ANNALS OF K-THEORY

no. 2 vol 1 2016

Revisiting Farrell's nonfiniteness of Nil

Jean-François Lafont, Stratos Prassidis and Kun Wang

msp

A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 1, No. 2, 2016
dx.doi.org/10.2140/akt.2016.1.209

Revisiting Farrell's nonfiniteness of Nil

Jean-François Lafont, Stratos Prassidis and Kun Wang

We study Farrell Nil-groups associated to a finite-order automorphism of a ring R. We show that any such Farrell Nil-group is either trivial or infinitely generated (as an abelian group). Building on this first result, we then show that any finite group that occurs in such a Farrell Nil-group occurs with infinite multiplicity. If the original finite group is a direct summand, then the countably infinite sum of the finite subgroup also appears as a direct summand. We use this to deduce a structure theorem for countable Farrell Nil-groups with finite exponent. Finally, as an application, we show that if V is any virtually cyclic group, then the associated Farrell or Waldhausen Nil-groups can always be expressed as a countably infinite sum of copies of a finite group, provided they have finite exponent (which is always the case in dimension zero).

1. Introduction

For a ring R and an automorphism $\alpha : R \to R$, one can form the twisted polynomial ring $R_\alpha[t]$, which as a left R-module coincides with the polynomial ring $R[t]$, but with product given by $rt = t\alpha(r)$. There is a natural augmentation map $\varepsilon : R_\alpha[t] \to R$ induced by setting $\varepsilon(t) = 0$. For $i \in \mathbb{Z}$, the *Farrell twisted Nil-groups* $\mathrm{NK}_i(R, \alpha) := \ker(\varepsilon_*)$ are defined to be the kernels of the induced K-theory map $\varepsilon_* : K_i(R_\alpha[t]) \to K_i(R)$. This induced map is split injective, hence $\mathrm{NK}_i(R, \alpha)$ can be viewed as a direct summand in $K_i(R_\alpha[t])$. In the special case where the automorphism α is the identity, the ring $R_\alpha[t]$ is just the ordinary polynomial ring $R[t]$, and the Farrell twisted Nil-groups reduce to the ordinary *Bass Nil-groups*, which we just denote by $\mathrm{NK}_i(R)$. We establish the following:

Theorem A. *Let R be a ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. Then* $\mathrm{NK}_i(R, \alpha)$ *is either trivial or infinitely generated as an abelian group.*

Lafont's research is partially supported by the NSF, under grants DMS-1207782, DMS-1510640. Prassidis's research has been cofinanced by the European Union (European Social Fund — ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) — Research Funding Program: THALIS. Wang's research is partially supported by an Ohio State University Presidential Fellowship award.
MSC2010: 18F25, 19D35, 18E10.
Keywords: Nil-groups, algebraic K-theory, Frobenius functors, Verschiebung functors.

The proof of this result relies heavily on a method developed by Farrell [1977], who first showed that the lower Bass Nil-groups $\mathrm{NK}_*(R)$ with $* \leq 1$ are always either trivial or infinitely generated. This result was subsequently extended to the higher Bass Nil-groups $\mathrm{NK}_*(R)$ with $* \geq 1$ by Prasolov [1982] (see also [van der Kallen 1980]). For Farrell's twisted Nils, when the automorphism α has finite order, Grunewald [2007] and Ramos [2007] independently established the corresponding result for $\mathrm{NK}_*(R, \alpha)$ when $* \leq 1$. All these papers used the same basic idea, which we call *Farrell's lemma*. We exploit the same idea, and establish our own version of Farrell's lemma (and prove the theorem) in Section 3.

Remark 1.1. Farrell's original proof of his lemma used the transfer map on K-theory. Naïvely, one might want to try to prove Theorem A as follows: choose m so that $\alpha^m = \alpha$. Then there is a ring homomorphism from $A = R_\alpha[t]$ to $B = R_\alpha[s]$ determined by $t \mapsto s^m$. Call the induced map on K-theory $F_m : K(A) \to K(B)$. Since B is a free (left) A-module of rank m, the transfer map V_m is also defined, and $G_m := V_m \circ F_m = \mu_m$ (multiplication by m). Then follow Farrell's original 1977 argument verbatim to conclude the proof. Unfortunately this approach does not work, for two reasons.

Firstly, the identity $G_m = \mu_m$ does not hold in the twisted case (basically due to the fact that $\bigoplus_m A$ and B are not isomorphic *as bimodules*). We do not explicitly know what the map G_m does on K-theory, but it is definitely *not* multiplication by an integer. Instead, we have the somewhat more complicated identity given in part (2) of our Lemma 3.1, but which is still sufficient to establish the theorem.

Secondly, while it is possible to derive the identity in part (2) of Lemma 3.1 using the transfer map as in Farrell's original argument, it is not at all clear how to obtain the analogue of part (3) in higher dimensions by working at the level of K-theory groups. Instead, we have to work at the level of categories, specifically, with the *Nil-category* $\mathrm{NIL}(R; \alpha)$ (see Section 2), in order to ensure property (3). The details are in [Grunewald 2008].

Next we refine somewhat the information we have on these Farrell Nils, by focusing on the finite subgroups arising as direct summands. In Section 4, we establish:

Theorem B. *Let R be a ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $H \leq \mathrm{NK}_i(R, \alpha)$ is a finite subgroup, then $\bigoplus_\infty H$ also appears as a subgroup of $\mathrm{NK}_i(R, \alpha)$. Moreover, if H is a direct summand in $\mathrm{NK}_i(R, \alpha)$, then so is $\bigoplus_\infty H$.*

In the statement above, and throughout the paper, $\bigoplus_\infty H$ denotes the direct sum of countably infinitely many copies of the group H. Theorem B together with some group-theoretic facts enable us to deduce a structure theorem for certain Farrell Nil-groups. In Section 5, we prove the following result:

Theorem C. *Let R be a countable ring, $\alpha : R \to R$ a ring automorphism of finite order, and $i \in \mathbb{Z}$. If $\mathrm{NK}_i(R, \alpha)$ has finite exponent, then there exists a finite abelian group H so that $\mathrm{NK}_i(R, \alpha) \cong \bigoplus_\infty H$.*

A straightforward corollary of Theorem C is the following:

Corollary 1.2. *Let G be a finite group, $\alpha \in \mathrm{Aut}(G)$. Then there exists a finite abelian group H, whose exponent divides some power of $|G|$, with the property that $\mathrm{NK}_0(\mathbb{Z}G, \alpha) \cong \bigoplus_\infty H$.*

Proof. Connolly and Prassidis [2002] proved that $\mathrm{NK}_0(\mathbb{Z}G, \alpha)$ has finite exponent when G is finite. KuKu and Tang [2003, Theorem 2.2] showed that $\mathrm{NK}_i(\mathbb{Z}G, \alpha)$ is $|G|$-primary torsion for all $i \geq 0$. These facts together with Theorem C above complete the proof. □

Remark 1.3. It is a natural question whether this corollary holds in dimensions other than zero. In negative dimensions $i < 0$, Farrell and Jones [1995] showed that $\mathrm{NK}_i(\mathbb{Z}G, \alpha)$ always vanishes when G is finite. In positive dimensions $i > 0$, there are partial results. As mentioned in the proof above, Kuku and Tang [2003, Theorem 2.2] showed that $\mathrm{NK}_i(\mathbb{Z}G, \alpha)$ is $|G|$-primary torsion. Grunewald [2008, Theorem 5.9] then generalized their result to *polycyclic-by-finite* groups in all dimensions. He showed that, for all $i \in \mathbb{Z}$, $\mathrm{NK}_i(\mathbb{Z}G, \alpha)$ is mn-primary torsion for every polycyclic-by-finite group G and every group automorphism $\alpha : G \to G$ of finite order, where $n = |\alpha|$ and m is the index of some polyinfinite cyclic subgroup of G (such a subgroup always exists). However, although we have these nice results on the possible orders of torsion elements, it seems there are no known results on the exponent of these Nil-groups. This is clearly a topic for future research.

Remark 1.4. As an example in dimension greater than zero, Weibel [2009] showed that $\mathrm{NK}_1(\mathbb{Z}D_4) \neq 0$, where D_4 denotes the dihedral group of order 8. He also constructed a surjection $\bigoplus_\infty(\mathbb{Z}_2 \oplus \mathbb{Z}_4) \to \mathrm{NK}_1(\mathbb{Z}D_4)$, showing that this group has exponent 2 or 4. It follows from our corollary that the group $\mathrm{NK}_1(\mathbb{Z}D_4)$ is isomorphic to one of the three groups $\bigoplus_\infty(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$, $\bigoplus_\infty \mathbb{Z}_4$, or $\bigoplus_\infty \mathbb{Z}_2$.

For our next application, we recall that for every group G there exists an assembly map $H_n^G(\underline{E}G; \boldsymbol{K}_\mathbb{Z}) \to K_n(\mathbb{Z}[G])$, where $H_*^?(-; \boldsymbol{K}_\mathbb{Z})$ denotes the specific equivariant generalized homology theory appearing in the K-theoretic Farrell–Jones isomorphism conjecture with coefficients in $\mathbb{Z}$, and $\underline{E}G$ is a model for the classifying space for proper G-actions. We refer the reader to Section 5 for a discussion of these notions, as well as for the proof of the following theorem:

Theorem D. *For any virtually cyclic group V, there exists a finite abelian group H with the property that there is an isomorphism*

$$\bigoplus_\infty H \cong \mathrm{coker}(H_0^V(\underline{E}V; \boldsymbol{K}_\mathbb{Z}) \to K_0(\mathbb{Z}[V])).$$

The same result holds in dimension n whenever $\operatorname{coker}(H_n^V(\underline{E}V; \boldsymbol{K}_{\mathbb{Z}}) \to K_n(\mathbb{Z}[V]))$ *has finite exponent.*

We conclude the paper with some general remarks and open questions in Section 6.

2. Some exact functors

In this section, we define various functors that will be used in our proofs. Let R be an associative ring with unit and $\alpha : R \to R$ be a ring automorphism of finite order, say $|\alpha| = n$. For each integer $i \in \mathbb{Z}$, denote by R_{α^i} the R-bimodule which coincides with R as an abelian group, but with bimodule structure given by $r \cdot x := rx$ and $x \cdot r := x\alpha^i(r)$ (where $x \in R_{\alpha^i}$ and $r \in R$). Note that, as left (or as right) R-modules, R_{α^i} and R are isomorphic, but they are in general *not* isomorphic as R-bimodules. For each right R-module M and integer i, define a new right R-module M_{α^i} as follows: as abelian groups, M_{α^i} is the same as M, however, the right R-module structure on M_{α^i} is given by $x \cdot r := x\alpha^i(r)$. Clearly $M_{\alpha^n} = M$ and $(M_{\alpha^i})_{\alpha^j} = M_{\alpha^{i+j}}$ as right R-modules. We could have defined $M_{\alpha^i} = M \otimes_R R_{\alpha^i}$; however, this has the slight disadvantage that the above equalities would not hold — we would only have natural isomorphisms between the corresponding functors.

Let $\boldsymbol{P}(R)$ denote the category of finitely generated right projective R-modules. There is an exact functor $S : \boldsymbol{P}(R) \to \boldsymbol{P}(R)$ given by $S(P) = P_\alpha$ on objects and $S(\phi) = \phi$ on morphisms. Observe that, if we forget about the right R-module structures, and just view these as abelian groups and group homomorphisms, then S is just the identity functor. By taking iterates S^i of the functor S, we obtain a functorial $\mathbb{Z}$-action on the category $\boldsymbol{P}(R)$, which factors through a functorial $\mathbb{Z}_n$-action (recall that n is the order of the ring automorphism α).

We are interested in the *Nil-category* $\mathrm{NIL}(R; \alpha)$. Recall that objects of this category are of the form (P, f), where P is an object in $\boldsymbol{P}(R)$ and $f : P \to P_\alpha = S(P)$ is a right R-module homomorphism which is *nilpotent*, in the sense that a high enough composite map of the following form is the zero map:

$$P \xrightarrow{S^{k-1}(f) \circ S^{k-2}(f) \circ \cdots \circ S^1(f) \circ f} P_{\alpha^k}.$$

A morphism $\phi : (P, f) \to (Q, g)$ in $\mathrm{NIL}(R; \alpha)$ is given by a morphism $\phi : P \to Q$ in $\boldsymbol{P}(R)$ which makes the obvious diagram commutative, i.e., $S(\phi) \circ f = g \circ \phi$. The exact structure on $\boldsymbol{P}(R)$ induces an exact structure on $\mathrm{NIL}(R; \alpha)$, and we have two exact functors

$$F : \mathrm{NIL}(R; \alpha) \to \boldsymbol{P}(R), \quad F(P, f) = P,$$
$$G : \boldsymbol{P}(R) \to \mathrm{NIL}(R; \alpha), \quad G(P) = (P, 0),$$

which give rise to a splitting of the K-theory groups

$$K_i(\mathrm{NIL}(R; \alpha)) = K_i(R) \oplus \mathrm{Nil}_i(R; \alpha),$$

where

$$\mathrm{Nil}_i(R;\alpha) := \mathrm{Ker}(K_i(\mathrm{NIL}(R;\alpha)) \to K_i(R)), \quad i \in \mathbb{N}$$

(natural numbers start with 0 in this paper).

Remark 2.1. The Farrell Nil-groups $\mathrm{NK}_*(R,\alpha)$ mentioned in the introduction coincide, with a dimension shift, with the groups $\mathrm{Nil}_*(R;\alpha^{-1})$ defined above. More precisely, one has for every $i \geq 1$ an isomorphism $\mathrm{NK}_i(R,\alpha) \cong \mathrm{Nil}_{i-1}(R;\alpha^{-1})$ [Grayson 1988, Theorem 2.1].

We now introduce two exact functors on the exact category $\mathrm{NIL}(R;\alpha)$ which will play an important role in our proofs. On the level of K-theory, one of these yields the twisted analogue of the Verschiebung operators, while the other gives the classical Frobenius operators.

Definition 2.2 (twisted Verschiebung functors). For each positive integer m, define the twisted Verschiebung functors $V_m : \mathrm{NIL}(R;\alpha) \to \mathrm{NIL}(R;\alpha)$ as follows. On objects, we set

$$\begin{aligned} V_m((P,f)) &= (P \oplus P_{\alpha^{-1}} \oplus P_{\alpha^{-2}} \oplus \cdots \oplus P_{\alpha^{-mn}}, \bar{f}) \\ &= \left(\bigoplus_{i=0}^{mn} P_{\alpha^{-i}}, \bar{f}\right) = \left(\bigoplus_{i=0}^{mn} S^{-i}(P), \bar{f}\right), \end{aligned}$$

where the morphism

$$\bar{f} : \bigoplus_{i=0}^{mn} P_{\alpha^{-i}} \longrightarrow \left(\bigoplus_{j=0}^{mn} P_{\alpha^{-j}}\right)_{\alpha} = \bigoplus_{j=0}^{mn} P_{\alpha^{-j+1}}$$

is defined componentwise by the maps $f_{ij} : P_{\alpha^{-i}} \to P_{\alpha^{-j+1}}$ given via the formula

$$f_{ij} = \begin{cases} \mathrm{id} & \text{if } 0 \leq i = j \leq mn-1, \\ f & \text{if } i = mn,\ j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the proof of Lemma 2.5 below, we will see that $\bar{f}$ is nilpotent, so that $V_m((P,f))$ does indeed define an object in the category $\mathrm{NIL}(R;\alpha)$. If $\phi : (P,f) \to (Q,g)$ is a morphism in the category $\mathrm{NIL}(R;\alpha)$, we define the morphism

$$V_m(\phi) : \left(\bigoplus_{i=0}^{mn} P_{\alpha^{-i}}, \bar{f}\right) \to \left(\bigoplus_{i=0}^{mn} Q_{\alpha^{-i}}, \bar{g}\right)$$

via the formula $V_m(\phi) = \bigoplus_{i=0}^{mn} S^{-i}(\phi)$. One checks that

(i) $\bar{g} \circ V_m(\phi) = S(V_m(\phi)) \circ \bar{f}$,

(ii) $V_m(\mathrm{id}) = \mathrm{id}$, and

(iii) $V_m(\phi \circ \psi) = V_m(\phi) \circ V_m(\psi)$, so that V_m is indeed a functor. Moreover, V_m is exact because each S^{-i} is exact.

Definition 2.3 (Frobenius functors). For positive integers m, define the Frobenius functors $F_m : \mathrm{NIL}(R;\alpha) \to \mathrm{NIL}(R;\alpha)$ as follows. On objects, set $F_m((P, f)) = (P, \tilde{f})$ where $\tilde{f}$ is the morphism defined by the composition

$$P \xrightarrow{S^{mn}(f)\circ S^{mn-1}(f)\circ\cdots\circ S^1(f)\circ f} P_{\alpha^{mn+1}} = P_\alpha$$

(recall that the ring automorphism α has order $|\alpha| = n$). It is immediate that the map $\tilde{f}$ is nilpotent, so that $F_m((P, f))$ is indeed an object in $\mathrm{NIL}(R;\alpha)$. Now if $\phi : (P, f) \to (Q, g)$ is a morphism in the category $\mathrm{NIL}(R;\alpha)$, we define the morphism $F_m(\phi) : (P, \tilde{f}) \to (Q, \tilde{g})$ to coincide with the morphism ϕ. It is obvious that $F_m(\mathrm{id}) = \mathrm{id}$ and $F_m(\phi \circ \psi) = F_m(\phi) \circ F_m(\psi)$, and one can easily check that $\tilde{g} \circ \phi = S(\phi) \circ \tilde{f}$, so that F_m is a genuine functor. Clearly F_m is exact.

Definition 2.4 (α-twisting functors). We define the exact functor $T : \mathrm{NIL}(R;\alpha) \to \mathrm{NIL}(R;\alpha)$ as follows. On objects, we set $T((P, f)) = (S^{-1}(P), S^{-1}(f))$, and if $\phi : (P, f) \to (Q, g)$ is a morphism, we set $T(\phi)$ to be the morphism $S^{-1}(\phi) : S^{-1}(P) \to S^{-1}(Q)$. Observe that, as with the functors S^i on the category $\boldsymbol{P}(R)$, the iterates T^i define a functorial $\mathbb{Z}$-action on the category $\mathrm{NIL}(R;\alpha)$, which factors through a functorial $\mathbb{Z}_n$-action.

The relationship between these various functors is described in the following lemma. We will write G_m for the composite exact functor $G_m = F_m \circ V_m$.

Lemma 2.5. *We have the equality* $G_m = \bigoplus_{i=0}^{mn} T^i$.

Proof. Let (P, f) be an object in $\mathrm{NIL}(R;\alpha)$. Then we have

$$G_m((P, f)) = \Bigl(\bigoplus_{i=0}^{mn} S^{-i}(P), \tilde{\bar{f}}\Bigr),$$

where

$$\tilde{\bar{f}} = S^{mn}(\bar{f}) \circ S^{mn-1}(\bar{f}) \circ \cdots \circ S^1(\bar{f}) \circ \bar{f}.$$

Note that, if we forget the right R-module structures, each S^i is the identity functor on abelian groups. So as a morphism of abelian groups, $\tilde{\bar{f}} = \bar{f}^{mn+1}$. Now recall that $\bar{f}$ is a morphism which cyclically permutes the $mn+1$ direct summands occurring in its source and target. Using this observation, it is then easy to see that $\tilde{\bar{f}} = \bar{f}^{mn+1}$ is diagonal and equal to $\bigoplus_{i=0}^{mn} S^{-i}(f)$. So on the level of objects, G_m and $\bigoplus_{i=0}^{mn} T_i$ agree. From this, we also see that $\bar{f}$ is nilpotent (as was indicated in Definition 2.2). It is obvious that they agree on morphisms. □

Remark 2.6. It is natural to consider the more general case when $\alpha : R \to R$ has finite order in the *outer automorphism group* of the ring R, i.e., there exists $n \geq 2$ and a unit $u \in R$ so that $\alpha^n(r) = uru^{-1}$ for all $r \in R$. In this situation, we have for any right R-module M and integer m an isomorphism $\tau_{m,M} : M_{\alpha^{mn}} \to M$,

$\tau_{m,M}(r) := ru^m$ of right R-modules. This gives rise to a natural isomorphism between the functors S_{mn} and $S_0 = \mathrm{Id}$. It is then easy to similarly define twisted Verschiebung functors and Frobenius functors, and to verify an analogue of Lemma 2.5. However, in this case, we generally do *not* have that T_n is naturally isomorphic to T_0, unless α fixes u. This key issue prevents our proof of Farrell's Lemma 3.1(2) below (which is the key to the proof of our main theorems) from going through in this more general setting.

3. Nonfiniteness of Farrell Nils

This section is devoted to proving Theorem A.

3A. ***A version of Farrell's lemma.*** We are now ready to establish our analogues of the key lemmas from [Farrell 1977].

Lemma 3.1. *The following results hold*:

(1) *For all $j \in \mathbb{N}$, the induced morphisms $K_j(V_m)$, $K_j(F_m) : K_j(\mathrm{NIL}(R;\alpha)) \to K_j(\mathrm{NIL}(R;\alpha))$ on K-theory map the summand $\mathrm{Nil}_j(R;\alpha)$ to itself.*

(2) *For all $j, m \in \mathbb{N}$, the identity $(2+mn)K_j(G_m) - K_j(G_m)^2 = \mu_{1+mn}$ holds, where the map μ_{1+mn} is multiplication by $1+mn$.*

(3) *For all $j \in \mathbb{N}$ and each $x \in \mathrm{Nil}_j(R;\alpha)$, there exists a positive integer $r(x)$ such that $K_j(F_m)(x) = 0$ for all $m \geq r(x)$.*

Proof. (1) Let $H_m := \bigoplus_{i=0}^{mn} S^{-i} : \boldsymbol{P}(R) \to \boldsymbol{P}(R)$; one then easily checks that $F \circ V_m = H_m \circ F$. We also have $F \circ F_m = F$. Statement (1) follows easily from these.

(2) By the additivity theorem for algebraic K-theory, Lemma 2.5 immediately gives us that

$$K_j(G_m) = \sum_{i=0}^{mn} K_j(T_i) = \mathrm{id} + m \sum_{i=1}^{n} K_j(T_i)$$

(recall that the functors T_i are n-periodic). Now let us evaluate the square of the map $K_j(G_m)$:

$$\begin{aligned} K_j(G_m)^2 &= \left(\mathrm{id} + m \sum_{i=1}^{n} K_j(T_i)\right)\left(\mathrm{id} + m \sum_{l=1}^{n} K_j(T_l)\right) \\ &= \mathrm{id} + 2m \sum_{i=1}^{n} K_j(T_i) + m^2 \sum_{i=1}^{n} \sum_{l=1}^{n} K_j(T_{i+l}) \\ &= \mathrm{id} + 2m \sum_{i=1}^{n} K_j(T_i) + m^2 \sum_{i=1}^{n} \sum_{l=1}^{n} K_j(T_l) \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{id} + 2m \sum_{i=1}^{n} K_j(T_i) + m^2 n \sum_{l=1}^{n} K_j(T_l) \\
&= \mathrm{id} + (2m + m^2 n) \sum_{i=1}^{n} K_j(T_i).
\end{aligned}
$$

In the third equality above, we used the fact that the T_i functors are n-periodic, so that shifting the index on the inner sum by i leaves the sum unchanged. Finally, substituting in the expression we have for $K_j(G_m)$ and the expression we derived for $K_j(G_m)^2$, we see that

$$
\begin{aligned}
&(2+mn)K_j(G_m) - K_j(G_m)^2 \\
&\quad = (2+mn)\Big(\mathrm{id} + m \sum_{i=1}^{n} K_j(T_i)\Big) - \Big(\mathrm{id} + (2m + m^2 n) \sum_{i=1}^{n} K_j(T_i)\Big) \\
&\quad = (2+mn)\,\mathrm{id} - \mathrm{id} = \mu_{(1+mn)},
\end{aligned}
$$

completing the proof of statement (2).

(3) This result is due to Grunewald [2008, Proposition 4.6]. □

3B. ***Proof of Theorem A.*** The theorem now follows easily. Let us focus on the case where $i \geq 1$, as the case $i \leq 1$ has already been established by Grunewald [2007] and Ramos [2007]. So let us assume that the Farrell Nil-group $\mathrm{NK}_i(R, \alpha) \cong \mathrm{Nil}_{i-1}(R; \alpha^{-1})$ is nontrivial and finitely generated, where $i \geq 1$. Then one can find arbitrarily large positive integers m with the property that the map $\mu_{(1+mn)}$ is an injective map from $\mathrm{Nil}_{i-1}(R; \alpha^{-1})$ to itself (for example, one can take m to be any multiple of the order of the torsion subgroup of $\mathrm{Nil}_{i-1}(R; \alpha^{-1})$). From Lemma 3.1(2), we can factor the map $\mu_{(1+mn)}$ as a composite

$$
\mu_{(1+mn)} = (\mu_{(2+mn)} - K_j(G_m)) \circ K_j(G_m),
$$

and hence there is an infinite sequence of integers m with the property that the corresponding maps $K_j(G_m) = K_j(F_m) \circ K_j(V_m)$ are injective. This implies that there are infinitely many integers m for which the map $K_j(F_m)$ is nonzero.

On the other hand, let $x_1, \ldots, x_k$ be a finite set of generators for the abelian group $\mathrm{Nil}_{i-1}(R; \alpha^{-1})$. Then from Lemma 3.1(3), we have that for any $m \geq \max\{r(x_i)\}$ the map $K_j(F_m)$ is identically zero, a contradiction. This completes the proof of Theorem A.

4. Finite subgroups of Farrell Nil-groups

4A. ***A lemma on splittings.*** In order to establish Theorem B, we will need an algebraic lemma for recognizing when two direct summands inside an ambient group jointly form a direct summand.

Lemma 4.1. *Let G be an abelian group and $H < G$, $K < G$ be a pair of subgroups. Suppose there are two retractions $\lambda : G \to H$ and $\rho : G \to K$ with the property that $\lambda(K) = \{0\}$. Then there exists a subgroup $L < G$ which is isomorphic to H and such that $L \oplus K$ is also a direct summand of G.*

Proof. Consider the morphism $(\lambda, \rho) : G \to H \times K$ given by $g \mapsto (\lambda(g), \rho(g))$. It is split by the morphism $\beta : H \times K \to G$ given by $(h, k) \mapsto h - \rho(h) + k$, since λ, ρ are retractions and $\lambda(K) = 0$. Therefore $\beta(H \times K)$ is a direct summand of G. Let $L < G$ be the image of $H \times \{0\}$ under β. By noting that $K = \beta(\{0\} \times K)$, we see that $L \oplus K < G$ is a direct summand. □

4B. ***Proof of Theorem B.*** In order to simplify the notation, we will simply write V_m for $K_j(V_m)$, and use a similar convention for F_m and G_m.

Case $i \geq 1$. We first consider the case when $i \geq 1$, and recall that $\mathrm{NK}_i(R, \alpha) \cong \mathrm{Nil}_{i-1}(R; \alpha^{-1})$. Let $H < \mathrm{Nil}_{i-1}(R; \alpha^{-1})$ be a finite subgroup. According to Lemma 3.1(3), since H is finite, there exists an integer $r(H) = \max_{x \in H}\{r(x)\}$, so that $F_m(H) = 0$ for all $m > r(H)$. Let S consist of all natural numbers $m > r(H)$ such that $\gcd(1 + mn, |H|) = 1$. Then S contains every multiple of $|H|$ which is greater than $r(H)$, so is an infinite set. By definition, the composite

$$\mathrm{Nil}_{i-1}(R; \alpha^{-1}) \xrightarrow{V_m} \mathrm{Nil}_{i-1}(R; \alpha^{-1}) \xrightarrow{F_m} \mathrm{Nil}_{i-1}(R; \alpha^{-1}),$$

is the morphism G_m, and we define the subgroup $H_m \leq \mathrm{Nil}_{i-1}(R; \alpha^{-1})$ to be $H_m := V_m(H)$. By the defining property of the set S we have that, for $m \in S$, $(\mu_{2+mn} - G_m) \circ G_m = \mu_{1+mn}$ is an isomorphism when restricted to H. Hence G_m is a monomorphism when restricted to H, forcing V_m to also be a monomorphism when restricted to H. So, for all $m \in S$, we see that $H_m \cong H$.

We now claim that $H_m \cap H = \{0\}$ for all $m \in S$. Indeed, since integers in S are larger than $r(H)$, we have $F_m(H) = 0$. But, for $m \in S$, the composite map $G_m = F_m \circ V_m$ is an isomorphism from H to $G_m(H) = F_m(H_m)$, so F_m must be injective on H_m. Putting these two statements together, we get that $H_m \cap H = \{0\}$. We conclude that $H \oplus H < \mathrm{Nil}_{i-1}(R; \alpha^{-1})$. Applying the same argument to $H \oplus H$ and so on, we conclude that $\oplus_\infty H < \mathrm{Nil}_{i-1}(R; \alpha^{-1})$.

Next, we argue that if the original subgroup H was a direct summand in the group $\mathrm{Nil}_{i-1}(R; \alpha^{-1})$ then we can find a copy of $H \oplus H$ which is also a direct summand in $\mathrm{Nil}_{i-1}(R; \alpha^{-1})$, and which *extends* the original direct summand (i.e., the first copy of H inside the direct summand $H \oplus H$ coincides with the original H).

To see this, let us assume that $H < \mathrm{Nil}_{i-1}(R; \alpha^{-1})$ is a direct summand, so there exists a retraction $\rho : \mathrm{Nil}_{i-1}(R; \alpha^{-1}) \to H$. Let H_m be obtained as above. We first construct a retraction $\lambda : \mathrm{Nil}_{i-1}(R; \alpha^{-1}) \to H_m$. Recall that μ_{1+mn} is an isomorphism on H_m, so there exists an integer l so that $\mu_l \circ \mu_{1+mn}$ is the identity on H_m.

We define $\lambda : \mathrm{Nil}_{i-1}(R;\alpha^{-1}) \to H_m$ to be the following composition of maps:

$$\mathrm{Nil}_{i-1}(R;\alpha^{-1}) \xrightarrow{F_m} \mathrm{Nil}_{i-1}(R;\alpha^{-1}) \xrightarrow{\mu_{2+mn}-G_m} \mathrm{Nil}_{i-1}(R;\alpha^{-1}) \xrightarrow{\rho} H \xrightarrow{V_m|_H} H_m \xrightarrow{\mu_l} H_m.$$

We claim that λ is a retraction. Note that for $x \in H_m$ there exists $y \in H$ with $V_m(y) = x$. We now evaluate

$$\begin{aligned}
\lambda(x) &= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ F_m)(x) \\
&= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ F_m)(V_m(y)) \\
&= (\mu_l \circ V_m \circ \rho \circ (\mu_{2+mn} - G_m) \circ G_m)(y) \\
&= (\mu_l \circ V_m \circ \rho \circ ((2+mn)G_m - G_m^2)))(y) \\
&= (\mu_l \circ V_m \circ \rho \circ \mu_{1+mn})(y) \\
&= (\mu_l \circ \mu_{1+mn})((V_m \circ \rho)(y)) \\
&= (\mu_l \circ \mu_{1+mn})(V_m(y)) \\
&= (\mu_l \circ \mu_{1+mn})(x) \\
&= x.
\end{aligned}$$

This verifies that λ is a retraction. Note also that $\lambda(H) = 0$, since $F_m(H) = 0$ follows from the fact that $m \in S$ (recall that integers in S are larger than $r(H)$). Hence we are in the situation of Lemma 4.1, and we can conclude that $H \oplus H$ also arises as a direct summand of $\mathrm{Nil}_{i-1}(R;\alpha^{-1})$. Note that, when applying our Lemma 4.1, we replaced the *second copy* H_m of H by some other (isomorphic) subgroup, but kept the *first copy* of H to be the original H. Hence the direct summand $H \oplus H$ does indeed extend the original summand H. Iterating the process, we obtain that $\bigoplus_\infty H$ is a direct summand of $\mathrm{Nil}_{i-1}(R;\alpha^{-1})$. This completes the proof of Theorem B in the case where $i \geq 1$.

Case $i \leq 1$. Next, let us consider the case of the Farrell Nil-groups $\mathrm{NK}_i(R,\alpha^{-1})$ where $i \leq 1$. For these, the proof of Theorem B proceeds via a (descending) induction on i, with the case $i = 1$ having been established above.

We remind the reader of the standard technique for extending results known for K_1 to lower K-groups. Take the ring $\Lambda\mathbb{Z}$ consisting of all $\mathbb{N} \times \mathbb{N}$ matrices with entries in $\mathbb{Z}$ which contain only finitely many nonzero entries in each row and each column, and quotient out by the ideal $I \lhd \Lambda\mathbb{Z}$ consisting of all matrices which vanish outside of a finite block. This gives the ring $\Sigma\mathbb{Z} = \Lambda\mathbb{Z}/I$, and we can now define the *suspension functor* on the category of rings by tensoring with the ring $\Sigma\mathbb{Z}$, i.e., sending a ring R to the ring $\Sigma(R) := \Sigma\mathbb{Z} \otimes R$, and a morphism $f : R \to S$ to the morphism $\mathrm{Id} \otimes f : \Sigma(R) \to \Sigma(S)$. The functor Σ has the property that there are natural isomorphisms $K_i(R) \cong K_{i+1}(\Sigma(R))$ (for all $i \in \mathbb{Z}$). Moreover, there is a natural isomorphism $\Sigma(R_\alpha[t]) \cong (\Sigma R)_{\mathrm{Id}\otimes\alpha}[t]$, which induces a commutative square

$$\begin{tikzcd} K_i(R_\alpha[t]) \arrow[rr] \arrow[d, "\cong"] & & K_i(R) \arrow[d, "\cong"] \\ K_{i+1}((\Sigma\mathbb{Z}\otimes R)_{\mathrm{Id}\otimes\alpha}[t]) \arrow[rr] & & K_{i+1}(\Sigma\mathbb{Z}\otimes R) \end{tikzcd}$$

By induction, this allows us to identify $\mathrm{NK}_{1-m}(R,\alpha)$ with $\mathrm{NK}_1(\Sigma^m R, \mathrm{Id}^{\otimes m}\otimes\alpha)$ for each $m \geq 1$, where Σ^m denotes the m-fold application of the functor Σ. Obviously, if the automorphism α has finite order in $\mathrm{Aut}(R)$, the induced automorphism $\mathrm{Id}^{\otimes m}\otimes\alpha$ will have finite order in $\mathrm{Aut}((\Sigma\mathbb{Z})^{\otimes m}\otimes R)$. So, for the Farrell Nil-groups $\mathrm{NK}_i(R,\alpha)$ with $i \leq 0$, the result immediately follows from the special case of NK_1 considered above. This completes the proof of Theorem B.

5. A structure theorem and Nils associated to virtually cyclic groups

In this section, we discuss some applications and prove Theorem C and Theorem D. For a general ring R, we know by Theorem A that a nontrivial Nil-group is an *infinitely generated* abelian group. While finitely generated abelian groups have a very nice structural theory, the picture is much more complicated in the infinitely generated case (the reader can consult [Robinson 1993, Chapter 4] for an overview of the theory). If one restricts to abelian (torsion) groups of *finite exponent*, then it is an old result of Prüfer [1923] that any such group is a direct sum of cyclic groups (see [Robinson 1993, item 4.3.5, p. 105] for a proof).

5A. ***Proof of Theorem C.*** We can now explain how our Theorem B allows us to obtain a structure theorem for certain Nil-groups. Let R be a countable ring and $\alpha : R \to R$ be an automorphism of finite order. It is well-known that the twisted Nil-groups $\mathrm{NK}_i(R,\alpha)$ are countable when R is a countable ring. [Sketch: $K_0(R)$ is countable since it is generated by $\mathrm{Idem}(R)$, the countable set consisting of all finite idempotent matrices with entries in R. When $i < 0$, $K_i(R)$ can be viewed as a quotient of $K_{i+1}(R[t,t^{-1}])$, so is countable by induction on i. And when $i > 0$, $K_i(R) := \pi_i(\mathrm{BGL}(R)^+)$, and countability follows from the fact that $\mathrm{BGL}(R)^+$ is a countable CW-complex. $\mathrm{NK}_i(R,\alpha)$ is then a subgroup of the countable group $K_i(R)$, so is itself countable.]

If in addition $\mathrm{NK}_i(R,\alpha)$ has finite exponent, then, by the result of Prüfer mentioned above, it follows that $\mathrm{NK}_i(R,\alpha)$ decomposes as a countable direct sum of cyclic groups of prime power order, each of which appears with some multiplicity. In view of our Theorem B, any summand which occurs must actually occur infinitely many times. Since the exponent of the Nil-group is finite, there is an upper bound on the prime power orders that can appear, and hence there are only finitely many possible isomorphism types of summands. Let H be the direct sum of a single copy of each cyclic group of prime power order which appear as a summand in $\mathrm{NK}_i(R,\alpha)$. It follows immediately that $\bigoplus_\infty H \cong \mathrm{NK}_i(R,\alpha)$. This completes the proof of Theorem C.

5B. ***Farrell–Jones isomorphism conjecture.*** In applications to geometric topology, the rings of interest are typically integral group rings $\mathbb{Z}G$. For computations of the K-theory of such rings, the key tool is provided by the (K-theoretic) *Farrell–Jones isomorphism conjecture* [Farrell and Jones 1993]. Davis and Lück [1998] gave a general framework for the formulations of various isomorphism conjectures. In particular, they constructed for any group G an OrG-spectrum, i.e., a functor $\boldsymbol{K}_{\mathbb{Z}} : \mathrm{Or}G \to \mathbf{Sp}$, where Or$G$ is the orbit category of G (objects are cosets G/H, $H < G$ and morphisms are G-maps) and $\mathbf{Sp}$ is the category of spectra. This functor has the property that $\pi_n(\boldsymbol{K}_{\mathbb{Z}}(G/H)) = K_n(\mathbb{Z}H)$. As an ordinary spectrum can be used to construct a generalized homology theory, this OrG-spectrum $\boldsymbol{K}_{\mathbb{Z}}$ was used to construct a G-equivariant homology theory $H_*^G(-;\boldsymbol{K}_{\mathbb{Z}})$. It has the property that

$$H_n^G(G/H;\boldsymbol{K}_{\mathbb{Z}}) = \pi_n(\boldsymbol{K}_{\mathbb{Z}}(G/H)) = K_n(\mathbb{Z}H)$$

(for all $H < G$ and $n \in \mathbb{Z}$). In particular, on a point,

$$H_n^G(*;\boldsymbol{K}_{\mathbb{Z}}) = H_n^G(G/G;\boldsymbol{K}_{\mathbb{Z}}) = K_n(\mathbb{Z}G).$$

Applying this homology theory to any G-CW-complex X, the obvious G-map $X \to *$ gives rise to an *assembly map*

$$H_n^G(X;\boldsymbol{K}_{\mathbb{Z}}) \to H_n^G(*;\boldsymbol{K}_{\mathbb{Z}}) \cong K_n(\mathbb{Z}G).$$

The Farrell–Jones isomorphism conjecture asserts that, when the space X is a model for the classifying space for G-actions with isotropy in the virtually cyclic subgroups of G, the above assembly map is an isomorphism. Thus, the conjecture roughly predicts that the K-theory of an integral group ring $\mathbb{Z}G$ is determined by the K-theory of the integral group rings of the virtually cyclic subgroups of G, assembled together in some homological fashion.

In view of this conjecture, one can view the K-theory of virtually cyclic groups as the "basic building blocks" for the K-theory of general groups. Focusing on such a virtually cyclic group V, one can consider the portion of the K-theory that comes from the finite subgroups of V. This would be the image of the assembly map

$$H_n^V(\underline{E}V;\boldsymbol{K}_{\mathbb{Z}}) \to H_n^V(*;\boldsymbol{K}_{\mathbb{Z}}) \cong K_n(\mathbb{Z}V),$$

where $\underline{E}V$ is a model for the classifying space for proper V-actions. While this map is always split injective (see [Bartels 2003]), it is not surjective in general. Thus, to understand the K-theory of a virtually cyclic group, we need to understand the K-theory of finite groups, and to understand the cokernel of the above assembly map.

The cokernel of this map can also be interpreted as the obstruction to reducing the family of virtually cyclic groups used in the Farrell–Jones isomorphism conjecture to the family of finite groups — this is the *transitivity principle* (see [Farrell and Jones 1993, Theorem A.10]). Our Theorem D gives some structure for the cokernel of the assembly map.

5C. *Proof of Theorem D.* Let V be a virtually cyclic group. Then one has that V is either of the form (i) $V = G \rtimes_\alpha \mathbb{Z}$, where G is a finite group and $\alpha \in \mathrm{Aut}(G)$, or is of the form (ii) $V = G_1 *_H G_2$, where G_i, H are finite groups and H is of index two in both G_i.

Let us first consider case (i). In this case, the integral group ring $\mathbb{Z}[V]$ is isomorphic to the ring $R_{\hat{\alpha}}[t, t^{-1}]$, the $\hat{\alpha}$-twisted ring of Laurent polynomials over the coefficient ring $R = \mathbb{Z}[G]$, where $\hat{\alpha} \in \mathrm{Aut}(\mathbb{Z}[G])$ is the ring automorphism canonically induced by the group automorphism α. Then it is known (see [Davis et al. 2011b, Lemma 3.1]) that the cokernel we are interested in consists of the direct sum of the Farrell Nil-group $\mathrm{NK}_n(\mathbb{Z}G, \hat{\alpha})$ and the Farrell Nil-group $\mathrm{NK}_n(\mathbb{Z}G, \hat{\alpha}^{-1})$. Applying Theorem C and Corollary 1.2 to these two Farrell Nil-groups, we are done.

In case (ii), we note that V has a canonical surjection onto the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, obtained by surjecting each G_i onto $G_i/H \cong \mathbb{Z}_2$. Taking the preimage of the canonical index-two subgroup $\mathbb{Z} \leq D_\infty$, we obtain a canonical index-two subgroup $W \leq V$. The subgroup W is a virtually cyclic group of type (i), and is of the form $H \rtimes_\alpha \mathbb{Z}$, where $\alpha \in \mathrm{Aut}(H)$. Hence it has associated Farrell Nil-groups $\mathrm{NK}_n(\mathbb{Z}H, \hat{\alpha})$.

The cokernel of the relative assembly map for the group V is a *Waldhausen Nil-group* associated to the splitting of V (see [Davis et al. 2011b, Lemma 3.1]). It was recently shown that this Waldhausen Nil-group is always isomorphic to a single copy of the Farrell Nil-group $\mathrm{NK}_n(\mathbb{Z}H, \hat{\alpha})$ associated to the canonical index-two subgroup $W \leq V$ (see for example [Davis et al. 2011a; 2011b], or for an earlier result in a similar vein [Lafont and Ortiz 2008]). Again, combining this with our Theorem C and Corollary 1.2, we are done, completing the proof of Theorem D.

6. Applications and concluding remarks

We conclude this short note with some further applications and remarks.

6A. *Waldhausen's A-theory.* Recall that Waldhausen [1985] introduced a notion of algebraic K-theory $A(X)$ of a topological space X. Once the K-theoretic contribution has been split off, one is left with the finitely dominated version of the algebraic K-theory $A^{\mathrm{fd}}(X)$. This finitely dominated version satisfies the "fundamental theorem of algebraic K-theory", in that one has a homotopy splitting

$$A^{\mathrm{fd}}(X \times S^1) \simeq A^{\mathrm{fd}}(X) \times BA^{\mathrm{fd}}(X) \times NA^{\mathrm{fd}}_+(X) \times NA^{\mathrm{fd}}_-(X); \tag{6.1}$$

see [Hüttemann et al. 2001] (the reader should compare this with the corresponding fundamental theorem of algebraic K-theory for rings; see [Grayson 1976]). The Nil-terms appearing in this splitting have been studied by Grunewald, Klein, and Macko [Grunewald et al. 2008], who defined Frobenius and Verschiebung operations F_n, V_n, on the homotopy groups $\pi_*(NA^{\mathrm{fd}}_{\pm}(X))$. In particular, they show that the composite $V_n \circ F_n$ is multiplication by n [ibid., Proposition 5.1], and that for any element $x \in \pi_i(NA^{\mathrm{fd}}_{\pm}(X))$ of finite order, one has $F_n(x) = 0$ for all sufficiently large n (see the discussion in [ibid., p. 334, Proof of Theorem 1.1]). Since these two properties are the only ones used in our proofs, an argument identical to the proof of Theorem B gives the following result:

Proposition 6.2. *Let X be an arbitrary space, and let $NA^{\mathrm{fd}}_{\pm}(X)$ be the associated Nil-spaces arising in the fundamental theorem of algebraic K-theory of spaces. If $H \leq \pi_i(NA^{\mathrm{fd}}_{\pm}(X))$ is any finite subgroup, then*

$$\bigoplus_{\infty} H \leq \pi_i(NA^{\mathrm{fd}}_{\pm}(X)).$$

Moreover, if H is a direct summand in $\pi_i(NA^{\mathrm{fd}}_{\pm}(X))$, then so is $\bigoplus_{\infty} H$.

Remark 6.3. An interesting question is whether there exists a "twisted" version of the splitting in (6.1), which applies to bundles $X \to W \to S^1$ over the circle (or more generally to approximate fibrations over the circle), and provides a homotopy splitting of the corresponding $A^{\mathrm{fd}}(W)$ in terms of spaces attached to X and the holonomy map.

6B. ***Cokernels of assembly maps.*** For a general group G, one would expect from the Farrell–Jones isomorphism conjectures that the cokernel of the relative assembly map for G should be "built up", in a homological manner, from the cokernels of the relative assembly maps of the various virtually cyclic subgroups of G (see for example [Lafont and Ortiz 2009] for an instance of this phenomenon). In view of our Theorem D, the following question seems relevant:

Question. Can one find a group G, an index $i \in \mathbb{Z}$, and a finite subgroup H, with the property that H embeds in $\mathrm{coker}(h_i^G(\underline{E}G) \to K_i(\mathbb{Z}[G]))$, but $\bigoplus_{\infty} H$ does not?

In other words, we are asking whether contributions from the various Nil-groups of the virtually cyclic subgroups of G could *partially cancel out in a cofinite manner*. Note the following special case of this question: is there an example for which this cokernel is a nontrivial finite group?

6C. ***Exotic Farrell Nil-groups.*** Our Theorem C shows that, for a countable *tame* ring (meaning the associated Farrell Nil-groups have finite exponent), the associated Farrell Nil-groups, while infinitely generated, still remain reasonably well behaved, i.e., are countable direct sums of a fixed finite group. In contrast, for a general ring R (or even a general integral group ring $\mathbb{Z}G$), all we know about the nontrivial Farrell Nil-groups is that they are infinitely generated abelian groups. Of course, the possibility of having infinite exponent *a priori* allows for many strange possibilities, e.g., the rationals $\mathbb{Q}$, or the Prüfer p-group $\mathbb{Z}(p^\infty)$ consisting of all complex p^i-th roots of unity ($i \geq 0$). We can ask:

Question. Can one find a ring R, automorphism $\alpha \in \mathrm{Aut}(R)$, and $i \in \mathbb{Z}$, so that $\mathrm{NK}_i(R, \alpha) \cong \mathbb{Q}$? How about $\mathrm{NK}_i(R, \alpha) \cong \mathbb{Z}(p^\infty)$? What if we require the ring to be an integral group ring $R = \mathbb{Z}G$?

Remark 6.4. Grunewald [2008, Theorem 5.10] proved that, for every group G and every group automorphism α of finite order, $\mathrm{NK}_i(\mathbb{Q}G, \alpha)$ (for all $i \in \mathbb{Z}$) is a vector space over the rationals after killing torsion elements. However this still leaves the possibility that they may vanish.

Or rather, in view of our results, the following question also seems natural:

Question. What conditions on the ring R, automorphism $\alpha \in \mathrm{Aut}(R)$, and $i \in \mathbb{Z}$, are sufficient to ensure $\mathrm{NK}_i(R, \alpha)$ is a torsion group of finite exponent? Does $\mathrm{NK}_i(\mathbb{Z}G; \alpha)$ have finite exponent for all polycyclic-by-finite groups when α is of finite order?

Finally, while this paper completes our understanding of the finiteness properties of Farrell Nil-groups associated with *finite-order* ring automorphisms, nothing seems to be known about the Nil-groups associated with *infinite-order* ring automorphisms. This seems like an obvious direction for further research.

Acknowledgments

We would like to thank Jim Davis, Crichton Ogle, and Chuck Weibel for helpful comments.

References

[Bartels 2003] A. C. Bartels, "On the domain of the assembly map in algebraic K-theory", *Algebr. Geom. Topol.* **3** (2003), 1037–1050. MR 2004h:19002 Zbl 1038.19001

[Connolly and Prassidis 2002] F. X. Connolly and S. Prassidis, "On the exponent of the NK_0-groups of virtually infinite cyclic groups", *Canad. Math. Bull.* **45**:2 (2002), 180–195. MR 2003d:19001 Zbl 1014.19001

[Davis and Lück 1998] J. F. Davis and W. Lück, "Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory", *K-Theory* **15**:3 (1998), 201–252. MR 99m:55004 Zbl 0921.19003

[Davis et al. 2011a] J. F. Davis, Q. Khan, and A. Ranicki, "Algebraic K-theory over the infinite dihedral group: an algebraic approach", *Algebr. Geom. Topol.* **11**:4 (2011), 2391–2436. MR 2012h:19008 Zbl 1236.19002

[Davis et al. 2011b] J. F. Davis, F. Quinn, and H. Reich, "Algebraic K-theory over the infinite dihedral group: a controlled topology approach", *J. Topol.* **4**:3 (2011), 505–528. MR 2012g:19006 Zbl 1227.19004

[Farrell 1977] F. T. Farrell, "The nonfiniteness of Nil", *Proc. Amer. Math. Soc.* **65**:2 (1977), 215–216. MR 56 #8624 Zbl 0365.18021

[Farrell and Jones 1993] F. T. Farrell and L. E. Jones, "Isomorphism conjectures in algebraic K-theory", *J. Amer. Math. Soc.* **6**:2 (1993), 249–297. MR 93h:57032 Zbl 0798.57018

[Farrell and Jones 1995] F. T. Farrell and L. E. Jones, "The lower algebraic K-theory of virtually infinite cyclic groups", *K-Theory* **9**:1 (1995), 13–30. MR 96e:19003 Zbl 0829.19002

[Grayson 1976] D. Grayson, "Higher algebraic K-theory, II (after Daniel Quillen)", pp. 217–240 in *Algebraic K-theory* (Evanston, IL 1976), Lecture Notes in Math. **551**, Springer, Berlin, 1976. MR 58 #28137 Zbl 0362.18015

[Grayson 1988] D. R. Grayson, "The K-theory of semilinear endomorphisms", *J. Algebra* **113**:2 (1988), 358–372. MR 89i:16021 Zbl 0656.16011

[Grunewald 2007] J. Grunewald, "Non-finiteness results for nil-groups", *Algebr. Geom. Topol.* **7** (2007), 1979–1986. MR 2009f:19001 Zbl 1127.19004

[Grunewald 2008] J. Grunewald, "The behavior of Nil-groups under localization and the relative assembly map", *Topology* **47**:3 (2008), 160–202. MR 2009i:19002 Zbl 1172.19002

[Grunewald et al. 2008] J. Grunewald, J. R. Klein, and T. Macko, "Operations on the A-theoretic nil-terms", *J. Topol.* **1**:2 (2008), 317–341. MR 2010c:19003 Zbl 1145.19001

[Hüttemann et al. 2001] T. Hüttemann, J. R. Klein, W. Vogell, F. Waldhausen, and B. Williams, "The "fundamental theorem" for the algebraic K-theory of spaces, I", *J. Pure Appl. Algebra* **160**:1 (2001), 21–52. MR 2002a:19003 Zbl 0982.19001

[van der Kallen 1980] W. van der Kallen, "Generators and relations in algebraic K-theory", pp. 305–310 in *Proceedings of the International Congress of Mathematicians* (Helsinki, 1978), edited by O. Lehto, Acad. Sci. Fennica, Helsinki, 1980. MR 81g:18017 Zbl 0426.18005

[Kuku and Tang 2003] A. O. Kuku and G. Tang, "Higher K-theory of group-rings of virtually infinite cyclic groups", *Math. Ann.* **325**:4 (2003), 711–726. MR 2004a:19001 Zbl 1043.19002

[Lafont and Ortiz 2008] J.-F. Lafont and I. J. Ortiz, "Relating the Farrell Nil-groups to the Waldhausen Nil-groups", *Forum Math.* **20**:3 (2008), 445–455. MR 2009g:19005 Zbl 1147.19005

[Lafont and Ortiz 2009] J.-F. Lafont and I. J. Ortiz, "Splitting formulas for certain Waldhausen Nil-groups", *J. Lond. Math. Soc.* (2) **79**:2 (2009), 309–322. MR 2009m:19002 Zbl 1176.19001

[Prasolov 1982] A. V. Prasolov, "The nonfiniteness of the group Nil", *Mat. Zametki* **32**:1 (1982), 9–12. In Russian; translated in *Math. Notes* **32**:1 (1982), 484–485. MR 83m:18013 Zbl 0512.18004

[Prüfer 1923] H. Prüfer, "Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen", *Math. Z.* **17**:1 (1923), 35–61. MR 1544601 Zbl 49.0084.03

[Ramos 2007] R. Ramos, "Non-finiteness of twisted nils", *Bol. Soc. Mat. Mexicana* (3) **13**:1 (2007), 55–64. MR 2009h:19002 Zbl 1178.19002

[Robinson 1993] D. J. S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics **80**, Springer, New York, 1993. MR 94m:20001 Zbl 0483.20001

[Waldhausen 1985] F. Waldhausen, "Algebraic K-theory of spaces", pp. 318–419 in *Algebraic and geometric topology* (New Brunswick, NJ, 1983), edited by A. Ranicki et al., Lecture Notes in Math. **1126**, Springer, Berlin, 1985. MR 86m:18011 Zbl 0579.18006

[Weibel 2009] C. Weibel, "NK_0 and NK_1 of the groups C_4 and D_4", *Comment. Math. Helv.* **84**:2 (2009), 339–349. MR 2010c:19005 Zbl 1167.19003

Received 23 Feb 2015. Accepted 31 Mar 2015.

JEAN-FRANÇOIS LAFONT: jlafont@math.ohio-state.edu
Department of Mathematics, The Ohio State University, 100 Math Tower, 231 West 18th Avenue, Columbus, OH 43210-1174, United States

STRATOS PRASSIDIS: prasside@aegean.gr
Department of Mathematics, University of the Aegean, Karlovassi, Samos, 83200, Greece

KUN WANG: kun.wang@vanderbilt.edu
Department of Mathematics, Vanderbilt University, Nashville, TN 37240, United States