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Multiplicative differential algebraic K-theory and applications

Ulrich Bunke and Georg Tamme

We construct a version of Beilinson's regulator as a map of sheaves of commutative ring spectra and use it to define a multiplicative variant of differential algebraic K-theory. We use this theory to give an interpretation of Bloch's construction of K_3-classes and the relation with dilogarithms. Furthermore, we provide a relation to Arakelov theory via the arithmetic degree of metrized line bundles, and we give a proof of the formality of the algebraic K-theory of number rings.

1. Introduction

Let X be an arithmetic scheme, i.e., a regular separated scheme of finite type over the integers. Its algebraic K-theory $K_*(X)$ is an object of fundamental interest in arithmetic. The algebraic K-theory of X is connected with the absolute Hodge cohomology $H^*_{\mathcal{AH}}(X, \mathbb{R}(\bullet))$ via a Chern character map

$$K_i(X) \to H^{2p-i}_{\mathcal{AH}}(X, \mathbb{R}(p)), \quad p, i \geq 0,$$

called the Beilinson regulator. An important but extremely difficult problem is to construct K-theory classes and to compute their images under the regulator map.

The papers [Bunke and Gepner 2013; Bunke and Tamme 2015] initiated a new approach to this problem. The idea is to represent algebraic K-theory classes of X by bundles on $M \times X$ for smooth manifolds M. In greater detail this goes as

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follows. The K-groups of X are the homotopy groups of the algebraic K-theory spectrum $\mathbf{K}(X)$. This spectrum defines a cohomology theory $\mathbf{K}(X)^*$ on topological spaces so that, e.g., $\mathbf{K}(X)^0(S^n) \cong K_0(X) \oplus K_n(X)$. The cohomology theory $\mathbf{K}(X)^*$ admits a differential refinement denoted by $\widehat{\mathbf{K}}^*(M \times X)$. This differential algebraic K-theory is a functor of two variables, a smooth manifold M and a scheme X as above. A class $\hat{x} \in \widehat{\mathbf{K}}^*(M \times X)$ combines the information of a class $x \in \mathbf{K}(X)^*(M)$ and a differential form on the manifold $M \times X(\mathbb{C})$ representing the image of x under Beilinson's regulator with secondary data. Thus, if we know a differential refinement $\hat{x}$ of x then, philosophically, it is easy to calculate the Beilinson regulator of x.

The tool to construct differential algebraic K-theory classes is the cycle map. It produces such classes from bundles on $M \times X$ equipped with additional geometric data. Here a bundle on $M \times X$ is a vector bundle on the ringed space $(M \times X, \mathtt{pr}_X^{-1} \mathcal{O}_X)$. The geometric extra structure is a hermitian metric and a connection on the associated complex vector bundle on $M \times X(\mathbb{C})$. The differential form representing the Beilinson regulator of the corresponding K-theory class is obtained using standard Chern–Weil theory.

The aim of the present paper is to develop a multiplicative version of differential algebraic K-theory and to illustrate it in some applications. The cup product allows us to construct new classes from given ones, but more interestingly, we will employ the secondary information captured by the differential algebraic K-theory in an essential way.

In order to achieve this goal we need a version of Beilinson's regulator on the level of ring spectra. Here our result is not completely satisfactory, as we have to replace absolute Hodge cohomology by the weaker analytic Deligne cohomology, which coincides with the former only for proper schemes. We construct a sheaf of ring spectra $\mathbf{K}$ on the site consisting of products of a smooth manifold and an arithmetic scheme such that $\pi_*(\mathbf{K}(M \times X)) \cong \mathbf{K}(X)^{-*}(M)$. To this end we apply a suitable group completion machine to the category of vector bundles on the ringed space $(M \times X, \mathtt{pr}_X^{-1} \mathcal{O}_X)$. We furthermore construct a sheaf of differential graded algebras $\mathbf{IDR}$ which computes analytic Deligne cohomology and use characteristic forms on vector bundles on the manifolds $M \times X(\mathbb{C})$ to construct a map of sheaves of ring spectra (H denotes the Eilenberg–MacLane functor)

$$\mathtt{r}^{\mathrm{Beil}} : \mathbf{K} \to H(\mathbf{IDR})$$

which on homotopy groups agrees with the Beilinson regulator. This is the main new contribution of the paper.

Once the multiplicative Beilinson regulator is established, we introduce the multiplicative differential algebraic K-theory and a multiplicative version of the cycle map in Section 3.

The remainder of the present paper is devoted to applications and illustrating how classical constructions from arithmetic fit into the framework of differential algebraic K-theory.

In Section 4 we use multiplicative differential algebraic K-theory in order to construct a secondary invariant from the Steinberg relation. As an application we give a conceptual explanation of Bloch's construction of elements in K_3 of a number ring from cycles in the Bloch complex, whose images under the regulator map can be described explicitly in terms of the dilogarithm function.

In Arakelov theory one studies metrized line bundles on number rings and their arithmetic degree. We explain in Section 5 how this can be understood entirely in the framework of differential algebraic K-theory.

Finally, in Section 6 we show that the real homotopy type of the algebraic K-theory spectrum $\mathbf{K}(\mathtt{Spec}(R))$ of rings of integers R in number fields is modeled by the commutative algebra $K_*(R)$ in a way which is natural in R. The precise formulation of this result is Theorem 6.3 and uses the notion of formality introduced in Definition 6.2.

2. Multiplicative theory

In this section we define algebraic K-theory as a sheaf $\mathbf{K}$ of commutative ring spectra on a site of products of a smooth manifold and a regular scheme (see Section 2A below). We furthermore define a sheaf of differential graded algebras $\mathbf{IDR}$ which calculates the analytic Deligne cohomology (Section 2B).

The main result is the construction of a version of Beilinson's regulator with values in analytic Deligne cohomology as a map between sheaves of ring spectra

$$\mathtt{r}^{\mathrm{Beil}} : \mathbf{K} \to H(\mathbf{IDR}),$$

where $H(\mathbf{IDR})$ is the Eilenberg–MacLane spectrum associated to the sheaf $\mathbf{IDR}$ (Theorem 2.31) using multiplicative characteristic forms (Section 2C).

Throughout the paper we use the language of $(\infty, 1)$-categories as developed by Lurie [2009] and simply called ∞-categories in the following. We view an ordinary category as an ∞-category by taking its nerve.

2A. ***The sites.*** We let $\mathbf{Mf}$ denote the category of smooth manifolds with the open covering topology. Here a smooth manifold is a smooth manifold with corners locally modeled on $[0, \infty)^n \subset \mathbb{R}^n$, $n \in \mathbb{N}$. The category $\mathbf{Mf}$ contains manifolds with boundary and is closed under products. $\mathbf{Mf}$ in particular contains the interval $I = \Delta^1 = [0, 1]$ and the standard simplices Δ^p for all $p \in \mathbb{N}$. We let $\mathbf{Reg}_{\mathbb{Z}}$ denote the category of regular separated schemes of finite type over $\mathtt{Spec}(\mathbb{Z})$ with the topology of Zariski open coverings. Manifolds and schemes are combined in the product $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ of these sites.

Let $\mathbf{C}$ be a presentable ∞-category [Lurie 2009, Chapter 5]. We can consider the ∞-category of functors $\mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})$. Objects in this ∞-category will also be referred to as *presheaves*.

Definition 2.1. An object $F \in \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})$ satisfies descent if F sends disjoint unions to products and for every covering $\mathcal{U}$ of an object $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ the natural map

$$F(M \times X) \to \mathtt{lim}_{\Delta^{\mathrm{op}}} F(\mathcal{U}_\bullet)$$

is an equivalence, where $\mathcal{U}_\bullet \in (\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\Delta^{\mathrm{op}}}$ denotes the Čech nerve of $\mathcal{U}$.

We write $\mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})$ for the full subcategory of objects satisfying descent. These objects will be called *sheaves*. The inclusion as a full subcategory admits a left adjoint L called sheafification [Lurie 2009, Lemma 6.2.2.7]. We express this by the diagram

$$L : \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}) \leftrightarrows \mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}).$$

We will also need the notion of homotopy invariance (in the manifold direction), which should not be confused with $\mathbb{A}^1$-homotopy invariance in the algebraic direction. Let $I := [0, 1]$ be the unit interval.

Definition 2.2. An object $F \in \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})$ is homotopy invariant (in the manifold direction) if the natural map

$$\mathtt{pr}^* : F(M \times X) \to F(I \times M \times X)$$

is an equivalence for every object $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$.

We write $\mathbf{Fun}^{I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})$ for the full subcategory of homotopy invariant objects. We again have an adjunction

$$\mathcal{H}^{\mathrm{pre}} : \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}) \leftrightarrows \mathbf{Fun}^{I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}),$$

and $\mathcal{H}^{\mathrm{pre}}$ is called the *homotopification*. We denote by $\mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})$ the full subcategory of presheaves satisfying both homotopy invariance and descent. Then we have a commutative diagram in ∞-categories

$$\begin{tikzcd}
\mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}) \arrow[r] \arrow[d] & \mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}) \arrow[d] \\
\mathbf{Fun}^{I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}) \arrow[r] & \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})
\end{tikzcd}$$

where all morphisms are inclusions of full subcategories. Taking adjoints, we get a commutative diagram of localizations,

$$\begin{tikzcd} \mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C}) & \mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C}) \arrow[l, "\mathcal{H}"'] \\ \mathbf{Fun}^{I}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C}) \arrow[u, "L_I"] & \mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C}) \arrow[l, "\mathcal{H}^{\mathrm{pre}}"'] \arrow[u, "L"'] \end{tikzcd} \tag{2.3}$$

In order to see that the horizontal adjunctions exists one can use identifications of the form

$$\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C})\simeq\mathbf{Fun}(\mathbf{Mf}^{\mathrm{op}},\mathbf{Fun}(\mathbf{Reg}_{\mathbb{Z}}^{\mathrm{op}},\mathbf{C}))$$

and refer to [Bunke et al. 2013, §2]. Then diagram (2.3) shows that sheafification commutes with homotopification in the sense that $L_I\circ\mathcal{H}^{\mathrm{pre}}\simeq\mathcal{H}\circ L$. Here L_I and $\mathcal{H}$ are the sheafification and the homotopification functors on the respective subcategories. It is not clear that $\mathcal{H}$ is the restriction of $\mathcal{H}^{\mathrm{pre}}$. Again, we refer to [Bunke et al. 2013, §2] for more details.

Note that any functor $\Phi:\mathbf{C}\to\mathbf{C}'$ between presentable ∞-categories induces a functor $\Phi_*:\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C})\to\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C}')$ which preserves homotopy invariant objects. In contrast, Φ_* preserves sheaves in general only if Φ commutes with limits. We will usually write Φ for Φ_* in order simplify the notation.

Later, we will need the following explicit description of the homotopification. We first define a functor

$$\mathbf{s}:\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C})\to\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{Fun}(\Delta^{\mathrm{op}},\mathbf{C}))$$

as the adjoint of

$$(\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}\times\Delta^{\mathrm{op}}\to(\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\quad (M\times X\times[p])\mapsto\Delta^p\times M\times X,$$

where $\Delta^p\in\mathbf{Mf}$ denotes the p-dimensional standard simplex. We further set

$$\bar{\mathbf{s}}:=\mathtt{colim}_{\Delta^{\mathrm{op}}}\circ\mathbf{s}:\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C})\to\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C}).\tag{2.4}$$

Lemma 2.5. (1) *There is a natural map* $\mathtt{id}\to\bar{\mathbf{s}}$.

(2) *If* $X\in\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C})$ *is homotopy invariant, then the natural map* $X\to\bar{\mathbf{s}}(X)$ *is an equivalence.*

(3) *If* f *is a morphism in* $\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C})$ *such that* $\bar{\mathbf{s}}(f)$ *is an equivalence, then* $\mathcal{H}^{\mathrm{pre}}(f)$ *is an equivalence.*

(4) *The map* $\mathtt{id}\to\bar{\mathbf{s}}$ *is equivalent to the unit of the homotopification* $\mathtt{id}\to\mathcal{H}^{\mathrm{pre}}$ *on* $\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{C})$.

Proof. The last statement implies the first three, which are exercises. Details can be found in [Bunke 2013, Problem 4.29]. For (4) we refer to [Bunke et al. 2013, Lemma 7.5]. □

2B. *The multiplicative Deligne complex.* We consider the site of smooth complex varieties $\mathbf{Sm}_{\mathbb{C}}$ with the Zariski topology and the product $\mathbf{Mf}\times\mathbf{Sm}_{\mathbb{C}}$. We denote by $\mathbf{Ch}$ the 1-category of complexes of abelian groups considered as ∞-category and by $\mathbf{Ch}[W^{-1}]$ its localization with quasi-isomorphisms inverted. We have the sheaf of complexes $A\in\mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf}\times\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}},\mathbf{Ch})$ of complex-valued smooth differential forms. It contains the subsheaf of complexes of real-valued forms $A_{\mathbb{R}}$. Obviously, $A\cong A_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$. The sheaf of complexes A furthermore has a decreasing Hodge filtration $\mathcal{F}$ such that elements in $\mathcal{F}^pA(M\times X)$ are locally of the form

$$\sum_{I,J,K,|J|\geq p}\omega_{I,J,K}\,dx^I\wedge dz^J\wedge d\bar{z}^K,$$

where the z_j are local holomorphic coordinates on X and the x_i are local coordinates on M (in contrast to [Bunke and Tamme 2015, §4.2], we forget the log-condition and the weight filtration). Since, degree-wise, these sheaves of complexes consist of modules over the sheaf of smooth functions, they satisfy descent, i.e., they are sheaves when considered as objects in $\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}},\mathbf{Ch}[W^{-1}])$ (see [Bunke et al. 2013, Lemma 7.12] for an argument). By the Poincaré lemma they are also homotopy invariant.

We let $B:\mathbf{Reg}_{\mathbb{Z}}\to\mathbf{Sm}_{\mathbb{C}}$ be the functor mapping a scheme X to the smooth complex variety $X\times_{\mathbb{Z}}\mathbb{C}$. Then $(\mathtt{id}\times B)^*A\in\mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{Ch})$ has a $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-action which preserves the Hodge filtration. The sheaf of complexes $\mathbf{DR}(p)\in\mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{Ch})$ is defined by

$$\mathbf{DR}(p):=[(\mathtt{id}\times B)^*\,\mathbf{DR}_{\mathbb{C}}(p)]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})},$$

where

$$\mathbf{DR}_{\mathbb{C}}(p):=\mathtt{Cone}\big((2\pi i)^pA_{\mathbb{R}}\oplus\mathcal{F}^pA\xrightarrow{\alpha\oplus\beta\mapsto\alpha-\beta}A\big)[2p-1].$$

Here $(\,\cdot\,)^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}$ denotes the object-wise fixed points under the group $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$. Note that all sheaves that appear above have in fact values in complexes of real vector spaces. Furthermore, taking invariants under the finite group $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$ is an exact functor on real vector spaces with $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-action. Therefore, taking $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-invariants preserves the descent and homotopy invariance conditions. Consequently, we can consider $\mathbf{DR}(p)\in\mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}},\mathbf{Ch}\,[W^{-1}])$.

Remark 2.6. For a smooth complex variety X, the complex $\mathbf{DR}_{\mathbb{C}}(p)(X)$ calculates the analytic Deligne cohomology $H^*_{\mathcal{D},\mathrm{an}}(X,\mathbb{R}(p))$ up to a shift of degrees by $2p$. If, in the definition of the cone, one replaces the complexes of smooth forms $A_{\mathbb{R}}$, A by their log-versions $A_{\mathbb{R},\log}$, $A_{\log}$ (consisting of forms which extend to some compactification of X with logarithmic poles along the boundary of X; see [Bunke and Tamme 2015, §4.2]) one obtains the so-called Beilinson–Deligne or weak absolute Hodge cohomology $H^*_{B\mathcal{D}}(X, \mathbb{R}(p))$. There is a natural map $H^*_{B\mathcal{D}}(X, \mathbb{R}(p)) \to H^*_{\mathcal{D},\mathrm{an}}(X, \mathbb{R}(p))$ which, in general, is neither injective nor surjective. It is an isomorphism if X is also proper over $\mathbb{C}$. If one moreover introduces the weight filtration $\hat{\mathcal{W}}$ and replaces $A_{\mathbb{R},\log}$, $A_{\log}$ by the subcomplexes $\hat{\mathcal{W}}_{2p}A_{\mathbb{R},\log}$, $\hat{\mathcal{W}}_{2p}A_{\log}$, one obtains the absolute Hodge cohomology $H^*_{\mathcal{A}\mathcal{H}}(X, \mathbb{R}(p))$ introduced by Beilinson [1986]. This is the cohomology theory used in [Bunke and Tamme 2015]. It follows from Deligne's theory of weights that the natural map $H^*_{\mathcal{A}\mathcal{H}}(X, \mathbb{R}(p)) \to H^*_{B\mathcal{D}}(X, \mathbb{R}(p))$ is an isomorphism in degrees $* \le p$, and in degrees $* \le 2p$ if X is proper.

In the following, we define a sheaf $\mathbf{IDR}(p) \in \mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Ch})$ which is object-wise quasi-isomorphic to $\mathbf{DR}(p)$, and which is better behaved with respect to the multiplicative structures. We define the morphism

$$\mathcal{I} : \mathbf{Mf} \to \mathbf{Mf}, \quad M \mapsto [0, 1] \times M.$$

It induces a corresponding morphism $\mathcal{I} \times \mathtt{id}_{\mathbf{Sm}_{\mathbb{C}}} : \mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}} \to \mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}}$. For a presheaf $\mathcal{F}$ on $\mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}}$ we define $\mathcal{I}\mathcal{F} := (\mathcal{I} \times \mathtt{id}_{\mathbf{Sm}_{\mathbb{C}}})^*\mathcal{F}$.

Definition 2.7. We define

$$\mathbf{IDR}_{\mathbb{C}}(p) \subseteq \mathcal{I}A[2p]$$

to be the subsheaf with values in $\mathbf{Ch}$ determined by the conditions that ω lies in $\mathbf{IDR}_{\mathbb{C}}(p)(M \times X)$ if and only if

(1) $\omega_{|\{0\}\times M\times X} \in (2\pi i)^p A_{\mathbb{R}}(M \times X)[2p]$,

(2) $\omega_{|\{1\}\times M\times X} \in \mathcal{F}^p A(M \times X)[2p]$.

We set $\mathbf{IDR}_{\mathbb{C}} := \prod_{p\ge 0} \mathbf{IDR}_{\mathbb{C}}(p)$ and define

$$\mathbf{IDR} := [(\mathtt{id} \times B)^* \, \mathbf{IDR}_{\mathbb{C}}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.$$

An algebraic analog of this complex was used by Burgos and Wang [1998].

Proposition 2.8. *There is an object-wise quasi-isomorphism*

$$q : \mathbf{IDR}(p) \to \mathbf{DR}(p). \tag{2.9}$$

Proof. We define a morphism of sheaves of complexes

$$q_{\mathbb{C}} : \mathbf{IDR}_{\mathbb{C}}(p) \to \mathbf{DR}_{\mathbb{C}}(p) \tag{2.10}$$

as follows. A form $\omega \in \mathbf{IDR}_{\mathbb{C}}(p)(M)$ gives rise to forms

(1) $\omega_{\mathbb{R}} := \omega_{|\{0\}\times M\times X} \in (2\pi i)^p A_{\mathbb{R}}(M \times X)[2p]$,

(2) $\omega_{\mathcal{F}} := \omega_{|\{1\}\times M\times X} \in \mathcal{F}^p A(M \times X)[2p]$,

(3) $\tilde{\omega} := \int_{[0,1]\times M\times X/M\times X} \omega \in A(M \times X)[2p-1]$.

We define

$$q_{\mathbb{C}}(\omega) := (\omega_{\mathbb{R}} \oplus \omega_{\mathcal{F}}, -\tilde{\omega}) \in \mathbf{DR}_{\mathbb{C}}(M \times X).$$

We have

$$dq_{\mathbb{C}}(\omega) = d(\omega_{\mathbb{R}} \oplus \omega_{\mathcal{F}}, -\tilde{\omega}) = (d\omega_{\mathbb{R}} \oplus d\omega_{\mathcal{F}}, d\tilde{\omega} + \omega_{\mathbb{R}} - \omega_{\mathcal{F}})$$

and

$$\begin{aligned} q_{\mathbb{C}}(d\omega) &= \left(d\omega_{\mathbb{R}} \oplus d\omega_{\mathcal{F}}, -\int_{[0,1]\times M/M} d\omega\right) \\ &= \left(d\omega_{\mathbb{R}} \oplus d\omega_{\mathcal{F}}, d\int_{[0,1]\times M/M} \omega + \omega_{\mathbb{R}} - \omega_{\mathcal{F}}\right) \\ &= (d\omega_{\mathbb{R}} \oplus d\omega_{\mathcal{F}}, d\tilde{\omega} + \omega_{\mathbb{R}} - \omega_{\mathcal{F}}), \end{aligned}$$

a calculation using Stokes' theorem. Hence $q_{\mathbb{C}}$ is a map of complexes.

Lemma 2.11. *For every $p \geq 0$ the map $q_{\mathbb{C}} : \mathbf{IDR}_{\mathbb{C}}(p) \to \mathbf{DR}_{\mathbb{C}}(p)$ is an object-wise quasi-isomorphism.*

Proof. We abbreviate

$$S := A/(2\pi i)^p A_{\mathbb{R}}[2p], \quad T := A/\mathcal{F}^p A_{\mathbb{R}}[2p].$$

Then we have an exact sequence

$$0 \to \mathbf{IDR}(p) \to \mathcal{I}A[2p] \to S \oplus T \to 0, \tag{2.12}$$

where the first map is the inclusion and the second is given by the evaluation at the endpoints of the interval. We further have a natural exact sequence

$$0 \to \mathbf{DR}(p) \to \mathtt{Cone}(A \oplus A \to A)[2p-1] \to \mathtt{Cone}(S \oplus T \to 0)[-1] \to 0. \tag{2.13}$$

We define a map of exact sequences (2.12) $\to$ (2.13) using the map $q_{\mathbb{C}}$ in the first entry, the same formula as for $q_{\mathbb{C}}$ in the second, and the obvious identity map at the last entry. Since the interval [0, 1] is contractible it follows from the relative Poincaré lemma that the middle map is a quasi-isomorphism. Since the last map is an isomorphism, it follows from the five lemma that $q_{\mathbb{C}}$ is a quasi-isomorphism, too. □

We observe that $(\mathtt{id} \times B)^* q_{\mathbb{C}}$ commutes with the $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-action and therefore induces an equivalence $q : \mathbf{IDR}(p) \to \mathbf{DR}(p)$, too. This finishes the proof of the proposition. □

It follows from Lemma 2.11 and the sheaf and homotopy invariance properties of **DR** that we can consider

$$\mathbf{IDR} \in \mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Ch}[W^{-1}]).$$

We now observe that the filtration $\mathcal{F}$ as well as the real subspaces are compatible with the multiplication

$$\wedge : A \otimes A \to A.$$

We therefore get products

$$\wedge : \mathbf{IDR}(p) \times \mathbf{IDR}(q) \to \mathbf{IDR}(p+q).$$

Taking the product over all p, we get as final result:

Corollary 2.14. *The product*

$$\mathbf{IDR} := \prod_{p \geq 0} \mathbf{IDR}(p)$$

has the structure of a sheaf of bigraded graded commutative d-algebras.

We denote the symmetric monoidal ∞-categories of chain complexes and chain complexes with quasi-isomorphisms inverted, with the tensor product, by $\mathbf{Ch}^{\otimes}$ and $\mathbf{Ch}[W^{-1}]^{\otimes}$, respectively. The notation for commutative algebra objects is **CAlg**. Commutative differential graded algebras are objects of $\mathbf{CAlg}(\mathbf{Ch}^{\otimes})$. They can be considered as objects in $\mathbf{CAlg}(\mathbf{Ch}[W^{-1}]^{\otimes})$. Since the forgetful functor

$$\mathbf{CAlg}(\mathbf{Ch}[W^{-1}]^{\otimes}) \to \mathbf{Ch}[W^{-1}]^{\otimes}$$

is a right adjoint, limits in commutative algebras are computed on underlying objects. Consequently, **IDR** can naturally be considered as an object

$$\mathbf{IDR} \in \mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{Ch}[W^{-1}]^{\otimes})). \tag{2.15}$$

2C. ***Geometries and characteristic forms.*** We first consider $M \times X \in \mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}}$. We view $M \times X$ as a locally ringed space with structure sheaf $\mathcal{O}_{M \times X} := \mathtt{pr}_X^{-1}\,\mathcal{O}_X$ given by the inverse image of the sheaf $\mathcal{O}_X$ under the projection to X. A sheaf of finitely generated locally free $\mathcal{O}_{M\times X}$-modules will be called a bundle on $M \times X$. If V is a bundle on $M \times X$ we have an associated complex vector bundle on $M \times X(\mathbb{C})$ which we abusively denote by the same symbol. It naturally carries a flat partial connection ∇^I in the M-direction and a holomorphic structure $\bar{\partial}$ in the X-direction, which is constant with respect to ∇^I, i.e., $[\nabla^I, \bar{\partial}] = 0$.

Definition 2.16 [Bunke and Tamme 2015, Definition 4.12]. A geometry on the bundle V is given by a pair (h^V, ∇^{II}) consisting of a hermitian metric h^V on V and a partial connection ∇^{II} in the X-direction that extends the holomorphic structure $\bar{\partial}$.

We form the connection $\nabla := \nabla^I + \nabla^{II}$ and let ∇^u be its unitarization with respect to h^V. In [Bunke and Tamme 2015], we use these connections in order to define a characteristic form in $\mathbf{DR}(M \times X)$. In the present paper we adjust the notion of a geometry such that we obtain a lift of the characteristic form to $\mathbf{IDR}(M \times X)$; see Lemma 2.22.

Let $\mathtt{pr} : I \times M \times X \to M \times X$ denote the projection.

Definition 2.17. An extended geometry g on V is a triple $g = ((h^V, \nabla^{II}), \tilde{\nabla})$ consisting of a geometry on V and a connection $\tilde{\nabla}$ on $\mathtt{pr}^* V$ such that

(1) $\tilde{\nabla}_{|\{0\}\times M\times X} = \nabla^u$,

(2) $\tilde{\nabla}_{|\{1\}\times M\times X} = \nabla$.

We now consider the arithmetic situation $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$. We keep calling a sheaf of finitely generated locally free $\mathcal{O}_{M\times X}$-modules a bundle. For the notion of $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-invariance in the following definition we refer to [Bunke and Tamme 2015, Definition 4.31].

Definition 2.18. An extended geometry g on a bundle V on $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ is a $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-invariant extended geometry g on the bundle $(\mathtt{id} \times B)^*(V)$.

Geometries and extended geometries exist and can be glued with partitions of unity on M. Compared with [Bunke and Tamme 2015] the situation is simplified since we drop the condition of being good. Examples are given by the canonical extensions:

Definition 2.19. Given a geometry (h^V, ∇^{II}) on the bundle V, we define the associated canonical extended geometry

$$\mathtt{can}(h^V, \nabla^{II}) := ((h^V, \nabla^{II}), \tilde{\nabla})$$

by taking for $\tilde{\nabla}$ the linear path from ∇^u to ∇.

For any $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ we denote the groupoid of bundles with extended geometry on $M \times X$ and isomorphisms respecting the extended geometry by $i\mathtt{Vect}^{\mathrm{exge}}(M \times X)$.

For a closed symmetric monoidal presentable ∞-category $\mathbf{C}^{\otimes}$ we denote by $\mathbf{Rig}(\mathbf{C}^{\otimes})$ the ∞-category of semiring objects in $\mathbf{C}$ (see [Gepner et al. 2013, Definition 7.1]). The typical example of a semiring in $\mathbf{Set}^{\times}$ is the semiring of integers $\mathbb{N}$. We let $\mathbf{Cat}[W^{-1}]^{\times}$ be the ∞-category of categories with categorical equivalences inverted, equipped with its cartesian symmetric monoidal structure. A semiring in $\mathbf{Cat}[W^{-1}]^{\times}$ will be called a Rig-category. Then a typical Rig-category is the category of vector spaces over some field with the operations $\oplus$ and $\otimes$. This follows from the recognition principle [Gepner et al. 2013, Theorem 8.8]. This principle implies that, using direct sum and tensor product of bundles with geometry, we can consider $i\,\mathtt{Vect}^{\mathrm{exge}}$ as a sheaf of Rig-categories

$$i\,\mathtt{Vect}^{\mathrm{exge}} \in \mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Rig}(\mathbf{Cat}[W^{-1}]^{\times})).$$

We furthermore interpret $\pi_0(i\,\mathtt{Vect}^{\mathrm{exge}})$ and $Z^0(\mathbf{IDR})$ as presheaves of semirings

$$\pi_0(i\,\mathtt{Vect}^{\mathrm{exge}}),\; Z^0(\mathbf{IDR}) \in \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Rig}(\mathbf{Set}^{\times})).$$

We let R^{∇} denote the curvature of a connection ∇. Furthermore, by

$$\mathbf{ch}_{2p}(\nabla) := [\mathtt{Tr}\exp(-R^{\nabla})]_{2p} = (-1)^p\; \mathtt{Tr}(R^{\nabla})^p$$

we denote the component of the unnormalized Chern character form in degree $2p$.

Definition 2.20. We define the transformation of presheaves of semirings

$$\widetilde{\omega} : \pi_0(i\,\mathtt{Vect}^{\mathrm{exge}}) \to Z^0(\mathbf{IDR})$$

by

$$\widetilde{\omega}(V, g) := \prod_{p \geq 0} \mathbf{ch}_{2p}(\widetilde{\nabla}).$$

A priori,

$$\prod_{p \geq 0} \mathbf{ch}_{2p}(\widetilde{\nabla}) \in \prod_{p \geq 0} \mathcal{I}A(M \times B(X)),$$

but the conditions for $\widetilde{\nabla}$ at the endpoints of the interval immediately imply that this product of forms belongs to the subcomplex $\mathbf{IDR}(X \times M)$ from Definition 2.7.

In [Bunke and Tamme 2015], for a bundle V with a geometry g we defined a characteristic form

$$\omega((V, (h^V, \nabla^{II}))) := \prod_p (\mathbf{ch}_{2p}(\nabla^u) \oplus \mathbf{ch}_{2p}(\nabla), \widetilde{\mathbf{ch}}_{2p-1}(\nabla^u, \nabla)), \tag{2.21}$$

where the last form denotes the transgression [Bunke and Tamme 2015, (66)]. This is compatible with our new construction in the sense of the lemma below. We let $i\,\mathtt{Vect}^{\mathrm{geom}}$ denote the symmetric monoidal stack of bundles with geometries on $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ and geometry-preserving isomorphisms.[1] Then the formula (2.21) gives a map $\omega : \pi_0(i\,\mathtt{Vect}^{\mathrm{geom}}) \to Z^0(\mathbf{DR})$. The construction of the canonical extended geometry in Definition 2.19 induces a map

$$\mathtt{can} : \pi_0(i\,\mathtt{Vect}^{\mathrm{geom}}) \to \pi_0(i\,\mathtt{Vect}^{\mathrm{exge}}),$$

which is additive, but not multiplicative.

[1] Note that in [Bunke and Tamme 2015] this symbol has a different meaning.

Lemma 2.22. *The diagram*

$$\begin{tikzcd} \pi_0(i\,\mathtt{Vect}^{\mathrm{exge}}) \arrow[r, "\widetilde{\omega}"] & Z^0(\mathbf{IDR}) \arrow[d, "q"] \\ \pi_0(i\,\mathtt{Vect}^{\mathrm{geom}}) \arrow[u, "\mathtt{can}"] \arrow[r, "\omega"] & Z^0(\mathbf{DR}) \end{tikzcd} \tag{2.23}$$

commutes.

Proof. This follows from the definition of q in (2.9), the construction of the transgression $\widetilde{\mathbf{ch}}_{2p-1}(\nabla^u, \nabla)$, and the definition of ω in (2.21). □

2D. ***The multiplicative K-theory sheaf and the regulator.*** In this section, we define algebraic K-theory as a sheaf of commutative ring spectra on $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$. To do so, we use the multiplicative version of group completion studied in [Gepner et al. 2013] (see in particular their Proposition 8.2). We denote by $\mathbf{Sp}^{\wedge}$ and $\mathbf{Sp}^{\geq 0, \wedge}$ the symmetric monoidal ∞-categories of spectra and connective spectra, respectively, with the smash product. The category $\mathbf{Sp}$ is the stable ∞-category generated by the sphere spectrum whose homotopy category is the stable homotopy category. For the purpose of the present paper we do not have to fix a particular model for $\mathbf{Sp}$. We will use the identification of ∞-categories

$$\mathbf{CommGroup}(\mathbf{sSet}[W^{-1}]^{\times}) \simeq \mathbf{Sp}^{\geq 0, \wedge}$$

which identifies a connective spectrum with its ∞-loop space. This equivalence refines to an equivalence of ∞-categories

$$\mathbf{Ring}(\mathbf{sSet}[W^{-1}]^{\times}) \simeq \mathbf{CAlg}(\mathbf{Sp}^{\geq 0, \wedge}). \tag{2.24}$$

Definition 2.25. We define the K-theory functor

$$K : \mathbf{Rig}(\mathbf{Cat}[W^{-1}]^{\times}) \to \mathbf{CAlg}(\mathbf{Sp}^{\wedge})$$

as the composition

$$\begin{aligned} \mathbf{Rig}(\mathbf{Cat}[W^{-1}]^{\times}) &\xrightarrow{\mathtt{N}} \mathbf{Rig}(\mathbf{sSet}[W^{-1}]^{\times}) && \text{(nerve)} \\ &\to \mathbf{Ring}(\mathbf{sSet}[W^{-1}]^{\times}) && \text{(ring completion)} \\ &\xrightarrow{\simeq} \mathbf{CAlg}(\mathbf{Sp}^{\geq 0, \wedge}) && \text{(using (2.24))} \\ &\to \mathbf{CAlg}(\mathbf{Sp}^{\wedge}) && \text{(forget connectivity).} \end{aligned}$$

We consider the sheaf

$$i\,\mathtt{Vect} \in \mathbf{Fun}^{\mathrm{desc}, I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Rig}(\mathbf{Cat}[W^{-1}]^{\times}))$$

which associates to each object $M \times X$ the Rig-category of bundles over $M \times X$ and isomorphisms.

Definition 2.26. We define the sheaf of K-theory spectra by

$$\mathbf{K} := L(K(i\,\mathtt{Vect})) \in \mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{Sp}^{\wedge})).$$

Remark 2.27. For $X \in \mathbf{Reg}_{\mathbb{Z}}$, the homotopy groups of the spectrum

$$\mathbf{K}(X) := \mathbf{K}(* \times X)$$

are the usual K-groups of X as defined by Quillen. This follows from the known facts that, for affine X, Quillen's K-theory coincides with K-theory defined by group completion and that, on $\mathbf{Reg}_{\mathbb{Z}}$, Quillen's K-theory satisfies Zariski-descent (see [Bunke and Tamme 2015, §3.3] for more details).

In general, the spectrum $\mathbf{K}(X)$ represents a generalized cohomology theory and, for a manifold M, we have

$$\pi_*(\mathbf{K}(M \times X)) \cong \mathbf{K}(X)^{-*}(M)$$

(see [Bunke and Tamme 2015, §4.5]).

Note that the homotopy invariance of $i\,\mathtt{Vect}$ implies the homotopy invariance of $K(i\,\mathtt{Vect})$. In contrast, $i\,\mathtt{Vect}^{\mathrm{exge}}$ is not homotopy invariant. But, applying the presheaf homotopification $\bar{\mathbf{s}} \simeq \mathcal{H}^{\mathrm{pre}}$ from (2.4), we get the following result:

Lemma 2.28. *The natural "forget the geometry" map*

$$\bar{\mathbf{s}}\,\mathtt{N}(i\,\mathtt{Vect}^{\mathrm{exge}}) \to \bar{\mathbf{s}}\,\mathtt{N}(i\,\mathtt{Vect}) \simeq \mathtt{N}(i\,\mathtt{Vect})$$

is an equivalence in $\mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Rig}(\mathbf{sSet}[W^{-1}]^{\times}))$.

Proof. Since the colimit over Δ^{op} appearing in the definition (2.4) of $\bar{\mathbf{s}}$ is sifted it commutes with the forgetful functor $\mathbf{Rig}(\mathbf{sSet}[W^{-1}]^{\times}) \to \mathbf{sSet}[W^{-1}]$. This follows from a two-fold application of [Lurie 2014, Corollary 3.2.3.2] to

$$\mathbf{Rig}(\mathbf{sSet}[W^{-1}]^{\times}) \simeq \mathbf{CAlg}(\mathbf{CAlg}(\mathbf{sSet}[W^{-1}]^{\times})^{\otimes}).$$

Since an equivalence in $\mathbf{Rig}(\mathbf{sSet}[W^{-1}]^{\times})$ is detected in $\mathbf{sSet}[W^{-1}]$ it suffices to show that the induced map in $\mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{sSet}[W^{-1}])$ is an equivalence.

We claim that for $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ the map of simplicial sets

$$\mathbf{s}\,\mathtt{N}(i\,\mathtt{Vect}^{\mathrm{exge}})(M \times X)_{\bullet,q} \to \mathbf{s}\,\mathtt{N}(i\,\mathtt{Vect})(M \times X)_{\bullet,q}$$

is a trivial Kan fibration. The result then follows by applying the colimit as in (2.4).

A p-simplex $x : \Delta^p \to \mathtt{N}(i\,\mathtt{Vect})(M \times X)_{\bullet,q}$ is given by a string of bundles and isomorphisms

$$V_0 \xrightarrow{\cong} V_1 \xrightarrow{\cong} \cdots \xrightarrow{\cong} V_q$$

on $\Delta^p \times M \times X$. A lifting of $x|_{\partial\Delta^p}$ is determined by an extended geometry on $V_0|_{\partial\Delta^p \times M \times X}$. Using the fact that extended geometries exist and can be glued using partitions of unity, we see that such a lifting can always be extended to a p-simplex of $\mathtt{sN}(i\mathtt{Vect}^{\mathrm{exge}})(M\times X)_{\bullet,q}$ lifting x. This implies the claim. $\square$

We now turn to the construction of the multiplicative version of Beilinson's regulator. We interpret a set as a discrete category. In this way we get a morphism

$$\iota : \mathbf{Rig}(\mathbf{Set}^{\times}) \to \mathbf{Rig}(\mathbf{Cat}[W^{-1}]^{\times}).$$

We have a commutative diagram (see [Bunke and Tamme 2015, Remark 2.13])

$$\begin{tikzcd} \mathbf{Ring}(\mathbf{Set}^{\times}) \arrow[r, "\iota"] \arrow[d, "S^0"] & \mathbf{Rig}(\mathbf{Cat}[W^{-1}]^{\times}) \arrow[d, "K"] \\ \mathbf{CAlg}(\mathbf{Ch}[W^{-1}]^{\otimes}) \arrow[r, "H"] & \mathbf{CAlg}(\mathbf{Sp}^{\wedge}) \end{tikzcd}$$

where S^0 interprets a commutative ring as a commutative monoid in chain complexes concentrated in degree zero, H is the Eilenberg–MacLane equivalence, and in the upper horizontal line we do not write the restriction of ι from semirings to rings explicitly. We write $r(\widetilde{\omega})$ for the composition

$$\begin{aligned} K(i\mathtt{Vect}^{\mathrm{exge}}) \to K(\iota(\pi_0(i\mathtt{Vect}^{\mathrm{exge}}))) &\xrightarrow{K(\iota(\widetilde{\omega}))} K(\iota(Z^0(\mathbf{IDR}))) \\ &\simeq H(S^0(Z^0(\mathbf{IDR}))) \to H(\mathbf{IDR}) \end{aligned}$$

in $\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{Sp}^{\wedge}))$.

In analogy with [Bunke and Tamme 2015, Definition 4.36] we adopt the following definition:

Definition 2.29. We define the multiplicative version of the naive Beilinson regulator

$$\mathtt{r}^{\mathrm{Beil}} : \mathbf{K} \to H(\mathbf{IDR})$$

as a morphism in $\mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{Sp}^{\wedge}))$ to be the sheafification of the composition

$$K(i\mathtt{Vect}) \xrightarrow{\simeq} \bar{\mathbf{s}}\,K(i\mathtt{Vect}) \xleftarrow[\text{Lemma 2.28}]{\simeq} \bar{\mathbf{s}}\,K(i\mathtt{Vect}^{\mathrm{exge}}) \xrightarrow{\bar{\mathbf{s}}(r(\widetilde{\omega}))} \bar{\mathbf{s}}\,H(\mathbf{IDR}) \xleftarrow{\simeq} H(\mathbf{IDR})$$

in $\mathbf{Fun}((\mathbf{Mf}\times\mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{Sp}^{\wedge}))$.

Here we use the fact that $H(\mathbf{IDR})$ is a sheaf (see (2.15)).

Remark 2.30. Since in the present paper we don't require geometries to be good in the sense of [Bunke and Tamme 2015, Definition 4.17] the characteristic forms don't necessarily satisfy a logarithmic growth condition at infinity. Therefore, we end up in analytic Deligne cohomology instead of absolute Hodge cohomology. The proof of Lemma 2.28 does not work for good geometries. In [Bunke and Tamme 2015] we found a way to avoid this problem using the Čechification of the de Rham complexes. At the moment we do not see how to refine this to a multiplicative version.

For $X \in \mathbf{Reg}_{\mathbb{Z}}$ Beilinson's regulator [1986] is a homomorphism from the K-theory of X to absolute Hodge cohomology (see Remark 2.6)

$$K_*(X) \to \prod_p H^{2p-*}_{\mathscr{A}\mathscr{H}}(X, \mathbb{R}(p)).$$

It is known to be multiplicative. We call its composition with the natural map

$$\prod_p H^{2p-*}_{\mathscr{A}\mathscr{H}}(X, \mathbb{R}(p)) \to H^{-*}(\mathbf{IDR}(* \times X))$$

the analytic version of Beilinson's regulator.

Theorem 2.31. *The naive Beilinson regulator*

$$\mathtt{r}^{\mathrm{Beil}} : \mathbf{K} \to H(\mathbf{IDR})$$

is a morphism of sheaves of ring spectra which, on the homotopy groups of its evaluation on $* \times X$, induces the analytic version of Beilinson's regulator.

Proof. The first assertion is true by construction. It is also immediate from the constructions and Lemma 2.22 that the map of sheaves of spectra underlying $\mathtt{r}^{\mathrm{Beil}}$ coincides with the one obtained in [Bunke and Tamme 2015, Definition 4.36] (after forgetting the logarithmic growth condition and using the equivalence $\mathbf{DR} \cong \mathbf{IDR}$). For the latter, the coincidence with Beilinson's regulator was proven in [Bunke and Tamme 2015, §4.7]. □

3. Multiplicative differential algebraic K-theory

3A. ***Basic definitions.*** The main goal of this section is the definition of a multiplicative version of differential algebraic K-theory for objects in $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ and the verification of its basic properties.

For a complex $C \in \mathbf{Ch}$ and an integer k we let $\sigma^{\geq k} C$ denote the naive truncation given by $\cdots \to 0 \to C^k \to C^{k+1} \to \cdots$. There is a natural inclusion morphism $\sigma^{\geq k} C \to C$.

Definition 3.1. For every integer $k \in \mathbb{Z}$, we define the sheaf of differential algebraic K-theory spectra

$$\widehat{\mathbf{K}}^{(k)} \in \mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Sp}^{\wedge})$$

by the pullback

$$\begin{tikzcd} \widehat{\mathbf{K}}^{(k)} \arrow[r, "R"] \arrow[d, "I"] & H(\sigma^{\geq k}\, \mathbf{IDR}) \arrow[d] \\ \mathbf{K} \arrow[r, "\mathtt{r}^{\mathrm{Beil}}"] & H(\mathbf{IDR}) \end{tikzcd}$$

We define the differential algebraic K-theory for objects in $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ as a presheaf of abelian groups

$$\widehat{\mathbf{K}}^k := \pi_{-k}(\widehat{\mathbf{K}}^{(k)}) \in \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Ab}).$$

Remark 3.2. The integer $k \in \mathbb{Z}$ determines that the homotopy group $\pi_{-d}(\widehat{\mathbf{K}}^k)$ for $d \in \mathbb{Z}$ captures interesting differential geometric information exactly if $d = k$.

In the following, we refine $\bigvee_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^{(k)}$ to a sheaf of commutative ring spectra (see [Bunke 2013, §4.6] for details). Using the symmetric monoidal functors

$$\mathbf{Set} \xrightarrow{\iota} \mathbf{sSet}[W^{-1}] \xrightarrow{\Sigma^\infty_+} \mathbf{Sp},$$

the abelian group $\mathbb{Z} \in \mathbf{CommMon}(\mathbf{Set})$ gives rise to the commutative ring spectrum $\Sigma^\infty_+\iota(\mathbb{Z}) \in \mathbf{CAlg}(\mathbf{Sp}^\wedge)$. For any commutative ring spectrum E we write $E[z, z^{-1}] := E \wedge \Sigma^\infty_+\iota(\mathbb{Z})$. We consider $\mathbf{IDR}[z, z^{-1}] := \mathbf{IDR} \otimes_{\mathbb{Z}} \mathbb{Z}[z, z^{-1}]$ as a sheaf of commutative differential graded algebras and define the subalgebra

$$\sigma^{\geq \bullet}\, \mathbf{IDR} := \bigoplus_{k\in\mathbb{Z}} z^k \sigma^{\geq k}\, \mathbf{IDR} \subseteq \mathbf{IDR}[z, z^{-1}].$$

We have a natural equivalence $H(\mathbf{IDR}[z, z^{-1}]) \simeq H(\mathbf{IDR})[z, z^{-1}]$.

Definition 3.3. We define differential algebraic K-theory as a sheaf of commutative ring spectra

$$\widehat{\mathbf{K}}^{(\bullet)} \in \mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{Sp}^\wedge))$$

by the pullback

$$\begin{tikzcd}
\widehat{\mathbf{K}}^{(\bullet)} \arrow[r, "R"] \arrow[d, "I"] & H(\sigma^{\geq \bullet}\, \mathbf{IDR}) \arrow[d] \\
\mathbf{K}[z, z^{-1}] \arrow[r, "\mathbf{r}^{\mathrm{Beil}}{[z,z^{-1}]}"] & H(\mathbf{IDR})[z, z^{-1}].
\end{tikzcd}$$

If we forget the ring spectrum structure, then we get a natural equivalence $\widehat{\mathbf{K}}^{(\bullet)} \simeq \bigvee_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^{(k)}$. In particular, we get a presheaf of graded commutative rings

$$\bigoplus_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^k \in \mathbf{Fun}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{GrRings}).$$

The maps R and I induce ring homomorphisms

$$R : \bigoplus_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^k \to \bigoplus_{k\in\mathbb{Z}} Z^k(\mathbf{IDR}), \quad I : \bigoplus_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^k \to \bigoplus_{k\in\mathbb{Z}} \mathbf{K}^k.$$

The map R is called the curvature. For any $k \in \mathbb{Z}$ we have exact sequences

$$\mathbf{K}^{k-1} \xrightarrow{\mathtt{r}^{\mathrm{Beil}}} H^{k-1}(\mathbf{IDR}) \xrightarrow{a} \widehat{\mathbf{K}}^k \xrightarrow{(I,R)} \mathbf{K}^k \times_{H^k(\mathbf{IDR})} Z^k(\mathbf{IDR}) \to 0$$

and

$$\mathbf{K}^{k-1} \xrightarrow{\mathtt{r}^{\mathrm{Beil}}} \mathbf{IDR}^{k-1} / \mathtt{im}(d) \xrightarrow{a} \widehat{\mathbf{K}}^k \xrightarrow{I} \mathbf{K}^k \to 0 \tag{3.4}$$

(see [Bunke and Tamme 2015, Proposition 5.4]). Moreover, we have the relation $R \circ a = d$.

3B. *Cycle maps.* We have the forgetful map

$$\pi_0(i\,\mathtt{Vect}^{\mathrm{exge}}) \to \pi_0(i\,\mathtt{Vect})$$

between the presheaves of semirings of isomorphism classes of bundles with and without extended geometries.

Proposition 3.5. *There are canonical cycle maps* $\mathtt{cycl}$ *and* $\widehat{\mathtt{cycl}}$ *fitting into the following diagram of presheaves of semirings on* $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$:

$$\begin{tikzcd}
\pi_0(i\,\mathtt{Vect}^{\mathrm{exge}}) \arrow[r, "\widehat{\mathtt{cycl}}"] \arrow[d] \arrow[rr, bend left, "\tilde{\omega}"] & \widehat{\mathbf{K}}^0 \arrow[r, "R"] \arrow[d, "I"] & Z^0(\mathbf{IDR}) \\
\pi_0(i\,\mathtt{Vect}) \arrow[r, "\mathtt{cycl}"] & \mathbf{K}^0 &
\end{tikzcd}$$

Proof. The construction is identical to that of [Bunke and Tamme 2015, Definitions 5.8, 5.9]. ☐

3C. S^1*-integration.* We consider $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$. Let

$$\mathbf{E} \in \mathbf{Fun}^{\mathrm{desc},I}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{Sp})$$

be a homotopy invariant sheaf of spectra. Then we have natural isomorphisms

$$\mathbf{E}^*(S^1 \times M \times X) \cong \mathbf{E}^*(M \times X) \oplus \mathbf{E}^{*-1}(M \times X).$$

The induced map $\mathbf{E}^*(S^1 \times M \times X) \to \mathbf{E}^{*-1}(M \times X)$ is called the desuspension map. This applies in particular to the K-theory sheaf $\mathbf{K}$ and the analytic Deligne cohomology $H(\mathbf{IDR})$.

On the other hand, on the level of differential forms we have the usual fiber integration along S^1, a map of complexes

$$\int_{S^1} : \mathbf{IDR}(S^1 \times M \times X) \to \mathbf{IDR}(M \times X)[-1].$$

It induces integration maps $\int_{S^1} : \sigma^{\geq k}\,\mathbf{IDR}(S^1 \times M \times X) \to \sigma^{\geq k-1}\,\mathbf{IDR}(M \times X)[-1]$ for any $k \in \mathbb{Z}$.

Proposition 3.6. *There exists a natural map*

$$\int_{S^1} : \widehat{\mathbf{K}}^*(S^1 \times M \times X) \to \widehat{\mathbf{K}}^{*-1}(M \times X)$$

of $\bigoplus_{k\in\mathbb{Z}} \widehat{\mathbf{K}}^k(M \times X)$*-modules which is compatible with the desuspension on* $\mathbf{K}^*$ *via the map* I *and with the integration* $\int_{S^1}$ *on* $Z^*(\mathbf{IDR})$ *via the curvature* R.

Proof. We define the endofunctor S^1 of $\mathbf{Fun}^{\mathrm{desc}}(\mathbf{Mf}^{\mathrm{op}}, \mathbf{C})$ for any presentable ∞-category $\mathbf{C}$ by $(S^1F)(M \times X) := F(S^1 \times M \times X)$. If $\mathbf{C}$ is symmetric monoidal and $F \in \mathbf{Fun}^{\mathrm{desc}}(\mathbf{Mf}^{\mathrm{op}}, \mathbf{CAlg}(\mathbf{C}))$, then the projection $\mathtt{pr} : S^1 \to *$ turns S^1F into an object of $\mathbf{Mod}(F)$.

We extend the endofunctor S^1 to $\mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C})$ using the identification

$$\mathbf{Fun}^{\mathrm{desc}}((\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})^{\mathrm{op}}, \mathbf{C}) \simeq \mathbf{Fun}^{\mathrm{desc}}(\mathbf{Mf}^{\mathrm{op}}, \mathbf{Fun}^{\mathrm{desc}}(\mathbf{Reg}_{\mathbb{Z}}^{\mathrm{op}}, \mathbf{C})).$$

The evaluation at the manifold $M = *$ provides an equivalence of ∞-categories

$$\mathtt{ev}_* : \mathbf{Fun}^{\mathrm{desc},I}(\mathbf{Mf}^{\mathrm{op}}, \mathbf{C}) \xrightarrow{\simeq} \mathbf{C}, \tag{3.7}$$

and we have an equivalence of functors $\mathbf{Fun}^{\mathrm{desc},I}(\mathbf{Mf}^{\mathrm{op}}, \mathbf{C}) \to \mathbf{C}$

$$\mathtt{ev}_* \circ S^1(-) \simeq (\mathtt{ev}_*(-))^{S^1}, \tag{3.8}$$

where $(-)^{S^1}$ is the cotensor structure. Let $\mathtt{pr} : S^1 \to *$ and $i : * \to S^1$ be the projection to a point and the inclusion of a base point. These maps induce a retraction

$$\mathtt{id}(-) \xrightarrow{\mathtt{pr}^*} (-)^{S^1} \xrightarrow{i^*} \mathtt{id}(-).$$

If $\mathbf{C}$ is stable, then we can naturally split off $\mathtt{id}(-)$ as a summand of $(-)^{S^1}$ and identify the complement with $\Omega(-)$. The desuspension map is by definition the projection

$$\mathrm{des} : (-)^{S^1} \to \Omega(-). \tag{3.9}$$

Under the equivalence (3.7) in the case $\mathbf{C} = \mathbf{Fun}^{\mathrm{desc}}(\mathbf{Reg}_{\mathbb{Z}}, \mathbf{Sp})$ it induces the desuspension map in cohomology mentioned above.

The integration of forms gives morphisms of sheaves with values in $\mathbf{Ch}$

$$\int_{S^1} : S^1\,\mathbf{IDR} \to \mathbf{IDR}[-1], \quad \int_{S^1} : S^1\sigma^{\geq k}\,\mathbf{IDR} \to \sigma^{\geq k-1}\,\mathbf{IDR}[-1]$$

which, when assembled for the various $k \in \mathbb{Z}$, after application of the Eilenberg–MacLane functor H, yield the commutative diagram

$$\begin{tikzcd} H(S^1\sigma^{\geq\bullet}\,\mathbf{IDR}) \arrow[r, "H(\int_{S^1})"] \arrow[d] & \Omega H(\sigma^{\geq\bullet-1}\,\mathbf{IDR}) \arrow[d] \\ H(S^1\,\mathbf{IDR}[z,z^{-1}]) \arrow[r, "H(\int_{S^1})"] & \Omega H(\mathbf{IDR}[z,z^{-1}]) \end{tikzcd} \tag{3.10}$$

in $\mathbf{Mod}(\widehat{\mathbf{K}}^{(\bullet)})$, where $\widehat{\mathbf{K}}^{(\bullet)}$ acts via the curvature map. From the naturality of the desuspension we get the commutative diagram

$$\begin{tikzcd} S^1\mathbf{K}[z,z^{-1}] \arrow[r, "\mathrm{des}"] \arrow[d, "\mathtt{r}^{\mathrm{Beil}}"] & \Omega\mathbf{K}[z,z^{-1}] \arrow[d, "\Omega\mathtt{r}^{\mathrm{Beil}}"] \\ S^1H(\mathbf{IDR}[z,z^{-1}]) \arrow[r, "\mathrm{des}"] & \Omega H(\mathbf{IDR}[z,z^{-1}]) \end{tikzcd} \tag{3.11}$$

in $\mathbf{Mod}(\widehat{\mathbf{K}}^{(\bullet)})$, where here $\widehat{\mathbf{K}}^{(\bullet)}$ acts via I.

Lemma 3.12. *We have a natural equivalence of morphisms*

$$\mathrm{des} \simeq H\big(\textstyle\int_{S^1}\big) : S^1H(\mathbf{IDR}[z,z^{-1}]) \to \Omega H(\mathbf{IDR}[z,z^{-1}])$$

in $\mathbf{Mod}(\widehat{\mathbf{K}}^{(\bullet)})$.

Before proving this lemma we finish the argument for Proposition 3.6. Together with (3.11), Lemma 3.12 provides the lower square of the following diagram in $\mathbf{Mod}(\widehat{\mathbf{K}}^{(\bullet)})$:

$$\begin{tikzcd} H(S^1\sigma^{\geq\bullet}\,\mathbf{IDR}) \arrow[r, "H(\int_{S^1})"] \arrow[d] & \Omega H(\sigma^{\geq\bullet-1}\,\mathbf{IDR}) \arrow[d] \\ H(S^1\,\mathbf{IDR}[z,z^{-1}]) \arrow[r, "H(\int_{S^1})"] & \Omega H(\mathbf{IDR}[z,z^{-1}]) \\ S^1\mathbf{K}[z,z^{-1}] \arrow[r, "\mathrm{des}"] \arrow[u, "\mathtt{r}^{\mathrm{Beil}}"] & \Omega\mathbf{K}[z,z^{-1}] \arrow[u, "\Omega\mathtt{r}^{\mathrm{Beil}}"'] \end{tikzcd}$$

The upper square is (3.10). In view of the definition of $\widehat{\mathbf{K}}^{(\bullet)}$ as a pullback, this diagram induces a map

$$\int_{S^1} : S^1\widehat{\mathbf{K}}^{(\bullet)} \to \Omega\,\widehat{\mathbf{K}}^{(\bullet)}$$

in $\mathbf{Mod}(\widehat{\mathbf{K}}^{(\bullet)})$. It induces the asserted integration map in cohomology. $\square$

Proof of Lemma 3.12. We have a natural equivalence in $\mathbf{Mod}(\mathbf{IDR}[z, z^{-1}])$

$$\mathbf{IDR}[z, z^{-1}][-1] \oplus \mathbf{IDR}[z, z^{-1}] \xrightarrow{\sim} S^1\, \mathbf{IDR}[z, z^{-1}],$$

given on $M \times X$ by $\omega \oplus \eta \mapsto dt \wedge \mathtt{pr}^* \omega + \mathtt{pr}^* \eta$, where t is the coordinate on S^1 and $\mathtt{pr} : S^1 \times M \times X \to M \times X$ is the projection. An explicit inverse is given by $\left(\int_{S^1}, i^*\right)$, where $i : M \times X \to S^1 \times M \times X$ is induced by the inclusion of a point in S^1. In view of the definition of the desuspension in (3.9) and the equivalence (3.8), we can identify the desuspension for $\mathbf{IDR}[z, z^{-1}]$ naturally with the map $\int_{S^1} : S^1\, \mathbf{IDR}[z, z^{-1}] \to \mathbf{IDR}[z, z^{-1}][-1]$. Now the result follows by applying H. □

4. A secondary Steinberg relation

4A. *Units.* Let R be a ring such that $X = \mathtt{Spec}(R) \in \mathbf{Reg}_{\mathbb{Z}}$. We have a natural homomorphism

$$c : R^\times \to \mathbf{K}^{-1}(X), \tag{4.1}$$

where we write $\mathbf{K}^{-1}(X)$ instead of $\mathbf{K}^{-1}(* \times X)$. Concretely, c is given as follows: For $\lambda \in R^\times$ we let $\mathcal{V}(\lambda)$ be the bundle on $S^1 \times X$ which restricts to the trivial bundle $\mathbb{O}_X$ at any point $t \in S^1$ and has holonomy λ along S^1. Then

$$\mathtt{cycl}(\mathcal{V}(\lambda)) = c(\lambda) \oplus 1 \in \mathbf{K}^0(S^1 \times X) \cong \mathbf{K}^{-1}(X) \oplus \mathbf{K}^0(X). \tag{4.2}$$

Since the kernel of the map $I : \widehat{\mathbf{K}}^{-1}(X) \to \mathbf{K}^{-1}(X)$ is a divisible abelian group, there exists a lift $\hat{c} : R^\times \to \widehat{\mathbf{K}}^{-1}(X)$ of c. In the following, we will fix a specific choice of this lift.

We first construct a geometry $(h^{(\lambda)}, \nabla^{(\lambda)})$ on $\mathcal{V}(\lambda)$. Abusing notation, we also denote the complex line bundle on $S^1 \times X(\mathbb{C})$ associated with $\mathcal{V}(\lambda)$ by the same symbol and view λ as a nowhere-vanishing function on $X(\mathbb{C})$. Let t be a parameter on S^1 and $\log(\lambda)$ a local choice of a logarithm of λ on $X(\mathbb{C})$. Then $\phi = \lambda^t$ is a local section of $\mathcal{V}(\lambda)$ which depends on the choice of logarithm. The metric and the connection are determined by their value on the local sections ϕ. We set

$$\begin{aligned} h^{(\lambda)}(\phi) &= 1, \\ \nabla^{(\lambda)}(\phi) &= \log(\lambda)\phi\, dt. \end{aligned} \tag{4.3}$$

These are well defined. Moreover, $\nabla^{(\lambda)}$ has holonomy λ along S^1 and $[\nabla, \bar{\partial}] = 0$. We equip $\mathcal{V}(\lambda)$ with the canonical extended geometry, denoted by $g(\lambda)$.

Definition 4.4. We define $\hat{c} : R^\times \to \widehat{\mathbf{K}}^{-1}(X)$ to be the composition

$$\hat{c} : R^\times \xrightarrow{\lambda \mapsto \widehat{\mathtt{cycl}}(\mathcal{V}(\lambda), g(\lambda))} \widehat{\mathbf{K}}^0(S^1 \times X) \xrightarrow{\int_{S^1}} \widehat{\mathbf{K}}^{-1}(X).$$

Lemma 4.5. *The curvature $R(\hat{c}(\lambda)) \in Z^{-1}(\mathbf{IDR}(X))$ is given by*

$$\begin{aligned} R(\hat{c}(\lambda)) &= R(\hat{c}(\lambda))(1) \\ &= i d\arg(\lambda) + d\log(|\lambda|^u) \in Z^{-1}(\mathbf{IDR}(1)(X)) \subset A^1(I \times X(\mathbb{C})), \end{aligned}$$

where u is the coordinate on the interval I. The induced map

$$\hat{c} : R^\times \to \widehat{\mathbf{K}}^{-1}(X)/a(H^{-2}(\mathbf{IDR}(X)))$$

is a homomorphism.

Proof. For the adjoint connection of $\nabla^{(\lambda)}$ we get

$$\nabla^{(\lambda),*}\phi = -\log(\bar{\lambda})\phi\, dt.$$

Hence the connection of the canonical extended geometry is given by

$$\widetilde{\nabla}^{(\lambda)}\phi = \bigl(\tfrac{1}{2}(1-u)(\log(\lambda) - \log(\bar{\lambda})) + u\log(\lambda)\bigr)\phi\, dt.$$

Together with (4.3) this implies that for two units $\lambda, \mu \in R^\times$ we have

$$(\mathcal{V}(\lambda\mu), g(\lambda\mu)) \cong (\mathcal{V}(\lambda), g(\lambda)) \otimes (\mathcal{V}(\mu), g(\mu)).$$

By the multiplicativity of the geometric cycle map we get

$$\widehat{\mathtt{cycl}}(\mathcal{V}(\lambda\mu), g(\lambda\mu)) = \widehat{\mathtt{cycl}}(\mathcal{V}(\lambda), g(\lambda)) \cup \widehat{\mathtt{cycl}}(\mathcal{V}(\mu), g(\mu)).$$

For the curvature we get

$$R^{\widetilde{\nabla}^{(\lambda)}} = -i dt \wedge d\arg(\lambda) - dt \wedge d\log(|\lambda|^u).$$

Hence

$$\begin{aligned} R(\widehat{\mathtt{cycl}}(\mathcal{V}(\lambda), g(\lambda))) &= 1 \oplus (i dt \wedge d\arg(\lambda) + dt \wedge d\log(|\lambda|^u)) \\ &\in Z^0(\mathbf{IDR}(0)(S^1 \times X)) \oplus Z^0(\mathbf{IDR}(1)(S^1 \times X)). \end{aligned}$$

Integration over S^1 kills the first summand and gives the statement about the curvature.

From the formula for the curvature and the fact that $c = I \circ \hat{c}$ (see (4.1)) is a homomorphism, we get

$$R(\hat{c}(\lambda\mu)) = R(\hat{c}(\lambda)) + R(\hat{c}(\mu)), \quad I(\hat{c}(\lambda\mu)) = I(\hat{c}(\lambda)) + I(\hat{c}(\mu)),$$

hence $\hat{c}(\lambda\mu) - \hat{c}(\lambda) - \hat{c}(\mu) \in a(H^{-2}(\mathbf{IDR}(X)))$. $\square$

4B. ***The Steinberg relation and the Bloch–Wigner function.*** In this subsection we explain how differential algebraic K-theory can be used to give a simple proof of a result of Bloch [2000] concerning the existence of classes in K_3 of a number ring whose regulator can be described in terms of the Bloch–Wigner dilogarithm function. The key ingredient is a secondary version of the Steinberg relation.

We begin by collecting some notation necessary to state the result. Recall the definition of the polylogarithm functions

$$\mathtt{Li}_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$$

for $k \geq 1$ and $|z| < 1$. They extend meromorphically to a covering of $\mathbb{C} \setminus \{1\}$.

Definition 4.6. The Bloch–Wigner function is the real-valued function on $\mathbb{C}$ given by

$$D^{\mathrm{BW}}(\lambda) := \log|\lambda| \arg(1-\lambda) + \operatorname{Im}\mathtt{Li}_2(\lambda)$$

(see [Zagier 2007, Chapter I, §3]).

Let R be a ring.

Definition 4.7. We write $R^\circ := \{\lambda \in R^\times \mid 1-\lambda \in R^\times\}$. The third *Bloch group* $\mathcal{B}_3(R)$ is defined as the kernel

$$\mathcal{B}_3(R) := \mathtt{ker}\big(\mathbb{Z}[R^\circ] \xrightarrow{\lambda\mapsto\lambda\wedge(1-\lambda)} R^\times \wedge R^\times\big).$$

Now let R be the ring of integers in a number field and $X := \mathtt{Spec}(R)$. The target of the regulator $\mathtt{r}^{\mathrm{Beil}}$ on $\mathbf{K}^{-3}(X)$ is $H^{-3}(\mathbf{IDR}(X))$. Since $X(\mathbb{C})$ is zero-dimensional we have

$$\begin{aligned} H^{-3}(\mathbf{IDR}(X)) &\cong H^{-3}(\mathbf{IDR}(2)(X)) \\ &\underset{q}{\cong} H^{-3}(\mathbf{DR}(2)(X)) \\ &\cong [2\pi i\mathbb{R}^{X(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}. \end{aligned} \tag{4.8}$$

Theorem 4.9 (Bloch). *For any* $x = \sum_{\lambda\in R^\circ} n_\lambda[\lambda] \in \mathcal{B}_3(R)$, *there exists an element* $\mathrm{bl}(x) \in \mathbf{K}^{-3}(X)$ *such that*, *under the identification* (4.8),

$$\mathtt{r}^{\mathrm{Beil}}(\mathrm{bl}(x)) = -\sum_\lambda n_\lambda \big(i D^{\mathrm{BW}}(\sigma(\lambda))\big)_{\sigma\in X(\mathbb{C})}.$$

Example 4.10. Assume that $n \in \mathbb{N}$, $n \geq 2$ and $\lambda \in R^*$ satisfies

$$\lambda^{n+1} - \lambda + 1 = 0.$$

Then $\frac{1}{1-\lambda} \in R^\circ$ and we consider the element $x := n[\lambda] + \left[\frac{1}{1-\lambda}\right] \in \mathbb{Z}[R^\circ]$. We claim that $x \in \mathcal{B}_3(R)$. Indeed,

$$\begin{aligned} n(\lambda \wedge (1-\lambda)) + \frac{1}{1-\lambda} \wedge \Big(1 - \frac{1}{1-\lambda}\Big) &= n(\lambda \wedge (1-\lambda)) + \frac{1}{1-\lambda} \wedge \frac{\lambda}{\lambda - 1} \\ &= \lambda^n \wedge (1-\lambda) + (1-\lambda) \wedge \frac{\lambda-1}{\lambda} \\ &= \frac{\lambda-1}{\lambda} \wedge (1-\lambda) + (1-\lambda) \wedge \frac{\lambda-1}{\lambda} \\ &= 0. \end{aligned}$$

We get an element $\mathrm{bl}(x) \in \mathbf{K}_3(R)$ such that

$$\mathtt{r}^{\mathrm{Beil}}(2)(\mathrm{bl}(x)) = (n+1)\big(-i D^{\mathrm{BW}}(\sigma(\lambda))\big)_{\sigma \in \mathtt{Spec}(R)(\mathbb{C})},$$

where we use that $D^{\mathrm{BW}}\big(\frac{1}{1-\lambda}\big) = D^{\mathrm{BW}}(\lambda)$. If $\sigma(\lambda)$ is not real, then $D^{\mathrm{BW}}(\sigma(\lambda))$ is not zero.

Proof of Theorem 4.9. Since $X(\mathbb{C})$ is zero-dimensional we have $H^{-2}(\mathbf{IDR}(X)) = 0$. Hence, by Lemma 4.5, the map $\hat{c} : R^\times \to \widehat{\mathbf{K}}^{-1}(X)$ is a homomorphism. Since $\bigoplus_{k \in \mathbb{Z}} \widehat{\mathbf{K}}^k(X)$ is graded commutative, we get an induced map $R^\times \wedge R^\times \to \widehat{\mathbf{K}}^{-2}(X)$, $\lambda \wedge \mu \mapsto \hat{c}(\lambda) \cup \hat{c}(\mu)$.

If $\lambda \in R^\circ$, then the Steinberg relation implies that

$$I(\hat{c}(\lambda) \cup \hat{c}(1-\lambda)) = c(\lambda) \cup c(1-\lambda) = 0 \quad \text{in } \mathbf{K}^{-2}(X).$$

Consider the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B}_3(R) \arrow[r] \arrow[d, dotted, "\mathrm{bl}"'] & \mathbb{Z}[R^\circ] \arrow[r, "\lambda \mapsto \lambda \wedge (1-\lambda)"] \arrow[d, dotted, "\mathcal{D}"'] & R^\times \wedge R^\times \arrow[d] & \\ 0 \arrow[r] & \mathbf{K}^{-3}(X)/\mathtt{ker}(\mathtt{r}^{\mathrm{Beil}}) \arrow[r, "\mathtt{r}^{\mathrm{Beil}}"] & \mathbf{IDR}^{-3}(X)/\mathtt{im}(d) \arrow[r, "a"] & \widehat{\mathbf{K}}^{-2}(X) \arrow[r, "I"] & \mathbf{K}^{-2}(X) \end{tikzcd} \tag{4.11}$$

The dotted arrow $\mathcal{D}$ exists by the Steinberg relation and since $\mathbb{Z}[R^\circ]$ is a free abelian group. The dotted arrow bl is the induced map on kernels.

We will now pin down a specific choice for $\mathcal{D}$ which will then imply the theorem. To do this, we consider the universal situation. Let

$$\mathbb{X} := \mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\} \cong \mathtt{Spec}(\mathbb{Z}[\lambda, \lambda^{-1}, (1-\lambda)^{-1}]).$$

We consider $\hat{c}(\lambda) \cup \hat{c}(1-\lambda) \in \widehat{\mathbf{K}}^{-2}(\mathbb{X})$. Again, by the Steinberg relation there exists $\mathcal{D}(\lambda) \in \mathbf{IDR}^{-3}(\mathbb{X})/\mathtt{im}(d)$ such that $a(\mathcal{D}(\lambda)) = \hat{c}(\lambda) \cup \hat{c}(1-\lambda)$. Since $R \circ a = d$, we must have

$$d(\mathcal{D}(\lambda)) = R(\hat{c}(\lambda)) \cup R(\hat{c}(1-\lambda)) \in \mathbf{IDR}^{-2}(\mathbb{X}). \tag{4.12}$$

Because we want to specialize to number rings later on, we are only interested in the component $\mathcal{D}(\lambda)(2) \in \mathbf{IDR}(2)^{-3}(\mathbb{X})$ (see (4.8)) This is determined by (4.12) up to elements in $H^{-3}(\mathbf{IDR}(2)(\mathbb{X}))$. Since $\mathcal{F}^2 A(I \times \mathbb{X}(\mathbb{C})) = 0$ we have quasi-isomorphisms

$$\begin{aligned}\mathbf{IDR}(2)(\mathbb{X}) &\underset{q}{\simeq} \mathbf{DR}(2)(\mathbb{X}) \\ &\cong \Big(\mathtt{Cone}\big((2\pi i)^2 A_{\mathbb{R}}(\mathbb{X}(\mathbb{C})) \to A(\mathbb{X}(\mathbb{C}))\big)[3]\Big)^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})} \\ &\cong \big((2\pi i) A_{\mathbb{R}}(\mathbb{X}(\mathbb{C}))[3]\big)^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})},\end{aligned} \tag{4.13}$$

where the last isomorphism is induced by taking i times the imaginary part on the second component of the cone. In particular,

$$H^{-3}(\mathbf{IDR}(2)(\mathbb{X})) = H^0(\mathbb{X}(\mathbb{C}), (2\pi i)\mathbb{R})^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})} = 0.$$

We now compute the right-hand side of (4.12). From Lemma 4.5 we get

$$\begin{aligned}&i \operatorname{Im}\big(R(\hat{c}(\lambda)) \cup R(\hat{c}(1-\lambda))\big) \\ &\qquad = i\, d \arg(\lambda) \wedge d \log(|1-\lambda|^u) + i\, d \log(|\lambda|^u) \wedge d \arg(1-\lambda).\end{aligned}$$

Hence, under the quasi-isomorphisms (4.13), $R(\hat{c}(\lambda)) \cup R(\hat{c}(1-\lambda))$ is mapped to

$$i \log(|1-\lambda|) d \arg(\lambda) - i \log(|\lambda|) d \arg(1-\lambda) \in \big((2\pi i) A^1_{\mathbb{R}}(\mathbb{X}(\mathbb{C}))\big)^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.$$

On the other hand, using $(d/dz)\,\mathtt{Li}_2(z) = (1/z)\,\mathtt{Li}_1(z) = -(1/z)\log(1-z)$ we get

$$\begin{aligned}dD^{\mathrm{BW}}(\lambda) &= \arg(1-\lambda) d \log(|\lambda|) + \log(|\lambda|) d \arg(1-\lambda) - \operatorname{Im} \log(1-\lambda) d \log(\lambda) \\ &= \log(|\lambda|) d \arg(1-\lambda) - \log(|1-\lambda|) d \arg(\lambda).\end{aligned}$$

It follows that, under the quasi-isomorphisms (4.13),

$$\mathcal{D}(\lambda)(2) = -i D^{\mathrm{BW}}(\lambda).$$

We now return to the number ring R. Note that in diagram (4.11) we may identify

$$\begin{aligned}\mathbf{IDR}^{-3}(X)/\operatorname{\mathtt{im}}(d) &= H^{-3}(\mathbf{IDR}(2)(X)) \\ &\cong \big((2\pi i) A^0_{\mathbb{R}}(X(\mathbb{C}))\big)^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})} \\ &= [2\pi i \mathbb{R}^{X(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.\end{aligned}$$

Any $\lambda \in R^\circ$ corresponds to a unique morphism $\lambda : X \to \mathbb{X}$, which on $\mathbb{C}$-valued points is given by $X(\mathbb{C}) \to \mathbb{X}(\mathbb{C}) = \mathbb{C}^\times \setminus \{1\}$, $\sigma \mapsto \sigma(\lambda)$. We construct $\mathcal{D}(\lambda) \in [2\pi i \mathbb{R}^{X(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}$ by pulling back along λ from the universal case on $\mathbb{X}$. Explicitly, we get

$$\mathcal{D}(\lambda) = \big(-i D^{\mathrm{BW}}(\sigma(\lambda))\big)_{\sigma \in X(\mathbb{C})}.$$

This implies the formula for bl stated in the theorem. $\square$

5. A height invariant for number rings

Let R be the ring of integers in a number field. We recall the following definition from Arakelov geometry:

Definition 5.1. A metrized line bundle $(\mathscr{L}, h^{\mathscr{L}})$ on $\mathtt{Spec}(R)$ is an invertible sheaf $\mathscr{L}$ on $\mathtt{Spec}(R)$ with a $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-invariant metric $h^{\mathscr{L}}$ on its complexification. We let $\widehat{\mathtt{Pic}}(\mathtt{Spec}(R))$ denote the multiplicative group of isomorphism classes of metrized line bundles under the tensor product and call it the arithmetic Picard group of R.

We may identify $\mathscr{L}$ with its R-module of global sections. A metric $h^{\mathscr{L}}$ is then given by a collection of metrics $h^{\mathscr{L}}_\sigma$ on $\mathscr{L} \otimes_{R,\sigma} \mathbb{C}$ for all $\sigma \in \mathtt{Spec}(R)(\mathbb{C})$ which is invariant under the $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-action.

An important invariant is the arithmetic degree

$$\widehat{\deg} : \widehat{\mathtt{Pic}}(\mathtt{Spec}(R)) \to \mathbb{R},$$

defined as follows (see [Lang 1988, IV, §3]): Let $(\mathscr{L}, h^{\mathscr{L}})$ be a metrized line bundle. Then

$$\widehat{\deg}((\mathscr{L}, h^{\mathscr{L}})) := \frac{1}{[K:\mathbb{Q}]}\Bigg(\log(\#(\mathscr{L}/R\cdot s)) - \frac{1}{2}\sum_{\sigma\in\mathtt{Spec}(R)(\mathbb{C})} \log(h_\sigma(s))\Bigg), \tag{5.2}$$

where $s \in \mathscr{L} \setminus \{0\}$ is any nonzero section.

The main aim of this section is to explain how the arithmetic Picard group and the arithmetic degree can be naturally understood in the framework of differential algebraic K-theory (see Theorem 5.8).

5A. ***Scaling the metric.*** Let M be a smooth manifold and $X \in \mathbf{Reg}_{\mathbb{Z}}$. We consider a geometric bundle (V, g), $g := (h^V, \nabla^{II})$, on $M \times X$ and let $f \in C^\infty(M \times X(\mathbb{C}))$ be a $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-invariant positive smooth function. Then we can consider the rescaled metric fh^V and geometry $g_f := (fh^V, \nabla^{II})$. In the following we work with the canonical extensions $\mathtt{can}(g)$ (see Definition 2.19) of the geometries. We are interested in the difference

$$\widehat{\mathtt{cycl}}(V, \mathtt{can}(g_f)) - \widehat{\mathtt{cycl}}(V, \mathtt{can}(g)) \in \widehat{\mathbf{K}}(X)^0(M).$$

Note that this difference is equal to $a(\alpha)$ for some $\alpha \in \mathbf{IDR}^{-1}(M \times X)/\operatorname{im}(d)$, where α is well-defined up to the image of $\mathtt{r}^{\mathrm{Beil}}$. We want to calculate α. To this end we use the homotopy formula [Bunke and Tamme 2015, Lemma 5.11]. We consider the bundle $\widehat{V} := \mathtt{pr}^* V$, where $\mathtt{pr} : [0, 1] \times M \times X \to M \times X$ is the projection. It is equipped with the geometry $\hat{g} := (\hat{h}, \mathtt{pr}^* \nabla^{II})$, $\hat{h} := (1 - x + xf)h$, where $x \in [0, 1]$ is the coordinate. By the homotopy formula we can take

$$\begin{aligned}\alpha &= \int_{[0,1]\times[0,1]\times M\times X/[0,1]\times M\times X} R(\widehat{\mathtt{cycl}}(\widehat{V}, \mathtt{can}(\hat{g}))) \\ &= \int_{[0,1]\times[0,1]\times M\times X/[0,1]\times M\times X} \widetilde{\omega}(\mathtt{can}(\hat{g})).\end{aligned}$$

For us, the most important case is the following (see [Bunke and Tamme 2015, Lemma 5.13]):

Lemma 5.3. *If* $\dim(M) = 0$ *and* $\dim(X(\mathbb{C})) = 0$, *we can take*

$$\alpha = \alpha(1) = -\tfrac{1}{2}\mathtt{rk}(V)\log(f)du.$$

Proof. We have $\widetilde{\omega}(\hat{g})(p) = 0$ for all p except $p = 0, 1$. In fact we have

$$\widetilde{\omega}(\hat{g})(0) \equiv \mathtt{rk}(V),$$

hence $\alpha(0) = 0$. In order to calculate $\widetilde{\omega}(\hat{g})(1)$, we first observe that

$$\widetilde{\nabla} = d + \frac{1-u}{2}\frac{(f-1)dx}{1+(f-1)x}.$$

We get

$$\widetilde{\omega}(\hat{h})(1) = \frac{\mathtt{rk}(V)}{2}\frac{(f-1)}{1+(f-1)x}du \wedge dx,$$

and therefore

$$\alpha = \alpha(1) = -\tfrac{1}{2}\mathtt{rk}(V)\log(f)du. \qquad \square$$

5B. ***The absolute height for number rings.*** We consider a ring of integers R in a number field K. Note that $\mathtt{Spec}(R)$ is regular, separated and of finite type over $\mathtt{Spec}(\mathbb{Z})$. We define the multiplicative subgroup

$$\mathbf{K}^0(\mathtt{Spec}(R))_{(1)} := \{x \in \mathbf{K}^0(\mathtt{Spec}(R)) \mid 1 - x \text{ is nilpotent}\}$$

of the group of units in the ring $\mathbf{K}^0(\mathtt{Spec}(R))$. It is known that

$$\mathbf{K}^0(\mathtt{Spec}(R)) \cong \mathbb{Z} \oplus \mathtt{Cl}(R),$$

where $\mathtt{Cl}(R)$ denotes the finite class group. Therefore

$$\mathbf{K}^0(\mathtt{Spec}(R))_{(1)} \cong \{1 + x \mid x \in \mathtt{Cl}(R)\} \cong \mathtt{Cl}(R)$$

is finite. We furthermore define

$$\widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)} := I^{-1}(\mathbf{K}^0(\mathtt{Spec}(R))_{(1)}) \subseteq \widehat{\mathbf{K}}^0(\mathtt{Spec}(R)).$$

If $x \in \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)}$, then necessarily $R(x) = R(\mathbf{1})$. Hence we have an exact sequence

$$0 \to H^{-1}(\mathbf{IDR}(\mathtt{Spec}(R)))/\,\mathtt{im}(\mathtt{r}^{\mathrm{Beil}}) \xrightarrow{1+a} \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)} \to \mathbf{K}^0(\mathtt{Spec}(R))_{(1)} \to 0. \quad (5.4)$$

We now define an absolute height function

$$h : \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)} \to \mathbb{R}$$

for number rings R. We will relate h with the arithmetic degree of metrized line bundles in the next subsection.

Note that

$$H^{-1}(\mathbf{IDR}(\mathtt{Spec}(R))) \cong H^{-1}(\mathbf{IDR}(1)(\mathtt{Spec}(R))) \cong [\mathbb{R}^{\mathtt{Spec}(R)(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.$$

Explicitly, a class $[\alpha] \in H^{-1}(\mathbf{IDR}(1)(\mathtt{Spec}(R)))$ which is represented by

$$\alpha \in \mathbf{IDR}(1)^{-1}(\mathtt{Spec}(R)) \subseteq A^1([0, 1] \times \mathtt{Spec}(R)(\mathbb{C}))$$

corresponds to the function

$$\mathtt{Spec}(R)(\mathbb{C}) \to \mathbb{R}, \quad \sigma \mapsto \mathrm{Re}\Big(\textstyle\int_{[0,1]} \sigma^*\alpha\Big). \quad (5.5)$$

We define a linear map

$$s : [\mathbb{R}^{\mathtt{Spec}(R)(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})} \to \mathbb{R}, \quad s(f) := \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma \in \mathtt{Spec}(R)(\mathbb{C})} f(\sigma).$$

Then $s \circ \mathtt{r}^{\mathrm{Beil}}(1) = 0$. In this way we get a homomorphism

$$h : H^{-1}(\mathbf{IDR}(\mathtt{Spec}(R)))/\,\mathtt{im}(\mathtt{r}^{\mathrm{Beil}}) \to \mathbb{R}, \quad h([f]) := s(f). \quad (5.6)$$

In view of (5.4) and since $\mathbf{K}^0(\mathtt{Spec}(R))_{(1)}$ is finite, the homomorphism (5.6) has a unique extension to $\widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)}$. Explicitly, if $x \in \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)}$, then there exists $N \in \mathbb{N}$ such that $x^N = 1 + a(f)$ for some $f \in H^{-1}(\mathbf{IDR}(\mathtt{Spec}(R)))$ and $h(x)$ is given by

$$h(x) = \frac{1}{N} h(1 + a(f)).$$

5C. ***The degree of metrized line bundles.*** We let R be the ring of integers in a number field K. We consider the trivial bundle $\mathcal{V} := \mathcal{O}_{\mathtt{Spec}(R)}$ with the canonical geometry g_0. Then

$$\widehat{\mathtt{cycl}}(\mathcal{V}, \mathtt{can}(g_0)) = \mathbf{1}.$$

Let $f : \mathtt{Spec}(R)(\mathbb{C}) \to \mathbb{R}^+$ be $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-invariant and form the geometry with rescaled metric $g_{0,f}$ as in Section 5A. Then

$$\widehat{\mathtt{cycl}}(\mathcal{V}, \mathtt{can}(g_{0,f})) \in \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)}.$$

Lemma 5.7. *We have*

$$h(\widehat{\mathtt{cycl}}(\mathcal{V}, \mathtt{can}(g_{0,f}))) = -\frac{1}{2[K:\mathbb{Q}]} \sum_{\sigma \in \mathtt{Spec}(R)(\mathbb{C})} \log(f(\sigma)).$$

Proof. Use (5.5) and Lemma 5.3. □

If $(\mathcal{L}, h^{\mathcal{L}}) \in \widehat{\mathtt{Pic}}(\mathtt{Spec}(R))$, then we have a canonical extended geometry $\mathtt{can}(h^{\mathcal{L}})$ on $\mathcal{L}$ and can form

$$\hat{c}(\mathcal{L}, h^{\mathcal{L}}) := \widehat{\mathtt{cycl}}(\mathcal{L}, \mathtt{can}(h^{\mathcal{L}})) \in \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)}.$$

Theorem 5.8. *The map* $\hat{c} : \widehat{\mathtt{Pic}}(\mathtt{Spec}(R)) \to \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)}$ *is an isomorphism. Furthermore, for any metrized line bundle* $(\mathcal{L}, h^{\mathcal{L}})$ *we have*

$$\widehat{\deg}(\mathcal{L}, h^{\mathcal{L}}) = h(\hat{c}(\mathcal{L}, h^{\mathcal{L}})).$$

Proof. Since all connections involved are trivial, we have

$$\mathtt{can}(h^{\mathcal{L}} \otimes h^{\mathcal{L}'}) = \mathtt{can}(h^{\mathcal{L}}) \otimes \mathtt{can}(h^{\mathcal{L}'}).$$

Thus $\hat{c}$ is a group homomorphism.

There is a natural map $[\mathbb{R}^{\mathtt{Spec}(R)(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})} \to \widehat{\mathtt{Pic}}(\mathtt{Spec}(R))$ which sends the tuple $\lambda = (\lambda_\sigma)$ to the trivial line bundle R with the metric $h^{(\lambda)}$ given by $h^{(\lambda)}_\sigma(1) = \exp(-2\lambda_\sigma)$. Recall that

$$H^{-1}(\mathbf{IDR}(\mathtt{Spec}(R))) \cong [\mathbb{R}^{\mathtt{Spec}(R)(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.$$

We claim that we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}(\mathbf{IDR}(\mathtt{Spec}(R)))/\mathtt{im}(\mathtt{r}^{\mathrm{Beil}}) \arrow[r] \arrow[d, equal] & \widehat{\mathtt{Pic}}(\mathtt{Spec}(R)) \arrow[r] \arrow[d, "\hat{c}"] & \mathtt{Pic}(\mathtt{Spec}(R)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H^{-1}(\mathbf{IDR}(\mathtt{Spec}(R)))/\mathtt{im}(\mathtt{r}^{\mathrm{Beil}}) \arrow[r] & \widehat{\mathbf{K}}^0(\mathtt{Spec}(R))_{(1)} \arrow[r] & \mathbf{K}^0(\mathtt{Spec}(R))_{(1)} \arrow[r] & 0
\end{tikzcd}$$

Indeed, the right vertical map is given by the topological cycle map, and it is known to be an isomorphism. The exactness of the upper row is straightforward, the lower row is (5.4). Finally, the commutativity of the left-hand square follows from Lemma 5.3.

In particular, $\hat{c}$ is an isomorphism.

For the second assertion, it suffices by the construction of h to check that for $\lambda = (\lambda_\sigma) \in [\mathbb{R}^{\mathtt{Spec}(R)(\mathbb{C})}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}$ we have

$$\widehat{\deg}(R, h^{(\lambda)}) = \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma \in \mathtt{Spec}(R)(\mathbb{C})} \lambda_\sigma .$$

But this is clear from the definition of $h^{(\lambda)}$ and (5.2) with $s = 1$. □

6. Formality of the algebraic K-theory of number rings

Let $M\mathbb{R}$ be the Moore spectrum of $\mathbb{R}$. For any spectrum E, we use the notation $E\mathbb{R} := E \wedge M\mathbb{R}$ for its realification.

Let $E \in \mathbf{CAlg}(\mathbf{Sp}^\wedge)$ be a commutative ring spectrum. Then we can form the differential graded commutative algebra $\pi_*(E\mathbb{R}) \in \mathbf{CAlg}(\mathbf{Ch}^\otimes)$ with trivial differentials. There is a unique equivalence class of maps

$$r : E \to H(\pi_*(E\mathbb{R}))$$

of spectra which induces the canonical realification map in homotopy.

Definition 6.1. The commutative ring spectrum E is called formal over $\mathbb{R}$ if r can be refined to a morphism of commutative ring spectra.

If $\pi_*(E\mathbb{R})$ is a free commutative $\mathbb{R}$-algebra, then E is formal over $\mathbb{R}$ (see [Bunke 2013] for an argument). This applies, e.g., to complex bordism **MU** or connective complex K-theory **ku**. From the formality of **ku** one can deduce the formality over $\mathbb{R}$ of periodic complex K-theory **KU**.

More generally, let $E \in \mathbf{Fun}(S, \mathbf{CAlg}(\mathbf{Sp}^\wedge))$ be a diagram of commutative ring spectra. It gives rise to a diagram $\pi_*(E\mathbb{R}) \in \mathbf{Fun}(S, \mathbf{CAlg}(\mathbf{Ch}^\otimes))$ of chain complexes with trivial differential.

Definition 6.2. We say that E is formal over $\mathbb{R}$ if there exists an equivalence $E\mathbb{R} \simeq H(\pi_*(E\mathbb{R}))$ of diagrams of commutative ring spectra which induces the identity on homotopy.

We let $\mathbf{S} \subseteq \mathbf{Reg}_\mathbb{Z}$ be the full subcategory whose objects are spectra of rings of integers in number fields.

Theorem 6.3. *The restriction of the sheaf of algebraic K-theory spectra* $\mathbf{K}$ *to* $\mathbf{S}$ *is formal over* $\mathbb{R}$.

Proof. We first show that the restriction of $H(\mathbf{IDR})$ to $\mathbf{S}$ is formal over $\mathbb{R}$. To this end we describe, for every ring of integers R in a number field K, canonical representatives of the cohomology of $\mathbf{IDR}(\mathtt{Spec}(R))$. We have

$$\mathbf{IDR}(\mathtt{Spec}(R))(p) \cong \left(\{\omega \in A(I)[2p] \mid \omega_{|\{0\}} \in (2\pi i)^p\mathbb{R}, \omega_{|\{1\}} = 0\}^{\mathtt{Spec}(R)(\mathbb{C})}\right)^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}$$

for $p \geq 1$, and

$$\mathbf{IDR}(\mathtt{Spec}(R))(0) \cong \big(\{\omega \in A(I) \mid \omega_{|\{0\}} \in \mathbb{R}\}^{\mathtt{Spec}(R)(\mathbb{C})}\big)^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}.$$

We have

$$H^*\big(\{\omega \in A(I)[2p] \mid \omega_{|\{0\}} \in (2\pi i)^p\mathbb{R}, \omega_{|\{1\}} = 0\}\big) \cong \begin{cases} i^{p+1}\mathbb{R}, & * = -2p+1, \\ 0, & \text{else,} \end{cases}$$

and

$$H^*\big(\{\omega \in A(I) \mid \omega_{|\{0\}} \in \mathbb{R}\}\big) \cong \begin{cases} \mathbb{R}, & * = 0, \\ 0, & \text{else.} \end{cases}$$

Explicit representatives of generators are given by $i^{p+1}dt$ (with t the coordinate of I) in the first case and 1 in the second. For real embeddings $\sigma \in \mathtt{Spec}(R)(\mathbb{C})$ and odd $p \in \mathbb{N}$, and for complex embeddings $\sigma \in \mathtt{Spec}(R)(\mathbb{C})$ and all $p \in \mathbb{N}_{>0}$, we define the following elements in $\mathbf{IDR}(\mathtt{Spec}(R))(p)$: for real σ,

$$x(\sigma)_{1-2p} := \left(\mathtt{Spec}(R)(\mathbb{C}) \ni \sigma' \mapsto \begin{cases} i^{p+1}dt, & \sigma' = \sigma, \\ 0, & \text{else,} \end{cases}\right) \in \mathbf{IDR}(\mathtt{Spec}(R))(p),$$

and for complex σ,

$$x(\sigma)_{1-2p} :=$$
$$\left(\mathtt{Spec}(R)(\mathbb{C}) \ni \sigma' \mapsto \begin{cases} i^{p+1}dt, & \sigma' = \sigma, \\ (-1)^{p+1}i^{p+1}dt, & \sigma' = \bar{\sigma}, \\ 0, & \text{else,} \end{cases}\right) \in \mathbf{IDR}(\mathtt{Spec}(R))(p).$$

We let $M'(R) \subseteq \mathbf{IDR}(\mathtt{Spec}(R))$ be the $\mathbb{R}$-submodule generated by the elements $x(\sigma)_{1-2p}$ for σ and p as above.

It is easy to see that the inclusion

$$H^*(\mathbf{IDR}(\mathtt{Spec}(R))) \cong \mathbb{R} \oplus M'(R) \subset \mathbf{IDR}(\mathtt{Spec}(R))$$

is a quasi-isomorphism of commutative differential graded algebras which is natural in R. We therefore get a morphism of diagrams of ring spectra

$$\mathtt{r}^{\mathrm{Beil}} : \mathbf{K}_{|\mathbf{S}} \to H(\mathbf{IDR})_{|\mathbf{S}} \simeq H(H^*(\mathbf{IDR})_{|\mathbf{S}}).$$

By Theorem 2.31 the induced map

$$\pi_*(\mathbf{K}_{|\mathbf{S}}) \otimes \mathbb{R} \to H^{-*}(\mathbf{IDR}_{|\mathbf{S}}) \tag{6.4}$$

coincides with Beilinson's regulator, which itself coincides up to a factor of 2 with Borel's regulator map [Burgos Gil 2002, Theorem 10.9]. By Borel's results [1974], (6.4) is injective, and the image is the kernel of the map

$$p : \mathbb{R} \oplus M'(R) \to \mathbb{R}, \quad b \mapsto \sum_{\sigma \in \mathtt{Spec}(R)(\mathbb{C})} n(\sigma)_{-1}(b),$$

where the $n(\sigma)_{-1}(b)$ are the coefficients of b in front of the generators $x(\sigma)_{-1}$. We define the subspace $M(R) := \mathtt{ker}(p) \cap M'(R)$. Then we can define a canonical splitting

$$M'(R) \to M(R), \quad b \mapsto b - \frac{p(b)}{[K:\mathbb{Q}]} \sum_{\sigma \in \mathtt{Spec}(R)(\mathbb{C})} x(\sigma)_{-1}.$$

It induces a canonical ring homomorphism $\mathbb{R} \oplus M'(R) \to \mathbb{R} \oplus M(R)$ which is left-inverse to the inclusion $\mathbb{R} \oplus M(R) \hookrightarrow \mathbb{R} \oplus M'(R)$ and therefore a map of diagrams of ring spectra $s : H(\mathbb{R} \oplus M') \to H(\mathbb{R} \oplus M)$ such that the composition

$$\mathbf{K}\mathbb{R}_{|\mathbf{S}} \xrightarrow{\mathtt{r}^{\mathrm{Beil}} \wedge M\mathbb{R}} H(\mathbb{R} \oplus M') \xrightarrow{s} H(\mathbb{R} \oplus M) \simeq H(\pi_*(\mathbf{K}_{|\mathbf{S}}))$$

is an equivalence of diagrams of commutative ring spectra. □

Observe that the structure of the homotopy groups of $\mathbf{K}(\mathtt{Spec}(R))\mathbb{R}$ implies that all Massey products are trivial. This can be considered as an A_∞-version of formality. The additional information given by Theorem 6.3 is that $\mathbf{K}(\mathtt{Spec}(R))$ is formal in the commutative sense and in a way which is natural in the ring R.

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ULRICH BUNKE: `ulrich.bunke@mathematik.uni-regensburg.de`
Fakultät für Mathematik, Universität Regensburg, D-93040 Regensburg, Germany

GEORG TAMME: `georg.tamme@mathematik.uni-regensburg.de`
Fakultät für Mathematik, Universität Regensburg, D-93040 Regensburg, Germany

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The Balmer spectrum of a tame stack

Jack Hall

Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. We classify the thick $\otimes$-ideals of $\mathsf{D}_{\mathrm{qc}}(X)^c$. If X is tame, then we also compute the Balmer spectrum of the $\otimes$-triangulated category of perfect complexes on X. In addition, if X admits a coarse space X_{cs}, then we prove that the Balmer spectra of X and X_{cs} are naturally isomorphic.

1. Introduction

Let X be a quasicompact and quasiseparated scheme. Let $\mathsf{Perf}(X)$ be the $\otimes$-triangulated category of perfect complexes on X. A celebrated result of Thomason [1997, Theorem 3.15], extending the work of Hopkins [1987, Section 4] and Neeman [1992a, Theorem 1.5], is a classification of the thick $\otimes$-ideals of $\mathsf{Perf}(X)$ in terms of the *Thomason* subsets of $|X|$, which are those subsets $Y \subseteq |X|$ expressible as a union $\cup_\alpha Y_\alpha$ such that $|X| \setminus Y_\alpha$ is quasicompact and open.

If X is a quasicompact and quasiseparated algebraic space, Deligne–Mumford stack, or algebraic stack, then it is also natural to consider the $\otimes$-triangulated category $\mathsf{Perf}(X)$ of perfect complexes on X (see [Hall and Rydh 2014, Section 4] for precise definitions).

In general, Thomason's classification of thick $\otimes$-ideals of $\mathsf{Perf}(X)$ fails for algebraic stacks (Example 3.2). If one instead works with the $\otimes$-ideal $\mathsf{D}_{\mathrm{qc}}(X)^c \subseteq \mathsf{Perf}(X)$ of *compact* perfect complexes, then the first main result of this article is that the classification goes through without change.

Theorem 1.1 (classification of thick $\otimes$-ideals). *If X is a quasicompact algebraic stack with quasifinite and separated diagonal, then there is a bijective and inclusion preserving correspondence between the thick $\otimes$-ideals of $\mathsf{D}_{\mathrm{qc}}(X)^c$ and the Thomason subsets of $|X|$.*

Some special cases of Theorem 1.1 are the following:

- If k is a field and G is a finite group, then $\mathsf{D}^b(\mathsf{Proj}\, kG)$ has no nontrivial $\otimes$-ideals.

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- If Y is a quasiprojective scheme over a field k with a proper action of an affine group scheme G, then the thick $\otimes$-ideals of $\mathsf{D}(\mathsf{QCoh}^G(Y))^c$ are in bijective correspondence with the G-invariant Thomason subsets of X.

The first special case is easy to prove directly and is well-known (for example, [Benson et al. 2011, Proposition 2.1]). In some sense, this makes our results orthogonal to those of [Benson et al. 2011]. The second special case was only known in characteristic 0 when Y was normal or quasi-affine [Krishna 2009, Theorem 7.8] or in characteristic p when G is finite of order prime to p and X is smooth [Dubey and Mallick 2012, Theorem 1.2].

We prove Theorem 1.1 using tensor nilpotence with parameters (Theorem 2.3), which extends [Thomason 1997, Theorem 3.8] and [Hopkins 1987, Theorem 10ii] (compare [Neeman 1992a, 1.1]) to quasicompact algebraic stacks with quasifinite and separated diagonal. As should be expected, stacks of the form $[Y/G]$, where Y is an affine variety over a field k and G is a finite group with order divisible by the characteristic of k, are the most troublesome. This is dealt with in Lemma 2.6, which relies on some results developed in Appendix A.

If $\mathcal{T}$ is a $\otimes$-triangulated category, then Balmer [2005] has functorially constructed from $\mathcal{T}$ a locally ringed space $\mathrm{Sp}_{\mathrm{Bal}}(\mathcal{T})$, the *Balmer spectrum*. A fundamental result of Balmer [2005, Theorem 5.5], which was extended by Buan, Krause and Solberg [Buan et al. 2007, Theorem 9.5] to the non-noetherian setting, is that if X is a quasicompact and quasiseparated scheme, then there is a naturally induced isomorphism

$$X \to \mathrm{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X)).$$

An algebraic stack is *tame* if its stabilizer groups at geometric points are finite linearly reductive group schemes [Abramovich et al. 2008, Definition 2.2]. Every scheme and algebraic space is tame. Moreover, in characteristic zero, a stack is Deligne–Mumford if and only if is tame. In characteristic $p > 0$, there are nontame Deligne–Mumford stacks (e.g., $B_{\mathbb{F}_p}(\mathbb{Z}/p\mathbb{Z})$) and tame stacks that are not Deligne–Mumford (e.g., $B_{\mathbb{F}_p}\mu_p$). Nagata's theorem [Hall and Rydh 2015, Theorem 1.2] provides a classification of finite linearly reductive group schemes over fields, which allows one to determine whether a given algebraic stack is tame. Our definition of tame stack is substantially weaker than that what appears in [Abramovich et al. 2008, Definition. 3.1] (see Appendix A).

Tame stacks are precisely those stacks with quasifinite diagonal such that the compact objects of $\mathsf{D}_{\mathrm{qc}}(X)$ coincide with the perfect complexes. In particular, for tame stacks $\mathsf{D}_{\mathrm{qc}}(X)^c$ contains a monoidal unit and so becomes a $\otimes$-triangulated category. Using Theorem 1.1, we extend the result of [Buan et al. 2007] to tame stacks.

Theorem 1.2. *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. If X is tame, then there is a natural isomorphism of locally ringed spaces*:

$$(|X|, \mathbb{O}_{X_{\mathrm{Zar}}}) \to \mathrm{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X)),$$

where $\mathbb{O}_{X_{\mathrm{Zar}}}$ is the Zariski sheaf $U \mapsto \Gamma(U, \mathbb{O}_X)$.

Theorem 1.2 implies that the Balmer spectrum cannot be used to reconstruct locally separated algebraic spaces [Knutson 1971, Example 2]. Balmer [2013] has recently initiated the study of unramified monoids in ⊗-triangulated categories and Neeman [2015] has classified them in the case of a separated noetherian scheme. It is hoped that a refinement of the Balmer spectrum can be constructed from unramified monoids, which would — at least — permit the reconstruction of algebraic spaces.

If X is an algebraic stack with finite inertia (e.g., a separated Deligne–Mumford stack), then X admits a *coarse space* $\pi : X \to X_{\mathrm{cs}}$ [Keel and Mori 1997; Rydh 2013], which is the universal map from X to an algebraic space. If X has finite inertia, then X has separated diagonal. Thus we can also establish the following.

Theorem 1.3. *Let X be a quasicompact, quasiseparated algebraic stack with finite inertia and coarse space $\pi : X \to X_{\mathrm{cs}}$. If X is tame, then*

$$\mathrm{Sp}_{\mathrm{Bal}}(\mathsf{L}\pi^*) : \mathrm{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X)) \to \mathrm{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X_{\mathrm{cs}}))$$

is an isomorphism of ringed spaces.

Krishna [2009, Theorem 7.10] proved Theorem 1.3 when X is of the form $[W/G]$, where W is quasiprojective and normal or quasi-affine, and G is a linear algebraic group in characteristic 0 acting properly on W. Dubey and Mallick [2012, Theorem 1.2] proved a similar result in positive characteristic, but required W to be smooth and G a finite group with order not divisible by the characteristic of the ground field. In particular, Theorem 1.3 is stronger than all existing results and Theorems 1.1 and 1.2 are new.

Assumptions and conventions. A priori, we make no separation assumptions on our algebraic stacks. However, all stacks used in this article will be, at the least, quasicompact and quasiseparated. Usually, they will also have separated diagonal. If X is an algebraic stack, then let $|X|$ denote its associated Zariski topological space [Laumon and Moret-Bailly 2000, Section 5]. For derived categories of algebraic stacks, we use the conventions and notations of [Hall and Rydh 2014, Section 1]. In particular, if X is an algebraic stack, then $\mathsf{Mod}(X)$ is the abelian category of $\mathbb{O}_X$-modules on the lisse-étale site of X and $\mathsf{D}_{\mathrm{qc}}(X)$ denotes the unbounded derived category of $\mathbb{O}_X$-modules with quasicoherent cohomology sheaves. If $f : X \to Y$ is a morphism of algebraic stacks, then there is always an adjoint pair of unbounded derived functors

$$\mathsf{D}_{\mathrm{qc}}(X) \underset{\mathsf{L}f_{\mathrm{qc}}^*}{\overset{\mathsf{R}(f_{\mathrm{qc}})_*}{\rightleftarrows}} \mathsf{D}_{\mathrm{qc}}(Y).$$

If f is quasicompact, quasiseparated and representable, then $\mathsf{R}(f_{\mathrm{qc}})_*$ agrees with $\mathsf{R}f_*$, the unbounded derived functor of $f_* : \mathsf{Mod}(X) \to \mathsf{Mod}(Y)$ [Hall and Rydh 2014, Lemma 2.5(3) and Theorem 2.6(2)]. If f is smooth, then $\mathsf{L}f_{\mathrm{qc}}^*$ agrees with the unique extension of the exact functor $f^* : \mathsf{Mod}(Y) \to \mathsf{Mod}(X)$ to the unbounded derived category.

2. Tensor nilpotence with parameters

Definition 2.1. Let X be an algebraic stack and let $\xi : M \to N$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(X)$. Let $Z \subseteq |X|$ be a subset. We say that ξ *vanishes at the points of* Z if for every algebraically closed field k and morphism $z : \operatorname{Spec} k \to X$ that factors through Z, then $\mathsf{L}z_{\mathrm{qc}}^*\xi$ is the zero map in $\mathsf{D}_{\mathrm{qc}}(\operatorname{Spec} k)$.

This definition is connected to a more familiar notion for schemes.

Lemma 2.2. *Let X be a scheme and let $\xi : M \to N$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(X)$. If $Z \subseteq |X|$ is a subset, then ξ* vanishes at the points of Z *if and only if $\xi \otimes^{\mathsf{L}}_{\mathcal{O}_X} \kappa(z)$ is the zero map in $\mathsf{D}(\kappa(z))$ for every $z \in Z$, where $\kappa(z)$ denotes the residue field of z.*

Proof. We immediately reduce to the situation where $X = \operatorname{Spec} \kappa$ and κ is a field. It now suffices to prove that if $\kappa \subseteq k$ is a field extension, where k is algebraically closed, then $\xi \otimes k$ is the zero map in $\mathsf{D}(k)$ if and only if ξ is the zero map in $\mathsf{D}(\kappa)$. This is obvious. □

If $K \in \mathsf{D}_{\mathrm{qc}}(X)$, then the *cohomological support* of K is defined to be the subset

$$\operatorname{supph}(K) = \bigcup_{n \in \mathbb{Z}} \operatorname{supp}(\mathcal{H}^n(K)) \subseteq |X|.$$

For the basic properties of cohomological support, see [Hall and Rydh 2014, Lemma 4.8], which extends [Thomason 1997, Lemma 3.3] to algebraic stacks. The main result of this section is the following theorem.

Theorem 2.3 (tensor nilpotence with parameters). *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. Let $\psi : E \to F$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(X)$, where $E \in \mathsf{D}_{\mathrm{qc}}(X)^c$. Let $K \in \mathsf{Perf}(X)$. If ψ vanishes at the points of* $\operatorname{supph}(K)$, *then there exists a positive integer n such that $K \otimes^{\mathsf{L}}_{\mathcal{O}_X} (\psi^{\otimes n}) = 0$ in $\mathsf{D}_{\mathrm{qc}}(X)$.*

The following example demonstrates that Theorem 2.3 cannot be weakened to the situation where $E \in \mathsf{Perf}(X)$.

Example 2.4. Let $X = B_{\mathbb{F}_2}(\mathbb{Z}/2\mathbb{Z})$, which is a quasicompact, nontame Deligne–Mumford stack with finite diagonal. Consider the adjunction morphism

$$\eta : \mathbb{O}_X \to x_* \mathbb{O}_{\mathbb{F}_2},$$

where $x : \operatorname{Spec} \mathbb{F}_2 \to X$ is the usual cover. Since $\operatorname{coker}(\eta) \cong \mathbb{O}_X$, the cone of η induces a natural map $\psi : \mathbb{O}_X \to \mathbb{O}_X[1]$. For all positive integers n, $\psi^{\otimes n} = \psi$. Clearly, ψ vanishes at the points of $|X|$ (because $x^*\eta$ is split). If $\psi = \psi^{\otimes n} = 0$ for some n, it is easily determined that this implies that $\mathbb{O}_X \in \mathsf{D}_{\mathrm{qc}}(X)^c$, which is false.

Proof of Theorem 2.3. Let $\mathbf{E}$ be the category of representable, quasifinite, flat and separated morphisms of finite presentation over X. Let $\mathbf{D} \subseteq \mathbf{E}$ be the full subcategory whose objects are those $(U \xrightarrow{p} X)$ such that there exists an integer $n > 0$ with $p^*(K \otimes^{\mathsf{L}}_{\mathbb{O}_X} (\psi^{\otimes n})) = 0$. It suffices to prove that $\mathbf{D} = \mathbf{E}$. By the induction principle (Theorem B.1), it is sufficient to verify the following three conditions:

(I1) If $(U \to W) \in \mathbf{E}$ is an open immersion and $W \in \mathbf{D}$, then $U \in \mathbf{D}$.

(I2) If $(V \to W) \in \mathbf{E}$ is finite and surjective, where V is an affine scheme, then $W \in \mathbf{D}$.

(I3) If $(U \xrightarrow{j} W)$, $(W' \xrightarrow{f} W) \in \mathbf{E}$, where j is an open immersion and f is étale and an isomorphism over $W \setminus U$, then $W \in \mathbf{D}$ whenever U, $W' \in \mathbf{D}$.

Now condition (I1) is trivial and condition (I3) is Lemma 2.5. For condition (I2), by Lemma 2.6, it remains to prove that every affine scheme belongs to $\mathbf{D}$. By Lemma 2.2 and [Thomason 1997, Lemma 3.14] (or [Neeman 1992a, Lemma 1.2]), the result follows. □

Lemma 2.5. *Consider a 2-cartesian diagram of algebraic stacks*

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "f_U"'] & W' \arrow[d, "f"] \\ U \arrow[r, hook, "j"] & W \end{tikzcd}$$

where W is quasicompact and quasiseparated, j is a quasicompact open immersion and f is representable, étale, finitely presented and an isomorphism over $W \setminus U$. Let $\psi : E \to F$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(W)$ and let $K \in \mathsf{D}_{\mathrm{qc}}(W)$. For each integer $n > 0$, let $\phi_n = K \otimes^{\mathsf{L}}_{\mathbb{O}_W} (\psi^{\otimes n})$. If $f^*\phi_n = 0$ and $j^*\phi_n = 0$, then $\phi_{2n} = 0$.

Proof. To simplify notation, we let $E_n = K \otimes^{\mathsf{L}}_{\mathbb{O}_W} E^{\otimes n}$ and $F_n = K \otimes^{\mathsf{L}}_{\mathbb{O}_W} F^{\otimes n}$. We will argue similarly to [Thomason 1997, Theorem 3.6], but using the Mayer–Vietoris triangle for étale neighbourhoods of stacks developed in [Hall and Rydh 2014, Lemma 5.7(1)] instead of [Thomason 1997, Lemma 3.5]. Let $k = f \circ j'$. By

[Hall and Rydh 2014, Lemma 5.7(1)], there is a distinguished triangle in $\mathsf{D}_{\mathrm{qc}}(W)$:

$$F_n \longrightarrow \mathsf{R}j_*j^*F_n \oplus \mathsf{R}f_*f^*F_n \longrightarrow \mathsf{R}k_*k^*F_n \xrightarrow{d} F_n[1].$$

By applying the homological functor $\mathrm{Hom}_{\mathcal{O}_W}(E_n, -)$ to the distinguished triangle above, we find that there exists a morphism $t : E_n \to \mathsf{R}k_*k^*F_n[-1]$ such that $\delta(t) = \phi_n$, where δ is the boundary map induced by d. But there is a commutative diagram

$$\begin{tikzcd} & (\mathsf{R}k_*k^*F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n} \arrow[dd, "\mathrm{Id}\otimes\psi^{\otimes n}"] \arrow[dr, "\delta\otimes\psi^{\otimes n}"] & \\ E_n \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n} \arrow[ur, "t\otimes\mathrm{Id}"] \arrow[rr, "\phi_{2n}"] \arrow[dr, "t\otimes\psi^{\otimes n}"'] & & F_n \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \\ & (\mathsf{R}k_*k^*F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \arrow[ur, "\delta\otimes\mathrm{Id}"'] & \end{tikzcd}$$

so it remains to prove that the vertical map above is zero. To see this, the projection formula [Hall and Rydh 2014, Corollary 4.12] implies that we have a commutative diagram

$$\begin{tikzcd} (\mathsf{R}k_*k^*F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n} \arrow[r, "\sim"] \arrow[d, "\mathrm{Id}\otimes\psi^{\otimes n}"'] & \mathsf{R}k_*k^*(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \otimes^{\mathsf{L}}_{\mathcal{O}_W} E^{\otimes n}[-1]) \arrow[d, "\mathsf{R}k_*k^*(F^{\otimes n}\otimes\phi_n[-1])"] \\ (\mathsf{R}k_*k^*F_n[-1]) \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \arrow[r, "\sim"] & \mathsf{R}k_*k^*(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n} \otimes^{\mathsf{L}}_{\mathcal{O}_W} F^{\otimes n}[-1]) \end{tikzcd}$$

Since $k^*\phi_n = 0$, the result follows. □

The following lemma is similar to a special case of [Elagin 2011, Theorem 7.3 and Corollary 9.6]. Also, see [Krishna 2009, proof of Proposition 7.6; Dubey and Mallick 2012, Lemma 3.8].

Lemma 2.6. *Let W be an algebraic stack and let $v : V \to W$ be a finite and faithfully flat morphism of finite presentation, where V is an affine scheme. Let $\psi : E \to F$ be a morphism in $\mathsf{D}_{\mathrm{qc}}(W)$, where $E \in \mathsf{D}_{\mathrm{qc}}(W)^c$. Let $K \in \mathsf{Perf}(W)$. If $v^*(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi) = 0$ in $\mathsf{D}_{\mathrm{qc}}(V)$, then $K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi = 0$ in $\mathsf{D}_{\mathrm{qc}}(W)$.*

Proof. By [Hall and Rydh 2014, Corollary 4.15], $\mathsf{R}(v_{\mathrm{qc}})_*$ admits a right adjoint $v^\times$ and there is a functorial isomorphism $v^\times(\mathcal{O}_W) \otimes^{\mathsf{L}}_{\mathcal{O}_V} \mathsf{L}v^*_{\mathrm{qc}}(M) \simeq v^\times(M)$ for every $M \in \mathsf{D}_{\mathrm{qc}}(W)$. In particular, if $v^*(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi) = 0$ in $\mathsf{D}_{\mathrm{qc}}(V)$, then $v^\times(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} \psi) = 0$ in $\mathsf{D}_{\mathrm{qc}}(V)$. By adjunction, it follows that the induced composition

$$\mathsf{R}(v_{\mathrm{qc}})_* v^\times(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} E) \to K \otimes^{\mathsf{L}}_{\mathcal{O}_W} E \to K \otimes^{\mathsf{L}}_{\mathcal{O}_W} F$$

vanishes in $\mathsf{D}_{\mathrm{qc}}(W)$. Thus it suffices to prove that

$$\mathsf{R}(v_{\mathrm{qc}})_* v^{\times}(K \otimes^{\mathsf{L}}_{\mathcal{O}_W} E) \to K \otimes^{\mathsf{L}}_{\mathcal{O}_W} E$$

admits a section. Since $E \in \mathsf{D}_{\mathrm{qc}}(W)^c$ and $K \in \mathsf{Perf}(W)$, it follows that $K \otimes^{\mathsf{L}}_{\mathcal{O}_W} E$ lies in $\mathsf{D}_{\mathrm{qc}}(W)^c$. Hence, we need only prove that if $M \in \mathsf{D}_{\mathrm{qc}}(W)^c$, the trace morphism $\mathrm{Tr}_M : \mathsf{R}(v_{\mathrm{qc}})_* v^{\times}(M) \to M$ admits a section. By Lemma A.1, M is quasi-isomorphic to a direct summand of $\mathsf{R}(v_{\mathrm{qc}})_* P$, where $P \in \mathsf{Perf}(V)$. Thus we are reduced to proving that $\mathrm{Tr}_{\mathsf{R}(v_{\mathrm{qc}})_* P}$ admits a section. This is trivial and the result follows. □

3. The classification of thick ⊗-ideals

If $\mathcal{T}$ is a ⊗-triangulated category and $S \subseteq \mathcal{T}$ is a subset, then define $\langle S \rangle_{\otimes} \subseteq \mathcal{T}$ to be the smallest thick ⊗-ideal of $\mathcal{T}$ containing S.

To prove Theorem 1.1, we require this analogue of [Thomason 1997, Lemma 3.14]:

Lemma 3.1. *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. If $P, Q \in \mathsf{D}_{\mathrm{qc}}(X)^c$ and* $\mathrm{supph}(P) \subseteq \mathrm{supph}(Q)$, *then* $\langle P \rangle_{\otimes} \subseteq \langle Q \rangle_{\otimes}$.

Proof. Argue exactly as in [Thomason 1997, Lemma 3.14] (cf. [Neeman 1992a, Lemma 1.2]), but using Theorem 2.3 instead of Thomason's Theorem 3.8. □

The following example shows Lemma 3.1 cannot be extended to $P, Q \in \mathsf{Perf}(X)$ when X is nontame. It also shows that Thomason's classification (Theorem 1.1) does not hold for $\mathsf{Perf}(X)$ in this case too.

Example 3.2. Let $x : \operatorname{Spec} \mathbb{F}_2 \to X$ be as in Example 2.4. Let $P = \mathcal{O}_X$ and let $Q = x_* \mathcal{O}_{\operatorname{Spec} \mathbb{F}_2}$. Then $P, Q \in \mathsf{Perf}(X)$ and $\mathrm{supph}(P) = \mathrm{supph}(Q)$. Note that $Q \in \mathsf{D}_{\mathrm{qc}}(X)^c$ and $P \notin \mathsf{D}_{\mathrm{qc}}(X)^c$. Since $\mathsf{D}_{\mathrm{qc}}(X)^c$ is a thick ⊗-ideal of $\mathsf{Perf}(X)$, it follows that $\langle Q \rangle_{\otimes} \subseteq \mathsf{D}_{\mathrm{qc}}(X)^c$. But if $\langle P \rangle_{\otimes} = \langle Q \rangle_{\otimes}$, then $P \in \mathsf{D}_{\mathrm{qc}}(X)^c$. But $P \notin \mathsf{D}_{\mathrm{qc}}(X)^c$; thus we have a contradiction.

Following Thomason [1997, Theorem 3.15] (or Neeman [1992a, Theorem 1.5]), given Lemma 3.1, we can prove Theorem 1.1.

Proof of Theorem 1.1. If $Y \subseteq |X|$ is a Thomason subset, then define

$$\mathcal{I}_Y = \{P \in \mathsf{D}_{\mathrm{qc}}(X)^c : \mathrm{supph}(P) \subseteq Y\}.$$

Clearly, $\mathcal{I}_Y$ is a thick ⊗-ideal of $\mathsf{D}_{\mathrm{qc}}(X)^c$. If $\mathcal{T}$ is a thick ⊗-ideal of $\mathsf{D}_{\mathrm{qc}}(X)^c$, then define

$$\varphi(\mathcal{T}) = \bigcup_{Q \in \mathcal{T}} \mathrm{supph}(Q).$$

By [Hall and Rydh 2014, Lemma 4.8(3)], $\varphi(\mathcal{T})$ is a Thomason subset of $|X|$. It suffices to prove that $\mathcal{I}_{\varphi(\mathcal{T})} = \mathcal{T}$ and $\varphi(\mathcal{I}_Y) = Y$.

Obviously, $\mathcal{T} \subseteq \mathcal{I}_{\varphi(\mathcal{T})}$. For the reverse inclusion, if $P \in \mathcal{I}_{\varphi(\mathcal{T})}$, then

$$\operatorname{supph}(P) \subseteq \bigcup_{Q \in \mathcal{T}} \operatorname{supph}(Q).$$

Since $\operatorname{supph}(P)$ and $\operatorname{supph}(Q)$ are constructible for every $Q \in \mathcal{T}$, it follows that there is a finite subset $J \subseteq \mathcal{T}$ such that

$$\operatorname{supph}(P) \subseteq \bigcup_{Q \in J} \operatorname{supph}(Q) = \operatorname{supph}(\oplus_{Q \in J} Q).$$

By Lemma 3.1, $\langle P \rangle_{\otimes} \subseteq \langle \oplus_{Q \in J} Q \rangle_{\otimes} \subseteq \mathcal{T}$. Thus $P \in \mathcal{T}$ and $\mathcal{I}_{\varphi(\mathcal{T})} = \mathcal{T}$.

Obviously, $Y \supseteq \varphi(\mathcal{I}_Y)$. Since Y is Thomason, it is expressible as a union $\cup_\alpha Y_\alpha$ such that $|X| \setminus Y_\alpha$ is quasicompact and open. By [Hall and Rydh 2014, Theorem A], for every α there is a compact complex Q_α with support Y_α. It follows that if $y \in Y$, then $y \in \operatorname{supph}(Q_\alpha) \subseteq Y$ for some α. In other words, $y \in \varphi(\mathcal{I}_Y)$, so $Y = \varphi(\mathcal{I}_Y)$. □

4. The Balmer spectrum of a tame stack

We will prove Theorem 1.2 using [Buan et al. 2007, Proposition 6.1].

Proof of Theorem 1.2. Let $s : (|X|, \operatorname{supph}) \to \big(|\operatorname{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X))|, \sigma_X\big)$ be the uniquely induced morphism of support data, where σ_X denotes the universal support datum. Since X is tame, it has finite cohomological dimension [Hall and Rydh 2015, Theorem 2.1(2)]; hence, $\mathsf{D}_{\mathrm{qc}}(X)^c = \mathsf{Perf}(X)$ [Hall and Rydh 2014, Remark 4.6]. By Theorem 1.1, $(|X|, \operatorname{supph})$ is classifying and by [Laumon and Moret-Bailly 2000, Corollaries 5.6.1 and 5.7.2] we know that $|X|$ is spectral. By [Buan et al. 2007, Proposition 6.1], s is a homeomorphism. By definition, $\mathcal{O}_{\operatorname{Sp}_{\mathrm{Bal}}(\mathsf{Perf}(X))}$ is the sheafification of the presheaf

$$(j : U \subseteq X) \mapsto \operatorname{End}_{\mathsf{Perf}(X)/\ker(j^*)\cap \mathsf{Perf}(X)}(j^*\mathcal{O}_X).$$

Since $|X|$ has a basis consisting of quasicompact open subsets, it is sufficient to identify $\operatorname{End}_{\mathsf{Perf}(X)/\ker(j^*)\cap\mathsf{Perf}(X)}(j^*\mathcal{O}_X)$ when j is a quasicompact open immersion. By [Hall and Rydh 2014, Lemma 6.7(2)], $\ker(j^*)$ is the localising envelope of a set of objects with compact image in $\mathsf{D}_{\mathrm{qc}}(X)$. By Thomason's localisation theorem (e.g., [Hall and Rydh 2014, Theorem 3.10] or [Neeman 1992b, Theorem 2.1]), $\mathsf{Perf}(U)$ is the thick closure of $\mathsf{Perf}(X)/\ker(j^*) \cap \mathsf{Perf}(X)$. Since there are natural isomorphisms

$$\operatorname{End}_{\mathsf{Perf}(X)/\ker(j^*)\cap\mathsf{Perf}(X)}(j^*\mathcal{O}_X) \cong \operatorname{End}_{\mathsf{Perf}(U)}(\mathcal{O}_U) \cong \operatorname{End}_{\mathcal{O}_U}(\mathcal{O}_U) = \Gamma(U, \mathcal{O}_X),$$

the result follows. □

Proof of Theorem 1.3. Since X has finite inertia, it has separated diagonal. By [Rydh 2013, Theorem 6.12], π is a separated universal homeomorphism, so X_{cs} is a quasicompact and quasiseparated algebraic space. By [Rydh 2013, Theorem 6.12], the natural map $(|X|, \mathcal{O}_{X_{\mathrm{Zar}}}) \to (|X_{\mathrm{cs}}|, \mathcal{O}_{(X_{\mathrm{cs}})_{\mathrm{Zar}}})$ is an isomorphism of locally ringed spaces. By Theorem 1.2, the result follows. □

Appendix A: Tame stacks and coarse spaces

We establish here some basic results about $\mathsf{R}(\pi_{\mathrm{qc}})_*$, where $\pi : X \to X_{\mathrm{cs}}$ is the coarse space of a quasiseparated algebraic stack X with finite inertia. Our first result, however, is a useful lemma that characterises the compact objects on a certain class of algebraic stacks, which includes BG for all finite groups G. This is likely known, though we are unaware of a reference for this result in the generality required.

Lemma A.1. *Let W be an algebraic stack and let $v : V \to W$ be a finite and faithfully flat morphism of finite presentation, where V is an affine scheme. If $M \in \mathsf{D}_{\mathrm{qc}}(W)^c$, then M is quasi-isomorphic to a direct summand of $\mathsf{R}(v_{\mathrm{qc}})_* P$ for some $P \in \mathsf{Perf}(V)$.*

Proof. If $P \in \mathsf{Perf}(V)$, then $\mathsf{R}(v_{\mathrm{qc}})_* P \in \mathsf{D}_{\mathrm{qc}}(W)^c$ [Hall and Rydh 2014, Corollary 4.15 and Example 3.8]. Thus, let $\mathcal{T} \subseteq \mathsf{D}_{\mathrm{qc}}(W)^c$ be the subcategory with objects those $N \in \mathsf{D}_{\mathrm{qc}}(W)^c$ that are quasi-isomorphic to direct summands of $\mathsf{R}(v_{\mathrm{qc}})_* P$ for some $P \in \mathsf{Perf}(V)$. Clearly, $\mathcal{T}$ is closed under shifts and direct summands. We now prove that $\mathcal{T}$ is triangulated. Thus let $f : N' \to N$ be a morphism in $\mathcal{T}$ and complete it to a distinguished triangle

$$N' \xrightarrow{f} N \xrightarrow{c} N'' \xrightarrow{\partial} N'[1].$$

We now prove that $N'' \in \mathcal{T}$. By assumption, there are P, $P' \in \mathsf{Perf}(V)$ and C, $C' \in \mathsf{D}_{\mathrm{qc}}(W)^c$ and quasi-isomorphisms $N \oplus C \simeq \mathsf{R}(v_{\mathrm{qc}})_* P$, $N' \oplus C' \simeq \mathsf{R}(v_{\mathrm{qc}})_* P'$. It follows that there is a distinguished triangle

$$N' \oplus C' \xrightarrow{f\oplus 0} N \oplus C \xrightarrow{c\oplus \mathrm{id}_C \oplus 0} N'' \oplus C \oplus C'[1] \xrightarrow{\partial \oplus p_{C'[1]}} N' \oplus C'[1],$$

where $p_{C'[1]} : C \oplus C'[1] \to C'[1]$ is the natural projection. In particular, we are reduced to the situation where $N' = \mathsf{R}(v_{\mathrm{qc}})_* P'$ and $N = \mathsf{R}(v_{\mathrm{qc}})_* P$. In this case, the morphism $f : N' \to N$ by duality induces a morphism $\tilde{f} : P' \to v^{\times} \mathsf{R}(v_{\mathrm{qc}})_* P$. It follows that the composition $\mathsf{R}(v_{\mathrm{qc}})_* P' \xrightarrow{f} \mathsf{R}(v_{\mathrm{qc}})_* P \to \mathsf{R}(v_{\mathrm{qc}})_* v^{\times} \mathsf{R}(v_{\mathrm{qc}})_* P$ is the map $\mathsf{R}(v_{\mathrm{qc}})_* \tilde{f}$. Now form a distinguished triangle

$$P' \xrightarrow{\tilde{f}} v^{\times} \mathsf{R}(v_{\mathrm{qc}})_* P \xrightarrow{k} K \xrightarrow{\delta} P'[1].$$

Since the morphism $\mathsf{R}(v_{\mathrm{qc}})_* P \to \mathsf{R}(v_{\mathrm{qc}})_* v^{\times} \mathsf{R}(v_{\mathrm{qc}})_* P$ admits a retraction, there exist a $Q \in \mathsf{D}_{\mathrm{qc}}(W)^c$ and a quasi-isomorphism $\mathsf{R}(v_{\mathrm{qc}})_* v^{\times} \mathsf{R}(v_{\mathrm{qc}})_* P \simeq \mathsf{R}(v_{\mathrm{qc}})_* P \oplus Q$.

There is an induced morphism of distinguished triangles

$$\begin{tikzcd}
\mathsf{R}(v_{\mathrm{qc}})_*P' \arrow[r, "\mathsf{R}(v_{\mathrm{qc}})_*\tilde{f}"] \arrow[d, equal] & \mathsf{R}(v_{\mathrm{qc}})_*v^{\times}\mathsf{R}(v_{\mathrm{qc}})_*P \arrow[r, "\mathsf{R}(v_{\mathrm{qc}})_*k"] \arrow[d, "\wr"'] & \mathsf{R}(v_{\mathrm{qc}})_*K \arrow[r, "\mathsf{R}(v_{\mathrm{qc}})_*\delta"] \arrow[d] & \mathsf{R}(v_{\mathrm{qc}})_*P'[1] \arrow[d] \\
\mathsf{R}(v_{\mathrm{qc}})_*P' \arrow[r, "f\oplus 0"] & \mathsf{R}(v_{\mathrm{qc}})_*P \oplus Q \arrow[r, "c\oplus \mathrm{id}_Q"] & N'' \oplus Q \arrow[r, "\partial+0"] & \mathsf{R}(v_{\mathrm{qc}})_*P'[1].
\end{tikzcd}$$

It follows that $\mathsf{R}(v_{\mathrm{qc}})_*K \simeq N'' \oplus Q$ and so $N'' \in \mathcal{T}$. By [Hall and Rydh 2014, Example 6.5 and Proposition 6.6], $\mathsf{D}_{\mathrm{qc}}(W)$ is compactly generated by $v_*\mathcal{O}_V$. But Thomason's Theorem [Neeman 1992b, Theorem 2.1] implies that $\mathsf{D}_{\mathrm{qc}}(W)^c$ is the smallest thick subcategory containing $v_*\mathcal{O}_V$. The result follows. □

Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor between triangulated categories. Assume that $\mathcal{S}$ and $\mathcal{T}$ admit t-structures. We say that F is *left* (resp. *right*) *t-exact* if $F(\mathcal{S}^{\geq 0}) \subseteq \mathcal{T}^{\geq 0}$ (resp. $F(\mathcal{S}^{\leq 0}) \subseteq \mathcal{T}^{\leq 0}$)). We say that F is *t-exact* if it is both left and right t-exact. The following result was suggested to us by David Rydh.

Theorem A.2. *If X be a quasiseparated algebraic stack with finite inertia and coarse space $\pi : X \to X_{\mathrm{cs}}$, then the restriction of $\mathsf{R}(\pi_{\mathrm{qc}})_*$ to $\mathsf{D}_{\mathrm{qc}}(X)^c$ is t-exact.*

Proof. By [Hall and Rydh 2014, Lemma 1.2(4)], this may be checked étale-locally on X_{cs}. Thus, we may assume that X_{cs} is an affine scheme. Since π is a universal homeomorphism, it follows that X is quasicompact. Also, since X has finite inertia, it has quasifinite and separated diagonal. By Theorem B.5, there exist morphisms of algebraic stacks $V \xrightarrow{v} W \xrightarrow{p} X$ such that V is an affine scheme, v is finite, faithfully flat and finitely represented and p is a representable, separated and finitely presented Nisnevich covering. By [Rydh 2013, Proposition 6.5], we may further assume that p is fixed-point reflecting. We now apply [Rydh 2013, Theorem 6.10] to conclude that the diagram

$$\begin{tikzcd}
W \arrow[r, "p"] \arrow[d, "\omega"'] & X \arrow[d, "\pi"] \\
W_{\mathrm{cs}} \arrow[r, "p_{\mathrm{cs}}"] & X_{\mathrm{cs}}
\end{tikzcd}$$

is cartesian and p_{cs} is representable, separated, étale and of finite presentation. Thus, it suffices to prove the result on W.

Clearly $\mathsf{R}(\pi_{\mathrm{qc}})_*$ is left t-exact, so it remains to address the right t-exactness. Take $M \in \mathsf{D}_{\mathrm{qc}}(W)^c \cap \mathsf{D}_{\mathrm{qc}}^{\leq 0}(W)$. By Lemma A.1, we may assume that there exists a map $i : M \to \mathsf{R}(v_{\mathrm{qc}})_*P$, where $P \in \mathsf{Perf}(V)$, that admits a retraction r. It follows that the composition $M \xrightarrow{i} \mathsf{R}(v_{\mathrm{qc}})_*P \to \tau^{>0}\mathsf{R}(v_{\mathrm{qc}})_*P$ is the zero map. Therefore the induced map $\mathsf{R}(\omega_{\mathrm{qc}})_*M \to \mathsf{R}(\omega_{\mathrm{qc}})_*\tau^{>0}\mathsf{R}(v_{\mathrm{qc}})_*P$ is the zero map. But v and

$\omega \circ v$ are affine, so there is a natural quasi-isomorphism $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*\mathsf{R}(v_{\mathrm{qc}})_*P \simeq \mathsf{R}(\omega_{\mathrm{qc}})_*\tau^{>0}\mathsf{R}(v_{\mathrm{qc}})_*P$. The resulting map

$$\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*M \to \tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*\mathsf{R}(v_{\mathrm{qc}})_*P$$

is 0 and also coincides with $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*(i)$, which admits a retraction $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*(r)$. In particular, $\tau^{>0}\mathsf{R}(\omega_{\mathrm{qc}})_*M \simeq 0$ and the result follows. □

Abramovich et al. [2008] work with a more restrictive definition of tame, rendering the following corollary a tautology. Indeed, they assume that X has finite inertia and is locally of finite presentation over a base scheme S and that $\pi : X \to X_{\mathrm{cs}}$ is such that π_* is exact on quasicoherent sheaves. In our case, we make none of these assumptions, rendering it nontrivial.

Corollary A.3. *Let X be a quasiseparated algebraic stack with finite inertia and coarse space $\pi : X \to X_{\mathrm{cs}}$. The following are equivalent*:

(1) *X is tame*;

(2) $\pi_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(X_{\mathrm{cs}})$ *is exact*;

(3) $\mathsf{R}\pi_* : \mathsf{D}^+_{\mathrm{qc}}(X) \to \mathsf{D}^+_{\mathrm{qc}}(X_{\mathrm{cs}})$ *is t-exact*;

(4) $\mathsf{R}(\pi_{\mathrm{qc}})_* : \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(X_{\mathrm{cs}})$ *is t-exact.*

Proof. We begin with some preliminary reductions. The morphism π is a separated universal homeomorphism [Rydh 2013, Theorem 6.12], so X_{cs} is a quasiseparated algebraic space and π is quasicompact and quasiseparated. Thus by Lemma 1.2(2) of [Hall and Rydh 2014] we get the implication (3) ⇒ (4), and by Theorem 2.6(2) of the same reference we have that (4) ⇒ (3). Clearly, item (1) may be verified after passing to an affine étale presentation of X_{cs}, and similarly for items (2) and (3) [Hall and Rydh 2014, Lemma 1.2(4) and Lemma 2.2(6)]. We may consequently assume that X_{cs} is an affine scheme. Since π has finite diagonal, it has affine diagonal, so we have (2)⇔(3) [Hall et al. 2014, Proposition 2.1]. By [Hall and Rydh 2015, Theorem C, (1) ⇒ (3)], we now obtain that (2) ⇒ (1). It remains to address (1) ⇒ (2).

Arguing exactly as in the proof of Theorem A.2, we may further assume that X admits a finite, faithfully flat and finitely presented cover $v : V \to X$, where V is an affine scheme. Since X is tame, $\mathbb{O}_X \in \mathsf{D}_{\mathrm{qc}}(X)^c$. By Theorem A.2, it follows that the induced morphism $\mathbb{O}_X \to v_*\mathbb{O}_V$ admits a retraction. If $M \in \mathsf{QCoh}(X)$, then it follows immediately that the natural map $M \to v_*v^*M$ admits a retraction. Thus, if $f : M \to N$ is a surjection in $\mathsf{QCoh}(X)$, then f is a retraction of the surjection v_*v^*f. Since $\pi \circ v$ is affine, $\pi_*v_*v^*f$ is surjective. In particular, π_*f is a retraction of a surjection, thus is surjective. The result follows. □

Appendix B: The induction principle

The *induction principle* [Stacks 2015, Tag 08GL] for algebraic spaces is closely related to the étale dévissage results of [Rydh 2011a]. When working with derived categories, where locality results are often quite subtle, it is often advantageous to have the strongest possible criteria at your disposal. In this appendix, we will prove the following induction principle for stacks with quasifinite and separated diagonal.

Before stating this result, we require some notation. Fix an algebraic stack S. If $P_1, \dots, P_r$ is a list of properties of morphisms of algebraic stacks over S, let $\mathbf{Stack}_{P_1,\dots,P_r/S}$ denote the full 2-subcategory of the category of algebraic stacks over S whose objects are those $(x : X \to S)$ such that x has properties $P_1, \dots, P_r$. The following abbreviations will be used: ét (étale), qff (quasifinite flat), sep (separated), fp (finitely presented) and rep (representable).

For example, $\mathbf{Stack}_{\mathrm{rep,sep,qff,fp}/S}$ consists of those algebraic stacks $x : X \to S$ such that x is representable, separated, quasifinite flat, and finitely presented. In a similar way, $\mathbf{Stack}_{\mathrm{rep,sep,\acute{e}t,fp}/S}$ consists of those algebraic stacks over S, $x : X \to S$, such that x is representable, separated, étale, and finitely presented. Note that while every morphism $(X' \to X)$ in $\mathbf{Stack}_{\mathrm{rep,sep,\acute{e}t,fp}/S}$ is representable, separated, étale, and finitely presented; in $\mathbf{Stack}_{\mathrm{rep,sep,qff,fp}/S}$ they can only be assumed to be representable, separated, quasifinite, and finitely presented (i.e., there are nonflat morphisms between objects).

Theorem B.1 (induction principle). *Let S be a quasicompact algebraic stack with quasicompact and separated diagonal. If S has quasifinite diagonal, let*

$$\mathbf{E} = \mathbf{Stack}_{\mathrm{rep,sep,qff,fp}/S};$$

or if S is Deligne–Mumford, let

$$\mathbf{E} = \mathbf{Stack}_{\mathrm{rep,sep,\acute{e}t,fp}/S}.$$

Let $\mathbf{D} \subseteq \mathbf{E}$ be a full subcategory satisfying the following properties:

(I1) *if $(X' \to X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$;*

(I2) *if $(X' \to X) \in \mathbf{E}$ is finite, flat, and surjective, where X' is an affine scheme, then $X \in \mathbf{D}$;*

(I3) *if $(U \xrightarrow{j} X)$, $(X' \xrightarrow{f} X) \in \mathbf{E}$, where j is an open immersion and f is étale and an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U, X' \in \mathbf{D}$.*

Then $\mathbf{D} = \mathbf{E}$. In particular, $S \in \mathbf{D}$.

Proof. Combine Lemma B.3 with Theorem B.5. $\square$

We wish to point out that Theorem B.1 relies on the existence of coarse spaces for stacks with finite inertia (i.e., the Keel–Mori theorem [Keel and Mori 1997; Rydh 2013]).

Nisnevich coverings. It will be useful to consider some variants and refinements of [Krishna and Østvær 2012, Sections 7–8].

If $p : W \to X$ is a representable morphism of algebraic stacks, then a *splitting sequence* for p is a sequence of quasicompact open immersions

$$\varnothing = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X,$$

such that p restricted to $X_i \setminus X_{i-1}$, when given the induced reduced structure, admits a section for each $i = 1, \dots, r$. In this situation, we say that p has a splitting sequence of length r. An étale and representable morphism of algebraic stacks $p : W \to X$ is a *Nisnevich covering* if it admits a splitting sequence.

Example B.2. Let X be a quasicompact and quasiseparated scheme. Then there exists an affine scheme W and a Nisnevich covering $p : W \to X$. Indeed, taking $W = \amalg_{i=1}^n U_i$, where the $\{U_i\}$ form a finite affine open covering of X gives the claim.

The following lemma is proved by a straightforward induction on the length of the splitting sequence.

Lemma B.3 (Nisnevich dévissage). *Let S be a quasicompact and quasiseparated algebraic stack. Let $\mathbf{E}$ be $\mathbf{Stack}_{\mathrm{rep,\acute{e}t,fp}/S}$ or $\mathbf{Stack}_{\mathrm{rep,sep,\acute{e}t,fp}/S}$. Let $\mathbf{D} \subseteq \mathbf{E}$ be a full 2-subcategory with the following properties*:

(N1) *if $(X' \to X) \in \mathbf{E}$ is an open immersion and $X \in \mathbf{D}$, then $X' \in \mathbf{D}$;*

(N2) *if $(U \xrightarrow{j} X)$, $(X' \xrightarrow{f} X) \in \mathbf{E}$, where j is an open immersion and f is an isomorphism over $X \setminus U$, then $X \in \mathbf{D}$ whenever $U, X' \in \mathbf{D}$.*

If $p : W \to X$ is a Nisnevich covering in $\mathbf{E}$ and $W \in \mathbf{D}$, then $X \in \mathbf{D}$.

The following lemma will also be useful.

Lemma B.4. *Let $p : W \to X$ be a Nisnevich covering of algebraic stacks.*

(1) *If $f : X' \to X$ is a morphism of algebraic stacks, then the pull back $p' : W' \to X'$ of p along f is a Nisnevich covering.*

(2) *Let $w : W' \to W$ be a Nisnevich covering of finite presentation. If p is of finite presentation and X is quasicompact and quasiseparated, then $p \circ w : W' \to X$ is a Nisnevich covering.*

Presentations. The following theorem refines [Rydh 2011a, Theorem 7.2] and will be crucial for the proof of Theorem B.1.

Theorem B.5. *Let X be a quasicompact algebraic stack with quasifinite and separated diagonal. Then there exist morphisms of algebraic stacks*

$$V \xrightarrow{v} W \xrightarrow{p} X$$

such that

- *V is an affine scheme*;
- *v is finite*, *flat*, *surjective and of finite presentation*;
- *p is a separated Nisnevich covering of finite presentation.*

In addition, *if S is a Deligne–Mumford stack*, *it can be arranged that v is also étale.*

Proof. The proof is similar to [Rydh 2013, Proposition 6.11; 2011a, Theorem 7.3].

By [Rydh 2011a, Theorem 7.1], there is an affine scheme U and a representable, separated, quasifinite, flat, and surjective morphism $u : U \to X$ of finite presentation. Let $W = \underline{\mathrm{Hilb}}^{\mathrm{open}}_{U/X} \to X$ be the subfunctor of the relative Hilbert scheme parametrising open and closed immersions to U over X. It follows that $p : W \to X$ is étale, representable and separated [Rydh 2011b, Corollary 6.2].

We now prove that p is a Nisnevich covering. To see this, we note that there exists a sequence of quasicompact open immersions

$$\varnothing = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X,$$

such that the restriction of u to $Z_i = (X_i \setminus X_{i-1})_{\mathrm{red}}$ for $i = 1, \ldots, r$ is finite, flat and finitely presented. By definition of $p : W \to X$, it follows immediately that $p\,|_{Z_i}$ admits a section corresponding to $u\,|_{Z_i}$ and so p is a separated Nisnevich covering.

Let $v : V \to W$ be the universal family, which is finite, flat, surjective and of finite presentation. Also, $V \to U$ is representable, étale and separated [Rydh 2011b, Corollary 6.2]. Suitably shrinking W, we obtain a separated Nisnevich covering $p : W \to X$ of finite presentation fitting into a 2-commutative diagram

$$\begin{tikzcd} V \arrow[r, "q"] \arrow[d, "v"'] & U \arrow[d, "u"] \\ W \arrow[r, "p"] & X \end{tikzcd} \tag{B.1}$$

and q is étale, separated and surjective. By Zariski's Main Theorem [Laumon and Moret-Bailly 2000, Theorem A.2], q is quasi-affine. By [Rydh 2013, Theorem 5.3], W has a coarse space $\pi : W \to W_{\mathrm{cs}}$ such that W_{cs} is a quasi-affine scheme and $\pi \circ v$

is affine. By Example B.2 and Lemma B.4, we may further reduce to the situation where W_{cs} is an affine scheme. Since $\pi \circ v$ is affine, the result follows. □

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JACK HALL: jack.hall@anu.edu.au
Mathematical Sciences Institute, The Australian National University, Acton ACT 2601, Australia

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Algebraic Kasparov K-theory, II

Grigory Garkusha

A kind of motivic stable homotopy theory of algebras is developed. Explicit fibrant replacements for the S^1-spectrum and $(S^1, \mathbb{G})$-bispectrum of an algebra are constructed. As an application, unstable, Morita stable and stable universal bivariant theories are recovered. These are shown to be embedded by means of contravariant equivalences as full triangulated subcategories of compact generators of some compactly generated triangulated categories. Another application is the introduction and study of the symmetric monoidal compactly generated triangulated category of K-motives. It is established that the triangulated category kk of Cortiñas and Thom (*J. Reine Angew. Math.* **610** (2007), 71–123) can be identified with the K-motives of algebras. It is proved that the triangulated category of K-motives is a localisation of the triangulated category of $(S^1, \mathbb{G})$-bispectra. Also, explicit fibrant $(S^1, \mathbb{G})$-bispectra representing stable algebraic Kasparov K-theory and algebraic homotopy K-theory are constructed.

1. Introduction

Throughout the paper k is a fixed commutative ring with unit and Alg_k is the category of nonunital k-algebras and nonunital k-homomorphisms. Also, F is a fixed field and Sm/F is the category of smooth algebraic varieties over F. If $\mathscr{C}$ is a category and A, B are objects of $\mathscr{C}$, we shall often write $\mathscr{C}(A, B)$ to denote the Hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$.

$\mathbb{A}^1$-homotopy theory is the homotopy theory of motivic spaces, i.e., presheaves of simplicial sets defined on Sm/F (see [Morel and Voevodsky 1999; Voevodsky 1998]). Each object $X \in \mathrm{Sm}/F$ is regarded as the motivic space $\mathrm{Hom}_{\mathrm{Sm}/F}(-, X)$. The affine line $\mathbb{A}^1$ plays the role of the interval.

$k[t]$-homotopy theory is the homotopy theory of simplicial functors defined on nonunital algebras, where each algebra A is regarded contravariantly as the space $rA = \mathrm{Hom}_{\mathrm{Alg}_k}(A, -)$ so that we can study algebras from a homotopy theoretic viewpoint (see [Garkusha 2007; 2014]). The role of the interval is played by the space $r(k[t])$ represented by the polynomial algebra $k[t]$. This theory borrows methods and approaches from $\mathbb{A}^1$-homotopy theory. Another source of ideas

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and techniques for $k[t]$-homotopy theory originates in Kasparov K-theory of C^*-algebras.

In [Garkusha 2007] a kind of unstable motivic homotopy theory of algebras was developed. In order to develop stable motivic homotopy theory of algebras and — most importantly — to make the explicit computations of fibrant replacements for suspension spectra $\Sigma^\infty rA$, $A \in \mathrm{Alg}_k$, presented in this paper, one first needs to introduce and study "unstable, Morita stable and stable Kasparov K-theory spectra" $\mathbb{K}(A, B)$, $\mathbb{K}^{\mathrm{mor}}(A, B)$ and $\mathbb{K}^{st}(A, B)$ respectively, where A, B are algebras. We refer the reader to [Garkusha 2014] for properties of the spectra. The aim of this paper is to develop stable motivic homotopy theory of algebras.

Throughout we work with a certain small subcategory $\Re$ of Alg_k and the category $U_\bullet\Re$ of certain pointed simplicial functors on $\Re$. $U_\bullet\Re$ comes equipped with a motivic model structure. We write $\mathrm{Sp}(\Re)$ to denote the stable model category of S^1-spectra associated with the model category $U_\bullet\Re$. $\mathbb{K}(A, -)$, $\mathbb{K}^{\mathrm{mor}}(A, -)$ and $\mathbb{K}^{st}(A, -)$ are examples of fibrant Ω-spectra in $\mathrm{Sp}(\Re)$ (see [Garkusha 2014]).

One of the main results of the paper says that $\mathbb{K}(A, -)$ is a fibrant replacement for the suspension spectrum $\Sigma^\infty rA \in \mathrm{Sp}(\Re)$ of an algebra $A \in \Re$. Namely, there is a natural weak equivalence of spectra

$$\Sigma^\infty rA \longrightarrow \mathbb{K}(A, -)$$

in $\mathrm{Sp}(\Re)$ (see Theorem 4.2).

This is an analog of a similar result by the author and Panin [Garkusha and Panin 2014a] computing a fibrant replacement of the suspension $\mathbb{P}^1$-spectrum $\Sigma^\infty_{\mathbb{P}^1} X_+$ of a smooth algebraic variety X. The main reason that computation of a fibrant replacement for $\Sigma^\infty_{\mathbb{P}^1} X_+$ is possible is the existence of framed correspondences of [Voevodsky 2001] on homotopy groups of (motivically fibrant) $\mathbb{P}^1$-spectra. In turn, the main reason why the computation of a fibrant replacement for $\Sigma^\infty rA$ is possible is that algebras have universal extensions.

Let $\mathrm{SH}_{S^1}(\Re)$ denote the homotopy category of $\mathrm{Sp}(\Re)$. $\mathrm{SH}_{S^1}(\Re)$ plays the same role as the stable homotopy category of motivic S^1-spectra $\mathrm{SH}_{S^1}(F)$ over a field F. It is a compactly generated triangulated category with compact generators $\{\Sigma^\infty rA[n]\}_{A \in \Re, n \in \mathbb{Z}}$. One of the important consequences of the above computation is that we are able to give an *explicit* description of the Hom-groups

$$\mathrm{SH}_{S^1}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA).$$

Precisely, there is an isomorphism of abelian groups (see Corollary 4.3)

$$\mathrm{SH}_{S^1}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n(A, B), \quad A, B \in \Re, n \in \mathbb{Z}.$$

It is important to note that the groups $\mathbb{K}_n(A, B)$ have an *explicit* description in terms of nonunital algebra homomorphisms (see [Garkusha 2014, Section 7] for details).

We also show in Theorem 4.4 that the full subcategory $\mathscr{S}$ of $\mathrm{SH}_{S^1}(\Re)$ spanned by the compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$ is triangulated and there is a contravariant equivalence of triangulated categories

$$D(\Re, \mathfrak{F}) \xrightarrow{\sim} \mathscr{S}$$

with $\Re \to D(\Re, \mathfrak{F})$ the universal unstable excisive homotopy invariant homology theory in the sense of [Garkusha 2013] with respect to the class of k-split surjective algebra homomorphisms $\mathfrak{F}$. This equivalence is an extension of the contravariant functor $A \in \Re \mapsto \Sigma^\infty rA \in \mathrm{SH}_{S^1}(\Re)$ to $D(\Re, \mathfrak{F})$. Thus $D(\Re, \mathfrak{F})$ is recovered from $\mathrm{SH}_{S^1}(\Re)$. It also follows that the *small* triangulated category $D(\Re, \mathfrak{F})^{\mathrm{op}}$ lives inside the "big" ambient triangulated category $\mathrm{SH}_{S^1}(\Re)$. This is reminiscent of Voevodsky's theorem [2000] saying that there is a full embedding of the *small* triangulated category $DM_{gm}^{eff}(F)$ of effective geometrical motives into the "big" triangulated category $DM^{eff}(F)$ of motivic complexes of Nisnevich sheaves with transfers.

Next, we introduce matrices into the game. Namely, if we localise $\mathrm{SH}_{S^1}(\Re)$ with respect to the family of compact objects

$$\{\mathrm{cone}(\Sigma^\infty r(M_nA) \to \Sigma^\infty rA)\}_{n>0},$$

we shall get a compactly generated triangulated category $\mathrm{SH}_{S^1}^{\mathrm{mor}}(\Re)$ with compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$. It is in fact the homotopy category of a model category $\mathrm{Sp}_{\mathrm{mor}}(\Re)$, which is the same category as $\mathrm{Sp}(\Re)$ but with a new model structure. We construct in a similar way a compactly generated triangulated category $\mathrm{SH}_{S^1}^{\infty}(\Re)$, obtained from $\mathrm{SH}_{S^1}(\Re)$ by localisation with respect to the family of compact objects

$$\{\mathrm{cone}(\Sigma^\infty r(M_\infty A) \to \Sigma^\infty rA)\},$$

where $M_\infty A = \cup_n M_nA$. It is also the homotopy category of a model category $\mathrm{Sp}_\infty(\Re)$, which is the same category as $\mathrm{Sp}(\Re)$ but with a new model structure.

We prove in Theorems 5.1 and 6.1 that for any algebra $A \in \Re$ there are natural weak equivalences of spectra

$$\Sigma^\infty rA \longrightarrow \mathbb{K}^{\mathrm{mor}}(A, -) \quad \text{and} \quad \Sigma^\infty rA \longrightarrow \mathbb{K}^{st}(A, -)$$

in $\mathrm{Sp}_{\mathrm{mor}}(\Re)$ and $\mathrm{Sp}_\infty(\Re)$, respectively. Also, for all $A, B \in \Re$ and $n \in \mathbb{Z}$ there are isomorphisms of abelian groups

$$\mathrm{SH}_{S^1}^{\mathrm{mor}}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n^{\mathrm{mor}}(A, B)$$

and

$$\mathrm{SH}^{\infty}_{S^1}(\Re)(\Sigma^\infty r B[n], \Sigma^\infty r A) \cong \mathbb{K}^{st}_n(A, B),$$

respectively. Furthermore, the full subcategories $\mathscr{S}_{\mathrm{mor}}$ and $\mathscr{S}_\infty$ of $\mathrm{SH}^{\mathrm{mor}}_{S^1}(\Re)$ and $\mathrm{SH}^{\infty}_{S^1}(\Re)$ spanned by the compact generators $\{\Sigma^\infty r A[n]\}_{A\in\Re, n\in\mathbb{Z}}$ are triangulated and there are contravariant equivalences of triangulated categories

$$D_{\mathrm{mor}}(\Re, \mathfrak{F}) \xrightarrow{\sim} \mathscr{S}_{\mathrm{mor}} \quad \text{and} \quad D_{st}(\Re, \mathfrak{F}) \xrightarrow{\sim} \mathscr{S}_\infty.$$

Here $\Re \to D_{\mathrm{mor}}(\Re, \mathfrak{F})$ (respectively $\Re \to D_{st}(\Re, \mathfrak{F})$) is the universal Morita stable (respectively stable) excisive homotopy invariant homology theory in the sense of [Garkusha 2013]. Thus $D_{\mathrm{mor}}(\Re, \mathfrak{F})$ and $D_{st}(\Re, \mathfrak{F})$ are recovered from $\mathrm{SH}^{\mathrm{mor}}_{S^1}(\Re)$ and $\mathrm{SH}^{st}_{S^1}(\Re)$, respectively. It also follows that the *small* triangulated categories $D_{\mathrm{mor}}(\Re, \mathfrak{F})^{\mathrm{op}}$, $D_{st}(\Re, \mathfrak{F})^{\mathrm{op}}$ live inside the ambient triangulated categories $\mathrm{SH}^{\mathrm{mor}}_{S^1}(\Re)$ and $\mathrm{SH}^{st}_{S^1}(\Re)$.

We next introduce a symmetric monoidal compactly generated triangulated category of K-motives $DK(\Re)$ together with a canonical contravariant functor

$$M_K : \Re \to DK(\Re).$$

The category $DK(\Re)$ is an analog of the triangulated category of K-motives for algebraic varieties introduced in [Garkusha and Panin 2012; 2014b].

For any algebra $A \in \Re$ its K-motive is, by definition, the object $M_K(A)$ of $DK(\Re)$. We have that

$$M_K(A) \otimes M_K(B) \cong M_K(A \otimes B)$$

for all $A, B \in \Re$ (see Proposition 7.1).

We prove in Theorem 7.2 that for any two algebras $A, B \in \Re$ and any integer n there is a natural isomorphism

$$DK(\Re)(M_K(B)[n], M_K(A)) \cong \mathbb{K}^{st}_n(A, B).$$

Moreover, if $\mathscr{T}$ is the full subcategory of $DK(\Re)$ spanned by K-motives of algebras $\{M_K(A)\}_{A\in\Re}$ then $\mathscr{T}$ is triangulated and there is an equivalence of triangulated categories

$$D_{st}(\Re, \mathfrak{F}) \to \mathscr{T}^{\mathrm{op}}$$

sending an algebra $A \in \Re$ to its K-motive $M_K(A)$ (see Theorem 7.2). It is also proved in Corollary 7.3 that for any algebra $A \in \Re$ and any integer n one has a natural isomorphism

$$DK(\Re)(M_K(A)[n], M_K(k)) \cong KH_n(A),$$

where the right hand side is the n-th homotopy K-theory group in the sense of Weibel [1989]. This result is reminiscent of a similar result for K-motives of algebraic varieties in the sense of [Garkusha and Panin 2012; 2014b] identifying the K-motive of the point with algebraic K-theory.

Cortiñas and Thom [2007] constructed a universal excisive homotopy invariant and M_∞-invariant homology theory on all k-algebras

$$j : \mathrm{Alg}_k \to kk.$$

The triangulated category kk is an analog of Cuntz's triangulated category kk^{lca} whose objects are the locally convex algebras [Cuntz 1997; 2005; Cuntz and Thom 2006].

We show in Theorem 7.4 that, if we denote by $kk(\Re)$ the full subcategory of kk spanned by the objects from $\Re$ and assume that the cone ring Γk in the sense of [Karoubi and Villamayor 1969] is in $\Re$, then there is a natural triangulated equivalence

$$kk(\Re) \xrightarrow{\sim} \mathscr{T}^{\mathrm{op}}$$

sending $A \in kk(\Re)$ to its K-motive $M_K(A)$. Thus we can identify $kk(\Re)$ with the K-motives of algebras. It also follows that the *small* triangulated category $kk(\Re)^{\mathrm{op}}$ lives inside the ambient triangulated category $DK(\Re)$.

One of the equivalent approaches to stable motivic homotopy theory in the sense of Morel and Voevodsky [1999] is the theory of $(S^1, \mathbb{G}_m)$-bispectra. The role of $\mathbb{G}_m$ in our context is played by the representable functor $\mathbb{G} := r(\sigma)$, where $\sigma = (t-1)k[t^{\pm 1}]$. We develop the motivic theory of $(S^1, \mathbb{G})$-bispectra. As usual they form a model category which we denote by $\mathrm{Sp}_{\mathbb{G}}(\Re)$. The homotopy category $\mathrm{SH}_{S^1,\mathbb{G}}(\Re)$ of $\mathrm{Sp}_{\mathbb{G}}(\Re)$ plays the same role as the stable motivic homotopy category $\mathrm{SH}(F)$ over a field F. We construct an explicit fibrant $(S^1, \mathbb{G})$-bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -)$, obtained from fibrant S^1-spectra $\mathbb{K}(\sigma^n A, -)$, $n \geqslant 0$, by stabilisation in the σ-direction.

The main computational result for bispectra, stated in Theorem 8.1, says that $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -)$ is a fibrant replacement of the suspension bispectrum associated with an algebra A. Namely, there is a natural weak equivalence of bispectra in $\mathrm{Sp}_{\mathbb{G}}(\Re)$

$$\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA \to \Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -),$$

where $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$ is the suspension bispectrum of rA.

Let k be the field of complex numbers $\mathbb{C}$ and let $\mathscr{K}^\sigma(A, -)$ be the (0,0)-space of the bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -)$. We raise a question whether there is a category $\Re$ of commutative $\mathbb{C}$-algebras such that the fibrant simplicial set $\mathscr{K}^\sigma(\mathbb{C}, \mathbb{C})$ has the homotopy type of $\Omega^\infty\Sigma^\infty S^0$. The question is justified by a recent result of Levine [2014] saying that over an algebraically closed field F of characteristic zero the homotopy groups of weight zero of the motivic sphere spectrum evaluated at F are isomorphic to the stable homotopy groups of the classical sphere spectrum. The role of the motivic sphere spectrum in our context is played by the bispectrum $\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty}r\mathbb{C}$.

We finish the paper by proving that the triangulated category $DK(\mathfrak{R})$ of K-motives is fully faithfully embedded into the homotopy category of $(S^1, \mathbb{G})$-bispectra. We also construct an explicit fibrant $(S^1, \mathbb{G})$-bispectrum $\mathbb{KG}^{st}(A, -)$ consisting of fibrant S^1-spectra $\mathbb{K}^{st}(\sigma^n A, -)$, $n \geqslant 0$. For this we prove the "cancellation theorem" for the spectra $\mathbb{K}^{st}(\sigma^n A, -)$ (see Theorem 9.5). It is reminiscent of the cancellation theorem proved by Voevodsky [2010a] for motivic cohomology. The same theorem was proved for K-theory of algebraic varieties in [Garkusha and Panin 2015].

We show in Theorem 9.7 that $\mathbb{KG}^{st}(A, -)$ is (2, 1)-periodic and represents stable algebraic Kasparov K-theory (cf. [Voevodsky 1998, Theorems 6.8 and 6.9]). Precisely, for any algebras $A, B \in \mathfrak{R}$ and any integers p, q there is an isomorphism

$$\pi_{p,q}(\mathbb{KG}^{st}(A, B)) \cong \mathbb{K}^{st}_{p-2q}(A, B).$$

As a consequence, one has that for any algebra $B \in \mathfrak{R}$ and any integers p, q there is an isomorphism

$$\pi_{p,q}(\mathbb{KG}^{st}(k, B)) \cong KH_{p-2q}(B).$$

Thus the bispectrum $\mathbb{KG}^{st}(k, B)$ yields an explicit model for homotopy K-theory.

We should stress that the term "motivic" is used in the paper only for the reason that the $k[t]$-homotopy theory of algebras shares many properties with Morel and Voevodsky's motivic homotopy theory of smooth schemes [1999] (see remarks on page 288 as well). If there is a likelihood of confusion with other motivic theories of commutative or noncommutative objects, the reader can just omit the term "motivic" everywhere.

In general, we shall not be very explicit about set-theoretical foundations, and we shall tacitly assume we are working in some fixed universe $\mathbb{U}$ of sets. Members of $\mathbb{U}$ are then called *small sets*, whereas a collection of members of $\mathbb{U}$ which does not itself belong to $\mathbb{U}$ will be referred to as a *large set* or a *proper class*. If there is no likelihood of confusion, we replace $\otimes_k$ by $\otimes$.

2. Preliminaries

In this section we collect basic facts about admissible categories of algebras and triangulated categories associated with them. We mostly follow [Garkusha 2007; 2013].

2.1. ***Algebraic homotopy.*** Following [Gersten 1971b] a category $\mathfrak{R}$ of k-algebras without unit is *admissible* if it is a full subcategory of Alg_k and

(1) if R is in $\mathfrak{R}$ and I is a (two-sided) ideal of R then I and R/I are in $\mathfrak{R}$;

(2) if R is in $\mathfrak{R}$, then so is $R[x]$, the polynomial algebra, in one variable;

(3) given a cartesian square

$$\begin{array}{ccc} D & \xrightarrow{\rho} & A \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

in Alg_k with A, B, C in $\mathfrak{R}$, then D is in $\mathfrak{R}$.

One may abbreviate (1)–(3) by saying that $\mathfrak{R}$ is closed under operations of taking ideals, homomorphic images, polynomial extensions in a finite number of variables, and fibre products. For instance, the *category of commutative k-algebras* CAlg_k is admissible.

Observe that every k-module M can be regarded as a nonunital k-algebra with trivial multiplication: $m_1 \cdot m_2 = 0$ for all $m_1, m_2 \in M$. Then $\mathrm{Mod}\, k$ is an admissible category of commutative k-algebras.

If R is an algebra then the polynomial algebra $R[x]$ admits two homomorphisms onto R

$$R[x] \overset{\partial_x^0}{\underset{\partial_x^1}{\rightrightarrows}} R \ ,$$

where

$$\partial_x^i|_R = 1_R, \quad \partial_x^i(x) = i, \qquad i = 0, 1.$$

Of course, $\partial_x^1(x) = 1$ has to be understood in the sense that $\Sigma r_n x^n \mapsto \Sigma r_n$.

Definition. Two homomorphisms f_0, $f_1 : S \to R$ are *elementary homotopic*, written $f_0 \sim f_1$, if there exists a homomorphism

$$f : S \to R[x]$$

such that $\partial_x^0 f = f_0$ and $\partial_x^1 f = f_1$. A map $f : S \to R$ is called an *elementary homotopy equivalence* if there is a map $g : R \to S$ such that fg and gf are elementary homotopic to id_R and id_S respectively.

For example, let A be a $\mathbb{Z}_{n\geqslant 0}$-graded algebra, then the inclusion $A_0 \to A$ is an elementary homotopy equivalence. The homotopy inverse is given by the projection $A \to A_0$. Indeed, the map $A \to A[x]$ sending a homogeneous element $a_n \in A_n$ to $a_n x^n$ is a homotopy between the composite $A \to A_0 \to A$ and the identity id_A.

The relation "elementary homotopic" is reflexive and symmetric [Gersten 1971b, p. 62]. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol "$\simeq$"). Following notation of [Gersten 1971a], the set of equivalence classes of morphisms $R \to S$ is written $[R, S]$.

Lemma 2.1 [Gersten 1971a]. *Given morphisms in* Alg_k

$$R \xrightarrow{f} S \underset{g'}{\overset{g}{\rightrightarrows}} T \xrightarrow{h} U$$

such that $g \simeq g'$, then $gf \simeq g'f$ and $hg \simeq hg'$.

Thus homotopy behaves well with respect to composition and we have category *Hotalg*, the *homotopy category of k-algebras*, whose objects are k-algebras and such that $\mathrm{Hotalg}(R, S) = [R, S]$. The homotopy category of an admissible category of algebras $\Re$ will be denoted by $\mathscr{H}(\Re)$. Call a homomorphism $s : A \to B$ an *I-weak equivalence* if its image in $\mathscr{H}(\Re)$ is an isomorphism. Observe that I-weak equivalences are those homomorphisms which have homotopy inverses.

A diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in Alg_k is a short exact sequence if f is injective, g is surjective, and the image of f is equal to the kernel of g.

Definition. An algebra R is *contractible* if $0 \sim 1$; that is, if there is a homomorphism $f : R \to R[x]$ such that $\partial_x^0 f = 0$ and $\partial_x^1 f = 1_R$.

For example, every square zero algebra $A \in \mathrm{Alg}_k$ is contractible by means of the homotopy $A \to A[x]$, $a \in A \mapsto ax \in A[x]$. In other words, every k-module, regarded as a k-algebra with trivial multiplication, is contractible.

Following [Karoubi and Villamayor 1969] we define ER, the *path algebra* on R, as the kernel of $\partial_x^0 : R[x] \to R$, so

$$ER \to R[x] \xrightarrow{\partial_x^0} R$$

is a short exact sequence in Alg_k. Also $\partial_x^1 : R[x] \to R$ induces a surjection $\partial_x^1 : ER \to R$ and we define the *loop algebra* ΩR of R to be its kernel, so we have a short exact sequence in Alg_k

$$\Omega R \to ER \xrightarrow{\partial_x^1} R.$$

We call it the *loop extension* of R. Clearly, ΩR is the intersection of the kernels of ∂_x^0 and ∂_x^1. By [Gersten 1971b, Lemma 3.3] ER is contractible for any algebra R.

2.2. *Categories of fibrant objects.*

Definition. Let $\mathscr{A}$ be a category with finite products and a final object e. Assume that $\mathscr{A}$ has two distinguished classes of maps, called *weak equivalences* and *fibrations*. A map is called a *trivial fibration* if it is both a weak equivalence and a fibration. We define a *path space* for an object B to be an object B^I together with maps

$$B \xrightarrow{s} B^I \xrightarrow{(d_0, d_1)} B \times B,$$

where s is a weak equivalence, (d_0, d_1) is a fibration, and the composite is the diagonal map.

Following [Brown 1973], we call $\mathscr{A}$ a *category of fibrant objects* or a *Brown category* if the following axioms are satisfied.

(A) Let f and g be maps such that gf is defined. If two of f, g, gf are weak equivalences then so is the third. Any isomorphism is a weak equivalence.

(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(C) Given a diagram

$$A \xrightarrow{u} C \xleftarrow{v} B,$$

with v a fibration (respectively a trivial fibration), the pullback $A \times_C B$ exists and the map $A \times_C B \to A$ is a fibration (respectively a trivial fibration).

(D) For any object B in $\mathscr{A}$ there exists at least one path space B^I (not necessarily functorial in B).

(E) For any object B the map $B \to e$ is a fibration.

2.3. ***The triangulated category $D(\mathfrak{R}, \mathfrak{F})$.*** In what follows we denote by $\mathfrak{F}$ the class of k-split surjective algebra homomorphisms. We shall also refer to $\mathfrak{F}$ as *fibrations*.

Let $\mathfrak{W}$ be a class of weak equivalences in an admissible category of algebras $\Re$ containing homomorphisms $A \to A[t]$, $A \in \Re$, such that the triple $(\Re, \mathfrak{F}, \mathfrak{W})$ is a Brown category.

Definition. The *left derived category* $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ of $\Re$ with respect to $(\mathfrak{F}, \mathfrak{W})$ is the category obtained from $\Re$ by inverting the weak equivalences.

By [Garkusha 2013] the family of weak equivalences in the category $\mathscr{H}\Re$ admits a calculus of right fractions. The left derived category $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ (possibly "large") is obtained from $\mathscr{H}\Re$ by inverting the weak equivalences. The left derived category $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ is left triangulated (see [Garkusha 2007; 2013] for details) with Ω a loop functor on it.

There is a general method of stabilising Ω (see Heller [Heller 1968]) and producing a triangulated (possibly "large") category $D(\Re, \mathfrak{F}, \mathfrak{W})$ from the left triangulated structure on $D^-(\Re, \mathfrak{F}, \mathfrak{W})$.

An object of $D(\Re, \mathfrak{F}, \mathfrak{W})$ is a pair (A, m) with $A \in D^-(\Re, \mathfrak{F}, \mathfrak{W})$ and $m \in \mathbb{Z}$. If $m, n \in \mathbb{Z}$ then we consider the directed set $I_{m,n} = \{k \in \mathbb{Z} \mid m, n \leqslant k\}$. The morphisms between (A, m) and $(B, n) \in D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ are defined by

$$D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})[(A, m), (B, n)] := \operatorname{colim}_{k \in I_{m,n}} D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})(\Omega^{k-m}(A), \Omega^{k-n}(B)).$$

Morphisms of $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ are composed in the obvious fashion. We define the *loop* automorphism on $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ by $\Omega(A, m) := (A, m-1)$. There is a natural functor $S : D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}) \to D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ defined by $A \longmapsto (A, 0)$.

$D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ is an additive category [Garkusha 2007; 2013]. We define a triangulation $\mathscr{T}r(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ of the pair $(D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}), \Omega)$ as follows. A sequence

$$\Omega(A, l) \to (C, n) \to (B, m) \to (A, l)$$

belongs to $\mathscr{T}r(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ if there is an even integer k and a left triangle of representatives $\Omega(\Omega^{k-l}(A)) \to \Omega^{k-n}(C) \to \Omega^{k-m}(B) \to \Omega^{k-l}(A)$ in $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$. Then the functor S takes left triangles in $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ to triangles in $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$. By [Garkusha 2007; 2013] $\mathscr{T}r(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ is a triangulation of $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ in the classical sense of [Verdier 1996].

By an $\mathfrak{F}$-*extension* or just *extension* in $\mathfrak{R}$ we mean a short exact sequence of algebras

$$(E) : A \to B \xrightarrow{\alpha} C$$

such that $\alpha \in \mathfrak{F}$. Let $\mathscr{E}$ be the class of all $\mathfrak{F}$-extensions in $\mathfrak{R}$.

Definition. Following [Cortiñas and Thom 2007] a *($\mathfrak{F}$-)excisive homology theory* on $\mathfrak{R}$ with values in a triangulated category $(\mathscr{T}, \Omega)$ consists of a functor $X : \mathfrak{R} \to \mathscr{T}$, together with a collection $\{\partial_E : E \in \mathscr{E}\}$ of maps $\partial_E^X = \partial_E \in \mathscr{T}(\Omega X(C), X(A))$. The maps ∂_E are to satisfy the following requirements.

(1) For all $E \in \mathscr{E}$ as above,

$$\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

is a distinguished triangle in $\mathscr{T}$.

(2) If

$$\begin{array}{lccccc} (E): & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ (E'): & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

is a map of extensions, then the following diagram commutes

$$\begin{tikzcd} \Omega X(C) \arrow[r, "\partial_E"] \arrow[d, "\Omega X(\gamma)"'] & X(A) \arrow[d, "X(\alpha)"] \\ \Omega X(C') \arrow[r, "\partial_{E'}"'] & X(A) \end{tikzcd}$$

We say that the functor $X : \Re \to \mathscr{T}$ is *homotopy invariant* if it maps homotopic homomorphisms to equal maps, or equivalently, if for every $A \in \mathrm{Alg}_k$, X maps the inclusion $A \subset A[t]$ to an isomorphism.

Denote by $\mathfrak{W}_\triangle$ the class of homomorphisms f such that $X(f)$ is an isomorphism for any excisive, homotopy invariant homology theory $X : \Re \to \mathscr{T}$. We shall refer to the maps from $\mathfrak{W}_\triangle$ as *stable weak equivalences*. The triple $(\Re, \mathfrak{F}, \mathfrak{W}_\triangle)$ is a Brown category. In what follows we shall write $D^-(\Re, \mathfrak{F})$ and $D(\Re, \mathfrak{F})$ to denote $D^-(\Re, \mathfrak{F}, \mathfrak{W}_\triangle)$ and $D(\Re, \mathfrak{F}, \mathfrak{W}_\triangle)$ respectively, dropping $\mathfrak{W}_\triangle$ from the notation.

By [Garkusha 2013] the canonical functor

$$\Re \to D(\Re, \mathfrak{F})$$

is the universal excisive, homotopy invariant homology theory on $\Re$.

3. Homotopy theory of algebras

Let $\Re$ be a *small* admissible category of algebras. We shall work with various model category structures for the category of simplicial functors on $\Re$. We mostly adhere to [Garkusha 2007; 2014].

3.1. ***The categories of pointed simplicial functors $U_\bullet\Re$.*** Throughout this paper we work with a model category $U_\bullet\Re$. To define it, we first enrich $\Re$ over pointed simplicial sets $\mathbb{S}_\bullet$. Given an algebra $A \in \Re$, denote by rA the representable functor $\mathrm{Hom}_\Re(A, -)$. Let $\Re_\bullet$ have the same objects as $\Re$ and have pointed simplicial sets of morphisms being the $rA(B) = \mathrm{Hom}_\Re(A, B)$ pointed at zero. Denote by $U_\bullet\Re$ the category of $\mathbb{S}_\bullet$-enriched functors from $\Re_\bullet$ to $\mathbb{S}_\bullet$. One easily checks that $U_\bullet\Re$ can be regarded as the category of covariant pointed simplicial functors $X : \Re \to \mathbb{S}_\bullet$ such that $X(0) = *$.

By [Dundas et al. 2003, Theorem 4.2] we define the projective model structure on $U_\bullet\Re$. This is a proper, simplicial, cellular model category with weak equivalences and fibrations being defined object-wise, and cofibrations being those maps having the left lifting property with respect to trivial fibrations.

The class of projective cofibrations for $U_\bullet\Re$ is generated by the set

$$I_{U_\bullet\Re} = \{rA \wedge (\partial\Delta^n \subset \Delta^n)_+\}^{n \geqslant 0},$$

indexed by $A \in \Re$. Likewise, the class of acyclic projective cofibrations is generated by

$$J_{U_\bullet \Re} = \{rA \wedge (\Lambda^k_n \subset \Delta^n)_+\}^{n>0}_{0\leqslant k\leqslant n}.$$

Given $\mathscr{X}, \mathscr{Y} \in U_\bullet \Re$ the pointed function complex $\operatorname{Map}_\bullet(\mathscr{X}, \mathscr{Y})$ is defined as

$$\operatorname{Map}_\bullet(\mathscr{X}, \mathscr{Y})_n = \operatorname{Hom}_{U_\bullet \Re}(\mathscr{X} \wedge \Delta^n_+, \mathscr{Y}), \quad n \geqslant 0.$$

By [Dundas et al. 2003, Lemma 2.1] there is a natural isomorphism of pointed simplicial sets

$$\operatorname{Map}_\bullet(rA, \mathscr{X}) \cong \mathscr{X}(A)$$

for all $A \in \Re$ and $\mathscr{X} \in U_\bullet \Re$.

Recall that the model category $U\Re$ of functors from $\Re$ to unpointed simplicial sets $\mathbb{S}$ is defined in a similar fashion (see [Garkusha 2007]). Since we mostly work with spectra in this paper, the category of spectra associated with $U_\bullet \Re$ is technically more convenient than the category of spectra associated with $U\Re$.

3.2. ***The model categories $U_\bullet \Re_I$, $U_\bullet \Re_J$, $U_\bullet \Re_{I,J}$.*** Let

$$I = \{i = i_A : r(A[t]) \to r(A) \mid A \in \Re\},$$

where each i_A is induced by the natural homomorphism $i : A \to A[t]$. Recall that a functor $F : \Re \to \mathbb{S}_\bullet/$ Spectra is *homotopy invariant* if $F(A) \to F(A[t])$ is a weak equivalence for all $A \in \Re$. Consider the projective model structure on $U_\bullet \Re$. We shall refer to the I-local equivalences as (projective) I-weak equivalences. Denote by $U_\bullet \Re_I$ the model category obtained from $U_\bullet \Re$ by Bousfield localisation with respect to the family I. Notice that any objectwise fibrant homotopy invariant functor $F \in U_\bullet \Re$ is an I-local object, hence fibrant in $U_\bullet \Re_I$.

Let us introduce the class of excisive functors on $\Re$. They look like flasque presheaves on a site defined by a cd-structure in the sense of [Voevodsky 2010b, Section 3].

Definition. A simplicial functor $\mathscr{X} \in U_\bullet \Re$ is called *excisive* with respect to $\mathfrak{F}$ if for any cartesian square in $\Re$

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & C \end{array}$$

with f a fibration (we call such squares *distinguished*), the square of simplicial sets

$$\begin{array}{ccc} \mathscr{X}(D) & \longrightarrow & \mathscr{X}(A) \\ \big\downarrow & & \big\downarrow \\ \mathscr{X}(B) & \longrightarrow & \mathscr{X}(C) \end{array}$$

is a homotopy pullback square. It immediately follows from the definition that every excisive object takes $\mathfrak{F}$-extensions in $\Re$ to homotopy fibre sequences of simplicial sets.

Let α denote a distinguished square in $\Re$ as shown:

$$\begin{array}{ccc} D & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & C \end{array}$$

Let us apply the simplicial mapping cylinder construction cyl to α and form the pushouts:

$$\begin{array}{ccccc} rC & \longrightarrow & \mathrm{cyl}(rC \to rA) & \longrightarrow & rA \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ rB & \longrightarrow & \mathrm{cyl}(rC \to rA) \sqcup_{rC} rB & \longrightarrow & rD \end{array}$$

Note that $rC \to \mathrm{cyl}(rC \to rA)$ is a projective cofibration between (projective) cofibrant objects of $U_\bullet\Re$. Thus $s(\alpha) = \mathrm{cyl}(rC \to rA) \sqcup_{rC} rB$ is (projective) cofibrant [Hovey 1999, 1.1.11]. For the same reasons, applying the simplicial mapping cylinder to $s(\alpha) \to rD$ and setting $t(\alpha) = \mathrm{cyl}(s(\alpha) \to rD)$ we get a projective cofibration

$$\mathrm{cyl}(\alpha)\colon s(\alpha) \longrightarrow t(\alpha).$$

Let $J_{U_\bullet\Re}^{\mathrm{cyl}(\alpha)}$ consists of all pushout product maps

$$s(\alpha) \wedge \Delta_+^n \sqcup_{s(\alpha)\wedge\partial\Delta_+^n} t(\alpha) \wedge \partial\Delta_+^n \longrightarrow t(\alpha) \wedge \Delta_+^n \ ,$$

and let $J = J_{U_\bullet\Re} \cup J_{U_\bullet\Re}^{\mathrm{cyl}(\alpha)}$. Denote by $U_\bullet\Re_J$ the model category obtained from $U_\bullet\Re$ by Bousfield localisation with respect to the family J. It is directly verified that $\mathscr{X} \in U_\bullet\Re$ is J-local if and only if it has the right lifting property with respect to J. Also, $\mathscr{X}$ is J-local if and only if it is objectwise fibrant and excisive [Garkusha 2007, Lemma 4.3].

Finally, let us introduce the model category $U_\bullet\Re_{I,J}$. It is, by definition, the Bousfield localisation of $U_\bullet\Re$ with respect to $I \cup J$. The weak equivalences (trivial cofibrations) of $U_\bullet\Re_{I,J}$ will be referred to as (projective) (I, J)-weak equivalences ((projective) (I, J)-trivial cofibrations). By [Garkusha 2007, Lemma 4.5] a functor $\mathscr{X} \in U_\bullet\mathfrak{R}$ is (I, J)-local if and only if it is objectwise fibrant, homotopy invariant and excisive.

Remark. The model category $U_\bullet\mathfrak{R}_{I,J}$ can also be regarded as a kind of unstable motivic model category associated with $\mathfrak{R}$. Indeed, the construction of $U_\bullet\mathfrak{R}_{I,J}$ is similar to Morel–Voevodsky's unstable motivic theory for smooth schemes Sm/F over a field F [Morel and Voevodsky 1999]. If we replace I by

$$I' = \{X \times \mathbb{A}^1 \xrightarrow{pr} X \mid X \in \mathrm{Sm}/F\},$$

and the family of distinguished squares by the family of elementary Nisnevich squares and get the corresponding family J' associated to it, then one of the equivalent models for Morel–Voevodsky's unstable motivic theory is obtained by Bousfield localisation of simplicial presheaves with respect to $I' \cup J'$.

For this reason, $U_\bullet\mathfrak{R}_{I,J}$ can also be called the category of (pointed) *motivic* spaces, where each algebra A is identified with the pointed motivic space rA. One can also refer to (I, J)-weak equivalences as *motivic weak equivalences*.

3.3. ***Monoidal structure on $U_\bullet\mathfrak{R}$.*** In this section we mostly follow [Østvær 2010, Section 2.1]. Suppose $\mathfrak{R}$ is tensor closed, that is $k \in \mathfrak{R}$ and $A \otimes B \in \mathfrak{R}$ for all $A, B \in \mathfrak{R}$. We introduce the monoidal product $\mathscr{X} \otimes \mathscr{Y}$ of $\mathscr{X}$ and $\mathscr{Y}$ in $U_\bullet\mathfrak{R}$ by the formulas

$$\mathscr{X} \otimes \mathscr{Y}(A) = \operatorname*{colim}_{A_1 \otimes A_2 \to A} \mathscr{X}(A_1) \wedge \mathscr{Y}(A_2).$$

The colimit is indexed on the category with objects $\alpha\colon A_1 \otimes A_2 \to A$ and maps the pairs of maps $(\varphi, \psi)\colon (A_1, A_2) \to (A_1', A_2')$ such that $\alpha'(\psi \otimes \varphi) = \alpha$. By functoriality of colimits it follows that $\mathscr{X} \otimes \mathscr{Y}$ is in $U_\bullet\mathfrak{R}$.

The tensor product can also be defined by the formula

$$\mathscr{X} \otimes \mathscr{Y}(A) = \int^{A_1, A_2 \in \mathfrak{R}} (\mathscr{X}(A_1) \wedge \mathscr{Y}(A_2)) \wedge \mathrm{Hom}_{\mathfrak{R}}(A_1 \otimes A_2, A).$$

This formula is obtained from a theorem of Day [1970], which also asserts that the triple $(U_\bullet\mathfrak{R}, \otimes, r(k))$ forms a closed symmetric monoidal category.

The internal Hom functor, right adjoint to $\mathscr{X} \otimes -$, is given by

$$\underline{\mathrm{Hom}}(\mathscr{X}, \mathscr{Y})(A) = \int_{B \in \mathfrak{R}} \mathrm{Map}_\bullet(\mathscr{X}(B), \mathscr{Y}(A \otimes B)),$$

where $\mathrm{Map}_\bullet$ stands for the function complex in $\mathbb{S}_\bullet$.

So there exist natural isomorphisms

$$\underline{\mathrm{Hom}}(\mathscr{X} \otimes \mathscr{Y}, \mathscr{Z}) \cong \underline{\mathrm{Hom}}(\mathscr{X}, \underline{\mathrm{Hom}}(\mathscr{Y}, \mathscr{Z}))$$

and

$$\underline{\mathrm{Hom}}(r(k), \mathscr{Z}) \cong \mathscr{Z}.$$

Concerning smash products of representable functors, one has a natural isomorphism

$$rA \otimes rB \cong r(A \otimes B), \quad A, B \in \Re.$$

Note as well that, for pointed simplicial sets K and L, one has $K \otimes L = K \wedge L$.

We recall a pointed simplicial set tensor and cotensor structure on $U_\bullet\Re$. If $\mathscr{X}$ and $\mathscr{Y}$ are in $U_\bullet\Re$ and K is a pointed simplicial set, the tensor $\mathscr{X} \otimes K$ is given by

$$\mathscr{X} \otimes K(A) = \mathscr{X}(A) \wedge K$$

and the cotensor $\mathscr{Y}^K$ is given in terms of the ordinary function complex:

$$\mathscr{Y}^K(A) = \mathrm{Map}_\bullet(K, \mathscr{Y}(A)).$$

The function complex $\mathrm{Map}_\bullet(\mathscr{X}, \mathscr{Y})$ of $\mathscr{X}$ and $\mathscr{Y}$ is defined by setting

$$\mathrm{Map}_\bullet(\mathscr{X}, \mathscr{Y})_n = \mathrm{Hom}_{U_\bullet\Re}(\mathscr{X} \otimes \Delta^n_+, \mathscr{Y}).$$

By the Yoneda lemma there exists a natural isomorphism of pointed simplicial sets

$$\mathrm{Map}_\bullet(rA, \mathscr{Y}) \cong \mathscr{Y}(A).$$

Using these definitions $U_\bullet\Re$ is enriched in pointed simplicial sets $\mathbb{S}_\bullet$. Moreover, there are natural isomorphisms of pointed simplicial sets

$$\mathrm{Map}_\bullet(\mathscr{X} \otimes K, \mathscr{Y}) \cong \mathrm{Map}_\bullet(K, \mathrm{Map}_\bullet(\mathscr{X}, \mathscr{Y})) \cong \mathrm{Map}_\bullet(\mathscr{X}, \mathscr{Y}^K).$$

It is also useful to note that

$$\underline{\mathrm{Hom}}(\mathscr{X}, \mathscr{Y})(A) = \mathrm{Map}_\bullet(\mathscr{X}, \mathscr{Y}(A \otimes -)) \quad \text{and} \quad \underline{\mathrm{Hom}}(rB, \mathscr{Y}) = \mathscr{Y}(- \otimes B).$$

It can be shown similarly to [Østvær 2010, Lemma 3.10; Propositions 3.43 and 3.89] that the model categories $U_\bullet\Re$, $U_\bullet\Re_I$, $U_\bullet\Re_J$, $U_\bullet\Re_{I,J}$ are monoidal.

4. Unstable algebraic Kasparov *K*-theory

Let $\mathscr{U}$ be an arbitrary category and let $\Re$ be an admissible category of k-algebras. Suppose that there are functors $F : \Re \to \mathscr{U}$ and $\widetilde{T} : \mathscr{U} \to \Re$ such that $\widetilde{T}$ is left adjoint to F. We denote $\widetilde{T}FA$, for $A \in \Re$, by TA and the counit map $\widetilde{T}FA \to A$ by η_A. If $X \in \mathrm{Ob}\,\mathscr{U}$ then the unit map $X \to F\widetilde{T}X$ is denoted by i_X. We note that the composition

$$FA \xrightarrow{i_{FA}} F\widetilde{T}FA \xrightarrow{F\eta_A} FA$$

equals 1_{FA} for every $A \in \Re$, and hence $F\eta_A$ splits in $\mathscr{U}$. We call an admissible category of k-algebras *T-closed* if $TA \in \Re$ for all $A \in \Re$.

Lemma 4.1. *Suppose $\mathscr{U}$ is either a full subcategory of the category of sets or a full subcategory of the category of k-modules. Suppose as well that $F : \Re \to \mathscr{U}$ is the forgetful functor. Then for every $A \in \Re$ the algebra TA is contractible, i.e., there is a contraction $\tau : TA \to TA[x]$ such that $\partial_x^0 \tau = 0$, $\partial_x^1 \tau = 1$. Moreover, the contraction is functorial in A.*

Proof. Consider a map $u : FTA \to FTA[x]$ sending an element $b \in FTA$ to $bx \in FTA[x]$. By assumption, u is a morphisms of $\mathscr{U}$. The desired contraction τ is uniquely determined by the map $u \circ i_{FA} : FA \to FTA[x]$. By using elementary properties of adjoint functors, one can show that $\partial_x^0 \tau = 0$ and $\partial_x^1 \tau = 1$. $\square$

Throughout this paper, whenever we deal with a T-closed admissible category of k-algebras $\Re$ we assume to be fixed an underlying category $\mathscr{U}$, which is a full subcategory of $\operatorname{Mod} k$.

Examples. (1) Let $\Re = \mathrm{Alg}_k$. Given an algebra A, consider the algebraic tensor algebra

$$TA = A \oplus A \otimes A \oplus A^{\otimes^3} \oplus \cdots,$$

with the usual product given by concatenation of tensors. In Cuntz's treatment of bivariant K-theory [Cuntz 1997; 2005; Cuntz and Thom 2006], tensor algebras play a prominent role.

There is a canonical k-linear map $A \to TA$ mapping A into the first direct summand. Every k-linear map $s : A \to B$ into an algebra B induces a homomorphism $\gamma_s : TA \to B$ defined by

$$\gamma_s(x_1 \otimes \cdots \otimes x_n) = s(x_1)s(x_2)\cdots s(x_n).$$

Plainly $\Re$ is T-closed.

(2) If $\Re = \mathrm{CAlg}_k$, then

$$T(A) = \mathrm{Sym}(A) = \bigoplus_{n \geqslant 1} S^n A,$$

the symmetric algebra of A, and $\Re$ is T-closed. Here

$$S^n A = A^{\otimes n} / \langle a_1 \otimes \cdots \otimes a_n - a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \rangle \quad \text{for } \sigma \in \Sigma_n.$$

We have a natural extension of algebras

$$0 \longrightarrow JA \xrightarrow{\iota_A} TA \xrightarrow{\eta_A} A \longrightarrow 0.$$

Here JA is defined as $\operatorname{Ker} \eta_A$. Clearly, JA is functorial in A.

Given a small T-closed admissible category of k-algebras $\Re$, we denote by $\mathrm{Sp}(\Re)$ the category of S^1-spectra in the sense of [Hovey 2001] associated with the model category $U_\bullet \Re_{I,J}$. Recall that a spectrum consists of sequences $\mathscr{E} = (\mathscr{E}_n)_{n \geqslant 0}$ of pointed simplicial functors in $U_\bullet\mathfrak{R}$ equipped with structure maps $\sigma_n^{\mathscr{E}}:\Sigma\mathscr{E}_n\to\mathscr{E}_{n+1}$, where $\Sigma=-\wedge S^1$ is the suspension functor. A map $f:\mathscr{E}\to\mathscr{F}$ of spectra consists of compatible maps $f_n:\mathscr{E}_n\to\mathscr{F}_n$ in the sense that the diagrams

$$\begin{tikzcd} \Sigma\mathscr{E}_n \arrow[r, "\sigma_n^{\mathscr{E}}"] \arrow[d, "\Sigma f_n"'] & \mathscr{E}_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma\mathscr{F}_n \arrow[r, "\sigma_n^{\mathscr{F}}"] & \mathscr{F}_{n+1} \end{tikzcd}$$

commute for all $n\geqslant 0$. The category $\mathrm{Sp}(\mathfrak{R})$ is endowed with the stable model structure (see [Hovey 2001] for details).

Given an algebra $A\in\mathfrak{R}$, we denote by $\Sigma^\infty rA$ the suspension spectrum associated with the functor rA pointed at zero. By definition, $(\Sigma^\infty rA)_n=rA\wedge S^n$ with obvious structure maps.

In order to define one of the main spectra of the paper $\mathscr{R}(A)$ associated to an algebra $A\in\mathfrak{R}$, we have to recall some definitions from [Garkusha 2014].

For any $B\in\mathfrak{R}$ we define a simplicial algebra

$$B^\Delta:[n]\mapsto B^{\Delta^n}:=B[t_0,\dots,t_n]/\Big\langle 1-\sum_i t_i\Big\rangle \quad (\cong B[t_1,\dots,t_n]).$$

Given a map of posets $\alpha:[m]\to[n]$, the map $\alpha^*:B^{\Delta^n}\to B^{\Delta^m}$ is defined by $\alpha^*(t_j)=\sum_{\alpha(i)=j}t_i$. We have that $B^\Delta\cong B\otimes k^\Delta$ and B^Δ is pointed at zero.

For any pointed simplicial set $X\in\mathbb{S}_\bullet$, we denote by $B^\Delta(X)$ the simplicial algebra $\mathrm{Map}_\bullet(X,B^\Delta)$. The simplicial algebra associated to any unpointed simplicial set and any simplicial algebra is defined in a similar way. By $\mathbb{B}^\Delta(X)$ we shall mean the pointed simplicial ind-algebra

$$B^\Delta(X)\to B^\Delta(\mathrm{sd}^1 X)\to B^\Delta(\mathrm{sd}^2 X)\to\cdots.$$

In particular, one defines the "path space" simplicial ind-algebra $P\mathbb{B}^\Delta$. We shall also write $\mathbb{B}^\Delta(\Omega^n)$ to denote $\mathbb{B}^\Delta(S^n)$, where $S^n=S^1\wedge\cdots\wedge S^1$ is the simplicial n-sphere. For any $A\in\mathfrak{R}$ we denote by $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A,\mathbb{B}^\Delta(\Omega^n))$ the colimit of the sequence in $\mathbb{S}_\bullet$

$$\mathrm{Hom}_{\mathrm{Alg}_k}(A,B^\Delta(S^n))\to\mathrm{Hom}_{\mathrm{Alg}_k}(A,B^\Delta(\mathrm{sd}^1 S^n))\to\mathrm{Hom}_{\mathrm{Alg}_k}(A,B^\Delta(\mathrm{sd}^2 S^n))\to\cdots.$$

The natural simplicial map $d_1:P\mathbb{B}^\Delta(\Omega^n)\to\mathbb{B}^\Delta(\Omega^n)$ has a natural k-linear splitting described below. Let $\mathbf{t}\in P\Bbbk^\Delta(\Delta^1\times\overset{n}{\cdots}\times\Delta^1)_0$ stand for the composite map

$$\mathrm{sd}^m(\Delta^1\times\overset{n+1}{\cdots}\times\Delta^1)\xrightarrow{pr}\mathrm{sd}^m\Delta^1\to\Delta^1\xrightarrow{t}k^\Delta,$$

where pr is the projection onto the $(n+1)$-th direct factor Δ^1 and $t = t_0 \in k^{\Delta^1}$. The element $\mathbf{t}$ can be regarded as a 1-simplex of the unital ind-algebra

$$\Bbbk^{\Delta}(\Delta^1 \times \overset{n}{\cdots} \times \Delta^1)$$

such that $\partial_0(\mathbf{t}) = 0$ and $\partial_1(\mathbf{t}) = 1$. Let $\iota : \mathbb{B}^{\Delta}(\Omega^n) \to (\mathbb{B}^{\Delta}(\Omega^n))^{\Delta^1}$ be the natural inclusion. Multiplication with $\mathbf{t}$ determines a k-linear map

$$(\mathbb{B}^{\Delta}(\Omega^n))^{\Delta^1} \xrightarrow{\mathbf{t}\cdot} P\mathbb{B}^{\Delta}(\Omega^n).$$

Now the desired k-linear splitting $\mathbb{B}^{\Delta}(\Omega^n) \xrightarrow{\upsilon} P\mathbb{B}^{\Delta}(\Omega^n)$ of simplicial ind-modules is defined as

$$\upsilon := \mathbf{t} \cdot \iota.$$

If we consider $\mathbb{B}^{\Delta}(\Omega^n)$ as a $(\mathbb{Z}_{\geqslant 0} \times \Delta)$-diagram in $\Re$, then there is a commutative diagram of extensions for $(\mathbb{Z}_{\geqslant 0} \times \Delta)$-diagrams

$$\begin{tikzcd}
J\mathbb{B}^{\Delta}(\Omega^n) \arrow[r] \arrow[d, "\xi_\upsilon"'] & T\mathbb{B}^{\Delta}(\Omega^n) \arrow[r] \arrow[d] & \mathbb{B}^{\Delta}(\Omega^n) \arrow[d, equal] \\
\mathbb{B}^{\Delta}(\Omega^{n+1}) \arrow[r] & P\mathbb{B}^{\Delta}(\Omega^n) \arrow[r, "d_1"] & \mathbb{B}^{\Delta}(\Omega^n)
\end{tikzcd}$$

where the map ξ_υ is uniquely determined by the k-linear splitting υ. For every element $f \in \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^{\Delta}(\Omega^n))$ one sets:

$$\varsigma(f) := \xi_\upsilon \circ J(f) \in \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, \mathbb{B}^{\Delta}(\Omega^{n+1})).$$

The spectrum $\mathscr{R}(A)$ is defined at every $B \in \Re$ as the sequence of spaces pointed at zero

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^{\Delta}), \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, \mathbb{B}^{\Delta}), \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^2A, \mathbb{B}^{\Delta}), \dots .$$

By [Garkusha 2014, Section 2] each $\mathscr{R}(A)_n(B)$ is a fibrant simplicial set and

$$\Omega^k \mathscr{R}(A)_0(B) = \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^{\Delta}(\Omega^k)).$$

Each structure map $\sigma_n : \mathscr{R}(A)_n \wedge S^1 \to \mathscr{R}(A)_{n+1}$ is defined at B as adjoint to the map $\varsigma : \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^{\Delta}) \to \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, \mathbb{B}^{\Delta}(\Omega))$.

For every $A \in \Re$ there is a natural map in $\mathrm{Sp}(\Re)$

$$i : \Sigma^\infty rA \to \mathscr{R}(A)$$

functorial in A.

Definition [Garkusha 2014]. (1) Given two k-algebras $A, B \in \Re$, the *unstable algebraic Kasparov K-theory space* $\mathscr{K}(A, B)$ is the fibrant space

$$\mathrm{colim}_n\, \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^{\Delta}(\Omega^n)),$$

where the colimit maps are given by ξ_υ-s and JA is as defined on page 290. Its homotopy groups will be denoted by $\mathscr{K}_n(A, B)$, $n \geqslant 0$. The simplicial functor $\mathscr{K}(A, -)$ is fibrant in $U_\bullet(\Re)_{I,J}$ by [Garkusha 2014, Section 4]. Also, there is a natural isomorphism of simplicial sets

$$\mathscr{K}(A, B) \cong \Omega\mathscr{K}(JA, B).$$

In particular, $\mathscr{K}(A, B)$ is an infinite loop space with $\mathscr{K}(A, B)$ which simplicially isomorphic to $\Omega^n\mathscr{K}(J^nA, B)$ (see [Garkusha 2014, Theorem 5.1]).

(2) The *unstable algebraic Kasparov KK-theory spectrum* of (A, B) consists of the sequence of spaces

$$\mathscr{K}(A, B), \mathscr{K}(JA, B), \mathscr{K}(J^2A, B), \dots,$$

together with the natural isomorphisms $\mathscr{K}(J^nA, B) \cong \Omega\mathscr{K}(J^{n+1}A, B)$. It forms an Ω-spectrum which we also denote by $\mathbb{K}(A, B)$. Its homotopy groups will be denoted by $\mathbb{K}_n(A, B)$, $n \in \mathbb{Z}$. Observe that $\mathbb{K}_n(A, B) \cong \mathscr{K}_n(A, B)$ for any $n \geqslant 0$ and $\mathbb{K}_n(A, B) \cong \mathscr{K}_0(J^{-n}A, B)$ for any $n < 0$.

There is a natural map of spectra

$$j : \mathscr{R}(A) \to \mathbb{K}(A, -).$$

By [Garkusha 2014, Section 6] this is a stable equivalence and $\mathbb{K}(A, -)$ is a fibrant object of $\mathrm{Sp}(\Re)$. In fact, for any algebra $B \in \Re$ the map

$$j : \mathscr{R}(A)(B) \to \mathbb{K}(A, B)$$

is a stable equivalence of ordinary spectra.

The following theorem is crucial in our analysis. It states that $\mathbb{K}(A, -)$ is a fibrant replacement of $\Sigma^\infty rA$ in $\mathrm{Sp}(\Re)$.

Theorem 4.2. *Given $A \in \Re$ the map $i : \Sigma^\infty rA \to \mathscr{R}(A)$ is a level (I, J)-weak equivalence, and therefore the composite map*

$$\Sigma^\infty rA \xrightarrow{i} \mathscr{R}(A) \xrightarrow{j} \mathbb{K}(A, -)$$

is a stable equivalence in $\mathrm{Sp}(\Re)$, *functorial in A.*

Proof. Recall that for any functor F from rings to simplicial sets, $\mathrm{Sing}(F)$ is defined at each ring R as the diagonal of the bisimplicial set $F(R[\Delta])$. The map

$$i_0 : (\Sigma^\infty rA)_0 \to \mathscr{R}(A)_0$$

equals $rA \to \mathrm{Ex}^\infty \circ \mathrm{Sing}(rA)$, which is an I-weak equivalence by [Garkusha 2007, Corollary 3.8]. Let us show that

$$i_1 : rA \wedge S^1 \to \mathscr{R}(A)_1 = \mathrm{Ex}^\infty \circ \mathrm{Sing}(r(JA))$$

is an $(I,\, J)$-weak equivalence. It is fully determined by the element $\rho_A : JA \to \Omega A$, which is a zero simplex of $\Omega(\mathrm{Ex}^\infty \circ \mathrm{Sing}(r(JA))(A))$, coming from the adjunction isomorphism

$$\mathrm{Map}_\bullet(rA \wedge S^1, \mathrm{Ex}^\infty \circ \mathrm{Sing}(r(JA))) \cong \Omega(\mathrm{Ex}^\infty \circ \mathrm{Sing}(r(JA))(A)).$$

Let $(I, 0)$ denote $\Delta[1]$ pointed at 0. Consider a commutative diagram of cofibrant objects in $U_\bullet\mathfrak{R}$

$$\begin{tikzcd}
rA \arrow[r, tail, "\nu"] \arrow[d, "\eta_A^*"'] & rA \wedge (I,0) \arrow[r, two heads] \arrow[d] & rA \wedge S^1 \arrow[d, equal] \\
r(TA) \arrow[r, tail] & \mathscr{X} \arrow[r, two heads, "\alpha"] & rA \wedge S^1
\end{tikzcd}$$

where the left square is pushout, the left map is induced by the canonical homomorphism $\eta_A : TA \to A$ and ν is induced by the natural inclusion $d^0 : \Delta[0] \to \Delta[1]$. Lemma 4.1 implies $r(TA)$ is weakly equivalent to zero in $U_\bullet\mathfrak{R}_I$. It follows that α is an I-weak equivalence.

By the universal property of pullback diagrams there is a unique morphism $\sigma : \mathscr{X} \to r(JA)$ whose restriction to $r(TA)$ equals ι_A^*, where $\iota_A = \mathrm{Ker}\,\eta_A$, which makes the diagram

$$\begin{tikzcd}
& rA \wedge (I,0) \arrow[rr] \arrow[dd] & & \mathscr{X} \arrow[dd, "\sigma"] \\
rA \arrow[ur] \arrow[rr] \arrow[dd, "1"'] & & r(TA) \arrow[ur] \arrow[dd] & \\
& pt \arrow[rr] & & r(JA) \\
rA \arrow[ur] \arrow[rr, "\eta_A^*"'] & & r(TA) \arrow[ur] &
\end{tikzcd}$$

commutative. Since the upper and the lower squares are homotopy pushouts in $U_\bullet\mathfrak{R}_{I,J}$ and $rA \wedge (I, 0)$ is weakly equivalent to zero, it follows from [Hirschhorn 2003, Proposition 13.5.10] that σ is an $(I,\, J)$-weak equivalence. Therefore the composite map, we shall denote it by ρ,

$$\mathscr{X} \xrightarrow{\sigma} r(JA) \to \mathscr{R}(A)_1$$

is an $(I,\, J)$-weak equivalence, where the right map is the natural I-weak equivalence.

Let $\mathscr{R}(A)_1[x] \in U_\bullet\mathfrak{R}$ be a simplicial functor defined as

$$\mathscr{R}(A)_1[x](B) = \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, \mathbb{B}^\Delta[x]) = Ex^\infty \circ \mathrm{Hom}_{\mathrm{Alg}_k}(JA, B[x]^\Delta), \qquad B \in \mathfrak{R}.$$

There is a natural map $s : \mathscr{R}(A)_1 \to \mathscr{R}(A)_1[x]$, induced by the monomorphism $B \to B[x]$ at each B. It follows from [Garkusha 2007, Proposition 3.2] that this map is a weak equivalence in $U_\bullet\mathfrak{R}$. The evaluation homomorphisms

$$\partial_x^0, \partial_x^1 : B[x] \to B$$

induce a map $(\partial_x^0, \partial_x^1) : \mathscr{R}(A)_1[x] \to \mathscr{R}(A)_1 \times \mathscr{R}(A)_1$, whose composition with s is the diagonal map $\mathscr{R}(A)_1 \to \mathscr{R}(A)_1 \times \mathscr{R}(A)_1$. We see that $\mathscr{R}(A)_1[x]$ is a path object for the projectively fibrant object $\mathscr{R}(A)_1$.

If we constructed a homotopy $H : \mathscr{X} \to \mathscr{R}(A)_1[x]$ such that $\partial_x^0 H = i_1\alpha$ and $\partial_x^1 H = \rho$ it would follow that $i_1\alpha$, being homotopic to the (I, J)-weak equivalence ρ, is an (I, J)-weak equivalence. Since also α is an (I, J)-weak equivalence, then so would be i_1.

The desired map H is uniquely determined by maps $h_1 : r(TA) \to \mathscr{R}(A)_1[x]$ and $h_2 : rA \wedge (I, 0) \to \mathscr{R}(A)_1[x]$ such that $h_1\eta_A^* = h_2\nu$ is defined as follows. The map h_1 is uniquely determined by the homomorphism $JA \to TA[x]$ which is the composition of ι_A and the contraction homomorphism $\tau : TA \to TA[x]$, functorial in A, that exists by Lemma 4.1. The map h_2 is uniquely determined by the one-simplex $JA \to A[\Delta^1][x]$ of $\mathrm{Ex}^\infty \circ \mathrm{Hom}_{\mathrm{Alg}_k}(JA, A[x]^\Delta)$ which is the composition of

$$\rho_A : JA \to \Omega A = (t^2 - t)A[t] \subset A[\Delta^1]$$

and the homomorphism $\omega : A[\Delta^1] \to A[\Delta^1][x]$ sending the variable t to

$$1 - (1 - t)(1 - x).$$

Thus we have shown that

$$i_1 : rA \wedge S^1 \to \mathscr{R}(A)_1$$

is an (I, J)-weak equivalence. It follows that the composite map

$$rA \wedge S^1 \xrightarrow{i_0 \wedge S^1} \mathscr{R}(A)_0 \wedge S^1 \xrightarrow{\sigma_0} \mathscr{R}(A)_1,$$

which is equal to i_1, is an (I, J)-weak equivalence. Hence σ_0 is an (I, J)-weak equivalence, because $i_0 \wedge S^1$ is an I-weak equivalence. More generally, one gets that every structure map

$$\mathscr{R}(A)_n \wedge S^1 \xrightarrow{\sigma_n} \mathscr{R}(A)_{n+1}$$

is an (I, J)-weak equivalence.

By induction, assume that $i_n : rA \wedge S^n \to \mathscr{R}(A)_n$ is an (I, J)-weak equivalence. Then $i_n \wedge S^1$ is an (I, J)-weak equivalence, and hence so is $i_{n+1} = \sigma_n \circ (i_n \wedge S^1)$. □

Denote by $\mathrm{SH}_{S^1}(\mathfrak{R})$ the stable homotopy category of $\mathrm{Sp}(\mathfrak{R})$. Since the endofunctor $- \wedge S^1$ is an equivalence on $\mathrm{SH}_{S^1}(\mathfrak{R})$ by [Hovey 2001], it follows from [Hovey 1999, Chapter 7] that $\mathrm{SH}_{S^1}(\mathfrak{R})$ is a triangulated category. Moreover, it is compactly generated with compact generators $\{(\Sigma^\infty rA)[n]\}_{A\in\mathfrak{R}, n\in\mathbb{Z}}$.

Corollary 4.3. *$\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$ is a family of compact generators for $\mathrm{SH}_{S^1}(\Re)$. Moreover, there is a natural isomorphism*

$$\mathrm{SH}_{S^1}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n(A, B)$$

for all $A, B \in \Re$ and $n \in \mathbb{Z}$.

Denote by $\mathscr{S}$ the full subcategory of $\mathrm{SH}_{S^1}(\Re)$ whose objects are

$$\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}.$$

The next statement gives another description of the triangulated category $D(\Re, \mathfrak{F})$.

Theorem 4.4. *The category $\mathscr{S}$ is triangulated. Moreover, there is a contravariant equivalence of triangulated categories*

$$T : D(\Re, \mathfrak{F}) \to \mathscr{S}.$$

Proof. By [Garkusha 2013] the natural functor

$$j : \Re \to D(\Re, \mathfrak{F})$$

is a universal excisive homotopy invariant homology theory. Consider the homology theory

$$t : \Re \to \mathrm{SH}_{S^1}(\Re)^{\mathrm{op}}$$

that takes an algebra $A \in \Re$ to $\Sigma^\infty rA$. It is homotopy invariant and excisive, hence there is a unique triangulated functor

$$T : D(\Re, \mathfrak{F}) \to \mathrm{SH}_{S^1}(\Re)^{\mathrm{op}},$$

such that $t = T \circ j$. If we apply T to the loop extension

$$\Omega A \to EA \to A,$$

we get an isomorphism

$$T(\Omega A) \cong \Sigma^\infty rA[1],$$

which is functorial in A.

It follows from Comparison Theorem B of [Garkusha 2014] and Corollary 4.3 that T is full and faithful. Every object of $\mathscr{S}$ is plainly equivalent to the image of an object in $D(\Re, \mathfrak{F})$. □

Remark. Suppose I is an infinite index set and $\{B_i\}_{i\in I}$ is a family of algebras from $\Re$ such that the algebra $B = \oplus_I B_i$ is in $\Re$. Then $\Sigma^\infty rB$ is a compact object of $\mathrm{SH}_{S^1}(\Re)$, but $\oplus_I \Sigma^\infty r(B_i)$ may not be compact. Furthermore, suppose $B = \oplus_I B_i$ is also a direct sum object of the B_i-s in the triangulated category $D(\Re, \mathfrak{F})$. Then $\mathrm{Hom}_{D(\Re,\mathfrak{F})}(B, \oplus_I C_i) \neq \oplus_I \mathrm{Hom}_{D(\Re,\mathfrak{F})}(B, C_i)$ in general, where $\{C_i\}_{i\in I}$ is a family of algebras from $\Re$ such that the algebra $\oplus_I C_i$ is in $\Re$.

For instance, consider the triangulated category KK of [Kasparov 1980], with which $D(\Re, \mathfrak{F})$ shares many properties. It follows from [Rosenberg and Schochet 1987, Theorem 1.12] that KK has countable coproducts given by $A = \oplus_I A_i$, where I is a countable set. However, the functor $KK(A, -)$ does not respect countable coproducts by [Rosenberg and Schochet 1987, Remark 7.12].

Recall from [Garkusha 2014] that we can vary $\Re$ in the following sense. If $\Re'$ is another T-closed admissible category of algebras containing $\Re$, then $D(\Re, \mathfrak{F})$ is a full subcategory of $D(\Re', \mathfrak{F})$.

5. Morita stable algebraic Kasparov *K*-theory

If A is an algebra and $n > 0$ is a positive integer, then there is a natural inclusion $\iota : A \to M_n A$ of algebras, sending A to the upper left corner of $M_n A$. Throughout this section $\Re$ is a small T-closed admissible category of k-algebras with $M_n A \in \Re$ for every $A \in \Re$ and $n \geqslant 1$.

Denote by $U_\bullet \Re^{\mathrm{mor}}_{I,J}$ the model category obtained from $U_\bullet \Re_{I,J}$ by Bousfield localisation with respect to the family of maps of cofibrant objects

$$\{r(M_n A) \to rA \mid A \in \Re, n > 0\}.$$

Let $\mathrm{Sp}_{\mathrm{mor}}(\Re)$ be the stable model category of S^1-spectra associated with $U_\bullet \Re^{\mathrm{mor}}_{I,J}$. Observe that it is also obtained from $\mathrm{Sp}(\Re)$ by Bousfield localisation with respect to the family of maps of cofibrant objects in $\mathrm{Sp}(\Re)$

$$\{F_s(r(M_n A)) \to F_s(rA) \mid A \in \Re, n > 0, s \geqslant 0\}.$$

Here $F_s : U_\bullet \Re^{\mathrm{mor}}_{I,J} \to \mathrm{Sp}_{\mathrm{mor}}(\Re)$ is the canonical functor adjoint to the evaluation functor $Ev_s : \mathrm{Sp}_{\mathrm{mor}}(\Re) \to U_\bullet \Re^{\mathrm{mor}}_{I,J}$.

Definition [Garkusha 2014]. (1) The *Morita stable algebraic Kasparov K-theory space* of two algebras $A, B \in \Re$ is the space

$$\mathscr{K}^{\mathrm{mor}}(A, B) = \mathrm{colim}(\mathscr{K}(A, B) \to \mathscr{K}(A, M_2 k \otimes B) \to \mathscr{K}(A, M_3 k \otimes B) \to \cdots).$$

Its homotopy groups will be denoted by $\mathscr{K}^{\mathrm{mor}}_n(A, B)$, $n \geqslant 0$.

(2) A functor $X : \Re \to \mathbb{S}/(\mathrm{Spectra})$ is *Morita invariant* if each morphism $X(A) \to X(M_n A)$, $A \in \Re$, $n > 0$, is a weak equivalence.

(3) An excisive, homotopy invariant homology theory $X : \Re \to \mathscr{T}$ is *Morita invariant* if each morphism $X(A) \to X(M_n A)$, $A \in \Re$, $n > 0$, is an isomorphism.

(4) The *Morita stable algebraic Kasparov K-theory spectrum* of $A, B \in \Re$ is the Ω-spectrum

$$\mathbb{K}^{\mathrm{mor}}(A, B) = (\mathscr{K}^{\mathrm{mor}}(A, B), \mathscr{K}^{\mathrm{mor}}(JA, B), \mathscr{K}^{\mathrm{mor}}(J^2 A, B), \ldots).$$

Denote by $\mathrm{SH}^{\mathrm{mor}}_{S^1}(\Re)$ the (stable) homotopy category of $\mathrm{Sp}_{\mathrm{mor}}(\Re)$. It is a compactly generated triangulated category with compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$. Let $\mathscr{S}_{\mathrm{mor}}$ be the full subcategory of $\mathrm{SH}^{\mathrm{mor}}_{S^1}(\Re)$ whose objects are $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$.

Recall the definition of the triangulated category $D_{\mathrm{mor}}(\Re,\mathfrak{F})$ from [Garkusha 2013]. Its objects are those of $\Re$ and the set of morphisms between two algebras $A, B\in\Re$ is defined as the colimit of the sequence of abelian groups

$$D(\Re,\mathfrak{F})(A,B)\to D(\Re,\mathfrak{F})(A,M_2B)\to D(\Re,\mathfrak{F})(A,M_3B)\to\cdots.$$

There is a canonical functor $\Re\to D_{\mathrm{mor}}(\Re,\mathfrak{F})$. It is a universal excisive, homotopy invariant and Morita invariant homology theory on $\Re$.

Theorem 5.1. *Given $A\in\Re$ the composite map*

$$\Sigma^\infty rA\xrightarrow{i}\mathscr{R}(A)\xrightarrow{j}\mathbb{K}(A,-)\to\mathbb{K}^{\mathrm{mor}}(A,-)\tag{5.1}$$

is a stable equivalence in $\mathrm{Sp}_{\mathrm{mor}}(\Re)$, *functorial in A. In particular, there is a natural isomorphism*

$$\mathrm{SH}^{\mathrm{mor}}_{S^1}(\Re)(\Sigma^\infty rB[n],\Sigma^\infty rA)\cong\mathbb{K}^{\mathrm{mor}}_n(A,B)$$

for all $A, B\in\Re$ and $n\in\mathbb{Z}$. Furthermore, the category $\mathscr{S}_{\mathrm{mor}}$ is triangulated and there is a contravariant equivalence of triangulated categories

$$T: D_{\mathrm{mor}}(\Re,\mathfrak{F})\to\mathscr{S}_{\mathrm{mor}}.$$

Proof. Let $\mathscr{S}^c$ and $\mathscr{S}^c_{\mathrm{mor}}$ be the categories of compact objects in $\mathrm{SH}_{S^1}(\Re)$ and $\mathrm{SH}^{\mathrm{mor}}_{S^1}(\Re)$ respectively. Denote by $\mathscr{R}$ the full triangulated subcategory of $\mathscr{S}$ generated by objects

$$\{\mathrm{cone}(\Sigma^\infty r(M_nA)\to\Sigma^\infty rA)[k]\mid A\in\Re, n>0, k\in\mathbb{Z}\}.$$

Let $\mathscr{R}^c$ be the thick closure of $\mathscr{R}$ in $\mathrm{SH}_{S^1}(\Re)$. It follows from [Neeman 1996, Theorem 2.1] that the natural functor

$$\mathscr{S}^c/\mathscr{R}^c\to\mathscr{S}^c_{\mathrm{mor}}$$

is full and faithful and $\mathscr{S}^c_{\mathrm{mor}}$ is the thick closure of $\mathscr{S}^c/\mathscr{R}^c$.

We claim that the natural functor

$$\mathscr{S}/\mathscr{R}\to\mathscr{S}^c/\mathscr{R}^c\tag{5.2}$$

is full and faithful. For this consider a map $\alpha: X\to Y$ in $\mathscr{S}^c$ such that its cone Z is in $\mathscr{R}^c$ and $Y\in\mathscr{S}$. We can find $Z'\in\mathscr{R}^c$ such that $Z\oplus Z'$ is isomorphic to an object $W \in \mathscr{R}$. Construct a commutative diagram in $\mathscr{S}^c$

$$\begin{array}{ccccccc} U & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & \Sigma U \\ {\scriptstyle s}\downarrow & & \| & & {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

where p is the natural projection. We see that αs is such that its cone W belongs to $\mathscr{R}$. Standard facts for Gabriel–Zisman localisation theory imply (5.2) is a fully faithful embedding. It also follows that

$$\mathscr{S}_{\mathrm{mor}} = \mathscr{S}/\mathscr{R}.$$

We want to compute Hom sets in $\mathscr{S}/\mathscr{R}$. For this observe first that there is a contravariant equivalence of triangulated categories

$$\tau : D(\Re, \mathfrak{F})/\mathfrak{U} \to \mathscr{S}_{\mathrm{mor}},$$

where $\mathfrak{U}$ is the smallest full triangulated subcategory of $D(\Re, \mathfrak{F})$ containing

$$\{\mathrm{cone}(A \xrightarrow{\iota} M_n A) \mid A \in \Re, n > 0\}.$$

This follows from Theorem 4.4.

By construction, every excisive homotopy invariant Morita invariant homology theory $\Re \to \mathscr{T}$ factors through $D(\Re, \mathfrak{F})/\mathfrak{U}$. Since $\Re \to D_{\mathrm{mor}}(\Re, \mathfrak{F})$ is a universal excisive homotopy invariant Morita invariant homology theory [Garkusha 2013], we see that there exists a triangle equivalence of triangulated categories

$$D_{\mathrm{mor}}(\Re, \mathfrak{F}) \simeq D(\Re, \mathfrak{F})/\mathfrak{U}.$$

So there is a natural contravariant triangle equivalence of triangulated categories

$$T : D_{\mathrm{mor}}(\Re, \mathfrak{F}) \to \mathscr{S}_{\mathrm{mor}}.$$

Using this and [Garkusha 2014, Theorem 9.8], there is a natural isomorphism

$$\mathscr{S}_{\mathrm{mor}}(\Sigma^\infty r B[n], \Sigma^\infty r A) \cong \mathbb{K}_n^{\mathrm{mor}}(A, B)$$

for all $A, B \in \Re$ and $n \in \mathbb{Z}$. The fact that (5.1) is a stable equivalence in $\mathrm{Sp}_{\mathrm{mor}}(\Re)$ is now obvious. □

6. Stable algebraic Kasparov *K*-theory

If A is an algebra set $M_\infty A = \cup_n M_n A$. There is a natural inclusion $\iota : A \to M_\infty A$ of algebras, sending A to the upper left corner of $M_\infty A$. Throughout the section $\Re$ is a small T-closed admissible category of k-algebras with $M_\infty(A) \in \Re$ for all $A \in \Re$.

Denote by $U_\bullet\mathfrak{R}^{\infty}_{I,J}$ the model category obtained from $U_\bullet\mathfrak{R}_{I,J}$ by Bousfield localisation with respect to the family of maps of cofibrant objects

$$\{r(M_\infty A) \to rA \mid A \in \mathfrak{R}\}.$$

Let $\mathrm{Sp}_\infty(\mathfrak{R})$ be the stable model category of S^1-spectra associated with $U_\bullet\mathfrak{R}^{\infty}_{I,J}$. Observe that it is also obtained from $\mathrm{Sp}(\mathfrak{R})$ by Bousfield localisation with respect to the family of maps of cofibrant objects in $\mathrm{Sp}(\mathfrak{R})$

$$\{F_s(r(M_\infty A)) \to F_s(rA) \mid A \in \mathfrak{R}, s \geqslant 0\}.$$

Definition [Garkusha 2014]. (1) The *stable algebraic Kasparov K-theory space* of two algebras $A, B \in \mathfrak{R}$ is the space

$$\mathscr{K}^{st}(A,B) = \operatorname{colim}(\mathscr{K}(A,B) \to \mathscr{K}(A, M_\infty k \otimes B) \to \mathscr{K}(A, M_\infty k \otimes M_\infty k \otimes B) \to \cdots).$$

Its homotopy groups will be denoted by $\mathscr{K}^{st}_n(A,B)$, $n \geqslant 0$.

(2) A functor $X : \mathfrak{R} \to \mathbb{S}/(\text{Spectra})$ is *stable* or *M_∞-invariant* if $X(A) \to X(M_\infty A)$ is a weak equivalence for all $A \in \mathfrak{R}$.

(3) An excisive, homotopy invariant homology theory $X : \mathfrak{R} \to \mathscr{T}$ is *stable* or *M_∞-invariant* if $X(A) \to X(M_\infty A)$ is an isomorphism for all $A \in \mathfrak{R}$.

(4) The *stable algebraic Kasparov K-theory spectrum* for $A, B \in \mathfrak{R}$ is the Ω-spectrum

$$\mathbb{K}^{st}(A,B) = (\mathscr{K}^{st}(A,B), \mathscr{K}^{st}(JA,B), \mathscr{K}^{st}(J^2A,B), \ldots).$$

Denote by $\mathrm{SH}^{\infty}_{S^1}(\mathfrak{R})$ the (stable) homotopy category of $\mathrm{Sp}_\infty(\mathfrak{R})$. It is a compactly generated triangulated category with compact generators $\{\Sigma^\infty rA[n]\}_{A\in\mathfrak{R}, n\in\mathbb{Z}}$. Let $\mathscr{S}_\infty$ be the full subcategory of $\mathrm{SH}^{\infty}_{S^1}(\mathfrak{R})$ whose objects are $\{\Sigma^\infty rA[n]\}_{A\in\mathfrak{R}, n\in\mathbb{Z}}$.

Recall from [Garkusha 2013] the definition of the triangulated category $D_{st}(\mathfrak{R}, \mathfrak{F})$. Its objects are those of $\mathfrak{R}$ and the set of morphisms between two algebras $A, B \in \mathfrak{R}$ is defined as the colimit of the sequence of abelian groups

$$D(\mathfrak{R},\mathfrak{F})(A,B) \to D(\mathfrak{R},\mathfrak{F})(A, M_\infty k \otimes_k B) \to D(\mathfrak{R},\mathfrak{F})(A, M_\infty k \otimes_k M_\infty k \otimes_k B) \to \cdots.$$

There is a canonical functor $\mathfrak{R} \to D_{st}(\mathfrak{R}, \mathfrak{F})$. It is the universal excisive, homotopy invariant and stable homology theory on $\mathfrak{R}$.

The proof of the next result literally repeats that of Theorem 5.1 if we replace the algebras M_nA with $M_\infty A$ and the categories $\mathscr{S}_{\mathrm{mor}}$ and $D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F})$ with $\mathscr{S}_\infty$ and $D_{st}(\mathfrak{R}, \mathfrak{F})$ respectively.

Theorem 6.1. *Given $A \in \mathfrak{R}$, the composite map*

$$\Sigma^\infty r A \xrightarrow{i} \mathscr{R}(A) \xrightarrow{j} \mathbb{K}(A, -) \to \mathbb{K}^{st}(A, -)$$

is a stable equivalence in $\mathrm{Sp}_\infty(\mathfrak{R})$, *functorial in A. In particular*, *there is a natural isomorphism*

$$\mathrm{SH}^\infty_{S^1}(\mathfrak{R})(\Sigma^\infty r B[n], \Sigma^\infty r A) \cong \mathbb{K}^{st}_n(A, B)$$

for all $A, B \in \mathfrak{R}$ and $n \in \mathbb{Z}$. Furthermore, *the category $\mathscr{S}_\infty$ is triangulated and there is a contravariant equivalence of triangulated categories*

$$T : D_{st}(\mathfrak{R}, \mathfrak{F}) \to \mathscr{S}_\infty.$$

Let ΓA, for $A \in \mathrm{Alg}_k$, be the algebra of $\mathbb{N} \times \mathbb{N}$-matrices which satisfy the following two properties.

(i) The set $\{a_{ij} \mid i, j \in \mathbb{N}\}$ is finite.

(ii) There exists a natural number $N \in \mathbb{N}$ such that each row and each column has at most N nonzero entries.

$M_\infty A \subset \Gamma A$ is an ideal. We put

$$\Sigma A = \Gamma A / M_\infty A.$$

We note that ΓA, ΣA are the cone and suspension rings of A considered by Karoubi and Villamayor [1969, p. 269], where a different but equivalent definition is given. By [Cortiñas and Thom 2007] there are natural ring isomorphisms

$$\Gamma A \cong \Gamma k \otimes A, \quad \Sigma A \cong \Sigma k \otimes A.$$

We call the short exact sequence

$$M_\infty A \rightarrowtail \Gamma A \twoheadrightarrow \Sigma A$$

the *cone extension*. By [Cortiñas and Thom 2007] $\Gamma A \twoheadrightarrow \Sigma A$ is a split surjection of k-modules.

Let τ be the k-algebra which is unital and free on two generators α and β satisfying the relation $\alpha\beta = 1$. By [Cortiñas and Thom 2007, Lemma 4.10.1] the kernel of the natural map

$$\tau \to k[t^{\pm 1}]$$

is isomorphic to $M_\infty k$. We set $\tau_0 = \tau \oplus_{k[t^{\pm 1}]} \sigma$.

Let A be a k-algebra. We get an extension

$$M_\infty A \longrightarrow \tau A \longrightarrow A[t^{\pm 1}],$$

and an analogous extension

$$M_\infty A \longrightarrow \tau_0 A \longrightarrow \sigma A.$$

Definition. We say that an admissible category of k-algebras $\Re$ is τ_0-*closed* (respectively Γ-*closed*) if $\tau_0 A \in \Re$ (respectively $\Gamma A \in \Re$) for all $A \in \Re$.

Cuntz [1997; 2005; Cuntz and Thom 2006] constructed a triangulated category kk^{lca} whose objects are the locally convex algebras. Later Cortiñas and Thom [2007] constructed in a similar fashion a triangulated category kk whose objects are all k-algebras Alg_k. If we suppose that $\Re$ is also Γ-closed, then one can define a full triangulated subcategory $kk(\Re)$ of kk whose objects are those of $\Re$.

It can be shown similar to [Garkusha 2007, Theorem 7.4] or [Garkusha 2013, Corollary 9.4] that there is an equivalence of triangulated categories

$$D_{st}(\Re, \mathfrak{F}) \xrightarrow{\sim} kk(\Re).$$

An important computational result of Cortiñas and Thom [2007] states that there is an isomorphism of graded abelian groups

$$\bigoplus_{n\in\mathbb{Z}} kk(\Re)(k, \Omega^n A) \cong \bigoplus_{n\in\mathbb{Z}} KH_n(A),$$

where the right hand side is the homotopy K-theory of $A \in \Re$ in the sense of [Weibel 1989].

Summarising the above arguments together with Theorem 6.1 we obtain the following:

Theorem 6.2. *Suppose $\Re$ is Γ-closed. Then there is a contravariant equivalence of triangulated categories*

$$kk(\Re) \to \mathscr{S}_\infty.$$

Moreover, *there is a natural isomorphism*

$$\mathrm{SH}^{\infty}_{S^1}(\Re)(\Sigma^\infty rA[n], \Sigma^\infty r(k)) \cong KH_n(A)$$

for any $A \in \Re$ and any integer n.

7. *K*-motives of algebras

Throughout the section we assume that $\Re$ is a small tensor closed and T-closed admissible category of k-algebras with $M_\infty(k) \in \Re$. It follows that

$$M_\infty A :\cong A \otimes M_\infty(k) \in \Re$$

for all $A \in \Re$.

In this section we define and study the triangulated category of K-motives. It shares many properties with the category of K-motives for algebraic varieties constructed in [Garkusha and Panin 2012; 2014b]

Since $\Re$ is tensor closed, it follows that $U_\bullet\Re^{\infty}_{I,J}$ is a monoidal model category. Let $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$ be the monoidal category of symmetric spectra in the sense of [Hovey 2001] associated to $U_\bullet\Re^{\infty}_{I,J}$.

Definition. The *category of K-motives* $DK(\Re)$ is the stable homotopy category of $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$. The *K-motive* $M_K(A)$ of an algebra $A \in \Re$ is the image of A in $DK(\Re)$, that is $M_K(A) = \Sigma^{\infty} rA$. Thus one has a canonical contravariant functor

$$M_K : \Re \to DK(\Re)$$

sending algebras to their K-motives.

The following proposition follows from standard facts for monoidal model categories.

Proposition 7.1. *$DK(\Re)$ is a symmetric monoidal compactly generated triangulated category with compact generators $\{M_K(A)\}_{A\in\Re}$. For any two algebras $A, B \in \Re$ one has a natural isomorphism*

$$M_K(A) \otimes M_K(B) \cong M_K(A \otimes B).$$

Furthermore, any extension of algebras in $\Re$

$$(E): \quad A \to B \to C$$

induces a triangle in $DK(\Re)$

$$M_K(E): \quad M_K(C) \to M_K(B) \to M_K(A) \xrightarrow{+} .$$

There is a pair of adjoint functors

$$V : \mathrm{Sp}_{\infty}(\Re) \leftrightarrows \mathrm{Sp}^{\Sigma}_{\infty}(\Re) : U,$$

where U is the right Quillen forgetful functor. These form a Quillen equivalence. In particular, the induced functors

$$V : \mathrm{SH}^{\infty}_{S^1}(\Re) \leftrightarrows DK(\Re) : U$$

are equivalences of triangulated categories. It follows from Proposition 7.1 that $\mathrm{SH}^{\infty}_{S^1}(\Re)$ is a symmetric monoidal category and

$$\Sigma^{\infty} rA \otimes \Sigma^{\infty} rB \cong \Sigma^{\infty} r(A \otimes B)$$

for all $A, B \in \Re$. Moreover,

$$V(\Sigma^{\infty} rA) \cong M_K(A)$$

for all $A \in \Re$.

Summarising the above arguments together with Theorem 6.1 we get the following:

Theorem 7.2. *For any two algebras A, $B \in \Re$ and any integer n one has a natural isomorphism of abelian groups*

$$DK(\Re)(M_K(B)[n], M_K(A)) \cong \mathbb{K}_n^{st}(A, B).$$

The full subcategory $\mathcal{T}$ of $DK(\Re)$ spanned by K-motives of algebras $\{M_K(A)\}_{A\in\Re}$ is triangulated and there is an equivalence of triangulated categories

$$D_{st}(\Re, \mathfrak{F}) \to \mathcal{T}^{\mathrm{op}}$$

sending an algebra $A \in \Re$ to its K-motive $M_K(A)$.

The next result is reminiscent of a similar result for K-motives of algebraic varieties in the sense of [Garkusha and Panin 2012; 2014b] identifying the K-motive of the point with algebraic K-theory.

Corollary 7.3. *Suppose $\Re$ is Γ-closed. Then for any algebra A and any integer n one has a natural isomorphism of abelian groups*

$$DK(\Re)(M_K(A)[n], M_K(k)) \cong KH_n(A),$$

where the right hand side is the n-th homotopy K-theory group in the sense of [Weibel 1989].

Proof. This follows from [Garkusha 2013, Theorem 10.6] and the preceding theorem. □

We finish the section by showing that the category $kk(\Re)$ of [Cortiñas and Thom 2007] can be identified with the K-motives of algebras.

Theorem 7.4. *Suppose $\Re$ is Γ-closed. Then there is a natural equivalence of triangulated categories*

$$kk(\Re) \xrightarrow{\sim} \mathcal{T}^{\mathrm{op}}$$

sending an algebra $A \in \Re$ to its K-motive $M_K(A)$.

Proof. This follows from Theorem 7.2 and the fact that $D_{st}(\Re, \mathfrak{F})$ and $kk(\Re)$ are triangle equivalent (see [Garkusha 2007, Theorem 7.4] or [Garkusha 2013, Corollary 9.4]). □

The latter theorem shows in particular that $kk(\Re)$ is embedded into the compactly generated triangulated category of K-motives $DK(\Re)$ and generates it.

8. The $\mathbb{G}$-stable theory

The stable motivic homotopy theory over a field is the homotopy theory of T-spectra, where $T = S^1 \wedge \mathbb{G}_m$ (see [Voevodsky 1998; Jardine 2000]). There are various equivalent definitions of the theory, one of which is given in terms of $(S^1, \mathbb{G}_m)$-bispectra. In our context the role of the motivic space $\mathbb{G}_m$ is played by $\sigma = (t-1)k[t^{\pm 1}]$. Its simplicial functor $r(\sigma)$ is denoted by $\mathbb{G}$. In this section we define the stable category of $(S^1, \mathbb{G})$-bispectra and construct an explicit fibrant replacement of the $(S^1, \mathbb{G})$-bispectrum $\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty} rA$ of an algebra A. One can also define a Quillen equivalent category of T-spectra, where $T = S^1 \wedge \mathbb{G}$, and compute an explicit fibrant replacement for the T-spectrum of an algebra. However we prefer to work with $(S^1, \mathbb{G})$-bispectra rather than T-spectra in order to study K-motives of algebras in terms of associated $(S^1, \mathbb{G})$-bispectra (see the next section).

Throughout the section we assume that $\Re$ is a small tensor closed and T-closed admissible category of k-algebras. We have that $\sigma A := A \otimes \sigma \in \Re$ for all $A \in \Re$.

Recall that $U_{\bullet}\Re_{I,J}$ is a monoidal model category. It follows from [Hovey 2001, Section 6.3] that $\mathrm{Sp}(\Re)$ is a $U_{\bullet}\Re_{I,J}$-model category. In particular

$$- \otimes \mathbb{G} : \mathrm{Sp}(\Re) \to \mathrm{Sp}(\Re)$$

is a left Quillen endofunctor.

By definition, a $(S^1, \mathbb{G})$-*bispectrum* or *bispectrum* $\mathscr{E}$ is given by a sequence $(E_0, E_1, \ldots)$, where each E_j is a S^1-spectrum of $\mathrm{Sp}(\Re)$, together with bonding morphisms $\varepsilon_n : E_n \wedge \mathbb{G} \to E_{n+1}$. Maps are sequences of maps in $\mathrm{Sp}(\Re)$ respecting the bonding morphisms. We denote the category of bispectra by $\mathrm{Sp}_{\mathbb{G}}(\Re)$. It can be regarded as the category of $\mathbb{G}$-spectra on $\mathrm{Sp}(\Re)$ in the sense of [Hovey 2001].

$\mathrm{Sp}_{\mathbb{G}}(\Re)$ is equipped with the stable $U_{\bullet}\Re_{I,J}$-model structure in which weak equivalences are defined by means of bigraded homotopy groups. The bispectrum object $\mathscr{E}$ determines a sequence of maps of S^1-spectra

$$E_0 \xrightarrow{\tilde{\varepsilon}_0} \Omega_{\mathbb{G}} E_1 \xrightarrow{\Omega_{\mathbb{G}}(\tilde{\varepsilon}_1)} \Omega_{\mathbb{G}}^2 E_2 \to \cdots,$$

where $\Omega_{\mathbb{G}}$ is the functor $\underline{\mathrm{Hom}}(\mathbb{G}, -)$ and $\tilde{\varepsilon}_n$-s are adjoint to the structure maps of $\mathscr{E}$. We define $\pi_{p,q}\mathscr{E}$ in A-sections as the colimit

$$\mathrm{colim}_l \big(\mathrm{Hom}_{\mathrm{SH}_{S^1}(\Re)}(S^{p-q}, \Omega_{\mathbb{G}}^{q+l} JE_l(A)) \\ \to \mathrm{Hom}_{\mathrm{SH}_{S^1}(\Re)}(S^{p-q}, \Omega_{\mathbb{G}}^{q+l+1} JE_{l+1}(A)) \to \cdots \big)$$

once $\mathscr{E}$ has been replaced up to levelwise equivalence by a levelwise fibrant object $J\mathscr{E}$ so that the "loop" constructions make sense. We also call $\pi_{*,q}\mathscr{E}$ the homotopy groups of weight q.

By definition, a map of bispectra is a weak equivalence in $Sp_{\mathbb{G}}(\Re)$ if it induces an isomorphism on bigraded homotopy groups. We denote the homotopy category of $Sp_{\mathbb{G}}(\Re)$ by $SH_{S^1,\mathbb{G}}(\Re)$. It is a compactly generated triangulated category.

To define the main $(S^1, \mathbb{G})$-bispectrum of this section, denoted by $\mathbb{KG}(A, -)$, we should first establish some facts for algebra homomorphisms.

Suppose $A, C \in \Re$, then one has a commutative diagram

$$\begin{tikzcd} J(A \otimes C) \arrow[r, tail] \arrow[d, "\gamma_{A,C}"'] & T(A \otimes C) \arrow[r, two heads, "\eta_{A\otimes C}"] \arrow[d] & A \otimes C \arrow[d, equal] \\ JA \otimes C \arrow[r, tail] & T(A) \otimes C \arrow[r, two heads, "\eta_A \otimes C"] & A \otimes C \end{tikzcd}$$

in which $\gamma_{A,C}$ is uniquely determined by the split monomorphism

$$i_A \otimes C : A \otimes C \to T(A) \otimes C.$$

One sets $\gamma^0_{A,C} := 1_{A\otimes C}$. We construct inductively

$$\gamma^n_{A,C} : J^n(A \otimes C) \to J^n(A) \otimes C, \quad n \geqslant 1.$$

Namely, $\gamma^{n+1}_{A,C}$ is the composite

$$J^{n+1}(A \otimes C) \xrightarrow{J(\gamma^n_{A,C})} J(J^n(A) \otimes C) \xrightarrow{\gamma_{J^n A,C}} J^{n+1}(A) \otimes C.$$

Given $n \geqslant 0$, we define a map

$$t_n = t_n^{A,C} : \mathscr{K}(J^n A, -) \to \mathscr{K}(J^n(A \otimes C), - \otimes C) = \underline{\mathrm{Hom}}(rC, \mathscr{K}(J^n(A \otimes C), -))$$

as follows. Let $B \in \Re$ and $(\alpha : J^{n+m} A \to \mathbb{B}(\Omega^m)) \in \mathscr{K}(J^n A, B)$. We set

$$t_n(\alpha) \in \mathscr{K}(J^n(A \otimes C), B \otimes C)$$

to be the composite

$$J^{n+m}(A \otimes C) \xrightarrow{\gamma^{n+m}_{A,C}} J^{n+m}(A) \otimes C \xrightarrow{\alpha\otimes C} \mathbb{B}^\Delta(\Omega^m) \otimes C \overset{\tau}{\cong} (\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^m).$$

Here τ is a canonical isomorphism (see [Cortiñas and Thom 2007, Proposition 3.1.3]) and $(\mathbb{B} \otimes \mathbb{C})^\Delta$ stands for the simplicial ind-algebra

$$[m, \ell] \mapsto \mathrm{Hom}_{\mathbb{S}}(\mathrm{sd}^m \Delta^\ell, (B \otimes C)^\Delta) = (B \otimes C)^{\mathrm{sd}^m \Delta^\ell} \cong k^{\mathrm{sd}^m \Delta^\ell} \otimes (B \otimes C).$$

One has to verify that t_n is consistent with maps

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+m} A, \mathbb{B}^\Delta(\Omega^m)) \xrightarrow{\varsigma} \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+m+1} A, \mathbb{B}^\Delta(\Omega^{m+1})).$$

More precisely, we must show that the map

$$J^{n+m+1}(A\otimes C)\xrightarrow{J(\gamma^{n+m}_{A,C})} J(J^{n+m}A\otimes C)$$
$$\xrightarrow{J(\alpha\otimes 1)} J(\mathbb{B}^{\Delta}(\Omega^m)\otimes C)\overset{J\tau}{\cong} J((\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^m))\xrightarrow{\xi_{\upsilon}}(\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^{m+1})$$

is equal to the map

$$J^{n+m+1}(A\otimes C)\xrightarrow{\gamma^{n+m+1}_{A,C}} J^{n+m+1}A\otimes C$$
$$\xrightarrow{J\alpha\otimes 1} J(\mathbb{B}^{\Delta}(\Omega^m))\otimes C\xrightarrow{\xi_{\upsilon}\otimes 1}\mathbb{B}^{\Delta}(\Omega^{m+1})\otimes C\overset{\tau}{\cong}(\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^{m+1}).$$

The desired property follows from commutativity of the diagram (we use [Garkusha 2014, Lemma 3.4] here)

$$\begin{tikzcd}
J^{n+m+1}(A\otimes C) \arrow[d, "J(\gamma^{n+m}_{A,C})"'] & & & \\
J(J^{n+m}A\otimes C) \arrow[r, "\gamma_{J^{n+m}A,C}"] \arrow[d, "J(\alpha\otimes 1)"'] & J^{n+m+1}A\otimes C \arrow[r, tail] \arrow[d, "J(\alpha)\otimes 1"'] & TJ^{n+m}A\otimes C \arrow[r, two heads] \arrow[d] & J^{n+m}A\otimes C \arrow[d, "\alpha\otimes 1"] \\
J(\mathbb{B}^{\Delta}(\Omega^m)\otimes C) \arrow[r] \arrow[d, equal] & J(\mathbb{B}^{\Delta}(\Omega^m))\otimes C \arrow[r, tail] \arrow[d, "\xi_{\upsilon}\otimes 1"'] & T(\mathbb{B}^{\Delta}(\Omega^m))\otimes C \arrow[r, two heads] \arrow[d] & \mathbb{B}^{\Delta}(\Omega^m)\otimes C \arrow[d, equal] \\
J(\mathbb{B}^{\Delta}(\Omega^m)\otimes C) \arrow[r] \arrow[d, "J\tau"'] & \mathbb{B}^{\Delta}(\Omega^{m+1})\otimes C \arrow[r, tail] \arrow[d, "\tau"'] & P(\mathbb{B}^{\Delta}(\Omega^m))\otimes C \arrow[r, two heads] \arrow[d, "\tau"] & \mathbb{B}^{\Delta}(\Omega^m)\otimes C \arrow[d, "\tau"] \\
J((\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^m)) \arrow[r, "\xi_{\upsilon}"'] & (\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^{m+1}) \arrow[r, tail] & P(\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^m) \arrow[r, two heads] & (\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^m)
\end{tikzcd}$$

We see that t_n is well defined. We claim that the collection of maps $(t_n)_n$ defines a map of S^1-spectra

$$t:\mathbb{K}(A,B)\to\mathbb{K}(A\otimes C,B\otimes C).$$

We have to check that for each $n\geqslant 0$ the diagram

$$\begin{tikzcd}
\mathscr{K}(J^nA,B) \arrow[r, "\cong"] \arrow[d, "t_n"'] & \Omega\mathscr{K}(J^{n+1}A,B) \arrow[d, "\Omega t_{n+1}"] \\
\mathscr{K}(J^n(A\otimes C),B\otimes C) \arrow[r, "\cong"] & \Omega\mathscr{K}(J^{n+1}(A\otimes C),B\otimes C)
\end{tikzcd}$$

is commutative. But this directly follows from the definition of the horizontal maps (see [Garkusha 2014, Theorem 5.1]) and arguments above made for the t_n.

If we replace C by σ we get that the array

$$\begin{array}{lcccc}
 & \vdots & \vdots & \vdots & \\
\mathbb{K}(\sigma^2 A, B): & \mathscr{K}(\sigma^2 A, B) & \mathscr{K}(J\sigma^2 A, B) & \mathscr{K}(J^2\sigma^2 A, B) & \cdots \\
\mathbb{K}(\sigma A, B): & \mathscr{K}(\sigma A, B) & \mathscr{K}(J\sigma A, B) & \mathscr{K}(J^2\sigma A, B) & \cdots \\
\mathbb{K}(A, B): & \mathscr{K}(A, B) & \mathscr{K}(JA, B) & \mathscr{K}(J^2 A, B) & \cdots
\end{array}$$

together with structure maps

$$\mathbb{K}(\sigma^n A, -) \otimes \mathbb{G} \to \mathbb{K}(\sigma^{n+1} A, -)$$

defined as adjoint maps to

$$t : \mathbb{K}(\sigma^n A, -) \to \underline{\mathrm{Hom}}(\mathbb{G}, \mathbb{K}(\sigma^{n+1} A, -))$$

forms a $(S^1, \mathbb{G})$-bispectrum, which we denote by $\mathbb{K}\mathbb{G}(A, -)$.

There is a natural map of $(S^1, \mathbb{G})$-bispectra

$$\Gamma : \Sigma^\infty_{\mathbb{G}} \Sigma^\infty rA \to \mathbb{K}\mathbb{G}(A, -),$$

where $\Sigma^\infty_{\mathbb{G}} \Sigma^\infty rA$ is the $(S^1, \mathbb{G})$-bispectrum represented by the array

$$\begin{array}{lccc}
 & \vdots & \vdots & \\
\Sigma^\infty rA \otimes \mathbb{G}^2 : & rA \otimes \mathbb{G}^2 \ (\cong r(\sigma^2 A)) & (rA \wedge S^1) \otimes \mathbb{G}^2 \ (\cong r(\sigma^2 A) \wedge S^1) & \cdots \\
\Sigma^\infty rA \otimes \mathbb{G} : & rA \otimes \mathbb{G} \ (\cong r(\sigma A)) & (rA \wedge S^1) \otimes \mathbb{G} \ (\cong r(\sigma A) \wedge S^1) & \cdots \\
\Sigma^\infty rA : & rA & rA \wedge S^1 & \cdots
\end{array}$$

with obvious structure maps.

By Theorem 4.2 each map

$$\Gamma_n : \Sigma^\infty rA \otimes \mathbb{G}^n \to \mathbb{K}\mathbb{G}(A, -)_n = \mathbb{K}(\sigma^n A, -)$$

is a stable weak equivalence in $\mathrm{Sp}(\mathfrak{R})$. By [Garkusha 2014] each $\mathbb{K}(\sigma^n A, -)$ is a fibrant object in $\mathrm{Sp}(\mathfrak{R})$. For each $n \geqslant 0$ we set

$$\begin{aligned}
&\Theta^\infty_{\mathbb{G}} \mathbb{K}\mathbb{G}(A, -)_n \\
&\quad = \mathrm{colim}(\mathbb{K}(\sigma^n A, -) \xrightarrow{t_0} \mathbb{K}(\sigma^{n+1} A, - \otimes \sigma) \xrightarrow{\Omega_{\mathbb{G}}(t_1)} \mathbb{K}(\sigma^{n+2} A, - \otimes \sigma^2) \to \cdots).
\end{aligned}$$

By specialising a collection of results in [Hovey 2001, Section 4] to our setting we have that $\Theta^\infty_{\mathbb{G}} \mathbb{K}\mathbb{G}(A, -)$ is a fibrant bispectrum and the natural map

$$j : \mathbb{K}\mathbb{G}(A, -) \to \Theta^\infty_{\mathbb{G}} \mathbb{K}\mathbb{G}(A, -)$$

is a weak equivalence in $\mathrm{Sp}_{\mathbb{G}}(\mathfrak{R})$.

We have thus shown that $\Theta^\infty_{\mathbb{G}}\mathbb{KG}(A,-)$ is an explicit fibrant replacement for the bispectrum $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$ of the algebra A. Denote by $\mathscr{K}^\sigma(A,B)$ the (0, 0)-space of the bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{KG}(A,B)$. It is, by construction, the colimit

$$\operatorname{colim}_n \mathscr{K}(\sigma^n A, \sigma^n B).$$

Its homotopy groups will be denoted by $\mathscr{K}^\sigma_n(A,B)$, $n \geqslant 0$.

Theorem 8.1. *Let A be an algebra in $\Re$; then the composite map*

$$j \circ \Gamma : \Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA \to \Theta^\infty_{\mathbb{G}}\mathbb{KG}(A,-)$$

is a fibrant replacement of $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$. In particular,

$$\mathrm{SH}_{S^1,\mathbb{G}}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rB, \Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA) = \mathscr{K}^\sigma_0(A,B)$$

for all $B \in \Re$.

Remark. Let $\mathrm{SH}(F)$ be the motivic stable homotopy category over a field F. The category $\mathrm{SH}_{S^1,\mathbb{G}}(\Re)$ shares many properties with $\mathrm{SH}(F)$. The author and Panin [Garkusha and Panin 2014a] have recently computed a fibrant replacement of $\Sigma^\infty_{s,t}X_+$, $X \in \mathrm{Sm}/F$, by developing the machinery of framed motives. The machinery is based on the theory of framed correspondences developed by Voevodsky [2001]. In turn, the computation of Theorem 8.1 is possible thanks to the existence of universal extensions of algebras.

Let F be an algebraically closed field of characteristic zero with an embedding $F \hookrightarrow \mathbb{C}$ and let SH be the stable homotopy category of ordinary spectra. Let $c : \mathrm{SH} \to \mathrm{SH}(F)$ be the functor induced by sending a space to the constant presheaf of spaces on Sm/F. Levine [2014] has recently shown that c is fully faithful, a fact implied by his result that the Betti realisation functor in the sense of [Ayoub 2010]

$$Re_B : \mathrm{SH}(F) \to \mathrm{SH}$$

gives an isomorphism

$$Re_{B*} : \pi_{n,0}\mathscr{S}_F(F) \to \pi_n(\mathscr{S})$$

for all $n \in \mathbb{Z}$. Here $\mathscr{S}_F$ is the motivic sphere spectrum in $\mathrm{SH}(F)$ and $\mathscr{S}$ is the classical sphere spectrum in SH. These results use recent developments for the spectral sequence associated with the slice filtration of the motivic sphere $\mathscr{S}_F$.

All this justifies raising the following questions.

Questions. (1) Is there an admissible category of commutative algebras $\Re$ over the field of complex numbers $\mathbb{C}$ such that the natural functor

$$c : \mathrm{SH} \to \mathrm{SH}_{S^1,\mathbb{G}}(\Re),$$

induced by the functor $\mathbb{S} \to U\mathfrak{R}$ sending a simplicial set to the constant simplicial functor on $\mathfrak{R}$, is fully faithful?

(2) Let $\mathfrak{R}$ be an admissible category of commutative $\mathbb{C}$-algebras and let $\mathscr{S}_{\mathbb{C}}$ be the bispectrum $\Sigma^{\infty}_{\mathbb{G}}\Sigma^{\infty} r\mathbb{C}$. Is it true that the homotopy groups of weight zero $\pi_{n,0}\mathscr{S}_{\mathbb{C}}(\mathbb{C}) = \mathscr{K}^{\sigma}_{n}(\mathbb{C}, \mathbb{C})$, $n \geqslant 0$, are isomorphic to the stable homotopy groups $\pi_n(\mathscr{S})$ of the classical sphere spectrum?

We should also mention that one can define $(S^1, \mathbb{G})$-bispectra by starting at the monoidal category of symmetric spectra $\mathrm{Sp}^{\Sigma}(\mathfrak{R})$ associated with the monoidal category $U_{\bullet}(\mathfrak{R})_{I,J}$ and then stabilising the left Quillen functor

$$- \otimes \mathbb{G} : \mathrm{Sp}^{\Sigma}(\mathfrak{R}) \to \mathrm{Sp}^{\Sigma}(\mathfrak{R}).$$

One produces a model category $\mathrm{Sp}^{\Sigma}_{\mathbb{G}}(\mathfrak{R})$ of (usual, nonsymmetric) $\mathbb{G}$-spectra in $\mathrm{Sp}^{\Sigma}(\mathfrak{R})$. Using Hovey's notation [2001], one has, by definition,

$$\mathrm{Sp}^{\Sigma}_{\mathbb{G}}(\mathfrak{R}) = \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}(\mathfrak{R}), - \otimes \mathbb{G}).$$

There is a Quillen equivalence

$$V : \mathrm{Sp}(\mathfrak{R}) \leftrightarrows \mathrm{Sp}^{\Sigma}(\mathfrak{R}) : U$$

as well as a Quillen equivalence

$$V : \mathrm{Sp}_{\mathbb{G}}(\mathfrak{R}) \leftrightarrows \mathrm{Sp}^{\Sigma}_{\mathbb{G}}(\mathfrak{R}) : U,$$

where U is the forgetful functor (see [Hovey 2001, Section 5.7]).

If we denote by $\mathrm{SH}^{\Sigma}_{S^1}(\mathfrak{R})$ and $\mathrm{SH}^{\Sigma}_{S^1,\mathbb{G}}(\mathfrak{R})$ the homotopy categories of $\mathrm{Sp}^{\Sigma}(\mathfrak{R})$ and $\mathrm{Sp}^{\Sigma}_{\mathbb{G}}(\mathfrak{R})$ respectively, then one has equivalences of categories

$$V : \mathrm{SH}_{S^1}(\mathfrak{R}) \leftrightarrows \mathrm{SH}^{\Sigma}_{S^1}(\mathfrak{R}) : U \quad \text{and} \quad V : \mathrm{SH}_{S^1,\mathbb{G}}(\mathfrak{R}) \leftrightarrows \mathrm{SH}^{\Sigma}_{S^1,\mathbb{G}}(\mathfrak{R}) : U.$$

We refer the interested reader to [Hovey 2001; Jardine 2000] for further details.

9. *K*-motives and $(S^1, \mathbb{G})$-bispectra

We prove in this section that the triangulated category of K-motives is fully faithfully embedded into the stable homotopy category of $(S^1, \mathbb{G})$-bispectra $\mathrm{SH}_{S^1,\mathbb{G}}(\mathfrak{R})$. In particular, the triangulated category $kk(\mathfrak{R})$ of [Cortiñas and Thom 2007] is fully faithfully embedded into $\mathrm{SH}_{S^1,\mathbb{G}}(\mathfrak{R})$ by means of a contravariant functor. As an application we construct an explicit fibrant $(S^1, \mathbb{G})$-bispectrum representing homotopy K-theory in the sense of [Weibel 1989].

Throughout this section we assume that $\mathfrak{R}$ is a small tensor closed, T-, Γ- and τ_0-closed admissible category of k-algebras. It follows that $\sigma A, \Sigma A, M_{\infty} A \in \mathfrak{R}$ for all $A \in \mathfrak{R}$.

Let $\mathrm{Sp}^{\Sigma}_{\infty,\mathbb{G}}(\Re)$ denote the model category of (usual, nonsymmetric) $\mathbb{G}$-spectra in $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$. Using Hovey's notation [2001], $\mathrm{Sp}^{\Sigma}_{\infty,\mathbb{G}}(\Re) = \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}_{\infty}(\Re), -\otimes \mathbb{G})$.

Proposition 9.1. *The functor*

$$-\otimes \mathbb{G} : \mathrm{Sp}^{\Sigma}_{\infty}(\Re) \to \mathrm{Sp}^{\Sigma}_{\infty}(\Re)$$

and the canonical functor

$$F_{0,\mathbb{G}} = \Sigma^{\infty}_{\mathbb{G}} : \mathrm{Sp}^{\Sigma}_{\infty}(\Re) \to \mathrm{Sp}^{\Sigma}_{\infty,\mathbb{G}}(\Re)$$

are left Quillen equivalences.

Proof. We first observe that $-\otimes \mathbb{G}$ is a left Quillen equivalence on $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$ if and only if so is $-\otimes \Sigma^{\infty}\mathbb{G}$. By [Cortiñas and Thom 2007, Section 4] there is an extension

$$M_{\infty}k \rightarrowtail \tau_0 \twoheadrightarrow \sigma.$$

It follows from [Cortiñas and Thom 2007, Lemma 7.3.2] that $\Sigma^{\infty}(r(\tau_0)) = 0$ in $DK(\Re)$, and hence $\Sigma^{\infty}(r(\tau_0))$ is weakly equivalent to zero in $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$.

The extension above yields therefore a zigzag of weak equivalences between cofibrant objects in $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$ from $\Sigma^{\infty}(r(M_{\infty}k))$ to $\Sigma^{\infty}\mathbb{G}\wedge S^1$. Since $\Sigma^{\infty}(r(M_{\infty}k))$ is weakly equivalent to the monoidal unit $\Sigma^{\infty}(r(k))$, we see that $\Sigma^{\infty}(r(k))$ is zigzag weakly equivalent to $(\Sigma^{\infty}\mathbb{G}) \wedge S^1$ in the category of cofibrant objects in $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$.

Since $\Sigma^{\infty}(r(k))$ is a monoidal unit in $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$, then $-\otimes \Sigma^{\infty}(r(k))$ is a left Quillen equivalence on $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$, and hence so is $-\otimes((\Sigma^{\infty}\mathbb{G}) \wedge S^1))$. But $-\wedge S^1$ is a left Quillen equivalence on $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$. Therefore $-\otimes \Sigma^{\infty}\mathbb{G}$ is a left Quillen equivalence by [Hovey 1999, Corollary 1.3.15].

The fact that the canonical functor

$$F_{0,\mathbb{G}} : \mathrm{Sp}^{\Sigma}_{\infty}(\Re) \to \mathrm{Sp}^{\Sigma}_{\infty,\mathbb{G}}(\Re)$$

is a left Quillen equivalence now follows from [Hovey 2001, Section 5.1]. $\square$

Denote the homotopy category of $\mathrm{Sp}^{\Sigma}_{\infty,\mathbb{G}}(\Re)$ by $\mathrm{SH}^{\Sigma,\infty}_{S^1,\mathbb{G}}(\Re)$.

Corollary 9.2. *The canonical functor*

$$F_{0,\mathbb{G}} = \Sigma^{\infty}_{\mathbb{G}} : DK(\Re) \to \mathrm{SH}^{\Sigma,\infty}_{S^1,\mathbb{G}}(\Re)$$

is an equivalence of triangulated categories.

Recall that $\mathrm{Sp}^{\Sigma}_{\infty}(\Re)$ is the Bousfield localisation of $\mathrm{Sp}^{\Sigma}(\Re)$ with respect to

$$\{F_s(r(M_{\infty}A)) \to F_s(rA) \mid A \in \Re, s \geqslant 0\}.$$

It follows that the induced triangulated functor is fully faithful

$$DK(\Re) \to \mathrm{SH}^{\Sigma}_{S^1}(\Re).$$

In a similar fashion, $\mathrm{Sp}^{\Sigma}_{\infty,\mathbb{G}}(\Re)$ can be obtained from $\mathrm{Sp}^{\Sigma}_{\mathbb{G}}(\Re)$ by Bousfield localisation with respect to

$$\{F_{k,\mathbb{G}}(F_s(r(M_\infty A))) \to F_{k,\mathbb{G}}(F_s(rA)) \mid A \in \Re, k, s \geqslant 0\}.$$

We summarise all of this together with Proposition 9.1 as follows.

Theorem 9.3. *There is an adjoint pair of triangulated functors*

$$\Phi : \mathrm{SH}^{\Sigma}_{S^1,\mathbb{G}}(\Re) \leftrightarrows DK(\Re) : \Psi$$

such that Ψ *is fully faithful. Moreover,* $\mathscr{T} = \operatorname{Ker}\Phi$ *is the localising subcategory of* $\mathrm{SH}^{\Sigma}_{S^1,\mathbb{G}}(\Re)$ *generated by the compact objects*

$$\{\mathrm{cone}(F_{k,\mathbb{G}}(F_s(r(M_\infty A))) \to F_{k,\mathbb{G}}(F_s(rA))) \mid A \in \Re\}$$

and $DK(\Re)$ *is triangle equivalent to* $\mathrm{SH}^{\Sigma}_{S^1,\mathbb{G}}(\Re)/\mathscr{T}$.

Corollary 9.4. *There is a contravariant fully faithful triangulated functor*

$$kk(\Re) \to \mathrm{SH}_{S^1,\mathbb{G}}(\Re).$$

Proof. This follows from Theorems 7.4 and 9.3. □

Let $\mathrm{Sp}_{\infty,\mathbb{G}}(\Re)$ denote the model category of $\mathbb{G}$-spectra in $\mathrm{Sp}_\infty(\Re)$. Using Hovey's notation [2001], we have

$$\mathrm{Sp}_{\infty,\mathbb{G}}(\Re) = \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}_\infty(\Re), -\otimes\mathbb{G}).$$

As above, there is a Quillen equivalence

$$V : \mathrm{Sp}_{\infty,\mathbb{G}}(\Re) \leftrightarrows \mathrm{Sp}^{\Sigma}_{\infty,\mathbb{G}}(\Re) : U,$$

where U is the forgetful functor. It induces an equivalence of triangulated categories

$$V : \mathrm{SH}^{\infty}_{S^1,\mathbb{G}}(\Re) \leftrightarrows \mathrm{SH}^{\Sigma,\infty}_{S^1,\mathbb{G}}(\Re) : U,$$

where the left hand side is the homotopy category of $\mathrm{Sp}_{\infty,\mathbb{G}}(\Re)$.

Given $A \in \Re$, consider a $(S^1, \mathbb{G})$-bispectrum $\mathbb{KG}^{st}(A, -)$ which we define at each $B \in \Re$ as

$$\mathrm{colim}_n(\mathbb{KG}(A, B) \to \mathbb{KG}(A, M_\infty k \otimes B)) \to \mathbb{KG}(A, M^2_\infty k \otimes B) \to \cdots).$$

It can also be presented as the array

$$\begin{array}{lcccc}
 & \vdots & \vdots & \vdots & \\
\mathbb{K}^{st}(\sigma^2 A, B): & \mathcal{K}^{st}(\sigma^2 A, B) & \mathcal{K}^{st}(J\sigma^2 A, B) & \mathcal{K}^{st}(J^2\sigma^2 A, B) & \cdots \\
\mathbb{K}^{st}(\sigma A, B): & \mathcal{K}^{st}(\sigma A, B) & \mathcal{K}^{st}(J\sigma A, B) & \mathcal{K}^{st}(J^2\sigma A, B) & \cdots \\
\mathbb{K}^{st}(A, B): & \mathcal{K}^{st}(A, B) & \mathcal{K}^{st}(JA, B) & \mathcal{K}^{st}(J^2 A, B) & \cdots
\end{array}$$

It follows from Theorem 6.1 that the canonical map of bispectra

$$\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty} rA \to \mathbb{K}\mathbb{G}^{st}(A, -)$$

is a level weak equivalence in $\mathrm{Sp}_{\infty,\mathbb{G}}(\Re)$. In fact we can say more. We shall show below that $\mathbb{K}\mathbb{G}^{st}(A, -)$ is a fibrant bispectrum and this arrow is a fibrant replacement of $\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty} rA$ in $\mathrm{Sp}_{\infty,\mathbb{G}}(\Re)$. To this end we have to prove the cancellation theorem for the S^1-spectrum $\mathbb{K}^{st}(A, -)$. The cancellation theorem for K-theory of algebraic varieties was proved in [Garkusha and Panin 2015]. It is also reminiscent of the cancellation theorem for motivic cohomology proved by Voevodsky [2010a].

Theorem 9.5 (cancellation for K-theory). *Each structure map of the bispectrum* $\mathbb{K}\mathbb{G}^{st}(A, -)$

$$\mathbb{K}^{st}(\sigma^n A, -) \to \Omega_{\mathbb{G}}\mathbb{K}^{st}(\sigma^{n+1} A, -), \quad n \geqslant 0,$$

is a weak equivalence of fibrant S^1-spectra.

Proof. It follows from Proposition 9.1 that the functor

$$- \otimes \mathbb{G} : \mathrm{Sp}_{\infty}(\Re) \to \mathrm{Sp}_{\infty}(\Re)$$

is a left Quillen equivalence. It remains to apply Theorem 6.1. □

Corollary 9.6. *For any $A \in \Re$ the bispectrum $\mathbb{K}\mathbb{G}^{st}(A, -)$ is fibrant in* $\mathrm{Sp}_{\infty,\mathbb{G}}(\Re)$. *Moreover, the canonical map of bispectra*

$$\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty} rA \to \mathbb{K}\mathbb{G}^{st}(A, -)$$

is a fibrant resolution for $\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty} rA$ in $\mathrm{Sp}_{\infty,\mathbb{G}}(\Re)$.

The following result says that the bispectrum $\mathbb{K}\mathbb{G}^{st}(A, -)$ is (2, 1)-periodic and represents stable algebraic Kasparov K-theory (cf. [Voevodsky 1998, Theorems 6.8 and 6.9]).

Theorem 9.7. *For any algebras $A, B \in \Re$ and any integers p, q there is an isomorphism of abelian groups*

$$\begin{aligned}
&\pi_{p,q}(\mathbb{K}\mathbb{G}^{st}(A, B)) \\
&\qquad \cong \mathrm{Hom}_{\mathrm{SH}_{S^1,\mathbb{G}}(\Re)}(\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty} rB \otimes S^{p-q} \otimes \mathbb{G}^q, \mathbb{K}\mathbb{G}^{st}(A, -)) \cong \mathbb{K}_{p-2q}^{st}(A, B).
\end{aligned}$$

In particular,

$$\pi_{p,q}(\mathbb{KG}^{st}(A, B)) \cong \pi_{p+2,q+1}(\mathbb{KG}^{st}(A, B)).$$

Proof. By Corollary 9.6 the bispectrum $\mathbb{KG}^{st}(A, -)$ is a fibrant replacement for $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$ in $\mathrm{Sp}_{\infty,\mathbb{G}}(\Re)$. Therefore,

$$\pi_{p,q}(\mathbb{KG}^{st}(A, B)) \cong \mathrm{Hom}_{\mathrm{SH}^\infty_{S^1,\mathbb{G}}(\Re)}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rB \otimes S^{p-q} \otimes \mathbb{G}^q, \Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA).$$

Corollary 9.2 implies that the right hand side is isomorphic to

$$DK(\Re)(M_K(B) \otimes S^{p-q} \otimes \mathbb{G}^q, M_K(A)).$$

On the other hand,

$$DK(\Re)(M_K(B) \otimes S^{p-q} \otimes \mathbb{G}^q, M_K(A)) \cong DK(\Re)(M_K(B) \otimes S^{p-2q} \otimes S^q \otimes \mathbb{G}^q, M_K(A)).$$

The proof of Proposition 9.1 implies $\Sigma^\infty(S^1 \otimes \mathbb{G})$ is isomorphic to the monoidal unit. Therefore,

$$DK(\Re)(M_K(B) \otimes S^{p-2q} \otimes S^q \otimes \mathbb{G}^q, M_K(A)) \cong DK(\Re)(M_K(B)[p-2q], M_K(A)).$$

Our statement now follows from Theorem 7.2. □

The next statement says that the bispectrum $\mathbb{KG}^{st}(k, B)$ gives a model for homotopy K-theory in the sense of [Weibel 1989] (compare [Voevodsky 1998, Theorem 6.9]).

Corollary 9.8. *For any algebra $B \in \Re$ and any integers p, q there is an isomorphism*

$$\pi_{p,q}(\mathbb{KG}^{st}(k, B)) \cong KH_{p-2q}(B).$$

Proof. This follows from the preceding theorem and [Garkusha 2014, 9.11]. □

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GRIGORY GARKUSHA: G.Garkusha@swansea.ac.uk
Department of Mathematics, Swansea University, Singleton Park, Swansea, SA2 8PP, United Kingdom

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The local symbol complex of a reciprocity functor

Evangelia Gazaki

For a reciprocity functor $\mathcal{M}$ we consider the local symbol complex

$$(\mathcal{M} \otimes^M \mathbb{G}_m)(\eta_C) \to \bigoplus_{P \in C} \mathcal{M}(k) \to \mathcal{M}(k),$$

where C is a smooth complete curve over an algebraically closed field k with generic point η_C and $\otimes^M$ is the product of Mackey functors. We prove that if $\mathcal{M}$ satisfies certain assumptions, then the homology of this complex is isomorphic to the K-group of reciprocity functors $T(\mathcal{M}, \underline{CH}_0(C)^0)(\operatorname{Spec} k)$.

1. Introduction

Let F be a perfect field. We consider the category $\mathcal{E}_F$ of finitely generated field extensions of F. F. Ivorra and K. Rülling [2015] created a theory of reciprocity functors. A reciprocity functor is a presheaf with transfers in the category $\mathrm{Reg}^{\leq 1}$ of regular schemes of dimension at most one over some field $k \in \mathcal{E}_F$ that satisfies various properties.

Some examples of reciprocity functors include commutative algebraic groups, homotopy invariant Nisnevich sheaves with transfers, Kähler differentials. Moreover, if $\mathcal{M}_1, \dots, \mathcal{M}_r$ are reciprocity functors, Ivorra and Rülling construct a K-group $T(\mathcal{M}_1, \dots, \mathcal{M}_r)$ which is itself a reciprocity functor.

One of the crucial properties of a reciprocity functor $\mathcal{M}$ is that it has local symbols. Namely, if C is a smooth, complete and geometrically connected curve over some field $k \in \mathcal{E}_F$ with generic point η, then at each closed point $P \in C$ there is a local symbol assignment

$$(.\,;\,.)_P : \mathcal{M}(\eta) \times \mathbb{G}_m(\eta) \to \mathcal{M}(k),$$

satisfying three characterizing properties, one of which is a reciprocity relation $\sum_{P \in C}(g; f)_P = 0$, for every $g \in \mathcal{M}(\eta)$ and $f \in \mathbb{G}_m(\eta)$. We note here that if G is a commutative algebraic group over an algebraically closed field k, then the local

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symbol of G coincides with the local symbol constructed by Rosenlicht–Serre in [Serre 1959]. The reciprocity relation induces a local symbol complex ($\underline{\underline{C}}$)

$$\left(\mathcal{M}\bigotimes^{M}\mathbb{G}_m\right)(\eta) \xrightarrow{((.;.)_P)_{P\in C}} \bigoplus_{P\in C}\mathcal{M}(k) \xrightarrow{\sum_P} \mathcal{M}(k),$$

where by $\otimes^M$ we denote the product of Mackey functors (see Definition 3.2). The main goal of this article is to give a description of the homology $H(\underline{\underline{C}})$ of the above complex in terms of K-groups of reciprocity functors. Our computations work well for curves C over an algebraically closed field k. In the last section we describe some special cases where the method could be refined to include nonalgebraically closed base fields. To obtain a concrete result, we need to impose two conditions on the reciprocity functor $T(\mathcal{M}, \underline{CH}_0(C)^0)$ (see Assumptions 3.3, 3.10). In Section 3 we prove the following theorem.

Theorem 1.1. *Let C be a smooth, complete curve over an algebraically closed field k. Let $\mathcal{M}$ be a reciprocity functor such that the K-group of reciprocity functors $T(\mathcal{M}, \underline{CH}_0(C)^0)$ satisfies the assumptions 3.3 and 3.10. Then the homology of the local symbol complex ($\underline{\underline{C}}$) is canonically isomorphic to the K-group $T(\mathcal{M}, \underline{CH}_0(C)^0)(\operatorname{Spec} k)$.*

Here $\underline{CH}_0(C)^0$ is a reciprocity functor that is identified with the Jacobian variety J of C.

In Section 4 we give some examples of reciprocity functors that satisfy the two assumptions. In particular, we prove the following theorem.

Theorem 1.2. *Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be homotopy invariant Nisnevich sheaves with transfers, and consider the reciprocity functor $\mathcal{M} = T(\mathcal{F}_1, \dots, \mathcal{F}_r)$. Let C be a smooth, complete curve over an algebraically closed field k. Then there is an isomorphism*

$$H(\underline{\underline{C}}) \simeq T(\mathcal{F}_1, \dots, \mathcal{F}_r, \underline{CH}_0(C)^0)(\operatorname{Spec} k).$$

In particular, if $G_1, \dots, G_r$ are semiabelian varieties over k, then we obtain an isomorphism

$$H(\underline{\underline{C}}) \simeq T(G_1, \dots, G_r, \underline{CH}_0(C)^0)(\operatorname{Spec} k) \simeq K(k; G_1, \dots, G_r, \underline{CH}_0(C)^0),$$

where

$$K(k; G_1, \dots, G_r, \underline{CH}_0(C)^0)$$

is the Somekawa K-group attached to

$$G_1, \dots, G_r.$$

Another case where the assumptions of Theorem 1.1 are satisfied is when $\mathcal{M} = T(\mathcal{M}_1, \dots, \mathcal{M}_r)$ such that $\mathcal{M}_i = \mathbb{G}_a$ for some $i \in \{1, \dots, r\}$. Using the main result of [Rülling and Yamazaki 2014] together with Theorem 5.4.7. of [Ivorra and Rülling 2015], we obtain the following corollary.

Corollary 1.3. *Let $\mathcal{M}_1, \dots, \mathcal{M}_r$ be reciprocity functors. Let*

$$\mathcal{M} = T(\mathbb{G}_a, \mathcal{M}_1, \dots, \mathcal{M}_r).$$

Then for any smooth complete curve C over k, $H(\underline{\underline{C}}) = 0$. In particular, if $\operatorname{char} k = 0$, *the complex*

$$\Omega^{n+1}_{k(C)} \xrightarrow{\mathrm{Res}_P} \bigoplus_{P \in C} \Omega^n_k \xrightarrow{\sum_P} \Omega^n_k$$

is exact.

The idea for Theorem 1.1 stems from the special case when $\mathcal{M} = \mathbb{G}_m$. In this case the local symbol

$$k(C)^\times \otimes^M k(C)^\times \xrightarrow{(.;.)_P} k^\times$$

at a closed point $P \in C$ factors through the group $T(\mathbb{G}_m, \mathbb{G}_m)(\eta_C)$. By a theorem in [Ivorra and Rülling 2015] this group is isomorphic to the usual Milnor K-group $K_2^M(k(C))$ and we recover the Milnor complex

$$K_2^M(k(C)) \to \bigoplus_{P \in C} k^\times \xrightarrow{\sum_P} k^\times.$$

This complex was studied by M. Somekawa [1990] and R. Akhtar [2000]. Using different methods, they both prove that the homology of the above complex is isomorphic to the Somekawa K-group $K(k; \mathbb{G}_m, \underline{CH}_0(C)^0)$. This group turns out to be isomorphic to the group $T(\mathbb{G}_m, \underline{CH}_0(C)^0)(\operatorname{Spec} k)$. (by [Ivorra and Rülling 2015, Theorem 5.1.8; Kahn and Yamazaki 2013, Theorem 11.14]). A similar result was proved by T. Hiranouchi [2014] for his Somekawa-type additive K-groups. Our method to prove Theorem 1.1 is similar to the method used by R. Akhtar and T. Hiranouchi.

Notation 1.4. For a smooth connected variety X over $k \in \mathcal{E}_F$, we denote by $k(X)$ the function field of X. Let C be a smooth complete curve over $k \in \mathcal{E}_F$ and $P \in C$ a closed point. We write ord_P for the normalized discrete valuation on $k(C)$ defined by the point P and for an integer $n \geq 1$, we put

$$U^{(n)}_{C,P} = \{f \in k(C)^\times : \mathrm{ord}_P(1-f) \geq n\}.$$

2. Review of definitions

Reciprocity functors. Let $\mathrm{Reg}^{\leq 1}$ be the category whose objects are regular F-schemes of dimension at most one which are separated and of finite type over some $k \in \mathcal{E}_F$. Let $\mathrm{Reg}^{\leq 1}$ Cor be the category with the same objects as $\mathrm{Reg}^{\leq 1}$ and with morphisms finite correspondences. A reciprocity functor $\mathcal{M}$ is a presheaf of abelian groups on $\mathrm{Reg}^{\leq 1}$ Cor which satisfies various properties. Here we only recall those properties that we will need later in the paper.

Notation 2.1. Let $\mathcal{M}$ be a reciprocity functor. For $k \in \mathcal{E}_F$ we will write

$$\mathcal{M}(k) := \mathcal{M}(\operatorname{Spec} k).$$

Let E/k be a finite extension of fields in $\mathcal{E}_F$. The morphism $\operatorname{Spec} E \to \operatorname{Spec} k$ induces a pull-back map $\mathcal{M}(k) \to \mathcal{M}(E)$, which we call restriction and will denote by $\mathrm{res}_{E/k}$. Moreover, there is a finite correspondence $\operatorname{Spec} k \to \operatorname{Spec} E$ which induces a push-forward $\mathcal{M}(E) \to \mathcal{M}(k)$, which we will call the trace and denote it by $\mathrm{Tr}_{E/k}$.

Injectivity. Let C be a smooth, complete curve over $k \in \mathcal{E}_F$. Each open set $U \subset C$ induces a pull-back map $\mathcal{M}(C) \to \mathcal{M}(U)$ that is required to be injective. Additionally, if η_C is the generic point of C, we have an isomorphism

$$\varinjlim \mathcal{M}(U) \xrightarrow{\simeq} \mathcal{M}(\eta_C),$$

where the limit extends over all open subsets $U \subset C$.

Specialization and trace maps. Let $P \in C$ be a closed point. For each open $U \subset C$ with $P \in U$, the closed immersion $P \hookrightarrow U$ induces $\mathcal{M}(U) \to \mathcal{M}(P)$. We consider the stalk $\mathcal{M}_{C,P} = \varinjlim \mathcal{M}(U)$, where the limit extends over all open $U \subset C$ with $P \in U$. The above morphisms induce a specialization map

$$s_P : \mathcal{M}_{C,P} \to \mathcal{M}(P).$$

Moreover, for every closed point $P \in C$ we obtain a Trace map, which we will denote by

$$\mathrm{Tr}_{P/k} : \mathcal{M}(P) \to \mathcal{M}(k).$$

The modulus condition and local symbols. Let $\mathcal{M}$ be a reciprocity functor. Let C be a smooth, projective and geometrically connected curve over $k \in \mathcal{E}_F$. The definition of a reciprocity functor imposes the existence for each section

$$g \in \mathcal{M}(\eta_C)$$

of a modulus $\mathfrak{m}$ corresponding to g. The modulus $\mathfrak{m}$ is an effective divisor

$$\mathfrak{m} = \sum_{P \in S} n_P P$$

on C, where S is a closed subset of C, such that $g \in \mathcal{M}_{C,P}$, for every $P \notin S$ and for every function $f \in k(C)^\times$ with $f \in \bigcap_{P\in S} U_{C,P}^{(n_P)}$, we have

$$\sum_{P\in C\setminus S} \operatorname{ord}_P(f)\operatorname{Tr}_{P/k}(s_P(g)) = 0.$$

Notation 2.2. Let $f \in k(C)^\times$ be such that $f \in \bigcap_{P\in S} U_{C,P}^{(n_P)}$. Then we will write $f \equiv 1 \bmod \mathfrak{m}$.

The modulus condition on $\mathcal{M}$ is equivalent to the existence, for each closed point $P \in C$, of a biadditive pairing called the local symbol at P

$$(.\,;.)_P : \mathcal{M}(\eta_C) \times \mathbb{G}_m(\eta_C) \to \mathcal{M}(k),$$

which satisfies the following three characterizing properties:

(1) $(g;\, f)_P = 0$, for $f \in U_{C,P}^{(n_P)}$, where $\mathfrak{m} = \sum_{P\in S} n_P P$ is a modulus corresponding to g.

(2) $(g;\, f)_P = \operatorname{ord}_P(f)\operatorname{Tr}_{P/k}(s_P(g))$, for all $g \in \mathcal{M}_{C,P}$ and $f \in k(C)^\times$.

(3) $\sum_{P\in C}(g;\, f)_P = 0$, for every $g \in \mathcal{M}(\eta_C)$ and $f \in k(C)^\times$.

The proof of existence and uniqueness of this local symbol is along the lines of [Serre 1959, Proposition 1, Chapter III]. In this paper we will use the precise definition of $(g;\, f)_P$, for $g \in \mathcal{M}(\eta_C)$ and $f \in k(C)^\times$, so we review it here.

Case 1: If $g \in \mathcal{M}_{C,P}$, property (2) forces us to define $(g;\, f)_P = \operatorname{ord}_P(f)\operatorname{Tr}_{P/k}(s_P(g))$.

Case 2: Let $P \in S$. Using the weak approximation theorem for valuations, we consider an auxiliary function f_P for f at P, i.e., a function $f_P \in k(C)^\times$ such that $f_P \in U_{C,P'}^{(n_{P'})}$ at every $P' \in S$, $P' \neq P$ and $f/f_P \in U_{C,P}^{(n_P)}$. Then we define

$$(g;\, f)_P = -\sum_{Q\notin S} \operatorname{ord}_Q(f_P)\operatorname{Tr}_{Q/k}(s_Q(g)).$$

Using the local symbol, one can define for each closed point $P \in C$,

$$\operatorname{Fil}_P^0 \mathcal{M}(\eta_C) := \mathcal{M}_{C,P}$$

and for $r \geq 1$

$$\operatorname{Fil}_P^r \mathcal{M}(\eta_C) := \bigl\{g \in \mathcal{M}(\eta_C) : (g;\, f)_P = 0, \text{ for all } f \in U_{C,P}^{(r)}\bigr\}.$$

Then $\{\operatorname{Fil}_P^r\}_{r\geq 0}$ form an increasing and exhaustive filtration of $\mathcal{M}(\eta_C)$.

The reciprocity functors $\mathcal{M}$ for which there exists an integer $n \geq 0$ such that $\mathcal{M}(\eta_C) = \operatorname{Fil}_P^n \mathcal{M}(\eta_C)$, for every smooth complete and geometrically connected curve C and every closed point $P \in C$, form a full subcategory of RF, which is denoted by RF_n. (see [Ivorra and Rülling 2015, Definition 1.5.7]).

K-group of reciprocity functors. Let $\mathcal{M}_1, \dots, \mathcal{M}_n$ be reciprocity functors. The K-group of reciprocity functors $T(\mathcal{M}_1, \dots, \mathcal{M}_n)$ is itself a reciprocity functor that satisfies various properties [Ivorra and Rülling 2015, Theorem 4.2.4]. We will not need the precise definition of $T(\mathcal{M}_1, \dots, \mathcal{M}_n)$, but only the following properties.

(a) For $k \in \mathcal{E}_F$, the group $T(\mathcal{M}_1, \dots, \mathcal{M}_n)(k)$ is a quotient of

$$\Big(\mathcal{M}_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathcal{M}_n\Big)(k),$$

where by $\bigotimes^M$ we denote the product of Mackey functors (see Definition 3.2). The group $T(\mathcal{M}_1, \dots, \mathcal{M}_n)(k)$ is generated by elements of the form

$$\mathrm{Tr}_{k'/k}(x_1 \otimes \cdots \otimes x_n),$$

with $x_i \in \mathcal{M}_i(k')$, where k'/k is any finite extension.

(b) Let C be a smooth, complete and geometrically connected curve over $L \in \mathcal{E}_k$ and let $P \in C$ be a closed point. Let $g_i \in \mathcal{M}_i(\eta_C)$. Then:

(i) If for some $r \geq 0$ we have $g_i \in \mathrm{Fil}^r_P\, \mathcal{M}_i(\eta_C)$ for $i = 1, \dots, n$, then

$$g_1 \otimes \cdots \otimes g_n \in \mathrm{Fil}^r_P\, T(\mathcal{M}_1, \dots, \mathcal{M}_n)(\eta_C).$$

Moreover, if the element g_i has modulus $\mathfrak{m}_i = \sum_{P \in S_i} n^i_P P$, for $i = 1, \dots, n$, then $\mathfrak{m} = \sum_{P \in \cup S_i} \max_{1 \leq i \leq n}\{n^i_P\} P$ is a modulus for $g_1 \otimes \cdots \otimes g_n$.

(ii) If $g_i \in \mathrm{Fil}^0_P\, \mathcal{M}_i(\eta_C)$, for $i = 1, \dots, n$, then we have an equality

$$s_P(g_1 \otimes \cdots \otimes g_n) = s_P(g_1) \otimes \cdots \otimes s_P(g_n).$$

Examples. Some examples of reciprocity functors include constant reciprocity functors, commutative algebraic groups, homotopy invariant Nisnevich sheaves with transfers. For an explicit description of each of these examples we refer to [Ivorra and Rülling 2015, Section 2]. The following example is of particular interest to us.

Let X be a smooth projective variety over $k \in \mathcal{E}_F$. Then there is a reciprocity functor $\underline{CH}_0(X)$ such that for any scheme $U \in \mathrm{Reg}^{\leq 1}$ over k we have

$$\underline{CH}_0(X)(U) = CH_0(X \times_k k(U)).$$

Since we assumed X is projective, the degree map $CH_0(X) \to \mathbb{Z}$ induces a map of reciprocity functors $\underline{CH}_0(X) \to \mathbb{Z}$ whose kernel will be denoted by $\underline{CH}_0(X)^0$. Both $\underline{CH}_0(X)$ and $\underline{CH}_0(X)^0$ are in RF_0.

Remark 2.3. If X has a k-rational point, we have a decomposition of reciprocity functors $\underline{CH}_0(X) \simeq \underline{CH}_0(X)^0 \oplus \mathbb{Z}$, where $\mathbb{Z}$ is the constant reciprocity functor.

Moreover, if $\mathcal{M}_1, \dots, \mathcal{M}_r$ are reciprocity functors, then by [Ivorra and Rülling 2015, Corollary 4.2.5(2)] we have a decomposition

$$T(\underline{CH}_0(X), \mathcal{M}_1, \dots, \mathcal{M}_r) \simeq T(\underline{CH}_0(X)^0, \mathcal{M}_1, \dots, \mathcal{M}_r) \oplus T(\mathbb{Z}, \mathcal{M}_1, \dots, \mathcal{M}_r).$$

Relation to Milnor K-theory and Kähler differentials. If we consider the reciprocity functor $T(\mathbb{G}_m^{\times n}) := T(\mathbb{G}_m, \dots, \mathbb{G}_m)$ attached to n copies of $\mathbb{G}_m$, then for every $k \in \mathcal{E}_F$ the group $T(\mathbb{G}_m^{\times n})(k)$ is isomorphic to the usual Milnor K-group $K_n^M(k)$ [Ivorra and Rülling 2015, Theorem 5.3.3].

Moreover, if k is of characteristic zero, then the group $T(\mathbb{G}_a, \mathbb{G}_m^{\times n-1})(k)$, $n \geq 1$, is isomorphic to the group of Kähler differentials $\Omega^{n-1}_{k/\mathbb{Z}}$ [Ivorra and Rülling 2015, Theorem 5.4.7].

3. The homology of the complex

Convention 3.1. From now on, unless otherwise mentioned, we will be working over an algebraically closed base field $k \in \mathcal{E}_F$.

Let $\mathcal{M}$ be a reciprocity functor. Let C be a smooth complete curve over k with generic point η_C. At each closed point $P \in C$ we have a local symbol $(.\,;.)_P$. We will denote by $(.\,;.)_C$ the collection of all symbols $\{(.\,;.)_P\}_{P\in C}$, namely

$$(.\,;.)_C : \mathcal{M}(\eta_C) \otimes \mathbb{G}_m(\eta_C) \to \bigoplus_{P\in C} \mathcal{M}(k).$$

We note here that a reciprocity functor $\mathcal{M}$ is also a Mackey functor. In what follows, we will need the definition of the product of Mackey functors $\mathcal{M}_1, \dots, \mathcal{M}_r$, evaluated at a finitely generated extension L of k. We review this definition here.

Definition 3.2. Let $\mathcal{M}_1, \dots, \mathcal{M}_r$ be Mackey functors over k. Let L be a finitely generated extension of k. Then,

$$\Bigl(\mathcal{M}_1 \overset{M}{\bigotimes} \cdots \overset{M}{\bigotimes} \mathcal{M}_r\Bigr)(L) := \Bigl(\bigoplus_{L'/L} \mathcal{M}_1(L') \bigotimes \cdots \bigotimes \mathcal{M}_r(L')\Bigr)\Big/ R,$$

where the sum is extended over all finite extensions L' of L and R is the subgroup generated by the following family of elements: If $L \subset K \subset E$ is a tower of finite field extensions and we have elements $x_i \in \mathcal{M}_i(E)$ for some $i \in \{1, \dots, r\}$ and $x_j \in \mathcal{M}_j(K)$, for every $j \neq i$, then

$$x_1 \otimes \cdots \otimes \mathrm{Tr}_{E/K}(x_i) \otimes \cdots \otimes x_r - \mathrm{res}_{E/K}(x_1) \otimes \cdots \otimes x_i \otimes \cdots \otimes \mathrm{res}_{E/K}(x_r) \in R.$$

The relation in R is known as the projection formula. Using the functoriality properties of the local symbol at each closed point $P \in C$ [Ivorra and Rülling 2015, Proposition 1.5.5], we obtain a complex

$$\Big(\mathcal{M}\bigotimes^{M}\mathbb{G}_m\Big)(\eta_C)\xrightarrow{(.\,;.)_C}\bigoplus_{P\in C}\mathcal{M}(k)\xrightarrow{\sum_P}\mathcal{M}(k).$$

Namely, if C' is a smooth complete curve over k with function field $k(C')\supset k(C)$ and we have a section $g\in\mathcal{M}(\eta_{C'})$ and a function $f\in k(C')^\times$, then we define

$$(g;f)_C=\Big(\sum_{\lambda(P')=P}(g;f)_{P'}\Big)_P\in\bigoplus_{P\in C}\mathcal{M}(k),$$

where $\lambda:C'\to C$ is the finite covering induced by the inclusion $k(C)\subset k(C')$.

We will denote this complex by $(\underline{\underline{C}})$ and its homology by $H(\underline{\underline{C}})$. We consider the reciprocity functor $\underline{CH}_0(C)$. Notice that the existence of a k-rational point $P_0\in C(k)$ yields a decomposition of reciprocity functors $\underline{CH}_0(C)\simeq\underline{CH}_0(C)^0\oplus\mathbb{Z}$. We make the following assumption on the K-group $T(\mathcal{M},\underline{CH}_0(C))$.

Assumption 3.3. Let $\mathcal{M}$ be a reciprocity functor. Let $g\in\mathcal{M}(\eta_C)$, $h\in\underline{CH}_0(C)(\eta_C)$ and $f\in k(C)^\times$. Let $P\in C$ be a closed point of C. Assume that the local symbol $(g\otimes h;f)_P\in T(\mathcal{M},\underline{CH}_0(C))(k)$ vanishes at every point P such that $s_P(h)=0$.

In the next section we will give examples where Assumption 3.3 is satisfied.

Proposition 3.4. *Let $\mathcal{M}$ be a reciprocity functor over k satisfying Assumption 3.3. Then there is a well defined map*

$$\Phi:\Big(\bigoplus_{P\in C}\mathcal{M}(k)\Big)\Big/\mathrm{Im}((.\,;.)_C)\to T(\mathcal{M},\underline{CH}_0(C))(k),$$
$$(a_P)_{P\in C}\to\sum_{P\in C}a_P\otimes[P].$$

Proof. First, we immediately observe that if $P\in C$ is any closed point of C, then the map $\phi_P:\mathcal{M}(k)\to T(\mathcal{M},\underline{CH}_0(C))(k)$ given by $a\to a\otimes[P]$ is well defined. In particular, the map

$$\Phi=\sum_P\phi_P:\bigoplus_{P\in C}\mathcal{M}(k)\to T(\mathcal{M},\underline{CH}_0(C))(k)$$

is well defined. Let D be a smooth complete curve over k with generic point η_D and assume there is a finite covering $\lambda:D\to C$. Let $g\in\mathcal{M}(\eta_D)$ and $f\in k(D)^\times$ be a function. For every closed point $P\in C$ we consider the element $(a_P)_P\in\bigoplus_{P\in C}\mathcal{M}(k)$ such that $a_P=(g;f)_P$. We are going to show that $\Phi\big(\sum_{P\in C}(g;f)_P\big)=0$.

First, we treat the case $D=C$ and $\lambda=1_C$. The element $g\in\mathcal{M}(\eta_C)$ admits a modulus $\mathfrak{m}$ with support S. We consider the zero-cycle $h=[\eta_C]\in\underline{CH}_0(C)(\eta_C)$. Notice that for a closed point $P\in C$, the specialization map

$$s_P:\underline{CH}_0(C)(\eta_C)\to\underline{CH}_0(C)(k)$$

has the property $s_P(h) = [P]$. We are going to show that

$$\Phi((g;\, f)_P) = (g \otimes h;\, f)_P \quad \text{for every} \;\; P \in C,$$

and the required property will follow from the reciprocity law of the local symbol. We consider the following cases.

(1) Let $P \notin S$. Then,

$$\begin{aligned}\Phi((g;\, f)_P) &= \phi_P(\mathrm{ord}_P(f) s_P(g)) = \mathrm{ord}_P(f) s_P(g) \otimes [P] \\ &= \mathrm{ord}_P(f) s_P(g) \otimes s_P(h) \\ &= \mathrm{ord}_P(f) s_P(g \otimes h) = (g \otimes h;\, f)_P.\end{aligned}$$

(2) Let $P \in S$ and $f \equiv 1 \bmod \mathfrak{m}$ at P. Since $\underline{CH}_0(C) \in RF_0$, h does not contribute to the modulus, and hence, by item (b)(ii) on page 322 we get

$$\Phi((g;\, f)_P) = \Phi(0) = 0 = (g \otimes h;\, f)_P.$$

(3) Let now $P \in S$ and $f \in K^\times$ be any function. We consider an auxiliary function f_P for f at P. By the definition of the local symbol, we have

$$\begin{aligned}\Phi((g;\, f)_P) = \phi_P\Big(-\sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g)\Big) &= -\sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g) \otimes [P] \\ = -\sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g) \otimes [Q] &+ \sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g) \otimes ([Q] - [P]).\end{aligned}$$

We observe that we have an equality

$$(g \otimes h;\, f)_P = -\sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g) \otimes [Q].$$

Next, notice that the flat embedding $k \hookrightarrow k(C)$ induces a restriction map $\mathrm{res}_{\eta/k} : CH_0(C) \to CH_0(C \times \eta_C)$. Let $h_0 = \mathrm{res}_{\eta/k}([P])$. We clearly have

$$\sum_{Q \notin S} \mathrm{ord}_Q(f_P) s_Q(g) \otimes ([P] - [Q]) = (g \otimes (h_0 - h);\, f)_P.$$

Since we assumed that the Assumption 3.3 is satisfied, we get that this last symbol vanishes. For, $s_P(h - h_0) = 0$.

The general case is treated in a similar way. Namely, if $\lambda : D \to C$ is a finite covering of smooth complete curves over k and $g \in \mathcal{M}(\eta_D)$, then the local symbol at a closed point $P \in C$ is defined to be $(g;\, f)_P = \sum_{\lambda(Q)=P} (g;\, f)_Q$. Considering the zero cycle $h = [\eta_D] \in \underline{CH}_0(C)(\eta_D)$, we can show that

$$\Phi_P((g;\, f)_P) = (g \otimes h;\, f)_P. \qquad \square$$

From now on we fix a k-rational point P_0 of C.

Corollary 3.5. *The map Φ of Proposition 3.4 induces a map*

$$\begin{aligned}\Phi : H(\underline{\underline{C}}) &\to T(\mathcal{M}, \underline{CH}_0(C)^0)(k),\\ (a_P)_{P\in C} &\to \sum_{P\in C} a\otimes([P]-[P_0]),\end{aligned}$$

which does not depend on the k-rational point P_0.

Proof. If $(a_P)_{P\in C} \in H(\underline{\underline{C}})$, then

$$\sum_P a_P = 0 \in \mathcal{M}(k),$$

and hence

$$\sum_P a_P \otimes [P_0] = 0 \in T(\mathcal{M}, \underline{CH}_0(C))(k).$$

We conclude that if $(a_P)_{P\in C} \in H(\underline{\underline{C}})$ then $\Phi((a_P)_{P\in C}) \in T(\mathcal{M}, \underline{CH}_0(C)^0)(k)$ and clearly the map does not depend on the k-rational point P_0. $\square$

Definition 3.6. Let $\mathcal{M}_1, \dots, \mathcal{M}_r$ be reciprocity functors over k. We consider the geometric K-group attached to $\mathcal{M}_1, \dots, \mathcal{M}_r$,

$$K^{\text{geo}}(k; \mathcal{M}_1, \dots, \mathcal{M}_r) = \Big(\mathcal{M}_1 \bigotimes^{M} \cdots \bigotimes^{M} \mathcal{M}_r\Big)(k)/R,$$

where the subgroup R is generated by the following family of elements. Let D be a smooth complete curve over k with generic point η_D. Let $g_i \in \mathcal{M}_i(\eta_D)$. Then each g_i admits a modulus $\mathfrak{m}_i$. Let $\mathfrak{m} = \sup_{1\le i\le r} \mathfrak{m}_i$ and S be the support of $\mathfrak{m}$. Let $f \in k(D)^\times$ be a function such that $f \equiv 1 \bmod \mathfrak{m}$. Then

$$\sum_{P\notin S} \operatorname{ord}_P(f) s_P(g_1) \otimes \cdots \otimes s_P(g_r) \in R.$$

Notation 3.7. The elements of the geometric K-group $K^{\text{geo}}(k; \mathcal{M}_1, \dots, \mathcal{M}_r)$ will be denoted as $\{x_1 \otimes \cdots \otimes x_r\}^{\text{geo}}$.

Remark 3.8. In the notation of [Ivorra and Rülling 2015] the group

$$K^{\text{geo}}(k; \mathcal{M}_1, \dots, \mathcal{M}_r)$$

is the same as the Lax Mackey functor $LT(\mathcal{M}_1, \dots, \mathcal{M}_r)$ evaluated at $\operatorname{Spec} k$ [Ivorra and Rülling 2015, Definition 3.1.2]. In general the group $T(\mathcal{M}_1, \dots, \mathcal{M}_r)(k)$ is a quotient of $K^{\text{geo}}(k; \mathcal{M}_1, \dots, \mathcal{M}_r)$. In the next section we give some examples where these two groups coincide.

Proposition 3.9. *Let P_0 be a fixed k-rational point of C. The map*

$$\Psi : K^{\mathrm{geo}}(k; \mathcal{M}, \underline{CH}_0(C)^0) \longrightarrow H(\underline{\underline{C}}),$$
$$\{x \otimes ([P] - [P_0])\}^{\mathrm{geo}} \longrightarrow (x_{P'})_{P' \in C},$$

with

$$x_{P'} = \begin{cases} x & \text{if } P' = P, \\ -x & \text{if } P' = P_0, \\ 0 & \text{otherwise}, \end{cases}$$

for $P \neq P_0$, is well defined and does not depend on the choice of the k-rational point P_0.

Proof. We start by defining the map $\Psi_{P_0} : \mathcal{M}(k) \otimes \underline{CH}_0(C)^0(k) \to H(\underline{\underline{C}})$ as in the statement of the proposition. To see that Ψ_{P_0} is well defined, let $f \in \bar{k}(C)^\times$. We need to verify that $\Psi_{P_0}(x \otimes \mathrm{div}(f)) = 0$ for every $x \in \mathcal{M}(k)$. Let $\pi : C \to \operatorname{Spec} k$ be the structure map. Consider the pull back

$$g = \pi^\star(x) \in \mathcal{M}(C).$$

Then $g \in \mathcal{M}(\eta_C)$ has modulus $\mathfrak{m} = 0$ and hence for a closed point $P \in C$ we have $(g, f)_P = \mathrm{ord}_P(f) s_P(\pi^\star(x)) = \mathrm{ord}_P(f)x$. Since

$$\Psi_{P_0}(x \otimes \mathrm{div}(f)) = (\mathrm{ord}_P(f)x)_{P \in C},$$

we conclude that $\Psi_{P_0}(x \otimes \mathrm{div}(f)) \in \mathrm{Im}(.\,,.)_C$.

Next, notice that Ψ_{P_0} does not depend on the base point P_0. For, if Q_0 is another base point, then

$$\begin{aligned}\Psi_{Q_0}(&\{x \otimes ([P] - [P_0])\}^{\mathrm{geo}}) \\ &= \Psi_{Q_0}(\{x \otimes ([P] - [Q_0])\}^{\mathrm{geo}}) - \Psi_{Q_0}(\{x \otimes ([P_0] - [Q_0])\}^{\mathrm{geo}}).\end{aligned}$$

Here $\Psi_{Q_0}(\{x \otimes ([P] - [Q_0])\}^{\mathrm{geo}})$ gives the element x at the coordinate P and $-x$ at the coordinate Q_0, while $-\Psi_{Q_0}(\{x \otimes ([P_0] - [Q_0])\}^{\mathrm{geo}})$ gives $-x$ at coordinate P_0 and x at Q_0. From now on we will denote this map by Ψ. In order to show that Ψ factors through $K^{\mathrm{geo}}(k; \mathcal{M}, \underline{CH}_0(C)^0)$, we consider a smooth complete curve D with generic point η_D. Let $g_1 \in \mathcal{M}(\eta_D)$ admitting a modulus $\mathfrak{m}$ with support S_D and $g_2 \in \underline{CH}_0(C)^0(\eta_D)$ having modulus $\mathfrak{m}_2 = 0$. Let moreover $f \in k(D)^\times$ be a function such that $f \equiv 1 \bmod \mathfrak{m}$. We need to show that

$$\Psi\Bigg(\sum_{R \notin S_D} \mathrm{ord}_R(f)\{s_R(g_1) \otimes s_R(g_2)\}^{\mathrm{geo}}\Bigg) = 0 \in H(\underline{\underline{C}}).$$

Since we assumed the existence of a k-rational point P_0, the group $\underline{CH}_0(C)^0(\eta_D)$ is generated by elements of the form $[h] - m[\mathrm{res}_{k(D)/k}(P_0)]$, where h is a closed point of $C \times k(D)$ having residue field of degree m over $k(D)$. Using the linearity of the symbol on the last coordinate, we may reduce to the case when g_2 is of the above form. Notice that $h = \operatorname{Spec} k(E) \hookrightarrow C \times \operatorname{Spec} k(D)$, where E is a smooth complete curve over k, and hence h induces two coverings

$$\begin{array}{ccc} E & \xrightarrow{\lambda} & D \\ {\scriptstyle\mu}\big\downarrow & & \\ C & & \end{array}$$

Let $S_E = \lambda^{-1}(S_D)$. For a closed point $R \in D$, we obtain an equality:

$$s_R([h]) = \sum_{\lambda(Q)=R} e(Q/R)[\mu(Q)],$$

where $e(Q/R)$ is the ramification index at the point $Q \in E$ lying over $R \in D$. Since $m = [k(E) : k(D)] = \sum_{\lambda(Q)=R} e(Q/R)$, we get

$$\begin{aligned}
&\Psi\Bigg(\sum_{R\notin S_D} \operatorname{ord}_R(f)\{s_R(g_1)\otimes s_R(g_2)\}^{\text{geo}}\Bigg)\\
&\qquad = \Psi\Bigg(\sum_{R\notin S_D} \operatorname{ord}_R(f)\Bigg\{s_R(g_1)\otimes\Bigg(\sum_{\lambda(Q)=R} e(Q/R)[\mu(Q)]-m[P_0]\Bigg)\Bigg\}^{\text{geo}}\Bigg)\\
&\qquad = \Psi\Bigg(\sum_{R\notin S_D}\sum_{\lambda(Q)=R} e(Q/R)\operatorname{ord}_R(f)\big\{s_R(g_1)\otimes([\mu(Q)]-[P_0])\big\}^{\text{geo}}\Bigg)\\
&\qquad = \Psi\Bigg(\sum_{Q\notin S_E} \operatorname{ord}_Q(\lambda^\star(f))\big\{s_Q(\lambda^\star(g_1))\otimes([\mu(Q)]-[P_0])\big\}^{\text{geo}}\Bigg).
\end{aligned}$$

Here we have used the equality $s_R(g_1) = s_Q(\lambda^\star(g_1))$, valid for a closed point $Q \in E$ lying over $R \in D$, and following from [Ivorra and Rülling 2015, Proposition 1.3.7($\mathcal{S}_2$)] and the assumption that the base field k is algebraically closed.

We conclude that

$$\begin{aligned}
&\Psi\Bigg(\sum_{Q\notin S_E} \operatorname{ord}_Q(\lambda^\star(f))\big\{s_Q(\lambda^\star(g_1))\otimes([\mu(Q)]-[P_0])\big\}_{Q/k}\Bigg)\\
&\qquad = \begin{cases} \sum\limits_{\mu(Q)=P} \operatorname{ord}_Q(\lambda^\star(f))s_Q(\lambda^\star(g_1)) & \text{at } P\neq P_0,\\ -\sum\limits_{P\neq P_0}\sum\limits_{\mu(Q)=P} \operatorname{ord}_Q(\lambda^\star(f))s_Q(\lambda^\star(g_1)) & \text{at } P_0.\end{cases}
\end{aligned}$$

This last computation completes the argument, after we notice that the reciprocity of the local symbol yields an equality

$$-\sum_{P\neq P_0}\sum_{\mu(Q)=P} \operatorname{ord}_Q(\lambda^\star(f))s_Q(\lambda^\star(g_1)) = (\lambda^\star(g_1);\lambda^\star(f))_{P_0}. \qquad \square$$

We make the following assumption on $T(\mathcal{M}, \underline{CH}_0(C)^0)$.

Assumption 3.10. Let $\mathcal{M}$ be a reciprocity functor. Assume that the K-group

$$T(\mathcal{M}, \underline{CH}_0(C)^0)(k)$$

coincides with the geometric K-group $K^{\mathrm{geo}}(k; \mathcal{M}, \underline{CH}_0(C)^0)$.

Theorem 3.11. *Let $\mathcal{M}$ be a reciprocity functor such that the group*

$$T(\mathcal{M}, \underline{CH}_0(C)^0)(k)$$

satisfies both Assumptions 3.3 and 3.10. Then we have an isomorphism

$$H(\underline{\underline{C}}) \simeq T(\mathcal{M}, \underline{CH}_0(C)^0)(k).$$

Proof. By Proposition 3.9 we obtain a homomorphism

$$\Psi : T(\mathcal{M}, \underline{CH}_0(C)^0)(k) \to H(\underline{\underline{C}}).$$

It is almost a tautology to check that Ψ is the inverse of Φ. Namely,

$$\Phi\Psi(x \otimes ([P]-[P_0])) = \Phi((x_{P'})_{P'}) = \sum_{P'} x_{P'} \otimes [P'] = x \otimes [P] - x \otimes [P_0],$$

and

$$\Psi\Phi((x_P)_P) = \Psi\biggl(\sum_{P\in C} x_P \otimes ([P]-[P_0])\biggr) = (x_P)_P.$$

Notice that for the last equality, we used the fact that $(x_P)_{P\in C} \in \ker\bigl(\sum_{P\in C}\bigr)$, and hence at coordinate P_0 we have $x_{P_0} = -\sum_{P\neq P_0} x_P$. □

4. Examples

In this section we give some examples of reciprocity functors $\mathcal{M}$ such that the K-group of reciprocity functors $T(\mathcal{M}, \underline{CH}_0(C)^0)$ satisfies Assumptions 3.3 and 3.10.

Homotopy invariant Nisnevich sheaves with transfers. We consider the category $\mathrm{HI}_{\mathrm{Nis}}$ of homotopy invariant Nisnevich sheaves with transfers over a perfect field F. Let $\mathcal{F}_1, \dots, \mathcal{F}_r \in \mathrm{HI}_{\mathrm{Nis}}$. Then each $\mathcal{F}_i$ induces a reciprocity functor $\widehat{\mathcal{F}}_i \in RF_1$ (see [Ivorra and Rülling 2015, Example 2.3]). The associated K-group of reciprocity functors $T(\widehat{\mathcal{F}}_1, \dots, \widehat{\mathcal{F}}_r)$ is also in RF_1. We claim that $T(T(\widehat{\mathcal{F}}_1, \dots, \widehat{\mathcal{F}}_r), \underline{CH}_0(C)^0)$ satisfies both assumptions of Theorem 3.11. The claim follows by the comparison of the K-group $T(T(\widehat{\mathcal{F}}_1, \dots, \widehat{\mathcal{F}}_r), \underline{CH}_0(C)^0)(k)$ with the Somekawa type K-group $K(k; \mathcal{F}_1 \dots, \mathcal{F}_r, \underline{CH}_0(C)^0)$ defined by B. Kahn and T. Yamazaki [2013, Definition 5.1].

Remark 4.1. If $\mathcal{M}_1, \dots, \mathcal{M}_r$ are reciprocity functors with $r \geq 3$, then F. Ivorra and K. Rülling in Corollary 4.2.5. of [Ivorra and Rülling 2015] prove that there is a functorial map

$$T(\mathcal{M}_1, \dots, \mathcal{M}_r) \to T(T(\mathcal{M}_1, \dots, \mathcal{M}_{r-1}), \mathcal{M}_r),$$

which is surjective as a map of Nisnevich sheaves. It is not clear whether this map is always an isomorphism which would imply that T is associative and we would call it a product. In the case $\mathcal{F}_i \in \mathrm{HI}_{\mathrm{Nis}}$, for every $i \in \{1, \dots, r\}$, associativity holds. In fact, in this case there is an isomorphism of reciprocity functors

$$T(\widehat{\mathcal{F}}_1, \dots, \widehat{\mathcal{F}}_r) \simeq \Bigl(\mathcal{F}_1 \bigotimes_{\mathrm{HI}_{\mathrm{Nis}}} \cdots \bigotimes_{\mathrm{HI}_{\mathrm{Nis}}} \dots \mathcal{F}_r\Bigr),$$

where $\mathcal{F}_1 \bigotimes_{\mathrm{HI}_{\mathrm{Nis}}} \cdots \bigotimes_{\mathrm{HI}_{\mathrm{Nis}}} \mathcal{F}_r$ is the product of homotopy invariant Nisnevich sheaves with transfers. (see [Kahn and Yamazaki 2013, Section 2.10] for the definition of the product and [Ivorra and Rülling 2015, Theorem 5.1.8] for the isomorphism).

Notation 4.2. By abuse of notation from now on we will write $T(\mathcal{F}_1, \dots, \mathcal{F}_r)$ for the K-group of reciprocity functors associated to $\widehat{\mathcal{F}}_1, \dots, \widehat{\mathcal{F}}_r$.

Remark 4.3. Let NST be the category of Nisnevich sheaves with transfers. We note here that there is a left adjoint to the inclusion functor $\mathrm{NST} \to \mathrm{HI}_{\mathrm{Nis}}$ which is denoted by h_0^{Nis} (see [Kahn and Yamazaki 2013, Section 2]). If U is a smooth curve over F, then there is a Nisnevich sheaf with transfers $L(U)$, where

$$L(U)(V) = \mathrm{Cor}(V, U)$$

is the group of finite correspondences for V smooth over F, i.e., the free abelian group on the set of closed integral subschemes of $V \times U$ which are finite and surjective over some irreducible component of V. Then the corresponding homotopy invariant Nisnevich sheaf with transfers $h_0^{\mathrm{Nis}}(U) := h_0^{\mathrm{Nis}}(L(U))$ is the sheaf associated to the presheaf of relative Picard groups

$$V \to \mathrm{Pic}(\overline{U} \times V, D \times V),$$

where $\overline{U}$ is the smooth compactification of U, $D = \overline{U} \setminus U$ and V runs through smooth F-schemes. When U is projective we have an isomorphism

$$h_0^{\mathrm{Nis}}(U) \simeq \underline{CH}_0(U)$$

(see [Kahn and Yamazaki 2013, Lemma 11.2]). In particular, $\underline{CH}_0(C)$ is homotopy invariant Nisnevich sheaf with transfers.

Let $\mathcal{F} \in \mathrm{HI}_{\mathrm{Nis}}$. If we are given a section $g \in \mathcal{F}(U)$ for some open dense $U \subset C$, then g induces a map of Nisnevich sheaves with transfers $\varphi : h_0^{\mathrm{Nis}}(U) \to \mathcal{F}$ such that

$$\varphi(U) : h_0^{\mathrm{Nis}}(U)(U) \to \mathcal{F}(U),$$
$$[\Delta] \to g,$$

where $[\Delta] \in h_0^{\mathrm{Nis}}(U)(U)$ is the class of the diagonal. The existence of the map φ follows by adjunction, since we have an obvious morphism $L(U) \to \mathcal{F}$ in NST.

Lemma 4.4. *Let $\mathcal{F}_1, \dots, \mathcal{F}_r \in \mathrm{HI}_{\mathrm{Nis}}$ be homotopy invariant sheaves with transfers. Then the K-group of reciprocity functors $T(T(\mathcal{F}_1, \dots, \mathcal{F}_r), \underline{CH}_0(C)^0)$ satisfies the assumptions of Theorem 3.11.*

Proof. By Remark 4.1 we get an isomorphism

$$T\big(T(\mathcal{F}_1, \dots, \mathcal{F}_r), \underline{CH}_0(C)^0\big)(k) \simeq T\big(\mathcal{F}_1, \dots, \mathcal{F}_r, \underline{CH}_0(C)^0\big)(k).$$

Moreover, by Theorem 5.1.8. of [Ivorra and Rülling 2015] we get that the groups $K^{\mathrm{geo}}(k; \mathcal{F}_1, \dots, \mathcal{F}_r, \underline{CH}_0(C)^0)$ and $T(\mathcal{F}_1, \dots, \mathcal{F}_r, \underline{CH}_0(C)^0)(k)$ are equal and they coincide with the Somekawa type K-group $K(k; \mathcal{F}_1, \dots, \mathcal{F}_r, \underline{CH}_0(C)^0)$. We conclude that Assumption 3.10 holds.

Regarding the Assumption 3.3, let $g_i \in \mathcal{F}_i(\eta_C)$ and $h \in \underline{CH}_0(C)^0(\eta_C)$ such that $s_P(h) = 0$ for some closed point $P \in C$. Let moreover $f \in k(C)^\times$. We need to verify that $(g_1 \otimes \cdots \otimes g_r \otimes h; f)_P = 0$. If $g_i \in \mathcal{F}_{i,C,P}$, for every $i \in \{1, \dots, r\}$, then

$$(g_1 \otimes \cdots \otimes g_r \otimes h; f)_P = \mathrm{ord}_P(f) s_P(g_1) \otimes \cdots \otimes s_P(g_r) \otimes s_P(h) = 0.$$

Assume P is in the support of g_i for some $i \in \{1, \dots, r\}$.

We first treat the case when $\mathcal{F}_i$ is curve-like (see [Kahn and Yamazaki 2013, Definition 11.1]), for $i = 1, \dots, r$. For such $\mathcal{F}_i$ it suffices to consider elements $g_i \in \mathcal{F}_i(\eta_C)$ with disjoint supports [Kahn and Yamazaki 2013, Proposition 11.11]. In this case the claim follows by the explicit computation of the local symbol [Kahn and Yamazaki 2013, Lemma 8.5, Proposition 11.6]. Namely, if $P \in \mathrm{supp}(g_i)$, then the local symbol at P is given by the formula

$$(g_1 \otimes \cdots \otimes g_r \otimes h; f)_P = s_P(g_1) \otimes \dots \partial_P(g_i, f) \otimes \cdots \otimes s_P(g_r) \otimes s_P(h) = 0,$$

where $\partial_P(g_i, f)$ is the symbol at P defined in [Kahn and Yamazaki 2013, Section 4.1].

Now assume that $\mathcal{F}_i$ is general, for $i = 1, \dots, r$. Since $g_i \in \mathcal{F}_i(\eta_C)$ and

$$\mathcal{F}_i(\eta_C) \simeq \varinjlim \mathcal{F}_i(U),$$

there is an open dense subset $U_i \subset C$ such that $g_i \in \mathcal{F}(U_i)$, for $i = 1, \dots, r$. By Remark 4.3 we get that the sections g_i induce morphisms in $\mathrm{HI}_{\mathrm{Nis}}$, $\varphi_i : h_0^{\mathrm{Nis}}(U_i) \to \mathcal{F}$.

In particular, we get a homomorphism

$$\varphi = \varphi_1 \otimes \cdots \otimes \varphi_r \otimes 1 : K\big(k; h_0^{\mathrm{Nis}}(U_1), \dots, h_0^{\mathrm{Nis}}(U_r), \underline{CH}_0(C)^0\big) \\ \to K(k; \mathcal{F}_1, \dots, \mathcal{F}_r, \underline{CH}_0(C)^0),$$

with the property

$$(g_1 \otimes \cdots \otimes g_r \otimes h;\, f)_P = \varphi(([\Delta_1] \otimes \cdots \otimes [\Delta_r] \otimes h;\, f)_P).$$

Notice that the latter element vanishes, because $h_0^{\mathrm{Nis}}(U_i)$ is curve-like, for $i = 1, \dots, r$ [Kahn and Yamazaki 2013, Lemma 11.2(c)] and hence

$$([\Delta_1] \otimes \cdots \otimes [\Delta_r] \otimes h;\, f)_P = 0. \qquad \square$$

Corollary 4.5. *Let $\mathcal{F}_1, \dots, \mathcal{F}_r \in \mathrm{HI}_{\mathrm{Nis}}$. Let $\mathcal{M} = T(\mathcal{F}_1, \dots, \mathcal{F}_r)$ and let $(\underline{\underline{C}})$ be the local symbol complex associated to $\mathcal{M}$ corresponding to the curve C. Then there is a canonical isomorphism*

$$H(\underline{\underline{C}}) \simeq T(\mathcal{F}_1, \dots, \mathcal{F}_r, \underline{CH}_0(C)^0)(k).$$

In particular, if $G_1, \dots, G_r$ are semiabelian varieties over k, then

$$H(\underline{\underline{C}}) \simeq T(G_1, \dots, G_r, \underline{CH}_0(C)^0)(k) \simeq K(k; G_1, \dots, G_r, \underline{CH}_0(C)^0),$$

where $K(k; G_1, \dots, G_r, \underline{CH}_0(C)^0)$ is the usual Somekawa K-group attached to semiabelian varieties [Somekawa 1990, Definition 1.2].

The $\mathbb{G}_a$-case. In this subsection we consider reciprocity functors

$$\mathcal{M}_1, \dots, \mathcal{M}_r, \quad r \geq 0,$$

and set $\mathcal{M}_0 = \mathbb{G}_a$. We consider the K-group of reciprocity functors

$$T(\mathbb{G}_a, \mathcal{M}_1, \dots, \mathcal{M}_r).$$

Lemma 4.6. *The K-group $T(T(\mathbb{G}_a, \mathcal{M}_1, \dots, \mathcal{M}_r), \underline{CH}_0(C)^0)$ satisfies Assumption 3.3.*

Proof. We have a functorial surjection

$$T(\mathbb{G}_a, \mathcal{M}_1, \dots, \mathcal{M}_r, \underline{CH}_0(C)^0)(k) \twoheadrightarrow T(T(\mathbb{G}_a, \mathcal{M}_1, \dots, \mathcal{M}_r), \underline{CH}_0(C)^0)(k).$$

The first group vanishes by the main result of [Rülling and Yamazaki 2014, Theorem 1.1]. Therefore, the second group vanishes as well. In particular, Assumption 3.3 is satisfied. $\square$

Lemma 4.7. *The K-group $T(T(\mathbb{G}_a, \mathcal{M}_1, \dots, \mathcal{M}_r), \underline{CH}_0(C)^0)$ satisfies Assumption 3.10.*

Proof. To prove the lemma, it suffices to show that

$$K^{\mathrm{geo}}(k; T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r), \underline{CH}_0(C)^0)$$

vanishes.

Claim. There is a well defined local symbol

$$T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\eta_C) \otimes \underline{CH}_0(C)^0(\eta_C) \otimes k(C)^\times \\ \to K^{\mathrm{geo}}(k; T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r), \underline{CH}_0(C)^0),$$

satisfying the unique properties (1)–(3) of page 321.

To have a well defined local symbol following [Serre 1959], we need for every closed point $P \in C$ the natural map

$$h : T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes \underline{CH}_0(C)^0(\mathcal{O}_{C,P}) \\ \to T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\eta_C) \otimes \underline{CH}_0(C)^0(\eta_C)$$

to be injective. For, if $g_1 \in T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\eta_C)$, $g_2 \in \underline{CH}_0(C)^0(\eta_C)$, then we say that $g_1 \otimes g_2$ is regular, if $g_1 \otimes g_2 = h(\tilde{g}_1 \otimes \tilde{g}_2)$, for some

$$\tilde{g}_1 \otimes \tilde{g}_2 \in T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes \underline{CH}_0(C)^0(\mathcal{O}_{C,P}).$$

For such $g_1 \otimes g_2$ we can define $(g_1 \otimes g_2; f)_P = \mathrm{ord}_P(f) s_P(\tilde{g}_1) \otimes s_P(\tilde{g}_2)$. For nonregular $g_1 \otimes g_2$ we define the local symbol using an auxiliary function f_P for f at P as usual (see page 321). The symbol $(. ; .)_P$ is well defined, since there is a unique lifting $\tilde{g}_1 \otimes \tilde{g}_2$ and the unique properties (1)–(3) of page 321 are satisfied by the very definition of the group $K^{\mathrm{geo}}(k; T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r), \underline{CH}_0(C)^0)$. Therefore to prove the claim, it suffices to show the injectivity of h.

Note that we have an equality

$$\underline{CH}_0(C)^0(\mathcal{O}_{C,P}) := \underline{CH}_0(C \times k(\mathrm{Spec}(\mathcal{O}_{C,P})))^0 = \underline{CH}_0(C)^0(\eta_C).$$

The map $T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \to T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\eta_C)$ is injective by the injectivity condition of reciprocity functors. Next, $T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)$ becomes a reciprocity functor of either $\mathbb{Q}$ or $\mathbb{F}_p$-vector spaces, depending on whether char F is 0 or $p > 0$. Setting $\kappa = \mathbb{Q}$ or $\mathbb{Z}/p$ depending on the case we have

$$T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes_{\mathbb{Z}} \underline{CH}_0(C)^0(\mathcal{O}_{C,P}) \\ = T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r)(\mathcal{O}_{C,P}) \otimes_\kappa (\kappa \otimes_{\mathbb{Z}} \underline{CH}_0(C)^0(\mathcal{O}_{C,P})).$$

Since the κ-module $\kappa \otimes_{\mathbb{Z}} \underline{CH}_0(C)^0(\mathcal{O}_{C,P})$ is flat, the claim follows.

To prove the lemma, we imitate the proof given in [Rülling and Yamazaki 2014] for the vanishing of $T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r, \underline{CH}_0(C)^0)(k)$. Let $\{(x_0, \ldots, x_r), \zeta\}^{\mathrm{geo}}$ be a generator of $K^{\mathrm{geo}}(k; T(\mathbb{G}_a, \mathcal{M}_1, \ldots, \mathcal{M}_r), \underline{CH}_0(C)^0)$. Since k is algebraically closed, we may assume $\zeta = [P_0] - [P_1]$, for some closed points $P_0, P_1 \in C$. Then we can show that

$$\{(x_0, \dots, x_r), \zeta\}^{\mathrm{geo}} = \sum_{P \in C} \big((x_0 g \otimes \mathrm{res}_{k(C)/k}(x_1) \cdots \otimes \mathrm{res}_{k(C)/k}(x_r)) \otimes \eta_C ; f\big)_P = 0,$$

where $f \in k(C)^\times$ is a function such that $\mathrm{ord}_{P_0}(f) = 1$ and $\mathrm{ord}_{P_1}(f) = -1$ and $g \in k(C)^\times$ is obtained using the exact sequence

$$\Omega^1_{k(C)/k} \longrightarrow \bigoplus_{P \in C} \frac{\Omega^1_{k(C)/k}}{\Omega^1_{C,P}} \xrightarrow{\sum \mathrm{Res}_P} k \longrightarrow 0.$$

For more details on this local symbol computation see Section 3 in [Rülling and Yamazaki 2014]. In particular, we refer to 3.2 and 3.4 there for the choice of the functions $f, g \in k(C)^\times$. $\square$

Corollary 4.8. *Let $\mathcal{M} = T(\mathbb{G}_a, \mathcal{M}_1, \dots, \mathcal{M}_r)$, where $\mathcal{M}_1, \dots, \mathcal{M}_r$ are reciprocity functors. For any smooth complete curve C over k, we have $H(\underline{\underline{C}}) = 0$. In particular, if* $\mathrm{char}\, F = 0$, *the complex*

$$\Omega^{n+1}_{k(C)} \xrightarrow{\mathrm{Res}_P} \bigoplus_{P \in C} \Omega^n_k \xrightarrow{\sum_P} \Omega^n_k$$

is exact.

Proof. When $\mathrm{char}\, F = 0$, Ivorra and Rülling [2015, Theorem 5.4.7] showed an isomorphism of reciprocity functors $\theta : \Omega^n \simeq T(\mathbb{G}_a, \mathbb{G}^{\times n})$. Moreover, the complex $(\underline{\underline{C}})$ factors through $\Omega^{n+1}_{k(C)}$. $\square$

5. The nonalgebraically closed case

In order to prove Theorem 3.11, we made the assumption that the curve C is over an algebraically closed field k. The reason this assumption was necessary is that for a general reciprocity functor $\mathcal{M}$ the local symbol at a closed point $P \in C$ does not have a local description, but rather depends on the other closed points. Namely, if P is in the support of the modulus $\mathfrak{m}$ corresponding to a section $g \in \mathcal{M}(\eta_C)$, then we have an equality

$$(g; f)_P = -\sum_{Q \notin S} \mathrm{ord}_Q(f_P)\, \mathrm{Tr}_{Q/k}(s_Q(g)),$$

where f_P is an auxiliary function for f at P. If for some reciprocity functor $\mathcal{M}$ we have a local description $(g; f)_P = \mathrm{Tr}_{P/k}(\partial_P(g; f)))$, where $\partial_P(g; f) \in \mathcal{M}(P)$, for every $P \in C$, then we can obtain a complex $(\underline{\underline{C}})'$

$$\Big(\mathcal{M} \bigotimes^{M} \mathbb{G}_m\Big)(\eta_C) \xrightarrow{\partial_C} \bigoplus_{P \in C} \mathcal{M}(P) \xrightarrow{\sum_P \mathrm{Tr}_{P/k}} \mathcal{M}(k).$$

For such a reciprocity functor $\mathcal{M}$, assuming the existence of a k-rational point $P_0 \in C(k)$, we can have a generalization of Theorem 3.11 for the complex $(\underline{\underline{C}})'$ by imposing the following two stronger conditions on $\mathcal{M}$. Namely, we make the following assumptions.

Assumption 5.1. Let $\mathcal{M}$ be a reciprocity functor for which we have a local description of the symbol $(g; f)_P = \mathrm{Tr}_{P/k}(\partial_P(g; f))$. Let $\lambda : D \to C$ be a finite morphism. Assume that for every $h \in \underline{CH}_0(C)(\eta_D)$ and every closed point $P \in C$ we have an equality

$$(g \otimes h; f)_P = \mathrm{Tr}_{P/k}(\partial_P(g, f) \otimes s_P(h)).$$

Assumption 5.2. We assume that for every finite extension L/k we have an equality

$$K^{\mathrm{geo}}(L; \mathcal{M}, \underline{CH}_0(C)^0) \simeq T(\mathcal{M}, \underline{CH}_0(C)^0)(L).$$

Notation 5.3. If E/L is a finite extension and $x \in \mathcal{M}(k)$, we will denote

$$x_E = \mathrm{res}_{E/L}(x).$$

Theorem 5.4. *Let $\mathcal{M}$ be a reciprocity functor that satisfies Assumptions 5.1 and 5.2. Then we have an isomorphism*

$$\begin{aligned}\Phi' : H(\underline{\underline{C}}') &\xrightarrow{\simeq} T(\mathcal{M}, \underline{CH}_0(C)^0)(k),\\ (a_P)_{P\in C} &\to \sum_{P\in C} \mathrm{Tr}_{P/k}(a_P \otimes ([P] - P_{0,k(P)})).\end{aligned}$$

Proof. We start by considering the map

$$\begin{aligned}\Phi' : \Big(\bigoplus_{P\in C} \mathcal{M}(P)\Big)\Big/ \operatorname{Im} \partial_C &\to T(\mathcal{M}, \underline{CH}_0(C))(k),\\ (a_P)_{P\in C} &\to \sum_{P\in C} \mathrm{Tr}_{P/k}(a_P \otimes [P]).\end{aligned}$$

The map Φ' is well defined because of Assumption 5.1. Restricting to $H(\underline{\underline{C}}')$, we obtain the map of the proposition. Moreover, we can consider the map

$$\begin{aligned}\Psi' : T(\mathcal{M}, \underline{CH}_0(C)^0)(k) &\to H(\underline{\underline{C}}')\\ Tr_{L/k}(x \otimes ([Q] - [L(Q) : L][P_{0,L}])) &\to (x_{P'})_{P'\in C},\end{aligned}$$

with $x_P = \mathrm{Tr}_{L(Q)/k(P)}(x)$, $x_{P_0} = -\mathrm{Tr}_{L(Q)/k}(x)$ and $x_{P'} = 0$ otherwise. Here L/k is a finite extension, $x \in \mathcal{M}(L)$, Q is a closed point of $C \times L$ having residue field $L(Q)$ that projects to $P \in C$ under the map $C \times L \to C$ with $P \neq P_0$. We denote by $k(P)$ the residue field of P. The map Ψ' will be well defined (using the same argument as in Proposition 3.9), as long as we check the following:

(1) If $k \subset L \subset E$ is a tower of finite extensions and we have elements $x \in \mathcal{M}(L)$, $y \in \underline{CH}_0(C)^0(E)$, then $\Psi'(\mathrm{Tr}_{L/k}(x \otimes \mathrm{Tr}_{E/L}(y))) = \Psi'(\mathrm{Tr}_{E/k}(x_E \otimes y))$.

(2) $\Psi'(\mathrm{Tr}_{L/k}(\mathrm{Tr}_{E/L}(x) \otimes y)) = \Psi'(\mathrm{Tr}_{E/k}(x \otimes y_E))$ for any $x \in \mathcal{M}(E)$ and $y \in \underline{CH}_0(C)^0(L)$.

For (1) we can reduce to the case when $y = [Q] - [E(Q) : E]P_{0,E}$, for some closed point Q of $C \times E$ with residue field $E(Q)$. Let Q' be the projection of Q in $C \times L$ and P the projection of Q' in C. Notice that we have an equality $\mathrm{Tr}_{E/L}([Q]) = [E(Q) : L(Q')][Q']$. We compute

$$\Psi'\big(\mathrm{Tr}_{E/k}(x_E \otimes ([Q] - [E(Q) : E][P_{0,E}]))\big) = \begin{cases} \mathrm{Tr}_{E(Q)/k(P)}(x) & \text{at } P, \\ -\mathrm{Tr}_{E(Q)/k}(x) & \text{at } P_0, \end{cases}$$

$$\begin{aligned} &\Psi'\big(\mathrm{Tr}_{L/k}(x \otimes \mathrm{Tr}_{E/L}([Q] - [E(Q) : E][P_{0,E}]))\big) \\ &\qquad = \Psi'\big(\mathrm{Tr}_{L/k}(x \otimes [E(Q) : L(Q')]([Q'] - [L(Q') : L][P_{0,L}]))\big) \\ &\qquad \begin{cases} [E(Q) : L(Q')]\,\mathrm{Tr}_{L(Q')/k(P)}(x) & \text{at } P, \\ -[E(Q) : L(Q')]\,\mathrm{Tr}_{L(Q')/k}(x) & \text{at } P_0. \end{cases} \end{aligned}$$

The claim then follows in view of the equality

$$\mathrm{Tr}_{E(Q)/k(P)}(x) = \mathrm{Tr}_{L(Q')/k(P)}\,\mathrm{Tr}_{E(Q)/L(Q')}(x) = [E(Q) : L(Q')]\,\mathrm{Tr}_{L(Q')/k(P)}(x).$$

For (2) we can again reduce to the case when $y = [Q] - [L(Q) : L][P_{0,L}]$ for some closed point Q of $C \times L$ with residue field $L(Q)$. Notice that we have an equality

$$[Q]_E = \sum_{Q' \to Q} e(Q'/Q)[Q'],$$

where the sum extends over all closed points Q' of $C \times E$ that project to Q. We compute:

$$\begin{aligned} &\Psi'(\mathrm{Tr}_{L/k}(\mathrm{Tr}_{E/L}(x) \otimes y)) \\ &\qquad = \begin{cases} \mathrm{Tr}_{L(Q)/k(P)}(\mathrm{Tr}_{E/L}(x)_{L(Q)}) & \text{at } P, \\ -\mathrm{Tr}_{L(Q)/k}(\mathrm{Tr}_{E/L}(x)_{L(Q)}) & \text{at } P_0, \end{cases} \\ &\qquad = \begin{cases} \mathrm{Tr}_{L(Q)/k(P)}(\sum_{Q' \to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/L(Q)}(x_{E(Q')})) & \text{at } P, \\ -\mathrm{Tr}_{L(Q)/k}(\sum_{Q' \to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/L(Q)}(x_{E(Q')})) & \text{at } P_0, \end{cases} \\ &\qquad = \begin{cases} \sum_{Q' \to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/k(P)}(x_{E(Q')}) & \text{at } P, \\ -\sum_{Q' \to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/k}(x_{E(Q')}) & \text{at } P_0. \end{cases} \end{aligned}$$

The equality $\mathrm{Tr}_{E/L}(x)_{L(Q)} = \sum_{Q' \to Q} e(Q'/Q)\,\mathrm{Tr}_{E(Q')/L(Q)}(x_{E(Q')})$ follows from Remark 1.3.3, Property (MF1) of [Ivorra and Rülling 2015] if we set

$$\varphi : \operatorname{Spec} E \to \operatorname{Spec} L \quad \text{and} \quad \psi : \operatorname{Spec} L(Q) \to \operatorname{Spec} L.$$

On the other hand we have

$$\begin{aligned}\Psi'(\mathrm{Tr}_{E/k}(x\otimes y_E)) &= \Psi'\Big(\sum_{Q'\to Q} e(Q'/Q)\Big)\mathrm{Tr}_{E/k}(x\otimes([Q']-[L(Q):L][P_{0,E}]))\\ &= \begin{cases}\sum_{Q'\to Q} e(Q'/Q))\,\mathrm{Tr}_{E(Q')/k(P)}(x_{E(Q')}) & \text{at } P,\\ -\sum_{Q'\to Q} e(Q'/Q))\,\mathrm{Tr}_{E(Q')/k}(x_{E(Q')}) & \text{at } P_0.\end{cases}\end{aligned}$$

Next we need to show that the maps Φ', Ψ' are each other's inverses. It is immediate that the composition $\Psi'\Phi'$ is the identity map. For the other composition, we consider an element $x\otimes([Q]-[L(Q):L][P_{0,L}])\in T(\mathcal{M},\underline{CH}_0(C)^0)(L)$. If $L(Q)$ is the residue field of Q, then Q induces an $L(Q)$-rational point $\widetilde{Q}$ of $C\times L(Q)$. Then we have an equality $\mathrm{Tr}_{L(Q)/L}([\widetilde{Q}])=[Q]$. By the projection formula we get an equality

$$x\otimes([Q]-[L(Q):L][P_{0,L}])=\mathrm{Tr}_{L(Q)/L}(x_{L(Q)}\otimes([\widetilde{Q}]-[P_{0,L(Q)}])),$$

we are therefore reduced to the case $L(Q)=L$. Then we have

$$\begin{aligned}\Phi'\Psi'\big(\mathrm{Tr}_{L/k}(x\otimes([Q]-[P_{0,L}]))\big) &= \mathrm{Tr}_{P/k}(\mathrm{Tr}_{L/P}(x)\otimes([P]-[P_{0,k(P)}]))\\ &= \mathrm{Tr}_{P/k}\,\mathrm{Tr}_{L/P}(x\otimes \mathrm{res}_{L/P}([P]-[P_{0,k(P)}]))\\ &= \mathrm{Tr}_{L/k}(x\otimes[Q]-[P_{0,L}]).\end{aligned}$$

This completes the proof of the theorem. □

Remark 5.5. We note here that for the algebraically closed field case if instead of the Assumption 3.3, we had made the stronger Assumption 5.1, the proof of the Proposition 3.4 would have become simpler. The only reason we used Assumption 3.3 is that in general the problem of computing the symbol $(g;f)_P$ locally is rather hard and is known only in very few cases, namely for homotopy invariant Nisnevich sheaves with transfers, as the next example indicates.

Example 5.6. Let $k\in\mathcal{E}_F$ be any perfect field. Let $\mathcal{F}_1,\dots,\mathcal{F}_r$ be homotopy invariant Nisnevich sheaves with transfers. Then as mentioned in the previous section, the main theorem of [Kahn and Yamazaki 2013] gives an isomorphism

$$T(\mathcal{F}_1,\dots,\mathcal{F}_r)(L)\simeq K^{\mathrm{geo}}(L;\mathcal{F}_1,\dots,\mathcal{F}_r)\simeq K(L;\mathcal{F}_1,\dots,\mathcal{F}_r),$$

where $K(L;\mathcal{F}_1,\dots,\mathcal{F}_r)$ is the Somekawa type K-group [Kahn and Yamazaki 2013, Definition 5.1] and L/k is any finite extension. In particular, let C be a smooth, complete and geometrically connected curve over k and $P\in C$ be a closed point. As in the proof of Lemma 4.4, we can reduce to the case when $\mathcal{F}_i$ is curve-like, for $i=1,\dots,r$. To describe the local symbol, it therefore suffices to consider sections $g_i\in\mathcal{F}_i(\eta_C)$ with disjoint supports. In this case, if P is in the support of

g_i for some $i \in \{1, \dots, r\}$ and $f \in k(C)^\times$ is a function, then we have the following explicit local description of $(g_1 \otimes \cdots \otimes g_r; f)_P$.

$$(g_1 \otimes \cdots \otimes g_r; f)_P = \mathrm{Tr}_{P/k}(s_P(g_1) \otimes \cdots \otimes \partial_P(g_i, f) \otimes \cdots \otimes s_P(g_r)).$$

Moreover, $\underline{CH}_0(C)^0$ is itself a homotopy invariant Nisnevich sheaf with transfers. Namely, $\underline{CH}_0(C)^0 \in RF_0$ and hence the above formula implies that the Assumption 5.1 is satisfied.

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EVANGELIA GAZAKI: valiagaz@math.uchicago.edu
Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637, United States

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