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For a smooth Deligne–Mumford stack \mathcal{X} , we describe a large number of *inertial products* on $K(I\mathcal{X})$ and $A^*(I\mathcal{X})$ and *inertial Chern characters*. We do this by developing a theory of *inertial pairs*. Each inertial pair determines an inertial product on $K(I\mathcal{X})$ and an inertial product on $A^*(I\mathcal{X})$ and Chern character ring homomorphisms between them. We show that there are many inertial pairs; indeed, every vector bundle V on \mathcal{X} defines two new inertial pairs. We recover, as special cases, the orbifold products considered by Chen and Ruan (2004), Abramovich, Graber and Vistoli (2002), Fantechi and Göttsche (2003), Jarvis, Kaufmann and Kimura (2007) and by the authors (2010), and the virtual product of González, Lupercio, Segovia, Uribe and Xicoténcatl (2007).

We also introduce an entirely new product we call the *localized orbifold product*, which is defined on $K(I\mathcal{X}) \otimes \mathbb{C}$.

The inertial products developed in this paper are used in a subsequent paper to describe a theory of inertial Chern classes and power operations in inertial K -theory. These constructions provide new manifestations of mirror symmetry, in the spirit of the hyper-Kähler resolution conjecture.

1. Introduction

The purpose of this note is to describe a large number of *inertial products* and Chern characters by developing a formalism of *inertial pairs*. An inertial pair for a Deligne–Mumford stack \mathcal{X} is a pair $(\mathcal{R}, \mathcal{S})$, where \mathcal{R} is a vector bundle on the double inertia stack $\mathbb{I}^2\mathcal{X}$ and \mathcal{S} is a nonnegative, rational K -theory class on the inertia stack $I\mathcal{X}$ satisfying certain compatibility conditions. For stacks with finite stabilizer, an inertial pair determines inertial products on cohomology, Chow groups, and K -theory of $I\mathcal{X}$. In the Chow group and cohomology, this product respects an orbifold grading equal to the ordinary grading corrected by the virtual rank of \mathcal{S} (or age). An inertial pair also allows us to define an inertial Chern character, which is a ring homomorphism for the new inertial products.

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The motivating example of an inertial pair is the orbifold pair $(\mathcal{R}, \mathcal{S})$, where \mathcal{R} is the obstruction bundle coming from orbifold Gromov–Witten theory, and \mathcal{S} is the class defined in [JKK 2007]. The corresponding product is the Chen–Ruan orbifold product, and the Chern character is the one defined in [loc. cit.]. One of the results of this paper is that every vector bundle V on a Deligne–Mumford stack determines two inertial pairs $(\mathcal{R}^+V, \mathcal{S}^+V)$ and $(\mathcal{R}^-V, \mathcal{S}^-V)$. The $+$ product corresponds to the orbifold product on the total space of the bundle V , but the $-$ product is twisted by an isomorphism and does not directly correspond to an orbifold product on a bundle. However, we prove (Theorem 4.2.2) that there is an automorphism of the total Chow group $A^*(I\mathcal{X}) \otimes \mathbb{C}$ which induces a ring isomorphism between the $-$ product for V and the $+$ product for V^* . A similar result also holds for cohomology.

When $V = \mathbb{T}$ is the tangent bundle of \mathcal{X} , we show that the virtual product considered in [González et al. 2007] is the product associated to the inertial pair $(\mathcal{R}^- \mathbb{T}, \mathcal{S}^- \mathbb{T})$. It follows, after tensoring with \mathbb{C} , that the virtual orbifold Chow ring is isomorphic (but not equal) to the Chen–Ruan orbifold Chow ring of the cotangent bundle \mathbb{T}^* . Our result also implies that there is a corresponding Chern character ring homomorphism for the virtual product.

In the final section we show that in certain cases, even if V is not a vector bundle but just an element of K -theory, we can still determine a product in localized K -theory. This allows us to define a new product on $K(I\mathcal{X}) \otimes \mathbb{C}$, which we call the *localized product*.

In a subsequent paper [Edidin et al. 2015] we will show that for *Gorenstein inertial pairs* (such as the one determining the virtual product) there is a theory of Chern classes and compatible power operations on inertial K -theory. This will be used to give further manifestations of mirror symmetry on hyper-Kähler Deligne–Mumford stacks.

Review of previous related work. Because there has been much work in this area by many authors from different areas of mathematics, we give a brief overview here of previous work to help put the current paper in context.

In 2000, inspired by physicists [Dixon et al. 1985; 1986], Chen and Ruan [2002] developed a new product on the cohomology of the inertia $I\mathcal{X}$ of an almost complex orbifold \mathcal{X} . In 2001, Fantechi and Göttsche [2003] showed that when the orbifold \mathcal{X} was a global quotient $[X/G]$ by a finite group, the Chen–Ruan orbifold cohomology ring $H_{\text{CR}}(\mathcal{X})$ was the G -invariant subring of $H_{\text{FG}}(X, G)$, the cohomology of the inertia manifold IX endowed with a certain noncommutative product. It followed that if X is the symmetric product of a surface with trivial canonical class, then the orbifold cohomology of \mathcal{X} is isomorphic to the cohomology ring of the Hilbert scheme, as predicted by the hyper-Kähler resolution conjecture [Ruan 2006].

At about the same time, Kaufmann [2002; 2003] presented an axiomatic approach to orbifolding Frobenius algebras, and described how the Fantechi–Göttsche construction fit into this framework (Kaufmann, personal communication, 2002).

Adem and Ruan [2003] then studied the K -theory $K(\mathcal{X})$ of a global quotient orbifold $\mathcal{X} = [X/G]$, where G is a Lie group, and they also studied the twisted K -theory of $[X/G]$. They did not construct a new “orbifold” product on $K(\mathcal{X})$, but they did show that there is a Chern character that gives a vector space isomorphism from $K(\mathcal{X})$ to $H_{\text{CR}}(\mathcal{X})$. This Chern character is *not* a ring homomorphism. Tu and Xu [2006] later extended this result to more general twistings and orbifolds.

Abramovich, Graber and Vistoli [Abramovich et al. 2002] constructed an algebraic version $A_{\text{AGV}}(\mathcal{X})$ of the Chen–Ruan cohomology, producing the corresponding product on the Chow group $A_{\text{AGV}}(\mathcal{X}) = A^*(I\mathcal{X})$ of the inertial stack $I\mathcal{X}$ of a (smooth) Deligne–Mumford stack with projective coarse moduli space.

In all of these constructions the basic idea is to use an analogue of the moduli space $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$ of genus-zero, three-pointed, orbifold (or G -equivariant) stable maps into \mathcal{X} . This space has three *evaluation maps* $e_i : \overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0) \rightarrow I\mathcal{X}$, and the structure constants $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ for the new product on $I\mathcal{X}$ are given by computing

$$\int_{\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)} \prod_{i=1}^3 e_i^*(\alpha_i) \cdot \text{eu}(\mathcal{R}),$$

where $\text{eu}(\mathcal{R})$ is the top Chern class of an obstruction bundle on $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, 0)$. The main difficulty in computing the new product was computing the obstruction bundle \mathcal{R} and its top Chern class.

In 2004, Chen and Hu [2006] produced a formula for the obstruction class in the case of *abelian* orbifolds and used it to describe a de Rham model for the Chen–Ruan product. In 2005, Jarvis, Kaufmann and Kimura [JKK 2007] proved a simple, intrinsic formula for the obstruction bundle \mathcal{R} for general (not just abelian) orbifolds, requiring no mention of stable curves or moduli spaces of maps. In the abelian case this formula reduces to Chen and Hu’s result. In [loc. cit.], that formula is used to do several things:

- (1) Create Chow- (respectively, K -) theoretic analogues of the Fantechi–Göttsche ring $H_{\text{FG}}(X, G)$ whose rings of invariants is the AGV ring $A_{\text{AGV}}(\mathcal{X})$ (respectively, a ring whose underlying vector space is $K(\mathcal{X})$ of Adem and Ruan). Corresponding products twisted by discrete torsion were also introduced.
- (2) Define a new (orbifold) product on the K -theory $K_{\text{orb}}(\mathcal{X})$ of the inertia $I\mathcal{X}$, for any smooth Deligne–Mumford stack \mathcal{X} .
- (3) Define an orbifold Chern character ring homomorphism from the new orbifold K -theory rings to the corresponding Chow or cohomology rings. This new

Chern character is a deformation of the ordinary Chern character, as the latter fails to preserve the orbifold multiplications.

- (4) Outline how the same formula and formalism may be used to give analogous results in other categories, e.g., equivariant structures on almost complex manifolds with a Lie group action.

At about the same time, Adem, Ruan and Zhang [Adem et al. 2008] independently defined an orbifold product on twisted $K_{\text{orb}}(\mathcal{X})$, and in the case of a global quotient by a finite group, Kaufmann and Pham [2009] connected this to the twisted Drinfel'd double of the group ring.

Becerra and Uribe [2009] extended these results to the equivariant setting for global quotients by infinite abelian groups, and in [EJK 2010], we extended these results to an equivariant setting for global quotients by general (nonabelian) infinite groups by introducing a variant of the formula for the obstruction bundle in [JKK 2007].

A recent paper [Hu and Wang 2013] repeats the description of the orbifold product of [JKK 2007; Adem et al. 2008], the formula of [JKK 2007; EJK 2010] for the obstruction class, and the Chern character ring homomorphism of those two papers in the almost-complex setting, as originally described in Section 10 of [JKK 2007].

In [Behrend et al. 2007; 2012; González et al. 2007; Lupercio et al. 2008], a different product in inertial Chow and inertial cohomology theory, analogous to the Chas–Sullivan product [1999] on loop spaces, was introduced. This so-called *virtual (orbifold) product* is a special case of the constructions of this paper; see Section 4.3. Surprisingly, it is *not* equal to the orbifold product for the cotangent bundle. We note, however, that after tensoring with \mathbb{C} , both the orbifold Chow and orbifold cohomology (but not orbifold K -theory) of the cotangent bundle are isomorphic to their virtual counterparts.

Kaufmann [2002; 2003; 2010] was the first to study the possibility of many stringy products on Frobenius algebras in settings involving functors other than just K -theory, cohomology, and Chow theory. He treated this primarily as an algebraic question and reformulated the problem of constructing a stringy product in terms of certain cocycles. It is not *a priori* clear that there should always exist a stringy product, but Kaufmann [2010] shows how to extend the ideas of [JKK 2007] to prove existence of at least one stringy product for his more general setting. In some cases he can also show uniqueness of the product [Kaufmann 2004].

Finally, we note that Pflaum, Postuma, Tang and Tseng [Pflaum et al. 2011] have shown that the Hochschild cohomology of a certain algebra attached to a groupoid presentation of a symplectic orbifold is isomorphic to the cohomology of the inertia orbifold as a vector space. The product in Hochschild cohomology

induces a product on the cohomology of the inertia orbifold. It would be interesting to understand the relation of that product to the products described in this paper.

2. Background material from [EJK 2010]

To make this paper self-contained, we recall some background material from the paper [EJK 2010].

2.1. Background notation. We work entirely in the complex algebraic category. We will work exclusively with smooth Deligne–Mumford stacks \mathcal{X} which have *finite stabilizer*, by which we mean the inertia map $I\mathcal{X} \rightarrow \mathcal{X}$ is finite. We will also assume that every stack \mathcal{X} has the *resolution property*. This means that every coherent sheaf is the quotient of a locally free sheaf. This assumption has two consequences. The first is that the natural map $K(\mathcal{X}) \rightarrow G(\mathcal{X})$ is an isomorphism, where $K(\mathcal{X})$ is the Grothendieck ring of vector bundles and $G(\mathcal{X})$ is the Grothendieck group of coherent sheaves. The second consequence is that \mathcal{X} is a *quotient stack* [Edidin et al. 2001]. This means that $\mathcal{X} = [X/G]$, where G is a linear algebraic group acting on a scheme or algebraic space X .

If \mathcal{X} is a smooth Deligne–Mumford stack, then we will implicitly choose a presentation $\mathcal{X} = [X/G]$. This allows us to identify the Grothendieck ring $K(\mathcal{X})$ with the equivariant Grothendieck ring $K_G(X)$, and the Chow ring $A^*(\mathcal{X})$ with the equivariant Chow ring $A_G^*(X)$. We will use the notation $K(\mathcal{X})$ and $K_G(X)$ (resp. $A^*(\mathcal{X})$ and $A_G^*(X)$) interchangeably.

Definition 2.1.1. Let G be an algebraic group acting on a scheme or algebraic space X . We define the *inertia space*

$$I_G X := \{(g, x) \mid gx = x\} \subset G \times X.$$

There is an induced action of G on $I_G X$ given by $g \cdot (m, x) = (gm g^{-1}, gx)$. The quotient stack $I\mathcal{X} := [I_G X/G]$ is the *inertia stack* of the quotient stack $\mathcal{X} := [X/G]$.

More generally, define the *higher inertia spaces* to be the k -fold fiber products $\mathbb{1}_G^k X = I_G X \times_X \cdots \times_X I_G X = \{(m_1, \dots, m_k, x) \mid m_i x = x \forall i = 1, \dots, k\} \subset G^k \times X$.

The quotient stack $\mathbb{1}^k \mathcal{X} := [\mathbb{1}_G^k X/G]$ is the corresponding *higher inertia stack*.

The assumption that \mathcal{X} has finite stabilizer means that the projection $I_G X \rightarrow X$ is a finite morphism. The composition $\mu: G \times G \rightarrow G$ induces a composition $\mu: I_G X \times_X I_G X \rightarrow I_G X$. This composition makes $I_G X$ into an X -group with identity section $X \rightarrow I_G X$ given by $x \mapsto (1, x)$.

Definition 2.1.2. Let G^ℓ be a G -space with the diagonal conjugation action. A *diagonal conjugacy class* is a G -orbit $\Phi \subset G^\ell$.

Definition 2.1.3. For all m in G , let $X^m = \{(m, x) \in I_G X\}$. For all (m_1, \dots, m_ℓ) in G^ℓ , let $X^{m_1, \dots, m_\ell} = \{(m_1, \dots, m_\ell, x) \in \mathbb{I}_G^\ell X\}$. For all conjugacy classes $\Psi \subset G$, let $I(\Psi) = \{(m, x) \in I_G X \mid m \in \Psi\}$. More generally, for all diagonal conjugacy classes $\Phi \subset G^\ell$, let $\mathbb{I}^\ell(\Phi) = \{(m_1, \dots, m_\ell, x) \in \mathbb{I}_G^\ell X \mid (m_1, \dots, m_\ell) \in \Phi\}$.

By definition, $I(\Psi)$ and $\mathbb{I}^\ell(\Phi)$ are G -invariant subsets of $I_G X$ and $\mathbb{I}_G^\ell X$, respectively. If G acts with finite stabilizer on X , then $I(\Psi)$ is empty unless Ψ consists of elements of finite order. Likewise, $\mathbb{I}^\ell(\Phi)$ is empty unless every ℓ -tuple $(m_1, \dots, m_\ell) \in \Phi$ generates a finite group. Since conjugacy classes of elements of finite order are closed, $I(\Psi)$ and $\mathbb{I}^\ell(\Phi)$ are closed.

Proposition 2.1.4 [EJK 2010, Propositions 2.11 and 2.17]. *If G acts properly on X , then $I(\Psi) = \emptyset$ for all but finitely many conjugacy classes Ψ and the $I(\Psi)$ are unions of connected components of $I_G X$. Likewise, $\mathbb{I}^\ell(\Phi)$ is empty for all but finitely many diagonal conjugacy classes $\Phi \subset G^\ell$ and each $\mathbb{I}^\ell(\Phi)$ is a union of connected components of $\mathbb{I}_G^\ell X$.*

We frequently work with a group G acting on a space X where the quotient stack $[X/G]$ is not connected. As a consequence, some care is required in the definition of the rank and Euler class of a vector bundle. Note that, for any X , the group $A_G^0(X)$ satisfies $A_G^0(X) = \mathbb{Z}^\ell$, where ℓ is the number of connected components of the quotient stack $\mathcal{X} = [X/G]$.

Definition 2.1.5. If E is an equivariant vector bundle on X , then we define the *rank* of E to be $\text{rk}(E) := \text{Ch}^0(E) \in \mathbb{Z}^\ell = A_G^0(X)$. Note that the rank of E lies in the semigroup \mathbb{N}^ℓ , where $\mathbb{N} = \{0, 1, 2, \dots\}$. If E_1, \dots, E_n are vector bundles, then the *virtual rank (or augmentation)* of the element $\sum_{i=1}^n n_i [E_i] \in K_G(X)$ is the weighted sum $\sum_i n_i \text{rk}(E_i) \in \mathbb{Z}^\ell$.

If E is a G -equivariant vector bundle on X , then the rank of E on the connected components of $\mathcal{X} = [X/G]$ is bounded (since we assume that \mathcal{X} has finite type).

Definition 2.1.6. If E is a G -equivariant vector bundle on X , we call the element $\lambda_{-1}(E^*) = \sum_{i=0}^{\infty} (-1)^i [\Lambda^i E^*] \in K_G(X)$ the *K -theoretic Euler class* of E . (Note that this sum is finite.)

Likewise, we define the element $c_{\text{top}}(E) \in A_G^*(X)$, corresponding to the sum of the top Chern classes of E on each connected component of $[X/G]$, to be the *Chow-theoretic Euler class* of E . These definitions can be extended to any nonnegative element by multiplicativity. It will be convenient to use the symbol $\text{eu}(\mathcal{F})$ to denote both of these Euler classes for a nonnegative element $\mathcal{F} \in K_G(X)$.

2.2. The logarithmic restriction and twisted pullback. We recall a construction from [EJK 2010] that will be used several times throughout the paper. However, to improve clarity, we use slightly different notation than in that paper.

Definition 2.2.1 [EJK 2010]. Let X be an algebraic space with an action of an algebraic group Z . Let E be a rank- n vector bundle on X and let g be a unitary automorphism of the fibers of $E \rightarrow X$. If we assume that the action of g commutes with the action of Z on E , the eigenbundles for the action of g are all Z -subbundles. Let $\exp(2\pi\sqrt{-1}\alpha_1), \dots, \exp(2\pi\sqrt{-1}\alpha_r)$ be the distinct eigenvalues of g acting on E , with $0 \leq \alpha_k < 1$ for all $k \in \{1 \dots, r\}$, and let E_1, \dots, E_r be the corresponding eigenbundles.

We define the *logarithmic trace of E* by the formula

$$L(g)(E) = \sum_{k=1}^r \alpha_k E_k \in K_Z(X) \otimes \mathbb{R} \tag{2.2.2}$$

on each connected component of X .

The next key fact about the logarithmic trace was proved in our previous paper.

Proposition 2.2.3 [EJK 2010, Proposition 4.6]. *Let $\mathbf{g} = (g_1, \dots, g_\ell)$ be an ℓ -tuple of elements of a compact subgroup of a reductive group H , satisfying $\prod_{i=1}^\ell g_i = 1$. Let X be an algebraic space with an action of an algebraic group Z , and let V be a $(Z \times H)$ -equivariant bundle on X , where H is assumed to act trivially on X . The element*

$$\sum_{i=1}^\ell L(g_i)(V) - V + V^{\mathbf{g}}$$

in $K_Z(X)$ is represented by a Z -equivariant vector bundle.

Using Proposition 2.2.3 we make the following definition.

Definition 2.2.4. Let G be an algebraic group acting quasifreely on an algebraic space, and let V be a G -equivariant vector bundle on X . Given $\mathbf{g} = (g_1, \dots, g_\ell) \in G^\ell$, if the g_i all lie in a common compact subgroup and satisfy $\prod_{i=1}^\ell g_i = 1$, then set

$$V(\mathbf{g}) = \sum_{i=1}^\ell L(g_i)(V|_{X^{\mathbf{g}}}) - V|_{X^{\mathbf{g}}} + V^{\mathbf{g}}|_{X^{\mathbf{g}}}.$$

We wish to extend this definition to give a map from $K_G(X)$ to $K_G(I_G X)$, but we must first understand the decompositions of $K_G(I_G X)$ and $A_G^*(I_G X)$ into conjugacy classes.

As a consequence of Proposition 2.1.4, we see that $K_G(I_G X)$ and $A_G^*(I_G X)$ are direct sums of the $K_G(I(\Psi))$ and $A_G^*(I(\Psi))$, respectively, as Ψ runs over conjugacy classes of elements of finite order in G . A similar statement holds for the equivariant K -theory and Chow groups of the higher inertia spaces as well.

Using Morita equivalence, we can give a more precise description of $K_G(I(\Psi))$. If $m \in \Psi$ is any element and $Z = Z_G(m)$ is the centralizer of m in G , then

$$K_G(I(\Psi)) = K_Z(X^m) \quad \text{and} \quad A_G^*(I(\Psi)) = A_Z^*(X^m).$$

Similarly, if $\Phi \subset G^\ell$ is a diagonal conjugacy class and $(m_1, \dots, m_\ell) \in \Phi$ and $Z = \bigcap_{i=1}^\ell Z_G(m_i)$, then

$$K_G(\mathbb{I}^\ell(\Phi)) = K_Z(X^{m_1, \dots, m_\ell}) \quad \text{and} \quad A_G^*(\mathbb{I}^\ell(\Phi)) = A_Z^*(X^{m_1, \dots, m_\ell}).$$

Definition 2.2.5. Define a map $L: K_G(X) \rightarrow K_G(I_G X) \otimes \mathbb{Q}$, as follows. For each conjugacy class $\Psi \subset G$ and each $V \in K_G(X)$, let $L(\Psi)(V)$ be the class in $K_Z(I(\Psi))$ which is Morita equivalent to $L(g)(V|_{X^g}) \in K_Z(X^g)_\mathbb{Q}$. Here g is any element of Ψ , and $Z = Z_G(g)$ is the centralizer of $g \in G$. The class $L(V)$ is the class whose restriction to $I(\Psi)$ is $L(\Psi)(V)$.

The proof of [EJK 2010, Lemma 5.4] shows that $L(\Psi)(V)$ (and thus $L(V)$) is independent of the choice of $g \in \Psi$.

Definition 2.2.6. If the diagonal conjugacy class $\Phi \subset G^\ell$ is represented by an ℓ -tuple (g_1, \dots, g_ℓ) such that $\prod_{i=1}^\ell g_i = 1$, then we define $V(\Phi)$ to be the class in $K_G(\mathbb{I}_G^\ell X)$ which is Morita equivalent to $V(\mathbf{g})$, where $\mathbf{g} = (g_1, \dots, g_\ell)$ is any element of Φ . Again, $V(\Phi)$ is independent of the choice of representative $\mathbf{g} \in \Phi$.

Definition 2.2.7. Identify $\mathbb{I}_G^\ell X$ with the closed and open subset of $\mathbb{I}_G^{\ell+1} X$ consisting of tuples $\{(g_1, \dots, g_{\ell+1}, x) \mid g_1 g_2 \dots g_{\ell+1} = 1\}$. If $V \in K_G(X)$, let $\text{LR}(V) \in K_G(\mathbb{I}_G^\ell X)$ be the class whose restriction to $\mathbb{I}^{\ell+1}(\Phi)$ is $V(\Phi)$, where the diagonal conjugacy class $\Phi \in G^{\ell+1}$ is represented by a tuple $(g_1, \dots, g_{\ell+1})$ satisfying $g_1 \dots g_{\ell+1} = 1$.

2.3. Orbifold products and the orbifold Chern character. Here we briefly review the construction and properties of orbifold products and orbifold Chern characters because they serve as a model for what we will do later.

Definition 2.3.1. For $i \in \{1, 2, 3\}$, let $e_i: \mathbb{I}_G^2 X \rightarrow I_G X$ be the evaluation morphism taking (m_1, m_2, m_3, x) to (m_i, x) and let $\mu: \mathbb{I}_G^2 X \rightarrow I_G X$ be the morphism taking (m_1, m_2, m_3, x) to $(m_1 m_2, x) = (m_3^{-1}, x)$.

Definition 2.3.2. Let \mathbb{T} be the equivariant bundle on X corresponding to the tangent bundle of \mathcal{X} , which satisfies $\mathbb{T} = TX - \mathfrak{g}$ in $K_G(X)$, where \mathfrak{g} is the Lie algebra of G .

Definition 2.3.3 [EJK 2010; JKK 2007; Kaufmann 2010]. The *orbifold product* on $K_G(I_G X)$ and $A_G^*(I_G X)$ is defined as

$$x \star y := \mu_*(e_1^* x \cdot e_2^* y \cdot \text{eu}(\text{LR}(\mathbb{T}))), \quad (2.3.4)$$

both for $x, y \in K_G(I_G X)$ and for $x, y \in A_G^*(I_G X)$.

Definition 2.3.5. We define the element $\mathcal{S} := L(\mathbb{T})$ in $K_G(I_G X)_{\mathbb{Q}}$ to be the logarithmic trace of \mathbb{T} , that is, for each m in G , we define \mathcal{S}_m in $K_{Z_G(m)}(X^m)_{\mathbb{Q}}$ by

$$\mathcal{S}_m := L(m)(\mathbb{T}).$$

The rank of \mathcal{S} is a \mathbb{Q} -valued, locally constant function on $I\mathcal{X} = [I_G X/G]$ called the *age*.

Remark 2.3.6. If the age of a connected component $[U/G]$ of $I\mathcal{X}$ is zero, then $[U/G]$ must be a connected component of $\mathcal{X} \subset I\mathcal{X}$.

Remark 2.3.7. In [Kaufmann 2010] the classes \mathcal{S} , manifestations of what physicists call *twist fields*, were interpreted in terms of cocycles which were then used to define stringy products. Our construction may be regarded as a realization of this procedure.

Definition 2.3.8. Given an element x in $A_G^*(I_G X)$ with ordinary Chow grading $\deg x$, the *orbifold degree (or grading) of x* is, like the ordinary Chow grading, constant on each component U of $I_G X$ corresponding to a connected component of $[U/G]$ of $[I_G X/G]$. On such a component U we define it to be the nonnegative rational number

$$\deg_{\text{orb}} x|_U = \deg x|_U + \text{age}[U/G]. \quad (2.3.9)$$

The induced grading on the group $A_G^*(I_G X)$ consists of summands $A_G^{\{q\}}(I_G X)$ of all elements with orbifold degree q .

Theorem 2.3.10 [JKK 2007; EJK 2010]. *The equivariant Chow group $(A_G^*(I_G X), \star, \deg_{\text{orb}})$ is a \mathbb{Q}^C -graded, commutative ring with unity $\mathbf{1}$, where $\mathbf{1}$ is the identity element in $A_G^*(X) = A_G^*(X^1) \subseteq A_G^*(I_G X)$ and C is the number of connected components of $[I_G X/G]$.*

Equivariant K-theory $(K_G(I_G X), \star)$ is a commutative ring with unity $\mathbf{1}$, where $\mathbf{1} := \mathcal{O}_X$ is the structure sheaf of $X = X^1 \subset I_G X$.

Definition 2.3.11. The *orbifold Chern character* $\mathcal{Ch} : K_G(I_G X) \rightarrow A_G^*(I_G X)_{\mathbb{Q}}$ is defined by the equation

$$\mathcal{Ch}(\mathcal{F}) := \text{Ch}(\mathcal{F}) \cdot \text{Td}(-\mathcal{S})$$

for all $\mathcal{F} \in K_G(I_G X)$, where Td is the usual Todd class. Moreover, for all $\alpha \in \mathbb{Q}$ we define $\mathcal{Ch}^\alpha(\mathcal{F})$ by the equation

$$\mathcal{Ch}(\mathcal{F}) = \sum_{\alpha \in \mathbb{Q}} \mathcal{Ch}^\alpha(\mathcal{F}),$$

where each $\mathcal{Ch}^\alpha(\mathcal{F})$ belongs to $A_G^{\{\alpha\}}(I_G X)$.

The *orbifold virtual rank (or orbifold augmentation)* is $\mathcal{Ch}^0 : K_G(I_G X) \rightarrow A_G^{\{0\}}(I_G X)_{\mathbb{Q}}$.

Theorem 2.3.12 [EJK 2010; JKK 2007]. *The orbifold Chern character*

$$\mathcal{Ch} : (K_G(I_G X), \star) \rightarrow (A_G^*(I_G X)_{\mathbb{Q}}, \star)$$

is a ring homomorphism.

In particular, if $[U/G]$ is a connected component of $[I_G X/G]$, then the virtual rank homomorphism restricted to the component $[U/G]$ gives a homomorphism $\mathcal{Ch}^0 : K_G(U) \rightarrow A_G^0(U)_{\mathbb{Q}} = \mathbb{Q}$, satisfying

$$\mathcal{Ch}^0(\mathcal{F}) = \begin{cases} 0 & \text{if } \text{age}[U/G] > 0, \\ \text{Ch}^0(\mathcal{F}) & \text{if } \text{age}[U/G] = 0 \end{cases}$$

for any $\mathcal{F} \in K_G(U)$.

3. Inertial products, Chern characters, and inertial pairs

In this section we generalize the ideas of orbifold cohomology, obstruction bundles, orbifold grading and the orbifold Chern character by defining *inertial products* on $K_G(I_G X)$ and $A_G^*(I_G X)$ using inertial bundles on $\mathbb{L}_G^2 X$. We further define a rational grading and a Chern character ring homomorphism via *Chern-compatible classes* of $K_G(I_G X)_{\mathbb{Q}}$.

The original example of an associative bundle is the obstruction bundle $\mathcal{R} = \text{LR}(\mathbb{T})$ of orbifold cohomology, and the original example of a Chern-compatible class is the logarithmic trace \mathcal{S} of \mathbb{T} , as described in Definition 2.3.5.

We show below that there are many *inertial pairs* of associative inertial bundles on $\mathbb{L}_G^2 X$ with Chern-compatible elements on $I_G X$, and hence there are many associative inertial products on $K_G(I_G X)$ and $A_G^*(I_G X)$ with rational gradings and Chern character ring homomorphisms.

3.1. Associative bundles and inertial products. We recall the following definition (and notation) from [EJK 2010]. It should be noted that a similar formalism also appeared in the paper [Kaufmann 2010].

Definition 3.1.1. Given a class $c \in A_G^*(\mathbb{L}_G^2 X)$ (resp. $K_G(\mathbb{L}_G^2 X)$), we define the *inertial product with respect to c* to be

$$x \star_c y := \mu_*(e_1^* x \cdot e_2^* y \cdot c), \quad (3.1.2)$$

where $x, y \in A_G^*(I_G X)$ (resp. $K_G(I_G X)$).

Given a vector bundle \mathcal{R} on $\mathbb{L}_G^2 X$ we define inertial products on $A_G^*(I_G X)$ and $K_G(I_G X)$ via formula (3.1.2), where $c = \text{eu}(\mathcal{R})$ is the Euler class of the bundle \mathcal{R} .

Definition 3.1.3. We say that \mathcal{R} is an *associative bundle* on $\mathbb{L}_G^2 X$ if the $\star_{\text{eu}(\mathcal{R})}$ products on both $A_G^*(I_G X)$ and $K_G(I_G X)$ are commutative and associative with

identity $\mathbf{1}$, where $\mathbf{1}$ is the identity class in $A_G^*(X)$ (resp. $K_G(X)$), viewed as a summand in $A_G^*(IGX)$ (resp. $K_G(IGX)$).

Proposition 3.1.4. *A sufficient condition for $\star_{\text{eu}(\mathcal{R})}$ to be commutative with identity $\mathbf{1}$ is that the following conditions be satisfied:*

(1) *For every conjugacy class $\Phi \subset G \times G$ with $e_1(\Phi) = 1$ or $e_2(\Phi) = 1$, we have*

$$\mathcal{R}|_{I(\Phi)} = \mathcal{O}. \quad (3.1.5)$$

(2) *If $i: \mathbb{L}_G^2 X \rightarrow \mathbb{L}_G^2 X$ denotes the isomorphism $i(m_1, m_2, x) = (m_1 m_2 m_1^{-1}, m_1, x)$, we have*

$$i^* \mathcal{R} = \mathcal{R}. \quad (3.1.6)$$

Proof. This is almost just a restatement of Propositions 3.7–3.9 in [EJK 2010]. However, we note that in Proposition 3.9 there is a slight error—that proposition incorrectly stated that the map $i: \mathbb{L}_G^2 X \rightarrow \mathbb{L}_G^2 X$ was the map induced by the naive involution $(m_1, m_2) \mapsto (m_2, m_1)$, rather than the correct “braiding map” $(m_1, m_2, x) \mapsto (m_1 m_2 m_1^{-1}, m_1, x)$. \square

A sufficient condition for associativity is also given in [EJK 2010]. In order to state the condition we require some notation, which we recall from that paper. Let $(m_1, m_2, m_3) \in G^3$ such that $m_1 m_2 m_3 = 1$, and let $\Phi_{1,2,3} \subset G^3$ be its diagonal conjugacy class. Let $\Phi_{12,3}$ be the conjugacy class of $(m_1 m_2, m_3)$ and $\Phi_{1,23}$ the conjugacy class of $(m_1, m_2 m_3)$. Let $\Phi_{i,j}$ be the conjugacy class of the pair (m_i, m_j) with $i < j$. Finally let Ψ_{123} be the conjugacy class of $m_1 m_2 m_3$; let Ψ_{ij} be the conjugacy class of $m_i m_j$; and let Ψ_i be the conjugacy class of m_i . There are evaluation maps

$$e_1: \mathbb{L}^2(\Phi_{a,b}) \rightarrow I(\Psi_a), \quad e_2: \mathbb{L}^2(\Phi_{a,b}) \rightarrow I(\Psi_b), \quad e_{i,j}: \mathbb{L}^3(\Phi_{1,2,3}) \rightarrow \mathbb{L}^2(\Psi_{i,j}),$$

and composition maps

$$\mu_{12,3}: \mathbb{L}^3(\Phi_{1,2,3}) \rightarrow \mathbb{L}^2(\Phi_{12,3}), \quad \mu_{1,23}: \mathbb{L}^3(\Phi_{1,2,3}) \rightarrow \mathbb{L}^2(\Phi_{1,23}).$$

The various maps we have defined are related by the following Cartesian diagrams of l.c.i. (local complete intersection) morphisms:

$$\begin{array}{ccc} \mathbb{L}^3(\Phi_{1,2,3}) & \xrightarrow{e_{1,2}} & \mathbb{L}^2(\Phi_{1,2}) & & \mathbb{L}^3(\Phi_{1,2,3}) & \xrightarrow{e_{2,3}} & \mathbb{L}^2(\Phi_{2,3}) \\ \downarrow \mu_{12,3} & & \downarrow \mu & & \downarrow \mu_{1,23} & & \downarrow \mu \\ \mathbb{L}^2(\Phi_{12,3}) & \xrightarrow{e_1} & I(\Psi_{12}) & & \mathbb{L}^2(\Phi_{1,23}) & \xrightarrow{e_2} & I(\Psi_{23}) \end{array} \quad (3.1.7)$$

Let $E_{1,2}$ and $E_{2,3}$ be the respective excess normal bundles of the two diagrams (3.1.7).

Proposition 3.1.8. *Let \mathcal{R} be a vector bundle on $\mathbb{L}_G^2 X$ satisfying (3.1.5) and (3.1.6). A sufficient condition for \mathcal{R} to be an associative bundle is if*

$$e_{1,2}^* \mathcal{R} + \mu_{12,3}^* \mathcal{R} + E_{1,2} = e_{2,3}^* \mathcal{R} + \mu_{1,23}^* \mathcal{R} + E_{2,3} \quad (3.1.9)$$

in $K_G(\mathbb{L}^3 X)$.

Proof. This follows from the proof of Proposition 3.12 of [EJK 2010], since the Euler class takes a sum of bundles to a product of Euler classes. \square

In practice, the only way we have to show that a bundle \mathcal{R} is associative is to show that it satisfies the identity (3.1.9). This leads to our next definition.

Definition 3.1.10. A bundle \mathcal{R} is *strongly associative* if it satisfies the identities (3.1.5), (3.1.6) and (3.1.9).

3.2. Chern characters, age, and inertial pairs. In many cases one can define a Chern character $K_G(I_G X)_{\mathbb{Q}} \rightarrow A_G^*(I_G X)_{\mathbb{Q}}$ which is a *ring homomorphism* with respect to the inertial product. To do this, however, we need to define a *Chern compatible class* $\mathcal{S} \in K_G(I_G X)$. As an added bonus, such a class will also allow us to define a new grading on $A_G^*(I_G X)$ compatible with the inertial product and analogous to the orbifold grading of orbifold cohomology.

Definition 3.2.1. Let \mathcal{R} be an associative bundle on $\mathbb{L}_G^2 X$. A nonnegative class $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$ is called *\mathcal{R} -Chern compatible* if the map

$$\tilde{\mathcal{C}h}: K_G(I_G X)_{\mathbb{Q}} \rightarrow A_G^*(I_G X)_{\mathbb{Q}}$$

defined by

$$\tilde{\mathcal{C}h}(V) = \text{Ch}(V) \cdot \text{Td}(-\mathcal{S})$$

is a ring homomorphism with respect to the \mathcal{R} -inertial products on $K_G(I_G X)$ and $A_G^*(I_G X)$.

Remark 3.2.2. Again, the original example of a Chern compatible class is the class \mathcal{S} defined in [JKK 2007], but we will see other examples below.

Proposition 3.2.3. *If \mathcal{R} is an associative vector bundle on $\mathbb{L}_G^2 X$, then a nonnegative class $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$ is \mathcal{R} -Chern compatible if the following identity holds in $K_G(\mathbb{L}_G^2 X)$:*

$$\mathcal{R} = e_1^* \mathcal{S} + e_2^* \mathcal{S} - \mu^* \mathcal{S} + T_{\mu}. \quad (3.2.4)$$

Proof. This follows from the same formal argument used in the proof of [EJK 2010, Theorem 7.3]. \square

Definition 3.2.5. A class $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$ is *strongly \mathcal{R} -Chern compatible* if it satisfies (3.2.4).

A pair $(\mathcal{R}, \mathcal{S})$ is an *inertial pair* if \mathcal{R} is a strongly associative bundle and \mathcal{S} is \mathcal{R} -strongly Chern compatible.

Definition 3.2.6. We define the \mathcal{S} -age on a connected component $[U/G]$ of $I\mathcal{X}$ to be the rational rank of \mathcal{S} on the component $[U/G]$:

$$\text{age}_{\mathcal{S}}[U/G] = \text{rk}(\mathcal{S})_{[U/G]}.$$

We define the \mathcal{S} -degree of an element $x \in A_G^*(I_G X)$ on such a component U of $I_G X$ to be

$$\text{deg}_{\mathcal{S}} x|_U = \text{deg } x|_U + \text{age}_{\mathcal{S}}[U/G],$$

where $\text{deg } x$ is the degree with respect to the usual grading by codimension on $A_G^*(I_G X)$. Similarly, if \mathcal{F} in $K_G(I_G X)$ is an element supported on U , then its \mathcal{S} -degree is

$$\text{deg}_{\mathcal{S}} \mathcal{F} = \text{age}_{\mathcal{S}} U \pmod{\mathbb{Z}}.$$

This yields a \mathbb{Q}/\mathbb{Z} -grading of the group $K_G(I_G X)$.

Proposition 3.2.7. *If \mathcal{R} is an associative vector bundle on $\mathbb{A}_G^2 X$ and $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$ is strongly \mathcal{R} -Chern compatible, then the \mathcal{R} -inertial products on $A_G^*(I_G X)$ and $K_G(I_G X)$ respect the \mathcal{S} -degrees. Furthermore, the inertial Chern character homomorphism $\tilde{\mathcal{C}h} : K_G(I_G X) \rightarrow A_G^*(I_G X)_{\mathbb{Q}}$ preserves the \mathcal{S} -degree modulo \mathbb{Z} .*

Proof. If $x, y \in A_G^*(I_G X)$, then the formula

$$x \star_{\text{eu}(\mathcal{R})} y = \mu_*(e_1^* x \cdot e_2^* y \cdot \text{eu}(\mathcal{R}))$$

implies that $\text{deg}(x \star_{\text{eu}(\mathcal{R})} y) = \text{deg } x + \text{deg } y + \text{rk } \mathcal{R} + \text{rk } T_{\mu}$. Since \mathcal{S} is strongly \mathcal{R} -Chern compatible, we know that $\mathcal{R} = e_1^* \mathcal{S} + e_2^* \mathcal{S} - \mu^* \mathcal{S} + T_{\mu}$. Comparing ranks shows that the \mathcal{S} -degree of $x \star_{\text{eu}(\mathcal{R})} y$ is the sum of the \mathcal{S} -degrees of x and y . The proof for $K_G(I_G X)$ follows from the fact that $\text{rk } \mathcal{R}$ and $\text{rk } T_{\mu}$ are integers. Finally, $\tilde{\mathcal{C}h}$ preserves the \mathcal{S} -degree mod \mathbb{Z} since if \mathcal{F} in $K_G(I_G X)$ is supported on U , where $[U/G]$ is a connected component of $[I_G X/G]$, then so is its inertial Chern character. \square

Definition 3.2.8. Let $A_G^{\{q\}}(I_G X)$ be the subspace in $A_G^*(I_G X)$ of elements with an \mathcal{S} -degree of q .

Definition 3.2.9. Given a class $\mathcal{S} \in K_G(I_G X)_{\mathbb{Q}}$, the restricted homomorphism $\tilde{\mathcal{C}h}^0 : K_G(I_G X) \rightarrow A_G^{\{0\}}(I_G X)$ is called the *inertial virtual rank (or inertial augmentation)* for \mathcal{S} .

Definition 3.2.10. An inertial pair $(\mathcal{R}, \mathcal{S})$ is called *Gorenstein* if \mathcal{S} has integral virtual rank and *strongly Gorenstein* if \mathcal{S} is represented by a vector bundle.

The Deligne–Mumford stack $\mathcal{X} = [X/G]$ is *strongly Gorenstein* if the inertial pair $(\mathcal{R} = \text{LR}(\mathbb{T}), \mathcal{S})$ associated to the orbifold product (as in Definitions 2.3.3 and 2.3.5) is strongly Gorenstein.

4. Inertial pairs associated to vector bundles

In this section we show how, for each choice of G -equivariant bundle V on X , we can use the methods of [EJK 2010] to define two new inertial pairs $(\mathcal{R}^+V, \mathcal{S}^+V)$ and $(\mathcal{R}^-V, \mathcal{S}^-V)$. We thus obtain corresponding inertial products and Chern characters. We denote the corresponding products associated to a vector bundle V as the \star_{V^+} and \star_{V^-} products. The \star_{V^+} product can be interpreted as an orbifold product on the total space of V , while the \star_{V^-} product on the Chow ring is a sign twist of the $\star_{(V^*)^+}$ product. Moreover, the two products induce isomorphic ring structures on $A^*(I\mathcal{X}) \otimes \mathbb{C}$. We prove that if $V = \mathbb{T}$ is the tangent bundle to $\mathcal{X} = [X/G]$, then the \star_{V^-} product agrees with the virtual orbifold product defined by [González et al. 2007].

To define the inertial pairs associated to a vector bundle, we introduce a variant of the logarithmic restriction introduced in [EJK 2010]. We begin with a simple proposition.

Proposition 4.0.1. *Let G be an algebraic group acting on a variety X and suppose that g_1, g_2 lie in a common compact subgroup. Let $Z = Z_G(g_1, g_2)$ be the centralizer of g_1 and g_2 in G .*

The virtual bundles

$$V^+(g_1, g_2) = L(g_1)(V|_{X^{g_1, g_2}}) + L(g_2)(V|_{X^{g_1, g_2}}) - L(g_1 g_2)(V|_{X^{g_1, g_2}}) \quad (4.0.2)$$

and

$$V^-(g_1, g_2) = L(g_1^{-1})(V|_{X^{g_1, g_2}}) + L(g_2^{-1})(V|_{X^{g_1, g_2}}) - L(g_2^{-1} g_1^{-1})(V|_{X^{g_1, g_2}}) \quad (4.0.3)$$

are represented by nonnegative integral elements in $K_Z(X^{g_1, g_2})$.

Proof. Since $X^g = X^{g^{-1}}$ and $V^-(g_1, g_2) = V^+(g_2^{-1}, g_1^{-1})$, it suffices to show that $V^+(g_1, g_2)$ is represented by a nonnegative integral element of $K_Z(X^{g_1, g_2})$. Let $g_3 = (g_1 g_2)^{-1}$. The identity $L(g)(V) + L(g^{-1})(V) = V - V^g$ implies that we can rewrite (4.0.2) as

$$\begin{aligned} V^+(g_1, g_2) &= L(g_1)(V|_{X^{g_1, g_2}}) + L(g_2)(V|_{X^{g_1, g_2}}) \\ &\quad + L(g_3)(V|_{X^{g_1, g_2}}) - V + V^{g_1 g_2} + V^{g_1 g_2} - V^{g_1, g_2}. \end{aligned}$$

Since $g_1 g_2 g_3 = 1$, by Proposition 2.2.3 the sum

$$L(g_1, g_2, g_3)(V) = L(g_1)(V|_{X^{g_1, g_2}}) + L(g_2)(V|_{X^{g_1, g_2}}) + L(g_3)(V|_{X^{g_1, g_2}}) - V + V^{g_1, g_2}$$

is represented by a nonnegative integral element of $K^Z(X_{g_1, g_2})$. Hence

$$V^+(g_1, g_2) = L(g_1, g_2, g_3)(V) + V^{g_1 g_2} - V^{g_1, g_2}$$

is represented by a nonnegative integral element of $K_Z(X^{g_1, g_2})$. \square

Let $\Phi \subset G \times G$ be a diagonal conjugacy class. As in [EJK 2010] we may identify $K_G(\mathbb{I}^2(\Phi))$ with $K_{Z_G(g_1, g_2)}(X^{g_1, g_2})$ for any $(g_1, g_2) \in \Phi$. Thanks to Proposition 4.0.1 we can define nonnegative classes $V^+(\Phi)$ and $V^-(\Phi)$ in $K_G(\mathbb{I}^2(\Phi))$. The argument used in the proof of [EJK 2010, Lemma 5.4] shows that the definitions of $V^+(\Phi)$ and $V^-(\Phi)$ are independent of the choice of $(g_1, g_2) \in G^2$. Thus we can make the following definition.

Definition 4.0.4. Define classes R^+V and R^-V in $K_G(\mathbb{I}_G^2 X)$ by setting the components of R^+V and R^-V in $K_G(\mathbb{I}^2(\Phi))$ to be $V^+(\Phi)$ and $V^-(\Phi)$, respectively. Similarly, we define classes $S^\pm V \in K_G(I_G X)_\mathbb{Q}$ by setting the restriction of $S^\pm V$ to a summand $K_G(I(\Psi))$ of $K_G(I_G X)$ to be the class Morita equivalent to $L(g^{\pm 1})(V) \in K_{Z_G(g)}(X^g)$, where $g \in \Psi$ is any element.

Theorem 4.0.5. *For any G -equivariant vector bundle V on X , the pairs*

$$\begin{aligned} (\mathcal{R}^+V, \mathcal{S}^+V) &= (\text{LR}(\mathbb{T}) + R^+V, \mathcal{S}\mathbb{T} + S^+V), \\ (\mathcal{R}^-V, \mathcal{S}^-V) &= (\text{LR}(\mathbb{T}) + R^-V, \mathcal{S}\mathbb{T} + S^-V) \end{aligned}$$

are inertial pairs. Hence they define associative inertial products with a Chern character homomorphism.

Proof. Since

$$\text{LR}(\mathbb{T}) = e_1^* \mathcal{S}\mathbb{T} + e_2^* \mathcal{S}\mathbb{T} - \mu^* \mathcal{S}\mathbb{T} + T_\mu \quad \text{and} \quad R^+V = e_1^* S^+V + e_2^* S^+V - \mu^* S^+V,$$

it follows that \mathcal{S}^+V is strongly \mathcal{R}^+V -Chern compatible.

To complete the proof we must show that $\text{LR}(\mathbb{T}) + R^+V$ is a strongly associative bundle. From their definitions we know that $\text{LR}(\mathbb{T})$ and R^+V satisfy the identities (3.1.5) and (3.1.6). We also know that $\text{LR}(\mathbb{T})$ satisfies (3.1.9). Thus, to prove that $\text{LR}(\mathbb{T}) + R^+V$, it suffices to show that R^+V satisfies the ‘‘cocycle’’ condition

$$e_{1,2}^* R^+V + \mu_{12,3}^* R^+V = e_{2,3}^* R^+V + \mu_{1,23}^* R^+V. \quad (4.0.6)$$

Now (4.0.6) follows from the following identity of bundles restricted to X^{m_1, m_2, m_3} :

$$V^+(m_1, m_2) + V^+(m_1 m_2, m_3) = V^+(m_2, m_3) + V^+(m_1, m_2 m_3). \quad (4.0.7)$$

Equation (4.0.7) is a formal consequence of the definition of the bundles V^+ . The result with R^+V and S^+V replaced by R^-V and S^-V , respectively, is proved analogously. \square

4.1. Geometric interpretation of the \star_{V^+} product. The \star_{V^+} has a relatively direct interpretation in terms of an orbifold product on the total space of the vector bundle $V \rightarrow X$.

Lemma 4.1.1. *Given a G -equivariant vector bundle $\pi : V \rightarrow X$, the inertia space $I_G V$ is a vector bundle (of nonconstant rank) on $I_G X$ with structure map $I\pi : I_G V \rightarrow I_G X$.*

Proof. Let $\Psi \subset G$ be a conjugacy class. Denote by $I_X(\Psi) \subset I_G X$ the component of $I_G X$ defined by $\{(g, x) \mid gx = x, g \in \Psi\}$. For any morphism $V \rightarrow X$ and any conjugacy class $\Psi \in G$, if $I_X(\Psi) = \emptyset$, then $I_V(\Psi)$ is also empty. Thus it suffices to show that $I_V(\Psi)$ is a vector bundle over $I_X(\Psi)$ for every conjugacy class $\Psi \subset G$ with $I_X(\Psi) \neq \emptyset$. Given $g \in \Psi$, the identification $I_X(\Psi) = G \times_{Z_G(g)} X^g$ reduces the problem to showing that for $g \in G$ the fixed locus V^g is a $Z_G(g)$ -equivariant vector bundle over X^g . Since the map $V \rightarrow X$ is G -equivariant, the map $V^g \rightarrow X$ has image X^g . The fiber over a point $x \in X^g$ is just $(V_x)^g$, where V_x is the fiber of $V \rightarrow X$ at x . \square

Since $I_G V \rightarrow I_G X$ is a vector bundle, the pullback maps

$$(I\pi)^* : K_G(I_G V) \rightarrow K_G(I_G X) \quad \text{and} \quad (I\pi)^* : A_G^*(I_G V) \rightarrow A_G^*(I_G X)$$

are isomorphisms. Both isomorphisms are compatible with the ordinary products on K -theory and equivariant Chow groups.

Theorem 4.1.2. *For $x, y \in A_G^*(I_G X)$ or $x, y \in K_G(X)$, we have*

$$x \star_{V^+} y = (Is)^*((I\pi)^* x \star (I\pi)^* y), \quad (4.1.3)$$

where \star is the usual orbifold product on the total space of the G -equivariant vector bundle $V \rightarrow X$ and Is^* is the Gysin map which is inverse to $I\pi^*$.

Proof. We give the proof only in equivariant Chow theory — the proof in equivariant K -theory is essentially identical. We compare the two sides of (4.1.3). If $\Psi_1, \Psi_2, \Psi_3 \subset G$ are conjugacy classes and $x \in A_G^*(I_X(\Psi_1))$, $y \in A_G^*(I_X(\Psi_2))$, then the contribution of $x \star_{V^+} y$ to $A_G^*(I_X(\Psi_3))$ is

$$\sum_{\Phi_{1,2}} \mu_*(e_1^* x \cdot e_2^* y \cdot \text{eu}(\text{LR}(\mathbb{T}) + R^+ V)), \quad (4.1.4)$$

where the sum is over all conjugacy classes $\Phi_{1,2} \subset G \times G$ satisfying

$$e_1(\Phi_{1,2}) = \Psi_1, \quad e_2(\Phi_{1,2}) = \Psi_2, \quad \mu(\Phi_{1,2}) = \Psi_3.$$

Since the class of tangent bundle of V equals $TX + V$, the tangent bundle to the stack $[V/G]$ is $TX + V - \mathfrak{g} = \mathbb{T} + V$. Thus, the contribution of the right-hand side of (4.1.3) is the sum

$$\sum_{\Phi_{1,2}} Is^*(\mu_{V^*}(I\pi)^*(e_1^* x \cdot e_2^* y \cdot \text{eu}(\text{LR}(\mathbb{T}) + \text{LR}(V)))), \quad (4.1.5)$$

where the map μ_V in (4.1.5) is understood to be the multiplication map $\mathbb{I}_G^2 V \rightarrow I_G V$. If Φ is a conjugacy class in $G \times G$ with $\mu(\Phi) = \Psi$, then the multiplication map $\mu_V : I_V(\Phi) \rightarrow I_V(\Psi)$ factors through the inclusion

$$I_V(\Phi) \hookrightarrow \mu^* I_V(\Psi) \xrightarrow{I\pi^* \mu} I_V(\Psi), \quad (4.1.6)$$

and we have the following diagram, with a Cartesian square on the right:

$$\begin{array}{ccccc} I_V(\Phi) & \hookrightarrow & \mu^* I_V(\Psi) & \xrightarrow{I\pi^* \mu} & I_V(\Psi) \\ & \searrow^{I\pi} & \downarrow & & \downarrow^{I\pi} \\ & & I(\Phi) & \xrightarrow{\mu} & I(\Psi) \end{array}$$

The normal bundle to the inclusion $I_V(\Phi) \hookrightarrow \mu^*(I_V(\Psi))$ is the pullback of the bundle V_Ψ/V_Φ on $\mathbb{I}_X^2(\Phi)$, where $V_\Phi \subset V|_{I_X(\Psi)}$ is the subbundle whose fiber over a point (g, x) is the subspace V^g , and the fiber of V_Φ over a point (g_1, g_2, x) is the subspace $V^{g_1, g_2} \subset V$. Using this information about the normal bundle we can rewrite (4.1.5) as

$$\mu_*(e_1^* x \cdot e_2^* y \cdot \text{eu}(\text{LR}(\mathbb{T} + V) + V_\Psi - V_\Phi)). \quad (4.1.7)$$

Finally, (4.1.7) can be identified with (4.1.4) by observing that if $g_1, g_2 \in G$ then

$$\begin{aligned} L(g_1)(V) + L(g_2)(V) + L((g_1 g_2)^{-1})(V) + V - V^{g_1, g_2} + V^{g_1, g_2} - V^{g_1 g_2} \\ = L(g_1)(V) + L(g_2)(V) - L(g_1 g_2)(V). \quad \square \end{aligned}$$

4.2. Geometric interpretation of the \star_V -product. The \star_V -product does not generally correspond to an orbifold product on a bundle. However, we will show that, after tensoring with \mathbb{C} , the inertial Chow (or cohomology) ring with the \star_V -product is isomorphic to the inertial Chow (or cohomology) ring coming from the total space of the dual bundle. The latter is isomorphic to the orbifold Chow (or cohomology) ring of the total space of the dual bundle.

Definition 4.2.1. Given a vector bundle V on a quotient stack $\mathcal{X} = [X/G]$, we define an automorphism Θ_V of $A^*(I\mathcal{X}) \otimes \mathbb{C}$ as follows. If x_Ψ is supported on a component $I(\Psi)$ of $I\mathcal{X}$ corresponding to a conjugacy class $\Psi \subset G$ then we set $\Theta_V(x_\Psi) = e^{i\pi a_\Psi} x_\Psi$, where a_Ψ is the virtual rank of the logarithmic trace $L(g^{-1})(V)$ for any representative element $g \in \Psi$. The same formula defines an automorphism of $H^*(I\mathcal{X}, \mathbb{C})$.

Theorem 4.2.2. For $x, y \in A_G^*(I_G X)$ we have

$$\begin{aligned} x \star_V y &= \pm (Is)^*((I\pi)^* x \star (I\pi)^* y) \\ &= \pm x \star_{V^{**}} y, \end{aligned} \quad (4.2.3)$$

where \star is the usual orbifold product on the total space of the G -equivariant vector bundle $V^* \rightarrow X$, and Is^* is the Gysin map which is inverse to $I\pi^*$, and the sign \pm is $(-1)^{a_{\Psi_1} + a_{\Psi_2} - a_{\Psi_{12}}}$ where $a_{\Psi_1} + a_{\Psi_2} - a_{\Psi_{12}}$ is a nonnegative integer. Moreover, if we tensor with \mathbb{C} , then we have the identity

$$\Theta_V(x \star_{V^-} y) = \Theta_V(x) \star_{V^{*+}} \Theta_V(y). \quad (4.2.4)$$

Remark 4.2.5. The \pm sign appearing in the previous theorem is an example of discrete torsion. Similar signs appear in the work of [Fantechi and Göttsche 2003].

Proof. Observe that if $g \in G$ acts on a representation V of $Z = Z_G(g)$ with weights $e^{i\theta_1}, \dots, e^{i\theta_r}$ then g naturally acts on V^* with weights $e^{-i\theta_1}, \dots, e^{-i\theta_r}$, and the $e^{i\theta_k}$ -eigenspace of V is dual to the $e^{-i\theta_k}$ -eigenspace of V^* . Hence $L(g^{-1})(V) = L(g)(V^*)^*$ as elements of $K(X^g) \otimes \mathbb{Q}$. Thus, given a pair $g_1, g_2 \in G$, we see that

$$V^-(g_1, g_2) = ((V^*)^+(g_1, g_2))^*$$

as $Z_G(g_1, g_2)$ -equivariant bundles on $X^{g_1, g_2} = X^{g_1^{-1}, g_2^{-1}}$. Hence, $\text{eu}(R^-V) = (-1)^{\text{rk } R^-V} \text{eu}(R^+V^*)$, so (4.2.3) holds. If x is supported in the component $I(\Psi_1)$ and y is supported in the component $I(\Psi_2)$, then $x \star_{V^-} y$ is supported at components $I(\Psi_{12})$, where Ψ_{12} is a conjugacy class of $g_1 g_2$ for some $g_1 \in \Psi_1$ and $g_2 \in \Psi_2$.

Now we have

$$\Theta_V(x \star_{V^-} y) = \sum_{\Psi_{12}} e^{i\pi a_{\Psi_{12}}} (-1)^{\text{rk } V^-(g_1, g_2)} x \star_{V^{*+}} y,$$

while

$$\Theta_V(x) \star_{V^{*+}} \Theta_V(y) = \sum_{\Psi_{12}} e^{i\pi(a_{\Psi_1} + a_{\Psi_2})} x \star_{V^{*+}} y.$$

Thus, (4.2.4) follows from the fact that $\text{rk } V^-(g_1, g_2) = a_{\Psi_1} + a_{\Psi_2} - a_{\Psi_{12}}$. \square

4.3. The virtual orbifold product is the \star_{TX^-} product. The virtual orbifold product was introduced in [González et al. 2007]. In our context it (or more precisely its algebraic analogue) can be defined as follows:

Definition 4.3.1. Let \mathbb{T}^{virt} be the class in $K_G(\mathbb{I}_G^2 X)$ defined by the formula

$$\mathbb{T}|_{\mathbb{I}_G^2 X} + \mathbb{T}|_{\mathbb{I}_G^2 X} - e_1^* \mathbb{T}|_{I_G X} - e_2^* \mathbb{T}|_{I_G X}, \quad (4.3.2)$$

where $\mathbb{T}|_{\mathbb{I}_G^2 X}$ refers to the pullback of the class \mathbb{T} to $\mathbb{I}_G^2 X$ via any of the three natural maps $\mathbb{I}_G^2 X \rightarrow X$, where $\mathbb{T}|_{I_G X}$ denotes the tangent bundle to the stack $I\mathcal{X} = [I_G X/G]$, and where $\mathbb{T}|_{\mathbb{I}_G^2 X}$ denotes the tangent bundle to the stack $I^2 \mathcal{X}$.

Proposition 4.3.3. *The identity $\mathbb{T}^{\text{virt}} = \text{LR}(\mathbb{T}) + R^-\mathbb{T}$ holds in $K_G(\mathbb{I}_G^2 X)$. In particular, \mathbb{T}^{virt} is represented by a nonnegative element of $K_G(\mathbb{I}_G^2 X)$ and the $\star_{\text{eu } \mathbb{T}^{\text{virt}}}$ product is commutative and associative. Moreover, $\mathcal{S}\mathbb{T} + S^-\mathbb{T} = N$, where N is*

the normal bundle of the canonical morphism $I_G X \rightarrow X$, so $(\mathbb{T}^{\text{virt}}, N)$ is a strongly Gorenstein inertial pair.

Proof. The proof follows from the identity

$$L(g)(\mathbb{T}) + L(g^{-1})(\mathbb{T}) = \mathbb{T}|_{X^s} - \mathbb{T}|_{X^s} = N|_{X^s}. \quad \square$$

Definition 4.3.4. Following [González et al. 2007], we define the *virtual orbifold product* to be the $\star_{\text{eu}(\mathbb{T}^{\text{virt}})}$ -product.

Corollary 4.3.5. *The virtual product \star_{virt} on $A_G^*(I_G X)$ agrees up to sign with the $\star_{\mathbb{T}^{*+}}$ inertial product on $A_G^*(I_G X)$ induced by the cotangent bundle \mathbb{T}^* of $\mathcal{X} = [X/G]$, and there is an isomorphism of rings*

$$(A_G^*(I_G X)_{\mathbb{C}}, \star_{\text{virt}}) \cong (A_G^*(I_G X)_{\mathbb{C}}, \star_{\mathbb{T}^{*+}}).$$

Remark 4.3.6. While in Chow theory these products differ by a simple discrete torsion (see Remark 4.2.5), in K -theory the virtual product is not so easily identified with the product $\star_{\mathbb{T}^{*+}}$, as can be seen from the fact that the gradings do not match (discrete torsion does not change the grading). If there is a connection in K -theory, it will have to be via something much more general, like a K -theoretic version of the matrix discrete torsion of [Kaufmann 2010].

4.4. An example with $\mathbb{P}(1, 3, 3)$. We illustrate the various inertial products in K -theory and Chow theory with the example of the weighted projective space $\mathcal{X} = \mathbb{P}(1, 3, 3) = [(\mathbb{A}^3 \setminus \{0\})/\mathbb{C}^*]$ where \mathbb{C}^* acts with weights $(1, 3, 3)$. The inertia $I\mathcal{X}$ has three sectors — the identity sector $\mathcal{X}^1 = \mathcal{X}$ and two twisted sectors \mathcal{X}^ω and $\mathcal{X}^{\omega^{-1}}$, where $\omega = e^{2\pi i/3}$. Both twisted sectors are isomorphic to a $\mathcal{B}\mu_3$ -gerbe over \mathbb{P}^1 . The K -theory of each sector is a quotient of the representation ring $R(\mathbb{C}^*)$. Precisely, we have

$$K(\mathcal{X}^1) = \mathbb{Z}[\chi]/\langle (\chi-1)(\chi^3-1)^2 \rangle \quad \text{and} \quad K(\mathcal{X}^\omega) = K(\mathcal{X}^{\omega^{-1}}) = \mathbb{Z}[\chi]/\langle (\chi^3-1)^2 \rangle,$$

where χ is the defining character of \mathbb{C}^* . The projection formula in equivariant K -theory implies that any inertial product is determined by the products $\mathbf{1}_{g_1} \star \mathbf{1}_{g_2} \in K(\mathcal{X}^{g_1 g_2})$, where $\mathbf{1}_g$ is the K -theoretic fundamental class on the sector \mathcal{X}^g .

The usual orbifold product is represented by the symmetric matrix

	\mathcal{X}^1	\mathcal{X}^ω	$\mathcal{X}^{\omega^{-1}}$
\mathcal{X}^1	1	1	1
\mathcal{X}^ω		1	$\text{eu}(\chi)$
$\mathcal{X}^{\omega^{-1}}$			$\text{eu}(\chi)$

The virtual and orbifold cotangent products are represented by the following matrices:

$$\begin{array}{c|ccc} & \mathcal{X}^1 & \mathcal{X}^\omega & \mathcal{X}^{\omega^{-1}} \\ \hline \mathcal{X}^1 & 1 & 1 & 1 \\ \mathcal{X}^\omega & & \text{eu}(\chi) & \text{eu}(\chi)^2 \\ \mathcal{X}^{\omega^{-1}} & & & \text{eu}(\chi) \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & \mathcal{X}^1 & \mathcal{X}^\omega & \mathcal{X}^{\omega^{-1}} \\ \hline \mathcal{X}^1 & 1 & 1 & 1 \\ \mathcal{X}^\omega & & \text{eu}(\chi^{-1}) & \text{eu}(\chi) \text{eu}(\chi^{-1}) \\ \mathcal{X}^{\omega^{-1}} & & & \text{eu}(\chi) \end{array}$$

If we define $t = c_1(\mathcal{X}) \in A^*(\mathcal{B}\mathbb{C}^*)$, then the inertial products on Chow and cohomology groups can also be represented by matrices, as above. After tensoring with \mathbb{C} , the Chow groups of the sectors are

$$A^*(\mathcal{X}^1)_{\mathbb{C}} = \mathbb{C}[t]/\langle t^3 \rangle \quad \text{and} \quad A^*(\mathcal{X}^\omega)_{\mathbb{C}} = A^*(\mathcal{X}^{\omega^{-1}})_{\mathbb{C}} = \mathbb{C}[t]/\langle t^2 \rangle.$$

The corresponding matrices for the virtual and cotangent orbifold products are

$$\begin{array}{c|ccc} & \mathcal{X}^1 & \mathcal{X}^\omega & \mathcal{X}^{\omega^{-1}} \\ \hline \mathcal{X}^1 & 1 & 1 & 1 \\ \mathcal{X}^\omega & & t & t^2 \\ \mathcal{X}^{\omega^{-1}} & & & t \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & \mathcal{X}^1 & \mathcal{X}^\omega & \mathcal{X}^{\omega^{-1}} \\ \hline \mathcal{X}^1 & 1 & 1 & 1 \\ \mathcal{X}^\omega & & -t & -t^2 \\ \mathcal{X}^{\omega^{-1}} & & & t \end{array}$$

The automorphism of $A^*(I\mathcal{X})_{\mathbb{C}}$ which is the identity on $A(\mathcal{X}^1)_{\mathbb{C}}$ and which acts by multiplication by $e^{2\pi i/3}$ on $A^*(\mathcal{X}^\omega)_{\mathbb{C}}$ and $e^{\pi i/3}$ on $A^*(\mathcal{X}^{\omega^{-1}})_{\mathbb{C}}$ defines a ring isomorphism between these products.

5. The localized orbifold product on $K(I\mathcal{X}) \otimes \mathbb{C}$

If an algebraic group G acts with finite stabilizer on smooth variety Y , then there is a decomposition of $K_G(Y) \otimes \mathbb{C}$ as a sum of localizations $\bigoplus_{\Psi} K_G(Y)_{\mathfrak{m}_{\Psi}}$. Here the sum is over conjugacy classes $\Psi \subset G$ such that $I(\Psi) \neq \emptyset$, and $\mathfrak{m}_{\Psi} \in \text{Spec } R(G)$ is the maximal ideal of class functions vanishing on the conjugacy class Ψ .

Given a conjugacy class $\Psi \subset G$ and a choice of $h \in \Psi$, denote the centralizer of h in G by $Z = Z_G(h)$. The conjugacy class of h in Z is just h alone, and there is a corresponding maximal ideal $\mathfrak{m}_h \in \text{Spec } R(Z)$. As described in [Edidin and Graham 2005, §4.3], the localization $K_G(I(\Psi))_{\mathfrak{m}_h}$ is a summand of the localization $K_G(I(\Psi))_{\mathfrak{m}_{\Psi}}$, and this summand is independent of the choice of h . This is called the *central summand* of Ψ and is denoted by $K_G(I(\Psi))_{c(\Psi)}$.

Since G acts with finite stabilizer, the projection $f_{\Psi} : I(\Psi) \rightarrow Y$ is a finite l.c.i. morphism. The nonabelian localization theorem of [Edidin and Graham 2005] states that the pullback $f_{\Psi}^* : K_G(Y) \otimes \mathbb{C} \rightarrow K_G(I(\Psi)) \otimes \mathbb{C}$ induces an isomorphism between the localization of $K_G(Y)$ at \mathfrak{m}_{Ψ} and the central summand $K_G(I(\Psi))_{c(\Psi)} \subset K_G(I(\Psi))_{\mathfrak{m}_{\Psi}}$. The inverse to f_{Ψ}^* is the map $\alpha \mapsto f_{\Psi*}(\alpha \cdot \text{eu}(N_{f_{\Psi}})^{-1})$. If we let f be the global stabilizer map $I_G Y \rightarrow Y$, then, after summing over all conjugacy classes Ψ in the support of $K_G(Y) \otimes \mathbb{C}$, we obtain an isomorphism

$$f^* : K_G(Y) \otimes \mathbb{C} \rightarrow K_G(I_G Y)_c,$$

where $K_G(I_G Y)_c = \bigoplus K_G(I(\Psi))_{c(\Psi)}$. The inverse is $f_*/\text{eu}(N_f)$.

Applying this construction with $Y = I_G X$ allows us to define a product we call *the localized orbifold product*.

Definition 5.0.1. The *localized orbifold product* on $K_G(I_G X) \otimes \mathbb{C}$ is defined by the formula

$$\alpha \star_{\text{LO}} \beta = If_*((If^* \alpha \star If^* \beta) \otimes \text{eu}(N_{If})^{-1}),$$

where \star is the usual orbifold product on $K_G(I_G(I_G X))_c$, and $If : I_G(I_G X) \rightarrow I_G X$ is the projection.

Remark 5.0.2. It should be noted that $I_G(I_G X)$ is not the same as $\mathbb{I}_G^2 X$. The inertia $I_G(I_G X) = \{(h, g, x) \mid hx = gx = x, hg = gh\}$ is a closed subspace of $\mathbb{I}_G^2 X$.

The localized product can be interpreted in the context of the \star_{V^+} product, where the vector bundle V is replaced by the virtual bundle $-N_f$. Observe that the pull-back of \mathbb{T} to $I_G X$ splits as $\mathbb{T} = \mathbb{T}_{I_G X} + N_f$, where N_f is the normal bundle to the finite l.c.i. map $I_G X \rightarrow X$. Although N_f is not a bundle on X , we can still compute $N_f^+(g_1, g_2)$ on $\mathbb{I}_G^2 X$.

The same formal argument used in the proof of Theorem 4.1.2 yields the following result.

Proposition 5.0.3. *The class $\text{eu}(\text{LR}(\mathbb{T}) + R^+(-N_f))$ is well-defined in localized K -theory, and*

$$\alpha \star_{\text{LO}} \beta = \alpha \star_{(-N_f)^+} \beta.$$

Remark 5.0.4. The inertial pair corresponding to the localized product is the formal pair $(\text{LR}(\mathbb{T}) + R^+(-N_f), \mathcal{S}\mathbb{T} + S^+(-N_f))$. However, the Chern character corresponding to this inertial pair is the usual orbifold Chern character and the corresponding product on $A^*(I\mathcal{X})$ is the usual orbifold product. The reason is that the orbifold Chern character isomorphism factors through $K_G(I_G X)_{(1)}$, the localization of $K_G(I_G X)$ at the augmentation ideal of $R(G)$. This localization corresponds to the untwisted sector of $I_G X$ where f restricts to the identity map.

Remark 5.0.5. Identifying $K_G(I_G X)_{\mathbb{C}}$ with the localization of $K_G(I_G(I_G X))_{\mathbb{C}}$ allows us to invert the class $\text{eu}(N_f)$. In [Kaufmann 2010, §3.4] the author gives a framework for defining similar products after formally inverting the Euler classes of normal bundles.

5.1. An example with $\mathbb{P}(1, 2)$. We consider the weighted projective line $\mathcal{X} = \mathbb{P}(1, 2) = [(\mathbb{A}^2 \setminus \{0\})/\mathbb{C}^*]$, where \mathbb{C}^* acts with weights $(1, 2)$. The inertia stack $I\mathcal{X}$ has two sectors, $\mathcal{X}^1 = \mathcal{X}$ and $\mathcal{X}^{-1} = \mathcal{B}\mu_2$. We have

$$K(\mathcal{X}^1) \otimes \mathbb{C} = \mathbb{C}[\chi]/\langle (\chi - 1)(\chi^2 - 1) \rangle \quad \text{and} \quad K(\mathcal{X}^{-1}) \otimes \mathbb{C} = \mathbb{C}[\chi]/\langle \chi^2 - 1 \rangle.$$

In particular, $K(\mathcal{IX}) \otimes \mathbb{C}$ is supported at $\pm 1 \in \mathbb{C}^*$. As was the case in Section 4.4, inertial ring structures are determined by the products $\mathbf{1}_{g_1} \star \mathbf{1}_{g_2} \in K(\mathcal{X}^{g_1 g_2})$. In terms of the localization decomposition, $K(\mathcal{IX}) \otimes \mathbb{C} = K(\mathcal{IX})_{(1)} \oplus K(\mathcal{IX})_{(-1)}$. The localized product is determined by computing the corresponding orbifold product on each localized piece using the decomposition of the element $\mathbf{1}_g$ into its localized pieces and the product $\mathbf{1}_{g_1} \star_{\text{LO}} \mathbf{1}_{g_2}$ decomposes as

$$(\mathbf{1}_{g_1})_{(1)} \star_{\text{LO}} (\mathbf{1}_{g_2})_{(1)} + (\mathbf{1}_{g_1})_{(-1)} \star_{\text{LO}} (\mathbf{1}_{g_2})_{(-1)}.$$

The multiplication matrix for $K(\mathcal{IX})_{(1)}$ is the usual orbifold matrix, which in this case is

$$\begin{array}{c|cc} & \mathcal{X}^1 & \mathcal{X}^{-1} \\ \hline \mathcal{X}^1 & 1 & 1 \\ \mathcal{X}^{-1} & 1 & \text{eu}(\chi) \end{array}$$

The multiplication matrix for the localized product on $K(\mathcal{IX})_{(-1)}$ is the same as the multiplication matrix for the orbifold product on $\mathcal{B}\mu_2$, which is

$$\begin{array}{c|cc} & \mathcal{X}^1 & \mathcal{X}^{-1} \\ \hline \mathcal{X}^1 & 1 & 1 \\ \mathcal{X}^{-1} & 1 & 1 \end{array}$$

Thus we see that the only nontrivial product is $\mathbf{1}_{(-1)} \star_{\text{LO}} \mathbf{1}_{(-1)}$. To obtain a single multiplication matrix we use the decomposition

$$\mathbf{1}_{(-1)} = \frac{1}{2}(1 + \chi) + \frac{1}{2}(1 - \chi) \in K(\mathcal{X}^{-1}) \otimes \mathbb{C},$$

where $\frac{1}{2}(1 + \chi)$ is supported at 1 and $\frac{1}{2}(1 - \chi)$ is supported at -1 . The final result is the matrix

$$\begin{array}{c|cc} & \mathcal{X}^1 & \mathcal{X}^{-1} \\ \hline \mathcal{X}^1 & 1 & 1 \\ \mathcal{X}^{-1} & 1 & \frac{(1+\chi)^2 \text{eu}(\chi) + (1-\chi)^2}{4} \end{array}$$

Because the twisted sector \mathcal{X}^{-1} has dimension 0, both the orbifold and usual Chern characters on this sector compute the virtual rank. The untwisted sector $\mathbb{P}(1, 2)$ has Chow ring $\mathbb{C}[t]/\langle t^2 \rangle$, where $t = c_1(\chi)$. Thus $\text{ch}(\text{eu}(\chi)) = t \in A^*(\mathbb{P}(1, 2)) \otimes \mathbb{C}$. Observe that

$$\text{ch}\left(\frac{(1 + \chi)^2 \text{eu}(\chi) + (1 - \chi)^2}{4}\right) = \frac{(2 + t)^2(t) + (-t)^2}{4} = t$$

in $\mathbb{C}[t]/\langle t^2 \rangle$ as well.

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vol. 1

no. 1

Statement of purpose Jonathan M. Rosenberg and Charles A. Weibel	1
On the Deligne–Beilinson cohomology sheaves Luca Barbieri-Viale	3
On some negative motivic homology groups Tohru Kohrita	19
The joint spectral flow and localization of the indices of elliptic operators Yosuke Kubota	43
A plethora of inertial products Dan Edidin, Tyler J. Jarvis and Takashi Kimura	85



2379-1683(2016)1:1;1-Y