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Jonathan M. Rosenberg
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On the Deligne–Beilinson cohomology sheaves

Luca Barbieri-Viale

We prove that the Deligne–Beilinson cohomology sheaves $\mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})$ are torsion-free as a consequence of the Bloch–Kato conjectures as proven by Rost and Voevodsky. This implies that $H^0(X, \mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})) = 0$ if X is unirational. For a surface X with $p_g = 0$ we show that the Albanese kernel, identified with $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$, can be characterized using the integral part of the sheaves associated to the Hodge filtration.

Introduction

For a compact algebraic manifold X over $\mathbb{C}$, the Deligne cohomology $H^*(X, \mathbb{Z}(\cdot)_{\mathcal{D}})$ is defined by taking the hypercohomology of the truncated de Rham complex augmented over $\mathbb{Z}$. The extension of such a cohomology theory to arbitrary algebraic complex varieties is usually called Deligne–Beilinson cohomology (for example, see [Gillet 1984; Esnault and Viehweg 1988] for definitions, properties and details). Since Deligne–Beilinson cohomology yields a Poincaré duality theory with supports, the associated Zariski sheaves $\mathcal{H}^*(\mathbb{Z}(\cdot)_{\mathcal{D}})$ have groups of global sections which are birational invariants of smooth complete varieties (see [Barbieri-Viale 1994; 1997]). The motivation for this paper is to start an investigation of these invariants.

We can show that the Deligne–Beilinson cohomology sheaves $\mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})$ are *torsion-free* (see Theorem 2.5) as a consequence of the Bloch–Kato isomorphisms, proven by Rost and Voevodsky (for example, see [Haesemeyer and Weibel 2014; Voevodsky 2011; Weibel 2008]). In particular, $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$ is torsion-free thanks to the Merkurjev–Suslin theorem [1983] on K_2. Thus, the corresponding invariants *vanish* for unirational varieties (see Corollary 2.8). Note that also the singular cohomology sheaves $\mathcal{H}^q(\mathbb{Z})$ are torsion-free. Indeed, a conditional proof (depending on the validity of the Bloch–Kato conjectures) of these properties has been known for a long time (see Remark 2.13).

Furthermore, if only $H^2(X, \mathcal{O}_X) = 0$, we can show that the group of global sections of $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$ is *exactly* the kernel of the cycle map $CH^2(X) \to H^4(X, \mathbb{Z}(2)_{\mathcal{D}})$ in Deligne cohomology, which contains the kernel of the Abel–Jacobi map (see

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Proposition 3.3 and Remark 3.6). This fact generalizes the result of H. Esnault [1990a, Theorem 2.5] for 0-cycles in the case of codimension-2 cycles to X of arbitrary dimension, and it is obtained by a different proof. Concerning the discrete part $\mathcal{F}^{2,2}_{\mathbb{Z}}$ of the Deligne–Beilinson cohomology sheaf $\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})$, we can describe, for any X proper and smooth, the torsion of $H^1(X, \mathcal{F}^{2,2}_{\mathbb{Z}})$ in terms of "transcendental cycles" and $H^3(X, \mathbb{Z})_{\mathrm{tors}}$, and we can see that it has no nonzero global sections (see Proposition 3.3). For surfaces with $p_g = 0$, we are able to compute the group of global sections of $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$ — Bloch's conjecture is that $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = 0$ — by means of some short exact sequences (described in Proposition 4.3) involving the discrete part $\mathcal{F}^{2,2}_{\mathbb{Z}}$ and the Hodge filtration. In order to do that, we argue first with arithmetic resolutions of the Zariski sheaves associated with the presheaves of mixed Hodge structures defined by singular cohomology: the Hodge and weight filtrations do have corresponding coniveau spectral sequences, the E_2 terms of which are given by the cohomology groups of the Zariski sheaves associated to such filtrations (see Proposition 1.2, Theorem 1.6 and compare with the local Hodge theory of [Barbieri-Viale 2002, §3]).

Notation. Throughout this note X is a complex algebraic variety. Let $H^*(X, A)$ and $H_*(X, A)$ be the singular cohomology and Borel–Moore homology of the associated analytic space X_{an} with coefficients in A, respectively, where A could be $\mathbb{Z}, \mathbb{Z}/n, \mathbb{C}$ or $\mathbb{C}^*$. Let $H^*(X)$ and $H_*(X)$ be the corresponding mixed Hodge structures, respectively (see [Deligne 1971]). Denote by $W_i H^*(X)$ and $W_{-i} H_*(X)$ the $\mathbb{Q}$-vector spaces given by the weight filtration, and by $F^i H^*(X)$ and $F^{-i} H_*(X)$ the complex vector spaces given by the Hodge filtration, respectively. For the ring $\mathbb{Z}$ of integers, we will denote by $\mathbb{Z}(r)$ the Tate twist in Hodge theory and by $H^*(X, \mathbb{Z}(r)_{\mathcal{D}})$ the Deligne–Beilinson cohomology groups (see [Esnault and Viehweg 1988; Gillet 1984]). The Tate twist induces the twist $A \otimes \mathbb{Z}(r)$ in the coefficients. which we shall denote by $A(r)$ for short. Denote by $\mathcal{H}^*(A(r))$ and $\mathcal{H}^*(\mathbb{Z}(r)_{\mathcal{D}})$ the Zariski sheaves on a given X associated to singular cohomology and Deligne–Beilinson cohomology, respectively.

1. Arithmetic resolutions in mixed Hodge theory

Let $Z \hookrightarrow X$ be a closed subscheme of the complex algebraic variety X. According to [Deligne 1974, (8.2.2) and (8.3.8)], the singular cohomology groups $H^*_Z(X, \mathbb{Z})$ carry a mixed Hodge structure fitting into long exact sequences

$$\cdots \to H^j_Z(X) \to H^j_T(X) \to H^j_{T-Z}(X - Z) \to H^{j+1}_Z(X) \to \cdots \tag{1.1}$$

for any pair $Z \subset T$ of closed subschemes of X. As has been remarked in [Jannsen 1990], the assignment

$$Z \subseteq X \rightsquigarrow (H^*_Z(X), H_*(Z))$$

yields a Poincaré duality theory with supports (see [Bloch and Ogus 1974]), and furthermore this theory is appropriate for algebraic cycles (in the sense of [Barbieri-Viale 1997]) with values in the abelian tensor category of mixed Hodge structures. In particular, sheafifying the presheaves of vector spaces

$$U \rightsquigarrow F^i H^j(U) \quad \text{and} \quad U \rightsquigarrow W_i H^j(U)$$

on a fixed variety X, we obtain Zariski sheaves $\mathcal{F}^i\mathcal{H}^j$ and $\mathcal{W}_i\mathcal{H}^j$, respectively, filtering the sheaves $\mathcal{H}^j(\mathbb{C})$. We then have:

Proposition 1.2. *Let X be smooth. The "arithmetic resolution"*

$$0 \to \mathcal{H}^q(\mathbb{C}) \to \coprod_{x\in X^0} (i_x)_* H^q(x) \to \coprod_{x\in X^1} (i_x)_* H^{q-1}(x) \to \cdots \to \coprod_{x\in X^q} (i_x)_*\mathbb{C} \to 0$$

is a bifiltered quasi-isomorphism

$$(\mathcal{H}^q(\mathbb{C}), \mathcal{F}, \mathcal{W}) \xrightarrow{\simeq} \Big(\coprod_{x\in X^\odot} (i_x)_* H^{q-\odot}(x),\ \coprod_{x\in X^\odot} (i_x)_* F,\ \coprod_{x\in X^\odot} (i_x)_* W\Big)$$

yielding flasque resolutions

$$\begin{aligned} 0 \to \mathrm{gr}^i_{\mathcal{F}}\, \mathrm{gr}^{\mathcal{W}}_j\, \mathcal{H}^q(\mathbb{C}) \to \coprod_{x\in X^0} (i_x)_*\, \mathrm{gr}^i_F\, \mathrm{gr}^W_j\, H^q(x) \to \cdots \\ \to \coprod_{x\in X^q} (i_x)_*\, \mathrm{gr}^{i-q}_F\, \mathrm{gr}^W_{j-2q}\, H^0(x) \to 0. \end{aligned}$$

Proof. By [Deligne 1971, Théorèmes 1.2.10 and 2.3.5] the functors F^n, W_n and gr^n_F (for any $n \in \mathbb{Z}$) from the category of mixed Hodge structures to that of vector spaces are exact; gr^W_n is exact as a functor from mixed Hodge structures to pure $\mathbb{Q}$-Hodge structures. So the claimed results are obtained via the "locally homologically effaceable" property (see [Bloch and Ogus 1974, Claim, p. 191]) by construction of the arithmetic resolution (given by [Bloch and Ogus 1974, Theorem 4.2]). For example, by applying F^i to the long exact sequences (1.1), taking direct limits over pairs $Z \subset T$ filtered by codimension and sheafifying, we obtain a flasque resolution

$$0 \to \mathcal{F}^i\mathcal{H}^q \to \coprod_{x\in X^0} (i_x)_* F^i H^q(x) \to \coprod_{x\in X^1} (i_x)_* F^{i-1} H^{q-1}(x) \to \cdots$$

of length q, where

$$F^* H^*(x) := \varinjlim_{U\subset\overline{\{x\}}} F^* H^*(U).$$

By this method we obtain as well a resolution of $\mathcal{W}_j$,

$$0 \to \mathcal{W}_j\mathcal{H}^q \to \coprod_{x\in X^0} (i_x)_* W_j H^q(x) \to \coprod_{x\in X^1} (i_x)_* W_{j-2} H^{q-1}(x) \to \cdots.$$

These resolutions give us the claimed bifiltered quasi-isomorphism. (Note: for X of dimension d, the fundamental class η_X belongs to $W_{-2d}H_{2d}(X) \cap F^{-d}H_{2d}(X)$, so that "local purity" yields the shift by two for the weight filtration and the shift by one for the Hodge filtration). In the same way we obtain resolutions of $\mathrm{gr}^i_{\mathcal{F}}$, $\mathrm{gr}^{\mathcal{W}}_j$ and $\mathrm{gr}^i_{\mathcal{F}}\,\mathrm{gr}^{\mathcal{W}}_j$. □

We may consider the twisted Poincaré duality theory $(F^nH^*, F^{-m}H_*)$ where the integers n and m play the role of twisting and indeed we have

$$F^{d-n}H^{2d-k}_Z(X) \cong F^{-n}H_k(Z)$$

for X smooth of dimension d. Via the arithmetic resolution of $\mathcal{F}^i\mathcal{H}^q$, we then have the following:

Corollary 1.3. *Let us assume that X is smooth, and let i be a fixed integer. We then have a "coniveau spectral sequence"*

$$E_2^{p,q} = H^p(X, \mathcal{F}^i\mathcal{H}^q) \Longrightarrow F^iH^{p+q}(X), \tag{1.4}$$

where $H^p(X, \mathcal{F}^i\mathcal{H}^q) = 0$ if $q < i$ or $q < p$.

Remark 1.5. Concerning the Zariski sheaves $\mathrm{gr}^i_{\mathcal{F}}\mathcal{H}^q$ and $\mathcal{H}^q/\mathcal{F}^i$, we indeed obtain corresponding coniveau spectral sequences as above.

Because of the maps of "Poincaré duality theories" $F^iH^*(-) \to H^*(-, \mathbb{C})$, we also have maps of coniveau spectral sequences; on the E_2-terms the map

$$H^p(X, \mathcal{F}^i\mathcal{H}^q) \to H^p(X, \mathcal{H}^q(\mathbb{C}))$$

is given by taking Zariski cohomology of $\mathcal{F}^i\mathcal{H}^q \hookrightarrow \mathcal{H}^q(\mathbb{C})$. For example, if $i < p$, we clearly have (by comparing the arithmetic resolutions) $H^p(X, \mathcal{F}^i\mathcal{H}^p) \cong H^p(X, \mathcal{H}^p(\mathbb{C}))$, and

$$H^p(X, \mathcal{H}^p(\mathbb{C})) \cong NS^p(X) \otimes \mathbb{C}$$

by [Bloch and Ogus 1974, Remark 7.6], where $NS^p(X)$ is the group of cycles of codimension p modulo algebraic equivalence. For $i = p$ we still have:

Theorem 1.6. *Let X be smooth. Then*

$$H^p(X, \mathcal{F}^p\mathcal{H}^p) \cong NS^p(X) \otimes \mathbb{C}.$$

Proof. By Proposition 1.2, we have

$$H^p(X, \mathcal{F}^p\mathcal{H}^p) \cong \mathrm{coker}\Big(\coprod_{x \in X^{p-1}} F^1H^1(x) \to \coprod_{x \in X^p} \mathbb{C}\Big),$$

whence the canonical map $H^p(X, \mathcal{F}^p\mathcal{H}^p) \to NS^p(X) \otimes \mathbb{C}$ is surjective. To show the injectivity, via the arithmetic resolution we see that

$$H^{2p-1}_{Z^{p-1}}(X, \mathbb{C}) \cong \operatorname{Ker}\Big(\coprod_{x \in X^{p-1}} H^1(x) \to \coprod_{x \in X^p} \mathbb{C} \Big),$$

where $H^*_{Z^i}$ denotes the direct limit of the cohomology groups with support on closed subsets of codimension $\geq i$; indeed, this formula is obtained by taking the direct limit of (1.1) over pairs $Z \subset T$ of codimension $\geq p$ and $\geq p-1$, respectively, since $H^{2p-1}_{Z^p} = 0$ and

$$H^{2p}_{Z^p}(X, \mathbb{C}) = \coprod_{x \in X^p} \mathbb{C}.$$

Furthermore,

$$F^p H^{2p-1}_{Z^{p-1}} \cong \operatorname{Ker}\Big(\coprod_{x \in X^{p-1}} F^1 H^1(x) \to \coprod_{x \in X^p} \mathbb{C} \Big)$$

and

$$H^{2p-1}_{Z^{p-1}}/F^p \cong \coprod_{x \in X^{p-1}} \operatorname{gr}^0_F H^1(x)$$

since the arithmetic resolution of $\mathcal{H}^p/\mathcal{F}^p$ has length $p-1$. Thus, we have

$$\operatorname{Im}\Big(\coprod_{x \in X^{p-1}} F^1 H^1(x) \to \coprod_{x \in X^p} \mathbb{C} \Big) = \operatorname{Im}\Big(\coprod_{x \in X^{p-1}} H^1(x) \to \coprod_{x \in X^p} \mathbb{C} \Big). \qquad \square$$

Remark 1.7. By considering the sheaf $\mathcal{H}^q(\mathbb{C})$ (which equals $\mathcal{F}^0\mathcal{H}^q$) on X filtered by the subsheaves $\mathcal{F}^i\mathcal{H}^q$ we have, as usual, (see [Deligne 1971, (1.4.5)]) a spectral sequence

$${}_{\mathcal{F}}E_1^{r,s} = H^{r+s}(X, \operatorname{gr}^s_{\mathcal{F}} \mathcal{H}^q) \Longrightarrow H^{r+s}(X, \mathcal{H}^q(\mathbb{C}))$$

with induced "aboutissement" filtration

$$F^i H^p(X, \mathcal{H}^q) := \operatorname{Im}\big(H^p(X, \mathcal{F}^i\mathcal{H}^q) \to H^p(X, \mathcal{H}^q(\mathbb{C}))\big).$$

The interested reader can check that this spectral sequence degenerates. This reproves Theorem 1.6, and also yields that the filtration $F^i H^p(X, \mathcal{H}^p)$ is the Néron–Severi group $NS^p(X) \otimes \mathbb{C}$ if $i \leq p$ and vanishes otherwise.

Remark 1.8. As an immediate consequence of this Theorem 1.6, via the coniveau spectral sequence (1.4), we see the well-known fact that the image of the cycle map $c\ell^p : NS^p(X) \otimes \mathbb{C} \to H^{2p}(X, \mathbb{C})$ is contained in $F^p H^{2p}(X)$.

For any X smooth and proper, we have $F^2H^2(X) = H^0(X, \mathcal{F}^2\mathcal{H}^2) = H^0(X, \Omega^2_X)$ and

$$H^0(X, \mathcal{H}^2/\mathcal{F}^2) \cong H^0(X, \mathcal{H}^2(\mathbb{C}))/H^0(X, \Omega^2_X) \cong \frac{H^2(X, \mathbb{C})}{H^0(X, \Omega^2_X) \oplus NS(X) \otimes \mathbb{C}}, \tag{1.9}$$

where $H^0(X,\mathcal{H}^2(\mathbb{C}))\cong H^0(X,\mathcal{H}^2(\mathbb{Z}))\otimes\mathbb{C}$ and

$$H^0(X,\mathcal{H}^2(\mathbb{Z}))=\operatorname{Im}\big(H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O}_X)\big)$$

is the lattice of "transcendental cycles". The formula (1.9) can be obtained, for example, by the exact sequence (given by the coniveau spectral sequence since $\mathcal{F}^2\mathcal{H}^1=0$)

$$0\to H^1(X,\mathcal{H}^1(\mathbb{C}))\to H^2(X)/F^2\to H^0(X,\mathcal{H}^2/\mathcal{F}^2)\to 0$$

since $H^1(X,\mathcal{H}^1(\mathbb{C}))=NS(X)\otimes\mathbb{C}$.

2. Deligne–Beilinson cohomology sheaves

Let X be smooth over $\mathbb{C}$. Let us consider the Zariski sheaf $\mathcal{H}^*(\mathbb{Z}(r)_{\mathcal{D}})$ associated to the presheaf of Deligne–Beilinson cohomology groups $U\rightsquigarrow H^*(U,\mathbb{Z}(r)_{\mathcal{D}})$ on X. We have canonical long exact sequences of sheaves

$$\cdots\to\mathcal{H}^q(\mathbb{Z}(r))\to\mathcal{H}^q(\mathbb{C})/\mathcal{F}^r\to\mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})\to\mathcal{H}^{q+1}(\mathbb{Z}(r))\to\cdots,\tag{2.1}$$

$$\cdots\to\mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}})\to\mathcal{H}^q(\mathbb{Z}(r))\oplus\mathcal{F}^r\mathcal{H}^q\to\mathcal{H}^q(\mathbb{C})\to\mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})\to\cdots,\tag{2.2}$$

$$\cdots\to\mathcal{F}^r\mathcal{H}^q\to\mathcal{H}^q(\mathbb{C}^*(r))\to\mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})\to\mathcal{F}^r\mathcal{H}^{q+1}\to\cdots\tag{2.3}$$

on X obtained by sheafifying the usual long exact sequences coming with Deligne–Beilinson cohomology (see [Esnault and Viehweg 1988, Corollary 2.10]).

Define the "discrete part" $\mathcal{F}^{r,q}_{\mathbb{Z}}$ (cf. [Esnault 1990a, §1]) of the Deligne–Beilinson cohomology sheaves by

$$\mathcal{F}^{r,q}_{\mathbb{Z}}:=\operatorname{Im}\big(\mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}})\to\mathcal{H}^q(\mathbb{Z}(r))\big),$$

or, equivalently by (2.1), $\mathcal{F}^{r,q}_{\mathbb{Z}}$ is the integral part of $\mathcal{F}^r\mathcal{H}^q$. Note that $\mathcal{F}^{r,q}_{\mathbb{Z}}$ is given by the inverse image of $\mathcal{F}^r\mathcal{H}^q$ under the canonical map $\mathcal{H}^q(\mathbb{Z})\to\mathcal{H}^q(\mathbb{C})$.

We may define the "transcendental part" of the Deligne–Beilinson cohomology sheaves to be

$$\mathcal{T}^{r,q}_{\mathcal{D}}:=\operatorname{Ker}\big(\mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}})\to\mathcal{H}^q(\mathbb{Z}(r))\big).$$

We then have the short exact sequence

$$0\to\mathcal{H}^q(\mathbb{Z}(r))/\mathcal{F}^{r,q}_{\mathbb{Z}}\to\mathcal{H}^q(\mathbb{C})/\mathcal{F}^r\to\mathcal{T}^{r,q+1}_{\mathcal{D}}\to 0\tag{2.4}$$

induced by (2.1). Note that if $r=0$ then $\mathcal{H}^q(\mathbb{C})/\mathcal{F}^0=0$ and (2.1) yields the isomorphism $\mathcal{H}^*(\mathbb{Z}(0)_{\mathcal{D}})\cong\mathcal{H}^*(\mathbb{Z})$, so that (2.2) splits in trivial short exact sequences.

Theorem 2.5. *Let X be smooth over $\mathbb{C}$ and assume $q\geq 0$.*

(i) *The sheaf $\mathcal{H}^q(\mathbb{Z})$ is torsion-free.*

(ii) *The sheaf $\mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})$ is torsion-free.*

(iii) *There is a canonical isomorphism* $\mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}}) \otimes \mathbb{Z}/n \cong \mathcal{H}^q(\mathbb{Z}/n)$

Proof. (i) In order to show that $\mathcal{H}^{q+1}(\mathbb{Z})$ is torsion-free it suffices to see that $\mathcal{H}^q(\mathbb{Z}) \to \mathcal{H}^q(\mathbb{Z}/n)$ is an epimorphism for any $n \in \mathbb{Z}$; via the canonical map $\mathcal{O}_X^* \to \mathcal{H}^1(\mathbb{Z})$ and cup product we obtain a map $(\mathcal{O}_X^*)^{\otimes q} \to \mathcal{H}^q(\mathbb{Z})$. The composition

$$(\mathcal{O}_X^*)^{\otimes q} \to \mathcal{H}^q(\mathbb{Z}) \to \mathcal{H}^q(\mathbb{Z}/n)$$

can be obtained as well as (cf. [Bloch and Srinivas 1983, p. 1240]) the composition

$$(\mathcal{O}_X^*)^{\otimes q} \xrightarrow{\text{sym}} \mathcal{K}_q^M \to \mathcal{H}^q(\mathbb{Z}/n),$$

where by definition of Milnor's K-theory sheaf the symbol map sym is an epimorphism. Thus it is left to show that the Galois symbol $\mathcal{K}_q^M \to \mathcal{H}^q(\mathbb{Z}/n)$ is an epimorphism (for the sake of exposition we are tacitly fixing an n-th root of unity, yielding a noncanonical isomorphism $\mathcal{H}^q_{\text{ét}}(\mu_n^{\otimes r}) \cong \mathcal{H}^q(\mathbb{Z}/n)$). The Galois symbol map can be obtained by mapping the Gersten resolution for Milnor's K-theory (for example, see [Kerz 2009, Theorem 7.1]) to the Bloch–Ogus arithmetic resolution of the sheaf $\mathcal{H}^q(\mathbb{Z}/n)$. In fact, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_q^M \arrow[r] \arrow[d] & \coprod_{\eta\in X^0} (i_\eta)_* K_q^M(k(\eta)) \arrow[r] \arrow[d] & \coprod_{x\in X^1} (i_x)_* K_{q-1}^M(k(x)) \arrow[d] \\
0 \arrow[r] & \mathcal{H}^q(\mathbb{Z}/n) \arrow[r] & \coprod_{\eta\in X^0} (i_\eta)_* H^q(\eta) \arrow[r] & \coprod_{x\in X^1} (i_x)_* H^{q-1}(x)
\end{tikzcd}$$

where $H^*(\text{point})$ is the Galois cohomology of $k(\text{point})$ with $\mathbb{Z}/n$-coefficients. Thus, the Bloch–Kato isomorphism $K_*^M(k(\text{point}))/n \xrightarrow{\sim} H^*(\text{point})$ (see [Haesemeyer and Weibel 2014, Theorem A]) and the exactness of the Gersten complex for Milnor's K-theory mod n (for example, see [Kerz 2009, Theorem 7.8]) yields the desired projection $\mathcal{K}_q^M \to \mathcal{K}_q^M/n \xrightarrow{\sim} \mathcal{H}^q(\mathbb{Z}/n)$.

(ii) & (iii) By considering the Bloch–Beilinson regulators

$$\mathcal{K}_q^M \to \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}})$$

(simply obtained by the fact that $\mathcal{K}_1^M = \mathcal{O}_X^* \cong \mathcal{H}^1(\mathbb{Z}(1)_{\mathcal{D}})$ and using the cup product) we have that the composition

$$\mathcal{K}_q^M \to \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}}) \to \mathcal{H}^q(\mathbb{Z}(q)) \to \mathcal{H}^q(\mathbb{Z}/n)$$

is the Galois symbol (see [Esnault 1990b, §0, p. 375]). Hence the composition

$$\mathcal{K}_q^M \to \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}})/n \to \mathcal{H}^q(\mathbb{Z}/n)$$

is an epimorphism. Therefore, by comparing with (2.7) below, the proof of the theorem is finished. □

Lemma 2.6. *We have a short exact sequence of sheaves*

$$0 \to \mathcal{H}^q(\mathbb{Z}(r)_{\mathcal{D}})/n \to \mathcal{H}^q(\mathbb{Z}/n) \to \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}} \to 0 \tag{2.7}$$

for all $q, r \geq 0$ *and* $n \in \mathbb{Z}$.

Proof. The sequence (2.7) is obtained from the long exact sequence (2.1) as follows. Since the sheaf $\mathcal{H}^q(\mathbb{Z}(r)) \cong \mathcal{H}^q(\mathbb{Z})$ for all $r \geq 0$ is torsion-free, we have $\mathcal{T}^{r,q+1}_{\mathcal{D},n\text{-tors}} = \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}}$. Using (2.4), since the sheaf $\mathcal{H}^q(\mathbb{C})/\mathcal{F}^r$ is uniquely divisible, we have that

$$\mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}} = (\mathcal{H}^q(\mathbb{Z}(r))/\mathcal{F}^{r,q}_{\mathbb{Z}}) \otimes \mathbb{Z}/n.$$

Thus we get a short exact sequence

$$0 \to \mathcal{F}^{r,q}_{\mathbb{Z}}/n \to \mathcal{H}^q(\mathbb{Z}/n) \to \mathcal{H}^{q+1}(\mathbb{Z}(r)_{\mathcal{D}})_{n\text{-tors}} \to 0$$

by tensoring with $\mathbb{Z}/n$ the canonical one induced by the subsheaf $\mathcal{F}^{r,q}_{\mathbb{Z}} \hookrightarrow \mathcal{H}^q(\mathbb{Z}(r))$. Since $\mathcal{T}^{r,q}_{\mathcal{D}}$ is divisible, we are done. □

By a standard argument (see [Barbieri-Viale 1994, §2; 1997]) we have:

Corollary 2.8. *Suppose that* X *is a smooth unirational complete variety. Then*

$$H^0(X, \mathcal{H}^{q+1}(\mathbb{Z}(q)_{\mathcal{D}})) = H^0(X, \mathcal{H}^q(\mathbb{Z})) = 0.$$

Remark 2.9. In particular, from Theorem 2.5(i) we get the short exact sequence

$$0 \to \mathcal{H}^q(\mathbb{Z}) \to \mathcal{H}^q(\mathbb{C}) \to \mathcal{H}^q(\mathbb{C}^*) \to 0. \tag{2.10}$$

Moreover, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathcal{H}^q(\mathbb{Z}(q))/\mathcal{F}^{q,q}_{\mathbb{Z}} \arrow[r] \arrow[u] & \mathcal{H}^q(\mathbb{C})/\mathcal{F}^q \arrow[r] \arrow[u] & \mathcal{T}^{q,q+1}_{\mathcal{D}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{H}^q(\mathbb{Z}(q)) \arrow[r] \arrow[u] & \mathcal{H}^q(\mathbb{C}) \arrow[r] \arrow[u] & \mathcal{H}^q(\mathbb{C}^*(q)) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{F}^{q,q}_{\mathbb{Z}} \arrow[r] \arrow[u] & \mathcal{F}^q \arrow[r] \arrow[u] & \mathcal{F}^q/\mathcal{F}^{q,q}_{\mathbb{Z}} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{2.11}$$

where the middle row is given by (2.10) and the top one is just given by (2.4). Finally, we have (see [Esnault 1990a, (1.3)α)]) the short exact sequence

$$0 \to \mathcal{H}^{q-1}(\mathbb{C}^*(q)) \to \mathcal{H}^q(\mathbb{Z}(q)_{\mathcal{D}}) \to \mathcal{F}_{\mathbb{Z}}^{q,q} \to 0 \tag{2.12}$$

given by (2.3) or (2.1), taking account of (2.10).

Remark 2.13. The argument in the proof of Theorem 2.5, assuming the validity of Bloch–Kato conjecture, was given in the previous version of this paper, available at arXiv:alg-geom/9412006v1. Indeed, O. Gabber announced (at the end of 1992) the (universal) exactness of the Gersten complex of Milnor's K-groups, and some discussions with B. Kahn directed my attention to Gabber's announcement. Actually, Theorem 2.5(i) was first considered in [Bloch and Srinivas 1983, p. 1240] for $q = 3$, it was conjectured in [Barbieri-Viale 1994, §7] in general, and a proof also appears in [Colliot-Thélène and Voisin 2012, Théorème 3.1].

3. Coniveau versus Hodge filtrations

Recall the existence of arithmetic resolutions of the sheaves $\mathcal{H}^*(\mathbb{Z}(\cdot)_{\mathcal{D}})$, the coniveau spectral sequence

$$_{\mathcal{D}}E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathbb{Z}(\cdot)_{\mathcal{D}})) \Longrightarrow H^{p+q}(X, \mathbb{Z}(\cdot)_{\mathcal{D}}), \tag{3.1}$$

and the formula $H^p(X, \mathcal{H}^p(\mathbb{Z}(p)_{\mathcal{D}})) \cong CH^p(X)$ (see [Gillet 1984]). By the spectral sequence (3.1), we have a long exact sequence

$$0 \to H^1(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \to H^3(X, \mathbb{Z}(2)_{\mathcal{D}}) \xrightarrow{\rho} H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \xrightarrow{\delta} CH^2(X). \tag{3.2}$$

The mapping δ is just a differential between $_{\mathcal{D}}E_2$-terms of the coniveau spectral sequence (3.1); we still have

$$\operatorname{Im}\delta = \operatorname{Ker}\big(CH^2(X) \xrightarrow{c\ell} H^4(X, \mathbb{Z}(2)_{\mathcal{D}})\big),$$

where $c\ell$ is the cycle class map in Deligne–Beilinson cohomology.

Proposition 3.3. *Let X be proper and smooth. Then*

$$H^0(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) = 0,$$

the group $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$ is infinitely divisible, and

$$H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})_{\text{tors}} \cong H^0(X, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}(2))).$$

If $H^2(X, \mathcal{O}_X) = 0$, then

$$H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \cong \operatorname{Ker}\big(CH^2(X) \xrightarrow{c\ell} H^4(X, \mathbb{Z}(2)_{\mathcal{D}})\big),$$

i.e., $\rho = 0$ in (3.2), and $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) \cong H^3(X, \mathbb{Z})_{\text{tors}}$.

Proof. By the canonical map of "Poincaré duality theories"

$$H^{\sharp-1}(-,\mathbb{C}^*(\cdot))\to H^{\sharp}(-,\mathbb{Z}(\cdot)_{\mathcal{D}}),$$

we obtain a map of coniveau spectral sequences and the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1(X,\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \arrow[r] & H^3(X,\mathbb{Z}(2)_{\mathcal{D}}) \arrow[r, "\rho"] & H^0(X,\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) & \\ 0 \arrow[r] & NS(X)\otimes\mathbb{C}^*(2) \arrow[r] \arrow[u, "\iota"] & H^2(X,\mathbb{C}^*(2)) \arrow[r] \arrow[u] & H^0(X,\mathcal{H}^2(\mathbb{C}^*(2))) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{3.4}$$

In fact, we have

$$H^1(X,\mathcal{H}^1(\mathbb{C}^*(2)))\cong NS(X)\otimes\mathbb{C}^*(2),$$

and the exactness on the right of the bottom exact sequence is provided by the vanishing $H^2(X,\mathcal{H}^1(\mathbb{C}^*(2)))=0$. The left-most map ι is induced by the corresponding map in the long exact sequence

$$\begin{aligned}\cdots\xrightarrow{\alpha} H^0(X,\mathcal{F}^{2,2}_{\mathbb{Z}})\to H^1(X,\mathcal{H}^1(\mathbb{C}^*(2)))\\ \to H^1(X,\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}}))\to H^1(X,\mathcal{F}^{2,2}_{\mathbb{Z}})\xrightarrow{\beta}\cdots\end{aligned} \tag{3.5}$$

obtained from the short exact sequence of sheaves (2.12) for $q=2$.

It is then easy to see that $H^0(X,\mathcal{F}^{2,2}_{\mathbb{Z}})$ and $H^1(X,\mathcal{F}^{2,2}_{\mathbb{Z}})$ are, respectively, the kernel and the cokernel of ι. In fact, $\beta=0$ in (3.5) because $H^2(X,\mathcal{H}^1(\mathbb{C}^*(2)))=0$. To see that $\alpha=0$ in (3.5), note that, by the coniveau spectral sequences,

$$H^0(X,\mathcal{H}^1(\mathbb{C}^*(2)))\cong H^1(X,\mathbb{C}^*(2)) \quad \text{and} \quad H^0(X,\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}}))\cong H^2(X,\mathbb{Z}(2)_{\mathcal{D}})$$

since $\mathbb{C}^*(2)\cong\mathcal{H}^1(\mathbb{Z}(2)_{\mathcal{D}})$ is Zariski constant; now, since X is proper,

$$H^1(X,\mathbb{C}^*(2))\cong H^2(X,\mathbb{Z}(2)_{\mathcal{D}}),$$

that is,

$$F^2H^1=0 \quad \text{and} \quad F^2H^2\hookrightarrow H^2(X,\mathbb{C}^*(2))$$

in the global version of (2.3), so that $\alpha=0$ in (3.5), as claimed. Furthermore, we have

$$H^3(X,\mathbb{Z}(2)_{\mathcal{D}})\cong H^2(X,\mathbb{C}^*(2))/F^2H^2, \quad \text{since } F^2H^3\hookrightarrow H^3(X,\mathbb{C}^*(2)).$$

We conclude that $H^0(X,\mathcal{F}^{2,2}_{\mathbb{Z}})$ vanishes because $F^2H^2\cap NS(X)\otimes\mathbb{C}^*(2)=0$. Moreover, $H^0(X,\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$ is torsion-free (by Theorem 2.5) so that $\operatorname{Im}\rho$ is torsion-free and $H^3(X,\mathbb{Z}(2)_{\mathcal{D}})\otimes\mathbb{Q}/\mathbb{Z}=0$ indeed; therefore, tensoring the top row of (3.4) with $\mathbb{Q}/\mathbb{Z}$, we get

$$H^1(X,\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}}))\otimes\mathbb{Q}/\mathbb{Z}=0.$$

Then the vanishing $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2}) \otimes \mathbb{Q}/\mathbb{Z} = 0$ follows from the description of the cokernel of ι. Further, by taking the torsion subgroups in the diagram (3.4), we obtain the assertion about the torsion of $H^1(X, \mathcal{F}_{\mathbb{Z}}^{2,2})$. If $H^2(X, \mathcal{O}_X) = 0$ then $NS(X) \cong H^2(X, \mathbb{Z})$. We then have (by the bottom row of the diagram (3.4) above) that $H^0(X, \mathcal{H}^2(\mathbb{C}^*(2))) \cong H^3(X, \mathbb{Z})_{\text{tors}}$, and its image in $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$ is equal to the image of ρ, whence the image of ρ is zero since it is torsion-free. Since $F^2H^2 = 0$ by a final diagram chase, we obtain the last claim. $\square$

Remark 3.6. $H^0(X, \mathcal{H}^2(\mathbb{Q}/\mathbb{Z}))$ is actually an extension of $H^0(X, \mathcal{H}^2(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z}$ by $H^3(X, \mathbb{Z})_{\text{tors}}$ because $H^0(X, \mathcal{H}^3(\mathbb{Z}))$ is torsion-free. Recall the commutative diagram (see [Barbieri-Viale 1994, 6.1; Esnault and Viehweg 1988, §7])

$$\begin{tikzcd}
0 \arrow[r] & A^2(X) \arrow[r] \arrow[d] & CH^2(X) \arrow[r] \arrow[d, "c\ell"'] & NS^2(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J^2(X) \arrow[r] & H^4(X, \mathbb{Z}(2)_{\mathcal{D}}) \arrow[r] & H_{\mathbb{Z}}^{2,2} \arrow[r] & 0
\end{tikzcd} \tag{3.7}$$

where $J^2(X)$ is the intermediate Jacobian, $A^2(X) \subset CH^2(X)$ is the subgroup of cycles which are algebraically equivalent to zero, $H_{\mathbb{Z}}^{2,2} \subset H^4(X, \mathbb{Z}(2))$ are integral Hodge cycles and the composition $NS^2(X) \to H_{\mathbb{Z}}^{2,2} \subset H^4(X, \mathbb{Z}(2))$ is the classical cycle class map in singular cohomology. Recall that we also have an exact sequence

$$H^0(X, \mathcal{H}^3(\mathbb{Z})) \to NS^2(X) \to H^4(X, \mathbb{Z}(2)).$$

The vanishing $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = H^0(X, \mathcal{H}^3(\mathbb{Z})) = 0$, e.g., if X is unirational by Corollary 2.8, would imply the finite generation of $NS^2(X)$ and the representability of $A^2(X)$.

In order to detect elements in the mysterious group $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$ of global sections, we dispose of the image of $H^0(X, \mathcal{H}^2/\mathcal{F}^2)$ (see (1.9)) which is the same (see the diagram (2.11)) as the image of

$$H^0(X, \mathcal{H}^2(\mathbb{Z})) \otimes \mathbb{C}/\mathbb{Q}(2) = H^0(X, \mathcal{H}^2(\mathbb{C}^*(2))) \otimes \mathbb{Q}.$$

Unfortunately these images cannot be the entire group, in general. Indeed, whenever the map

$$H^0(X, \mathcal{H}^2(\mathbb{C}^*(2))) \to H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$$

is surjective then ρ is surjective in (3.2) (because of (3.4)), whence the cycle map is injective, which is not the case in general (indeed, for any surface with $p_g \neq 0$ the cycle map is not injective, by [Mumford 1968]).

4. Surfaces with $p_g = 0$

In the following we let X denote a complex algebraic surface which is smooth and complete. Let $A_0(X)$ be the subgroup of $CH^2(X)$ of cycles of degree zero. Let

$$\phi : A_0(X) \to J^2(X)$$

be induced by the canonical mapping to the Albanese variety. It is well known (see [Gillet 1984, Theorem 2 and Corollary]) that $c\ell|_{A_0(X)} = \phi$, where $c\ell$ is the cycle map in Deligne cohomology, i.e., in the diagram (3.7) we have that $NS^2(X) \cong H^4(X, \mathbb{Z}(2))$ and $A^2(X) \cong A_0(X)$ under our assumptions. Actually, it is known that (see [Barbieri-Viale and Srinivas 1995]) on such a surface X the sheaf $\mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$ is *flasque* and

$$H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \cong \varinjlim_{U \subset X} \frac{H^2(U, \mathbb{C})}{F^2H^2(U) + H^2(U, \mathbb{Z}(2))},$$

where the limit is taken over the nonempty Zariski open subsets of X. We then have

$$H^1(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = 0. \tag{4.1}$$

Moreover, the sheaf $\mathcal{H}^4(\mathbb{Z}(2)_{\mathcal{D}})$ vanishes on a surface (as it is easy to see via the exact sequence (2.1)). Thus, via the spectral sequence (3.1), the vanishing (4.1) corresponds to the vanishing of the cokernel of the cycle map

$$CH^2(X) \xrightarrow{c\ell} H^4(X, \mathbb{Z}(2)_{\mathcal{D}}),$$

which is equivalent (using the diagram (3.7)) to the well-known surjectivity of ϕ. Finally, we have that $H^2(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \cong H^5(X, \mathbb{Z}(2)_{\mathcal{D}}) = 0$. In conclusion, the only possibly nonzero terms in the spectral sequence (3.1) are: those giving the exact sequence (3.2), $H^0(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \cong H^2(X, \mathbb{Z}(2)_{\mathcal{D}})$ and $H^0(X, \mathcal{H}^1(\mathbb{Z}(2)_{\mathcal{D}})) = \mathbb{C}^*$. Since $\operatorname{Ker}\phi = \operatorname{Ker} c\ell = \operatorname{Im}\delta$ in (3.2), we obtain:

Lemma 4.2. *The group $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$ is uniquely divisible for any surface X which is smooth and complete.*

Proof. Note that $A^2(X)$ is always divisible (for example, see [Bloch and Ogus 1974, Lemma 7.10]) and $A_0(X)_{\text{tors}} \cong J^2(X)_{\text{tors}}$ by [Rojtman 1980], so $\operatorname{Ker}\phi \otimes \mathbb{Q}/\mathbb{Z} = 0$. Now (3.2) yields $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \otimes \mathbb{Q}/\mathbb{Z} = 0$ since $\operatorname{Im}\delta = \operatorname{Ker}\phi$ and $\operatorname{Ker}\delta = \operatorname{Im}\rho$ both vanish when tensored with $\mathbb{Q}/\mathbb{Z}$. Using Theorem 2.5 we are done. □

We know (see [Mumford 1968]) that $A_0(X) \cong J^2(X)$ implies that $p_g = 0$. Conversely, Bloch's conjecture is that if $p_g = 0$ then $A_0(X) \cong J^2(X)$. Therefore, by Proposition 3.3, we obtain that $p_g = 0$ if and only if $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) = 0$, assuming the validity of Bloch's conjecture (see [Barbieri-Viale and Srinivas 1995;

Rosenschon 1999; Esnault 1990a; Gillet 1984]). A first characterization of the uniquely divisible group $H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}}))$ is given by the following:

Proposition 4.3. *Let X be a smooth complete surface with $p_g = 0$. We then have the following canonical short exact sequences*

$$0 \to H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \to H^1(X, \mathcal{F}^2/\mathcal{F}^{2,2}_{\mathbb{Z}}) \to H^3(X, \mathbb{C}^*(2)) \to 0, \tag{4.4}$$

$$0 \to H^0(X, \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})) \to H^1(X, \mathcal{H}^2(\mathbb{Z}(2))/\mathcal{F}^{2,2}_{\mathbb{Z}})) \to H^3/F^2 \to 0, \tag{4.5}$$

where

$$0 \to F^2H^3 \to H^1(X, \mathcal{F}^2/\mathcal{F}^{2,2}_{\mathbb{Z}}) \to A_0(X) \to 0, \tag{4.6}$$

$$0 \to H^3(X, \mathbb{Z})/\operatorname{tors} \to H^1(X, \mathcal{H}^2(\mathbb{Z}(2))/\mathcal{F}^{2,2}_{\mathbb{Z}}) \to A_0(X) \to 0 \tag{4.7}$$

are also exact.

Proof. All these exact sequences are obtained by considering the exact diagram of cohomology groups associated with the diagram of sheaves (2.11) (where $\mathcal{T}^{2,3}_{\mathcal{D}} = \mathcal{H}^3(\mathbb{Z}(2)_{\mathcal{D}})$ on a surface) taking account of Theorems 2.5 and 1.6, Proposition 3.3 and the coniveau spectral sequence (1.4). For example, the sequence (4.4) is obtained by taking the long exact sequence of cohomology groups associated with the right-most column of (2.11), the fact that $H^1(X, \mathcal{H}^2(\mathbb{C}^*(2))) \cong H^3(X, \mathbb{C}^*(2))$ on a surface and the formula (4.1). For (4.5) one has to use the top row of (2.11), the formulas (1.9) and (4.1), and the fact that $H^3/F^2 \cong H^1(X, \mathcal{H}^2/\mathcal{F}^2)$. The left-most column of (2.11) yields (4.7), since

$$H^2(X, \mathcal{F}^{2,2}_{\mathbb{Z}}) \cong H^2(X, \mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}})) \cong CH^2(X)$$

by (2.12) (see [Esnault 1990a, Theorem 1.3]) and the map of sheaves $\mathcal{H}^2(\mathbb{Z}(2)_{\mathcal{D}}) \to \mathcal{H}^2(\mathbb{Z}(2))$ induces the degree map on H^2. For (4.6) one has to argue with the commutative square in the left bottom corner of (2.11) and the isomorphism

$$H^2(X, \mathcal{F}^2) \cong H^2(X, \mathcal{H}^2(\mathbb{C})) \cong \mathbb{C};$$

remember that $H^1(X, \mathcal{F}^{2,2}_{\mathbb{Z}}) = H^3(X, \mathbb{Z})_{\text{tors}}$, whence it goes to zero in $F^2H^3 \cong H^1(X, \mathcal{F}^2\mathcal{H}^2)$. □

Remark 4.8. Because of Proposition 4.3, Bloch's conjecture is equivalent to showing that the canonical injections of sheaves

$$\mathcal{F}^2/\mathcal{F}^{2,2}_{\mathbb{Z}} \hookrightarrow \mathcal{H}^2(\mathbb{C}^*(2)) \quad \text{and} \quad \mathcal{H}^2(\mathbb{Z}(2))/\mathcal{F}^{2,2}_{\mathbb{Z}} \hookrightarrow \mathcal{H}^2/\mathcal{F}^2$$

remain injections on H^1. It would be very nice to know of any reasonable description of the Zariski cohomology classes of these subsheaves.

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On some negative motivic homology groups

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For an arbitrary separated scheme X of finite type over a finite field $\mathbb{F}_q$ and a negative integer j, we prove, under the assumption of resolution of singularities, that $H_{-1}(X, \mathbb{Z}(j))$ is canonically isomorphic to $H_{-1}(\pi_0(X), \mathbb{Z}(j))$ if $j = -1$ or -2, and $H_i(X, \mathbb{Z}(j))$ vanishes if $i \leq -2$ and $i - j \leq 1$. As the group $H_{-1}(\pi_0(X), \mathbb{Z}(j))$ is explicitly known, this gives a explicit calculation of motivic homology of degree -1 and weight -1 or -2 of an arbitrary scheme over a finite field.

1. Introduction

In this paper, we assume that schemes are separated and of finite type over a perfect field. The finite field with q elements is written as $\mathbb{F}_q$. For a scheme X, $\pi_0(X)$ denotes the spectrum of $\mathcal{O}_X(X)^{\text{ét}}$, the largest étale k-algebra contained in $\mathcal{O}_X(X)$ that is finite over k. The properties of $\pi_0(X)$ relevant to us can be found in [Liu 2002, pp. 495–496].

The aim of this paper is to prove the following theorem on motivic homology.

Theorem 1.1. *Assume that resolution of singularities holds over $\mathbb{F}_q$. Let i and j be negative integers. Then, for all schemes X over $\mathbb{F}_q$,*

$$H_i(X, \mathbb{Z}(j)) = 0$$

if $i \leq -2$ and $i - j \leq 1$. In degree $i = -1$, the canonical map

$$\alpha_X : H_{-1}(X, \mathbb{Z}(j)) \longrightarrow H_{-1}(\pi_0(X), \mathbb{Z}(j))$$

is an isomorphism if $i - j \leq 1$, i.e., $j = -1$ or -2.

Since $\pi_0(X)$ is finite étale over $\mathbb{F}_q$, it is a finite disjoint union of spectra of finite fields. Hence, the isomorphism $H_{-1}(X, \mathbb{Z}(j)) \cong H_{-1}(\pi_0(X), \mathbb{Z}(j))$ of Theorem 1.1 and the explicit computation of negative motivic homology groups of finite fields as in Lemma 2.1 give an explicit computation of $H_{-1}(X, \mathbb{Z}(j))$ ($j = -1$ or -2) for an arbitrary scheme X over $\mathbb{F}_q$. In particular, if X is geometrically connected over $\mathbb{F}_q$ (this is equivalent to requiring that X be connected and $\pi_0(X) = \operatorname{Spec} \mathbb{F}_q$ [Liu 2002, Chapter 10, Corollary 2.21(a)]), we have the following corollary.

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Corollary 1.2. *Under resolution of singularities, if X is a geometrically connected scheme over $\mathbb{F}_q$ and $j=-1$ or -2, there is a canonical isomorphism*

$$H_{-1}(X,\mathbb{Z}(j)) \xrightarrow{\sim} H_{-1}(\operatorname{Spec}\mathbb{F}_q,\mathbb{Z}(j)) \cong \mathbb{F}_{q^{-j}}^{\times}.$$

Remark 1.3. It is worth noting that if one assumes Parshin's conjecture the statement in Theorem 1.1 holds for all negative integers i and j without the bound $i-j\le 1$. One only needs to invoke [Kondo and Yasuda 2013, Proposition 4.1] instead of Proposition 2.7 in order to prove the claims of Lemma 4.1, Proposition 4.2 and Proposition 4.3 without the bounding conditions on $i-j$. One may similarly prove Proposition 4.7, Lemma 4.9 and Lemma 4.10 for all negative integers j.

Theorem 1.1 is a version in the context of motivic homology of the following theorem of Kondo and Yasuda on Borel–Moore motivic homology. In fact, if the scheme X is proper, our Theorem 1.1 is due to Kondo and Yasuda:

Theorem 1.4 [Kondo and Yasuda 2013, Theorem 1.1]. *Let $j=-1$ or -2 and let X be a connected scheme over a finite field $\mathbb{F}_q$. If X is not proper,*

$$H_{-1}^{\mathrm{BM}}(X,\mathbb{Z}(j))=0.$$

If X is proper, the pushforward maps

$$H_{-1}^{\mathrm{BM}}(X,\mathbb{Z}(j)) \longrightarrow H_{-1}^{\mathrm{BM}}(\operatorname{Spec}\mathcal{O}_X(X),\mathbb{Z}(j))$$

are isomorphisms.

Theorem 1.4 itself is a generalization of [Akhtar 2004, Proposition 3.1], where the claim for $j=-1$ was proved for smooth projective schemes X.

The case $i\le -2$ of our Theorem 1.1 is also due to Kondo and Yasuda if the scheme X is proper (see Proposition 2.7).

Theorem 1.1 and Theorem 1.4 are related as follows. If X is a scheme over $\mathbb{F}_q$, $\operatorname{Spec}\mathcal{O}_X(X)$ is also a scheme over $\mathbb{F}_q$. (Recall our convention on schemes.) Thus, the canonical factorization

$$X \longrightarrow \operatorname{Spec}\mathcal{O}_X(X) \longrightarrow \pi_0(X) \longrightarrow \operatorname{Spec}\mathbb{F}_q$$

of the structure morphism of X gives, on applying Theorem 1.1, the isomorphisms

$$H_i(X,\mathbb{Z}(j)) \xrightarrow{\sim} H_i(\operatorname{Spec}\mathcal{O}_X(X),\mathbb{Z}(j)) \xrightarrow{\sim} H_i(\pi_0(X),\mathbb{Z}(j))$$

because $\pi_0(\operatorname{Spec}\mathcal{O}_X(X))=\pi_0(X)$ by definition. It is trivially true that the first map is an isomorphism if X is affine and so is the second if X is proper. Since motivic homology and Borel–Moore homology agree for proper schemes, the theorem of Kondo and Yasuda says that the first map is an isomorphism when X is proper. They proved this without assuming resolution of singularities. Our Theorem 1.1

claims that both maps are always isomorphisms if we assume the existence of resolution of singularities.

Let us end this introduction with a summary of the properties of motivic homology and cohomology theories which we shall use freely in the subsequent sections.

By motivic (co)homology (with compact supports) with coefficients in an abelian group A, we mean the following four theories defined for schemes X over a perfect field k:

- motivic homology,
$$H_i(X, A(j)) := \mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(k)}(A(j)[i], M(X));$$
- motivic cohomology,
$$H^i(X, A(j)) := \mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(k)}(M(X), A(j)[i]);$$
- motivic homology with compact supports,
$$H_i^{\mathrm{BM}}(X, A(j)) := \mathrm{CH}_j(X, i-2j; A);$$
- motivic cohomology with compact supports,
$$H_c^i(X, A(j)) := \mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(k)}(M^c(X), A(j)[i]).$$

Here $\mathrm{DM}^-_{\mathrm{Nis}}(k)$ is Voevodsky's triangulated category of motives [Voevodsky et al. 2000, Chapter 5] and $\mathrm{CH}_j(X, i-2j; A)$ is Bloch's [1986] higher Chow group. We refer to motivic homology with compact supports as Borel–Moore homology. We index higher Chow groups "homologically", by dimension of cycles, contrary to the more common indexing by codimension of cycles. With this indexing $\mathrm{CH}_r(X, s; A)$ is a subquotient of the group of cycles of dimension $r+s$ in $X \times \Delta^s$ that intersect properly with all faces. The advantage of this convention is that we do not need to require X to be equidimensional. If X is equidimensional, we have $\mathrm{CH}_r(X, s; A) = \mathrm{CH}^{\dim X - r}(X, s; A)$.

There is a canonical isomorphism [Voevodsky et al. 2000, Chapter 5, Proposition 4.2.9; Mazza et al. 2006, Proposition 19.18]
$$\mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(k)}(\mathbb{Z}(j)[i], M^c(X)) \xrightarrow{\sim} \mathrm{CH}_j(X, i-2j)$$
if X is quasiprojective and k admits resolution of singularities in the sense of [Voevodsky et al. 2000, Chapter 4, Definition 3.4]. We chose to define Borel–Moore homology by Bloch's higher Chow groups mainly because they have localization sequences without assuming resolution of singularities. For motivic cohomology, it is known that if X is smooth and of pure dimension d, there is a canonical isomorphism [Mazza et al. 2006, Theorem 19.1]
$$H^i(X, A(j)) \xrightarrow{\sim} \mathrm{CH}_{d-j}(X, 2j-i; A) \overset{\mathrm{def}}{=} H^{\mathrm{BM}}_{2d-i}(X, A(d-j)).$$

The theories with and without compact supports agree for proper schemes, as the canonical morphism $M(X) \to M^c(X)$ becomes the identity. Moreover, if X is a smooth scheme of pure dimension d and k admits resolution of singularities, there is a canonical isomorphism [Voevodsky et al. 2000, Chapter 5, Theorem 4.3.7(3)]

$$H_i(X, \mathbb{Z}(j)) \cong H_c^{2d-i}(X, \mathbb{Z}(d-j)).$$

2. Borel–Moore homology

In this preliminary section, we review some results on Borel–Moore homology groups. Let us begin with an explicit computation of all negative motivic homology groups of a finite field, following [Kondo and Yasuda 2013, Remark 2.6].

Lemma 2.1. *Let i and j be negative integers. Then*

$$H_i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(j)) = \begin{cases} \mathbb{F}_{q^{-j}}^\times & \text{if } i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $i < j \le -1$,

$$H_i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(j)) \cong \mathrm{CH}_j(\operatorname{Spec} \mathbb{F}_q, i-2j) = 0$$

for dimension reasons. For $j \le i$, consider the long exact sequence

$$\cdots \to H^{-i-1}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}(-j)) \to H^{-i-1}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}/\mathbb{Z}(-j)) \\ \to H^{-i}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(-j)) \to H^{-i}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}(-j)) \to \cdots .$$

The first and the last terms vanish because

$$H^t(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}(-j)) = \mathrm{CH}_j(\mathbb{F}_q, -t-2j)_{\mathbb{Q}} \hookrightarrow K_{-t-2j}(\mathbb{F}_q)_{\mathbb{Q}} = 0$$

if $-t-2j \neq 0$. The embedding follows from Bloch's Riemann–Roch theorem [1986, Theorem 9.1] and the last equality by Quillen's calculation [1972, Theorem 8] of K-groups of finite fields. Now, because we are in the range $-i-1 < -j$, we may apply [Geisser and Levine 2000, Theorem 8.4] and [Geisser and Levine 2001, Corollary 1.2], whose hypotheses are satisfied by the theorem of Rost and Voevodsky, to obtain

$$\begin{aligned} H_i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(j)) &\cong H^{-i}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(-j)) \\ &\cong H^{-i-1}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}/\mathbb{Z}(-j)) \\ &\cong \bigoplus_l H^{-i-1}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j)) \\ &\cong \bigoplus_{l \neq p} H_{\text{ét}}^{-i-1}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j)). \end{aligned}$$

Hence, if $i \le -3$,

$$H_i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(j)) \cong \bigoplus_{l \neq p} H_{\text{ét}}^{-i-1}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j)) = 0,$$

for $\mathbb{F}_q$ has cohomological dimension 1.

If $i = -1$,

$$\begin{aligned} H_{-1}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Z}(j)) &\cong \bigoplus_{l \neq p} H_{\text{ét}}^{0}(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j)) \\ &\cong \bigoplus_{l \neq p} \mathbb{Z}/l^{r_l} \cong \mathbb{Z}/(q^{-j}-1) \cong \mathbb{F}_{q^{-j}}^{\times}, \end{aligned}$$

where r_l is the number such that $q^{-j} - 1 = \prod_l l^{r_l}$.

Finally, for $i = -2$, we need to show that the group $H_{\text{ét}}^1(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j))$ vanishes for an arbitrary prime $l \neq p$. Since this is a Galois cohomology group of a finite field with torsion coefficients, there is an exact sequence

$$\begin{aligned} 0 \to H_{\text{ét}}^0(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j)) \to \mathbb{Q}_l/\mathbb{Z}_l(-j) \\ \xrightarrow{\text{id} - \text{Frob}} \mathbb{Q}_l/\mathbb{Z}_l(-j) \to H_{\text{ét}}^1(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j)) \to 0. \end{aligned}$$

Since $\mathbb{Q}_l/\mathbb{Z}_l(-j)$ is a divisible group, one can easily see that the homomorphism $\text{id} - \text{Frob}$ is either zero or surjective. As we have seen above, the first term of this exact sequence is a finite group. Thus, $\text{id} - \text{Frob}$ is not the zero map, so it must be surjective. This shows the vanishing of $H_{\text{ét}}^1(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}_l/\mathbb{Z}_l(-j))$. □

Remark 2.2. Since motivic homology is defined for schemes of finite type over some base field, for Lemma 2.1 to make sense, we need to specify the base field of $\operatorname{Spec} \mathbb{F}_q$. However, as the proof shows, the lemma holds for any choice of the base field. More generally, see Lemma 4.6.

Let us further evaluate other relevant motivic invariants for later use.

Lemma 2.3. (i) $H^i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}(j)) = 0$ *unless* $(i, j) = (0, 0)$.

(ii) *If K is a finitely generated field of transcendence degree* 1 *over* $\mathbb{F}_q$, *then* $H^i(\operatorname{Spec} K, \mathbb{Q}(j)) = 0$ *unless* $(i, j) = (0, 0)$ *or* $(1, 1)$.

Proof. (i) For dimension reasons, the cohomology group in question vanishes unless $0 \le j$ and $i \le j$. By Bloch's Riemann–Roch theorem [1986, Theorem 9.1], there is an embedding $H^i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}(j)) \hookrightarrow K_{-i+2j}(\mathbb{F}_q)_{\mathbb{Q}}$. Since the positive degree K-groups of a finite field are torsion [Quillen 1972, Theorem 8], this implies $H^i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}(j)) = 0$ when $-i + 2j \ge 1$. Hence, the group $H^i(\operatorname{Spec} \mathbb{F}_q, \mathbb{Q}(j))$ vanishes unless $0 \le j$, $i \le j$ and $-i + 2j \le 0$, i.e., $(i, j) = (0, 0)$.

(ii) Let us first note that, for dimension reasons, the cohomology group in question vanishes unless $i \leq j$ and $j \geq 0$.

Let X be a smooth projective curve over $\mathbb{F}_q$ with function field K. For a nonempty open subscheme U of X, the localization sequence for higher Chow groups yields an exact sequence (because X, U and $X \setminus U$ are all smooth)

$$H^{i-2}(X\setminus U, \mathbb{Q}(j-1)) \to H^i(X, \mathbb{Q}(j)) \to H^i(U, \mathbb{Q}(j)) \to H^{i-1}(X\setminus U, \mathbb{Q}(j-1)).$$

If $j \neq 1$ or $i < j = 1$, the above two cohomology groups of $X \setminus U$ vanish by (i). Thus we obtain an isomorphism

$$H^i(X, \mathbb{Q}(j)) \xrightarrow{\sim} H^i(U, \mathbb{Q}(j)).$$

Taking the colimit over nonempty open subschemes U of X and applying [Mazza et al. 2006, Lemma 3.9], we obtain an isomorphism

$$H^i(X, \mathbb{Q}(j)) \xrightarrow{\sim} H^i(\operatorname{Spec} K, \mathbb{Q}(j)). \tag{2.4}$$

By Bloch's Riemann–Roch theorem, $H^i(X, \mathbb{Q}(j))$ is a subgroup of $K_{-i+2j}(X)_{\mathbb{Q}}$, and this group vanishes if $-i + 2j > 0$, by Harder's theorem [1977]. This means that $H^i(\operatorname{Spec} K, \mathbb{Q}(j)) = 0$ unless $(i, j) = (0, 0)$ or $(1, 1)$. □

Lemma 2.5. *Let X be a smooth curve over $\mathbb{F}_q$. Then $H^i(X, \mathbb{Q}(j)) = 0$ unless $(i, j) = (0, 0)$, $(1, 1)$ or $(2, 1)$.*

Proof. Let K be the function field of X. By the same argument used to construct the isomorphism (2.4) in the proof of Lemma 2.3(ii), we obtain for an arbitrary smooth curve X a canonical isomorphism

$$H^i(X, \mathbb{Q}(j)) \xrightarrow{\sim} H^i(\operatorname{Spec} K, \mathbb{Q}(j))$$

if $j \neq 1$ or $j = 1$ but $i \neq 1, 2$.

Thus, by Lemma 2.3(ii), $H^i(X, \mathbb{Q}(j)) = 0$ when $j \neq 1$ unless $(i, j) = (0, 0)$, and $H^i(X, \mathbb{Q}(j))$ also vanishes when $j = 1$ and $i \neq 1, 2$. Hence, the lemma follows. □

The next two lemmas are special cases of [Kondo and Yasuda 2013, Proposition 4.1, Lemma 4.2]. In that paper, a more general claim is proved under the assumption of Parshin's conjecture. The lemmas below are the part where Parshin's conjecture is not necessary. We include their proofs for the convenience of the reader.

Lemma 2.6 [Kondo and Yasuda 2013, Lemma 4.2]. *Let X be an irreducible scheme of dimension $d \geq 1$ over $\mathbb{F}_q$. Then, for $i, j \leq -1$ with $i - j \leq 1$,*

$$\operatorname{colim}_U H_i^{\mathrm{BM}}(U, \mathbb{Z}(j)) = 0,$$

where U runs through the set of all nonempty open subschemes of X.

Proof. Let K denote the function field of X. By definition, we have

$$\operatorname{colim}_U H_i^{\mathrm{BM}}(U, \mathbb{Z}(j)) = \operatorname{colim}_U \mathrm{CH}^{d-j}(U, i-2j) \cong \mathrm{CH}^{d-j}(\operatorname{Spec} K, i-2j).$$

Hence, for dimension reasons, $\operatorname{colim}_U H_i^{\mathrm{BM}}(U, \mathbb{Z}(j)) = 0$ if $d-j > i-2j$, i.e., $d > i-j$.

It remains to prove that $\mathrm{CH}^{d-j}(\operatorname{Spec} K, i-2j) = 0$ for $d \leq i-j$, i.e., when $d = 1$. In this case, observe that we have the equality $i = j+1$ and the inequality $j = i-1 \leq -2$. Hence,

$$\mathrm{CH}^{1-j}(\operatorname{Spec} K, i-2j) = \mathrm{CH}^{1-j}(\operatorname{Spec} K, 1-j) \cong K^M_{1-j}(K),$$

but the last group vanishes by the calculation of Milnor K-groups of degree ≥ 3 of a global field [Bass and Tate 1973, II, Theorem 2.1(3)]. □

Proposition 2.7 [Kondo and Yasuda 2013, Proposition 4.1]. *Let X be a scheme over $\mathbb{F}_q$. Then we have $H_i^{\mathrm{BM}}(X, \mathbb{Z}(j)) = 0$ if $i \leq -2$, $j \leq -1$ and $i - j \leq 1$.*

Proof. We prove this by induction on the dimension of X. We may suppose that X is a reduced scheme because the Borel–Moore homology groups of a scheme and its reduction are the same. Thus, when $\dim X = 0$, it is enough to show the claim for $\operatorname{Spec} \mathbb{F}_{q^n}$. This case was treated in Lemma 2.1 and Remark 2.2.

Suppose, now, that the proposition is true for dimension $\leq d-1$. Let us first prove the claim for an *irreducible* scheme X of dimension d. The localization sequence for Borel–Moore homology gives the exact sequence

$$\operatorname{colim}_Y H_i^{\mathrm{BM}}(Y, \mathbb{Z}(j)) \longrightarrow H_i^{\mathrm{BM}}(X, \mathbb{Z}(j)) \longrightarrow \operatorname{colim}_Y H_i^{\mathrm{BM}}(X \setminus Y, \mathbb{Z}(j)),$$

where Y runs through the set of reduced closed subschemes of X whose underlying sets are proper subsets of that of X. Since the first term is zero by the induction hypothesis and the last vanishes by Lemma 2.6, we obtain $H_i^{\mathrm{BM}}(X, \mathbb{Z}(j)) = 0$.

Now, for a general X, consider the abstract blowup

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & \coprod_n X_n \arrow[d, "f"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

where the X_n are the irreducible components of X and Z is the reduced closed subscheme of X where f is not an isomorphism. This gives rise to an exact sequence

$$H_i^{\mathrm{BM}}(Z, \mathbb{Z}(j)) \oplus \bigoplus_n H_i^{\mathrm{BM}}(X_n, \mathbb{Z}(j)) \xrightarrow{i_* - f_*} H_i^{\mathrm{BM}}(X, \mathbb{Z}(j)) \xrightarrow{\delta} H_{i-1}^{\mathrm{BM}}(Z', \mathbb{Z}(j)),$$

where δ denotes the connecting map. Hence, the vanishing of $H_i^{\mathrm{BM}}(X, \mathbb{Z}(j))$ follows from the induction hypothesis and the case of irreducible schemes. □

3. With $\mathbb{Q}$-coefficients

Before proving Theorem 1.1 in Section 4 under the assumption of resolution of singularities, we shall prove a weaker but unconditional result without assuming any conjectures. We use de Jong's alteration [1996] in place of smooth compactification and use results of Kelly [2012], in particular the existence of a localization sequence for motivic cohomology with compact supports with $\mathbb{Z}[1/p]$-coefficients.

Theorem 3.1. *For a smooth scheme X over $\mathbb{F}_q$ and $i, j \leq -1$ with $i - j \leq 1$,*

$$H_i(X, \mathbb{Q}(j)) = 0.$$

We need a lemma.

Lemma 3.2. *Suppose X is a scheme over $\mathbb{F}_q$ of dimension at most $d-1$. Then*

$$H_c^{2d-i}(X, \mathbb{Q}(d-j)) = 0$$

if $i, j \leq 0$ and $i - j \leq 2$.

Proof. First, observe that the result is true for all $d \geq 1$ if $\dim X = 0$, by Lemma 2.3(i).

We shall prove the lemma for a fixed $d = d_0$ by induction on $\dim X$. In doing so, we may suppose that the lemma is true for $d \leq d_0 - 1$ (by induction on d). Suppose that the lemma is true for schemes of dimension $\leq n-1$, and let X be a scheme of dimension n. There is a localization sequence [Kelly 2012, Proposition 5.5.5]

$$H_c^{2d_0-i}(X_{\mathrm{sm}}, \mathbb{Q}(d_0-j)) \to H_c^{2d_0-i}(X, \mathbb{Q}(d_0-j)) \to H_c^{2d_0-i}(X \setminus X_{\mathrm{sm}}, \mathbb{Q}(d_0-j)),$$

where X_{sm} is the smooth locus of X. The last term vanishes by the induction hypothesis. As for the first term, if we write $X_{\mathrm{sm}} = \coprod X_i$ (where the X_i are the connected components of X), we have

$$H_c^{2d_0-i}(X_{\mathrm{sm}}, \mathbb{Q}(d_0-j)) \cong \bigoplus_i H_c^{2d_0-i}(X_i, \mathbb{Q}(d_0-j)).$$

So, in order to show that the first term is zero, it suffices to show that

$$H_c^{2d_0-i}(X, \mathbb{Q}(d_0-j)) = 0$$

for a smooth integral scheme X of dimension n. (For schemes of smaller dimensions, the vanishing statement follows from the induction hypothesis.)

Now, let U be a nonempty open subscheme of X, and consider the localization sequence [Kelly 2012, Proposition 5.5.5]

$$\cdots \to H_c^{2d_0-i-1}(X \setminus U, \mathbb{Q}(d_0-j)) \to H_c^{2d_0-i}(U, \mathbb{Q}(d_0-j)) \\ \to H_c^{2d_0-i}(X, \mathbb{Q}(d_0-j)) \to H_c^{2d_0-i}(X \setminus U, \mathbb{Q}(d_0-j)) \to \cdots.$$

The last term vanishes by the induction hypothesis, and the first term too since the lemma is known for $d = d_0 - 1$ and since

$$H_c^{2d_0-i-1}(X \setminus U, \mathbb{Q}(d_0 - j)) = H_c^{2(d_0-1)-(i-1)}(X \setminus U, \mathbb{Q}((d_0 - 1) - (j - 1))).$$

Thus, we see that there is an isomorphism

$$H_c^{2d_0-i}(U, \mathbb{Q}(d_0 - j)) \xrightarrow{\sim} H_c^{2d_0-i}(X, \mathbb{Q}(d_0 - j)).$$

This means that in order to show the claim for X it is enough to prove it for some open subscheme U of X.

By de Jong's theorem [1996, Theorem 4.1], there is an alteration $\phi : X' \longrightarrow X$ and an open immersion $X' \hookrightarrow \widehat{X}'$ into a smooth projective integral scheme $\widehat{X}'$. There is a nonempty open subscheme U of X such that the projection

$$U' := U \times_X X' \xrightarrow{f} U$$

is finite and étale of degree $\delta = [k(U') : k(U)]$. The composition

$$H_c^{2d_0-i}(U, \mathbb{Q}(d_0 - j)) \xrightarrow{f^*} H_c^{2d_0-i}(U', \mathbb{Q}(d_0 - j)) \xrightarrow{f_*} H_c^{2d_0-i}(U, \mathbb{Q}(d_0 - j))$$

is multiplication by $\delta (\neq 0)$, so it is an isomorphism. In particular, f^* is injective. On the other hand, $H_c^{2d_0-i}(U', \mathbb{Q}(d_0 - j)) = 0$ because U' is an open subscheme of a smooth projective integral scheme $\widehat{X}'$ and

$$H_c^{2d_0-i}(\widehat{X}', \mathbb{Q}(d_0-j)) \cong H^{2d_0-i}(\widehat{X}', \mathbb{Q}(d_0-j)) \cong H^{\mathrm{BM}}_{2n-2d_0+i}(\widehat{X}', \mathbb{Q}(n-d_0+j)) = 0$$

by Proposition 2.7 (we used $i, j \leq 0$ and $i - j \leq 2$ here). Hence, by the injectivity of f^*, we conclude that $H_c^{2d_0-i}(U, \mathbb{Q}(d_0 - j)) = 0$. The lemma is proved. □

Proof of Theorem 3.1. We may assume that X is an integral scheme. Let us write $d := \dim X$. If U is an open subscheme of X, the associated localization sequence for motivic cohomology with compact supports [Kelly 2012, Proposition 5.5.5] gives an exact sequence

$$\begin{aligned} H_c^{2d-i-1}(X \setminus U, \mathbb{Q}(d - j)) &\longrightarrow H_i(U, \mathbb{Q}(j)) \\ &\longrightarrow H_i(X, \mathbb{Q}(j)) \longrightarrow H_c^{2d-i}(X \setminus U, \mathbb{Q}(d - j)). \end{aligned}$$

Here, we used [Kelly 2012, Theorem 5.5.14(3)]. By Lemma 3.2, the first and the last terms vanish, so we have an isomorphism

$$H_i(U, \mathbb{Q}(j)) \xrightarrow{\sim} H_i(X, \mathbb{Q}(j)).$$

As before, by de Jong's theorem, there is an alteration $\phi : X' \longrightarrow X$ and a nonempty open immersion $X' \hookrightarrow \widehat{X}'$ into a smooth projective integral scheme $\widehat{X}'$. There is an open subscheme U of X such that the projection

$$U' := U \times_X X' \xrightarrow{f} U$$

is a finite étale morphism of degree δ. The composition

$$H_i(U, \mathbb{Q}(j)) \xrightarrow{f^*} H_i(U', \mathbb{Q}(j)) \xrightarrow{f_*} H_i(U, \mathbb{Q}(j))$$

is multiplication by δ, so it is an isomorphism. In particular, f^* is injective. But $H_i(U', \mathbb{Q}(j)) = 0$ because U' is an open subscheme of a smooth projective integral scheme $\widehat{X}'$, for which we know $H_i(\widehat{X}', \mathbb{Q}(j)) = 0$ by Theorem 1.4 and Lemma 2.1. Hence, we obtain $H_i(X, \mathbb{Q}(j)) \cong H_i(U, \mathbb{Q}(j)) = 0$. This proves the theorem. □

4. Proof of Theorem 1.1

In the rest of this paper, we assume the existence of resolution of singularities in the sense of [Voevodsky et al. 2000, Chapter 4, Definition 3.4]. This assumption is needed even to deal with smooth schemes because our argument depends on the existence of smooth compactification. Alternatively, the reader may choose to assume that schemes have dimension at most 3.

Our proof of Theorem 1.1 goes as follows. We first prove the vanishing statement for $i \leq -2$. For smooth schemes, this is done by showing that a smooth compactification induces an isomorphism of motivic homology groups of certain indices (Proposition 4.2) and applying Kondo and Yasuda's result (Proposition 2.7). For a singular scheme, the statement is proved by induction on dimension using the abstract blowup sequence associated with a desingularization of the scheme (Proposition 4.3). In order to prove the statement for $i = -1$, we first deal with the smooth case by taking smooth compactification (Proposition 4.7). Then, combining these results, we construct the inverse to the canonical map $\alpha_X : H_{-1}(X, \mathbb{Z}(j)) \to H_{-1}(\pi_0(X), \mathbb{Z}(j))$ by using the universal property of a certain pushout diagram of motivic homology groups. We show that this diagram is indeed cocartesian by means of Galois cohomology, following the method of [Kondo and Yasuda 2013].

Lemma 4.1. *Let X be a scheme over $\mathbb{F}_q$ of dimension at most $d-1$. If* $\dim X = 0$,

$$H^{2d-i}(X, \mathbb{Z}(d-j)) = 0$$

for $i, j \leq 0$. If $\dim X \geq 1$, *a desingularization $\widetilde{X} \longrightarrow X$ of X induces isomorphisms*

$$H^{2d-i}(X, \mathbb{Z}(d-j)) \xrightarrow{\sim} H^{2d-i}(\widetilde{X}, \mathbb{Z}(d-j))$$

for $i, j \leq 1$ with $i - j \leq 2$.

Proof. If $\dim X = 0$, X_{red} is a finite disjoint union of spectra of finite fields over $\mathbb{F}_q$. Thus,

$$H^{2d-i}(X, \mathbb{Z}(d-j)) \cong H^{2d-i}(X_{\mathrm{red}}, \mathbb{Z}(d-j)) \cong H^{\mathrm{BM}}_{-2d+i}(X_{\mathrm{red}}, \mathbb{Z}(-d+j)) = 0.$$

The first isomorphism follows because the motive $M(X)$ is isomorphic to the motive $M(X_{\mathrm{red}})$, and the second because X_{red} is smooth. The last equality follows from Lemma 2.1 because $d \geq 1$ implies that $-2d+i \leq -2$ and $-d+j \leq -1$.

We prove the second assertion by induction on the dimension of X. Let Z be the closed subscheme of X on which $\widetilde{X} \longrightarrow X$ is not an isomorphism. Then the abstract blowup

$$\begin{tikzcd} Z' \arrow[r, "\mathrm{inc}'"] \arrow[d, "f'"'] & \widetilde{X} \arrow[d, "f"] \\ Z \arrow[r, "\mathrm{inc}"'] & X \end{tikzcd}$$

gives rise to a long exact sequence of motivic cohomology groups

$$\begin{aligned} \cdots \longrightarrow H^{2d-i-1}(Z', \mathbb{Z}(d-j)) \longrightarrow H^{2d-i}(X, \mathbb{Z}(d-j)) \\ \xrightarrow{(f^*,\mathrm{inc}^*)} H^{2d-i}(\widetilde{X}, \mathbb{Z}(d-j)) \oplus H^{2d-i}(Z, \mathbb{Z}(d-j)) \\ \xrightarrow{\mathrm{inc}'^* - f'^*} H^{2d-i}(Z', \mathbb{Z}(d-j)) \longrightarrow \cdots . \end{aligned}$$

In order to show that f^* is an isomorphism, we shall prove that the three cohomology groups of Z and Z' vanish.

If $\dim X = 1$, then $\dim Z = \dim Z' = 0$; so Z_{red} and Z'_{red} are finite disjoint unions of spectra of finite fields over $\mathbb{F}_q$. Hence, it suffices to observe that the first claim of the lemma implies, for any finite field $\mathbb{F}$ over $\mathbb{F}_q$, the following (note that we are in the range $d-1 \geq 1$):

$$\begin{aligned} H^{2d-i-1}(\operatorname{Spec}\mathbb{F}, \mathbb{Z}(d-j)) = H^{2(d-1)-(i-1)}(\operatorname{Spec}\mathbb{F}, \mathbb{Z}((d-1)-(j-1))) = 0, \\ H^{2d-i}(\operatorname{Spec}\mathbb{F}, \mathbb{Z}(d-j)) = H^{2(d-1)-(i-2)}(\operatorname{Spec}\mathbb{F}, \mathbb{Z}((d-1)-(j-1))) = 0. \end{aligned}$$

Now suppose that $\dim X \geq 2$ and assume that the lemma is known for schemes of smaller dimension. We shall again prove that the cohomology groups of Z and Z' in the above long exact sequence vanish. Let us prove that $H^{2d-i}(Z, \mathbb{Z}(d-j)) = 0$. Since $\dim Z < \dim X$, by the induction hypothesis, we have an isomorphism

$$H^{2d-i}(Z, \mathbb{Z}(d-j)) \xrightarrow{\simeq} H^{2d-i}(\widetilde{Z}, \mathbb{Z}(d-j)),$$

where $\widetilde{Z}$ is a desingularization of Z. Since $\widetilde{Z}$ is a smooth scheme, every connected component Z_r $(r = 1, \cdots, r_0)$ of $\widetilde{Z}$ is also smooth. By [Mazza et al. 2006, Theorem 19.1],

$$H^{2d-i}(\widetilde{Z}, \mathbb{Z}(d-j)) \cong \bigoplus_{r=1}^{r_0} H^{2d-i}(Z_r, \mathbb{Z}(d-j))$$

$$\cong \bigoplus_{r=1}^{r_0} H^{\mathrm{BM}}_{2\dim Z_r - 2d + i}(Z_r, \mathbb{Z}(\dim Z_r - d + j)).$$

The last group vanishes if $i \leq 2$, $j \leq 1$ and $i - j \leq 3$. Indeed, since $\dim Z_r \leq d-2$, when i and j satisfy these inequalities, we have

$$\begin{aligned} 2\dim Z_r - 2d + i &\leq 2(d-2) - 2d + i = i - 4 \leq -2, \\ \dim Z_r - d + j &\leq (d-2) - d + j = j - 2 \leq -1, \end{aligned}$$

and

$$\begin{aligned} (2\dim Z_r - 2d + i) - (\dim Z_r - d + j) &= \dim Z_r - d + i - j \\ &\leq i - j - 2 \leq 1. \end{aligned}$$

Hence, by Proposition 2.7, we obtain

$$\bigoplus_{r=1}^{r_0} H^{\mathrm{BM}}_{2\dim Z_r - 2d + i}(Z_r, \mathbb{Z}(\dim Z_r - d + j)) = 0.$$

Similarly, we can calculate $H^{2d-i}(Z', \mathbb{Z}(d-j)) = 0$ if $i \leq 2$, $j \leq 1$ and $i - j \leq 3$, and $H^{2d-i-1}(Z', \mathbb{Z}(d-j)) = 0$ if $i \leq 1$, $j \leq 1$ and $i - j \leq 2$. Therefore, the lemma follows. $\square$

Proposition 4.2. *Suppose X is a connected smooth scheme of dimension d over $\mathbb{F}_q$. Then a smooth compactification $X \hookrightarrow X'$ of X induces isomorphisms*

$$H_i(X, \mathbb{Z}(j)) \xrightarrow{\sim} H_i(X', \mathbb{Z}(j))$$

for all $i \leq -1$ and $j \leq 0$ with $i - j \leq 1$.

Proof. There is nothing to prove if $d = 0$, so we deal with the case where $d \geq 1$. Since there is a localization sequence

$$\begin{aligned} H^{2d-i-1}_c(X' \setminus X, \mathbb{Z}(d-j)) &\longrightarrow H_i(X, \mathbb{Z}(j)) \\ &\longrightarrow H_i(X', \mathbb{Z}(j)) \longrightarrow H^{2d-i}_c(X' \setminus X, \mathbb{Z}(d-j)) \end{aligned}$$

and $X' \setminus X$ is proper, it suffices to show $H^{2d-i-1}(X' \setminus X, \mathbb{Z}(d-j)) = 0$ and $H^{2d-i}(X' \setminus X, \mathbb{Z}(d-j)) = 0$. We prove this for the first group. The proof for the second group is identical. First, note that by the first assertion of Lemma 4.1, we may assume that the irreducible components of $X' \setminus X$ have nonzero dimension. Now, let $\widetilde{X' \setminus X}$ be a desingularization of $X' \setminus X$ and write its decomposition into connected components as $\widetilde{X' \setminus X} = \coprod_{s=1}^{s_0} X_s$. With the second assertion of

Lemma 4.1, we can calculate

$$\begin{aligned} H^{2d-i-1}(X'\setminus X,\mathbb{Z}(d-j)) &\cong H^{2d-i-1}(\widetilde{X'\setminus X},\mathbb{Z}(d-j)) \\ &\cong \bigoplus_{s=1}^{s_0} H^{2d-i-1}(X_s,\mathbb{Z}(d-j)) \\ &\cong \bigoplus_{s=1}^{s_0} H^{\mathrm{BM}}_{2\dim X_s-2d+i+1}(X_s,\mathbb{Z}(\dim X_s-d+j)), \end{aligned}$$

and the last group vanishes if $i\le -1$, $j\le 0$ and $i-j\le 1$ by Proposition 2.7. $\square$

We are now able to prove the first half of Theorem 1.1.

Proposition 4.3. *Let X be a scheme over $\mathbb{F}_q$. Then $H_i(X,\mathbb{Z}(j))=0$ if $i\le -2$, $j\le -1$ and $i-j\le 1$.*

Proof. If $\dim X=0$, the proposition holds by Lemma 2.1. Let us assume that $\dim X\ge 1$ and prove the proposition by induction on $\dim X$. Let Z be a closed subscheme of X which contains all singular points of X and has dimension less than that of X. The abstract blowup

$$\begin{tikzcd} Z' \arrow[r, "\mathrm{inc}'"] \arrow[d, "f'"'] & \widetilde{X} \arrow[d, "f"] \\ Z \arrow[r, "\mathrm{inc}"'] & X \end{tikzcd}$$

gives rise to a long exact sequence

$$\begin{aligned} H_i(Z',\mathbb{Z}(j)) \xrightarrow{(f'_*,\mathrm{inc}'_*)} H_i(Z,\mathbb{Z}(j))\oplus H_i(\widetilde{X},\mathbb{Z}(j)) \\ \xrightarrow{\mathrm{inc}_*-f_*} H_i(X,\mathbb{Z}(j)) \longrightarrow H_{i-1}(Z',\mathbb{Z}(j)). \end{aligned}$$

By the induction hypothesis,

$$H_i(Z',\mathbb{Z}(j))=0,\quad H_i(Z,\mathbb{Z}(j))=0\quad\text{and}\quad H_{i-1}(Z',\mathbb{Z}(j))=0.$$

Hence,

$$H_i(X,\mathbb{Z}(j))\cong H_i(\widetilde{X},\mathbb{Z}(j))\cong H_i(\widetilde{X}',\mathbb{Z}(j))=0,$$

where $\widetilde{X}'$ denotes a smooth compactification of $\widetilde{X}$, the second isomorphism follows from Proposition 4.2, and the last group vanishes by Proposition 2.7. $\square$

Next we shall consider the case where $i=-1$.

Lemma 4.4. *Let X be a geometrically connected scheme over a field k and $i: X\hookrightarrow X'$ be a compactification, i.e., an open immersion into a proper scheme X' with dense image. Then X' is geometrically connected over k.*

Proof. Since X' is connected, it is enough to show that $\pi_0(X')$ has a k-rational point [Liu 2002, Chapter 10, Corollary 2.21(a)]. Now, i induces a k-morphism $\pi_0(X) \longrightarrow \pi_0(X')$. Since X is geometrically connected over a field, $\pi_0(X) = \operatorname{Spec} k$. So this morphism defines a k-rational point on $\pi_0(X')$. □

Remark 4.5. With the same notation, the above proof shows that $\pi_0(X) = \pi_0(X')$.

We need the independence of motivic homology from the choice of the base field.

Lemma 4.6. *If l/k is a finite extension of fields and X is a scheme of finite type over l, we have a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(k)}(\mathbb{Z}(j)[i], M(X)) \cong \mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(l)}(\mathbb{Z}(j)[i], M(X))$$

for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\leq 0}$, where, on the left-hand side, X is regarded as a scheme over k by the composition $X \xrightarrow{\mathrm{str}} \operatorname{Spec} l \to \operatorname{Spec} k$.

Proof. For $j < 0$, by [Mazza et al. 2006, Corollary 15.3] and [Voevodsky 2010, Corollary 4.10], there is an isomorphism

$$\mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(l)}(\mathbb{Z}(j)[i], M(X)) \cong H_{i-2j-1}\left(\frac{\mathrm{Cor}_l(\Delta^*_l, X \times_l (\mathbb{A}^{-j}_l - \{0\}))}{\mathrm{Cor}_l(\Delta^*_l, X \times_l \{1\})}\right),$$

where Cor denotes the group of finite correspondences. Since l is a finite extension of k, if S is a scheme over l and T is over k, we have $\mathrm{Cor}_l(T \times_k l, S) \cong \mathrm{Cor}_k(T, S)$. Hence, the right-hand side is isomorphic to

$$H_{i-2j-1}\left(\frac{\mathrm{Cor}_k(\Delta^*_k, X \times_k (\mathbb{A}^{-j}_k - \{0\}))}{\mathrm{Cor}_k(\Delta^*_k, X \times_k \{1\})}\right),$$

which is, in turn, isomorphic to $\mathrm{Hom}_{\mathrm{DM}^-_{\mathrm{Nis}}(k)}(\mathbb{Z}(j)[i], M(X))$. □

Proposition 4.7. *If X is smooth over $\mathbb{F}_q$, there are canonical isomorphisms*

$$\phi : H_{-1}(X, \mathbb{Z}(j)) \xrightarrow{\sim} H_{-1}(\pi_0(X), \mathbb{Z}(j))$$

for $j = -1$ and -2.

Proof. We may assume that X is connected and regard it as a scheme over $\pi_0(X)$ by Lemma 4.6. Now X is geometrically connected as a scheme over $\pi_0(X)$, so its smooth compactification X' is also smooth over $\pi_0(X)$ by Lemma 4.4. Now, the map ϕ fits in the commutative diagram

$$\begin{tikzcd}
H_{-1}(X, \mathbb{Z}(j)) \arrow[r, "\phi"] \arrow[d, "\sim"'] & H_{-1}(\pi_0(X), \mathbb{Z}(j)) \arrow[d, equal] \\
H_{-1}(X', \mathbb{Z}(j)) \arrow[r, "\sim"] & H_{-1}(\pi_0(X'), \mathbb{Z}(j))
\end{tikzcd}$$

where the left vertical map is an isomorphism by Proposition 4.2, the bottom horizontal map is an isomorphism by Theorem 1.4 and the right vertical equality follows from Remark 4.5. Thus, ϕ is an isomorphism. □

In order to compute motivic homology of singular schemes, we shall now study how motivic homology groups behave under resolution of singularities.

Lemma 4.8 [Kondo and Yasuda 2013, Lemma 2.7]. *For two finite fields $\mathbb{F}_{q^n} \subset \mathbb{F}_{q^m}$, the canonical map*

$$H_{-1}(\operatorname{Spec}\mathbb{F}_{q^m}, \mathbb{Z}(j)) \longrightarrow H_{-1}(\operatorname{Spec}\mathbb{F}_{q^n}, \mathbb{Z}(j))$$

is surjective if $j < 0$.

Proof. As we have seen in the proof of Lemma 2.1, the cycle class map gives an isomorphism

$$H_{-1}(\operatorname{Spec}\mathbb{F}, \mathbb{Z}(j)) \cong \bigoplus_{l \neq p} H^0_{\text{ét}}(\operatorname{Spec}\mathbb{F}, \mathbb{Q}_l/\mathbb{Z}_l(-j))$$

for $j \leq -1$ and a finite field $\mathbb{F}$. Now, the cycle class map is compatible with the pushforward along a finite morphism [Geisser and Levine 2001, Lemma 3.5(2)], so the surjectivity follows from the corresponding statement for étale cohomology [Soulé 1979, Lemma 6(iii), p. 269 and IV.1.7, p. 283]. □

Lemma 4.9. *Let X be a scheme over $\mathbb{F}_q$, $f : \widetilde{X} \longrightarrow X$ be a desingularization, and $j = -1$ or -2. Then the map*

$$f_* : H_{-1}(\widetilde{X}, \mathbb{Z}(j)) \longrightarrow H_{-1}(X, \mathbb{Z}(j))$$

is surjective.

Proof. We prove this by induction on the dimension of X. Let Z be a closed subscheme of X which contains all singularities of X and has dimension less than that of X. The abstract blowup

$$\begin{tikzcd} Z' \arrow[r, "\mathrm{inc}'"] \arrow[d, "f'"'] & \widetilde{X} \arrow[d, "f"] \\ Z \arrow[r, "\mathrm{inc}"'] & X \end{tikzcd}$$

gives rise to a long exact sequence

$$H_{-1}(Z', \mathbb{Z}(j)) \xrightarrow{(f'_*, \mathrm{inc}'_*)} H_{-1}(Z, \mathbb{Z}(j)) \oplus H_{-1}(\widetilde{X}, \mathbb{Z}(j)) \xrightarrow{\mathrm{inc}_* - f_*} H_{-1}(X, \mathbb{Z}(j)) \xrightarrow{\delta} H_{-2}(Z', \mathbb{Z}(j)) = 0,$$

where δ is the connecting map. The last term vanishes by Proposition 4.3. By an easy diagram chase, in order to show the surjectivity of f_*, it is enough to show the surjectivity of f'_*. Let us write $Z = \bigcup Z_i$, where Z_i are the irreducible components of Z, and let $\widetilde{Z}_i$ be a desingularization of Z_i and $p : \coprod \widetilde{Z}_i \longrightarrow \bigcup Z_i$ be the morphism induced by the desingularizations. Note that p is then a desingularization of Z. For each index i, choose a closed point $x_i \in \widetilde{Z}_i$. Let $y_i := p(x_i) \in Z_i \subset Z$ be the image of x_i under p. Since f' is surjective, there is a closed point $z_i \in Z'$ with $f'(z_i) = y_i$ for each i. Choose some finite field extension $\mathbb{F}$ of $\mathbb{F}_q$ which contains all residue fields $k(x_i)$, $k(y_i)$ and $k(z_i)$. The inclusions of these residue fields into $\mathbb{F}$ give rise to $\mathbb{F}$-rational points

$$\begin{aligned} x_i &: \operatorname{Spec} \mathbb{F} \longrightarrow \operatorname{Spec} k(x_i) \longrightarrow X, \\ y_i &: \operatorname{Spec} \mathbb{F} \longrightarrow \operatorname{Spec} k(y_i) \longrightarrow X, \\ z_i &: \operatorname{Spec} \mathbb{F} \longrightarrow \operatorname{Spec} k(z_i) \longrightarrow X, \end{aligned}$$

which are, with an abuse of notation, denoted by the same letters x_i, y_i and z_i. These points give the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "f'"] & Z = \bigcup Z_i \\ \coprod_i \operatorname{Spec} \mathbb{F} \arrow[u, "\coprod z_i"] \arrow[ur, "\coprod y_i"] \arrow[r, "\coprod x_i"'] & \coprod \widetilde{Z}_i \arrow[u, "p"'] \end{tikzcd}$$

Taking homology groups, we obtain

$$\begin{tikzcd} H_{-1}(Z', \mathbb{Z}(j)) \arrow[r, "f'_*"] & H_{-1}(Z, \mathbb{Z}(j)) \\ \bigoplus H_{-1}(\operatorname{Spec} \mathbb{F}, \mathbb{Z}(j)) \arrow[u, "\oplus z_{i*}"] \arrow[ur, "\oplus y_{i*}"] \arrow[r, "\oplus x_{i*}"'] \arrow[d, "\text{Lemma 4.8}"'] & \bigoplus H_{-1}(\widetilde{Z}_i, \mathbb{Z}(j)) \arrow[u, "\text{induction hypothesis}"'] \arrow[d, "\text{Proposition 4.7}", "\sim"'] \\ \bigoplus H_{-1}(\operatorname{Spec} k(x_i), \mathbb{Z}(j)) \arrow[r, two heads, "\text{Lemma 4.8}"'] & \bigoplus H_i(\pi_0(\widetilde{Z}_i), \mathbb{Z}(j)) \end{tikzcd}$$

Hence f'_* is surjective. $\square$

The next lemma compares the motivic homology of a given scheme with the motivic homology of one of its irreducible components.

Lemma 4.10. *Let X be a connected scheme over $\mathbb{F}_q$ and X_1 be an irreducible component. If $j = -1$ or -2, the inclusion of X_1 into X induces a surjection*

$$H_{-1}(X_1, \mathbb{Z}(j)) \longrightarrow H_{-1}(X, \mathbb{Z}(j)).$$

Proof. Let us write $X = X_1 \cup X_2 \cup \cdots \cup X_r$, where the X_i are the irreducible components of X. Since the lemma is obvious for $r = 1$, we assume that $r > 1$ below. The abstract blowup

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[d] & X_r \arrow[d] \\ \bigcup_{i \le r-1} X_i \arrow[r, "\psi"'] & X \end{tikzcd}$$

(all the maps in the diagram are inclusions) gives an exact sequence

$$\begin{aligned} H_{-1}(Z, \mathbb{Z}(j)) \longrightarrow H_{-1}(X_r, \mathbb{Z}(j)) \oplus H_{-1}\big(\textstyle\bigcup_{i \le r-1} X_i, \mathbb{Z}(j)\big) \\ \longrightarrow H_{-1}(X, \mathbb{Z}(j)) \longrightarrow H_{-2}(Z, \mathbb{Z}(j)) = 0, \end{aligned}$$

where the last equality comes from Proposition 4.3. By induction on the number of irreducible components of X, it suffices to prove the surjectivity of $\psi_* : H_{-1}\big(\bigcup_{i \le r-1} X_i, \mathbb{Z}(j)\big) \longrightarrow H_{-1}(X, \mathbb{Z}(j))$, which, in turn, follows from the surjectivity of $\phi_* : H_{-1}(Z, \mathbb{Z}(j)) \longrightarrow H_{-1}(X_r, \mathbb{Z}(j))$.

Since X is connected, Z is not empty. In particular, it has a closed point, say $z \in Z$. Choose a desingularization $\pi : \widetilde{X}_r \longrightarrow X_r$ and let a closed point $\widetilde{w} \in \widetilde{X}_r$ be a preimage of $w := \phi(z) \in X_r$. Choose some finite field extension $\mathbb{F}$ of $\mathbb{F}_q$ containing all the residue fields $k(z)$, $k(w)$ and $k(\widetilde{w})$, and regard z, w and $\widetilde{w}$ as $\mathbb{F}$-rational points. Now, there is a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\phi"] & X_r \\ \operatorname{Spec} \mathbb{F} \arrow[u, "z"] \arrow[ur, "w"] \arrow[r, "\widetilde{w}"'] & \widetilde{X}_r \arrow[u, "\pi"'] \end{tikzcd}$$

Passing to homology groups, we obtain (noting that $\mathcal{O}(\pi_0(\widetilde{X}_r)) \subset k(\widetilde{w}) \subset \mathbb{F}$) the commutative diagram

$$\begin{tikzcd} H_{-1}(Z, \mathbb{Z}(j)) \arrow[r, "\phi_*"] & H_{-1}(X_r, \mathbb{Z}(j)) \\ H_{-1}(\operatorname{Spec} \mathbb{F}, \mathbb{Z}(j)) \arrow[u, "z_*"] \arrow[r, "\widetilde{w}_*"] \arrow[dr, two heads, "\text{Lemma 4.8}"'] & H_{-1}(\widetilde{X}_r, \mathbb{Z}(j)) \arrow[u, two heads, "\text{Lemma 4.9}"'] \arrow[d, "\sim"', "\text{Proposition 4.7}"] \\ & H_{-1}(\pi_0(\widetilde{X}_r), \mathbb{Z}(j)) \end{tikzcd}$$

Hence, ϕ_* is surjective. $\square$

Proof of Theorem 1.1. We have already proved the first half in Proposition 4.3. It remains to prove the second half, i.e., the following statement:

> *Let X be an arbitrary scheme over $\mathbb{F}_q$ and $j = -1$ or -2. Then the canonical map $\alpha_X : H_{-1}(X, \mathbb{Z}(j)) \longrightarrow H_{-1}(\pi_0(X), \mathbb{Z}(j))$ is an isomorphism.*

If $\dim X = 0$, then α_X is clearly an isomorphism (because we may assume X to be a disjoint union of reduced schemes, i.e., a union of spectra of finite fields).

We prove the theorem by induction on the dimension of X. Assume that the theorem holds for schemes of dimension at most $d-1$. We prove the assertion for a d-dimensional scheme X. By Lemma 4.6, we may assume without loss of generality that X is geometrically connected and reduced. Choose a nonempty closed subscheme Z of X such that $X \setminus Z$ is smooth and $\dim Z < \dim X$. First, we claim that the inclusion $Z \hookrightarrow X$ induces a surjection

$$\beta : H_{-1}(Z, \mathbb{Z}(j)) \longrightarrow H_{-1}(X, \mathbb{Z}(j)).$$

Indeed, there is some irreducible component, call it X_1, of X such that $Z \cap X_1 \neq \varnothing$. Let $\widetilde{X}_1$ be a desingularization of X_1. Choose a closed point $x \in Z \cap X_1$ and its preimage $\tilde{x} \in \widetilde{X}_1$. Let $\mathbb{F}$ be a sufficiently large finite field that contains both residue fields $k(x)$ and $k(\tilde{x})$. (Note that $k(\tilde{x})$ contains $\pi_0(\widetilde{X}_1)$.) Then there is a commutative diagram

$$\begin{tikzcd}
 & H_{-1}(\pi_0(\widetilde{X}_1), \mathbb{Z}(j)) \\
 & H_{-1}(\widetilde{X}_1, \mathbb{Z}(j)) \arrow[u, "\sim"', "\text{Proposition 4.7}"] \arrow[d, "\text{Lemma 4.9}"] \\
H_{-1}(\operatorname{Spec} \mathbb{F}, \mathbb{Z}(j)) \arrow[uur, "\text{Lemma 4.8}"] \arrow[ur, "\tilde{x}_*"] \arrow[r, "x_*"] \arrow[d, "x_*"'] & H_{-1}(X_1, \mathbb{Z}(j)) \arrow[d, "\text{Lemma 4.10}"] \\
H_{-1}(Z, \mathbb{Z}(j)) \arrow[r, "\beta"'] & H_{-1}(X, \mathbb{Z}(j))
\end{tikzcd}$$

The commutativity of the diagram implies the surjectivity of β.

Next, consider the commutative diagram

$$\begin{tikzcd}
H_{-1}(Z, \mathbb{Z}(j)) \arrow[r, two heads, "\beta"] \arrow[d, "\alpha_Z"', "\sim,\ \text{ind. hypothesis}"] & H_{-1}(X, \mathbb{Z}(j)) \arrow[d, "\alpha_X"] \\
H_{-1}(\pi_0(Z), \mathbb{Z}(j)) \arrow[r, two heads, "\gamma", "\text{Lemma 4.8}"'] & H_{-1}(\pi_0(X), \mathbb{Z}(j))
\end{tikzcd}$$

In order to show that α_X is an isomorphism, it is enough to show its injectivity, for the surjectivity is obvious from the diagram. The injectivity follows once one constructs a group homomorphism

$$l : H_{-1}(\pi_0(X), \mathbb{Z}(j)) \longrightarrow H_{-1}(X, \mathbb{Z}(j))$$

such that $l \circ \gamma \circ \alpha_Z = \beta$, because then the surjectivity of β and the equalities $l \circ \alpha_X \circ \beta = l \circ \gamma \circ \alpha_Z = \beta$ imply that $l \circ \alpha_X = \mathrm{id}$. (In fact, l is the inverse to α_X because we also have $\alpha_X \circ l \circ \gamma = \alpha_X \circ \beta \circ \alpha_Z^{-1} = \gamma$, and the surjectivity of γ implies that $\alpha_X \circ l = \mathrm{id}$.)

The existence of such a map l follows if one shows that the square in the following diagram is cocartesian:

$$\begin{tikzcd}
H_{-1}(\pi_0(Y), \mathbb{Z}(j)) \arrow[r] \arrow[d] & H_{-1}(\pi_0(\widetilde{X}), \mathbb{Z}(j)) \arrow[d, "p_*"] \arrow[ddr, "p_* \circ \alpha_{\widetilde{X}}^{-1}"] & \\
H_{-1}(\pi_0(Z), \mathbb{Z}(j)) \arrow[r, "\gamma"'] \arrow[drr, bend right, "\beta \circ \alpha_Z^{-1}"'] & H_{-1}(\pi_0(X), \mathbb{Z}(j)) \arrow[dr, dotted, "\exists! l"] & \\
& & H_{-1}(X, \mathbb{Z}(j))
\end{tikzcd}$$

Here, $p : \widetilde{X} \longrightarrow X$ is a desingularization of X, $Y := (\widetilde{X} \times_X Z)_{\mathrm{red}}$ and γ is the map induced by the canonical morphism $\pi_0(Z) \longrightarrow \pi_0(X)$. Indeed, the map l defined by universality in the above diagram satisfies $l \circ \gamma \circ \alpha_Z = \beta$ by its definition. Note that $\alpha_{\widetilde{X}}^{-1}$ makes sense because $\alpha_{\widetilde{X}}$ is an isomorphism by Proposition 4.7, and so does α_Z^{-1} by the induction hypothesis.

Since for a zero-dimensional $\mathbb{F}_q$-scheme S and $j \le -1$ there is an isomorphism

$$H_{-1}(S, \mathbb{Z}(j)) \cong \bigoplus_{l \neq p} H^0_{\text{ét}}(S, \mathbb{Q}_l/\mathbb{Z}_l(-j))$$

that is functorial with respect to pushforward along finite morphisms [Geisser and Levine 2001, Lemma 3.5(2)] induced by the Geisser–Levine cycle map, it is enough to show that the diagram

$$\begin{tikzcd}
H^0_{\text{ét}}(\pi_0(Y), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \arrow[r] \arrow[d] & H^0_{\text{ét}}(\pi_0(\widetilde{X}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \arrow[d] \\
H^0_{\text{ét}}(\pi_0(Z), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \arrow[r] & H^0_{\text{ét}}(\pi_0(X), \mathbb{Q}_l/\mathbb{Z}_l(-j))
\end{tikzcd} \tag{$*$}$$

is cocartesian for all primes $l \neq p$. (Here, the arrows are pushforward maps along finite morphisms.)

Now, consider the diagram

$$\begin{tikzcd} H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \arrow[r, "a"] \arrow[d, "b"'] & H^0_{\text{ét}}(\pi_0(\bar{\widetilde{X}}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \arrow[d, "c"] \\ H^0_{\text{ét}}(\pi_0(\bar{Z}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \arrow[r, "d"'] & H^0_{\text{ét}}(\pi_0(\bar{X}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \end{tikzcd} \tag{$**$}$$

where $\bar{}$ indicates the base change to the algebraic closure $\bar{\mathbb{F}}_q$; for example, $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.

Let us for the moment assume that the diagram ($**$) is cocartesian in the category of $G(\bar{\mathbb{F}}_q/\mathbb{F}_q)$-modules and that the module

$$N := \ker\{H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \xrightarrow{(a,b)} H^0_{\text{ét}}(\pi_0(\bar{\widetilde{X}}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \oplus H^0_{\text{ét}}(\pi_0(\bar{Z}), \mathbb{Q}_l/\mathbb{Z}_l(-j))\}$$

is divisible. We shall show that the diagram ($*$) is cocartesian under these assumptions. Since the diagram ($**$) is a pushout, there is an exact sequence of $G(\bar{\mathbb{F}}_q/\mathbb{F}_q)$-modules

$$0 \longrightarrow H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N \xrightarrow{(a,b)} H^0_{\text{ét}}(\pi_0(\bar{\widetilde{X}}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \oplus H^0_{\text{ét}}(\pi_0(\bar{Z}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \xrightarrow{c-d} H^0_{\text{ét}}(\pi_0(\bar{X}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \longrightarrow 0,$$

where (a, b) is, of course, the quotient map induced by the map (a, b) defined on $H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))$. Taking Galois cohomology of $G(\bar{\mathbb{F}}_q/\mathbb{F}_q)$-modules, we obtain the long exact sequence

$$0 \to (H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N)^{G(\bar{\mathbb{F}}_q/\mathbb{F}_q)} \to H^0_{\text{ét}}(\pi_0(\widetilde{X}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \oplus H^0_{\text{ét}}(\pi_0(Z), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \to H^0_{\text{ét}}(\pi_0(X), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \to H^1(G(\bar{\mathbb{F}}_q/\mathbb{F}_q), H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N) \to \cdots .$$

Since $j \neq 0$, the Frobenius automorphism acts nontrivially on the divisible group $H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N$, which is just a direct sum of copies of the divisible group $\mathbb{Q}_l/\mathbb{Z}_l(-j)/N$. By the same reasoning as in the last part of the proof of Lemma 2.1, we conclude that

$$H^1(G(\bar{\mathbb{F}}_q/\mathbb{F}_q), H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N) = 0.$$

Similarly, the short exact sequence

$$0 \longrightarrow N \longrightarrow H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \longrightarrow H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N \longrightarrow 0$$

gives rise to a long exact sequence in Galois cohomology

$$
\begin{aligned}
0 \longrightarrow N^{G(\overline{\mathbb{F}}_q/\mathbb{F}_q)} &\longrightarrow H^0_{\text{ét}}(\pi_0(Y), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \\
&\longrightarrow (H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N)^{G(\overline{\mathbb{F}}_q/\mathbb{F}_q)} \longrightarrow H^1(G(\overline{\mathbb{F}}_q/\mathbb{F}_q), N) = 0.
\end{aligned}
$$

The last term vanishes because N is assumed divisible and the Galois action is nontrivial if $N \neq 0$. (Since N must have infinite cardinality, a trivial Galois action would imply that $N' = N^{G(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$ is infinite, but this would contradict the fact that N' is a subgroup of the finite group $H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))^{G(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$.) In particular, the map

$$
H^0_{\text{ét}}(\pi_0(Y), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \longrightarrow (H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Q}_l/\mathbb{Z}_l(-j))/N)^{G(\overline{\mathbb{F}}_q/\mathbb{F}_q)}
$$

is surjective.

Combining all these, we obtain an exact sequence

$$
\begin{aligned}
H^0_{\text{ét}}(\pi_0(Y), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \longrightarrow H^0_{\text{ét}}(\pi_0(\widetilde{X}), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \oplus H^0_{\text{ét}}(\pi_0(Z), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \\
\longrightarrow H^0_{\text{ét}}(\pi_0(X), \mathbb{Q}_l/\mathbb{Z}_l(-j)) \longrightarrow 0.
\end{aligned}
$$

This means that the diagram $(*)$ is cocartesian.

It now remains to prove that the diagram $(**)$ is cocartesian and N is a divisible group. Using the Pontryagin duality

$$
H^0_{\text{ét}}(T, \mathbb{Z}_l(j)) \cong \operatorname{Hom}_{\mathbb{Z}}(H^0_{\text{ét}}(T, \mathbb{Q}_l/\mathbb{Z}_l(-j)), \mathbb{Q}/\mathbb{Z})
$$

for a zero-dimensional scheme T over $\overline{\mathbb{F}}_q$, obtained by taking the inverse limit over r of the duality

$$
H^0_{\text{ét}}(T, \mathbb{Z}/l^r(j)) \cong \operatorname{Hom}_{\mathbb{Z}}(H^0_{\text{ét}}(T, \mathbb{Z}/l^r(-j)), \mathbb{Q}/\mathbb{Z}),
$$

we can see that it suffices to prove that the diagram with pullback homomorphisms

$$
\begin{tikzcd}
H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Z}_l(j)) & H^0_{\text{ét}}(\pi_0(\bar{\bar{X}}), \mathbb{Z}_l(j)) \arrow[l, "a'"'] \\
H^0_{\text{ét}}(\pi_0(\bar{Z}), \mathbb{Z}_l(j)) \arrow[u, "b'"] & H^0_{\text{ét}}(\pi_0(\bar{X}), \mathbb{Z}_l(j)) \arrow[l] \arrow[u]
\end{tikzcd}
$$

is cartesian and the cokernel of

$$
H^0_{\text{ét}}(\pi_0(\bar{\bar{X}}), \mathbb{Z}_l(j)) \oplus H^0_{\text{ét}}(\pi_0(\bar{Z}), \mathbb{Z}_l(j)) \xrightarrow{a'+b'} H^0_{\text{ét}}(\pi_0(\bar{Y}), \mathbb{Z}_l(j))
$$

is torsion-free. But, since there are canonical isomorphisms

$$
H^0_{\text{ét}}(\bar{X}, \mathbb{Z}_l(j)) \cong \operatorname{Hom}_{\text{Set}}(\pi_0(\bar{X}), \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(j)
$$

of $G(\bar{\mathbb{F}}_q/\mathbb{F}_q)$-modules (observe that these groups are just direct sums of $\mathbb{Z}_l(j)$ with one summand for each connected component of $\bar{X}$), it boils down to showing that the diagram

$$\begin{tikzcd} \pi_0(\bar{Y}) \arrow[r, "\phi"] \arrow[d, "\psi"'] & \pi_0(\bar{\bar{X}}) \arrow[d] \\ \pi_0(\bar{Z}) \arrow[r] & \pi_0(\bar{X}) \end{tikzcd}$$

where ϕ and ψ are the canonical maps, is cocartesian in the category of sets and the cokernel of the map

$$\mathrm{Hom}_{\mathrm{Set}}(\pi_0(\bar{\bar{X}}), \mathbb{Z}_l) \oplus \mathrm{Hom}_{\mathrm{Set}}(\pi_0(\bar{Z}), \mathbb{Z}_l) \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(\pi_0(\bar{Y}), \mathbb{Z}_l)$$

sending (f, g) to $f \circ \phi + g \circ \psi$ is torsion-free. The claim about the cokernel is straightforward. (The proof can be found in [Kondo and Yasuda 2013, Lemma 3.3].)

Let us prove the assertion on the square diagram. Because the map ψ is surjective and $\pi_0(\bar{X})$ consists of one element as we are working with a geometrically connected scheme X, it suffices to show that any two elements x_1 and x_2 in $\pi_0(\bar{\bar{X}})$ are related by the equivalence relation generated by the relation $\sim$ on $\pi_0(\bar{\bar{X}})$ defined by $s \sim s'$ if there are $t, t' \in \pi_0(\bar{Y})$ such that $\phi(t) = s$, $\phi(t') = s'$ and $\psi(t) = \psi(t')$. In order to prove this, we may assume that x_1 and x_2 in $\pi_0(\bar{\bar{X}})$ correspond to irreducible components C_1 and C_2 of $\bar{X}$ with nonempty intersection $C_1 \cap C_2$. (If x_1 and x_2 correspond to irreducible elements C_1 and C_2 with empty intersection, choose a sequence of elements $x_1 = s_1, s_2, \ldots, s_{r-1}, s_r = x_2 \in \pi_0(\bar{\bar{X}})$ such that their corresponding irreducible components $C_1 = S_1, S_2, \ldots, S_{r-1}, S_r = C_2$ of $\bar{X}$ have the property that $S_i \cap S_{i+1} \neq \varnothing$ for $i = 1, \ldots, r-1$. Then apply the above case successively to s_i and s_{i+1}.) Now, since C_1 and C_2 intersect, choose $y \in C_1 \cap C_2$. Clearly, $\bar{X}$ is not smooth at y. Since $\bar{X} \setminus \bar{Z}$ is smooth, it follows that $y \in \bar{Z}$. Choose $y_1, y_2 \in \bar{\bar{X}}$ lying above y such that y_1 belongs to C_1 and y_2 to C_2. By definition of Y, y_1 and y_2 belong to $\bar{Y}$ and the connected components to which they belong are denoted by the same letters. We then have $\phi(y_1) = x_1$, $\phi(y_2) = x_2$ and $\psi(y_1) = y = \psi(y_2)$. This proves the theorem. □

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The joint spectral flow and localization of the indices of elliptic operators

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We introduce the notion of the joint spectral flow, which is a generalization of the spectral flow, by using Segal's model of the connective K-theory spectrum. We apply it for some localization results of indices motivated by Witten's deformation of Dirac operators, and rephrase some analytic techniques in terms of topology.

1. Introduction

We give a topological viewpoint for the index and localization phenomena for elliptic operators on certain fiber bundles, using the notion of the joint spectral flow, which is a generalization of the spectral flow introduced by Atiyah, Patodi and Singer [Atiyah et al. 1976]. The spectral flow has various generalizations: for example, the higher spectral flow given by Dai and Zhang [1998] and the noncommutative spectral flow by Leichtnam and Piazza [2003] and Wahl [2007]. However, what we introduce here is a completely different new generalization.

The spectral flow for a one-parameter family of self-adjoint operators is an integer counting the number of eigenvalues with multiplicity crossing over zero. In geometric situations, it is related to the index of some Fredholm operators, as shown in [Atiyah et al. 1976] as follows. For a one-parameter family of self-adjoint elliptic differential operators D_t of first order ($t \in S^1$) on $\Gamma(Y, E)$, where Y is a closed manifold and E is a hermitian vector bundle on Y, the first-order differential operator $d/dt + D_t$ on $\Gamma(Y \times S^1, \pi^* E)$ is also elliptic, and its index coincides with

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the spectral flow. The proof is given essentially by the index theorem for families over the closed 1-dimensional manifold S^1.

The joint spectral flow deals with an n-parameter family of n-tuples of mutually commuting self-adjoint operators and their joint spectra. We deal with continuous or smooth families of commuting Fredholm n-tuples, which are defined in Definition 2.3, and the “Dirac operators” associated with them. In the special case $n = 1$, the joint spectral flow coincides with the usual spectral flow. We also relate it with the index of some elliptic operators, as in the case of the ordinal spectral flow.

Theorem 3.19. *Let B be a closed n-dimensional Spin^c manifold, $Z \to M \to B$ a smooth fiber bundle over B such that the total space M is also a Spin^c-manifold, E a smooth complex vector bundle over M, and V an n-dimensional Spin^c vector bundle over B. For a bundle map $\{D_v(x)\}$ from $V \setminus \{0\}$ to the bundle of fiberwise pseudodifferential operators $\Psi^1_f(M, E)$ satisfying Condition 3.18, we have*

$$\operatorname{ind}(\pi^* \not{\mathfrak{D}}_B + D(x)) = \operatorname{jsf}\big(\{D(x)\}\big).$$

The proof also works in a similar way to the original one. The crucial theorem introduced by Segal [1977] is that the space of n-tuples of mutually commuting compact self-adjoint operators is a model for the spectrum of connective K-theory.

The joint spectral flow and its index formula imply some localization results. E. Witten [1982] reinterpreted and reproved some localization formulas for the indices of Dirac operators from the viewpoint of supersymmetry. He deformed Dirac operators by adding potential terms coming from Morse functions or Killing vectors. Recently, Fujita, Furuta and Yoshida [2010] used an infinite-dimensional analogue to localize the Riemann–Roch numbers of certain completely integrable systems and their prequantum data on their Bohr–Sommerfeld fibers. Here a fiber of a Lagrangian fiber bundle is Bohr–Sommerfeld if the restriction of the prequantum line bundle to it is trivially flat (flat with trivial monodromy). In this case the indices of Dirac operators on fiber bundles localize on some special fibers instead of points. Here we relate them with our joint spectral flow and give a topological viewpoint for this analytic way of localization. A strong point of our method is that we give a precise way to compute the multiplicity at each point on which the index localizes. As a consequence, we reprove and generalize theorems of Witten and Fujita, Furuta and Yoshida.

Corollary 4.3 [Andersen 1997; Fujita et al. 2010]. *Let (M, ω) be a symplectic manifold of dimension $2n$, $\mathbb{T}^n \to M \to B$ a Lagrangian fiber bundle, and (L, ∇^L, h) its prequantum data. Then its Riemann–Roch number $\mathrm{RR}(M, L)$ coincides with the number of Bohr–Sommerfeld fibers.*

Finally we consider an operator-theoretic problem.

Unfortunately, there are not many examples of geometrically important operators (for example Dirac operators) represented as Dirac operators associated with commuting Fredholm n-tuples coming from differential operators. Compared with the case where their principal symbols "decompose" as the sum of commuting n-tuples, which is the easiest case because this occurs when when their tangent bundles decompose, the case where the Dirac operators themselves decompose is much more difficult because it requires some integrability of decompositions of tangent bundles. However, the bounded operators $\not{D}(1+\not{D}^2)^{-1/2}$ associated with the Dirac operators $\not{D}$ and zeroth-order pseudodifferential operators are much easier to deal with than first-order differential operators. We glue two commuting n-tuples of pseudodifferential operators by using topological methods to show that the indices for families are complete obstructions to decomposing families of Dirac operators. Here the theories of extensions of C^*-algebras and of Cuntz's quasihomomorphisms play an important role.

Theorem 5.3. *Let $Z \to M \to B$ be a fiber bundle. We assume that there are vector bundles $V_1, \dots, V_l$ on B and $E_1, \dots, E_l$ on M such that the vertical tangent bundle $T_V M$ is isomorphic to $\pi^* V_1 \otimes E_1 \oplus \dots \oplus \pi^* V_l \otimes E_l$. Then its fiberwise Dirac operator $\not{D}_f^E$ is n-decomposable* (in the sense of Definition 5.2) *if and only if the index for a family* $\mathrm{ind}(\not{D}_f^E)$ *is in the image of $K^n(B, B^{(n-1)}) \to K^n(B)$, or equivalently the image of $\tilde{k}^n(B) \to K^n(B)$.*

This paper is organized as follows. In Section 2, we relate Segal's description of the connective K-theory with the theory of Fredholm operators. In Section 3, we introduce the notion of the joint spectral flow and prove its index formula. In Section 4, we apply the theory and reprove or generalize some classical facts. In Section 5 we deal with the problem of decomposing Dirac operators and give an index-theoretic complete obstruction.

Conventions. We use the following notation throughout this paper:

First, any topological space is assumed to be locally compact and Hausdorff unless otherwise noted (there are exceptions, which are mentioned individually).

Second, we use some topological terms as follows. For a based space $(X, *)$, we denote by ΣX the suspension $X \times S^1/(X \times *_{S^1} \cup *_X \times S^1)$ and by ΩX the reduced loop space $\mathrm{Map}((S^1, *), (X, *))$. On the other hand, for an unbased space X we denote by ΣX and IX the direct sums $X \times (0,1)$ and $X \times [0,1]$, respectively. Similarly, for a C^*-algebra A we denote by ΣA and IA its suspensions $A \otimes C_0(0,1)$ and $A \otimes C[0,1]$. In particular, we denote by just Σ (resp. I) the topological space $(0,1)$ or the C^*-algebra $C_0(0,1)$ (resp. $[0,1]$ or $C[0,1]$).

2. Fredholm picture of the connective K-theory

In this section, we first summarize the notion of connective K-theory and its relation to operator algebras according to [Segal 1977] and [Dădărlat and Némethi 1990]. Then we connect it with a model of the K-theory spectrum that is related to the space of Fredholm operators. Finally we generalize the theory for the twisted case. This is fundamental to describing the joint spectral flow.

Let $\{H^i\}_{i\in\mathbb{Z}}$ be a generalized cohomology theory. We say $\{h_i\}_{i\in\mathbb{Z}}$ is the connective cohomology theory associated to $\{H_i\}$ if it is a generalized cohomology theory satisfying the following properties:

(1) There is a canonical natural transformation $h^i \to H^i$ that induces an isomorphism $h^i(\mathrm{pt}) \to H^i(\mathrm{pt})$ for $i \leq 0$.

(2) We have $h^i(\mathrm{pt}) = 0$ for $i > 0$.

Then (reduced) connective K-theory is the connective cohomology theory that is associated to (reduced) K-theory.

Segal [1977] gave an explicit realization of connective K-theory spectra by using operator-algebraic methods.

For a pair of compact Hausdorff spaces (X, A), we denote by $F(X, A)$ the configuration space with labels in finite-dimensional subspaces of a fixed (separable infinite-dimensional) Hilbert space. More precisely, an element of $F(X, A)$ is a pair $(S, \{V_x\}_{x\in S})$, where S is a countable subset of $X \setminus A$ whose cluster points are all in A and each V_x is a nonzero finite-dimensional subspace of a Hilbert space $\mathcal{H}$ such that V_x and V_y are orthogonal if $x \neq y$. It is a non-locally compact topological space with canonical topology satisfying the following:

(1) When two sequences $\{x_i\}$, $\{y_i\}$ converge to the same point z and V_z is the limit of $\{V_{i,x_i} \oplus V_{i,y_i}\}$, the limit of $(\{x_i, y_i\}, \{V_{i,x_i}, V_{i,y_i}\})$ is $(\{z\}, \{V_z\})$.

(2) When all cluster points of a sequence $\{x_i\}$ are in A, the limit of $(\{x_i\}, \{V_{i,x_i}\})$ is $(\varnothing, \varnothing)$.

Then the following holds for this topological space:

Proposition 2.1. *Let (X, A) be a pair of compact Hausdorff spaces. We assume that X is connected, A is path-connected, and A is a neighborhood deformation retract in X. Then the space $F(X, A)$ is homotopy-equivalent to its subspace $F_{\mathrm{fin}}(X, A) := \{(S, \{V_x\}_{x\in S}) \in F(X, A) \mid \#S < \infty\}$ and a sequence $F_{\mathrm{fin}}(A, *) \to F_{\mathrm{fin}}(X, *) \to F_{\mathrm{fin}}(X, A)$ is a quasifibration. Here morphisms are induced by continuous maps $(A, *) \to (X, *) \to (X, A)$. Hence the map $F(X, A) \to \Omega F(SX, SA)$ induces a homotopy equivalence.*

Proof. See [Segal 1977, Proposition 1.3; Dădărlat and Némethi 1990, Section 3.1]. □

This means that $\{F(S^n, *)\}_{n=1,2,\dots}$ is an Ω-spectrum, and hence homotopy classes of continuous maps to it realize some cohomology theory.

Now we introduce two other non-locally compact spaces. First, let $F_n(\mathcal{H})$ be the space of $(n+1)$-tuples $\{T_i\}_{i=0,\dots,n}$ of self-adjoint bounded operators on $\mathcal{H}$ that satisfy the following:

(1) The operator $T^2 := \sum T_i^2$ is equal to the identity.

(2) The operator T_i commutes with T_j for any i and j.

(3) The operators T_i $(i = 1, 2, \dots, n)$ and $T_0 - 1$ are compact.

Then there is a canonical one-to-one correspondence between $F_n(\mathcal{H})$ and $F(S^n, *)$. If we have an element $(S, \{V_x\})$ of $F(S^n, *)$, then we obtain an $(n+1)$-tuple $(T_0, \dots, T_n)$ by setting $T_i := \sum_{x \in S} x_i P_{V_x}$, where P_V is the orthogonal projection onto V and x_i the i-th coordinate of x in $S^n \subset \mathbb{R}^{n+1}$. Conversely, if we have an element $(T_0, \dots, T_n)$ in $F_n(\mathcal{H})$, then we obtain data of joint spectra and the eigenspaces because the T_i are simultaneously diagonalizable. Actually, this correspondence is homeomorphic.

On the other hand, if we have an element $(T_0, \dots, T_n) \in F_n(\mathcal{H})$, then there is a canonical inclusion from the spectrum of the abelian C^*-algebra $C^*(T_0, \dots, T_n)$ into the unit sphere of $\mathbb{R}^{n+1}$ according to condition (1). It gives a $*$-homomorphism $C(S^n) \to \mathbb{B}(\mathcal{H})$ sending x_i to T_i. Now, by virtue of condition (3), the image of its restriction to $C_0(S^n \setminus \{*\})$ (where $* = (1, 0, \dots, 0)$) is in the compact operator algebra $\mathbb{K} = \mathbb{K}(\mathcal{H})$. Conversely, if we have a $*$-homomorphism $\varphi : C_0(\mathbb{R}^n) \to \mathbb{K}$, then we obtain an element $(\varphi(x_0 - 1) + 1, \varphi(x_1), \dots, \varphi(x_n))$ in $F_n(\mathcal{H})$. This gives a canonical one-to-one correspondence between $F_n(\mathcal{H})$ and $\mathrm{Hom}(C_0(\mathbb{R}^n), \mathbb{K})$. This correspondence is also a homeomorphism when we equip $\mathrm{Hom}(C_0(\mathbb{R}^n), \mathbb{K})$ with the strong topology. Moreover, a continuous family of $*$-homomorphisms $\{\varphi_x\}_{x \in X}$ parametrized by a finite CW-complex X is regarded as a $*$-homomorphism $C_0(\mathbb{R}^n) \to C(X) \otimes \mathbb{K} \cong C(X, \mathbb{K})$.

Proposition 2.2 [Segal 1977; Dădărlat and Némethi 1990]. *Let X be a finite CW-complex and $n \in \mathbb{Z}_{>0}$. The three sets*

(1) $[X, F(S^n, *)]$,

(2) $[X, F_n(\mathcal{H})]$,

(3) $[C_0(\mathbb{R}^n), C(X) \otimes \mathbb{K}]$

are canonically mutually isomorphic and form the n-th reduced connective K-group $\tilde{k}^n(X)$. Here the first two are the sets of homotopy classes of continuous maps and the third is that of homotopy classes of $*$-homomorphisms.

Proof. We have already seen that these three sets are canonically isomorphic and $\{F(S^n,*)\}_{n=1,2,\dots}$ is an Ω-spectrum. The desired canonical natural transform is a canonical map Φ from $[C_0(\mathbb{R}^n), C(X)\otimes\mathbb{K}]$ to $KK(C_0(\mathbb{R}^n), C(X)\otimes\mathbb{K})\cong K^n(X)$ that sends a homotopy class $[\varphi]$ to $[\mathcal{H}\otimes C(X),\varphi,0]$. Hence we only have to compute $\pi_i(F(S^n,*))$. First for a general C^*-algebra A, the map $[C_0(\mathbb{R}),A]\to K_1(A)$ is an isomorphism because a $*$-homomorphism from $C_0(\mathbb{R})$ to A is determined by a unitary operator. Hence $[X,F(S^1,*)]$ is isomorphic to $K^1(X)$. In the case $i\geq n$ we have

$$\pi_i(F(S^n,*))\cong\pi_{i-n+1}(F(S^1,*))\cong K^1(\mathbb{R}^{i-n+1}),$$

that is, $\mathbb{Z}$ when $i-n$ is even and 0 when $i-n$ is odd. In the case $i<n$ we have $\pi_i(F(S^n,*))\cong\pi_0(F(S^{n-i},*))\cong 0$ because $F(S^{n-i},*)$ is connected. □

Next we relate this picture to a realization of K-theory that uses the space of Fredholm operators.

Atiyah and Singer [1969] gave a realization of the K-theory spectrum. Let $\mathbb{C}\ell_n$ be the complex Clifford algebra associated to $\mathbb{C}^n$ and its canonical inner product, $e_1,\dots,e_n$ its canonical self-adjoint generators with relations $e_ie_j+e_je_i=2\delta_{ij}$ and $\mathcal{H}$ a Hilbert space with a $\mathbb{Z}/2$-grading and a $\mathbb{Z}/2$-graded $\mathbb{C}\ell_n$-action c. Then the (non-locally compact) space of odd bounded self-adjoint Fredholm operators T that commute with the $\mathbb{C}\ell_n$-action, and such that $c(e_1)\cdots c(e_n)T|_{\mathcal{H}^0}$ is neither positive- nor negative-definite modulo compact operators if n is odd, represents the K^{-n}-functor.

Similarly, we represent the K^n-functor for $n>0$ as a space of Fredholm operators. For an ungraded separable infinite-dimensional Hilbert space $\mathcal{H}$, let $\mathcal{H}_{\mathbb{C}\ell_n}$ be the $\mathbb{Z}/2$-graded Hilbert $\mathbb{C}\ell_n$-module $\mathcal{H}\mathbin{\hat{\otimes}}\mathbb{C}\ell_n$. Now, for $n>0$, let $\mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})$ be the (non-locally compact) space of odd bounded self-adjoint operators in $\mathbb{B}(\mathcal{H}_{\mathbb{C}\ell_n})$ that is Fredholm, that is, invertible modulo $\mathbb{K}(\mathcal{H}_{\mathbb{C}\ell_n})$. Moreover, if n is odd, we additionally assume that $c(e_1)\cdots c(e_n)T|_{\mathcal{H}\otimes\mathbb{C}\ell_n^0}$ is neither positive- nor negative-definite. Then it represents the K^n-functor. It can be understood from the viewpoint of Kasparov's [1980b] KK-theory (or bivariant K-theory). As is well-known, the KK-theory has various formulations, and the original one of Kasparov is deeply related to the theory of Fredholm operators and their indices (see also [Blackadar 1998]). For separable $\mathbb{Z}/2$-graded C^*-algebras A and B, a cycle in $KK(A,B)$ is of the form $[E,\varphi,F]$, where E is a countably generated $\mathbb{Z}/2$-graded Hilbert B-module, φ a $*$-homomorphism from A to $\mathbb{B}(E)$ and F an odd self-adjoint "Fredholm" operator on E relative to A. More precisely, F is an operator in $\mathbb{B}(E)$ such that $[\varphi(a),F]$, $\varphi(a)(F^2-1)$ and $\varphi(a)(F-F^*)$ are in $\mathbb{K}(E)$ for any $a\in A$. A continuous family (in the norm topology) of $\mathbb{C}\ell_n$-equivariant odd Fredholm operators $F(x)$ ($x\in X$) gives a cycle $[\mathcal{H}_{\mathbb{C}\ell_n}\mathbin{\hat{\otimes}}C(X),1,F]$ in $KK(\mathbb{C},C(X)\mathbin{\hat{\otimes}}\mathbb{C}\ell_n)$ by regarding F as an element in $\mathbb{B}(\mathcal{H}_{\mathbb{C}\ell_n}\otimes C(X))$ by pointwise multiplication. Because this KK-cycle depends only on its homotopy class, this correspondence gives a map from $[X, \mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})]$ to $KK(\mathbb{C}, C_0(X) \hat{\otimes} \mathbb{C}\ell_n)$. We can see that it is actually an isomorphism by using the equivalence relations called the operator homotopy [Kasparov 1980b]. Here we do not have to care about additions of degenerate cycles by virtue of the Kasparov stabilization theorem [Kasparov 1980a].

Now we have shown that there is some operator-theoretic description of the connective K-theory, but it is not consistent to the Fredholm picture of KK-theory and our construction of the K-theory spectrum. Next we see that these two are canonically related.

Both of the two groups $KK(C_0(\mathbb{R}^n), C(X))$ and $KK(\mathbb{C}, C(X) \hat{\otimes} \mathbb{C}\ell_n)$ are isomorphic to $K^n(X)$. The canonical isomorphism

$$KK(C_0(\mathbb{R}^n), C(X)) \to KK(\mathbb{C}, C(X) \hat{\otimes} \mathbb{C}\ell_n)$$

is given by taking the Kasparov product [1980b] with the canonical generator of $KK(\mathbb{C}, C_0(\mathbb{R}^n) \otimes \mathbb{C}\ell_n)$ from the left. This canonical generator also has many identifications, and here we use the one in [Kasparov 1980b]. It is based on the Fredholm picture and is of the form $[C_0(\mathbb{R}^n) \hat{\otimes} \mathbb{C}\ell_n, 1, C]$, where $C := \sum c_i x_i (1 + |x|^2)^{-1/2}$. Here $c_i := c(e_i)$ is left multiplication of e_i on $\mathbb{C}\ell_n$, which is a $\mathbb{C}\ell_n$-module by right multiplication.

Now we apply it for cycles that come from $\varphi \in \mathrm{Hom}(C_0(\mathbb{R}^n), C(X) \otimes \mathbb{K})$. We then have

$$\begin{aligned}[][C_0(\mathbb{R}^n) \hat{\otimes} \mathbb{C}\ell_n, 1, C] \otimes_{C_0(\mathbb{R}^n)} [\mathcal{H} \hat{\otimes} C(X), \varphi, 0] \\ &= \big[C_0(\mathbb{R}^n) \otimes_\varphi (\mathcal{H} \otimes C(X)) \hat{\otimes} \mathbb{C}\ell_n, 1, C \otimes_\varphi \mathrm{id}\big] \\ &= \Big[\mathcal{E}(\varphi) \hat{\otimes} \mathbb{C}\ell_n, 1, \sum c_i T_i\Big].\end{aligned}$$

Here we denote by $\mathcal{E}(\varphi)$ the Hilbert $C(X)$-module $\{\overline{\varphi_x(C_0(\mathbb{R}^n))\mathcal{H}}\}_{x \in X}$ (more precisely, the submodule of $C(X) \otimes \mathcal{H}$ that consists of $\mathcal{H}$-valued functions on X whose evaluations at x are in $\overline{\varphi_x(C_0(\mathbb{R}^n))\mathcal{H}}$). A $*$-homomorphism $\varphi : C_0(\mathbb{R}^n) \to \mathbb{B}(\mathcal{E}(\varphi))$ uniquely extends to $\tilde{\varphi} : C_b(\mathbb{R}^n) \to \mathbb{B}(\mathcal{E}(\varphi))$ because φ is nondegenerate onto $\mathbb{B}(\mathcal{E}(\varphi))$ (see Section 5 of [Lance 1995]). We set $T_i := \tilde{\varphi}(x_i(1 + |x|^2)^{-1/2})$.

This can be regarded as the Fredholm picture of connective K-theory. However, unfortunately it is not useful for our purpose because $\mathcal{E}(\varphi)$ may not be locally trivial and hence not a bundle of Hilbert spaces in general. Nonetheless, cycles arising in geometric contexts, which are our main interest, have a better description, as follows:

Definition 2.3. • An n-tuple of bounded self-adjoint operators $(T_1, \dots, T_n)$ on $\mathcal{H}$ is called a *bounded commuting Fredholm n-tuple* if it satisfies the following:

(1) The operator $T^2 := \sum T_i^2$ is in $1 + \mathbb{K}(\mathcal{H})$.

(2) The operator T_i commutes with T_j for any i and j.

We denote by $\mathcal{F}_n(\mathcal{H})$ the set of bounded commuting Fredholm n-tuples equipped with the norm topology.

• An n-tuple of unbounded self-adjoint operators $(D_1, \dots, D_n)$ on $\mathcal{H}$ is an *unbounded commuting Fredholm n-tuple* if it satisfies the following:

(1) The operator $D^2 := \sum D_i^2$ is densely defined, Fredholm, and has compact resolvents.

(2) The operator D_i commutes with D_j for any i and j on $\mathrm{dom}(D^2)^2$.

We denote the set of unbounded commuting Fredholm n-tuples by $\mathscr{F}_n(\mathcal{H})$. It is equipped with the strongest topology such that the map

$$(D^1, \dots, D^n) \mapsto (D^1(1+D^2)^{-1/2}, \dots, D^n(1+D^2)^{-1/2})$$

is continuous. This definition is an analogue of the Riesz topology on the space of self-adjoint operators.

• For a bounded (resp. unbounded) commuting Fredholm n-tuple $(T_1, \dots, T_n)$ (resp. $(D_1, \dots, D_n)$), we say that an odd self-adjoint operator $T := c_1T_1 + \cdots + c_nT_n$ on $\mathcal{H} \hat{\otimes} \mathbb{C}\ell_n$ (resp. $D := c_1D_1 + \cdots + c_nD_n$ with the domain $\mathrm{dom}(D^2)^{1/2}$) is the *Dirac operator* associated with $(T_1, \dots, T_n)$. For simplicity of notation, hereafter we use the same letter T (resp. D) for commuting Fredholm n-tuples and the Dirac operators associated with them.

The continuous map $(\overline{\mathbb{D}^n}, \partial\mathbb{D}^n) \to (S^n, *)$ that collapses the boundary, more precisely of the form

$$(T_1, \dots, T_n) \mapsto \big(2T^2 - 1, 2(1-T^2)^{1/2}T_1, \dots, 2(1-T^2)^{1/2}T_n\big),$$

which is the unique continuous extension of the composition map of the canonical isomorphism between $\mathbb{D}^n$ and $\mathbb{R}^n$ and the stereographic projection, induces a continuous map $\iota : \mathcal{F}_n(\mathcal{H}) \to F_n(\mathcal{H})$ by functional calculus and definition of the topology on $\mathscr{F}_n(\mathcal{H})$. On the other hand, for $(T_1, \dots, T_n) \in \mathcal{F}_n(\mathcal{H})$, the Dirac operator T is in $\mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})$. This correspondence gives a map from $[X, \mathcal{F}_n(\mathcal{H})]$ to $[X, \mathcal{F}_{\mathbb{C}\ell_n}(\mathcal{H})] \cong KK(\mathbb{C}, C(X) \otimes \mathbb{C}\ell_n)$; the interpretation, in a geometric context, is taking the index bundle with $\mathbb{C}\ell_n$-module structure for the continuous family of Dirac operators associated with $(T_1, \dots, T_n)$. Hence we denote it by ind.

Theorem 2.4. *The following diagram commutes*:

$$\begin{tikzcd} {[X, \mathcal{F}_n(\mathcal{H})]} \arrow[r, "\mathrm{ind}"] \arrow[d, "\iota"'] & KK(\mathbb{C}, C(X) \hat{\otimes} \mathbb{C}\ell_n) \\ {[X, F_n(\mathcal{H})]} \arrow[r, "\Phi"] & KK(C_0(\mathbb{R}^n), C(X)). \arrow[u, "\sim"'] \end{tikzcd}$$

Proof. Let $\{T(x)\}_{x\in X} := \{(T_1(x), \dots, T_n(x))\}_{x\in X}$ be a continuous family of bounded commuting Fredholm n-tuples and φ^T its image under ι. Then $\Phi \circ \iota[\{T(x)\}]$ is of the form $[\mathcal{E}(\varphi^T) \hat{\otimes} \mathbb{C}\ell_n, 1, T]$. Now we give a homotopy connecting $[\mathrm{ind}\, T] = \big[(\mathcal{H} \otimes C(X)) \hat{\otimes} \mathbb{C}\ell_n, 1, T(x)\big]$ and $[\mathcal{E}(\varphi^T) \hat{\otimes} \mathbb{C}\ell_n, 1, T(x)]$ directly. It is given by the Kasparov $\mathbb{C}$-$IC(X)$-bimodule

$$\big[\mathcal{E}(\varphi^T) \oplus_{\mathrm{ev}_0} (\mathcal{H}_{C(X)} \otimes I), 1, T\big],$$

where $\mathcal{E}(\varphi^T) \oplus_{\mathrm{ev}_0} (\mathcal{H}_{C(X)} \otimes I) := \big\{(x, f) \in \mathcal{E}(\varphi^T) \oplus (\mathcal{H}_{C(X)} \otimes I) \mid f(0) = x\big\}$. □

Remark 2.5. For a general locally compact CW-complex we have an analogue of K-theory with compact support. The K-group with compact support $K^n_{\mathrm{cpt}}(X)$ is defined as the kernel of the canonical morphism $K^n(X^+) \to K^n(x_0)$, where X^+ is the one-point compactification of X and $\{x_0\} = X^+ \setminus X$. It coincides with the K-group of the nonunital C^*-algebra $C_0(X)$ by definition. Similarly, we write $k^n_{\mathrm{cpt}}(X)$ for the kernel of $k^n(X^+) \to k^n(x_0)$. When X^+ has a relatively compact deformation retract of $\{x_0\}$, $\tilde{k}^n_{\mathrm{cpt}}(X)$ is isomorphic to the set of compactly supported homotopy classes of maps from X to $F(S^n, *)$ with compact support, $F_n(\mathcal{H})$, or $\mathrm{Hom}(C_0(\mathbb{R}^n, \mathbb{K}))$. Hence it is also isomorphic to $\mathrm{Hom}(C_0(\mathbb{R}^n), C_0(X) \otimes \mathbb{K})$. In terms of our Fredholm picture, a continuous family of Fredholm n-tuples on X which is *bounded below* by some $\kappa > 0$ (i.e., $D(x)^2 \geq \kappa$) outside some compact subset $K \subset X$ determines a k^n-cycle on X. For simplicity we write just $\tilde{k}(X)$ instead of $\tilde{k}_{\mathrm{cpt}}(X)$ in this paper.

Remark 2.6. The above formulation is compatible with the product of cohomology theories. We define the product of continuous families of bounded commuting Fredholm n-tuples $T(x) = (T_1(x), \dots, T_n(x))$ in $\mathrm{Map}(X, \mathcal{F}_n(\mathcal{H}))$ and m-tuples $S(x) = (S_1(x), \dots, S_m(x))$ in $\mathrm{Map}(X, \mathcal{F}_m(\mathcal{H}'))$ by

$$\begin{aligned} T(x) \times S(x) &= (T_1(x), \dots, T_n(x)) \times (S_1(x), \dots, S_m(x)) \\ &:= \big(T_1(x) \otimes 1, \dots, T_n(x) \otimes 1, 1 \otimes S_1(x), \dots, 1 \otimes S_m(x)\big) \\ &\qquad\qquad\qquad \in \mathrm{Map}(X, \mathcal{F}_{n+m}(\mathcal{H} \otimes \mathcal{H}')). \end{aligned}$$

Then the homotopy class of $T(x) \times S(x)$ depends only on the homotopy classes of $T(x)$ and $S(x)$. Consequently $[\{T(x)\}] \cup [\{S(x)\}] := [\{T(x) \times S(x)\}]$ gives a well-defined product $[X, \mathcal{F}_n(\mathcal{H})] \times [X, \mathcal{F}_m(\mathcal{H})] \to [X, \mathcal{F}_{n+m}(\mathcal{H})]$ that is compatible with the product of connective K-groups, which is induced from the canonical map $(S^n, *) \times (S^m, *) \to (S^n, *) \wedge (S^m, *) \cong (S^{n+m}, *)$. By a similar argument we can define the product for unbounded commuting Fredholm n-tuples.

The twisted case. Next, we generalize the above theory to twisted connective K-theory. In the above argument, we have used the action of the Clifford algebra $\mathbb{C}\ell_n$ as the coefficients to construct a Dirac operator associated with a family of commuting Fredholm n-tuples. Now we regard it as the Clifford algebra bundle $\mathbb{C}\ell(\underline{\mathbb{C}^n})$ associated with the trivial bundle. We generalize the notion of commuting Fredholm n-tuples and apply the general Clifford algebra bundles $\mathbb{C}\ell(V_{\mathbb{C}})$ associated with Spin^c vector bundles V for the coefficients of the Dirac operators associated with them.

We consider the canonical actions of $\mathrm{GL}(n; \mathbb{R})$ on the spaces $F(S^n, *)$, $F_n(\mathcal{H})$ and $\mathrm{Hom}(C_0(\mathbb{R}^n), C(X) \otimes \mathbb{K})$. For example, on $F_{n+m}(\mathcal{H})$ the action is of the form

$$g \cdot (T_0, T_1, \dots, T_n) := \Big(\sum g_{1j} T_j, \dots, \sum g_{nj} T_j \Big).$$

Let V be a real vector bundle over X. We denote a fiber bundle

$$\mathrm{GL}(V) \times_{\mathrm{GL}(n;\mathbb{R})} F(S^n, *) \quad (\text{resp. } \mathrm{GL}(V) \times_{\mathrm{GL}(n;\mathbb{R})} F_n(\mathcal{H}))$$

by F_V (resp. $F_V(\mathcal{H})$). Similarly, $\mathrm{GL}(n, \mathbb{R})$ acts on the space of bounded (resp. unbounded) commuting Fredholm n-tuples $\mathcal{F}_n(\mathcal{H})$ (resp. $\mathscr{F}_n(\mathcal{H})$), and we denote by $\mathcal{F}_V(\mathcal{H})$ (resp. $\mathscr{F}_V(\mathcal{H})$) the corresponding fiber bundle.

Definition 2.7. A V-twisted family of bounded (resp. unbounded) commuting Fredholm n-tuples is a continuous section $T = T(x)$ in $\Gamma(X, \mathcal{F}_V(\mathcal{H}))$ (resp. in $\Gamma(X, \mathscr{F}_V(\mathcal{H}))$).

In the similar way as in the above argument, the space of continuous sections $\Gamma\mathbb{C}\ell(V) = \Gamma(X, \mathbb{C}\ell(V))$ is a C^*-algebra and a continuous section $T \in \Gamma(X, \mathcal{F}_V(\mathcal{H}))$ defines a Kasparov $\mathbb{C}$-$\Gamma\mathbb{C}\ell(V)$-bimodule

$$\big[\mathcal{H} \hat{\otimes} \mathbb{C}\ell(V), 1, c(e_1)T_{e_1}(x) + \cdots + c(e_n)T_{e_n}(x)\big],$$

which is independent of the choice of an orthonormal basis $\{e_1, \dots, e_n\} \in V_x$. Therefore we obtain a map $\pi_0(\Gamma(X, \mathcal{F})) \to KK(\mathbb{C}, \Gamma\mathbb{C}\ell(V))$.

Proposition 2.8. *Let X be a finite CW-complex and V a real vector bundle. The three sets*

(1) $\Gamma(X, F_V)$,

(2) $\Gamma(X, F_V(\mathcal{H}))$,

(3) $\mathrm{Hom}_{C(X)}(C_0(V), C(X) \otimes \mathbb{K})$

are canonically mutually homeomorphic, and their connected components form the twisted reduced connective K-group associated with the principal bundle

$$\mathrm{GL}(V) \times_{\mathrm{GL}(n,\mathbb{R})} \mathcal{G}_k^{\mathrm{mod}},$$

which we denote by $\tilde{k}^V(X)$ (see Section 3 of [Atiyah and Segal 2004]). *Here* $\mathrm{Hom}_{C(X)}(C_0(V \times \mathbb{R}^k), C(X) \otimes \mathbb{K})$ *is the set of* $C(X)$*-homomorphisms, that is,* $*$*-homomorphisms that are compatible with their* $C(X)$*-module structures.*

Theorem 2.9. *Let X be a finite CW-complex. Then the following diagram commutes*:

$$\begin{tikzcd}
\pi_0\big(\Gamma(X, \mathcal{F}_V(\mathcal{H}))\big) \arrow[r, "\mathrm{ind}"] \arrow[d, "\iota"'] & KK(\mathbb{C}, \Gamma\mathbb{C}\ell(V)) \\
\pi_0\big(\Gamma(X, F_V(\mathcal{H}))\big) \arrow[r, "\Phi"] & \mathcal{R}KK(X; C_0(V), C(X)). \arrow[u, "\sim"]
\end{tikzcd}$$

Here $\mathcal{R}KK(X; C_0(V), C(X))$ *is the representable* KK*-group* [Kasparov 1988].

In the same way as in K-theory, the Thom isomorphism holds for twisted connective K-theory.

Proposition 2.10. *The following isomorphism holds*:

$$k^W(X) \cong k^{\pi^* V \oplus \pi^* W}(V)$$

Proof. Let F be a closed subspace of X and denote by V_F the restriction $V|_F$ of a vector bundle V. Then there is a morphism

$$\mathrm{Hom}_{C(F)}(C_0(W_F), C(F) \otimes \mathbb{K}) \to \mathrm{Hom}_{C_0(V_F)}(C_0(\pi^*(V \oplus W)_{V_F}), C_0(V_F) \otimes \mathbb{K}),$$
$$\varphi \mapsto \mathrm{id}_V \otimes \varphi,$$

which is an isomorphism if V is trivial on F, and functorial with respect to inclusions. The Mayer–Vietoris exact sequence implies the global isomorphism. □

In particular, combining with the Thom isomorphism of connective K-theory, we obtain that the twist associated with V is trivial if V has a Spin^c structure.

3. The joint spectral flow

Now we give the precise definition of the joint spectral flow by using the notions introduced in Section 2. Next, we prove an index theorem generalizing the spectral flow index theorem of Atiyah, Patodi and Singer [Atiyah et al. 1976]. Finally we generalize it to the case in which the coefficients c_i are globally twisted by a Spin^c vector bundle.

3A. *Definitions and an index theorem.* In the previous section we have seen that $F(S^n, *)$ represents connective K-theory. Now we introduce another configuration space $P(X, A)$ with labels in positive integers on X relative to A. More precisely, an element of $P(X, A)$ is a pair $(S, \{n_x\}_{x \in S})$, where S is a countable subset of $X \setminus A$ whose cluster points are all in A and each n_x is a positive integer. The topology is defined in the same way as that of $F(X, A)$. Then $P(S^n, *)$ is canonically homotopy equivalent to the infinite symmetric product of $(S^n, *)$, which is a model of the Eilenberg–MacLane space $K(\mathbb{Z}, n)$ by virtue of the Dold–Thom theorem [1958]. There is a canonical continuous map j from $F(S^n, *)$ to $P(S^n, *)$ "forgetting" data about vector spaces except their dimensions, which is given more precisely by

$$(S, \{V_x\}_{x \in S}) \longmapsto (S, \{\dim V_x\}_{x \in S}).$$

In the viewpoint of commuting Fredholm n-tuples, this map forgets their eigenspaces and keeps only their joint spectra with multiplicity. It induces a group homomorphism

$$j_* : \tilde{k}^n(X) \longrightarrow H^n(X; \mathbb{Z}).$$

Now we introduce the notion of the joint spectral flow:

Definition 3.1. Let X be an oriented closed manifold of dimension n. For a continuous family $\{T(x)\} = \{(T_0(x), \dots, T_n(x))\}_{x \in X}$ of elements in $F_n(\mathcal{H})$ parametrized by X, we say that $\langle j_*[\{T(x)\}], [X] \rangle \in \mathbb{Z}$ is its *joint spectral flow*, which we denote by $\mathrm{jsf}(\{T(x)\})$. For a continuous family of bounded or unbounded commuting Fredholm n-tuple $\{T_1, \dots, T_n\}$, we say $\mathrm{jsf}(\iota\{T(x)\})$ is its joint spectral flow, denoted simply by $\mathrm{jsf}(\{T(x)\})$.

Example 3.2 (the case $n = 1$). According to Section 7 of [Atiyah et al. 1976], the spectral flow is defined as the canonical group isomorphism $\mathrm{sf} : \pi_1(F_1(\mathcal{H})) \to \mathbb{Z}$ as follows. For a continuous map $T : S^1 \to F_1(\mathcal{H})$ such that the essential spectrum of each T_t is $\{-1, 1\}$, there is a family of continuous functions $j_i : [0, 1] \to [-1, 1]$ such that $-1 = j = 0 \leq j_1 \leq \cdots \leq j_m = 1$ and $\sigma(T(t)) = \{j_0(t), \dots, j_m(t)\}$ for any $t \in [0, 1]$. Then we obtain the integer l such that $j_k(1) = j_{k+l}(0)$ for any k. This l is called the spectral flow. Now let $\{T(t)\}$ be a continuous family of bounded self-adjoint Fredholm operators such that $\sigma(T(t)) = \{0, (t+1)/2, 1\}$ and the eigenspace $E_{(t+1)/2}$ is of dimension 1. Then by definition its spectral flow $\mathrm{sf}(\{T(t)\})$ is equal to 1. On the other hand, we obtain $j_*(\{T(t)\}) = 1 \in H^1(S^1; \mathbb{Z})$ since the canonical inclusion $S^1 \to \mathrm{Sym}^\infty(S^1, *)$ gives a generator $1 \in H^1(S^1; \mathbb{Z}) \cong [S^1, \mathrm{Sym}^\infty(S^1, *)]$ (see [Dold and Thom 1958] or Proposition 5.2.23 of [Aguilar et al. 2002]). This means that the joint spectral flow coincides with the ordinary spectral flow in the case $X = S^1$.

Proposition 3.3. *The homomorphism j_* is a natural transform of multiplicative cohomology theories.*

Proof. According to Section 3 of [Dădărlat and Némethi 1990],

$$S : \mathrm{Hom}(C_0(\mathbb{R}^n), \mathbb{K}) \to \mathrm{Hom}(C_0(\mathbb{R}^{n+1}), C_0(\mathbb{R}) \otimes \mathbb{K}),$$
$$\varphi \mapsto \mathrm{id}_{\mathbb{R}} \otimes \varphi$$

or equivalently

$$S : F(S^n, *) \to \Omega F(S^n \times I, S^n \times \{0, 1\} \cup \{*\} \times I),$$
$$(S, \{V_x\}_{x \in S}) \mapsto \{t \mapsto ((x, t), \{V_x\}_{x \in S})\}$$

gives the structure map $F(S^n, *) \to \Omega F(S^{n+1}, *)$. By the same argument we obtain that

$$S : P(S^n, *) \to \Omega P(S^n \times I, S^n \times \{0, 1\} \cup \{*\} \times I),$$
$$(S, \{n_x\}_{x \in S}) \mapsto \{t \mapsto ((x, t), \{n_x\}_{x \in S})\}$$

gives the structure map $P(S^n, *) \to \Omega P(S^{n+1}, *)$. Now, by definition the following diagram commutes:

$$\begin{tikzcd}
F(S^n, *) \arrow[r, "S"] \arrow[d, "j"'] & \Omega F(S^{n+1}, *) \arrow[d, "j"] \\
P(S^n, *) \arrow[r, "S"] & \Omega P(S^{n+1}, *).
\end{tikzcd}$$

The multiplicativity of j_* follows immediately since the multiplicative structure on $\{F(S^n, *)\}_{n=0,1,2,\ldots}$ and $P(S^n, *)_{n=0,1,2,\ldots}$ are induced from the map

$$(S^n, *) \times (S^m, *) \to (S^{n+m}, *)$$

coming from the wedge product. □

To prove a generalization of the spectral flow index theorem, we note the relation between the joint spectral flow and the Chern character. The Chern character is a natural transform from the K-functor to the rational cohomology functor. Here there is a generalization of the Chern character for a general cohomology theory, which was introduced by Dold [1962] and is called the Chern–Dold character.

Now we identify $k^*(X)$ with $\tilde{k}^{*+1}(SX)$ to extend j_* to a natural transform between unreduced cohomology theories $k^*(X) \to H^*(X; \mathbb{Z})$. It is compatible with the original j_* according to Proposition 3.3.

Proposition 3.4. *The n-th Chern–Dold character* $\mathrm{ch}_n : k^n(X) \otimes \mathbb{Q} \to H^n(X; \mathbb{Q})$ *coincides with j_* rationally.*

Proof. The diagram

$$\begin{tikzcd} k^n(X)\otimes\mathbb{Q} \arrow[r, "\mathrm{ch}"'] \arrow[d, "j_*"'] & H^n(X;k^*(\mathrm{pt})\otimes\mathbb{Q}) \arrow[d, "1\otimes j_*"'] \\ H^n(X;\mathbb{Q}) \arrow[r, "\sim", "\mathrm{ch}=\mathrm{id}"'] & H^n(X;H^*(\mathrm{pt})\otimes\mathbb{Q}) \end{tikzcd}$$

commutes by Proposition 3.3 and naturality of the Chern–Dold character. In fact, Dold [1962] proved that there is a one-to-one correspondence between natural transforms of multiplicative cohomology theories $h\to h'$ and graded ring homomorphisms $h(\mathrm{pt})\to h'(\mathrm{pt})$ if $h'(\mathrm{pt})$ is a graded vector space over $\mathbb{Q}$. The Chern–Dold character is induced from the ring homomorphism $h^*(\mathrm{pt})\to\mathbb{Q}\otimes_{\mathbb{Z}}h^*(\mathrm{pt})$. Naturality follows from uniqueness.

Now $k^*(\mathrm{pt})\cong\mathbb{Z}[\beta]$ (β is of degree -2), $H^*(\mathrm{pt})\cong\mathbb{Z}$ and the ring homomorphism j_* from $\mathbb{Z}[\beta]$ to $\mathbb{Z}$ is given by $1\mapsto 1$ and $\beta\mapsto 0$. Hence $(1\otimes j_*)\circ\mathrm{ch}$ coincides with the n-th Chern–Dold character ch_n. This implies that $j_*=\mathrm{ch}_n$. □

Let X be a closed Spin^c manifold, $\mathfrak{S}_{\mathbb{C}}(X)$ the associated $\mathbb{C}\ell_n$-module bundle of $\mathrm{Spin}^c(X)$ by the left multiplication on $\mathbb{C}\ell_n$ as a right $\mathbb{C}\ell_n$-module and $\mathfrak{D}_X$ the $\mathbb{C}\ell_n$-Dirac operator on $\mathfrak{S}_{\mathbb{C}}(X)$. Now $\mathfrak{S}_{\mathbb{C}}(X)$ is equipped with the canonical $\mathbb{Z}/2$-grading and $\mathfrak{D}_X$ is an odd operator. Then it gives an element of $K_n(X)\cong KK(C(X)\hat{\otimes}\mathbb{C}\ell_n,\mathbb{C})$

$$[\mathfrak{D}_X]:=\big[L^2(X,\mathfrak{S}_{\mathbb{C}}(X)),m,\mathfrak{D}_X(1+\mathfrak{D}_X^2)^{-1/2}\big],$$

which is the fundamental class of K-theory. Here $m:C(X)\hat{\otimes}\mathbb{C}\ell_n\to\mathbb{B}\big(L^2(\mathfrak{S}_{\mathbb{C}}(X))\big)$ is given by Clifford multiplication.

Lemma 3.5. *Let $\{T(x)\}_{x\in X}$ be a continuous family of commuting Fredholm n-tuples. Then*

$$\langle[\mathrm{ind}\,T],[\mathfrak{D}_X]\rangle_n=\mathrm{jsf}\{T(x)\}.$$

Here $\langle\,\cdot\,,\cdot\,\rangle_n$ is the canonical pairing between $K^n(X)$ and $K_n(X)$.

Proof. First, we prove the lemma in the case of even n. In this case we have a unique irreducible representation Δ_n of $\mathbb{C}\ell_n$ and the Dirac operator $\not{D}_X$ on $\not{S}_{\mathbb{C}}(X):=\mathrm{Spin}^c(X)\times_{\mathbb{C}\ell_n}\Delta_n$. Now Δ_n is equipped with a canonical $\mathbb{Z}/2$-grading and $\not{D}$ is an odd operator. It defines a KK-cycle

$$[\not{D}_X]:=\big[L^2(X,\not{S}_{\mathbb{C}}(X)),m,\not{D}_X(1+\not{D}_X^2)^{-1/2}\big]\in KK(C(X),\mathbb{C}).$$

We denote by $[\![\mathrm{ind}\,T]\!]$ a KK-cycle $[\mathcal{H}\otimes\Delta_n,1,T]\in KK(\mathbb{C},C(X))$. Since $\mathbb{C}\ell_n\cong\Delta_n\otimes\Delta_n^*$ as $\mathbb{C}\ell_n$-$\mathbb{C}\ell_n$-bimodules, we have the equalities $[\mathfrak{D}_X]=[\not{D}_X]\otimes\Delta_n$ and

$[\operatorname{ind} T] = [\![\operatorname{ind} T]\!] \hat{\otimes} \Delta_n$ (in particular $\operatorname{ch}[\operatorname{ind} T] = \operatorname{ch}[\![\operatorname{ind} T]\!]$). Here Δ^* is a Hilbert $\mathbb{C}\ell_n$-module by the inner product $\langle x, y\rangle := x^* y$.

The pairing $\langle \cdot, \cdot \rangle_n$ is given by the Kasparov product

$$KK(\mathbb{C}, C(X) \otimes \mathbb{C}\ell_n) \otimes KK(C(X) \otimes \mathbb{C}\ell_n, \mathbb{C}) \to \mathbb{Z}.$$

Therefore

$$\begin{aligned}\langle [\operatorname{ind} T], [\mathfrak{D}]\rangle_n &= [\operatorname{ind} T] \otimes_{C(X)\otimes\mathbb{C}\ell_n} [\mathfrak{D}_X] \\ &= ([\![\operatorname{ind} T]\!] \otimes_{C(X)} [\not{D}_X]) \otimes (\Delta^* \otimes_{\mathbb{C}\ell_n} \Delta) = [\![\operatorname{ind} T]\!] \otimes_{C(X)} [\not{D}_X].\end{aligned}$$

Now we use the Chern character for K-homology that is compatible with pairing. The Chern character of the Spin^c Dirac operator $\not{D}_X$ is given by the Todd class associated with the Spin^c structure of TX. Hence

$$\begin{aligned}\langle [\{T(x)\}], [\not{D}_X]\rangle &= \langle \operatorname{ch}([\![\operatorname{ind} T]\!]), \operatorname{ch}([\not{D}_X])\rangle \\ &= \langle \operatorname{ch}([\operatorname{ind} T]), \operatorname{Td}(X) \cap [X]\rangle \\ &= \langle \operatorname{ch}_n([\operatorname{ind} T]), [X]\rangle = \operatorname{jsf}\{T(x)\}.\end{aligned}$$

Here the third equality holds because $\operatorname{ch}([\operatorname{ind} T])$ is in $\bigoplus_{k\geq 0} H^{n+2k}(X; \mathbb{Q}) \cong H^n(X; \mathbb{Q})$ and the zeroth Todd class $\operatorname{Td}_0(X)$ is equal to 1. The last equality holds by Proposition 3.4.

Finally, we prove the lemma in the case of odd n. We can reduce the problem to the case $n = 1$ because, for a family of self-adjoint operators $S(t)$ parametrized by S^1 whose spectral flow is 1 (hence $[\operatorname{ind} S] = 1 \in K^1(S^1) \cong \mathbb{Z}$), we have

$$\begin{aligned}\langle [\operatorname{ind} T], [\mathfrak{D}]\rangle_n &= \big\langle [\operatorname{ind} T] \cup [\operatorname{ind} S], [\mathfrak{D}_X] \otimes [\mathfrak{D}_{S^1}]\big\rangle_{n+1} \\ &= \operatorname{jsf}(\{T(x)\} \times \{S(t)\}) = \operatorname{jsf}\{T(x)\}.\end{aligned}$$

Here we use the fact that the joint spectral flow of the product family $\{T(x)\} \times \{S(t)\}$ coincides with the product $\operatorname{jsf}(\{T(x)\}) \cdot \operatorname{jsf}(\{S(t)\})$. $\square$

Now we give an index theorem that is a generalization of the spectral flow index theorem in [Atiyah et al. 1976].

Let B be a closed n-dimensional Spin^c manifold, $Z \to M \to B$ a smooth fiber bundle over B and E a smooth complex vector bundle over M. We fix a decomposition $TM = T_V M \oplus T_H M$ of the tangent bundle, where $T_V M := \{v \in TM \mid \pi_* v = 0\}$ is the vertical tangent bundle. For a hermitian vector bundle E, we denote by $\pi^* \mathfrak{S}^E_{\mathbb{C}}(B)$ the $\mathbb{C}\ell_n$-module bundle $\pi^* \mathfrak{S}_{\mathbb{C}}(B) \otimes E$ on M. Now we define the *pull-back* of the $\mathbb{C}\ell_n$-Dirac operator $\mathfrak{D}_B$ on B twisted by E as

$$\begin{aligned}\pi^* \mathfrak{D}_B : \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B)) &\xrightarrow{\nabla} \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B) \otimes T^* M) \\ &\xrightarrow{p_{T^*_H M}} \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B) \otimes T^*_H M) \xrightarrow{h} \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B)).\end{aligned}$$

Here, h is the left Clifford action of $\mathbb{C}\ell(TB) \cong \mathbb{C}\ell(T_H M)$ on $\pi^* \mathfrak{S}^E_{\mathbb{C}}(B)$. This operator is expressed using an orthogonal basis $\{e_1, \dots, e_n\}$ of $T_{\pi(x)} B \cong T^*_{\pi(x)} B$ as

$$\pi^* \mathfrak{D}_B = \sum h(\pi^* e_i) \nabla^{\pi^* \mathfrak{S}^E_{\mathbb{C}}(B)}_{\pi^* e_i}.$$

Now it satisfies

$$\pi^* \mathfrak{D}_B (\pi^* \varphi) = \pi^* (\mathfrak{D}_B \varphi).$$

Let $\{D_1, \dots, D_n\}$ be an n-tuple of fiberwise first-order pseudodifferential operators on E, that is, a smooth family $\{D_1(x), \dots, D_n(x)\}$ of pseudodifferential operators on $\Gamma(Z_x, E|_{Z_x})$. Moreover, we assume two conditions:

Condition 3.6. (1) The operators D_i and D_j commute for any i and j.

(2) The square sum $\sum_{i=1}^n D_i^2$ is fiberwise elliptic, that is, its principal symbol is invertible on $S(T_V M)$.

Then, by taking a trivialization of the Hilbert bundle of fiberwise L^2-sections $\mathcal{L}^2_f(M, E \hat{\otimes} \mathbb{C}\ell_n) := \{L^2(Z_x, E_x \hat{\otimes} \mathbb{C}\ell_n)\}_{x \in B}$,

$$\{D(x)\} = \{(D_1(x), \dots, D_n(x))\}$$

forms a continuous family of unbounded commuting Fredholm n-tuples parametrized by B.

Indeed, according to Kuiper's theorem, any Hilbert space bundles are trivial and $[D(x)]$ is independent of the choice of a trivialization. The second assertion holds because a trivialization of Hilbert bundle $\mathcal{V}$ gives a unitary $U \in \mathrm{Hom}_{C(X)}(C(X) \otimes \mathcal{H}, \Gamma(X, \mathcal{V}))$, and hence two trivializations U and U' give a norm-continuous unitary-valued function $U^{-1}U'$, which is homotopic to the identity. Combining with a connection on $\pi^* \mathfrak{S}_{\mathbb{C}}(B)$, which is fiberwise flat, the Dirac operator $D(x) = c_1 D_1(x) + \dots + c_n D_n(x)$ associated with $\{D(x)\}$ (here we denote by c the $\mathbb{C}\ell_n$-action on $\mathfrak{S}_{\mathbb{C}}(B)$ and write $c_i := c(e_i)$ for an orthonormal basis $\{e_i\}$) also defines a first-order pseudodifferential operator on $\pi^* \mathfrak{S}^E_{\mathbb{C}}(B)$.

Now we state our main theorem:

Theorem 3.7. *Let B, M, E, and $\{D(x)\}$ be as above. Then*

$$\mathrm{ind}_0(\pi^* \mathfrak{D}_B + D(x)) = \mathrm{jsf}\{D(x)\}.$$

Here, for an odd self-adjoint operator D on $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, we denote by $\mathrm{ind}_0 D$ *the Fredholm index of $D^0 : \mathcal{H}^0 \to \mathcal{H}^1$.*

To prove this theorem, we use a lemma about an operator inequality. In this section we denote $D(x)$ and $\pi^* \mathfrak{D}_B$ simply by D_f and D_b.

Lemma 3.8. *For any $\alpha \geq 0$ there is a constant $C > 0$ such that, for any $\xi \in \Gamma(M, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B))$,*

$$\langle [D_b, D_f]\xi, \xi\rangle \geq -\alpha \|D_f \xi\|^2 - C\|\xi\|^2. \tag{3.9}$$

Proof. First, we observe that $[D_b, D_f]$ is a fiberwise first-order pseudodifferential operator as well. Let $(V, x_b^1, \dots, x_b^n)$ be a local coordinate of $x \in B$ and $(U, x_b^1, \dots, x_b^n, x_f^1, \dots, x_f^m)$ a local coordinate in $\pi^{-1}(V)$ such that the tangent vectors $\partial_{x_b^i}(p)$ are in $(T_H M)_p$ for any $p \in \pi^{-1}(x)$. We get such a coordinate by identifying a neighborhood of the zero section of $T_H M|_{\pi^{-1}(x)} \cong N\pi^{-1}(x)$ with a tubular neighborhood of $\pi^{-1}(x)$. We assume that $\pi^* \mathfrak{S}^E_{\mathbb{C}}(B)$ is trivial on U, and fix a trivialization. Then, for any fiberwise pseudodifferential operator P supported in U, the operator $[\partial_{x_b^i}, P]$ is also fiberwise pseudodifferential. Indeed, when we write down a fiberwise pseudodifferential operator P on a bounded open subset of $\mathbb{R}^{n+m} = \mathbb{R}^n_{x_b} \times \mathbb{R}^m_{x_f}$ as

$$Pu(x_b, x_f) = \int_{(y_f, \xi_f) \in \mathbb{R}^m \times \mathbb{R}^m} e^{i\langle x_f - y_f, \xi_f\rangle} a(x_b, x_f, y_f, \xi_f) u(x_b, y_f)\, dy_f\, d\xi_f,$$

we have

$$\begin{aligned}
[\partial_{x_b^i}, P]u(x_b, x_f) &= \int \partial_{x_b^i}\big(e^{i\langle x_f - y_f, \xi_f\rangle} a(x_b, x_f, y_f, \xi_f) u(x_b, y_f)\big)\, dy_f\, d\xi_f \\
&\qquad - \int e^{i\langle x_f - y_f, \xi_f\rangle} a(x_b, x_f, y_f, \xi_f)\, \partial_{x_b^i} u(x_b, y_f)\, dy_f\, d\xi_f \\
&= \int e^{i\langle x_f - y_f, \xi_f\rangle} (\partial_{x_b^i}(a(x_b, x_f, y_f, \xi_f)) u(x_b, y_f))\, dy_f\, d\xi_f.
\end{aligned}$$

Let $D_b' := \sum g^{ij} h(\partial_{x_b^i}) \nabla_{\partial_{x_b^j}}$. Since the Riemannian metric g^{ij} on $T_H M$ only depends on the local coordinate of B (i.e., is a function on B), an operator $[D_b', P] = \big[\sum g^{ij} h(\partial_{x_b^i})(\partial_{x_b^j} + \omega(\partial_{x_b^j})), P\big]$ is also fiberwise pseudodifferential.

For any $\xi \in \Gamma(U, \pi^* \mathfrak{S}^E_{\mathbb{C}}(B)|_U)$, the section $[D_b, P]\xi|_{\pi^{-1}(x)}$ depends only on the restriction of ξ and its differentials in normal direction on $\pi^{-1}(x)$. Since the Dirac operator D_b coincides with D_b' on $U_0 := U \cap \pi^{-1}(x)$ and $[P, D_b']$ is fiberwise pseudodifferential, $[D_b, P]\xi|_{\pi^{-1}(x)} = [D_b', P]\xi|_{\pi^{-1}(x)}$ does not depend on the differentials of ξ. Now, the above argument is independent of the choice of $x \in B$. As a consequence, $[D_b, P]$ is also fiberwise pseudodifferential. By using a partition of unity, we can see that $[D_b, D_f]$ is also a fiberwise pseudodifferential operator.

As a consequence, we obtain that $[D_b, D_f](1 + D_f^2)^{-1/2}$ is a zeroth order pseudodifferential operator. In particular, it is bounded. Now, for any $\lambda > 0$, we obtain the inequality

$$\begin{aligned}\langle [D_b, D_f]\xi, \xi\rangle &= \langle \lambda[D_b, D_f]\xi, \lambda^{-1}\xi\rangle \\ &\geq -\tfrac{1}{2}\lambda^2\langle [D_b, D_f]\xi, [D_b, D_f]\xi\rangle - \tfrac{1}{2}\lambda^{-2}\langle \xi, \xi\rangle \\ &\geq -\tfrac{1}{2}\lambda^2\|[D_b, D_f](1+D_f^2)^{-1/2}\|^2\langle (1+D_f^2)\xi, \xi\rangle - \tfrac{1}{2}\lambda^{-2}\langle \xi, \xi\rangle \\ &= -\tfrac{1}{2}\lambda^2\|[D_b, D_f](1+D_f^2)^{-1/2}\|^2\langle D_f\xi, D_f\xi\rangle \\ &\quad - \tfrac{1}{2}\big(\lambda^2\|[D_b, D_f](1+D_f^2)^{-1/2}\|^2 + \lambda^{-2}\big)\langle \xi, \xi\rangle,\end{aligned}$$

as is introduced in Lemma 7.5 of [Kaad and Lesch 2012]. Now, by choosing $\lambda := \sqrt{2\alpha}\|[D_b, D_f](1+D_f^2)^{-1/2}\|^{-1}$ and $C := \alpha + \lambda^{-2}/2$, we show this C satisfies the above condition. □

Now we use the Connes–Skandalis-type sufficiency condition to realize the Kasparov product of unbounded Kasparov bimodules introduced by Kucerovsky [1997].

Theorem 3.10. *Suppose that* (E_1, φ_1, D_1), (E_2, φ_2, D_2), *and* $(E_1 \hat{\otimes} E_2, \varphi_1 \hat{\otimes} 1, D)$ *are unbounded Kasparov bimodules for* (A, B), (B, C), *and* (A, C) *such that the following conditions hold*:

(1) *For all x in some dense subset of $\varphi_1(A)E_1$, the operator*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2\end{pmatrix}, \begin{pmatrix} 0 & T_x \\ T_x^* & 0\end{pmatrix}\right]$$

is bounded on $\operatorname{dom}(D \oplus D_2)$.

(2) *The resolvent of D is compatible with $D_1 \hat{\otimes} 1$.*

(3) *For all x in the domain, $\langle D_1 x, Dx\rangle + \langle Dx, D_1 x\rangle \geq \kappa\langle x, x\rangle$.*

Here $x \in E_1$ is homogeneous and $T_x : E_2 \to E$ maps $e \mapsto x \hat{\otimes} e$. Then

$$[E_1 \hat{\otimes} E_1, \varphi_1 \hat{\otimes} 1, D] \in KK(A, C)$$

represents the Kasparov product of $[E_1, \varphi_1, D_1] \in KK(A, C)$ *and* $[E_2, \varphi_2, D_2] \in KK(B, C)$.

Here the resolvent of D is said to be *compatible* with D' if there is a dense submodule $\mathcal{W} \subset E_1 \hat{\otimes} E_2$ such that $D'(i\mu + D)^{-1}(i\mu' + D')^{-1}$ is defined on $\mathcal{W}$ for any $\mu, \mu' \in \mathbb{R} \setminus \{0\}$. It holds, for example, in the case that $\operatorname{dom} D \subset \operatorname{dom} D'$.

Proof of Theorem 3.7. According to Lemma 3.5, the remaining part for the proof is that the left-hand side coincides with the pairing $\langle [\operatorname{ind} D], [\mathfrak{D}_B]\rangle_n$. Here this pairing is given by the Kasparov product $KK(\mathbb{C}, C(B) \hat{\otimes} \mathbb{C}\ell_n) \otimes KK(C(B) \hat{\otimes} \mathbb{C}\ell_n, \mathbb{C}) \to \mathbb{Z}$.

It is computed as follows:

$$\begin{aligned}[][\mathcal{L}^2(M, E \hat{\otimes} \mathbb{C}\ell_n), 1, D] \otimes_{C(B)\hat{\otimes}\mathbb{C}\ell_n} [L^2(B, \mathfrak{S}_{\mathbb{C}}(B)), m, \mathfrak{D}_B] \\ = \big[L^2(M, (E \hat{\otimes} \mathbb{C}\ell_n) \hat{\otimes}_{\mathbb{C}\ell_n} \pi^*\mathfrak{S}_{\mathbb{C}}(B)), 1, \mathfrak{D}_B \times D\big] \\ = [L^2(M, \pi^*\mathfrak{S}_{\mathbb{C}}(B)^E), 1, \mathfrak{D}_B \times D].\end{aligned}$$

Now it remains to prove that $D_b + D_f$ satisfies conditions (1), (2), and (3) of Theorem 3.10.

For any $\sigma \in C^\infty(M, E)$ and $\xi \in C^\infty(B, \mathfrak{S}_{\mathbb{C}}(B))$, the Leibniz rule of $\pi^*\mathfrak{D}_B$ implies that

$$\begin{aligned}(D_b + D_f)T_\sigma \xi &= (D_b + D_f)(\sigma \cdot \pi^*\xi) = (D_b + D_f)x \cdot \pi^*\xi + \sigma \cdot D_b \pi^*\xi \\ &= T_{(D_b+D_f)\sigma}\xi + \sigma \cdot \pi^*(\mathfrak{D}_B \xi).\end{aligned}$$

Therefore $(D_b + D_f)T_\sigma - T_\sigma \mathfrak{D}_B = T_{(D_b+D_f)\sigma}$ is a bounded operator and hence condition (1) holds. Condition (2) holds since $\operatorname{dom}(D_b + D_f) \subset \operatorname{dom} D_f$. For any $\xi \in C^\infty(M, \mathfrak{S}_{\mathbb{C}}^E(M))$, which is dense in the domain,

$$\langle D_f \xi, (D_b + D_f)\xi \rangle + \langle (D_b + D_f)\xi, D_f \xi \rangle = \langle [D_b, D_f]\xi, \xi \rangle + \|D_f \xi\|^2.$$

Condition (3) follows from this and Lemma 3.8. □

Remark 3.11. The calculus above is motivated by that of [Connes and Skandalis 1984], in which the authors dealt with principal symbols and zeroth order pseudo-differential operators. Here we use the unbounded operators directly in order to apply it for more general cases. For example, by the same argument we obtain a similar formula

$$\operatorname{ind}_0(D + A(x)) = \operatorname{jsf}(\{A(x)\})$$

for a smooth family of mutually commuting self-adjoint complex coefficient matrices $A(x) = (A_1(x), \dots, A_n(x))$. Other examples are given in the next section.

3B. ***A Callias-type index theorem for open manifolds.*** Now we consider generalizing our index theorem to the case of noncompact base spaces. The pairing of homology and cohomology works in the noncompact case if the cohomology is replaced with the one with compact support. We can deal with it in the context of an infinite-dimensional analogue of Callias-type [1978] operators. Here we use fiber-wise elliptic operators as the potential term in the original theory of Callias. First we define the admissibility of a connective K-cocycle (see also [Bunke 1995]).

Definition 3.12. We call a continuous family of commuting Fredholm n-tuples $\{D_1, \dots, D_n\}$ parametrized by a complete Riemannian manifold B *admissible* if there is a constant $\kappa > 0$ such that:

(1) $D(x)^2 \geq \kappa > 0$ for $x \in X \setminus K$.

(2) There are $C_1, C_2 > 0$ such that $\langle([D_b, D_f] + D_f^2)\xi, \xi\rangle \geq C_1\|D_f\xi\|^2 - C_2\|\xi\|^2$ and $\kappa C_1 > C_2$.

Actually the second condition is not essential.

Lemma 3.13. *For any continuous family $\{D_1, \dots, D_n\}$ of commuting Fredholm n-tuples parametrized by a complete n-dimensional Riemannian manifold B that satisfies condition* (1) *above, there is some $t > 0$ such that $tD := (tD_1, \dots, tD_n)$ is admissible.*

Proof. By a similar calculus to the one in Lemma 3.8 (we replace D_f in the first term with tD_f but do not replace the one that arises in $(1 + D_f^2)$ in the middle part) we obtain that, for any $\lambda > 0$,

$$\langle[D_b, tD_f]\xi, \xi\rangle = -\tfrac{1}{2}\lambda^2 R\langle D_f\xi, D_f\xi\rangle - \tfrac{1}{2}(\lambda^2 R + \lambda^{-2})\langle\xi, \xi\rangle,$$

where we set $R := \|[D_b, D_f](1 + D_f^2)^{-1/2}\|^2$. Now, if we choose $\lambda = R^{-1/2}$, then

$$\langle([D_b, tD_f] + (tD_f)^2)\xi, \xi\rangle \geq \frac{t^2}{2}\|D_f\xi\|^2 - \Big(\frac{t^2}{2} + R\Big)\|\xi\|^2.$$

Now we can take a constant κ in condition (1) for tD_f to be $t\kappa$. When we set $C_1 = t^2/2$ and $C_2 = t^2/2$, for sufficiently large $t > 0$ the inequality $(t\kappa)C_1 \geq C_2$ holds and hence the constants $t\kappa$, C_1, and C_2 satisfies condition (2). $\square$

Now we introduce a geometric setting and an index theorem for the noncompact case.

Let B be a complete n-dimensional manifold, $Z \to M \to B$ a smooth fiber bundle over B with fixed decomposition of the tangent bundle $TM \cong T_V M \oplus T_H M$, E a smooth complex vector bundle over M and $\{D_1, \dots, D_n\}$ an n-tuple of fiberwise first-order pseudodifferential operators on E that satisfies Condition 3.6. Moreover, we assume that $\{D_1, \dots, D_n\}$ is admissible.

Theorem 3.14. *In the above situation, the operator $\pi^*\mathfrak{D}_B + D(x)$ is Fredholm and*

$$\mathrm{ind}_0(\pi^*\mathfrak{D}_B + D(x)) = \mathrm{jsf}\{D(x)\}.$$

Proof. The proof is essentially the same as for Theorem 3.7; the remaining part is to show that $\mathfrak{D}_B + D(x)$ is a Fredholm operator. We prove this by using an estimate motivated by Theorem 3.7 of [Gromov and Lawson 1983]. Here we use the notation D_b and D_f again. Let E_λ ($\lambda \in \mathbb{R}$) be the λ-eigenspace for the self-adjoint operator $D_b + D_f$. Now we fix an $\alpha > 0$. Then, for any $\sigma \in \bigoplus_{|\lambda|<\alpha} E_\lambda$,

$$\begin{aligned} 0 \le \|D_b\sigma\|^2 &\le \|(D_b + D_f)\sigma\|^2 - (([D_b, D_f] + D_f^2)\sigma, \sigma) \\ &\le \alpha\|\sigma\|^2 - C_1\|D_f\sigma\|^2 + C_2\|\sigma\|^2 \\ &\le (\alpha + C_2)\|\sigma\|^2 - C_1\|D_f\sigma\|^2_{B\setminus K} \\ &\le (\alpha - \kappa C_1 + C_2)\|\sigma\|^2 + \kappa C_1\|\sigma\|^2_K. \end{aligned}$$

By assumption we can retake $\alpha > 0$ such that $\kappa C_1 - C_2 > \alpha$. Then there is a constant $C > 0$ such that

$$C\|\sigma\| \le \|\sigma\|_K.$$

Now we take a parametrix Q of the elliptic operator $D_b + D_f$ and set $\mathcal{S} := 1 - QD$. Take P to be the projection from $L^2(M, \pi^*\mathfrak{S}^E_{\mathbb{C}}(B))$ to the subspace $L^2(\pi^{-1}(K), \pi^*\mathfrak{S}^E_{\mathbb{C}}(B)|_{\pi^{-1}(K)})$. Then $P\mathcal{S}$ is a compact operator and

$$\|P\mathcal{S}\sigma\| \ge \|P\sigma\| - \|PDQ\sigma\| \ge C\|\sigma\| - \alpha\|PQ\|\|\sigma\|.$$

Taking $\alpha > 0$ sufficiently small, we see that $P\mathcal{S}$ is bounded below by $C - \alpha\|PQ\| > 0$. This implies that $\bigoplus E_\lambda$ is finite-dimensional, since a compact operator on it is bounded below by some positive number. □

Example 3.15 (the case $B = \mathbb{R}$). Let $\{A(t)\}_{t\in\mathbb{R}}$ be a continuous family of self-adjoint matrices such that there is a $\lambda > 0$ and two self-adjoint invertible matrices A_+, A_- such that $A_t = A_-$ for $t \le -\lambda$ and $A_t = A_+$ for $\lambda \le t$. Now, as is noted in Remark 3.11, we have a finite-dimensional analogue of Theorem 3.14. In the 1-dimensional case it is of the form

$$\operatorname{ind}\left(\frac{d}{dt} + cA_t\right) = \operatorname{sf}(\{A_t\})$$

for sufficiently large $c > 0$. Now obviously its right-hand side is given by the difference

$$\#\{\text{negative eigenvalues of } A_-\} - \#\{\text{negative eigenvalues of } A_+\}.$$

It is nonzero in general, whereas in the case that the parameter space is a circle we have to deal with operators on an infinite-dimensional Hilbert space to obtain examples of nontrivial indices.

Example 3.16. Let B be a complete Spin^c manifold with compactification $\overline{B}$, $Z_1, \dots, Z_n$ closed odd-dimensional Spin^c manifolds and $\{g_x^1, \dots, g_x^n\}_{x\in\overline{B}}$ a smooth family of metrics on $Z_1, \dots, Z_n$ such that the scalar curvature of the product manifold $Z := Z_1 \times \cdots \times Z_n$ is uniformly strictly positive outside a compact subset $K \subset B$. We denote by $\not{D}_{i,x}$ the Dirac operator on Z_i with respect to the metric

g^i_x. Then there is a constant $\lambda > 0$ such that $(\lambda \not{D}_{1,x}, \dots, \lambda \not{D}_{n,x})$ is an admissible family of commuting Fredholm n-tuples and the Fredholm index of the Spin^c Dirac operator on $M := B \times Z$ with respect to the product metric coincides with its joint spectral flow. This gives a map

$$\mathrm{ind} : \big[(\bar{B}, \partial B), (\mathcal{R}(Z_1, \dots, Z_n), \mathcal{R}(Z_1, \dots, Z_n)_{\geq \lambda})\big] \to \mathbb{Z},$$

where $\mathcal{R}(Z_1, \dots, Z_n)$ is the product of spaces of Riemannian metrics $\mathcal{R}(Z_1) \times \cdots \times \mathcal{R}(Z_n)$ and $\mathcal{R}(Z_1, \dots, Z_n)_{\geq \lambda}$ is the subspace of $\mathcal{R}(Z_1, \dots, Z_n)$ such that the scalar curvature of the product metric $(Z_1, g_1) \times \cdots \times (Z_n, g_n)$ is larger than $\lambda > 0$ (its homotopy type is independent of the choice of λ). In particular, when we choose B to be $\mathbb{R}^n$, the left-hand side is isomorphic to $\pi_{n-1}(\mathcal{R}(Z_1, \dots, Z_n)_{\geq \lambda})$ because $\mathcal{R}(Z_1, \dots, Z_n)$ is contractible.

3C. *Families twisted by a vector bundle.* In this section we generalize the joint spectral flow and its index theorem to the case of V-twisted families of commuting Fredholm n-tuples introduced at the end of Section 2. It is essential in Section 4A.

Let V be a real vector bundle. Denote the fiber bundle $\mathrm{GL}(V) \times_{\mathrm{GL}(n,\mathbb{R})} P(S^n, *)$ by P_V. The set of homotopy classes of continuous sections $\pi_0 \Gamma(X, P_V)$ forms the twisted cohomology group $H^V(X; \mathbb{Z})$. Now, twists of the ordinary cohomology theory are classified by $H^1(X, \mathbb{Z}/2)$, and in our case the corresponding cohomology classes are determined by the orientation bundle of V. As in Definition 3.1, the continuous map $j : F_V(\mathcal{H}) \to P_V$ induces the natural transform $j_* : k^V \to H^V$.

Definition 3.17. Let X be an oriented closed manifold of dimension n and V an n-dimensional oriented vector bundle. For a V-twisted continuous family $\{T(x)\}_{x \in X}$ of commuting Fredholm n-tuple, we say that the integer $\langle j_*[\{T(x)\}], [X] \rangle \in \mathbb{Z}$ is its *joint spectral flow*, denoted by $\mathrm{jsf}(\{T(x)\})$. Here we identify the two groups $H^V(X; \mathbb{Z})$ and $H^n(X; \mathbb{Z})$ in the canonical way. For a V-twisted continuous family of bounded or unbounded commuting Fredholm n-tuple $\{T(x)\}$, we say $\mathrm{jsf}(\iota\{T(x)\})$ is its joint spectral flow, denoted simply by $\mathrm{jsf}(\{T(x)\})$.

Now we introduce the corresponding geometric setting and prove a generalization of the joint spectral flow index theorem (Theorem 3.7) for a family twisted by a Spin^c vector bundle.

Let B be a closed n-dimensional Spin^c manifold, $Z \to M \to B$ a smooth fiber bundle over B such that the total space M is also a Spin^c manifold, V an n-dimensional Spin^c vector bundle over B and E a smooth complex vector bundle over M. We denote by $\Psi^1_f(M, E)$ the fiber bundle over B whose fiber on $x \in B$ is the space of first-order pseudodifferential operators on $\Gamma(Z_x, E|_{Z_x})$. We consider a map of B-bundles $\{D_v(x)\}_{(x,v) \in V \setminus \{0\}} : V \setminus \{0\} \to \Psi^1_f(M, E)$ that satisfies the following conditions:

Condition 3.18. (1) The operators $D_v(x)$ and $D_w(x)$ commute for any $v, w \in V_x \setminus \{0\}$.

(2) The equality $g \cdot (D_{v_1}(x), \dots, D_{v_n}(x)) = (D_{g\cdot v_1}(x), \dots, D_{g\cdot v_n}(x))$ holds for any $g \in \mathrm{GL}(n;\mathbb{R})$ and basis $(v_1, \dots, v_n)$ of V_x.

(3) The square sum $\sum_{i=1}^n D_{v_i}^2$ is fiberwise elliptic, that is, its principal symbol is invertible on $S(T_V M)$, for an orthonormal basis $\{v_1, \dots, v_n\}$.

Then it forms a V-twisted continuous family of unbounded commuting Fredholm n-tuples $\{D(x)\}$.

Next, we replace the fundamental KK-class on B with the one that is compatible with $\{D(x)\}$. Instead of $\mathfrak{S}_{\mathbb{C}}(M)$, we consider the spinor bundle $\mathfrak{S}_{\mathbb{C}}(B; V) := \$_{\mathbb{C}}(TB \oplus V)$ for an even-dimensional Spin^c vector bundle $TB \oplus V$. It is equipped with the action of $\mathbb{C}\ell(TB) \hat{\otimes} \mathbb{C}\ell(V)$. Here we denote by c and h its restriction on $\mathbb{C}\ell(V) \hat{\otimes} 1$ and $1 \hat{\otimes} \mathbb{C}\ell(TB)$ respectively. Now we define a pullback of the Dirac operator $\pi^*\mathfrak{D}_B^V$ twisted by E in a similar way to the one in Section 3A.

Theorem 3.19. *Let B, M and $D(x)$ be as above. Then*

$$\operatorname{ind}(\pi^*\mathfrak{D}_B^V + D(x)) = \operatorname{jsf}\{D(x)\}.$$

Proof. First we embed V into a trivial real vector bundle $\underline{\mathbb{R}}^p$ linearly, and denote its orthogonal complement by W.

We define the KK-classes

$$[D_W] := \Big[\mathcal{L}_f^2(W, \mathbb{C}\ell(\pi^*W)), m, D_W := \sum h(e_i)\frac{\partial}{\partial w_i}\Big] \in KK(\Gamma_0\mathbb{C}\ell(\pi^*W), C(B)),$$
$$[C_W] := \Big[\Gamma_0\mathbb{C}\ell(\pi^*W), m, C_W := \sum c(e_i)w_i\Big] \in KK(C(B), \Gamma_0\mathbb{C}\ell(\pi^*W)),$$

where $\{e_i\}$ is an orthonormal basis on W_x and $w_i = \langle w, e_i\rangle$ the coordinate functions with respect to $\{e_i\}$. We mention that D_W and C_W are independent of the choice of $\{e_i\}$, and hence are well-defined. Then, the theory of harmonic oscillators (see, e.g., [Higson and Guentner 2004, Section 1.13]) shows that $[D_W] \otimes_{\Gamma_0\mathbb{C}\ell(\pi^*W)} [C_W] = [D_W + C_W] = 1 \in KK(C(B), C(B))$ because the kernel of the harmonic oscillator is 1-dimensional and $O(n)$-invariant. Now

$$D \times C_W = (D_{v_1}, \dots, D_{v_n}, c_{w_1}, \dots, c_{w_k})$$

is a smooth family of commuting Fredholm n-tuples twisted by $V \oplus W \cong \underline{\mathbb{R}}^p$. Moreover it is admissible on W because $(D \times C_W)^2 = D^2 + \|w\|^2$. According to Theorem 3.14,

$$\operatorname{ind}(D_b + D_f + D_W + C_W) = \operatorname{jsf}(\{D \otimes C_W(x, w)\}) = \operatorname{jsf}(\{D(x)\}).$$

On the other hand, by the associativity of the Kasparov product

$$\begin{aligned}\operatorname{ind}(D_b+D_f+D_W+C_W) &= [D_f+C_W]\otimes_{\Gamma_0(\pi^*W)}[D_W+D_b] \\ &= ([D_f]\otimes_{C(B)}[C_W])\otimes_{\Gamma_0(\pi^*W)}([D_W]\otimes_{C(B)}[D_f]) \\ &= [D_f]\otimes_{C(B)}[D_b] = \operatorname{ind}(D_b+D_f). \qquad \square\end{aligned}$$

Some examples of geometric situations in which this theorem is applied are introduced in Section 4A.

4. Applications

In this section we introduce some applications of the joint spectral flow and its index theorem.

4A. ***Witten deformation and localization.*** It is easy to obtain the joint spectral flow of a continuous family of commuting Fredholm n-tuples when their joint spectra intersect with zero transversally. In such cases we often reduce the problem of computing the index (which usually requires solving some linear partial differential equations or integrating some characteristic classes) to that of counting the number of points with multiplicity.

The most typical example is the classical Poincaré–Hopf theorem.

Corollary 4.1 (the Poincaré–Hopf theorem). *Let M be a closed* Spin^c *manifold and X a vector field on M whose null points $M^X := \{p \in M \mid X(p) = 0\}$ are isolated. Then*

$$\chi(M) = \sum_{p\in M^X} \nu_p.$$

This proof is essentially the same as that of [Witten 1982]. Here we restrict M to Spin^c manifolds, but this is not an essential assumption.

Proof. By the Hodge–Kodaira decomposition, the Euler number $\chi(M)$ can be computed as the index of the de Rham operator $D_{\mathrm{dR}} := d+d^* : \Gamma\big(\bigwedge^{\mathrm{even/odd}} TM\big) \to \Gamma\big(\bigwedge^{\mathrm{odd/even}} TM\big)$. Now $\mathbb{C}\ell(TM)$ acts on $\mathbb{C}\ell(TM)$ in two ways: $c(v)\xi := v\cdot\xi$ and $h(v)\xi := \gamma(\xi)\cdot v$ (for $v \in TM$ and $\xi \in \mathbb{C}\ell(TM)$), where γ is the grading operator on $\mathbb{C}\ell(TM) \cong \mathbb{C}\ell(TM)^0 \oplus \mathbb{C}\ell(TM)^1$. They induce the $\mathbb{C}\ell(TM) \hat{\otimes} \mathbb{C}\ell(TM)$-action on $\mathbb{C}\ell(TM)$ because $c(v)$ and $h(v)$ anticommute. Because M is a Spin^c manifold, it is a unique irreducible $\mathbb{C}\ell(TM \oplus TM)$-module $\$_{\mathbb{C}}(TM \oplus TM)$. By Leibniz's rule,

$$D_{\mathrm{dR}}(\gamma(\xi)\cdot X) = -\gamma(D_{\mathrm{dR}}\xi)\cdot X + (-1)^{\partial\gamma(\xi)}\gamma(\xi)\cdot D_{\mathrm{dR}}(X)$$

where we use the fact that D_{dR} is an odd operator. This means that D_{dR} and $h(X)$ anticommute modulo the bounded operator $(-1)^{\partial\xi+1}h(D_{\mathrm{dR}}(X))$. This shows that $D_{\mathrm{dR}}+th(X)$ is Fredholm for any $t>0$, because $(D_{\mathrm{dR}}+th(X))^2 = D_{\mathrm{dR}}^2 + t^2\|X\|^2 + t[D_{\mathrm{dR}}, h(X)]$ is a bounded perturbation of the Laplace operator D_{dR}^2, which is positive with compact resolvent. On the other hand, $h(X)=\sum\langle e_i, X\rangle h(e_i)$ is a commuting n-tuple of Fredholm operators twisted by TM (now we regard $\langle e_i, X\rangle$ as Fredholm operators on the 1-dimensional vector space). As a consequence of Theorem 3.19 (and Remark 3.11), we have

$$\chi(M) = \operatorname{ind}(D_{\mathrm{dR}}) = \operatorname{ind}(D_{\mathrm{dR}}+h(X)) = \operatorname{jsf}(\{\langle e_i, X\rangle\}) = \sum_{p\in M^X} \nu_p.$$

The last equation follows from the definition of the joint spectral flow. □

Now we consider an infinite-dimensional analogue of this approach for localization problems of indices.

Let B be an n-dimensional closed Spin^c manifold, $M_1, \dots, M_n \to B$ fiber bundles such that each fiber $Z_1, \dots, Z_n$ is an odd-dimensional closed manifold and the $T_V M_i$ are equipped with Spin^c structures, and E a complex vector bundle on $M := M_1 \times_B \cdots \times_B M_n$. Now $TB \oplus \underline{\mathbb{R}}^n$ is a $2n$-dimensional vector bundle and hence there is a unique $\mathbb{C}\ell(TB\oplus\underline{\mathbb{R}}^n)$-module bundle $\$_{\mathbb{C}}(T_V M\oplus\underline{\mathbb{R}}^n)$. We denote by $\$_{\mathbb{C}}(T_V M_i) \cong \$^0_{\mathbb{C}}(T_V M_i) \oplus \$^1_{\mathbb{C}}(T_V M_i)$ the unique $\mathbb{Z}/2$-graded $\mathbb{C}\ell(T_V M_i)$-module bundle, which is isomorphic to $\$_{\mathbb{C}}(T_V M_i \oplus \underline{\mathbb{R}})$. Then it decomposes into tensor products as

$$\begin{aligned} \$_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) &\cong \$_{\mathbb{C}}(T_V M_1 \oplus \underline{\mathbb{R}}) \mathbin{\hat\otimes} \cdots \mathbin{\hat\otimes} \$_{\mathbb{C}}(T_V M_n \oplus \underline{\mathbb{R}}) \\ &\cong (\$^0_{\mathbb{C}}(T_V M_1) \mathbin{\hat\otimes} \mathbb{C}\ell_1) \mathbin{\hat\otimes} \cdots \mathbin{\hat\otimes} (\$^0_{\mathbb{C}}(T_V M_n) \mathbin{\hat\otimes} \mathbb{C}\ell_1) \\ &\cong (\$^0_{\mathbb{C}}(T_V M_1) \otimes \cdots \otimes \$^0_{\mathbb{C}}(T_V M_n)) \mathbin{\hat\otimes} \mathbb{C}\ell_n. \end{aligned}$$

Hereafter we set $\mathfrak{S}_{\mathbb{C},f}(M;\underline{\mathbb{R}}^n) := \$_{\mathbb{C}}(T_V M\oplus\underline{\mathbb{R}}^n)$ and $\$^0_{\mathbb{C},f}(M;\underline{\mathbb{R}}^n) := \$^0_{\mathbb{C}}(T_V M_1)\otimes \cdots \otimes \$^0_{\mathbb{C}}(T_V M_n)$. The inclusions $T_V M \subset T_V M \oplus \underline{\mathbb{R}}^n$ and $\underline{\mathbb{R}}^n \subset T_V M \oplus \underline{\mathbb{R}}^n$ induce the actions of $\mathbb{C}\ell(TM)$ and $\mathbb{C}\ell_n$ on $\mathfrak{S}_{\mathbb{C},f}(M;\underline{\mathbb{R}}^n)$. Under the above identification, a vector $v = v_1 \oplus \cdots \oplus v_n \in T_V M$ acts as

$$(c(v_1)\otimes 1 \otimes \cdots \otimes 1)\otimes c_1 + \cdots + (1\otimes \cdots \otimes 1 \otimes c(v_n))\otimes c_n$$

and $\mathbb{C}\ell_n$ acts as $1\otimes h$ (here we denote the left and twisted right actions of $\mathbb{C}\ell_n$ on $\mathbb{C}\ell_n$ by c and h). Hence the fiberwise Dirac operator $\not{D}_f$ decomposes as

$$\not{D}_f = c_1 \not{D}_1 + \cdots + c_n \not{D}_n,$$

where the $\not{D}_i$ are Dirac operators for the M_i direction

$$\not{D}_i : \Gamma(M, \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) \xrightarrow{d} \Gamma(M, \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) \otimes T^*M))$$
$$\xrightarrow{p_{T_V M_i}} \Gamma(\not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n)) \xrightarrow{c} \Gamma(M, \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n)).$$

Similarly, the twisted spinor bundle $\mathfrak{S}^E_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n) := \not{S}_{\mathbb{C}}(T_V M \oplus \underline{\mathbb{R}}^n) \otimes E$ is isomorphic to $\not{S}^{0,E}_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n) \hat{\otimes} \mathbb{C}\ell_n$. Moreover if E is equipped with a connection ∇^E whose curvature R^E satisfies $R^E(X, Y) = 0$ for any $X \in T_V M_i$ and $Y \in T_V M_j$ ($i \neq j$), then the Dirac operator twisted by E decomposes as $\not{D}^E = c_1 \not{D}^E_1 + \cdots + c_n \not{D}^E_n$, such that $\not{D}_i$ commutes with $\not{D}_j$. Now $(\not{D}^E_1, \cdots, \not{D}^E_n)$ forms a smooth family of unbounded commuting Fredholm n-tuples, and $\not{D}^E_f$ is the smooth family of the Dirac operators associated with it.

More generally, we obtain some examples of twisted commuting Fredholm n-tuples. Let V be a real vector bundle whose structure group is a discrete subgroup G of $\mathrm{GL}(n, \mathbb{R})$, $B' = G(V)$ the frame bundle of V, $M'_1, \dots, M'_n$ fiber bundles over B' with a G-action on $M' := M_1 \times_{B'} \cdots \times_{B'} M'_n$ that is compatible with the projection $M' \to B'$ and E a G-equivariant vector bundle on M' whose connection ∇ is G-equivariant and satisfies the above assumption on the curvature. The G-action on M' induces a unitary representation U_x of G on $L^2(Z'_x, \mathfrak{S}^E_{\mathbb{C}}(Z'_x))$, where $Z'_x := \pi'^{-1}(x)$ (π' is the projection from M' to B). We assume that

$$U_x(g) \not{D}^E_i U_x(g)^* = \sum g_{ij} \not{D}^E_j .$$

Then $\big(g = (v_1, \dots, v_n), (\not{D}^E_1(x, g), \dots, \not{D}^E_n(x, g))\big) \in B' \times \mathcal{F}_n(\mathcal{H})$ is G-invariant, and hence the map $x \mapsto \not{D}^E_v(x)$ is well-defined and determines a V-twisted smooth family of commuting Fredholm n-tuples.

There are two fundamental examples. The first is the $\mathrm{SL}(n, \mathbb{Z})$-action on $\mathbb{T}^n = (S^1)^n$ or the product bundle $\mathbb{T}^n \times B$. The second is the $\mathfrak{S}_n$-action on the bundle $M' \times_B \cdots \times_B M'$. Then the fiberwise Dirac operator on the fiber bundle $M := M'/G \to B$ coincides with the Dirac operator associated with $\{\not{D}^E_v(x)\}_{x \in B}$.

Theorem 4.2. *Let B, M, V, E, and ∇ be as above. Then*

$$\mathrm{ind}_0(\not{D}^E_M) = \mathrm{jsf}\{\not{D}^E_v(x)\}.$$

This theorem is a direct consequence of Theorem 3.7 since the Dirac operator $\not{D}^E_M$ has the same principal symbol as $\pi^* \mathfrak{D}_B + \not{D}^E_f(x)$. As a special case, we can show localization of the Riemann–Roch number for Lagrangian fiber bundles on their Bohr–Sommerfeld fibers:

Corollary 4.3. *Let (M, ω) be a symplectic manifold of dimension $2n$, $\mathbb{T}^n \to M \to B$ a Lagrangian fiber bundle, and (L, ∇^L, h) its prequantum data, that is, (L, h) is a hermitian line bundle over M with connection ∇^L that is compatible with h*

whose curvature form R^L coincides with $-2\pi i\omega$. Then the Riemann–Roch number $\mathrm{RR}(M, L) := \mathrm{ind}_0 \, \not{D}_M^{\lambda^{1/2}\otimes L}$ (*where λ is the determinant line bundle* $\det T^{(1,0)}M$) *coincides with the number of fibers $\mathbb{T}_x$ such that $\nabla^L|_{\mathbb{T}_x}$ is flat with trivial monodromy*, *which are called the Bohr–Sommerfeld fibers.*

Proof. The structure of Lagrangian fiber bundles is studied in Section 2 of [Duistermaat 1980], and the following are known:

- Fact 1: There is a lattice bundle $P \subset TB$, which induces a flat metric on TB.
- Fact 2: If P is trivial, M is actually a principal $\mathbb{T}^n$-bundle.

We denote the $\mathrm{GL}(n, \mathbb{Z})$-frame bundle of TB by B' and the pullback of M by the quotient $B' \to B$ by M'. The manifold M' has a canonical symplectic structure and $M' \to B'$ is also a Lagrangian fiber bundle. By Fact 2, M' is a principal $\mathbb{T}^n$-bundle on B'. We identify the space of constant vector fields on a fiber M'_x with the Lie algebra $\mathfrak{t} = \mathrm{Lie}(\mathbb{T}^n)$.

The free $\mathrm{GL}(n, \mathbb{Z})$-action on B' extends to an action on M' preserving its symplectic form and affine structure on each fiber. Therefore it induces an action on $\mathfrak{t}$ as $g \cdot X_i = g_{ij} X_j$ for some fixed basis $X_1, \dots, X_n$ of $\mathfrak{t}$. Indeed, by considering the canonical trivialization of the tangent bundle $TB' \cong B' \times \underline{\mathbb{R}}^n$ that is compatible with the isomorphism $\mathfrak{t} \cong T_x B'$ given by a fixed almost-complex structure J, we obtain an isomorphism $\mathfrak{t} \cong T_x B' \cong \mathbb{R}^n$ that is independent of the choice of $x \in B'$. Under this identification, $g\cdot : \mathbb{R}^n \cong T_x B' \to T_{g\cdot x} B' \cong \mathbb{R}^n$ is represented by (g_{ij}) as a matrix. Hence g also acts on $\mathfrak{t}$ as (g_{ij}).

Next, we construct some flat connections. The isomorphism $T_V M \cong T_H M \cong \pi^* TB$ induced by J implies the isomorphism $\not{\mathfrak{S}}_{\mathbb{C},f}(M; \underline{\mathbb{R}}^n) \cong \not{S}_{\mathbb{C}}(M) \cong \pi^* \not{\mathfrak{S}}_{\mathbb{C}}(B)$. Moreover, it induces a flat metric on TM that is trivially flat on each fiber $\mathbb{T}^n$, and so are associated bundles with TM, in particular $\lambda^{1/2}$ and $\not{S}_{\mathbb{C}}^{\lambda^{1/2}}(M)$. Since R^L is equal to 0 when it is restricted on each fiber, ∇^L is also fiberwise flat and the product connection $\nabla = \nabla^{\not{S}_{\mathbb{C}}^{\lambda^{1/2}\otimes L}(M)}$ is trivially flat if and only if ∇^L is trivially flat.

Finally we see that B, M, $V = TB$, $E = \lambda^{1/2} \otimes L$ and $\nabla^{\lambda^{1/2}\otimes L}$ satisfy the assumptions of Theorem 4.2. Then $\{\nabla_v(x)\}$ forms a family of commuting Fredholm n-tuples twisted by TB and the index of the Dirac operator $\not{D}_M^{\lambda^{1/2}\otimes L}$ coincides with the joint spectral flow.

The kernel of $\Delta_f := \nabla_{e_1}^2 + \cdots + \nabla_{e_n}^2$ is not zero if and only if ∇ is (and hence ∇^L is) trivially flat. This means that the joint spectra of $\{\nabla(x)\}$ cross over zero only on the Bohr–Sommerfeld fibers. The remaining part is that the multiplicity of eigenvalues crossing zero on each Bohr–Sommerfeld fiber is equal to 1. This follows from the fact in symplectic geometry that the tubular neighborhood of a Lagrangian submanifold is isomorphic to its cotangent bundle as a symplectic manifold, and that $T^*\mathbb{T}^n$ is actually the product space $(T^*S^1)^n$. More detail is found in Section 6.4 of [Fujita et al. 2010]. $\square$

4B. ***Generalized Toeplitz index theorem.*** In this section we introduce a generalization of a classical theorem relating the index of Toeplitz operators with the winding numbers.

Definition 4.4. Let Y be an n-dimensional closed manifold with $n = 2m - 1$. For $\varphi : Y \to U(k)$ the generalized Toeplitz operator T_φ is defined by

$$Pm_\varphi P : PL^2(Y, \mathbin{\$}_{\mathbb{C}}(Y))^{\oplus k} \longrightarrow PL^2(Y, \mathbin{\$}_{\mathbb{C}}(Y))^{\oplus k},$$

where P is the orthogonal projection onto $\overline{\operatorname{span}}\{\varphi \mid D\!\!\!/\varphi = \lambda\varphi \text{ for some } \lambda \geq 0\}$.

Example 4.5 ($Y = S^1$). When $Y = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (and hence when $\mathbin{\$}_{\mathbb{C}}(Y)$ associated with the canonical Spin^c-structure on Y is a trivial bundle), we can identify its Dirac operator as d/dt. Its spectrum coincides with $\mathbb{Z}$, and the eigenspaces E_n are the 1-dimensional complex vector spaces $\mathbb{C} \cdot e^{int}$. Therefore $PH = \overline{\operatorname{span}}\{e^{int} \mid n \in \mathbb{Z}_{\geq 0}\}$, and the corresponding generalized Toeplitz operators T_φ are simply the ordinary ones. Its index is obtained from the winding number by $\operatorname{ind} T_\varphi = -\operatorname{winding} \varphi$.

Now we generalize this index theorem for generalized Toeplitz operators in a special case. Let $\Delta_n = \Delta_n^0 \oplus \Delta_n^1$ be a unique irreducible $\mathbb{Z}/2$-graded $\mathbb{C}\ell_n$-module and γ the grading operator on it. When we have a continuous map $\varphi = (\varphi_0, \ldots, \varphi_n) : Y \to S^n$, we obtain an even unitary $\varphi_0 + \gamma c_1\varphi_1 + \cdots + \gamma c_n\varphi_n$, where c_i $(i = 1, \ldots, n)$ are Clifford multiplications of an orthonormal basis $e_1, \ldots, e_n$. For simplicity of notation, we use the same letter φ for its restriction to Δ_n^0.

Theorem 4.6. *Let Y and φ be as above. Then*

$$\operatorname{ind} T_\varphi = -\deg(\varphi : Y \to S^n).$$

Proof. Baum and Douglas [1982] proved the cohomological formula for this index, which is analogous to the Atiyah–Singer formula. As a consequence, we have that

$$\operatorname{ind} T_\varphi = -\langle \operatorname{ch}(\varphi)\operatorname{Td}(X), [X]\rangle.$$

Actually we can give a proof of Theorem 4.6 by using this and the description of the Chern character in Lemma 3.5. □

4C. ***Localization of APS index for families and eta-form.*** We can also apply our joint spectral flow index theorem for fiber bundles whose fibers are compact manifolds with boundary. A main reference for this section is [Melrose and Piazza 1997].

Let B be a closed n-dimensional Spin^c manifold and $Z \to M \to B$ a smooth fiber bundle over B whose boundary also forms a fiber bundle $\partial Z \to \partial M \to B$. Here we also assume that M is Spin^c. The Riemannian metric g on TM is introduced by the direct sum decomposition $g_f \oplus \pi^* g_B$ on $T_V M \oplus T_H M$, where g_B is a Riemannian metric on $TB \cong T_H M$ and g_f is a smooth family of Riemannian metrics on the fibers Z_x that are exact b-metrics near the boundaries ∂Z_x. We assume that $\$_{\mathbb{C}}(T_V M)$ has a $\mathbb{C}\ell(V)$-action and the fiberwise Dirac operator $\not{D}^E_f$ on $\$^E_{\mathbb{C}}(M)$ coincides with the Dirac operator $c(v_1)D_{v_1}+\cdots+c(v_n)D_{v_n}$ associated with a V-twisted n-tuple $\{D_v\}$ of fiberwise first-order pseudodifferential operators that satisfies Condition 3.18. We denote by $H^{1,0}(M,E)$ the fiberwise Sobolev space, that is, the completion of $C^\infty(M,E)$ by the inner product $\langle\,\cdot\,,\cdot\,\rangle_{L^2}+\langle\nabla^E_f\,\cdot\,,\nabla^E_f\,\cdot\rangle$, where $\nabla^E_f := p_{T_V M}\circ\nabla^E$. Then an element in $H^{1,0}(M,E)$ is fiberwise continuous and there is a bounded operator

$$\partial : H^{1,0}(M,E)\to L^2(\partial M, E|_{\partial M}),\quad \sigma\mapsto\sigma|_{\partial M}.$$

Now we fix a spectral section $P\in C(B,\{\Psi_0(\partial Z_x, E|_{\partial Z_x})\}_{x\in B})$. Here a spectral section P is a projection such that there is a smooth function $R : B\to\mathbb{R}$ and the condition $D_f(x)\sigma=\lambda\sigma$ implies $P(x)\sigma=\sigma$ if $\lambda>R(x)$ and $P(x)\sigma=0$ if $\lambda<-R(x)$ for any $x\in B$. Then this P determines an elliptic boundary condition at each fiber and

$$\begin{gathered}\not{D}_f : L^2(M,E)\to L^2(M,E),\\ \operatorname{dom}\not{D}_f := \{\sigma\in H^{(1,0)}(M,E)\mid P(\partial\sigma)=0\}\end{gathered}$$

is a fiberwise Fredholm self-adjoint operator. Hence it forms a V-twisted continuous family of unbounded commuting Fredholm n-tuples $\{D_v(x)\}$ parametrized by B.

Theorem 4.7. $$\operatorname{ind}_P(\not{D}) = \operatorname{jsf}(\{D(x)\}).$$

The same proof that we gave for Theorem 3.7 and 3.19 also works here. This is because we deal with operators directly, instead of the topology of their principal symbols. We only remark that in this situation D_b and $D_f(1+D_f^2)^{-1/2}$ commute modulo bounded operators. Furthermore, we obtain an analogue of Theorem 3.14.

Now we introduce its application for a geometric problem.

Let B be an n-dimensional closed manifold, $V\to B$ a real vector bundle of dimension n and $Y\to N\to B$ a fiber bundle with $\dim Z=n-1$. We assume that there is an oriented embedding of M into V as a fiber bundle. Then there is a fiber bundle $Z\to M\to B$ of manifolds whose boundary is isomorphic to $Y\to N\to B$ as fiber bundles. Now we define the eta-form [Bismut and Cheeger 1989] for N by

$$\begin{aligned}\hat{\eta}_P &= \int_0^\infty \hat{\eta}_P(t)\,dt,\\ \hat{\eta}_P(t) &= \frac{1}{\sqrt{\pi}}\operatorname{Str}_{\mathbb{C}\ell_1}\Big(\frac{d\tilde{\mathbb{B}}_t}{dt}e^{-\tilde{\mathbb{B}}_t^2}\Big),\end{aligned}$$

where $\tilde{\mathbb{B}}_t$ is the deformed $\mathbb{C}\ell_1$-superconnection. This differential form is closed and used for the Atiyah–Patodi–Singer index theorem for families.

On the other hand, the canonical metric on V induces a smooth family of exact b-metrics on $T_V M$. Therefore, first-order differential operators $\partial/\partial v_i$ on V_x (where $v_1, \dots, v_n$ is a basis of V_x) form a V-twisted commuting Fredholm n-tuple.

Theorem 4.8. *Let $Z \to M \to B$ and V be as above. If there is an oriented embedding of M into V as fiber bundles, its eta-form $\hat{\eta}_P$ is in the image of $H^n(B; \mathbb{Z})$. Moreover, in that case*

$$\int_B \hat{\eta}_P = \mathrm{ind}_P(\not{D}_M) = \mathrm{jsf}\{D(x)\}.$$

Proof. From Theorem 4.7 we have $j_*\{D(x)\} = \mathrm{ch}(\mathrm{ind}_P(\not{D}_f))$. Now the Atiyah–Patodi–Singer index theorem for families [Melrose and Piazza 1997] says that $\mathrm{ch}(\mathrm{ind}_P(\not{D}_f)) = \pi_!(\hat{A}(T_V M)) + \hat{\eta}_P$. In our case $T_V M$ is trivial and hence the first term of the above equality vanishes. $\square$

In particular, in the case $Y = S^{n-1}$, we get an obstruction for an oriented sphere bundle to be isomorphic to the unit sphere bundle of a vector bundle. This is related to the comparison of the homotopy types of $\mathrm{Diff}_+(S^{n-1})$ and $\mathrm{SO}(n)$, which is the Smale conjecture.

5. Decomposing Dirac operators

Now the converse problem arises. When do geometric Dirac operators "decompose" into Dirac operators associated with commuting Fredholm n-tuples? In this section we deal with zeroth-order pseudodifferential operators to obtain a complete obstruction from its index by using the theory of C^*-algebra extensions, which is related to KK^1-theory and index theory in [Kasparov 1980b].

We start with some folklore. Let T_φ be a Toeplitz operator associated with $\varphi \in C(S^1)^\times$. Then T_φ is not a normal operator in general, and $\mathrm{Re}\, T_\varphi$ can only commute with $\mathrm{Im}\, T_\varphi$ if $\mathrm{ind}\, T_\varphi$ is equal to 0. In this situation, the index of the operator $T_\varphi = \mathrm{Re}\, T_\varphi + i\, \mathrm{Im}\, T_\varphi$ gives a complete obstruction to the existence of mutually commuting self-adjoint operators A and B such that $(A - \mathrm{Re}\, T_\varphi)$ and $(B - \mathrm{Im}\, T_\varphi)$ are compact. Our purpose in this section is to give an analogue and a generalization of this for the bounded operators associated with Dirac operators.

Before we consider the case of families, we deal with a single Dirac operator. First of all, we assume that its principal symbol decomposes. This is interpreted as a geometric condition as follows: Let M be a closed Spin^c manifold and $H_1, \dots, H_n$ mutually orthogonal odd-dimensional subbundles of TM whose direct sum spans TM. As is argued in Section 4A, $\not{\mathfrak{S}}_{\mathbb{C}}(M; \underline{\mathbb{R}}^n) := \not{S}_{\mathbb{C}}(TM \oplus \underline{\mathbb{R}}^n)$ decomposes as

$$\not{\mathfrak{S}}_{\mathbb{C}}(M; \underline{\mathbb{R}}^n) \cong (\not{S}^0_{\mathbb{C}}(H_1) \otimes \cdots \otimes \not{S}^0_{\mathbb{C}}(H_n)) \hat{\otimes} \mathbb{C}\ell_n.$$

Hereafter we set $\slashed{S}^0_{\mathbb{C}}(M;\underline{\mathbb{R}}^n) := \slashed{S}^0_{\mathbb{C}}(H_1)\otimes\cdots\otimes\slashed{S}^0_{\mathbb{C}}(H_n)$. Under this identification, the principal symbol of the Dirac-type operator $\slashed{D}^E$ on $\slashed{S}^E_{\mathbb{C}}(M;\underline{\mathbb{R}}^n)$ is interpreted as

$$\sigma(\slashed{D}^E) = \sum_{i=1}^{k}\left(\sum_{j=1}^{\dim H_i} 1\otimes\cdots\otimes c(e_{i,j})\xi_{i,j}\otimes\cdots\otimes 1\right)\hat{\otimes}\, c_i,$$

where each $\{e_{i,j}\}_{j=1,\dots,\dim H_i}$ is an orthonormal basis on H_i and the $\xi_{i,j} := \langle \xi, e_{i,j}\rangle$ are coordinate functions on each cotangent space. Then we can construct a commuting n-tuple on the symbol level. This also works for the Dirac operator $\slashed{D}^E$ twisted by a complex vector bundle E. We say the Dirac operator $\slashed{D}^E$ is said to be n-*decomposable* if there is a bounded commuting Fredholm n-tuple $(T_1,\dots,T_n)$ such that each T_i is a zeroth-order pseudodifferential operator on $\Gamma(M, \slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))$ whose principal symbol is of the form $\sigma(T_i) = \sum_j 1\otimes\cdots\otimes c(e_{i,j})\xi_{i,j}\otimes\cdots\otimes 1$. In that case, the bounded operator $\slashed{D}^E(1+(\slashed{D}^E)^2)^{-1/2}$ associated with $\slashed{D}^E$ coincides modulo compact operators with the Dirac operator associated with the bounded commuting Fredholm n-tuple T.

In fact, n-decomposability is a K-theoretic property and is determined by the index:

Proposition 5.1. *Let M, $H_1,\dots,H_n$, and E be as above. Then the Dirac operator $\slashed{D}^E$ is n-decomposable if and only if* $\operatorname{ind}(\slashed{D}^E) = 0$.

Proof. A decomposition of the principal symbol gives a $*$-homomorphism $\sigma(\slashed{D}^E) : C(S^{n-1}) \to A := \Gamma\big(S(TM), \operatorname{End}(\pi^*\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))\big)$ that maps the coordinate function x_i $(i = 1,\dots,n)$ of $\mathbb{R}^n$, which contains S^{n-1} as the unit sphere, to an element $\sum_j c(e_{i,j})\xi_{i,j}$. It is well-defined because the square sum $\sum_i\big(\sum_j c(e_{i,j})\xi_{i,j}\big)^2$ is equal to 1. Hence we can replace the problem of obtaining a decomposition of $\slashed{D}^E$ with that of obtaining a lift, as is shown in the following diagram by the dotted arrow, of $\sigma(\slashed{D}^E)$:

$$\begin{tikzcd}
 & & & C(S^{n-1}) \arrow[dll, dotted] \arrow[d, "\varphi"] & \\
0 \arrow[r] & \Psi^{-1}(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n)) \arrow[r] \arrow[d, equal] & \Psi^{0}(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n)) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & \mathbb{K}(\mathcal{H}) \arrow[r] & \mathbb{B}(\mathcal{H}) \arrow[r] & Q(\mathcal{H}) \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{H} := L^2(M, \slashed{S}^{E,0}_{\mathbb{C}}(M,\underline{\mathbb{R}}^n))$ and $\Psi^0(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))$ (resp. $\Psi^{-1}(\slashed{S}^{E,0}_{\mathbb{C}}(M;\underline{\mathbb{R}}^n))$) is the norm closure of the space of pseudodifferential operators of order 0 (resp. -1). In terms of the extension theory, it means that the extension $\varphi^*\tau = \tau\circ\varphi$ is trivial. Now, as mentioned above, the theory of C^*-algebra extension is translated into

KK^1-theory. In particular, a semisplit extension φ has a lift after stabilizing by the trivial extension if and only if the KK^1-class $[\varphi]$ is zero. Moreover, in our case we do not have to care for the stabilization of φ because the Voiculescu theorem [1976] ensures that φ absorbs any trivial extensions.

In the case of odd n, ind $\not{D}^E$ is immediately 0 because $KK^1(C(S^{n-1}), \mathbb{K})$ itself is 0. On the other hand, ind $\not{D}^E$ is also 0 because $\dim M$ is odd.

In the case of even n, we obtain an integer $\varphi^*[\tau] \in KK^1(C(S^{n-1}), \mathbb{K}) \cong \mathbb{Z}$ as the Fredholm index of $\tau \circ \varphi(u) \in Q(\mathcal{H})$ by Theorem 18.10.2 of [Blackadar 1998]. Here u is the canonical generator of $KK^1(\mathbb{C}, C(S^{n-1})) \cong K_1(C(S^{n-1}))$, and its additive inverse is represented by a family of unitary matrices $u := \sum c_1 c_i x_i \in C(S^{n-1}, \operatorname{End}(\Delta_n^0))$ (it is a consequence of Theorem 4.6). Now $\cdot\, \tau \circ \varphi(u)$ coincides with the principal symbol of the Dirac operator $c_1 \cdot (\not{D}^E)^0$ on $\Gamma(M, \not{S}^{E,0}(M))$ because $\not{S}^0_{\mathbb{C}}(M) \cong \not{S}^0_{\mathbb{C}}(M; \underline{\mathbb{R}}^n) \mathbin{\hat{\otimes}} \Delta_n^0$. $\square$

We now turn to the case of families of Dirac operators, which is our main interest.

Let $Z \to M \to B$ be a fiber bundle and set $n := \dim B$. We assume that there are Spin^c vector bundles $V_1, \dots, V_l$ on B and $H_1, \dots, H_l$ on M such that $\pi^* V_i \otimes H_i$ are also Spin^c and the vertical tangent bundle $T_V M$ is isomorphic to their direct sum $\pi^* V_1 \otimes H_1 \oplus \cdots \oplus \pi^* V_l \otimes H_l$. We denote the direct sum $V_1 \oplus \cdots \oplus V_l$ by V and assume that $\dim V = n$. Moreover, we assume that each H_i is odd-dimensional and decomposes as $H_i \cong H_i^0 \oplus \underline{\mathbb{R}}$. Now, as is in Section 4A, the spinor bundle $\not{\mathfrak{S}}_{\mathbb{C},f}(M; V) := \not{S}_{\mathbb{C}}(T_V M \oplus V)$ decomposes as

$$\not{\mathfrak{S}}_{\mathbb{C},f}(M; V) \cong \big(\not{S}_{\mathbb{C}}(\pi^* V_1 \otimes H_1^0) \otimes \cdots \otimes \not{S}_{\mathbb{C}}(\pi^* V_n \otimes H_n^0)\big) \mathbin{\hat{\otimes}} \mathbb{C}\ell(\pi^* V).$$

Hereafter we set $\not{S}^{E,0}_{\mathbb{C},f}(M; V) := \not{S}_{\mathbb{C}}(\pi^* V_1 \otimes H_1^0) \otimes \cdots \otimes \not{S}_{\mathbb{C}}(\pi^* V_n \otimes H_n^0)$. The principal symbol of the fiberwise Dirac operator $\not{D}_f^E$ on the twisted fiberwise spinor bundle $\not{\mathfrak{S}}^E_{\mathbb{C},f}(M; V) := \not{\mathfrak{S}}_{\mathbb{C},f}(M; V) \otimes E$ also decomposes as a V-twisted continuous family of commuting n-tuples. Indeed, for $v = v_1 \oplus \cdots \oplus v_l$, the explicit decomposition is given by the correspondence

$$\sigma(\not{D}_f^E)_v = \sum \big(c(v_1 \otimes e_{1,j}) \xi_{e_{1,j}}\big) + \cdots + \sum \big(c(v_1 \otimes e_{l,j}) \xi_{e_{l,j}}\big).$$

This gives a $*$-homomorphism $\sigma(\not{D}_f^E)_v : C(S(V)) \to C(B) \otimes Q(\mathcal{H})$ that is compatible with the inclusions $C(B) \subset C(S(V))$ and $C(B) \otimes 1 \subset C(B) \otimes Q(\mathcal{H})$. In particular, when V is trivial it reduces to the $*$-homomorphism $\sigma(\not{D}_f^E)_v : C(S^{n-1}) \to C(B) \otimes Q(\mathcal{H})$.

Definition 5.2. The fiberwise Dirac operator $\not{D}_f^E$ is said to be *n-decomposable* if there is a V-twisted bounded commuting Fredholm n-tuple $\{T_v(x)\}$ such that each T_v is a zeroth-order pseudodifferential operator on $\Gamma(\not{S}^{E,0}_{\mathbb{C},f}(M; V))$ whose principal symbol is $\sigma(T_v) = \sum \big(c(v_1 \otimes e_{1,j}) \xi_{e_{1,j}}\big) + \cdots + \sum \big(c(v_1 \otimes e_{l,j}) \xi_{e_{l,j}}\big)$.

In this case, $\not{D}^E_f(1+(\not{D}^E_f)^2)^{-1/2}$ coincides modulo compact operators with the smooth family of Dirac operators associated with the V-twisted bounded commuting Fredholm n-tuples $\{T_v(x)\}$. Hence the K-class $[\operatorname{ind} \not{D}^E_f]$ is in the image of the canonical natural transform from $\tilde{k}^n(B)$ to $K^n(B)$. Moreover, the index of the Dirac operator $\not{D}^E_M$ on M twisted by E, which coincides with that of $\pi^*\not{\mathfrak{D}}_B + \not{D}^E_f$, can be obtained from the joint spectral flow $\operatorname{jsf}\{T_v(x)\}$.

Theorem 5.3. *Let $Z \to M \to B$, $V_1, \dots, V_l$, $H_1, \dots, H_l$, and E be as above. Then $\not{D}^E_f$ is n-decomposable if and only if $\operatorname{ind}(\not{D}^E_f)$ is in the image of $K^n(B, B^{(n-1)}) \to K^n(B)$, or equivalently the image of $\tilde{k}^n(B) \to K^n(B)$. In that case, the equality $\operatorname{ind} \not{D}^E_M = \operatorname{jsf}\{\not{D}^E_f\}$ holds.*

Here $B^{(n-1)}$ is the $(n-1)$-skeleton of a cellular decomposition of B. The image of $K(B, B^{(n-1)}) \to K^n(B)$, which is the Atiyah–Hirzebruch filtered K-group $F^{n-1}K^n(B)$, is independent of the choice of decompositions and coincides with the image of $\tilde{k}^n(B) \to K^n(B)$ because of the functoriality of $\tilde{k}^* \to K^*$ and the fact that $\tilde{k}^n(B^{(n-1)}) = 0$.

Remark 5.4. In the proof, except for the last part, the condition that B is an n-dimensional closed manifold is not necessary. Actually it is sufficient to be a finite CW-complex. Moreover, if B is an n-dimensional CW-complex, the last part also holds.

The proof is divided into several steps. First, we show that $\not{D}^E_f$ is locally n-decomposable.

Lemma 5.5. *Let $M = B \times Z$ and $TZ \cong H_1 \oplus \cdots \oplus H_n$. If the index of the fiberwise Dirac operator $\not{D}^E_f$ on $\not{S}^E_{\mathbb{C}}(M; \mathbb{R}^n)$ is zero, then $\not{D}^E_f$ is n-decomposable.*

Proof. As in Proposition 5.1, it suffices to find a lift of the extension $\sigma(\not{D}^E_f)_v : C(S^{k-1}) \to C(B) \otimes C(S(TZ)) \subset C(B) \otimes Q(\mathcal{H})$. This exists when the metrics on fibers are constant because $\sigma(\not{D}^E_f)_v$ is trivial and absorbable by Kasparov's [1980b] generalized Voiculescu theorem. In the general case, it exists because $\sigma(\not{D}^E_f)_v|_{M_y} = u_y(\sigma(\not{D}^E_f)_v|_{M_x})u_y^*$, where $u_y : \pi^*\not{S}^E_{\mathbb{C}}(M_x; \mathbb{R}^n) \to \pi^*\not{S}^E_{\mathbb{C}}(M_y; \mathbb{R}^n)$ is the isometry induced from the polar part of the identity map $\mathrm{id} : TM_x \to TM_y$. □

Next we introduce a technique for gluing two decompositions. We can deal with the problem cohomologically by using Cuntz's [1983] notion of quasihomomorphisms. The "difference" of two lifts $\varphi_0, \varphi_1 : C(S(V)) \to C(B) \otimes \mathbb{B}(\mathcal{H})$ of $\sigma(\not{D}^E_v)$ gives an element of the representable KK-group [Kasparov 1988]

$$[\varphi_0, \varphi_1] := \left[\hat{\mathcal{H}} \hat{\otimes} C(B), \begin{pmatrix} \varphi_0 & 0 \\ 0 & \varphi_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \in \mathcal{R}KK\big(B; C(S(V)), C(B) \otimes \mathbb{K}\big).$$

In particular, in the case that V is trivial, we can reduce the representable KK-group $\mathcal{R}KK\big(B; C(S(V)), C(B)\big)$ by $KK(C(S^{n-1}), C(B) \otimes \mathbb{K})$. Then the split exact sequence

$$0 \longrightarrow C_0(S^{n-1} \setminus \{*\}) \longrightarrow C(S^{n-1}) \xrightarrow{p} \mathbb{C} \longrightarrow 0$$

gives an isomorphism

$$KK(C(S^{n-1}), C(F)) \cong KK\big(C_0(S^{n-1} \setminus \{*\}), C(F)\big) \oplus KK(\mathbb{C}, C(F)).$$

When both of φ_0 and φ_1 are unital, $[\varphi_0, \varphi_1]$ corresponds to $[\varphi_0, \varphi_1]|_{C(S^{n-1}\setminus\{*\})} \oplus 0$ under the above identification because $p^*[\varphi_0, \varphi_1] = [1, 1] = 0$.

Lemma 5.6. *Let F_0, F_1 be closed subsets of B such that $B = (F_0)^\circ \cup (F_1)^\circ$ and $F := F_0 \cap F_1$. We assume that M and E are trivial on F and $\sigma(\not{D}^E_f)$ has lifts φ_0 and φ_1 on F_0 and F_1. Then the image of $[\varphi_0, \varphi_1] \in KK(C_0(S^{n-1} \setminus \{*\}, \mathbb{K} \otimes C(F)) \cong K^{n-1}(F)$ under the boundary map of the Mayer–Vietoris sequence coincides with $[\operatorname{ind} \not{D}^E_f] \in K^n(B)$.*

Proof. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathbb{D}^n \setminus \{0\}) \arrow[r] \arrow[d, "\iota"] & C_0(\overline{\mathbb{D}^n} \setminus \{0\}) \arrow[r] \arrow[d] & C(S^{n-1}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_0(\mathbb{D}^n) \arrow[r] \arrow[d, equal] & C_0(\overline{\mathbb{D}^n}) \arrow[r] & C(S^{n-1}) \arrow[r] & 0 \\
0 \arrow[r] & C_0(\mathbb{D}^n) \arrow[r] & C_0(\overline{\mathbb{D}^n} \setminus \{*\}) \arrow[r] \arrow[u] & C_0(S^{n-1} \setminus \{*\}) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

we obtain a diagram of KK-groups

$$\begin{tikzcd}
KK^1(C_0(\mathbb{D}^n \setminus \{0\}), C(F)) \arrow[r, "\sim", "\partial_1"'] & KK^0(C(S^{n-1}), C(F)) \arrow[d, equal] \\
KK^1(C(\mathbb{D}^n), C(F)) \arrow[r, "\partial_2"'] \arrow[u, "\iota^*"] \arrow[d, equal] & KK^0(C(S^{n-1}), C(F)) \arrow[d] \\
KK^1(C(\mathbb{D}^n), C(F)) \arrow[r, "\sim", "\partial_3"'] & KK^0(C_0(S^{n-1} \setminus \{*\}), C(F))
\end{tikzcd}$$

Here, for a C^*-algebra A, the group $KK^1(A, C(F))$ is canonically isomorphic to $KK(A, \Sigma C(F)) \cong KK(A, C_0(\Sigma F))$. One can see that the inverse $(\partial_1)^{-1}$ of the boundary map coincide with the product with $[\mathrm{id}_\Sigma] \in KK(\Sigma, \Sigma)$.

As a consequence we obtain that

$$\iota^* \partial_3^{-1}[\varphi_0, \varphi_1] = \partial_1^{-1}[\varphi_0, \varphi_1] = [\varphi_0 \otimes \mathrm{id}_\Sigma, \varphi_1 \otimes \mathrm{id}_\Sigma].$$

Next, we consider the isomorphism between

$$KK(C_0(\mathbb{D}^n), C_0(\Sigma F)) \quad \text{and} \quad KK(\mathbb{C}, C_0(\Sigma F) \hat{\otimes} \mathbb{C}\ell_n).$$

As is in Section 2, this correspondence is given by taking a product with the canonical generator

$$[C_{\mathbb{D}^n}] := \Big[C_0(\mathbb{D}^n) \hat{\otimes} \mathbb{C}\ell_n, 1, C_{\mathbb{R}^n} := \sum x_i \cdot c_i \Big]$$

of $KK(\mathbb{C}, C_0(\mathbb{D}^n) \hat{\otimes} \mathbb{C}\ell_n)$. Restricting to $C_0(\mathbb{D}^n \setminus \{0\}) \hat{\otimes} \mathbb{C}\ell_n$, the operator $C_{\mathbb{D}^n}$ also defines an element $[C_{\mathbb{D}^n \setminus \{0\}}]$ in $KK(\mathbb{C}, C(\mathbb{D}^n \setminus \{0\}) \hat{\otimes} \mathbb{C}\ell_n)$. When we regard the topological space $\mathbb{D}^n \setminus \{0\}$ as ΣS^{n-1}, the operator $C_{\mathbb{D}^n}$ is of the form $tC_{S^{n-1}}$, where $C_{S^{n-1}} := \sum c_i \cdot x_i \in C(S^{n-1}) \hat{\otimes} \mathbb{C}\ell_n$ and t is the identity function on $(0, 1)$. Now the diagram

$$\begin{tikzcd}
KK(C_0(\mathbb{D}^n), C_0(\Sigma F)) \arrow[d, "\iota^*"] \arrow[dr, "{[C_{\mathbb{D}^n}]}"] & \\
& KK(\mathbb{C}, C_0(\Sigma F) \hat{\otimes} \mathbb{C}\ell_n) \\
KK(C_0(\mathbb{D}^n \setminus \{0\}), C_0(\Sigma F)) \arrow[ur, "{[C_{\mathbb{D}^n \setminus \{0\}}]}"'] &
\end{tikzcd}$$

commutes. As a consequence, we can compute $[C_{\mathbb{D}^n}] \otimes_{C_0(\mathbb{D}^n)} \partial_3^{-1}[\varphi_0, \varphi_1]$ by using Proposition 18.10.1 of [Blackadar 1998] as follows:

$$\begin{aligned}
&[C_{\mathbb{D}^n}] \otimes_{C_0(\mathbb{D}^n)} \partial_3^{-1}[\varphi_0, \varphi_1] \\
&\quad = [C_{\mathbb{D}^n \setminus \{0\}}] \otimes_{C_0(\mathbb{D}^n \setminus \{0\})} \iota^* \partial_3^{-1}[\varphi_0, \varphi_1] \\
&\quad = [tC_{S^{n-1}}] \otimes_{C_0(\Sigma S^{n-1})} [\varphi_0 \otimes \mathrm{id}_\Sigma, \varphi_1 \otimes \mathrm{id}_\Sigma] \\
&\quad = \left[\hat{\mathcal{H}}_{C_0(\Sigma F)} \hat{\otimes} \mathbb{C}\ell_n, 1, \begin{pmatrix} \varphi_0(tC_{S^{n-1}}) & 0 \\ 0 & \varphi_1(tC_{S^{n-1}}) \end{pmatrix} \right. \\
&\qquad\qquad \left. + \begin{pmatrix} 1 - \varphi_0(tC_{S^{n-1}})^2 & 0 \\ 0 & 1 - \varphi_1(tC_{S^{n-1}})^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\
&\quad = \left[\hat{\mathcal{H}}_{C_0(\Sigma F)} \hat{\otimes} \mathbb{C}\ell_n, 1, \begin{pmatrix} \varphi_0(tC_{S^{n-1}}) & 1 - \varphi_0(tC_{S^{n-1}})^2 \\ 1 - \varphi_1(tC_{S^{n-1}})^2 & \varphi_1(tC_{S^{n-1}}) \end{pmatrix} \right] \\
&\quad = [\hat{\mathcal{H}}_{C_0(\Sigma F)} \hat{\otimes} \mathbb{C}\ell_n, 1, T].
\end{aligned}$$

Here

$$T=\{T_t\}_{t\in[0,1]}:=\begin{cases}\begin{pmatrix}\varphi_0((1-2t)C_{S^{n-1}}) & 1-(1-2t)^2\\ 1-(1-2t)^2 & \varphi_0((1-2t)C_{S^{n-1}})\end{pmatrix} & 0\le t\le 1/2,\\ \begin{pmatrix}\varphi_0((2t-1)C_{S^{n-1}}) & 1-(2t-1)^2\\ 1-(2t-1)^2 & \varphi_1((2t-1)C_{S^{n-1}})\end{pmatrix} & 1/2\le t\le 1.\end{cases}$$

Now we claim that this KK-class coincides with that coming from the cycle

$$\big[\mathcal{H}_{C_0(\Sigma F)}\hat{\otimes}\mathbb{C}\ell_n,\,1,\,t\varphi_0(C_{S^{n-1}})+(1-t)\varphi_1(C_{S^{n-1}})\big].$$

Indeed, because T_t-T_{1-t} is compact for any $t\in[0,1/2]$, the homotopy of continuous families of Fredholm operators

$$\mathfrak{T}_{s,t}:=\begin{cases}T_t & 0\le t\le s/2,\\ \dfrac{t-s/2}{1-s}T_{s/2}+\dfrac{1-t-s/2}{1-s}T_{1-s/2} & s/2\le t\le 1-s/2,\\ T_t & 1-s/2\le t\le 1\end{cases}$$

connects $\mathfrak{T}_0=T$ with

$$\mathfrak{T}_1=\begin{pmatrix}\varphi_0(C_{S^{n-1}}) & 0\\ 0 & t\varphi_0(C_{S^{n-1}})+(1-t)\varphi_1(C_{S^{n-1}})\end{pmatrix}.$$

Finally we obtain that $[\varphi_0,\varphi_1]$ coincides with $[t\varphi_0(C_{S^{n-1}})+(1-t)\varphi_1(C_{S^{n-1}})]$ in $KK(C_0(S^{n-1}\setminus\{*\}),C(F))\cong K^n(\Sigma F)$. Next we map it by the boundary map δ_{MV} of the Mayer–Vietoris exact sequence.

We denote by $I(F_0,F_1;F)$ the space $F_0\sqcup IF\sqcup F_1$. The image of δ_{MV} is induced from the map $I(F_0,F_1;F)\to(I(F_0,F_1;F),F_0\cup F_1)$ and excision. Therefore $\delta_{MV}[t\varphi_0(C_{S^{n-1}})+(1-t)\varphi_1(C_{S^{n-1}})]$ is of the form

$$\begin{cases}\varphi_0(C_{S^{n-1}})_x & x\in F_0,\\ t\varphi_0(C_{S^{n-1}})_x+(1-t)\varphi_1(C_{S^{n-1}})_x & (x,t)\in IF,\\ \varphi_1(C_{S^{n-1}})_x & x\in F_1.\end{cases}$$

It is a lift of the pullback of the principal symbol $\sigma(\not{D}^E_v)$ by the canonical projection $I(F_0,F_1;F)\to B$, which introduces the homotopy equivalence. As a consequence the above operator coincides with $\not{D}^E_f(1+(\not{D}^E_f)^2)^{-1/2}$ modulo compact operators and hence defines the same KK-class. □

Lemma 5.7. *If* $[\operatorname{ind}\not{D}^E_f]=0\in K^n(B)$, *then* $\not{D}^E_f$ *is n-decomposable.*

Proof. Let $U_1,\dots,U_m$ be a local trivialization of the fiber bundle $M\to B$ and the vector bundles $V_1,\dots,V_l\to B$ such that M is also trivial on $F_i:=\overline{U_i}$. By assumption and Lemma 5.5, $\not{D}^E_f$ is n-decomposable on each F_i.

We start with the case that $B = F_0 \cup F_1$, and set $F := F_0 \cap F_1$. First, we fix a trivial and absorbable extension $\pi : C(S^{n-1}) \to Q(\mathcal{H}_\pi)$ of $\mathbb{K}$ by $C(S^{n-1})$ and denote by π_A an extension $C(S^{n-1}) \to Q(\mathcal{H}_\pi) \to Q(\mathcal{H}_\pi) \otimes A$ of $C(S^{n-1})$ by $A \otimes \mathbb{K}$ for a unital C^*-algebra A.

Now we choose lifts φ_0 and φ_1 of $\sigma(\not{D}^E_v)$ on F_0 and F_1. By Kasparov's generalized Voiculescu theorem, the φ_i are approximately equivalent to $\varphi_i \oplus \pi_{C(F_i)}$. More precisely, there are continuous families of unitaries $u_i : \mathcal{L}^2_f(\$^E_{\mathbb{C}}(M; V)) \to \mathcal{L}^2_f(\$^E_{\mathbb{C}}(M; V)) \oplus \mathcal{H}_\pi \otimes C(B)$ such that $u_i(\varphi_i \oplus \pi_{C(F_i)})u_i^* \equiv \varphi_i$ modulo compact operators. According to Lemma 5.6, $\delta_{MV}([\varphi_0, \varphi_1]) = [D_f] = 0$. Hence, by exactness of the Mayer–Vietoris sequence, we have quasihomomorphisms $[\alpha_i, \beta_i]$ ($i = 0, 1$) such that $[\alpha_0, \beta_0]|_F - [\alpha_1, \beta_1]|_F = [\varphi_0, \varphi_1]|$. Now there are unitaries v_i such that $v_i(\pi_{C(F_i)} \oplus \alpha_i \oplus \alpha_i^\perp)v_i^* \equiv \pi_{C(F_i)}$ modulo compact operators. We set

$$\psi_i := u_i\big(\varphi \oplus v_i(\pi_{C(F_i)} \oplus \beta_i \oplus \alpha_i^\perp)v_i^*\big)u_i^*.$$

Then the $[\varphi_i, \psi_i]$ are quasihomomorphisms and $[\varphi_i, \psi_i] = [\alpha_i, \beta_i]$ for $i = 0, 1$ in $KK(C(S^{n-1}), C(F_i))$.

Now $[\varphi_0, \psi_0]|_F - [\varphi_1, \psi_1]|_F = [\varphi_0, \varphi_1]$, which implies $[\psi_0, \psi_1]|_F = 0$. As a consequence, there is a homotopy of quasihomomorphisms $[\Psi_0^t, \Psi_1^t]$ ($t \in [0, 1]$) from $C(S^{n-1})$ to $C(F) \otimes \mathbb{B}(\mathcal{H})$ connecting $[\psi_0|_F, \psi_1|_F]|$ and $[\theta, \theta]$ for some θ. Here we use the fact that the extensions $\psi_i|_F$ contain $\pi_{C(F)}$ and hence are absorbable. Finally we get a homotopy

$$\tilde{\Psi}_t := \begin{cases} \Psi_0^{2t} & 0 \le t \le 1/2, \\ \Psi_1^{2-2t} & 1/2 \le t \le 1 \end{cases}$$

of $*$-homomorphisms from $C(S^{n-1})$ to $C(F) \otimes \mathbb{B}(\mathcal{H})$ connecting ψ_0 and ψ_1.

Now we denote by D the fiber product of C^*-algebras

$$\begin{array}{ccc} D & \longrightarrow & C(F) \otimes (B(\mathcal{H}) \oplus \mathbb{B}(\mathcal{H})) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id} \otimes (p \oplus p)} \\ C(IF) \otimes Q(\mathcal{H}) & \xrightarrow{\mathrm{ev}_0 \oplus \mathrm{ev}_1} & C(F) \otimes (Q(\mathcal{H}) \oplus Q(\mathcal{H})) \end{array}$$

and by τ the extension

$$0 \longrightarrow C_0(SF) \otimes \mathbb{K} \longrightarrow C(IF) \otimes \mathbb{B}(\mathcal{H}) \longrightarrow D \longrightarrow 0.$$

Then $\sigma(\not{D}^E_f)_v$ and $(\psi_0 \oplus \psi_1)$ determine a $*$-homomorphism $\sigma : C(S^{n-1}) \to D$. Because the C^*-algebra $C(S^{n-1})$ is nuclear, the Choi–Effros theorem [1976] implies that the pullback $\sigma^*\tau$ is an invertible extension and hence defines an element $[\sigma^*\tau]$ in $KK^1(C(S^{n-1}), C_0(SF) \otimes \mathbb{K})$. By the construction of $\tilde{\Phi}$, $\sigma^*\tau$ is homotopic to the trivial extension $\pi \circ \tilde{\Psi}$, which implies that $[\sigma^*\tau] = 0$. Consequently, σ itself has a lift $C(S^{n-1}) \to IC(F) \otimes \mathbb{B}(\mathcal{H})$.

Finally, we obtain a lift φ of $\sigma(\not{D}^E_f)_v$ on $I(F_0, F_1; F)$. Its pullback by a continuous section $B \to I(F_0, F_1; F)$ given by a partition of unity is a desired lift of $\sigma(\not{D}^E_f)_v$.

In general case we apply induction on the number of covers. We assume that there is a trivialization $B = F_1 \cup \cdots \cup F_n \cup F_{n+1}$ and set $G_0 := F_1 \cup \cdots \cup F_n$ and $G_1 := F_{n+1}$. By the inductive assumption, we obtain lifts φ_0 and φ_1 on G_0 and G_1. First we may assume that V is trivial by restricting φ_0 to the closure of an open neighborhood of $G := G_0 \cap G_1 \subset G_0$. Now each φ_i contains $\pi_{C(G_i)}$ by its construction. Moreover, because M and V are trivial on IG by assumption, we can take a lift of σ containing $\pi_{C(IG)}$. Now, the precise assertion obtained from the above argument is that if (1) M and V are trivial on G, (2) there are lifts φ_i on $C(G_i)$ ($i = 0, 1$), and (3) each φ_i is absorbable (and hence contains $\pi_{C(G_i)}$), then there is a lift φ on G containing $\pi_{C(B)}$. Hence the induction process works. $\square$

Finally we prove our main theorem. Here we mention that in the above argument we restrict the case that the lifts can be taken to be invertible operators.

Proof of Theorem 5.3. We assume that $[\operatorname{ind} \not{D}^E_f]$ is in the image of $K^n(B, B^{(n-1)})$. Let $U \subset V$ be an inclusion of small open balls in B, $F_0 := U^c$ and $F_1 := \overline{V}$. Then $[\operatorname{ind} \not{D}^E_f|_{F_0}]$ and $[\operatorname{ind} \not{D}^E_f|_{F_1}]$ are 0 by assumption, and hence, according to Lemma 5.7, $\not{D}^E_f$ is n-decomposable on F_0 and F_1. Now, because $F := F_0 \cap F_1$ is homotopic to S^{n-1}, the group $KK(C_0(S^{n-1} \setminus \{*\}), C(F))$ is isomorphic to $\tilde{k}^{n-1}(F) = [C_0(\mathbb{R}^{n-1}), C(F)]$. This implies that there is a $*$-homomorphism $\psi : C_0(S^{n-1} \setminus \{*\}) \to C(F) \otimes \mathbb{K}$ such that $[\varphi_0, \varphi_1] = \Phi[\psi]$. Since φ_1 is absorbable, there is a unitary u from $\mathcal{H}_{C(F)}$ to $\mathcal{H}_{C(F)} \oplus \mathcal{H}_{C(F)}$ such that $u(\varphi_1 \oplus \mathrm{ev}_* \cdot 1)u^* \equiv \varphi_1$ modulo compact operators. Moreover, by an argument similar to Lemma 5.7, we obtain a lift of $\sigma(\not{D}^E_f)$ on IF that coincides with φ_0 on $F \times \{0\}$ and $u(\varphi_1 \oplus \tilde{\psi})u^*$ on $F \times \{1\}$, where $\tilde{\psi}$ is a unital extension of ψ.

The remaining part is to construct a homotopy connecting $\varphi \oplus ev_* \cdot 1$ with $\varphi \oplus \tilde{\psi}$. This is not realized as a family of $*$-homomorphisms on $C(S^{n-1})$ but a continuous family of bounded commuting Fredholm n-tuples. Let ι^* be the canonical $*$-homomorphism $C(\overline{\mathbb{D}^n} \setminus \{*\}) \to C_0(S^{n-1} \setminus \{*\})$. Then we can take a homotopy connecting $\psi \circ \iota^*$ and 0 since $\mathbb{D}^n$ is contractible.

Finally, in the same way as in the proof of Lemma 5.7, we obtain a $*$-homomorphism T that makes the following diagram commute:

$$\begin{CD}
C(\overline{\mathbb{D}}(V)) @>{T}>> C(B) \otimes \mathbb{B}(\mathcal{H}) \\
@VVV @VVV \\
C(S(V)) @>{\sigma(\not{D}^E_f)_v}>> C(B) \otimes Q(\mathcal{H})
\end{CD}$$

Now $\{T(x)\}_v := T(x, v)$ gives a decomposition of $\not{D}^E_f$. $\square$

As a concluding remark, we introduce a corollary of Theorem 5.3:

Corollary 5.8. *If $\not{D}_f^E$ is n-decomposable, then $\not{D}_f^{E\otimes\pi^*F}$ is also n-decomposable for a complex vector bundle F on B. Moreover, in this case*

$$\mathrm{ind}_0(\not{D}_M^{E\otimes\pi^*F}) = \mathrm{jsf}\{\not{D}_f^{E\otimes\pi^*F}\} = \dim F\cdot\mathrm{jsf}\{\not{D}_f^E\} = \dim F\cdot\mathrm{ind}_0(\not{D}_M^E).$$

Proof. This follows from the fact that the connective K-group gives a multiplicative filtration in the K-group. □

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A plethora of inertial products

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For a smooth Deligne–Mumford stack $\mathscr{X}$, we describe a large number of *inertial products* on $K(I\mathscr{X})$ and $A^*(I\mathscr{X})$ and *inertial Chern characters*. We do this by developing a theory of *inertial pairs*. Each inertial pair determines an inertial product on $K(I\mathscr{X})$ and an inertial product on $A^*(I\mathscr{X})$ and Chern character ring homomorphisms between them. We show that there are many inertial pairs; indeed, every vector bundle V on $\mathscr{X}$ defines two new inertial pairs. We recover, as special cases, the orbifold products considered by Chen and Ruan (2004), Abramovich, Graber and Vistoli (2002), Fantechi and Göttsche (2003), Jarvis, Kaufmann and Kimura (2007) and by the authors (2010), and the virtual product of González, Lupercio, Segovia, Uribe and Xicoténcatl (2007).

We also introduce an entirely new product we call the *localized orbifold product*, which is defined on $K(I\mathscr{X}) \otimes \mathbb{C}$.

The inertial products developed in this paper are used in a subsequent paper to describe a theory of inertial Chern classes and power operations in inertial K-theory. These constructions provide new manifestations of mirror symmetry, in the spirit of the hyper-Kähler resolution conjecture.

1. Introduction

The purpose of this note is to describe a large number of *inertial products* and Chern characters by developing a formalism of *inertial pairs*. An inertial pair for a Deligne–Mumford stack $\mathscr{X}$ is a pair $(\mathscr{R}, \mathscr{S})$, where $\mathscr{R}$ is a vector bundle on the double inertia stack $\mathbb{I}^2\mathscr{X}$ and $\mathscr{S}$ is a nonnegative, rational K-theory class on the inertia stack $I\mathscr{X}$ satisfying certain compatibility conditions. For stacks with finite stabilizer, an inertial pair determines inertial products on cohomology, Chow groups, and K-theory of $I\mathscr{X}$. In the Chow group and cohomology, this product respects an orbifold grading equal to the ordinary grading corrected by the virtual rank of $\mathscr{S}$ (or age). An inertial pair also allows us to define an inertial Chern character, which is a ring homomorphism for the new inertial products.

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The motivating example of an inertial pair is the orbifold pair $(\mathscr{R}, \mathscr{S})$, where $\mathscr{R}$ is the obstruction bundle coming from orbifold Gromov–Witten theory, and $\mathscr{S}$ is the class defined in [JKK 2007]. The corresponding product is the Chen–Ruan orbifold product, and the Chern character is the one defined in [loc. cit.]. One of the results of this paper is that every vector bundle V on a Deligne–Mumford stack determines two inertial pairs $(\mathscr{R}^+V, \mathscr{S}^+V)$ and $(\mathscr{R}^-V, \mathscr{S}^-V)$. The $+$ product corresponds to the orbifold product on the total space of the bundle V, but the $-$ product is twisted by an isomorphism and does not directly correspond to an orbifold product on a bundle. However, we prove (Theorem 4.2.2) that there is an automorphism of the total Chow group $A^*(I\mathscr{X}) \otimes \mathbb{C}$ which induces a ring isomorphism between the $-$ product for V and the $+$ product for V^*. A similar result also holds for cohomology.

When $V = \mathbb{T}$ is the tangent bundle of $\mathscr{X}$, we show that the virtual product considered in [González et al. 2007] is the product associated to the inertial pair $(\mathscr{R}^-\mathbb{T}, \mathscr{S}^-\mathbb{T})$. It follows, after tensoring with $\mathbb{C}$, that the virtual orbifold Chow ring is isomorphic (but not equal) to the Chen–Ruan orbifold Chow ring of the cotangent bundle $\mathbb{T}^*$. Our result also implies that there is a corresponding Chern character ring homomorphism for the virtual product.

In the final section we show that in certain cases, even if V is not a vector bundle but just an element of K-theory, we can still determine a product in localized K-theory. This allows us to define a new product on $K(I\mathscr{X}) \otimes \mathbb{C}$, which we call the *localized product.*

In a subsequent paper [Edidin et al. 2015] we will show that for *Gorenstein inertial pairs* (such as the one determining the virtual product) there is a theory of Chern classes and compatible power operations on inertial K-theory. This will be used to give further manifestations of mirror symmetry on hyper-Kähler Deligne–Mumford stacks.

Review of previous related work. Because there has been much work in this area by many authors from different areas of mathematics, we give a brief overview here of previous work to help put the current paper in context.

In 2000, inspired by physicists [Dixon et al. 1985; 1986], Chen and Ruan [2002] developed a new product on the cohomology of the inertia $I\mathscr{X}$ of an almost complex orbifold $\mathscr{X}$. In 2001, Fantechi and Göttsche [2003] showed that when the orbifold $\mathscr{X}$ was a global quotient $[X/G]$ by a finite group, the Chen–Ruan orbifold cohomology ring $H_{\mathrm{CR}}(\mathscr{X})$ was the G-invariant subring of $H_{\mathrm{FG}}(X, G)$, the cohomology of the inertia manifold IX endowed with a certain noncommutative product. It followed that if X is the symmetric product of a surface with trivial canonical class, then the orbifold cohomology of $\mathscr{X}$ is isomorphic to the cohomology ring of the Hilbert scheme, as predicted by the hyper-Kähler resolution conjecture [Ruan 2006].

At about the same time, Kaufmann [2002; 2003] presented an axiomatic approach to orbifolding Frobenius algebras, and described how the Fantechi–Göttsche construction fit into this framework (Kaufmann, personal communication, 2002).

Adem and Ruan [2003] then studied the K-theory $K(\mathscr{X})$ of a global quotient orbifold $\mathscr{X} = [X/G]$, where G is a Lie group, and they also studied the twisted K-theory of $[X/G]$. They did not construct a new "orbifold" product on $K(\mathscr{X})$, but they did show that there is a Chern character that gives a vector space isomorphism from $K(\mathscr{X})$ to $H_{\mathrm{CR}}(\mathscr{X})$. This Chern character is *not* a ring homomorphism. Tu and Xu [2006] later extended this result to more general twistings and orbifolds.

Abramovich, Graber and Vistoli [Abramovich et al. 2002] constructed an algebraic version $A_{\mathrm{AGV}}(\mathscr{X})$ of the Chen–Ruan cohomology, producing the corresponding product on the Chow group $A_{\mathrm{AGV}}(\mathscr{X}) = A^*(I\mathscr{X})$ of the inertial stack $I\mathscr{X}$ of a (smooth) Deligne–Mumford stack with projective coarse moduli space.

In all of these constructions the basic idea is to use an analogue of the moduli space $\overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0)$ of genus-zero, three-pointed, orbifold (or G-equivariant) stable maps into $\mathscr{X}$. This space has three *evaluation maps* $e_i : \overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0) \to I\mathscr{X}$, and the structure constants $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ for the new product on $I\mathscr{X}$ are given by computing

$$\int_{\overline{\mathscr{M}}_{0,3}(\mathscr{X},0)} \prod_{i=1}^{3} e_i^*(\alpha_i) \cdot \mathrm{eu}(\mathscr{R}),$$

where $\mathrm{eu}(\mathscr{R})$ is the top Chern class of an obstruction bundle on $\overline{\mathscr{M}}_{0,3}(\mathscr{X}, 0)$. The main difficulty in computing the new product was computing the obstruction bundle $\mathscr{R}$ and its top Chern class.

In 2004, Chen and Hu [2006] produced a formula for the obstruction class in the case of *abelian* orbifolds and used it to describe a de Rham model for the Chen–Ruan product. In 2005, Jarvis, Kaufmann and Kimura [JKK 2007] proved a simple, intrinsic formula for the obstruction bundle $\mathscr{R}$ for general (not just abelian) orbifolds, requiring no mention of stable curves or moduli spaces of maps. In the abelian case this formula reduces to Chen and Hu's result. In [loc. cit.], that formula is used to do several things:

(1) Create Chow- (respectively, K-) theoretic analogues of the Fantechi–Göttsche ring $H_{\mathrm{FG}}(X, G)$ whose rings of invariants is the AGV ring $A_{\mathrm{AGV}}(\mathscr{X})$ (respectively, a ring whose underlying vector space is $K(\mathscr{X})$ of Adem and Ruan). Corresponding products twisted by discrete torsion were also introduced.

(2) Define a new (orbifold) product on the K-theory $K_{\mathrm{orb}}(\mathscr{X})$ of the inertia $I\mathscr{X}$, for any smooth Deligne–Mumford stack $\mathscr{X}$.

(3) Define an orbifold Chern character ring homomorphism from the new orbifold K-theory rings to the corresponding Chow or cohomology rings. This new

Chern character is a deformation of the ordinary Chern character, as the latter fails to preserve the orbifold multiplications.

(4) Outline how the same formula and formalism may be used to give analogous results in other categories, e.g., equivariant structures on almost complex manifolds with a Lie group action.

At about the same time, Adem, Ruan and Zhang [Adem et al. 2008] independently defined an orbifold product on twisted $K_{\mathrm{orb}}(\mathscr{X})$, and in the case of a global quotient by a finite group, Kaufmann and Pham [2009] connected this to the twisted Drinfel'd double of the group ring.

Becerra and Uribe [2009] extended these results to the equivariant setting for global quotients by infinite abelian groups, and in [EJK 2010], we extended these results to an equivariant setting for global quotients by general (nonabelian) infinite groups by introducing a variant of the formula for the obstruction bundle in [JKK 2007].

A recent paper [Hu and Wang 2013] repeats the description of the orbifold product of [JKK 2007; Adem et al. 2008], the formula of [JKK 2007; EJK 2010] for the obstruction class, and the Chern character ring homomorphism of those two papers in the almost-complex setting, as originally described in Section 10 of [JKK 2007].

In [Behrend et al. 2007; 2012; González et al. 2007; Lupercio et al. 2008], a different product in inertial Chow and inertial cohomology theory, analogous to the Chas–Sullivan product [1999] on loop spaces, was introduced. This so-called *virtual (orbifold) product* is a special case of the constructions of this paper; see Section 4.3. Surprisingly, it is *not* equal to the orbifold product for the cotangent bundle. We note, however, that after tensoring with $\mathbb{C}$, both the orbifold Chow and orbifold cohomology (but not orbifold K-theory) of the cotangent bundle are isomorphic to their virtual counterparts.

Kaufmann [2002; 2003; 2010] was the first to study the possibility of many stringy products on Frobenius algebras in settings involving functors other than just K-theory, cohomology, and Chow theory. He treated this primarily as an algebraic question and reformulated the problem of constructing a stringy product in terms of certain cocycles. It is not *a priori* clear that there should always exist a stringy product, but Kaufmann [2010] shows how to extend the ideas of [JKK 2007] to prove existence of at least one stringy product for his more general setting. In some cases he can also show uniqueness of the product [Kaufmann 2004].

Finally, we note that Pflaum, Postuma, Tang and Tseng [Pflaum et al. 2011] have shown that the Hochschild cohomology of a certain algebra attached to a groupoid presentation of a symplectic orbifold is isomorphic to the cohomology of the inertia orbifold as a vector space. The product in Hochschild cohomology induces a product on the cohomology of the inertia orbifold. It would be interesting to understand the relation of that product to the products described in this paper.

2. Background material from [EJK 2010]

To make this paper self-contained, we recall some background material from the paper [EJK 2010].

2.1. ***Background notation.*** We work entirely in the complex algebraic category. We will work exclusively with smooth Deligne–Mumford stacks $\mathscr{X}$ which have *finite stabilizer*, by which we mean the inertia map $I\mathscr{X} \to \mathscr{X}$ is finite. We will also assume that every stack $\mathscr{X}$ has the *resolution property*. This means that every coherent sheaf is the quotient of a locally free sheaf. This assumption has two consequences. The first is that the natural map $K(\mathscr{X}) \to G(\mathscr{X})$ is an isomorphism, where $K(\mathscr{X})$ is the Grothendieck ring of vector bundles and $G(\mathscr{X})$ is the Grothendieck group of coherent sheaves. The second consequence is that $\mathscr{X}$ is a *quotient stack* [Edidin et al. 2001]. This means that $\mathscr{X} = [X/G]$, where G is a linear algebraic group acting on a scheme or algebraic space X.

If $\mathscr{X}$ is a smooth Deligne–Mumford stack, then we will implicitly choose a presentation $\mathscr{X} = [X/G]$. This allows us to identify the Grothendieck ring $K(\mathscr{X})$ with the equivariant Grothendieck ring $K_G(X)$, and the Chow ring $A^*(\mathscr{X})$ with the equivariant Chow ring $A^*_G(X)$. We will use the notation $K(\mathscr{X})$ and $K_G(X)$ (resp. $A^*(\mathscr{X})$ and $A^*_G(X)$) interchangeably.

Definition 2.1.1. Let G be an algebraic group acting on a scheme or algebraic space X. We define the *inertia space*

$$I_G X := \{(g, x) \mid gx = x\} \subset G \times X.$$

There is an induced action of G on $I_G X$ given by $g \cdot (m, x) = (gmg^{-1}, gx)$. The quotient stack $I\mathscr{X} := [I_G X/G]$ is the *inertia stack* of the quotient stack $\mathscr{X} := [X/G]$.

More generally, define the *higher inertia spaces* to be the k-fold fiber products

$$\mathbb{I}^k_G X = I_G X \times_X \cdots \times_X I_G X = \{(m_1, \dots, m_k, x) \mid m_i x = x \ \forall i = 1, \dots, k\} \subset G^k \times X.$$

The quotient stack $\mathbb{I}^k\mathscr{X} := [\mathbb{I}^k_G X/G]$ is the corresponding *higher inertia stack.*

The assumption that $\mathscr{X}$ has finite stabilizer means that the projection $I_G X \to X$ is a finite morphism. The composition $\mu\colon G \times G \to G$ induces a composition $\mu\colon I_G X \times_X I_G X \to I_G X$. This composition makes $I_G X$ into an X-group with identity section $X \to I_G X$ given by $x \mapsto (1, x)$.

Definition 2.1.2. Let G^ℓ be a G-space with the diagonal conjugation action. A *diagonal conjugacy class* is a G-orbit $\Phi \subset G^\ell$.

Definition 2.1.3. For all m in G, let $X^m = \{(m, x) \in I_G X\}$. For all $(m_1, \ldots, m_\ell)$ in G^ℓ, let $X^{m_1,\ldots,m_\ell} = \{(m_1, \ldots, m_\ell, x) \in \mathbb{I}^\ell_G X\}$. For all conjugacy classes $\Psi \subset G$, let $I(\Psi) = \{(m, x) \in I_G X \mid m \in \Psi\}$. More generally, for all diagonal conjugacy classes $\Phi \subset G^\ell$, let $\mathbb{I}^\ell(\Phi) = \{(m_1, \ldots m_\ell, x) \in \mathbb{I}^\ell_G X \mid (m_1, \ldots, m_\ell) \in \Phi\}$.

By definition, $I(\Psi)$ and $\mathbb{I}^\ell(\Phi)$ are G-invariant subsets of $I_G X$ and $\mathbb{I}^\ell_G X$, respectively. If G acts with finite stabilizer on X, then $I(\Psi)$ is empty unless Ψ consists of elements of finite order. Likewise, $\mathbb{I}^\ell(\Phi)$ is empty unless every ℓ-tuple $(m_1, \ldots, m_\ell) \in \Phi$ generates a finite group. Since conjugacy classes of elements of finite order are closed, $I(\Psi)$ and $\mathbb{I}^\ell(\Phi)$ are closed.

Proposition 2.1.4 [EJK 2010, Propositions 2.11 and 2.17]. *If G acts properly on X, then $I(\Psi) = \varnothing$ for all but finitely many conjugacy classes Ψ and the $I(\Psi)$ are unions of connected components of $I_G X$. Likewise, $\mathbb{I}^l(\Phi)$ is empty for all but finitely many diagonal conjugacy classes $\Phi \subset G^\ell$ and each $\mathbb{I}^l(\Phi)$ is a union of connected components of $\mathbb{I}^l_G X$.*

We frequently work with a group G acting on a space X where the quotient stack $[X/G]$ is not connected. As a consequence, some care is required in the definition of the rank and Euler class of a vector bundle. Note that, for any X, the group $A^0_G(X)$ satisfies $A^0_G(X) = \mathbb{Z}^\ell$, where ℓ is the number of connected components of the quotient stack $\mathscr{X} = [X/G]$.

Definition 2.1.5. If E is an equivariant vector bundle on X, then we define the *rank* of E to be $\mathrm{rk}(E) := \mathrm{Ch}^0(E) \in \mathbb{Z}^\ell = A^0_G(X)$. Note that the rank of E lies in the semigroup $\mathbb{N}^\ell$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. If $E_1, \ldots, E_n$ are vector bundles, then the *virtual rank (or augmentation)* of the element $\sum_{i=1}^n n_i[E_i] \in K_G(X)$ is the weighted sum $\sum_i n_i \mathrm{rk}(\mathscr{E}_i) \in \mathbb{Z}^\ell$.

If E is a G-equivariant vector bundle on X, then the rank of E on the connected components of $\mathscr{X} = [X/G]$ is bounded (since we assume that $\mathscr{X}$ has finite type).

Definition 2.1.6. If E is a G-equivariant vector bundle on X, we call the element $\lambda_{-1}(E^*) = \sum_{i=0}^\infty (-1)^i [\Lambda^i E^*] \in K_G(X)$ the *K-theoretic Euler class* of E. (Note that this sum is finite.)

Likewise, we define the element $c_{\mathrm{top}}(E) \in A^*_G(X)$, corresponding to the sum of the top Chern classes of E on each connected component of $[X/G]$, to be the *Chow-theoretic Euler class* of E. These definitions can be extended to any nonnegative element by multiplicativity. It will be convenient to use the symbol $\mathrm{eu}(\mathscr{F})$ to denote both of these Euler classes for a nonnegative element $\mathscr{F} \in K_G(X)$.

2.2. ***The logarithmic restriction and twisted pullback.*** We recall a construction from [EJK 2010] that will be used several times throughout the paper. However, to improve clarity, we use slightly different notation than in that paper.

Definition 2.2.1 [EJK 2010]. Let X be an algebraic space with an action of an algebraic group Z. Let E be a rank-n vector bundle on X and let g be a unitary automorphism of the fibers of $E \to X$. If we assume that the action of g commutes with the action of Z on E, the eigenbundles for the action of g are all Z-subbundles. Let $\exp(2\pi\sqrt{-1}\alpha_1), \dots, \exp(2\pi\sqrt{-1}\alpha_r)$ be the distinct eigenvalues of g acting on E, with $0 \le \alpha_k < 1$ for all $k \in \{1 \dots, r\}$, and let $E_1, \dots, E_r$ be the corresponding eigenbundles.

We define the *logarithmic trace of E* by the formula

$$L(g)(E) = \sum_{k=1}^{r} \alpha_k E_k \in K_Z(X) \otimes \mathbb{R} \tag{2.2.2}$$

on each connected component of X.

The next key fact about the logarithmic trace was proved in our previous paper.

Proposition 2.2.3 [EJK 2010, Proposition 4.6]. *Let $\boldsymbol{g} = (g_1, \dots, g_\ell)$ be an ℓ-tuple of elements of a compact subgroup of a reductive group H, satisfying $\prod_{i=1}^{\ell} g_i = 1$. Let X be an algebraic space with an action of an algebraic group Z, and let V be a $(Z \times H)$-equivariant bundle on X, where H is assumed to act trivially on X. The element*

$$\sum_{i=1}^{\ell} L(g_i)(V) - V + V^{\boldsymbol{g}}$$

in $K_Z(X)$ is represented by a Z-equivariant vector bundle.

Using Proposition 2.2.3 we make the following definition.

Definition 2.2.4. Let G be an algebraic group acting quasifreely on an algebraic space, and let V be a G-equivariant vector bundle on X. Given $\boldsymbol{g} = (g_1, \dots g_\ell) \in G^\ell$, if the g_i all lie in a common compact subgroup and satisfy $\prod_{i=1}^{\ell} g_i = 1$, then set

$$V(\boldsymbol{g}) = \sum_{i=1}^{\ell} L(g_i)(V|_{X^{\boldsymbol{g}}}) - V|_{X^{\boldsymbol{g}}} + V^{\boldsymbol{g}}|_{X^{\boldsymbol{g}}}.$$

We wish to extend this definition to give a map from $K_G(X)$ to $K_G(I_G X)$, but we must first understand the decompositions of $K_G(I_G X)$ and $A^*_G(I_G X)$ into conjugacy classes.

As a consequence of Proposition 2.1.4, we see that $K_G(I_G X)$ and $A^*_G(I_G X)$ are direct sums of the $K_G(I(\Psi))$ and $A^*_G(I(\Psi))$, respectively, as Ψ runs over conjugacy classes of elements of finite order in G. A similar statement holds for the equivariant K-theory and Chow groups of the higher inertia spaces as well.

Using Morita equivalence, we can give a more precise description of $K_G(I(\Psi))$. If $m \in \Psi$ is any element and $Z = Z_G(m)$ is the centralizer of m in G, then

$$K_G(I(\Psi)) = K_Z(X^m) \quad \text{and} \quad A^*_G(I(\Psi)) = A^*_Z(X^m).$$

Similarly, if $\Phi \subset G^\ell$ is a diagonal conjugacy class and $(m_1, \dots, m_\ell) \in \Phi$ and $Z = \bigcap_{i=1}^{\ell} Z_G(m_i)$, then

$$K_G(\mathbb{I}^l(\Phi)) = K_Z(X^{m_1,\dots,m_\ell}) \quad \text{and} \quad A^*_G(\mathbb{I}^l(\Phi)) = A^*_Z(X^{m_1,\dots,m_\ell}).$$

Definition 2.2.5. Define a map $L\colon K_G(X) \to K_G(I_G X) \otimes \mathbb{Q}$, as follows. For each conjugacy class $\Psi \subset G$ and each $V \in K_G(X)$, let $L(\Psi)(V)$ be the class in $K_Z(I(\Psi))$ which is Morita equivalent to $L(g)(V|_{X^g}) \in K_Z(X^g)_{\mathbb{Q}}$. Here g is any element of Ψ, and $Z = Z_G(g)$ is the centralizer of $g \in G$. The class $L(V)$ is the class whose restriction to $I(\Psi)$ is $L(\Psi)(V)$.

The proof of [EJK 2010, Lemma 5.4] shows that $L(\Psi)(V)$ (and thus $L(V)$) is independent of the choice of $g \in \Psi$.

Definition 2.2.6. If the diagonal conjugacy class $\Phi \subset G^\ell$ is represented by an ℓ-tuple $(g_1, \dots, g_\ell)$ such that $\prod_{i=1}^{\ell} g_i = 1$, then we define $V(\Phi)$ to be the class in $K_G(\mathbb{I}^l_G X)$ which is Morita equivalent to $V(\boldsymbol{g})$, where $\boldsymbol{g} = (g_1, \dots, g_\ell)$ is any element of Φ. Again, $V(\Phi)$ is independent of the choice of representative $\boldsymbol{g} \in \Phi$.

Definition 2.2.7. Identify $\mathbb{I}^\ell_G X$ with the closed and open subset of $\mathbb{I}^{\ell+1}_G X$ consisting of tuples $\{(g_1, \dots, g_{\ell+1}, x) \mid g_1 g_2 \dots g_{\ell+1} = 1\}$. If $V \in K_G(X)$, let $\mathrm{LR}(V) \in K_G(\mathbb{I}^\ell_G X)$ be the class whose restriction to $\mathbb{I}^{l+1}(\Phi)$ is $V(\Phi)$, where the diagonal conjugacy class $\Phi \in G^{\ell+1}$ is represented by a tuple $(g_1, \dots, g_{\ell+1})$ satisfying $g_1 \cdots g_{\ell+1} = 1$.

2.3. *Orbifold products and the orbifold Chern character.* Here we briefly review the construction and properties of orbifold products and orbifold Chern characters because they serve as a model for what we will do later.

Definition 2.3.1. For $i \in \{1, 2, 3\}$, let $e_i : \mathbb{I}^2_G X \to I_G X$ be the evaluation morphism taking (m_1, m_2, m_3, x) to (m_i, x) and let $\mu : \mathbb{I}^2_G X \to I_G X$ be the morphism taking (m_1, m_2, m_3, x) to $(m_1 m_2, x) = (m_3^{-1}, x)$.

Definition 2.3.2. Let $\mathbb{T}$ be the equivariant bundle on X corresponding to the tangent bundle of $\mathscr{X}$, which satisfies $\mathbb{T} = TX - \mathfrak{g}$ in $K_G(X)$, where $\mathfrak{g}$ is the Lie algebra of G.

Definition 2.3.3 [EJK 2010; JKK 2007; Kaufmann 2010]. The *orbifold product* on $K_G(I_G X)$ and $A^*_G(I_G X)$ is defined as

$$x \star y := \mu_*(e_1^* x \cdot e_2^* y \cdot \mathrm{eu}(\mathrm{LR}(\mathbb{T}))), \tag{2.3.4}$$

both for $x, y \in K_G(I_G X)$ and for $x, y \in A^*_G(I_G X)$.

Definition 2.3.5. We define the element $\mathscr{S} := L(\mathbb{T})$ in $K_G(I_G X)_{\mathbb{Q}}$ to be the logarithmic trace of $\mathbb{T}$, that is, for each m in G, we define $\mathscr{S}_m$ in $K_{Z_G(m)}(X^m)_{\mathbb{Q}}$ by

$$\mathscr{S}_m := L(m)(\mathbb{T}).$$

The rank of $\mathscr{S}$ is a $\mathbb{Q}$-valued, locally constant function on $I\mathscr{X} = [I_G X/G]$ called the *age*.

Remark 2.3.6. If the age of a connected component $[U/G]$ of $I\mathscr{X}$ is zero, then $[U/G]$ must be a connected component of $\mathscr{X} \subset I\mathscr{X}$.

Remark 2.3.7. In [Kaufmann 2010] the classes $\mathscr{S}$, manifestations of what physicists call *twist fields*, were interpreted in terms of cocycles which were then used to define stringy products. Our construction may be regarded as a realization of this procedure.

Definition 2.3.8. Given an element x in $A^*_G(I_G X)$ with ordinary Chow grading $\deg x$, the *orbifold degree (or grading) of* x is, like the ordinary Chow grading, constant on each component U of $I_G X$ corresponding to a connected component of $[U/G]$ of $[I_G X/G]$. On such a component U we define it to be the nonnegative rational number

$$\deg_{\mathrm{orb}} x|_U = \deg x|_U + \mathrm{age}[U/G]. \tag{2.3.9}$$

The induced grading on the group $A^*_G(I_G X)$ consists of summands $A^{\{q\}}_G(I_G X)$ of all elements with orbifold degree q.

Theorem 2.3.10 [JKK 2007; EJK 2010]. *The equivariant Chow group* $(A^*_G(I_G X), \star, \deg_{\mathrm{orb}})$ *is a* $\mathbb{Q}^C$*-graded, commutative ring with unity* $\mathbf{1}$, *where* $\mathbf{1}$ *is the identity element in* $A^*_G(X) = A^*_G(X^1) \subseteq A^*_G(I_G X)$ *and* C *is the number of connected components of* $[I_G X/G]$.

Equivariant K-theory $(K_G(I_G X), \star)$ *is a commutative ring with unity* $\mathbf{1}$, *where* $\mathbf{1} := \mathscr{O}_X$ *is the structure sheaf of* $X = X^1 \subset I_G X$.

Definition 2.3.11. The *orbifold Chern character* $\mathscr{C}h : K_G(I_G X) \to A^*_G(I_G X)_{\mathbb{Q}}$ is defined by the equation

$$\mathscr{C}h(\mathscr{F}) := \mathrm{Ch}(\mathscr{F}) \cdot \mathrm{Td}(-\mathscr{S})$$

for all $\mathscr{F} \in K_G(I_G X)$, where Td is the usual Todd class. Moreover, for all $\alpha \in \mathbb{Q}$ we define $\mathscr{C}h^\alpha(\mathscr{F})$ by the equation

$$\mathscr{C}h(\mathscr{F}) = \sum_{\alpha \in \mathbb{Q}} \mathscr{C}h^\alpha(\mathscr{F}),$$

where each $\mathscr{C}h^\alpha(\mathscr{F})$ belongs to $A^{\{\alpha\}}_G(I_G X)$.

The *orbifold virtual rank (or orbifold augmentation)* is $\mathscr{C}h^0 : K_G(I_G X) \to A^{\{0\}}_G(I_G X)_{\mathbb{Q}}$.

Theorem 2.3.12 [EJK 2010; JKK 2007]. *The orbifold Chern character*

$$\mathscr{C}h : (K_G(I_G X), \star) \to (A^*_G(I_G X)_{\mathbb{Q}}, \star)$$

is a ring homomorphism.

In particular, if $[U/G]$ *is a connected component of* $[I_G X/G]$, *then the virtual rank homomorphism restricted to the component* $[U/G]$ *gives a homomorphism* $\mathscr{C}h^0 : K_G(U) \to A^0_G(U)_{\mathbb{Q}} = \mathbb{Q}$, *satisfying*

$$\mathscr{C}h^0(\mathscr{F}) = \begin{cases} 0 & \text{if } \operatorname{age}[U/G] > 0, \\ \operatorname{Ch}^0(\mathscr{F}) & \text{if } \operatorname{age}[U/G] = 0 \end{cases}$$

for any $\mathscr{F} \in K_G(U)$.

3. Inertial products, Chern characters, and inertial pairs

In this section we generalize the ideas of orbifold cohomology, obstruction bundles, orbifold grading and the orbifold Chern character by defining *inertial products* on $K_G(I_G X)$ and $A^*_G(I_G X)$ using inertial bundles on $\mathbb{I}^2_G X$. We further define a rational grading and a Chern character ring homomorphism via *Chern-compatible classes of* $K_G(I_G X)_{\mathbb{Q}}$.

The original example of an associative bundle is the obstruction bundle $\mathscr{R} = \operatorname{LR}(\mathbb{T})$ of orbifold cohomology, and the original example of a Chern-compatible class is the logarithmic trace $\mathscr{S}$ of $\mathbb{T}$, as described in Definition 2.3.5.

We show below that there are many *inertial pairs* of associative inertial bundles on $\mathbb{I}^2_G X$ with Chern-compatible elements on $I_G X$, and hence there are many associative inertial products on $K_G(I_G X)$ and $A^*_G(I_G X)$ with rational gradings and Chern character ring homomorphisms.

3.1. *Associative bundles and inertial products.*

We recall the following definition (and notation) from [EJK 2010]. It should be noted that a similar formalism also appeared in the paper [Kaufmann 2010].

Definition 3.1.1. Given a class $c \in A^*_G(\mathbb{I}^2_G X)$ (resp. $K_G(\mathbb{I}^2_G X)$), we define the *inertial product with respect to c* to be

$$x \star_c y := \mu_*(e_1^* x \cdot e_2^* y \cdot c), \tag{3.1.2}$$

where $x, y \in A^*_G(I_G X)$ (resp. $K_G(I_G X)$).

Given a vector bundle $\mathscr{R}$ on $\mathbb{I}^2_G X$ we define inertial products on $A^*_G(I_G X)$ and $K_G(I_G X)$ via formula (3.1.2), where $c = \operatorname{eu}(\mathscr{R})$ is the Euler class of the bundle $\mathscr{R}$.

Definition 3.1.3. We say that $\mathscr{R}$ is an *associative bundle* on $\mathbb{I}^2_G X$ if the $\star_{\operatorname{eu}(\mathscr{R})}$ products on both $A^*_G(I_G X)$ and $K_G(I_G X)$ are commutative and associative with identity $\mathbf{1}$, where $\mathbf{1}$ is the identity class in $A^*_G(X)$ (resp. $K_G(X)$), viewed as a summand in $A^*_G(I_G X)$ (resp. $K_G(I_G X)$).

Proposition 3.1.4. *A sufficient condition for $\star_{\mathrm{eu}(\mathscr{R})}$ to be commutative with identity $\mathbf{1}$ is that the following conditions be satisfied*:

(1) *For every conjugacy class $\Phi \subset G \times G$ with $e_1(\Phi) = 1$ or $e_2(\Phi) = 1$, we have*

$$\mathscr{R}|_{I(\Phi)} = \mathscr{O}. \tag{3.1.5}$$

(2) *If $i : \mathbb{I}^2_G X \to \mathbb{I}^2_G X$ denotes the isomorphism $i(m_1, m_2, x) = (m_1 m_2 m_1^{-1}, m_1, x)$, we have*

$$i^*\mathscr{R} = \mathscr{R}. \tag{3.1.6}$$

Proof. This is almost just a restatement of Propositions 3.7–3.9 in [EJK 2010]. However, we note that in Proposition 3.9 there is a slight error — that proposition incorrectly stated that the map $i : \mathbb{I}^2_G X \to \mathbb{I}^2_G X$ was the map induced by the naive involution $(m_1, m_2) \mapsto (m_2, m_1)$, rather than the correct “braiding map” $(m_1, m_2, x) \mapsto (m_1 m_2 m_1^{-1}, m_1, x)$. □

A sufficient condition for associativity is also given in [EJK 2010]. In order to state the condition we require some notation, which we recall from that paper. Let $(m_1, m_2, m_3) \in G^3$ such that $m_1 m_2 m_3 = 1$, and let $\Phi_{1,2,3} \subset G^3$ be its diagonal conjugacy class. Let $\Phi_{12,3}$ be the conjugacy class of $(m_1 m_2, m_3)$ and $\Phi_{1,23}$ the conjugacy class of $(m_1, m_2 m_3)$. Let $\Phi_{i,j}$ be the conjugacy class of the pair (m_i, m_j) with $i < j$. Finally let Ψ_{123} be the conjugacy class of $m_1 m_2 m_3$; let Ψ_{ij} be the conjugacy class of $m_i m_j$; and let Ψ_i be the conjugacy class of m_i. There are evaluation maps

$$e_1 : \mathbb{I}^2(\Phi_{a,b}) \to I(\Psi_a), \quad e_2 : \mathbb{I}^2(\Phi_{a,b}) \to I(\Psi_b), \quad e_{i,j} : \mathbb{I}^3(\Phi_{1,2,3}) \to \mathbb{I}^2(\Psi_{i,j}),$$

and composition maps

$$\mu_{12,3} : \mathbb{I}^3(\Phi_{1,2,3}) \to \mathbb{I}^2(\Phi_{12,3}), \quad \mu_{1,23} : \mathbb{I}^3(\Phi_{1,2,3}) \to \mathbb{I}^2(\Phi_{1,23}).$$

The various maps we have defined are related by the following Cartesian diagrams of l.c.i. (local complete intersection) morphisms:

$$\begin{tikzcd}
\mathbb{I}^3(\Phi_{1,2,3}) \arrow[r, "e_{1,2}"] \arrow[d, "\mu_{12,3}"'] & \mathbb{I}^2(\Phi_{1,2}) \arrow[d, "\mu"] & \mathbb{I}^3(\Phi_{1,2,3}) \arrow[r, "e_{2,3}"] \arrow[d, "\mu_{1,23}"'] & \mathbb{I}^2(\Phi_{2,3}) \arrow[d, "\mu"] \\
\mathbb{I}^2(\Phi_{12,3}) \arrow[r, "e_1"] & I(\Psi_{12}) & \mathbb{I}^2(\Phi_{1,23}) \arrow[r, "e_2"] & I(\Psi_{23})
\end{tikzcd} \tag{3.1.7}$$

Let $E_{1,2}$ and $E_{2,3}$ be the respective excess normal bundles of the two diagrams (3.1.7).

Proposition 3.1.8. *Let $\mathscr{R}$ be a vector bundle on $\mathbb{I}^2_G X$ satisfying (3.1.5) and (3.1.6). A sufficient condition for $\mathscr{R}$ to be an associative bundle is if*

$$e^*_{1,2}\mathscr{R} + \mu^*_{12,3}\mathscr{R} + E_{1,2} = e^*_{2,3}\mathscr{R} + \mu^*_{1,23}\mathscr{R} + E_{2,3} \tag{3.1.9}$$

in $K_G(\mathbb{I}^3 X)$.

Proof. This follows from the proof of Proposition 3.12 of [EJK 2010], since the Euler class takes a sum of bundles to a product of Euler classes. □

In practice, the only way we have to show that a bundle $\mathscr{R}$ is associative is to show that it satisfies the identity (3.1.9). This leads to our next definition.

Definition 3.1.10. A bundle $\mathscr{R}$ is *strongly associative* if it satisfies the identities (3.1.5), (3.1.6) and (3.1.9).

3.2. ***Chern characters, age, and inertial pairs.*** In many cases one can define a Chern character $K_G(I_G X)_\mathbb{Q} \to A^*_G(I_G X)_\mathbb{Q}$ which is a *ring homomorphism* with respect to the inertial product. To do this, however, we need to define a *Chern compatible class* $\mathscr{S} \in K_G(I_G X)$. As an added bonus, such a class will also allow us to define a new grading on $A^*_G(I_G X)$ compatible with the inertial product and analogous to the orbifold grading of orbifold cohomology.

Definition 3.2.1. Let $\mathscr{R}$ be an associative bundle on $\mathbb{I}^2_G X$. A nonnegative class $\mathscr{S} \in K_G(I_G X)_\mathbb{Q}$ is called $\mathscr{R}$-Chern compatible if the map

$$\widetilde{\mathscr{C}h} \colon K_G(I_G X)_\mathbb{Q} \to A^*_G(I_G X)_\mathbb{Q}$$

defined by

$$\widetilde{\mathscr{C}h}(V) = \mathrm{Ch}(V) \cdot \mathrm{Td}(-\mathscr{S})$$

is a ring homomorphism with respect to the $\mathscr{R}$-inertial products on $K_G(I_G X)$ and $A^*_G(I_G X)$.

Remark 3.2.2. Again, the original example of a Chern compatible class is the class $\mathscr{S}$ defined in [JKK 2007], but we will we see other examples below.

Proposition 3.2.3. *If $\mathscr{R}$ is an associative vector bundle on $\mathbb{I}^2_G X$, then a nonnegative class $\mathscr{S} \in K_G(I_G X)_Q$ is $\mathscr{R}$-Chern compatible if the following identity holds in $K_G(\mathbb{I}^2_G X)$:*

$$\mathscr{R} = e^*_1\mathscr{S} + e^*_2\mathscr{S} - \mu^*\mathscr{S} + T_\mu. \tag{3.2.4}$$

Proof. This follows from the same formal argument used in the proof of [EJK 2010, Theorem 7.3]. □

Definition 3.2.5. A class $\mathscr{S} \in K_G(I_G X)_\mathbb{Q}$ is *strongly $\mathscr{R}$*-Chern compatible if it satisfies (3.2.4).

A pair $(\mathscr{R}, \mathscr{S})$ is an *inertial pair* if $\mathscr{R}$ is a strongly associative bundle and $\mathscr{S}$ is $\mathscr{R}$-strongly Chern compatible.

Definition 3.2.6. We define the *$\mathscr{S}$-age* on a connected component $[U/G]$ of $I\mathscr{X}$ to be the rational rank of $\mathscr{S}$ on the component $[U/G]$:

$$\mathrm{age}_{\mathscr{S}}[U/G] = \mathrm{rk}(\mathscr{S})_{[U/G]}.$$

We define the $\mathscr{S}$-degree of an element $x \in A^*_G(I_G X)$ on such a component U of $I_G X$ to be

$$\deg_{\mathscr{S}} x|_U = \deg x|_U + \mathrm{age}_{\mathscr{S}}[U/G],$$

where $\deg x$ is the degree with respect to the usual grading by codimension on $A^*_G(I_G X)$. Similarly, if $\mathscr{F}$ in $K_G(I_G X)$ is an element supported on U, then its $\mathscr{S}$-degree is

$$\deg_{\mathscr{S}} \mathscr{F} = \mathrm{age}_{\mathscr{S}} U \mod \mathbb{Z}.$$

This yields a $\mathbb{Q}/\mathbb{Z}$-grading of the group $K_G(I_G X)$.

Proposition 3.2.7. *If $\mathscr{R}$ is an associative vector bundle on $\mathbb{I}^2_G X$ and $\mathscr{S} \in K_G(I_G X)_{\mathbb{Q}}$ is strongly $\mathscr{R}$-Chern compatible, then the $\mathscr{R}$-inertial products on $A^*_G(I_G X)$ and $K_G(I_G X)$ respect the $\mathscr{S}$-degrees. Furthermore, the inertial Chern character homomorphism $\widetilde{\mathscr{C}h} : K_G(I_G X) \to A^*_G(I_G X)_{\mathbb{Q}}$ preserves the $\mathscr{S}$-degree modulo $\mathbb{Z}$.*

Proof. If $x, y \in A^*_G(I_G X)$, then the formula

$$x \star_{\mathrm{eu}(\mathscr{R})} y = \mu_*(e_1^* x \cdot e_2^* y \cdot \mathrm{eu}(\mathscr{R}))$$

implies that $\deg(x \star_{\mathrm{eu}(\mathscr{R})} y) = \deg x + \deg y + \mathrm{rk}\mathscr{R} + \mathrm{rk} T_\mu$. Since $\mathscr{S}$ is strongly $\mathscr{R}$-Chern compatible, we know that $\mathscr{R} = e_1^*\mathscr{S} + e_2^*\mathscr{S} - \mu^*\mathscr{S} + T_\mu$. Comparing ranks shows that the $\mathscr{S}$-degree of $x \star_{\mathrm{eu}(\mathscr{R})} y$ is the sum of the $\mathscr{S}$-degrees of x and y. The proof for $K_G(I_G X)$ follows from the fact that $\mathrm{rk}\mathscr{R}$ and $\mathrm{rk} T_\mu$ are integers. Finally, $\widetilde{\mathscr{C}h}$ preserves the $\mathscr{S}$-degree mod $\mathbb{Z}$ since if $\mathscr{F}$ in $K_G(I_G X)$ is supported on U, where $[U/G]$ is a connected component of $[I_G X/G]$, then so is its inertial Chern character. □

Definition 3.2.8. Let $A^{\{q\}}_G(I_G X)$ be the subspace in $A^*_G(I_G X)$ of elements with an $\mathscr{S}$-degree of q.

Definition 3.2.9. Given a class $\mathscr{S} \in K_G(I_G X)_{\mathbb{Q}}$, the restricted homomorphism $\widetilde{\mathscr{C}h}^0 : K_G(I_G X) \to A^{\{0\}}_G(I_G X)$ is called the *inertial virtual rank (or inertial augmentation)* for $\mathscr{S}$.

Definition 3.2.10. An inertial pair $(\mathscr{R}, \mathscr{S})$ is called *Gorenstein* if $\mathscr{S}$ has integral virtual rank and *strongly Gorenstein* if $\mathscr{S}$ is represented by a vector bundle.

The Deligne–Mumford stack $\mathscr{X} = [X/G]$ is *strongly Gorenstein* if the inertial pair $(\mathscr{R} = \mathrm{LR}(\mathbb{T}), \mathscr{S})$ associated to the orbifold product (as in Definitions 2.3.3 and 2.3.5) is strongly Gorenstein.

4. Inertial pairs associated to vector bundles

In this section we show how, for each choice of G-equivariant bundle V on X, we can use the methods of [EJK 2010] to define two new inertial pairs $(\mathscr{R}^+V, \mathscr{S}^+V)$ and $(\mathscr{R}^-V, \mathscr{S}^-V)$. We thus obtain corresponding inertial products and Chern characters. We denote the corresponding products associated to a vector bundle V as the $\star_{V^+}$ and $\star_{V^-}$ products. The $\star_{V^+}$ product can be interpreted as an orbifold product on the total space of V, while the $\star_{V^-}$ product on the Chow ring is a sign twist of the $\star_{(V^*)^+}$ product. Moreover, the two products induce isomorphic ring structures on $A^*(I\mathscr{X})\otimes\mathbb{C}$. We prove that if $V=\mathbb{T}$ is the tangent bundle to $\mathscr{X}=[X/G]$, then the $\star_{V^-}$ product agrees with the virtual orbifold product defined by [González et al. 2007].

To define the inertial pairs associated to a vector bundle, we introduce a variant of the logarithmic restriction introduced in [EJK 2010]. We begin with a simple proposition.

Proposition 4.0.1. *Let G be an algebraic group acting on a variety X and suppose that g_1, g_2 lie in a common compact subgroup. Let $Z=Z_G(g_1,g_2)$ be the centralizer of g_1 and g_2 in G.*

The virtual bundles

$$V^+(g_1,g_2)=L(g_1)(V|_{X^{g_1,g_2}})+L(g_2)(V|_{X^{g_1,g_2}})-L(g_1g_2)(V|_{X^{g_1,g_2}}) \quad (4.0.2)$$

and

$$V^-(g_1,g_2)=L(g_1^{-1})(V|_{X^{g_1,g_2}})+L(g_2^{-1})(V|_{X^{g_1,g_2}})-L(g_2^{-1}g_1^{-1})(V|_{X^{g_1,g_2}}) \quad (4.0.3)$$

are represented by nonnegative integral elements in $K_Z(X^{g_1,g_2})$.

Proof. Since $X^g=X^{g^{-1}}$ and $V^-(g_1,g_2)=V^+(g_2^{-1},g_1^{-1})$, it suffices to show that $V^+(g_1,g_2)$ is represented by a nonnegative integral element of $K_Z(X^{g_1,g_2})$. Let $g_3=(g_1g_2)^{-1}$. The identity $L(g)(V)+L(g^{-1})(V)=V-V^g$ implies that we can rewrite (4.0.2) as

$$\begin{aligned}V^+(g_1,g_2)=L(g_1)(V|_{X^{g_1,g_2}})&+L(g_2)(V|_{X^{g_1,g_2}})\\ &+L(g_3)(V|_{X^{g_1,g_2}})-V+V^{g_1,g_2}+V^{g_1g_2}-V^{g_1,g_2}.\end{aligned}$$

Since $g_1g_2g_3=1$, by Proposition 2.2.3 the sum

$$L(g_1,g_2,g_3)(V)=L(g_1)(V|_{X^{g_1,g_2}})+L(g_2)(V|_{X^{g_1,g_2}})+L(g_3)(V|_{X^{g_1,g_2}})-V+V^{g_1,g_2}$$

is represented by a nonnegative integral element of $K^Z(X_{g_1,g_2})$. Hence

$$V^+(g_1,g_2)=L(g_1,g_2,g_3)(V)+V^{g_1g_2}-V^{g_1,g_2}$$

is represented by a nonnegative integral element of $K_Z(X^{g_1,g_2})$. $\square$

Let $\Phi \subset G \times G$ be a diagonal conjugacy class. As in [EJK 2010] we may identify $K_G(\mathbb{I}^2(\Phi))$ with $K_{Z_G(g_1,g_2)}(X^{g_1,g_2})$ for any $(g_1, g_2) \in \Phi$. Thanks to Proposition 4.0.1 we can define nonnegative classes $V^+(\Phi)$ and $V^-(\Phi)$ in $K_G(\mathbb{I}^2(\Phi))$. The argument used in the proof of [EJK 2010, Lemma 5.4] shows that the definitions of $V^+(\Phi)$ and $V^-(\Phi)$ are independent of the choice of $(g_1, g_2) \in G^2$. Thus we can make the following definition.

Definition 4.0.4. Define classes R^+V and R^-V in $K_G(\mathbb{I}^2_G X)$ by setting the components of R^+V and R^-V in $K_G(\mathbb{I}^2(\Phi))$ to be $V^+(\Phi)$ and $V^-(\Phi)$, respectively. Similarly, we define classes $S^{\pm}V \in K_G(I_G X)_{\mathbb{Q}}$ by setting the restriction of $S^{\pm}V$ to a summand $K_G(I(\Psi))$ of $K_G(I_G X)$ to be the class Morita equivalent to $L(g^{\pm 1})(V) \in K_{Z_G(g)}(X^g)$, where $g \in \Psi$ is any element.

Theorem 4.0.5. *For any G-equivariant vector bundle V on X, the pairs*

$$(\mathscr{R}^+V, \mathscr{S}^+V) = (\mathrm{LR}(\mathbb{T}) + R^+V, \mathscr{S}\mathbb{T} + S^+V),$$
$$(\mathscr{R}^-V, \mathscr{S}^-V) = (\mathrm{LR}(\mathbb{T}) + R^-V, \mathscr{S}\mathbb{T} + S^-V)$$

are inertial pairs. Hence they define associative inertial products with a Chern character homomorphism.

Proof. Since

$$\mathrm{LR}(\mathbb{T}) = e_1^*\mathscr{S}\mathbb{T} + e_2^*\mathscr{S}\mathbb{T} - \mu^*\mathscr{S}\mathbb{T} + T_\mu \quad \text{and} \quad R^+V = e_1^*S^+V + e_2^*S^+V - \mu^*S^+V,$$

it follows that $\mathscr{S}^+V$ is strongly $\mathscr{R}^+V$-Chern compatible.

To complete the proof we must show that $\mathrm{LR}(\mathbb{T}) + R^+V$ is a strongly associative bundle. From their definitions we know that $\mathrm{LR}(\mathbb{T})$ and R^+V satisfy the identities (3.1.5) and (3.1.6). We also know that $\mathrm{LR}(\mathbb{T})$ satisfies (3.1.9). Thus, to prove that $\mathrm{LR}(\mathbb{T}) + R^+V$, it suffices to show that R^+V satisfies the “cocycle” condition

$$e_{1,2}^*R^+V + \mu_{12,3}^*R^+V = e_{2,3}^*R^+V + \mu_{1,23}^*R^+V. \tag{4.0.6}$$

Now (4.0.6) follows from the following identity of bundles restricted to X^{m_1,m_2,m_3}:

$$V^+(m_1, m_2) + V^+(m_1m_2, m_3) = V^+(m_2, m_3) + V^+(m_1, m_2m_3). \tag{4.0.7}$$

Equation (4.0.7) is a formal consequence of the definition of the bundles V^+. The result with R^+V and S^+V replaced by R^-V and S^-V, respectively, is proved analogously. □

4.1. ***Geometric interpretation of the $\star_{V^+}$ product.*** The $\star_{V^+}$ has a relatively direct interpretation in terms of an orbifold product on the total space of the vector bundle $V \to X$.

Lemma 4.1.1. *Given a G-equivariant vector bundle $\pi : V \to X$, the inertia space $I_G V$ is a vector bundle (of nonconstant rank) on $I_G X$ with structure map $I\pi : I_G V \to I_G X$.*

Proof. Let $\Psi \subset G$ be a conjugacy class. Denote by $I_X(\Psi) \subset I_G X$ the component of $I_G X$ defined by $\{(g, x) \mid gx = x,\ g \in \Psi\}$. For any morphism $V \to X$ and any conjugacy class $\Psi \in G$, if $I_X(\Psi) = \varnothing$, then $I_V(\Psi)$ is also empty. Thus it suffices to show that $I_V(\Psi)$ is a vector bundle over $I_X(\Psi)$ for every conjugacy class $\Psi \subset G$ with $I_X(\Psi) \neq \varnothing$. Given $g \in \Psi$, the identification $I_X(\Psi) = G \times_{Z_G(g)} X^g$ reduces the problem to showing that for $g \in G$ the fixed locus V^g is a $Z_G(g)$-equivariant vector bundle over X^g. Since the map $V \to X$ is G-equivariant, the map $V^g \to X$ has image X^g. The fiber over a point $x \in X^g$ is just $(V_x)^g$, where V_x is the fiber of $V \to X$ at x. □

Since $I_G V \to I_G X$ is a vector bundle, the pullback maps

$$(I\pi)^*\colon K_G(I_G V) \to K_G(I_G X) \quad \text{and} \quad (I\pi)^*\colon A^*_G(I_G V) \to A^*_G(I_G X)$$

are isomorphisms. Both isomorphisms are compatible with the ordinary products on K-theory and equivariant Chow groups.

Theorem 4.1.2. *For $x, y \in A^*_G(I_G X)$ or $x, y \in K_G(X)$, we have*

$$x \star_{V^+} y = (Is)^*((I\pi)^* x \star (I\pi)^* y), \tag{4.1.3}$$

where $\star$ is the usual orbifold product on the total space of the G-equivariant vector bundle $V \to X$ and Is^* is the Gysin map which is inverse to $I\pi^*$.

Proof. We give the proof only in equivariant Chow theory — the proof in equivariant K-theory is essentially identical. We compare the two sides of (4.1.3). If $\Psi_1, \Psi_2, \Psi_3 \subset G$ are conjugacy classes and $x \in A^*_G(I_X(\Psi_1))$, $y \in A^*_G(I_X(\Psi_2))$, then the contribution of $x \star_{V^+} y$ to $A^*_G(I_X(\Psi_3))$ is

$$\sum_{\Phi_{1,2}} \mu_*\big(e_1^* x \cdot e_2^* y \cdot \mathrm{eu}(\mathrm{LR}(\mathbb{T}) + R^+ V)\big), \tag{4.1.4}$$

where the sum is over all conjugacy classes $\Phi_{1,2} \subset G \times G$ satisfying

$$e_1(\Phi_{1,2}) = \Psi_1, \quad e_2(\Phi_{1,2}) = \Psi_2, \quad \mu(\Phi_{1,2}) = \Psi_3.$$

Since the class of tangent bundle of V equals $TX + V$, the tangent bundle to the stack $[V/G]$ is $TX + V - \mathfrak{g} = \mathbb{T} + V$. Thus, the contribution of the right-hand side of (4.1.3) is the sum

$$\sum_{\Phi_{1,2}} Is^*\big(\mu_{V*}(I\pi)^*(e_1^* x \cdot e_2^* y \cdot \mathrm{eu}(\mathrm{LR}(\mathbb{T}) + \mathrm{LR}(V)))\big), \tag{4.1.5}$$

where the map μ_V in (4.1.5) is understood to be the multiplication map $\mathbb{I}^2_G V \to I_G V$. If Φ is a conjugacy class in $G \times G$ with $\mu(\Phi) = \Psi$, then the multiplication map $\mu_V : I_V(\Phi) \to I_V(\Psi)$ factors through the inclusion

$$I_V(\Phi) \hookrightarrow \mu^* I_V(\Psi) \xrightarrow{I\pi^*\mu} I_V(\Psi), \tag{4.1.6}$$

and we have the following diagram, with a Cartesian square on the right:

$$\begin{tikzcd}
I_V(\Phi) \arrow[r, hook] \arrow[dr, "I\pi"] & \mu^* I_V(\Psi) \arrow[r, "I\pi^*\mu"] \arrow[d] & I_V(\Psi) \arrow[d, "I\pi"] \\
& I(\Phi) \arrow[r, "\mu"] & I(\Psi)
\end{tikzcd}$$

The normal bundle to the inclusion $I_V(\Phi) \hookrightarrow \mu^*(I_V(\Psi))$ is the pullback of the bundle V_Ψ / V_Φ on $\mathbb{I}^2_X(\Phi)$, where $V_\Phi \subset V|_{I_X(\Psi)}$ is the subbundle whose fiber over a point (g, x) is the subspace V^g, and the fiber of V_Φ over a point (g_1, g_2, x) is the subspace $V^{g_1, g_2} \subset V$. Using this information about the normal bundle we can rewrite (4.1.5) as

$$\mu_*\big(e_1^* x \cdot e_2^* y \cdot \mathrm{eu}(\mathrm{LR}(\mathbb{T} + V) + V_\Psi - V_\Phi)\big). \tag{4.1.7}$$

Finally, (4.1.7) can be identified with (4.1.4) by observing that if $g_1, g_2 \in G$ then

$$\begin{aligned} L(g_1)(V) + L(g_2)(V) + L((g_1 g_2)^{-1})(V) + V - V^{g_1, g_2} + V^{g_1, g_2} - V^{g_1 g_2} \\ = L(g_1)(V) + L(g_2)(V) - L(g_1 g_2)(V). \quad \square \end{aligned}$$

4.2. *Geometric interpretation of the $\star_{V^-}$ product.* The $\star_{V^-}$ product does not generally correspond to an orbifold product on a bundle. However, we will show that, after tensoring with $\mathbb{C}$, the inertial Chow (or cohomology) ring with the $\star_{V^-}$ product is isomorphic to the inertial Chow (or cohomology) ring coming from the total space of the dual bundle. The latter is isomorphic to the orbifold Chow (or cohomology) ring of the total space of the dual bundle.

Definition 4.2.1. Given a vector bundle V on a quotient stack $\mathscr{X} = [X/G]$, we define an automorphism Θ_V of $A^*(I\mathscr{X}) \otimes \mathbb{C}$ as follows. If x_Ψ is supported on a component $I(\Psi)$ of $I\mathscr{X}$ corresponding to a conjugacy class $\Psi \subset G$ then we set $\Theta_V(x_\Psi) = e^{i\pi a_\Psi} x_\Psi$, where a_Ψ is the virtual rank of the logarithmic trace $L(g^{-1})(V)$ for any representative element $g \in \Psi$. The same formula defines an automorphism of $H^*(I\mathscr{X}, \mathbb{C})$.

Theorem 4.2.2. *For $x, y \in A^*_G(I_G X)$ we have*

$$\begin{aligned} x \star_{V^-} y &= \pm (Is)^*((I\pi)^* x \star (I\pi)^* y) \\ &= \pm x \star_{V^{*+}} y, \end{aligned} \tag{4.2.3}$$

where $\star$ is the usual orbifold product on the total space of the G-equivariant vector bundle $V^* \to X$, and Is^* is the Gysin map which is inverse to $I\pi^*$, and the sign $\pm$ is $(-1)^{a_{\Psi_1}+a_{\Psi_2}-a_{\Psi_{12}}}$ where $a_{\Psi_1}+a_{\Psi_2}-a_{\Psi_{12}}$ is a nonnegative integer. Moreover, if we tensor with $\mathbb{C}$, then we have the identity

$$\Theta_V(x \star_{V^-} y) = \Theta_V(x) \star_{V^{*+}} \Theta_V(y). \tag{4.2.4}$$

Remark 4.2.5. The $\pm$ sign appearing in the previous theorem is an example of discrete torsion. Similar signs appear in the work of [Fantechi and Göttsche 2003].

Proof. Observe that if $g \in G$ acts on a representation V of $Z = Z_G(g)$ with weights $e^{i\theta_1}, \dots, e^{i\theta_r}$ then g naturally acts on V^* with weights $e^{-i\theta_1}, \dots, e^{-i\theta_r}$, and the $e^{i\theta_k}$-eigenspace of V is dual to the $e^{-i\theta_k}$-eigenspace of V^*. Hence $L(g^{-1})(V) = L(g)(V^*)^*$ as elements of $K(X^g) \otimes \mathbb{Q}$. Thus, given a pair $g_1, g_2 \in G$, we see that

$$V^-(g_1, g_2) = ((V^*)^+(g_1, g_2))^*$$

as $Z_G(g_1, g_2)$-equivariant bundles on $X^{g_1,g_2} = X^{g_1^{-1},g_2^{-1}}$. Hence, $\mathrm{eu}(R^-V) = (-1)^{\mathrm{rk}\, R^-V}\, \mathrm{eu}(R^+V^*)$, so (4.2.3) holds. If x is supported in the component $I(\Psi_1)$ and y is supported in the component $I(\Psi_2)$, then $x \star_{V^-} y$ is supported at components $I(\Psi_{12})$, where Ψ_{12} is a conjugacy class of $g_1 g_2$ for some $g_1 \in \Psi_1$ and $g_2 \in \Psi_2$.

Now we have

$$\Theta_V(x \star_{V^-} y) = \sum_{\Psi_{12}} e^{i\pi a_{\Psi_{12}}} (-1)^{\mathrm{rk}\, V^-(g_1,g_2)} x \star_{V^{*+}} y,$$

while

$$\Theta_V(x) \star_{V^{*+}} \Theta_V(y) = \sum_{\Psi_{12}} e^{i\pi(a_{\Psi_1}+a_{\Psi_2})} x \star_{V^{*+}} y.$$

Thus, (4.2.4) follows from the fact that $\mathrm{rk} V^-(g_1, g_2) = a_{\Psi_1} + a_{\Psi_2} - a_{\Psi_{12}}$. □

4.3. ***The virtual orbifold product is the $\star_{TX^-}$ product.*** The virtual orbifold product was introduced in [González et al. 2007]. In our context it (or more precisely its algebraic analogue) can be defined as follows:

Definition 4.3.1. Let $\mathbb{T}^{\mathrm{virt}}$ be the class in $K_G(\mathbb{I}^2_G X)$ defined by the formula

$$\mathbb{T}|_{\mathbb{I}^2_G X} + \mathbb{T}_{\mathbb{I}^2_G X} - e_1^*\mathbb{T}_{I_G X} - e_2^*\mathbb{T}_{I_G X}, \tag{4.3.2}$$

where $\mathbb{T}|_{\mathbb{I}^2_G X}$ refers to the pullback of the class $\mathbb{T}$ to $\mathbb{I}^2_G X$ via any of the three natural maps $\mathbb{I}^2_G X \to X$, where $\mathbb{T}_{I_G X}$ denotes the tangent bundle to the stack $I\mathscr{X} = [I_G X/G]$, and where $\mathbb{T}_{\mathbb{I}^2_G X}$ denotes the tangent bundle to the stack $I^2\mathscr{X}$.

Proposition 4.3.3. *The identity $\mathbb{T}^{\mathrm{virt}} = \mathrm{LR}(\mathbb{T}) + R^-\mathbb{T}$ holds in $K_G(\mathbb{I}^2_G X)$. In particular, $\mathbb{T}^{\mathrm{virt}}$ is represented by a nonnegative element of $K_G(\mathbb{I}^2_G X)$ and the $\star_{\mathrm{eu}\,\mathbb{T}^{\mathrm{virt}}}$-product is commutative and associative. Moreover, $\mathscr{S}\mathbb{T} + S^-\mathbb{T} = \boldsymbol{N}$, where $\boldsymbol{N}$ is* *the normal bundle of the canonical morphism $I_G X \to X$, so $(\mathbb{T}^{\mathrm{virt}}, \boldsymbol{N})$ is a strongly Gorenstein inertial pair.*

Proof. The proof follows from the identity

$$L(g)(\mathbb{T}) + L(g^{-1})(\mathbb{T}) = \mathbb{T}|_{X^g} - \mathbb{T}_{X^g} = \boldsymbol{N}|_{X^g}. \qquad \square$$

Definition 4.3.4. Following [González et al. 2007], we define the *virtual orbifold product* to be the $\star_{\mathrm{eu}(\mathbb{T}^{\mathrm{virt}})}$-product.

Corollary 4.3.5. *The virtual product $\star_{\mathrm{virt}}$ on $A^*_G(I_G X)$ agrees up to sign with the $\star_{\mathbb{T}^{*+}}$ inertial product on $A^*_G(I_G X)$ induced by the cotangent bundle $\mathbb{T}^*$ of $\mathscr{X} = [X/G]$, and there is an isomorphism of rings*

$$(A^*_G(I_G X)_{\mathbb{C}}, \star_{\mathrm{virt}}) \cong (A^*_G(I_G X)_{\mathbb{C}}, \star_{\mathbb{T}^{*+}}).$$

Remark 4.3.6. While in Chow theory these products differ by a simple discrete torsion (see Remark 4.2.5), in K-theory the virtual product is not so easily identified with the product $\star_{\mathbb{T}^{*+}}$, as can be seen from the fact that the gradings do not match (discrete torsion does not change the grading). If there is a connection in K-theory, it will have to be via something much more general, like a K-theoretic version of the matrix discrete torsion of [Kaufmann 2010].

4.4. ***An example with*** $\mathbb{P}(\mathbf{1}, \mathbf{3}, \mathbf{3})$**.** We illustrate the various inertial products in K-theory and Chow theory with the example of the weighted projective space $\mathscr{X} = \mathbb{P}(1, 3, 3) = [(\mathbb{A}^3 \setminus \{0\})/\mathbb{C}^*]$ where $\mathbb{C}^*$ acts with weights $(1, 3, 3)$. The inertia $I\mathscr{X}$ has three sectors — the identity sector $\mathscr{X}^1 = \mathscr{X}$ and two twisted sectors $\mathscr{X}^{\omega}$ and $\mathscr{X}^{\omega^{-1}}$, where $\omega = e^{2\pi i/3}$. Both twisted sectors are isomorphic to a $\mathscr{B}\mu_3$-gerbe over $\mathbb{P}^1$. The K-theory of each sector is a quotient of the representation ring $R(\mathbb{C}^*)$. Precisely, we have

$$K(\mathscr{X}^1) = \mathbb{Z}[\chi]/\langle(\chi-1)(\chi^3-1)^2\rangle \quad \text{and} \quad K(\mathscr{X}^{\omega}) = K(\mathscr{X}^{\omega^{-1}}) = \mathbb{Z}[\chi]/\langle(\chi^3-1)^2\rangle,$$

where χ is the defining character of $\mathbb{C}^*$. The projection formula in equivariant K-theory implies that any inertial product is determined by the products $\mathbf{1}_{g_1} \star \mathbf{1}_{g_2} \in K(\mathscr{X}^{g_1 g_2})$, where $\mathbf{1}_g$ is the K-theoretic fundamental class on the sector $\mathscr{X}^g$.

The usual orbifold product is represented by the symmetric matrix

	$\mathscr{X}^1$	$\mathscr{X}^{\omega}$	$\mathscr{X}^{\omega^{-1}}$
$\mathscr{X}^1$	1	1	1
$\mathscr{X}^{\omega}$		1	$\mathrm{eu}(\chi)$
$\mathscr{X}^{\omega^{-1}}$			$\mathrm{eu}(\chi)$

The virtual and orbifold cotangent products are represented by the following matrices:

$$\begin{array}{c|ccc} & \mathscr{X}^1 & \mathscr{X}^\omega & \mathscr{X}^{\omega^{-1}} \\ \hline \mathscr{X}^1 & 1 & 1 & 1 \\ \mathscr{X}^\omega & & \mathrm{eu}(\chi) & \mathrm{eu}(\chi)^2 \\ \mathscr{X}^{\omega^{-1}} & & & \mathrm{eu}(\chi) \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & \mathscr{X}^1 & \mathscr{X}^\omega & \mathscr{X}^{\omega^{-1}} \\ \hline \mathscr{X}^1 & 1 & 1 & 1 \\ \mathscr{X}^\omega & & \mathrm{eu}(\chi^{-1}) & \mathrm{eu}(\chi)\,\mathrm{eu}(\chi^{-1}) \\ \mathscr{X}^{\omega^{-1}} & & & \mathrm{eu}(\chi) \end{array}$$

If we define $t = c_1(\mathscr{X}) \in A^*(\mathscr{B}\mathbb{C}^*)$, then the inertial products on Chow and cohomology groups can also be represented by matrices, as above. After tensoring with $\mathbb{C}$, the Chow groups of the sectors are

$$A^*(\mathscr{X}^1)_{\mathbb{C}} = \mathbb{C}[t]/\langle t^3\rangle \quad \text{and} \quad A^*(\mathscr{X}^\omega)_{\mathbb{C}} = A^*(\mathscr{X}^{\omega^{-1}})_{\mathbb{C}} = \mathbb{C}[t]/\langle t^2\rangle.$$

The corresponding matrices for the virtual and cotangent orbifold products are

$$\begin{array}{c|ccc} & \mathscr{X}^1 & \mathscr{X}^\omega & \mathscr{X}^{\omega^{-1}} \\ \hline \mathscr{X}^1 & 1 & 1 & 1 \\ \mathscr{X}^\omega & & t & t^2 \\ \mathscr{X}^{\omega^{-1}} & & & t \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & \mathscr{X}^1 & \mathscr{X}^\omega & \mathscr{X}^{\omega^{-1}} \\ \hline \mathscr{X}^1 & 1 & 1 & 1 \\ \mathscr{X}^\omega & & -t & -t^2 \\ \mathscr{X}^{\omega^{-1}} & & & t \end{array}$$

The automorphism of $A^*(I\mathscr{X})_{\mathbb{C}}$ which is the identity on $A(\mathscr{X}^1)_{\mathbb{C}}$ and which acts by multiplication by $e^{2\pi i/3}$ on $A^*(\mathscr{X}^\omega)_{\mathbb{C}}$ and $e^{\pi i/3}$ on $A^*(\mathscr{X}^{\omega^{-1}})_{\mathbb{C}}$ defines a ring isomorphism between these products.

5. The localized orbifold product on $K(I\mathscr{X}) \otimes \mathbb{C}$

If an algebraic group G acts with finite stabilizer on smooth variety Y, then there is a decomposition of $K_G(Y) \otimes \mathbb{C}$ as a sum of localizations $\bigoplus_\Psi K_G(Y)_{\mathfrak{m}_\Psi}$. Here the sum is over conjugacy classes $\Psi \subset G$ such that $I(\Psi) \neq \varnothing$, and $\mathfrak{m}_\Psi \in \operatorname{Spec} R(G)$ is the maximal ideal of class functions vanishing on the conjugacy class Ψ.

Given a conjugacy class $\Psi \subset G$ and a choice of $h \in \Psi$, denote the centralizer of h in G by $Z = Z_G(h)$. The conjugacy class of h in Z is just h alone, and there is a corresponding maximal ideal $\mathfrak{m}_h \in \operatorname{Spec} R(Z)$. As described in [Edidin and Graham 2005, §4.3], the localization $K_G(I(\Psi))_{\mathfrak{m}_h}$ is a summand of the localization $K_G(I(\Psi))_{\mathfrak{m}_\Psi}$, and this summand is independent of the choice of h. This is called the *central summand* of Ψ and is denoted by $K_G(I(\Psi))_{c(\Psi)}$.

Since G acts with finite stabilizer, the projection $f_\Psi : I(\Psi) \to Y$ is a finite l.c.i. morphism. The nonabelian localization theorem of [Edidin and Graham 2005] states that the pullback $f_\Psi^*\colon K_G(Y)\otimes\mathbb{C} \to K_G(I(\Psi))\otimes\mathbb{C}$ induces an isomorphism between the localization of $K_G(Y)$ at $\mathfrak{m}_\Psi$ and the central summand $K_G(I(\Psi))_{c(\Psi)} \subset K_G(I(\Psi))_{\mathfrak{m}_\Psi}$. The inverse to f_Ψ^* is the map $\alpha \mapsto f_{\Psi *}(\alpha \cdot \mathrm{eu}(N_{f_\Psi})^{-1})$. If we let f be the global stabilizer map $I_G Y \to Y$, then, after summing over all conjugacy classes Ψ in the support of $K_G(Y) \otimes \mathbb{C}$, we obtain an isomorphism

$$f^*\colon K_G(Y) \otimes \mathbb{C} \to K_G(I_G Y)_c,$$

where $K_G(I_G Y)_c = \bigoplus K_G(I(\Psi))_{c(\Psi)}$. The inverse is $f_*/\mathrm{eu}(N_f)$.

Applying this construction with $Y = I_G X$ allows us to define a product we call *the localized orbifold product*.

Definition 5.0.1. The *localized orbifold product* on $K_G(I_G X) \otimes \mathbb{C}$ is defined by the formula

$$\alpha \star_{\mathrm{LO}} \beta = If_*\big((If^*\alpha \star If^*\beta) \otimes \mathrm{eu}(N_{If})^{-1}\big),$$

where $\star$ is the usual orbifold product on $K_G(I_G(I_G X))_c$, and $If : I_G(I_G X) \to I_G X$ is the projection.

Remark 5.0.2. It should be noted that $I_G(I_G X)$ is not the same as $\mathbb{I}^2_G X$. The inertia $I_G(I_G X) = \{(h, g, x) \mid hx = gx = x, hg = gh\}$ is a closed subspace of $\mathbb{I}^2_G X$.

The localized product can be interpreted in the context of the $\star_{V^+}$ product, where the vector bundle V is replaced by the virtual bundle $-N_f$. Observe that the pull-back of $\mathbb{T}$ to $I_G X$ splits as $\mathbb{T} = \mathbb{T}_{I_G X} + N_f$, where N_f is the normal bundle to the finite l.c.i. map $I_G X \to X$. Although N_f is not a bundle on X, we can still compute $N_f^+(g_1, g_2)$ on $\mathbb{I}^2_G X$.

The same formal argument used in the proof of Theorem 4.1.2 yields the following result.

Proposition 5.0.3. *The class* $\mathrm{eu}(\mathrm{LR}(\mathbb{T}) + R^+(-N_f))$ *is well-defined in localized K-theory, and*

$$\alpha \star_{\mathrm{LO}} \beta = \alpha \star_{(-N_f)^+} \beta.$$

Remark 5.0.4. The inertial pair corresponding to the localized product is the formal pair $(\mathrm{LR}(\mathbb{T}) + R^+(-N_f), \mathscr{S}\mathbb{T} + S^+(-N_f))$. However, the Chern character corresponding to this inertial pair is the usual orbifold Chern character and the corresponding product on $A^*(I\mathscr{X})$ is the usual orbifold product. The reason is that the orbifold Chern character isomorphism factors through $K_G(I_G X)_{(1)}$, the localization of $K_G(I_G X)$ at the augmentation ideal of $R(G)$. This localization corresponds to the untwisted sector of $I_G X$ where f restricts to the identity map.

Remark 5.0.5. Identifying $K_G(I_G X)_{\mathbb{C}}$ with the localization of $K_G(I_G(I_G X))_{\mathbb{C}}$ allows us to invert the class $\mathrm{eu}(N_f)$. In [Kaufmann 2010, §3.4] the author gives a framework for defining similar products after formally inverting the Euler classes of normal bundles.

5.1. ***An example with*** $\mathbb{P}(\mathbf{1}, \mathbf{2})$**.** We consider the weighted projective line $\mathscr{X} = \mathbb{P}(1, 2) = [(\mathbb{A}^2 \smallsetminus \{0\})/\mathbb{C}^*]$, where $\mathbb{C}^*$ acts with weights $(1, 2)$. The inertia stack $I\mathscr{X}$ has two sectors, $\mathscr{X}^1 = \mathscr{X}$ and $\mathscr{X}^{-1} = \mathscr{B}\mu_2$. We have

$$K(\mathscr{X}^1) \otimes \mathbb{C} = \mathbb{C}[\chi]/\langle(\chi - 1)(\chi^2 - 1)\rangle \quad \text{and} \quad K(\mathscr{X}^{-1}) \otimes \mathbb{C} = \mathbb{C}[\chi]/\langle\chi^2 - 1\rangle.$$

In particular, $K(I\mathscr{X})\otimes\mathbb{C}$ is supported at $\pm 1 \in \mathbb{C}^*$. As was the case in Section 4.4, inertial ring structures are determined by the products $\mathbf{1}_{g_1} \star \mathbf{1}_{g_2} \in K(\mathscr{X}^{g_1 g_2})$. In terms of the localization decomposition, $K(I\mathscr{X}) \otimes \mathbb{C} = K(I\mathscr{X})_{(1)} \oplus K(I\mathscr{X})_{(-1)}$. The localized product is determined by computing the corresponding orbifold product on each localized piece using the decomposition of the element $\mathbf{1}_g$ into its localized pieces and the product $\mathbf{1}_{g_1} \star_{\mathrm{LO}} \mathbf{1}_{g_2}$ decomposes as

$$(\mathbf{1}_{g_1})_{(1)} \star_{\mathrm{LO}} (\mathbf{1}_{g_2})_{(1)} + (\mathbf{1}_{g_1})_{(-1)} \star_{\mathrm{LO}} (\mathbf{1}_{g_2})_{(-1)}.$$

The multiplication matrix for $K(I\mathscr{X})_{(1)}$ is the usual orbifold matrix, which in this case is

	$\mathscr{X}^1$	$\mathscr{X}^{-1}$
$\mathscr{X}^1$	1	1
$\mathscr{X}^{-1}$	1	$\mathrm{eu}(\chi)$

The multiplication matrix for the localized product on $K(I\mathscr{X})_{(-1)}$ is the same as the multiplication matrix for the orbifold product on $\mathscr{B}\mu_2$, which is

	$\mathscr{X}^1$	$\mathscr{X}^{-1}$
$\mathscr{X}^1$	1	1
$\mathscr{X}^{-1}$	1	1

Thus we see that the only nontrivial product is $\mathbf{1}_{(-1)} \star_{\mathrm{LO}} \mathbf{1}_{(-1)}$. To obtain a single multiplication matrix we use the decomposition

$$\mathbf{1}_{(-1)} = \tfrac{1}{2}(1+\chi) + \tfrac{1}{2}(1-\chi) \in K(\mathscr{X}^{-1}) \otimes \mathbb{C},$$

where $\frac{1}{2}(1+\chi)$ is supported at 1 and $\frac{1}{2}(1-\chi)$ is supported at -1. The final result is the matrix

	$\mathscr{X}^1$	$\mathscr{X}^{-1}$
$\mathscr{X}^1$	1	1
$\mathscr{X}^{-1}$	1	$\dfrac{(1+\chi)^2\,\mathrm{eu}(\chi)+(1-\chi)^2}{4}$

Because the twisted sector $\mathscr{X}^{-1}$ has dimension 0, both the orbifold and usual Chern characters on this sector compute the virtual rank. The untwisted sector $\mathbb{P}(1,2)$ has Chow ring $\mathbb{C}[t]/\langle t^2\rangle$, where $t = c_1(\chi)$. Thus $\mathrm{ch}(\mathrm{eu}(\chi)) = t \in A^*(\mathbb{P}(1,2)) \otimes \mathbb{C}$. Observe that

$$\mathrm{ch}\left(\frac{(1+\chi)^2\,\mathrm{eu}(\chi)+(1-\chi)^2}{4}\right) = \frac{(2+t)^2(t)+(-t)^2}{4} = t$$

in $\mathbb{C}[t]/\langle t^2\rangle$ as well.

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