case that $\mathrm{dom}\, D \subset \mathrm{dom}\, D'$.

Proof of Theorem 3.7. According to Lemma 3.5, the remaining part for the proof is that the left-hand side coincides with the pairing $\langle [\mathrm{ind}\, D], [\not{\mathfrak{D}}_B]\rangle_n$. Here this pairing is given by the Kasparov product $KK(\mathbb{C}, C(B) \hat{\otimes} \mathbb{C}\ell_n) \otimes KK(C(B) \hat{\otimes} \mathbb{C}\ell_n, \mathbb{C}) \to \mathbb{Z}$.

It is computed as follows:

$$
\begin{aligned}
[\mathcal{L}^2(M, E \hat{\otimes} \mathbb{C}\ell_n), 1, D] \otimes_{C(B)\hat{\otimes}\mathbb{C}\ell_n} [L^2(B, \mathfrak{S}_{\mathbb{C}}(B)), m, \mathfrak{D}_B]& \\
= \big[L^2(M, (E \hat{\otimes} \mathbb{C}\ell_n) \hat{\otimes}_{\mathbb{C}\ell_n} \pi^* \mathfrak{S}_{\mathbb{C}}(B)), 1, \mathfrak{D}_B \times D\big]& \\
= [L^2(M, \pi^* \mathfrak{S}_{\mathbb{C}}(B)^E), 1, \mathfrak{D}_B \times D].&
\end{aligned}
$$

Now it remains to prove that $D_b + D_f$ satisfies conditions (1), (2), and (3) of Theorem 3.10.

For any $\sigma \in C^\infty(M, E)$ and $\xi \in C^\infty(B, \mathfrak{S}_{\mathbb{C}}(B))$, the Leibniz rule of $\pi^*\mathfrak{D}_B$ implies that

$$
\begin{aligned}
(D_b + D_f) T_\sigma \xi = (D_b + D_f)(\sigma \cdot \pi^* \xi) &= (D_b + D_f)x \cdot \pi^*\xi + \sigma \cdot D_b \pi^* \xi \\
&= T_{(D_b + D_f)\sigma} \xi + \sigma \cdot \pi^*(\mathfrak{D}_B \xi).
\end{aligned}
$$

Therefore $(D_b + D_f)T_\sigma - T_\sigma \mathfrak{D}_B = T_{(D_b+D_f)\sigma}$ is a bounded operator and hence condition (1) holds. Condition (2) holds since $\operatorname{dom}(D_b + D_f) \subset \operatorname{dom} D_f$. For any $\xi \in C^\infty(M, \mathfrak{S}^E_{\mathbb{C}}(M))$, which is dense in the domain,

$$
\langle D_f \xi, (D_b + D_f)\xi \rangle + \langle (D_b + D_f)\xi, D_f \xi \rangle = \langle [D_b, D_f]\xi, \xi \rangle + \| D_f \xi \|^2.
$$

Condition (3) follows from this and Lemma 3.8. □

Remark 3.11. The calculus above is motivated by that of [Connes and Skandalis 1984], in which the authors dealt with principal symbols and zeroth order pseudodifferential operators. Here we use the unbounded operators directly in order to apply it for more general cases. For example, by the same argument we obtain a similar formula

$$
\operatorname{ind}_0(D + A(x)) = \operatorname{jsf}(\{A(x)\})
$$

for a smooth family of mutually commuting self-adjoint complex coefficient matrices $A(x) = (A_1(x), \dots, A_n(x))$. Other examples are given in the next section.

3B. ***A Callias-type index theorem for open manifolds.*** Now we consider generalizing our index theorem to the case of noncompact base spaces. The pairing of homology and cohomology works in the noncompact case if the cohomology is replaced with the one with compact support. We can deal with it in the context of an infinite-dimensional analogue of Callias-type [1978] operators. Here we use fiberwise elliptic operators as the potential term in the original theory of Callias. First we define the admissibility of a connective K-cocycle (see also [Bunke 1995]).

Definition 3.12. We call a continuous family of commuting Fredholm n-tuples $\{D_1, \dots, D_n\}$ parametrized by a complete Riemannian manifold B *admissible* if there is a constant $\kappa > 0$ such that:

(1) $D(x)^2 \geq \kappa > 0$ for $x \in X \setminus K$.

(2) There are $C_1, C_2 > 0$ such that $\langle([D_b, D_f] + D_f^2)\xi, \xi\rangle \geq C_1\|D_f\xi\|^2 - C_2\|\xi\|^2$ and $\kappa C_1 > C_2$.

Actually the second condition is not essential.

Lemma 3.13. *For any continuous family $\{D_1, \dots, D_n\}$ of commuting Fredholm n-tuples parametrized by a complete n-dimensional Riemannian manifold B that satisfies condition* (1) *above, there is some $t > 0$ such that $tD := (tD_1, \dots, tD_n)$ is admissible.*

Proof. By a similar calculus to the one in Lemma 3.8 (we replace D_f in the first term with tD_f but do not replace the one that arises in $(1 + D_f^2)$ in the middle part) we obtain that, for any $\lambda > 0$,

$$\langle[D_b, tD_f]\xi, \xi\rangle = -\tfrac{1}{2}\lambda^2 R\langle D_f\xi, D_f\xi\rangle - \tfrac{1}{2}(\lambda^2 R + \lambda^{-2})\langle\xi, \xi\rangle,$$

where we set $R := \|[D_b, D_f](1 + D_f^2)^{-1/2}\|^2$. Now, if we choose $\lambda = R^{-1/2}$, then

$$\langle([D_b, tD_f] + (tD_f)^2)\xi, \xi\rangle \geq \frac{t^2}{2}\|D_f\xi\|^2 - \Big(\frac{t^2}{2} + R\Big)\|\xi\|^2.$$

Now we can take a constant κ in condition (1) for tD_f to be $t\kappa$. When we set $C_1 = t^2/2$ and $C_2 = t^2/2$, for sufficiently large $t > 0$ the inequality $(t\kappa)C_1 \geq C_2$ holds and hence the constants $t\kappa$, C_1, and C_2 satisfies condition (2). □

Now we introduce a geometric setting and an index theorem for the noncompact case.

Let B be a complete n-dimensional manifold, $Z \to M \to B$ a smooth fiber bundle over B with fixed decomposition of the tangent bundle $TM \cong T_V M \oplus T_H M$, E a smooth complex vector bundle over M and $\{D_1, \dots, D_n\}$ an n-tuple of fiberwise first-order pseudodifferential operators on E that satisfies Condition 3.6. Moreover, we assume that $\{D_1, \dots, D_n\}$ is admissible.

Theorem 3.14. *In the above situation, the operator $\pi^*\mathfrak{D}_B + D(x)$ is Fredholm and*

$$\mathrm{ind}_0(\pi^*\mathfrak{D}_B + D(x)) = \mathrm{jsf}\{D(x)\}.$$

Proof. The proof is essentially the same as for Theorem 3.7; the remaining part is to show that $\mathfrak{D}_B + D(x)$ is a Fredholm operator. We prove this by using an estimate motivated by Theorem 3.7 of [Gromov and Lawson 1983]. Here we use the notation D_b and D_f again. Let E_λ ($\lambda \in \mathbb{R}$) be the λ-eigenspace for the self-adjoint operator $D_b + D_f$. Now we fix an $\alpha > 0$. Then, for any $\sigma \in \bigoplus_{|\lambda|<\alpha} E_\lambda$,

$$\begin{aligned}0 \le \|D_b\sigma\|^2 &\le \|(D_b + D_f)\sigma\|^2 - (([D_b, D_f] + D_f^2)\sigma, \sigma)\\ &\le \alpha\|\sigma\|^2 - C_1\|D_f\sigma\|^2 + C_2\|\sigma\|^2\\ &\le (\alpha + C_2)\|\sigma\|^2 - C_1\|D_f\sigma\|^2_{B\setminus K}\\ &\le (\alpha - \kappa C_1 + C_2)\|\sigma\|^2 + \kappa C_1\|\sigma\|^2_K.\end{aligned}$$

By assumption we can retake $\alpha > 0$ such that $\kappa C_1 - C_2 > \alpha$. Then there is a constant $C > 0$ such that

$$C\|\sigma\| \le \|\sigma\|_K.$$

Now we take a parametrix Q of the elliptic operator $D_b + D_f$ and set $\mathcal{S} := 1 - QD$. Take P to be the projection from $L^2(M, \pi^*\mathfrak{S}^E_{\mathbb{C}}(B))$ to the subspace $L^2(\pi^{-1}(K), \pi^*\mathfrak{S}^E_{\mathbb{C}}(B)|_{\pi^{-1}(K)})$. Then $P\mathcal{S}$ is a compact operator and

$$\|P\mathcal{S}\sigma\| \ge \|P\sigma\| - \|PDQ\sigma\| \ge C\|\sigma\| - \alpha\|PQ\|\|\sigma\|.$$

Taking $\alpha > 0$ sufficiently small, we see that $P\mathcal{S}$ is bounded below by $C-\alpha\|PQ\| > 0$. This implies that $\bigoplus E_\lambda$ is finite-dimensional, since a compact operator on it is bounded below by some positive number. □

Example 3.15 (the case $B = \mathbb{R}$). Let $\{A(t)\}_{t\in\mathbb{R}}$ be a continuous family of self-adjoint matrices such that there is a $\lambda > 0$ and two self-adjoint invertible matrices A_+, A_- such that $A_t = A_-$ for $t \le -\lambda$ and $A_t = A_+$ for $\lambda \le t$. Now, as is noted in Remark 3.11, we have a finite-dimensional analogue of Theorem 3.14. In the 1-dimensional case it is of the form

$$\operatorname{ind}\left(\frac{d}{dt} + cA_t\right) = \operatorname{sf}(\{A_t\})$$

for sufficiently large $c > 0$. Now obviously its right-hand side is given by the difference

$$\#\{\text{negative eigenvalues of } A_-\} - \#\{\text{negative eigenvalues of } A_+\}.$$

It is nonzero in general, whereas in the case that the parameter space is a circle we have to deal with operators on an infinite-dimensional Hilbert space to obtain examples of nontrivial indices.

Example 3.16. Let B be a complete Spin^c manifold with compactification $\overline{B}$, $Z_1, \dots, Z_n$ closed odd-dimensional Spin^c manifolds and $\{g^1_x, \dots, g^n_x\}_{x\in\overline{B}}$ a smooth family of metrics on $Z_1, \dots, Z_n$ such that the scalar curvature of the product manifold $Z := Z_1 \times \cdots \times Z_n$ is uniformly strictly positive outside a compact subset $K \subset B$. We denote by $\not{D}_{i,x}$ the Dirac operator on Z_i with respect to the metric

g^i_x. Then there is a constant $\lambda > 0$ such that $(\lambda \not{D}_{1,x}, \dots, \lambda \not{D}_{n,x})$ is an admissible family of commuting Fredholm n-tuples and the Fredholm index of the Spin^c Dirac operator on $M := B \times Z$ with respect to the product metric coincides with its joint spectral flow. This gives a map

$$\mathrm{ind} : \big[(\overline{B}, \partial B), (\mathcal{R}(Z_1, \dots, Z_n), \mathcal{R}(Z_1, \dots, Z_n)_{\geq \lambda})\big] \to \mathbb{Z},$$

where $\mathcal{R}(Z_1, \dots, Z_n)$ is the product of spaces of Riemannian metrics $\mathcal{R}(Z_1) \times \cdots \times \mathcal{R}(Z_n)$ and $\mathcal{R}(Z_1, \dots, Z_n)_{\geq \lambda}$ is the subspace of $\mathcal{R}(Z_1, \dots, Z_n)$ such that the scalar curvature of the product metric $(Z_1, g_1) \times \cdots \times (Z_n, g_n)$ is larger than $\lambda > 0$ (its homotopy type is independent of the choice of λ). In particular, when we choose B to be $\mathbb{R}^n$, the left-hand side is isomorphic to $\pi_{n-1}(\mathcal{R}(Z_1, \dots, Z_n)_{\geq \lambda})$ because $\mathcal{R}(Z_1, \dots, Z_n)$ is contractible.

3C. *Families twisted by a vector bundle.* In this section we generalize the joint spectral flow and its index theorem to the case of V-twisted families of commuting Fredholm n-tuples introduced at the end of Section 2. It is essential in Section 4A.

Let V be a real vector bundle. Denote the fiber bundle $\mathrm{GL}(V) \times_{\mathrm{GL}(n,\mathbb{R})} P(S^n, *)$ by P_V. The set of homotopy classes of continuous sections $\pi_0\Gamma(X, P_V)$ forms the twisted cohomology group $H^V(X; \mathbb{Z})$. Now, twists of the ordinary cohomology theory are classified by $H^1(X, \mathbb{Z}/2)$, and in our case the corresponding cohomology classes are determined by the orientation bundle of V. As in Definition 3.1, the continuous map $j : F_V(\mathcal{H}) \to P_V$ induces the natural transform $j_* : k^V \to H^V$.

Definition 3.17. Let X be an oriented closed manifold of dimension n and V an n-dimensional oriented vector bundle. For a V-twisted continuous family $\{T(x)\}_{x\in X}$ of commuting Fredholm n-tuple, we say that the integer $\langle j_*[\{T(x)\}], [X]\rangle \in \mathbb{Z}$ is its *joint spectral flow*, denoted by $\mathrm{jsf}(\{T(x)\})$. Here we identify the two groups $H^V(X; \mathbb{Z})$ and $H^n(X; \mathbb{Z})$ in the canonical way. For a V-twisted continuous family of bounded or unbounded commuting Fredholm n-tuple $\{T(x)\}$, we say $\mathrm{jsf}(\iota\{T(x)\})$ is its joint spectral flow, denoted simply by $\mathrm{jsf}(\{T(x)\})$.

Now we introduce the corresponding geometric setting and prove a generalization of the joint spectral flow index theorem (Theorem 3.7) for a family twisted by a Spin^c vector bundle.

Let B be a closed n-dimensional Spin^c manifold, $Z \to M \to B$ a smooth fiber bundle over B such that the total space M is also a Spin^c manifold, V an n-dimensional Spin^c vector bundle over B and E a smooth complex vector bundle over M. We denote by $\Psi^1_f(M, E)$ the fiber bundle over B whose fiber on $x \in B$ is the space of first-order pseudodifferential operators on $\Gamma(Z_x, E|_{Z_x})$. We consider a map of B-bundles $\{D_v(x)\}_{(x,v)\in V\setminus\{0\}} : V \setminus \{0\} \to \Psi^1_f(M, E)$ that satisfies the following conditions:

Condition 3.18. (1) The operators $D_v(x)$ and $D_w(x)$ commute for any $v, w \in V_x \setminus \{0\}$.

(2) The equality $g \cdot (D_{v_1}(x), \dots, D_{v_n}(x)) = (D_{g\cdot v_1}(x), \dots, D_{g\cdot v_n}(x))$ holds for any $g \in \mathrm{GL}(n; \mathbb{R})$ and basis $(v_1, \dots, v_n)$ of V_x.

(3) The square sum $\sum_{i=1}^n D_{v_i}^2$ is fiberwise elliptic, that is, its principal symbol is invertible on $S(T_V M)$, for an orthonormal basis $\{v_1, \dots, v_n\}$.

Then it forms a V-twisted continuous family of unbounded commuting Fredholm n-tuples $\{D(x)\}$.

Next, we replace the fundamental KK-class on B with the one that is compatible with $\{D(x)\}$. Instead of $\not{S}_{\mathbb{C}}(M)$, we consider the spinor bundle $\not{S}_{\mathbb{C}}(B; V) := \not{S}_{\mathbb{C}}(TB \oplus V)$ for an even-dimensional Spin^c vector bundle $TB \oplus V$. It is equipped with the action of $\mathbb{C}\ell(TB) \hat{\otimes} \mathbb{C}\ell(V)$. Here we denote by c and h its restriction on $\mathbb{C}\ell(V) \hat{\otimes} 1$ and $1 \hat{\otimes} \mathbb{C}\ell(TB)$ respectively. Now we define a pullback of the Dirac operator $\pi^*\not{\mathfrak{D}}_B^V$ twisted by E in a similar way to the one in Section 3A.

Theorem 3.19. *Let B, M and $D(x)$ be as above. Then*

$$\mathrm{ind}(\pi^*\not{\mathfrak{D}}_B^V + D(x)) = \mathrm{jsf}\{D(x)\}.$$

Proof. First we embed V into a trivial real vector bundle $\underline{\mathbb{R}}^p$ linearly, and denote its orthogonal complement by W.

We define the KK-classes

$$[D_W] := \Big[\mathcal{L}_f^2(W, \mathbb{C}\ell(\pi^*W)), m, D_W := \sum h(e_i)\frac{\partial}{\partial w_i}\Big] \in KK(\Gamma_0\mathbb{C}\ell(\pi^*W), C(B)),$$
$$[C_W] := \Big[\Gamma_0\mathbb{C}\ell(\pi^*W), m, C_W := \sum c(e_i)w_i\Big] \in KK(C(B), \Gamma_0\mathbb{C}\ell(\pi^*W)),$$

where $\{e_i\}$ is an orthonormal basis on W_x and $w_i = \langle w, e_i\rangle$ the coordinate functions with respect to $\{e_i\}$. We mention that D_W and C_W are independent of the choice of $\{e_i\}$, and hence are well-defined. Then, the theory of harmonic oscillators (see, e.g., [Higson and Guentner 2004, Section 1.13]) shows that $[D_W] \otimes_{\Gamma_0\mathbb{C}\ell(\pi^*W)} [C_W] = [D_W + C_W] = 1 \in KK(C(B), C(B))$ because the kernel of the harmonic oscillator is 1-dimensional and $O(n)$-invariant. Now

$$D \times C_W = (D_{v_1}, \dots, D_{v_n}, c_{w_1}, \dots, c_{w_k})$$

is a smooth family of commuting Fredholm n-tuples twisted by $V \oplus W \cong \underline{\mathbb{R}}^p$. Moreover it is admissible on W because $(D \times C_W)^2 = D^2 + \|w\|^2$. According to Theorem 3.14,

$$\mathrm{ind}(D_b + D_f + D_W + C_W) = \mathrm{jsf}(\{D \otimes C_W(x, w)\}) = \mathrm{jsf}(\{D(x)\}).$$

On the other hand, by the associativity of the Kasparov product

$$\begin{aligned}\operatorname{ind}(D_b+D_f+D_W+C_W) &= [D_f+C_W]\otimes_{\Gamma_0(\pi^*W)}[D_W+D_b]\\ &= ([D_f]\otimes_{C(B)}[C_W])\otimes_{\Gamma_0(\pi^*W)}([D_W]\otimes_{C(B)}[D_f])\\ &= [D_f]\otimes_{C(B)}[D_b] = \operatorname{ind}(D_b+D_f). \qquad \square\end{aligned}$$

Some examples of geometric situations in which this theorem is applied are introduced in Section 4A.

4. Applications

In this section we introduce some applications of the joint spectral flow and its index theorem.

4A. ***Witten deformation and localization.*** It is easy to obtain the joint spectral flow of a continuous family of commuting Fredholm n-tuples when their joint spectra intersect with zero transversally. In such cases we often reduce the problem of computing the index (which usually requires solving some linear partial differential equations or integrating some characteristic classes) to that of counting the number of points with multiplicity.

The most typical example is the classical Poincaré–Hopf theorem.

Corollary 4.1 (the Poincaré–Hopf theorem). *Let M be a closed* Spin^c *manifold and X a vector field on M whose null points $M^X := \{p \in M \mid X(p) = 0\}$ are isolated. Then*

$$\chi(M) = \sum_{p \in M^X} \nu_p.$$

This proof is essentially the same as that of [Witten 1982]. Here we restrict M to Spin^c manifolds, but this is not an essential assumption.

Proof. By the Hodge–Kodaira decomposition, the Euler number $\chi(M)$ can be computed as the index of the de Rham operator $D_{\mathrm{dR}} := d+d^* : \Gamma\bigl(\bigwedge^{\text{even/odd}} TM\bigr) \to \Gamma\bigl(\bigwedge^{\text{odd/even}} TM\bigr)$. Now $\mathbb{C}\ell(TM)$ acts on $\mathbb{C}\ell(TM)$ in two ways: $c(v)\xi := v \cdot \xi$ and $h(v)\xi := \gamma(\xi)\cdot v$ (for $v \in TM$ and $\xi \in \mathbb{C}\ell(TM)$), where γ is the grading operator on $\mathbb{C}\ell(TM) \cong \mathbb{C}\ell(TM)^0 \oplus \mathbb{C}\ell(TM)^1$. They induce the $\mathbb{C}\ell(TM) \hat{\otimes} \mathbb{C}\ell(TM)$-action on $\mathbb{C}\ell(TM)$ because $c(v)$ and $h(v)$ anticommute. Because M is a Spin^c manifold, it is a unique irreducible $\mathbb{C}\ell(TM \oplus TM)$-module $\$_{\mathbb{C}}(TM \oplus TM)$. By Leibniz's rule,

$$D_{\mathrm{dR}}(\gamma(\xi)\cdot X) = -\gamma(D_{\mathrm{dR}}\xi)\cdot X + (-1)^{\partial\gamma(\xi)}\gamma(\xi)\cdot D_{\mathrm{dR}}(X)$$

where we use the fact that D_{dR} is an odd operator. This means that D_{dR} and $h(X)$ anticommute modulo the bounded operator $(-1)^{\partial\xi+1}h(D_{\mathrm{dR}}(X))$. This shows that $D_{\mathrm{dR}} + th(X)$ is Fredholm for any $t > 0$, because $(D_{\mathrm{dR}} + th(X))^2 = D_{\mathrm{dR}}^2 + t^2\|X\|^2 + t[D_{\mathrm{dR}}, h(X)]$ is a bounded perturbation of the Laplace operator D_{dR}^2, which is positive with compact resolvent. On the other hand, $h(X) = \sum \langle e_i, X\rangle h(e_i)$ is a commuting n-tuple of Fredholm operators twisted by TM (now we regard $\langle e_i, X\rangle$ as Fredholm operators on the 1-dimensional vector space). As a consequence of Theorem 3.19 (and Remark 3.11), we have

$$\chi(M) = \mathrm{ind}(D_{\mathrm{dR}}) = \mathrm{ind}(D_{\mathrm{dR}} + h(X)) = \mathrm{jsf}(\{\langle e_i, X\rangle\}) = \sum_{p\in M^X} \nu_p.$$

The last equation follows from the definition of the joint spectral flow. □

Now we consider an infinite-dimensional analogue of this approach for localization problems of indices.

Let B be an n-dimensional closed Spin^c manifold, $M_1, \ldots, M_n \to B$ fiber bundles such that each fiber $Z_1, \ldots, Z_n$ is an odd-dimensional closed manifold and the $T_V M_i$ are equipped with Spin^c structures, and E a complex vector bundle on $M := M_1 \times_B \cdots \times_B M_n$. Now $TB \oplus \underline{\mathbb{R}}^n$ is a $2n$-dimensional vector bundle and hence there is a unique $\mathbb{C}\ell(TB \oplus \underline{\mathbb{R}}^n)$-module bundle $\$_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n)$. We denote by $\$_{\mathbb{C}}(T_V M_i) \cong \$^0_{\mathbb{C}}(T_V M_i) \oplus \$^1_{\mathbb{C}}(T_V M_i)$ the unique $\mathbb{Z}/2$-graded $\mathbb{C}\ell(T_V M_i)$-module bundle, which is isomorphic to $\$_{\mathbb{C}}(T_V M_i \oplus \underline{\mathbb{R}})$. Then it decomposes into tensor products as

$$\begin{aligned}\$_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) &\cong \$_{\mathbb{C}}(T_V M_1 \oplus \underline{\mathbb{R}}) \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} \$_{\mathbb{C}}(T_V M_n \oplus \underline{\mathbb{R}}) \\ &\cong (\$^0_{\mathbb{C}}(T_V M_1) \mathbin{\hat{\otimes}} \mathbb{C}\ell_1) \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} (\$^0_{\mathbb{C}}(T_V M_n) \mathbin{\hat{\otimes}} \mathbb{C}\ell_1) \\ &\cong (\$^0_{\mathbb{C}}(T_V M_1) \otimes \cdots \otimes \$^0_{\mathbb{C}}(T_V M_n)) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n.\end{aligned}$$

Hereafter we set $\not{\mathfrak{S}}_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n) := \$_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n)$ and $\$^0_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n) := \$^0_{\mathbb{C}}(T_V M_1) \otimes \cdots \otimes \$^0_{\mathbb{C}}(T_V M_n)$. The inclusions $T_V M \subset T_V M \oplus \underline{\mathbb{R}}^n$ and $\underline{\mathbb{R}}^n \subset T_V M \oplus \underline{\mathbb{R}}^n$ induce the actions of $\mathbb{C}\ell(TM)$ and $\mathbb{C}\ell_n$ on $\not{\mathfrak{S}}_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n)$. Under the above identification, a vector $v = v_1 \oplus \cdots \oplus v_n \in T_V M$ acts as

$$(c(v_1) \otimes 1 \otimes \cdots \otimes 1) \otimes c_1 + \cdots + (1 \otimes \cdots \otimes 1 \otimes c(v_n)) \otimes c_n$$

and $\mathbb{C}\ell_n$ acts as $1 \otimes h$ (here we denote the left and twisted right actions of $\mathbb{C}\ell_n$ on $\mathbb{C}\ell_n$ by c and h). Hence the fiberwise Dirac operator $\not{D}_f$ decomposes as

$$\not{D}_f = c_1 \not{D}_1 + \cdots + c_n \not{D}_n,$$

where the $\not{D}_i$ are Dirac operators for the M_i direction

$$\not{D}_i : \Gamma(M, \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) \xrightarrow{d} \Gamma(M, \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) \otimes T^*M)) \xrightarrow{p_{T_V M_i}} \Gamma(\not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n)) \xrightarrow{c} \Gamma(M, \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n)).$$

Similarly, the twisted spinor bundle $\mathfrak{S}^E_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n) := \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) \otimes E$ is isomorphic to $\not{S}^{0,E}_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n) \hat{\otimes} \mathbb{C}\ell_n$. Moreover if E is equipped with a connection ∇^E whose curvature R^E satisfies $R^E(X, Y) = 0$ for any $X \in T_V M_i$ and $Y \in T_V M_j$ ($i \neq j$), then the Dirac operator twisted by E decomposes as $\not{D}^E = c_1 \not{D}^E_1 + \cdots + c_n \not{D}^E_n$, such that $\not{D}_i$ commutes with $\not{D}_j$. Now $(\not{D}^E_1, \cdots, \not{D}^E_n)$ forms a smooth family of unbounded commuting Fredholm n-tuples, and $\not{D}^E_f$ is the smooth family of the Dirac operators associated with it.

More generally, we obtain some examples of twisted commuting Fredholm n-tuples. Let V be a real vector bundle whose structure group is a discrete subgroup G of $\mathrm{GL}(n, \mathbb{R})$, $B' = G(V)$ the frame bundle of V, $M'_1, \dots, M'_n$ fiber bundles over B' with a G-action on $M' := M_1 \times_{B'} \cdots \times_{B'} M'_n$ that is compatible with the projection $M' \to B'$ and E a G-equivariant vector bundle on M' whose connection ∇ is G-equivariant and satisfies the above assumption on the curvature. The G-action on M' induces a unitary representation U_x of G on $L^2(Z'_x, \mathfrak{S}^E_{\mathbb{C}}(Z'_x))$, where $Z'_x := \pi'^{-1}(x)$ (π' is the projection from M' to B). We assume that

$$U_x(g) \not{D}^E_i U_x(g)^* = \sum g_{ij} \not{D}^E_j.$$

Then $\big(g = (v_1, \dots, v_n), (\not{D}^E_1(x, g), \dots, \not{D}^E_n(x, g))\big) \in B' \times \mathcal{F}_n(\mathcal{H})$ is G-invariant, and hence the map $x \mapsto \not{D}^E_v(x)$ is well-defined and determines a V-twisted smooth family of commuting Fredholm n-tuples.

There are two fundamental examples. The first is the $\mathrm{SL}(n, \mathbb{Z})$-action on $\mathbb{T}^n = (S^1)^n$ or the product bundle $\mathbb{T}^n \times B$. The second is the $\mathfrak{S}_n$-action on the bundle $M' \times_B \cdots \times_B M'$. Then the fiberwise Dirac operator on the fiber bundle $M := M'/G \to B$ coincides with the Dirac operator associated with $\{\not{D}^E_v(x)\}_{x \in B}$.

Theorem 4.2. *Let B, M, V, E, and ∇ be as above. Then*

$$\mathrm{ind}_0(\not{D}^E_M) = \mathrm{jsf}\{\not{D}^E_v(x)\}.$$

This theorem is a direct consequence of Theorem 3.7 since the Dirac operator $\not{D}^E_M$ has the same principal symbol as $\pi^* \not{\mathfrak{D}}_B + \not{D}^E_f(x)$. As a special case, we can show localization of the Riemann–Roch number for Lagrangian fiber bundles on their Bohr–Sommerfeld fibers:

Corollary 4.3. *Let (M, ω) be a symplectic manifold of dimension $2n$, $\mathbb{T}^n \to M \to B$ a Lagrangian fiber bundle, and (L, ∇^L, h) its prequantum data, that is, (L, h) is a hermitian line bundle over M with connection ∇^L that is compatible with h*

whose curvature form R^L coincides with $-2\pi i\omega$. Then the Riemann–Roch number $\mathrm{RR}(M, L) := \mathrm{ind}_0 \not{D}_M^{\lambda^{1/2}\otimes L}$ (*where λ is the determinant line bundle* $\det T^{(1,0)}M$) *coincides with the number of fibers $\mathbb{T}_x$ such that $\nabla^L|_{\mathbb{T}_x}$ is flat with trivial monodromy, which are called the Bohr–Sommerfeld fibers.*

Proof. The structure of Lagrangian fiber bundles is studied in Section 2 of [Duistermaat 1980], and the following are known:

- Fact 1: There is a lattice bundle $P \subset TB$, which induces a flat metric on TB.
- Fact 2: If P is trivial, M is actually a principal $\mathbb{T}^n$-bundle.

We denote the $\mathrm{GL}(n, \mathbb{Z})$-frame bundle of TB by B' and the pullback of M by the quotient $B' \to B$ by M'. The manifold M' has a canonical symplectic structure and $M' \to B'$ is also a Lagrangian fiber bundle. By Fact 2, M' is a principal $\mathbb{T}^n$-bundle on B'. We identify the space of constant vector fields on a fiber M'_x with the Lie algebra $\mathfrak{t} = \mathrm{Lie}(\mathbb{T}^n)$.

The free $\mathrm{GL}(n, \mathbb{Z})$-action on B' extends to an action on M' preserving its symplectic form and affine structure on each fiber. Therefore it induces an action on $\mathfrak{t}$ as $g \cdot X_i = g_{ij} X_j$ for some fixed basis $X_1, \dots, X_n$ of $\mathfrak{t}$. Indeed, by considering the canonical trivialization of the tangent bundle $TB' \cong B' \times \underline{\mathbb{R}}^n$ that is compatible with the isomorphism $\mathfrak{t} \cong T_x B'$ given by a fixed almost-complex structure J, we obtain an isomorphism $\mathfrak{t} \cong T_x B' \cong \mathbb{R}^n$ that is independent of the choice of $x \in B'$. Under this identification, $g\cdot : \mathbb{R}^n \cong T_x B' \to T_{g\cdot x} B' \cong \mathbb{R}^n$ is represented by (g_{ij}) as a matrix. Hence g also acts on $\mathfrak{t}$ as (g_{ij}).

Next, we construct some flat connections. The isomorphism $T_V M \cong T_H M \cong \pi^* TB$ induced by J implies the isomorphism $\not{\mathfrak{S}}_{\mathbb{C}, f}(M; \underline{\mathbb{R}}^n) \cong \not{S}_{\mathbb{C}}(M) \cong \pi^* \not{\mathfrak{S}}_{\mathbb{C}}(B)$. Moreover, it induces a flat metric on TM that is trivially flat on each fiber $\mathbb{T}^n$, and so are associated bundles with TM, in particular $\lambda^{1/2}$ and $\not{S}_{\mathbb{C}}^{\lambda^{1/2}}(M)$. Since R^L is equal to 0 when it is restricted on each fiber, ∇^L is also fiberwise flat and the product connection $\nabla = \nabla^{\not{S}_{\mathbb{C}}^{\lambda^{1/2}\otimes L}(M)}$ is trivially flat if and only if ∇^L is trivially flat.

Finally we see that B, M, $V = TB$, $E = \lambda^{1/2} \otimes L$ and $\nabla^{\lambda^{1/2}\otimes L}$ satisfy the assumptions of Theorem 4.2. Then $\{\nabla_v(x)\}$ forms a family of commuting Fredholm n-tuples twisted by TB and the index of the Dirac operator $\not{D}_M^{\lambda^{1/2}\otimes L}$ coincides with the joint spectral flow.

The kernel of $\Delta_f := \nabla_{e_1}^2 + \cdots + \nabla_{e_n}^2$ is not zero if and only if ∇ is (and hence ∇^L is) trivially flat. This means that the joint spectra of $\{\nabla(x)\}$ cross over zero only on the Bohr–Sommerfeld fibers. The remaining part is that the multiplicity of eigenvalues crossing zero on each Bohr–Sommerfeld fiber is equal to 1. This follows from the fact in symplectic geometry that the tubular neighborhood of a Lagrangian submanifold is isomorphic to its cotangent bundle as a symplectic manifold, and that $T^*\mathbb{T}^n$ is actually the product space $(T^*S^1)^n$. More detail is found in Section 6.4 of [Fujita et al. 2010]. $\square$

4B. ***Generalized Toeplitz index theorem.*** In this section we introduce a generalization of a classical theorem relating the index of Toeplitz operators with the winding numbers.

Definition 4.4. Let Y be an n-dimensional closed manifold with $n = 2m - 1$. For $\varphi : Y \to U(k)$ the generalized Toeplitz operator T_φ is defined by

$$Pm_\varphi P : PL^2(Y, \not{S}_{\mathbb{C}}(Y))^{\oplus k} \longrightarrow PL^2(Y, \not{S}_{\mathbb{C}}(Y))^{\oplus k},$$

where P is the orthogonal projection onto $\overline{\operatorname{span}}\{\varphi \mid \not{D}\varphi = \lambda\varphi \text{ for some } \lambda \geq 0\}$.

Example 4.5 ($Y = S^1$). When $Y = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (and hence when $\not{S}_{\mathbb{C}}(Y)$ associated with the canonical Spin^c-structure on Y is a trivial bundle), we can identify its Dirac operator as d/dt. Its spectrum coincides with $\mathbb{Z}$, and the eigenspaces E_n are the 1-dimensional complex vector spaces $\mathbb{C} \cdot e^{int}$. Therefore $PH = \overline{\operatorname{span}}\{e^{int} \mid n \in \mathbb{Z}_{\geq 0}\}$, and the corresponding generalized Toeplitz operators T_φ are simply the ordinary ones. Its index is obtained from the winding number by $\operatorname{ind} T_\varphi = -\operatorname{winding} \varphi$.

Now we generalize this index theorem for generalized Toeplitz operators in a special case. Let $\Delta_n = \Delta_n^0 \oplus \Delta_n^1$ be a unique irreducible $\mathbb{Z}/2$-graded $\mathbb{C}\ell_n$-module and γ the grading operator on it. When we have a continuous map $\varphi = (\varphi_0, \dots, \varphi_n) : Y \to S^n$, we obtain an even unitary $\varphi_0 + \gamma c_1 \varphi_1 + \cdots + \gamma c_n \varphi_n$, where c_i $(i = 1, \dots, n)$ are Clifford multiplications of an orthonormal basis $e_1, \dots, e_n$. For simplicity of notation, we use the same letter φ for its restriction to Δ_n^0.

Theorem 4.6. *Let Y and φ be as above. Then*

$$\operatorname{ind} T_\varphi = -\deg(\varphi : Y \to S^n).$$

Proof. Baum and Douglas [1982] proved the cohomological formula for this index, which is analogous to the Atiyah–Singer formula. As a consequence, we have that

$$\operatorname{ind} T_\varphi = -\langle \operatorname{ch}(\varphi) \operatorname{Td}(X), [X] \rangle.$$

Actually we can give a proof of Theorem 4.6 by using this and the description of the Chern character in Lemma 3.5. □

4C. ***Localization of APS index for families and eta-form.*** We can also apply our joint spectral flow index theorem for fiber bundles whose fibers are compact manifolds with boundary. A main reference for this section is [Melrose and Piazza 1997].

Let B be a closed n-dimensional Spin^c manifold and $Z \to M \to B$ a smooth fiber bundle over B whose boundary also forms a fiber bundle $\partial Z \to \partial M \to B$. Here we also assume that M is Spin^c. The Riemannian metric g on TM is introduced by the direct sum decomposition $g_f \oplus \pi^* g_B$ on $T_V M \oplus T_H M$, where g_B is a Riemannian metric on $TB \cong T_H M$ and g_f is a smooth family of Riemannian metrics on the fibers Z_x that are exact b-metrics near the boundaries ∂Z_x. We assume that $\$_{\mathbb{C}}(T_V M)$ has a $\mathbb{C}\ell(V)$-action and the fiberwise Dirac operator $\not{D}_f^E$ on $\$_{\mathbb{C}}^E(M)$ coincides with the Dirac operator $c(v_1)D_{v_1}+\cdots+c(v_n)D_{v_n}$ associated with a V-twisted n-tuple $\{D_v\}$ of fiberwise first-order pseudodifferential operators that satisfies Condition 3.18. We denote by $H^{1,0}(M, E)$ the fiberwise Sobolev space, that is, the completion of $C^\infty(M, E)$ by the inner product $\langle \cdot\,, \cdot \rangle_{L^2} + \langle \nabla_f^E \cdot\,, \nabla_f^E \cdot \rangle$, where $\nabla_f^E := p_{T_V M} \circ \nabla^E$. Then an element in $H^{1,0}(M, E)$ is fiberwise continuous and there is a bounded operator

$$\partial : H^{1,0}(M, E) \to L^2(\partial M, E|_{\partial M}), \quad \sigma \mapsto \sigma|_{\partial M}.$$

Now we fix a spectral section $P \in C(B, \{\Psi_0(\partial Z_x, E|_{\partial Z_x})\}_{x\in B})$. Here a spectral section P is a projection such that there is a smooth function $R : B \to \mathbb{R}$ and the condition $D_f(x)\sigma = \lambda\sigma$ implies $P(x)\sigma = \sigma$ if $\lambda > R(x)$ and $P(x)\sigma = 0$ if $\lambda < -R(x)$ for any $x \in B$. Then this P determines an elliptic boundary condition at each fiber and

$$\begin{gathered} \not{D}_f : L^2(M, E) \to L^2(M, E), \\ \operatorname{dom} \not{D}_f := \{\sigma \in H^{(1,0)}(M, E) \mid P(\partial\sigma) = 0\} \end{gathered}$$

is a fiberwise Fredholm self-adjoint operator. Hence it forms a V-twisted continuous family of unbounded commuting Fredholm n-tuples $\{D_v(x)\}$ parametrized by B.

Theorem 4.7. $$\operatorname{ind}_P(\not{D}) = \operatorname{jsf}(\{D(x)\}).$$

The same proof that we gave for Theorem 3.7 and 3.19 also works here. This is because we deal with operators directly, instead of the topology of their principal symbols. We only remark that in this situation D_b and $D_f(1+D_f^2)^{-1/2}$ commute modulo bounded operators. Furthermore, we obtain an analogue of Theorem 3.14.

Now we introduce its application for a geometric problem.

Let B be an n-dimensional closed manifold, $V \to B$ a real vector bundle of dimension n and $Y \to N \to B$ a fiber bundle with $\dim Z = n-1$. We assume that there is an oriented embedding of M into V as a fiber bundle. Then there is a fiber bundle $Z \to M \to B$ of manifolds whose boundary is isomorphic to $Y \to N \to B$ as fiber bundles. Now we define the eta-form [Bismut and Cheeger 1989] for N by

$$\begin{aligned} \hat{\eta}_P &= \int_0^\infty \hat{\eta}_P(t)\,dt, \\ \hat{\eta}_P(t) &= \frac{1}{\sqrt{\pi}} \operatorname{Str}_{\mathbb{C}\ell_1}\left(\frac{d\tilde{\mathbb{B}}_t}{dt} e^{-\tilde{\mathbb{B}}_t^2}\right), \end{aligned}$$

where $\tilde{\mathbb{B}}_t$ is the deformed $\mathbb{C}\ell_1$-superconnection. This differential form is closed and used for the Atiyah–Patodi–Singer index theorem for families.

On the other hand, the canonical metric on V induces a smooth family of exact b-metrics on $T_V M$. Therefore, first-order differential operators $\partial/\partial v_i$ on V_x (where $v_1, \dots, v_n$ is a basis of V_x) form a V-twisted commuting Fredholm n-tuple.

Theorem 4.8. *Let $Z \to M \to B$ and V be as above. If there is an oriented embedding of M into V as fiber bundles, its eta-form $\hat{\eta}_P$ is in the image of $H^n(B; \mathbb{Z})$. Moreover, in that case*

$$\int_B \hat{\eta}_P = \operatorname{ind}_P(\not{D}_M) = \operatorname{jsf}\{D(x)\}.$$

Proof. From Theorem 4.7 we have $j_*\{D(x)\} = \operatorname{ch}(\operatorname{ind}_P(\not{D}_f))$. Now the Atiyah–Patodi–Singer index theorem for families [Melrose and Piazza 1997] says that $\operatorname{ch}(\operatorname{ind}_P(\not{D}_f)) = \pi_!(\hat{A}(T_V M)) + \hat{\eta}_P$. In our case $T_V M$ is trivial and hence the first term of the above equality vanishes. □

In particular, in the case $Y = S^{n-1}$, we get an obstruction for an oriented sphere bundle to be isomorphic to the unit sphere bundle of a vector bundle. This is related to the comparison of the homotopy types of $\operatorname{Diff}_+(S^{n-1})$ and $\mathrm{SO}(n)$, which is the Smale conjecture.

5. Decomposing Dirac operators

Now the converse problem arises. When do geometric Dirac operators "decompose" into Dirac operators associated with commuting Fredholm n-tuples? In this section we deal with zeroth-order pseudodifferential operators to obtain a complete obstruction from its index by using the theory of C^*-algebra extensions, which is related to KK^1-theory and index theory in [Kasparov 1980b].

We start with some folklore. Let T_φ be a Toeplitz operator associated with $\varphi \in C(S^1)^\times$. Then T_φ is not a normal operator in general, and $\operatorname{Re} T_\varphi$ can only commute with $\operatorname{Im} T_\varphi$ if $\operatorname{ind} T_\varphi$ is equal to 0. In this situation, the index of the operator $T_\varphi = \operatorname{Re} T_\varphi + i \operatorname{Im} T_\varphi$ gives a complete obstruction to the existence of mutually commuting self-adjoint operators A and B such that $(A - \operatorname{Re} T_\varphi)$ and $(B - \operatorname{Im} T_\varphi)$ are compact. Our purpose in this section is to give an analogue and a generalization of this for the bounded operators associated with Dirac operators.

Before we consider the case of families, we deal with a single Dirac operator. First of all, we assume that its principal symbol decomposes. This is interpreted as a geometric condition as follows: Let M be a closed Spin^c manifold and $H_1, \dots, H_n$ mutually orthogonal odd-dimensional subbundles of TM whose direct sum spans TM. As is argued in Section 4A, $\not{\mathfrak{S}}_{\mathbb{C}}(M; \underline{\mathbb{R}}^n) := \not{S}_{\mathbb{C}}(TM \oplus \underline{\mathbb{R}}^n)$ decomposes as

$$\not{\mathfrak{S}}_{\mathbb{C}}(M; \underline{\mathbb{R}}^n) \cong (\not{S}^0_{\mathbb{C}}(H_1) \otimes \cdots \otimes \not{S}^0_{\mathbb{C}}(H_n)) \hat{\otimes} \mathbb{C}\ell_n.$$

Hereafter we set $\slashed{S}^0_{\mathbb{C}}(M;\underline{\mathbb{R}}^n) := \slashed{S}^0_{\mathbb{C}}(H_1) \otimes \cdots \otimes \slashed{S}^0_{\mathbb{C}}(H_n)$. Under this identification, the principal symbol of the Dirac-type operator $\slashed{D}^E$ on $\slashed{S}^E_{\mathbb{C}}(M;\underline{\mathbb{R}}^n)$ is interpreted as

$$\sigma(\slashed{D}^E) = \sum_{i=1}^{k} \Biggl(\sum_{j=1}^{\dim H_i} 1 \otimes \cdots \otimes c(e_{i,j})\xi_{i,j} \otimes \cdots \otimes 1 \Biggr) \hat{\otimes}\, c_i ,$$

where each $\{e_{i,j}\}_{j=1,\dots,\dim H_i}$ is an orthonormal basis on H_i and the $\xi_{i,j} := \langle \xi, e_{i,j} \rangle$ are coordinate functions on each cotangent space. Then we can construct a commuting n-tuple on the symbol level. This also works for the Dirac operator $\slashed{D}^E$ twisted by a complex vector bundle E. We say the Dirac operator $\slashed{D}^E$ is said to be *n-decomposable* if there is a bounded commuting Fredholm n-tuple $(T_1, \dots, T_n)$ such that each T_i is a zeroth-order pseudodifferential operator on $\Gamma(M, \slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))$ whose principal symbol is of the form $\sigma(T_i) = \sum_j 1 \otimes \cdots \otimes c(e_{i,j})\xi_{i,j} \otimes \cdots \otimes 1$. In that case, the bounded operator $\slashed{D}^E(1+(\slashed{D}^E)^2)^{-1/2}$ associated with $\slashed{D}^E$ coincides modulo compact operators with the Dirac operator associated with the bounded commuting Fredholm n-tuple T.

In fact, n-decomposability is a K-theoretic property and is determined by the index:

Proposition 5.1. *Let M, $H_1, \dots, H_n$, and E be as above. Then the Dirac operator $\slashed{D}^E$ is n-decomposable if and only if* $\operatorname{ind}(\slashed{D}^E) = 0$.

Proof. A decomposition of the principal symbol gives a $*$-homomorphism $\sigma(\slashed{D}^E) : C(S^{n-1}) \to A := \Gamma\bigl(S(TM), \operatorname{End}(\pi^* \slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))\bigr)$ that maps the coordinate function x_i $(i = 1, \dots, n)$ of $\mathbb{R}^n$, which contains S^{n-1} as the unit sphere, to an element $\sum_j c(e_{i,j})\xi_{i,j}$. It is well-defined because the square sum $\sum_i \bigl(\sum_j c(e_{i,j})\xi_{i,j}\bigr)^2$ is equal to 1. Hence we can replace the problem of obtaining a decomposition of $\slashed{D}^E$ with that of obtaining a lift, as is shown in the following diagram by the dotted arrow, of $\sigma(\slashed{D}^E)$:

$$\begin{tikzcd}
 & & & C(S^{n-1}) \arrow[dl, dotted] \arrow[d, "\varphi"] & \\
0 \arrow[r] & \Psi^{-1}(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n)) \arrow[r] \arrow[d, equal] & \Psi^{0}(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n)) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & \mathbb{K}(\mathcal{H}) \arrow[r] & \mathbb{B}(\mathcal{H}) \arrow[r] & Q(\mathcal{H}) \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{H} := L^2(M, \slashed{S}^{E,0}_{\mathbb{C}}(M,\underline{\mathbb{R}}^n))$ and $\Psi^0(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))$ (resp. $\Psi^{-1}(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))$) is the norm closure of the space of pseudodifferential operators of order 0 (resp. -1). In terms of the extension theory, it means that the extension $\varphi^*\tau = \tau \circ \varphi$ is trivial. Now, as mentioned above, the theory of C^*-algebra extension is translated into

KK^1-theory. In particular, a semisplit extension φ has a lift after stabilizing by the trivial extension if and only if the KK^1-class $[\varphi]$ is zero. Moreover, in our case we do not have to care for the stabilization of φ because the Voiculescu theorem [1976] ensures that φ absorbs any trivial extensions.

In the case of odd n, ind $\not{D}^E$ is immediately 0 because $KK^1(C(S^{n-1}), \mathbb{K})$ itself is 0. On the other hand, ind $\not{D}^E$ is also 0 because $\dim M$ is odd.

In the case of even n, we obtain an integer $\varphi^*[\tau] \in KK^1(C(S^{n-1}), \mathbb{K}) \cong \mathbb{Z}$ as the Fredholm index of $\tau \circ \varphi(u) \in Q(\mathcal{H})$ by Theorem 18.10.2 of [Blackadar 1998]. Here u is the canonical generator of $KK^1(\mathbb{C}, C(S^{n-1})) \cong K_1(C(S^{n-1}))$, and its additive inverse is represented by a family of unitary matrices $u := \sum c_1 c_i x_i \in C(S^{n-1}, \mathrm{End}(\Delta_n^0))$ (it is a consequence of Theorem 4.6). Now $\cdot\, \tau \circ \varphi(u)$ coincides with the principal symbol of the Dirac operator $c_1 \cdot (\not{D}^E)^0$ on $\Gamma(M, \not{S}^{E,0}(M))$ because $\not{S}^0_{\mathbb{C}}(M) \cong \not{S}^0_{\mathbb{C}}(M; \underline{\mathbb{R}}^n) \hat{\otimes} \Delta^0_n$. □

We now turn to the case of families of Dirac operators, which is our main interest.

Let $Z \to M \to B$ be a fiber bundle and set $n := \dim B$. We assume that there are Spin^c vector bundles $V_1, \dots, V_l$ on B and $H_1, \dots, H_l$ on M such that $\pi^* V_i \otimes H_i$ are also Spin^c and the vertical tangent bundle $T_V M$ is isomorphic to their direct sum $\pi^* V_1 \otimes H_1 \oplus \dots \oplus \pi^* V_l \otimes H_l$. We denote the direct sum $V_1 \oplus \dots \oplus V_l$ by V and assume that $\dim V = n$. Moreover, we assume that each H_i is odd-dimensional and decomposes as $H_i \cong H_i^0 \oplus \underline{\mathbb{R}}$. Now, as is in Section 4A, the spinor bundle $\not{\mathfrak{S}}_{\mathbb{C},f}(M; V) := \not{S}_{\mathbb{C}}(T_V M \oplus V)$ decomposes as

$$\not{\mathfrak{S}}_{\mathbb{C},f}(M; V) \cong \big(\not{S}_{\mathbb{C}}(\pi^* V_1 \otimes H_1^0) \otimes \dots \otimes \not{S}_{\mathbb{C}}(\pi^* V_n \otimes H_n^0)\big) \hat{\otimes} \mathbb{C}\ell(\pi^* V).$$

Hereafter we set $\not{S}^{E,0}_{\mathbb{C},f}(M; V) := \not{S}_{\mathbb{C}}(\pi^* V_1 \otimes H_1^0) \otimes \dots \otimes \not{S}_{\mathbb{C}}(\pi^* V_n \otimes H_n^0)$. The principal symbol of the fiberwise Dirac operator $\not{D}^E_f$ on the twisted fiberwise spinor bundle $\not{\mathfrak{S}}^E_{\mathbb{C},f}(M; V) := \not{\mathfrak{S}}_{\mathbb{C},f}(M; V) \otimes E$ also decomposes as a V-twisted continuous family of commuting n-tuples. Indeed, for $v = v_1 \oplus \dots \oplus v_l$, the explicit decomposition is given by the correspondence

$$\sigma(\not{D}^E_f)_v = \sum \big(c(v_1 \otimes e_{1,j}) \xi_{e_{1,j}}\big) + \dots + \sum \big(c(v_1 \otimes e_{l,j}) \xi_{e_{l,j}}\big).$$

This gives a $*$-homomorphism $\sigma(\not{D}^E_f)_v : C(S(V)) \to C(B) \otimes Q(\mathcal{H})$ that is compatible with the inclusions $C(B) \subset C(S(V))$ and $C(B) \otimes 1 \subset C(B) \otimes Q(\mathcal{H})$. In particular, when V is trivial it reduces to the $*$-homomorphism $\sigma(\not{D}^E_f)_v : C(S^{n-1}) \to C(B) \otimes Q(\mathcal{H})$.

Definition 5.2. The fiberwise Dirac operator $\not{D}^E_f$ is said to be *n-decomposable* if there is a V-twisted bounded commuting Fredholm n-tuple $\{T_v(x)\}$ such that each T_v is a zeroth-order pseudodifferential operator on $\Gamma(\not{S}^{E,0}_{\mathbb{C},f}(M; V))$ whose principal symbol is $\sigma(T_v) = \sum \big(c(v_1 \otimes e_{1,j}) \xi_{e_{1,j}}\big) + \dots + \sum \big(c(v_1 \otimes e_{l,j}) \xi_{e_{l,j}}\big)$.

In this case, $\not{D}_f^E(1+(\not{D}_f^E)^2)^{-1/2}$ coincides modulo compact operators with the smooth family of Dirac operators associated with the V-twisted bounded commuting Fredholm n-tuples $\{T_v(x)\}$. Hence the K-class $[\mathrm{ind}\,\not{D}_f^E]$ is in the image of the canonical natural transform from $\tilde{k}^n(B)$ to $K^n(B)$. Moreover, the index of the Dirac operator $\not{D}_M^E$ on M twisted by E, which coincides with that of $\pi^*\not{\mathfrak{D}}_B+\not{D}_f^E$, can be obtained from the joint spectral flow $\mathrm{jsf}\{T_v(x)\}$.

Theorem 5.3. *Let $Z\to M\to B$, $V_1,\dots,V_l$, $H_1,\dots,H_l$, and E be as above. Then $\not{D}_f^E$ is n-decomposable if and only if $\mathrm{ind}(\not{D}_f^E)$ is in the image of $K^n(B,B^{(n-1)})\to K^n(B)$, or equivalently the image of $\tilde{k}^n(B)\to K^n(B)$. In that case, the equality $\mathrm{ind}\,\not{D}_M^E=\mathrm{jsf}\{\not{D}_f^E\}$ holds.*

Here $B^{(n-1)}$ is the $(n-1)$-skeleton of a cellular decomposition of B. The image of $K(B,B^{(n-1)})\to K^n(B)$, which is the Atiyah–Hirzebruch filtered K-group $F^{n-1}K^n(B)$, is independent of the choice of decompositions and coincides with the image of $\tilde{k}^n(B)\to K^n(B)$ because of the functoriality of $\tilde{k}^*\to K^*$ and the fact that $\tilde{k}^n(B^{(n-1)})=0$.

Remark 5.4. In the proof, except for the last part, the condition that B is an n-dimensional closed manifold is not necessary. Actually it is sufficient to be a finite CW-complex. Moreover, if B is an n-dimensional CW-complex, the last part also holds.

The proof is divided into several steps. First, we show that $\not{D}_f^E$ is locally n-decomposable.

Lemma 5.5. *Let $M=B\times Z$ and $TZ\cong H_1\oplus\cdots\oplus H_n$. If the index of the fiberwise Dirac operator $\not{D}_f^E$ on $\not{S}_{\mathbb{C}}^E(M;\mathbb{R}^n)$ is zero, then $\not{D}_f^E$ is n-decomposable.*

Proof. As in Proposition 5.1, it suffices to find a lift of the extension $\sigma(\not{D}_f^E)_v: C(S^{k-1})\to C(B)\otimes C(S(TZ))\subset C(B)\otimes Q(\mathcal{H})$. This exists when the metrics on fibers are constant because $\sigma(\not{D}_f^E)_v$ is trivial and absorbable by Kasparov's [1980b] generalized Voiculescu theorem. In the general case, it exists because $\sigma(\not{D}_f^E)_v|_{M_y}=u_y(\sigma(\not{D}_f^E)_v|_{M_x})u_y^*$, where $u_y:\pi^*\not{S}_{\mathbb{C}}^E(M_x;\mathbb{R}^n)\to\pi^*\not{S}_{\mathbb{C}}^E(M_y;\mathbb{R}^n)$ is the isometry induced from the polar part of the identity map $\mathrm{id}:TM_x\to TM_y$. $\square$

Next we introduce a technique for gluing two decompositions. We can deal with the problem cohomologically by using Cuntz's [1983] notion of quasihomomorphisms. The "difference" of two lifts $\varphi_0,\varphi_1: C(S(V))\to C(B)\otimes\mathbb{B}(\mathcal{H})$ of $\sigma(\not{D}_v^E)$ gives an element of the representable KK-group [Kasparov 1988]

$$[\varphi_0,\varphi_1]:=\left[\hat{\mathcal{H}}\,\hat{\otimes}\,C(B),\begin{pmatrix}\varphi_0&0\\0&\varphi_1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\in\mathcal{R}KK\big(B;C(S(V)),C(B)\otimes\mathbb{K}\big).$$

In particular, in the case that V is trivial, we can reduce the representable KK-group $\mathcal{R}KK\big(B; C(S(V)), C(B)\big)$ by $KK(C(S^{n-1}), C(B) \otimes \mathbb{K})$. Then the split exact sequence

$$0 \longrightarrow C_0(S^{n-1} \setminus \{*\}) \longrightarrow C(S^{n-1}) \xrightarrow{p} \mathbb{C} \longrightarrow 0$$

gives an isomorphism

$$KK(C(S^{n-1}), C(F)) \cong KK\big(C_0(S^{n-1} \setminus \{*\}), C(F)\big) \oplus KK(\mathbb{C}, C(F)).$$

When both of φ_0 and φ_1 are unital, $[\varphi_0, \varphi_1]$ corresponds to $[\varphi_0, \varphi_1]|_{C(S^{n-1}\setminus\{*\})} \oplus 0$ under the above identification because $p^*[\varphi_0, \varphi_1] = [1, 1] = 0$.

Lemma 5.6. *Let F_0, F_1 be closed subsets of B such that $B = (F_0)^\circ \cup (F_1)^\circ$ and $F := F_0 \cap F_1$. We assume that M and E are trivial on F and $\sigma(\not{D}^E_f)$ has lifts φ_0 and φ_1 on F_0 and F_1. Then the image of $[\varphi_0, \varphi_1] \in KK(C_0(S^{n-1} \setminus \{*\}, \mathbb{K} \otimes C(F)) \cong K^{n-1}(F)$ under the boundary map of the Mayer–Vietoris sequence coincides with $[\operatorname{ind} \not{D}^E_f] \in K^n(B)$.*

Proof. From the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\mathbb{D}^n \setminus \{0\}) & \longrightarrow & C_0(\overline{\mathbb{D}^n} \setminus \{0\}) & \longrightarrow & C(S^{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow & & \| & & \\
0 & \longrightarrow & C_0(\mathbb{D}^n) & \longrightarrow & C_0(\overline{\mathbb{D}^n}) & \longrightarrow & C(S^{n-1}) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0(\mathbb{D}^n) & \longrightarrow & C_0(\overline{\mathbb{D}^n} \setminus \{*\}) & \longrightarrow & C_0(S^{n-1} \setminus \{*\}) & \longrightarrow & 0
\end{array}$$

we obtain a diagram of KK-groups

$$\begin{array}{ccc}
KK^1(C_0(\mathbb{D}^n \setminus \{0\}), C(F)) & \xrightarrow[\partial_1]{\sim} & KK^0(C(S^{n-1}), C(F)) \\
{\scriptstyle \iota^*}\uparrow & & \| \\
KK^1(C(\mathbb{D}^n), C(F)) & \xrightarrow[\partial_2]{} & KK^0(C(S^{n-1}), C(F)) \\
\| & & \downarrow \\
KK^1(C(\mathbb{D}^n), C(F)) & \xrightarrow[\partial_3]{\sim} & KK^0(C_0(S^{n-1} \setminus \{*\}), C(F))
\end{array}$$

Here, for a C^*-algebra A, the group $KK^1(A, C(F))$ is canonically isomorphic to $KK(A, \Sigma C(F)) \cong KK(A, C_0(\Sigma F))$. One can see that the inverse $(\partial_1)^{-1}$ of the boundary map coincide with the product with $[\mathrm{id}_\Sigma] \in KK(\Sigma, \Sigma)$.

As a consequence we obtain that

$$\iota^* \partial_3^{-1}[\varphi_0, \varphi_1] = \partial_1^{-1}[\varphi_0, \varphi_1] = [\varphi_0 \otimes \mathrm{id}_\Sigma, \varphi_1 \otimes \mathrm{id}_\Sigma].$$

Next, we consider the isomorphism between

$$KK(C_0(\mathbb{D}^n), C_0(\Sigma F)) \quad \text{and} \quad KK(\mathbb{C}, C_0(\Sigma F) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n).$$

As is in Section 2, this correspondence is given by taking a product with the canonical generator

$$[C_{\mathbb{D}^n}] := \Big[C_0(\mathbb{D}^n) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n, 1, C_{\mathbb{R}^n} := \sum x_i \cdot c_i\Big]$$

of $KK(\mathbb{C}, C_0(\mathbb{D}^n) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n)$. Restricting to $C_0(\mathbb{D}^n \setminus \{0\}) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n$, the operator $C_{\mathbb{D}^n}$ also defines an element $[C_{\mathbb{D}^n \setminus \{0\}}]$ in $KK(\mathbb{C}, C(\mathbb{D}^n \setminus \{0\}) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n)$. When we regard the topological space $\mathbb{D}^n \setminus \{0\}$ as ΣS^{n-1}, the operator $C_{\mathbb{D}^n}$ is of the form $tC_{S^{n-1}}$, where $C_{S^{n-1}} := \sum c_i \cdot x_i \in C(S^{n-1}) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n$ and t is the identity function on $(0, 1)$. Now the diagram

$$\begin{tikzcd}
KK(C_0(\mathbb{D}^n), C_0(\Sigma F)) \arrow[d, "\iota^*"] \arrow[dr, "{[C_{\mathbb{D}^n}]}"] & \\
& KK(\mathbb{C}, C_0(\Sigma F) \mathbin{\hat{\otimes}} \mathbb{C}\ell_n) \\
KK(C_0(\mathbb{D}^n \setminus \{0\}), C_0(\Sigma F)) \arrow[ur, "{[C_{\mathbb{D}^n \setminus \{0\}}]}"'] &
\end{tikzcd}$$

commutes. As a consequence, we can compute $[C_{\mathbb{D}^n}] \otimes_{C_0(\mathbb{D}^n)} \partial_3^{-1}[\varphi_0, \varphi_1]$ by using Proposition 18.10.1 of [Blackadar 1998] as follows:

$$\begin{aligned}
&[C_{\mathbb{D}^n}] \otimes_{C_0(\mathbb{D}^n)} \partial_3^{-1}[\varphi_0, \varphi_1] \\
&\qquad = [C_{\mathbb{D}^n \setminus \{0\}}] \otimes_{C_0(\mathbb{D}^n \setminus \{0\})} \iota^* \partial_3^{-1}[\varphi_0, \varphi_1] \\
&\qquad = [tC_{S^{n-1}}] \otimes_{C_0(\Sigma S^{n-1})} [\varphi_0 \otimes \mathrm{id}_\Sigma, \varphi_1 \otimes \mathrm{id}_\Sigma] \\
&\qquad = \left[\hat{\mathcal{H}}_{C_0(\Sigma F)} \mathbin{\hat{\otimes}} \mathbb{C}\ell_n, 1, \begin{pmatrix} \varphi_0(tC_{S^{n-1}}) & 0 \\ 0 & \varphi_1(tC_{S^{n-1}}) \end{pmatrix} \right. \\
&\qquad\qquad\qquad \left. + \begin{pmatrix} 1 - \varphi_0(tC_{S^{n-1}})^2 & 0 \\ 0 & 1 - \varphi_1(tC_{S^{n-1}})^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\
&\qquad = \left[\hat{\mathcal{H}}_{C_0(\Sigma F)} \mathbin{\hat{\otimes}} \mathbb{C}\ell_n, 1, \begin{pmatrix} \varphi_0(tC_{S^{n-1}}) & 1 - \varphi_0(tC_{S^{n-1}})^2 \\ 1 - \varphi_1(tC_{S^{n-1}})^2 & \varphi_1(tC_{S^{n-1}}) \end{pmatrix} \right] \\
&\qquad = [\hat{\mathcal{H}}_{C_0(\Sigma F)} \mathbin{\hat{\otimes}} \mathbb{C}\ell_n, 1, T].
\end{aligned}$$

Here

$$T = \{T_t\}_{t\in[0,1]} := \begin{cases} \begin{pmatrix} \varphi_0((1-2t)C_{S^{n-1}}) & 1-(1-2t)^2 \\ 1-(1-2t)^2 & \varphi_0((1-2t)C_{S^{n-1}}) \end{pmatrix} & 0 \le t \le 1/2, \\ \begin{pmatrix} \varphi_0((2t-1)C_{S^{n-1}}) & 1-(2t-1)^2 \\ 1-(2t-1)^2 & \varphi_1((2t-1)C_{S^{n-1}}) \end{pmatrix} & 1/2 \le t \le 1. \end{cases}$$

Now we claim that this KK-class coincides with that coming from the cycle

$$\big[\mathcal{H}_{C_0(\Sigma F)} \hat{\otimes} \mathbb{C}\ell_n, 1, t\varphi_0(C_{S^{n-1}}) + (1-t)\varphi_1(C_{S^{n-1}})\big].$$

Indeed, because $T_t - T_{1-t}$ is compact for any $t \in [0, 1/2]$, the homotopy of continuous families of Fredholm operators

$$\mathfrak{T}_{s,t} := \begin{cases} T_t & 0 \le t \le s/2, \\ \dfrac{t-s/2}{1-s} T_{s/2} + \dfrac{1-t-s/2}{1-s} T_{1-s/2} & s/2 \le t \le 1-s/2, \\ T_t & 1-s/2 \le t \le 1 \end{cases}$$

connects $\mathfrak{T}_0 = T$ with

$$\mathfrak{T}_1 = \begin{pmatrix} \varphi_0(C_{S^{n-1}}) & 0 \\ 0 & t\varphi_0(C_{S^{n-1}}) + (1-t)\varphi_1(C_{S^{n-1}}) \end{pmatrix}.$$

Finally we obtain that $[\varphi_0, \varphi_1]$ coincides with $[t\varphi_0(C_{S^{n-1}}) + (1-t)\varphi_1(C_{S^{n-1}})]$ in $KK(C_0(S^{n-1} \setminus \{*\}), C(F)) \cong K^n(\Sigma F)$. Next we map it by the boundary map δ_{MV} of the Mayer–Vietoris exact sequence.

We denote by $I(F_0, F_1; F)$ the space $F_0 \sqcup IF \sqcup F_1$. The image of δ_{MV} is induced from the map $I(F_0, F_1; F) \to (I(F_0, F_1; F), F_0 \cup F_1)$ and excision. Therefore $\delta_{MV}[t\varphi_0(C_{S^{n-1}}) + (1-t)\varphi_1(C_{S^{n-1}})]$ is of the form

$$\begin{cases} \varphi_0(C_{S^{n-1}})_x & x \in F_0, \\ t\varphi_0(C_{S^{n-1}})_x + (1-t)\varphi_1(C_{S^{n-1}})_x & (x,t) \in IF, \\ \varphi_1(C_{S^{n-1}})_x & x \in F_1. \end{cases}$$

It is a lift of the pullback of the principal symbol $\sigma(\not{D}^E_v)$ by the canonical projection $I(F_0, F_1; F) \to B$, which introduces the homotopy equivalence. As a consequence the above operator coincides with $\not{D}^E_f(1 + (\not{D}^E_f)^2)^{-1/2}$ modulo compact operators and hence defines the same KK-class. □

Lemma 5.7. *If* $[\operatorname{ind} \not{D}^E_f] = 0 \in K^n(B)$, *then* $\not{D}^E_f$ *is* n*-decomposable.*

Proof. Let $U_1, \dots, U_m$ be a local trivialization of the fiber bundle $M \to B$ and the vector bundles $V_1, \dots, V_l \to B$ such that M is also trivial on $F_i := \overline{U_i}$. By assumption and Lemma 5.5, $\not{D}^E_f$ is n-decomposable on each F_i.

We start with the case that $B = F_0 \cup F_1$, and set $F := F_0 \cap F_1$. First, we fix a trivial and absorbable extension $\pi : C(S^{n-1}) \to Q(\mathcal{H}_\pi)$ of $\mathbb{K}$ by $C(S^{n-1})$ and denote by π_A an extension $C(S^{n-1}) \to Q(\mathcal{H}_\pi) \to Q(\mathcal{H}_\pi) \otimes A$ of $C(S^{n-1})$ by $A \otimes \mathbb{K}$ for a unital C^*-algebra A.

Now we choose lifts φ_0 and φ_1 of $\sigma(\not{D}_v^E)$ on F_0 and F_1. By Kasparov's generalized Voiculescu theorem, the φ_i are approximately equivalent to $\varphi_i \oplus \pi_{C(F_i)}$. More precisely, there are continuous families of unitaries $u_i : \mathcal{L}^2_f(\$^E_{\mathbb{C}}(M; V)) \to \mathcal{L}^2_f(\$^E_{\mathbb{C}}(M; V)) \oplus \mathcal{H}_\pi \otimes C(B)$ such that $u_i(\varphi_i \oplus \pi_{C(F_i)})u_i^* \equiv \varphi_i$ modulo compact operators. According to Lemma 5.6, $\delta_{MV}([\varphi_0, \varphi_1]) = [D_f] = 0$. Hence, by exactness of the Mayer–Vietoris sequence, we have quasihomomorphisms $[\alpha_i, \beta_i]$ $(i = 0, 1)$ such that $[\alpha_0, \beta_0]|_F - [\alpha_1, \beta_1]|_F = [\varphi_0, \varphi_1]|$. Now there are unitaries v_i such that $v_i(\pi_{C(F_i)} \oplus \alpha_i \oplus \alpha_i^\perp)v_i^* \equiv \pi_{C(F_i)}$ modulo compact operators. We set

$$\psi_i := u_i\big(\varphi \oplus v_i(\pi_{C(F_i)} \oplus \beta_i \oplus \alpha_i^\perp)v_i^*\big)u_i^*.$$

Then the $[\varphi_i, \psi_i]$ are quasihomomorphisms and $[\varphi_i, \psi_i] = [\alpha_i, \beta_i]$ for $i = 0, 1$ in $KK(C(S^{n-1}), C(F_i))$.

Now $[\varphi_0, \psi_0]|_F - [\varphi_1, \psi_1]|_F = [\varphi_0, \varphi_1]$, which implies $[\psi_0, \psi_1]|_F = 0$. As a consequence, there is a homotopy of quasihomomorphisms $[\Psi_0^t, \Psi_1^t]$ $(t \in [0, 1])$ from $C(S^{n-1})$ to $C(F) \otimes \mathbb{B}(\mathcal{H})$ connecting $[\psi_0|_F, \psi_1|_F]|$ and $[\theta, \theta]$ for some θ. Here we use the fact that the extensions $\psi_i|_F$ contain $\pi_{C(F)}$ and hence are absorbable. Finally we get a homotopy

$$\tilde{\Psi}_t := \begin{cases} \Psi_0^{2t} & 0 \le t \le 1/2, \\ \Psi_1^{2-2t} & 1/2 \le t \le 1 \end{cases}$$

of $*$-homomorphisms from $C(S^{n-1})$ to $C(F) \otimes \mathbb{B}(\mathcal{H})$ connecting ψ_0 and ψ_1.

Now we denote by D the fiber product of C^*-algebras

$$\begin{array}{ccc} D & \longrightarrow & C(F) \otimes (B(\mathcal{H}) \oplus \mathbb{B}(\mathcal{H})) \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id} \otimes (p \oplus p)} \\ C(IF) \otimes Q(\mathcal{H}) & \xrightarrow{\mathrm{ev}_0 \oplus \mathrm{ev}_1} & C(F) \otimes (Q(\mathcal{H}) \oplus Q(\mathcal{H})) \end{array}$$

and by τ the extension

$$0 \longrightarrow C_0(SF) \otimes \mathbb{K} \longrightarrow C(IF) \otimes \mathbb{B}(\mathcal{H}) \longrightarrow D \longrightarrow 0.$$

Then $\sigma(\not{D}_f^E)_v$ and $(\psi_0 \oplus \psi_1)$ determine a $*$-homomorphism $\sigma : C(S^{n-1}) \to D$. Because the C^*-algebra $C(S^{n-1})$ is nuclear, the Choi–Effros theorem [1976] implies that the pullback $\sigma^*\tau$ is an invertible extension and hence defines an element $[\sigma^*\tau]$ in $KK^1(C(S^{n-1}), C_0(SF) \otimes \mathbb{K})$. By the construction of $\tilde{\Phi}$, $\sigma^*\tau$ is homotopic to the trivial extension $\pi \circ \tilde{\Psi}$, which implies that $[\sigma^*\tau] = 0$. Consequently, σ itself has a lift $C(S^{n-1}) \to IC(F) \otimes \mathbb{B}(\mathcal{H})$.

Finally, we obtain a lift φ of $\sigma(\not{D}^E_f)_v$ on $I(F_0, F_1; F)$. Its pullback by a continuous section $B \to I(F_0, F_1; F)$ given by a partition of unity is a desired lift of $\sigma(\not{D}^E_f)_v$.

In general case we apply induction on the number of covers. We assume that there is a trivialization $B = F_1 \cup \cdots \cup F_n \cup F_{n+1}$ and set $G_0 := F_1 \cup \cdots \cup F_n$ and $G_1 := F_{n+1}$. By the inductive assumption, we obtain lifts φ_0 and φ_1 on G_0 and G_1. First we may assume that V is trivial by restricting φ_0 to the closure of an open neighborhood of $G := G_0 \cap G_1 \subset G_0$. Now each φ_i contains $\pi_{C(G_i)}$ by its construction. Moreover, because M and V are trivial on IG by assumption, we can take a lift of σ containing $\pi_{C(IG)}$. Now, the precise assertion obtained from the above argument is that if (1) M and V are trivial on G, (2) there are lifts φ_i on $C(G_i)$ ($i = 0, 1$), and (3) each φ_i is absorbable (and hence contains $\pi_{C(G_i)}$), then there is a lift φ on G containing $\pi_{C(B)}$. Hence the induction process works. $\square$

Finally we prove our main theorem. Here we mention that in the above argument we restrict the case that the lifts can be taken to be invertible operators.

Proof of Theorem 5.3. We assume that $[\operatorname{ind} \not{D}^E_f]$ is in the image of $K^n(B, B^{(n-1)})$. Let $U \subset V$ be an inclusion of small open balls in B, $F_0 := U^c$ and $F_1 := \overline{V}$. Then $[\operatorname{ind} \not{D}^E_f|_{F_0}]$ and $[\operatorname{ind} \not{D}^E_f|_{F_1}]$ are 0 by assumption, and hence, according to Lemma 5.7, $\not{D}^E_f$ is n-decomposable on F_0 and F_1. Now, because $F := F_0 \cap F_1$ is homotopic to S^{n-1}, the group $KK(C_0(S^{n-1} \setminus \{*\}), C(F))$ is isomorphic to $\tilde{k}^{n-1}(F) = [C_0(\mathbb{R}^{n-1}), C(F)]$. This implies that there is a $*$-homomorphism $\psi : C_0(S^{n-1} \setminus \{*\}) \to C(F) \otimes \mathbb{K}$ such that $[\varphi_0, \varphi_1] = \Phi[\psi]$. Since φ_1 is absorbable, there is a unitary u from $\mathcal{H}_{C(F)}$ to $\mathcal{H}_{C(F)} \oplus \mathcal{H}_{C(F)}$ such that $u(\varphi_1 \oplus \mathrm{ev}_* \cdot 1)u^* \equiv \varphi_1$ modulo compact operators. Moreover, by an argument similar to Lemma 5.7, we obtain a lift of $\sigma(\not{D}^E_f)$ on IF that coincides with φ_0 on $F \times \{0\}$ and $u(\varphi_1 \oplus \tilde{\psi})u^*$ on $F \times \{1\}$, where $\tilde{\psi}$ is a unital extension of ψ.

The remaining part is to construct a homotopy connecting $\varphi \oplus ev_* \cdot 1$ with $\varphi \oplus \tilde{\psi}$. This is not realized as a family of $*$-homomorphisms on $C(S^{n-1})$ but a continuous family of bounded commuting Fredholm n-tuples. Let ι^* be the canonical $*$-homomorphism $C(\overline{\mathbb{D}^n} \setminus \{*\}) \to C_0(S^{n-1} \setminus \{*\})$. Then we can take a homotopy connecting $\psi \circ \iota^*$ and 0 since $\mathbb{D}^n$ is contractible.

Finally, in the same way as in the proof of Lemma 5.7, we obtain a $*$-homomorphism T that makes the following diagram commute:

$$\begin{tikzcd}
C(\overline{\mathbb{D}}(V)) \arrow[r, "T"] \arrow[d] & C(B) \otimes \mathbb{B}(\mathcal{H}) \arrow[d] \\
C(S(V)) \arrow[r, "\sigma(\not{D}^E_f)_v"] & C(B) \otimes Q(\mathcal{H})
\end{tikzcd}$$

Now $\{T(x)\}_v := T(x, v)$ gives a decomposition of $\not{D}^E_f$. $\square$

As a concluding remark, we introduce a corollary of Theorem 5.3:

Corollary 5.8. *If $\not{D}_f^E$ is n-decomposable, then $\not{D}_f^{E\otimes\pi^*F}$ is also n-decomposable for a complex vector bundle F on B. Moreover, in this case*

$$\operatorname{ind}_0(\not{D}_M^{E\otimes\pi^*F}) = \operatorname{jsf}\{\not{D}_f^{E\otimes\pi^*F}\} = \dim F\cdot\operatorname{jsf}\{\not{D}_f^E\} = \dim F\cdot\operatorname{ind}_0(\not{D}_M^E).$$

Proof. This follows from the fact that the connective K-group gives a multiplicative filtration in the K-group. □

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