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# Algebraic Kasparov K-theory, II

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A kind of motivic stable homotopy theory of algebras is developed. Explicit fibrant replacements for the  $S^1$ -spectrum and  $(S^1, \mathbb{G})$ -bispectrum of an algebra are constructed. As an application, unstable, Morita stable and stable universal bivariant theories are recovered. These are shown to be embedded by means of contravariant equivalences as full triangulated subcategories of compact generators of some compactly generated triangulated categories. Another application is the introduction and study of the symmetric monoidal compactly generated triangulated category of  $K$ -motives. It is established that the triangulated category  $kk$  of Cortiñas and Thom (*J. Reine Angew. Math.* **610** (2007), 71–123) can be identified with the  $K$ -motives of algebras. It is proved that the triangulated category of  $K$ -motives is a localisation of the triangulated category of  $(S^1, \mathbb{G})$ -bispectra. Also, explicit fibrant  $(S^1, \mathbb{G})$ -bispectra representing stable algebraic Kasparov  $K$ -theory and algebraic homotopy  $K$ -theory are constructed.

## 1. Introduction

Throughout the paper  $k$  is a fixed commutative ring with unit and  $\text{Alg}_k$  is the category of nonunital  $k$ -algebras and nonunital  $k$ -homomorphisms. Also,  $F$  is a fixed field and  $\text{Sm}/F$  is the category of smooth algebraic varieties over  $F$ . If  $\mathcal{C}$  is a category and  $A, B$  are objects of  $\mathcal{C}$ , we shall often write  $\mathcal{C}(A, B)$  to denote the Hom-set  $\text{Hom}_{\mathcal{C}}(A, B)$ .

$\mathbb{A}^1$ -homotopy theory is the homotopy theory of motivic spaces, i.e., presheaves of simplicial sets defined on  $\text{Sm}/F$  (see [Morel and Voevodsky 1999; Voevodsky 1998]). Each object  $X \in \text{Sm}/F$  is regarded as the motivic space  $\text{Hom}_{\text{Sm}/F}(-, X)$ . The affine line  $\mathbb{A}^1$  plays the role of the interval.

$k[t]$ -homotopy theory is the homotopy theory of simplicial functors defined on nonunital algebras, where each algebra  $A$  is regarded contravariantly as the space  $rA = \text{Hom}_{\text{Alg}_k}(A, -)$  so that we can study algebras from a homotopy theoretic viewpoint (see [Garkusha 2007; 2014]). The role of the interval is played by the space  $r(k[t])$  represented by the polynomial algebra  $k[t]$ . This theory borrows methods and approaches from  $\mathbb{A}^1$ -homotopy theory. Another source of ideas

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and techniques for  $k[t]$ -homotopy theory originates in Kasparov  $K$ -theory of  $C^*$ -algebras.

In [Garkusha 2007] a kind of unstable motivic homotopy theory of algebras was developed. In order to develop stable motivic homotopy theory of algebras and — most importantly — to make the explicit computations of fibrant replacements for suspension spectra  $\Sigma^\infty_r A$ ,  $A \in \text{Alg}_k$ , presented in this paper, one first needs to introduce and study “unstable, Morita stable and stable Kasparov  $K$ -theory spectra”  $\mathbb{K}(A, B)$ ,  $\mathbb{K}^{\text{mor}}(A, B)$  and  $\mathbb{K}^{st}(A, B)$  respectively, where  $A, B$  are algebras. We refer the reader to [Garkusha 2014] for properties of the spectra. The aim of this paper is to develop stable motivic homotopy theory of algebras.

Throughout we work with a certain small subcategory  $\mathfrak{A}$  of  $\text{Alg}_k$  and the category  $U_\bullet \mathfrak{A}$  of certain pointed simplicial functors on  $\mathfrak{A}$ .  $U_\bullet \mathfrak{A}$  comes equipped with a motivic model structure. We write  $\text{Sp}(\mathfrak{A})$  to denote the stable model category of  $S^1$ -spectra associated with the model category  $U_\bullet \mathfrak{A}$ .  $\mathbb{K}(A, -)$ ,  $\mathbb{K}^{\text{mor}}(A, -)$  and  $\mathbb{K}^{st}(A, -)$  are examples of fibrant  $\Omega$ -spectra in  $\text{Sp}(\mathfrak{A})$  (see [Garkusha 2014]).

One of the main results of the paper says that  $\mathbb{K}(A, -)$  is a fibrant replacement for the suspension spectrum  $\Sigma^\infty_r A \in \text{Sp}(\mathfrak{A})$  of an algebra  $A \in \mathfrak{A}$ . Namely, there is a natural weak equivalence of spectra

$$\Sigma^\infty_r A \longrightarrow \mathbb{K}(A, -)$$

in  $\text{Sp}(\mathfrak{A})$  (see Theorem 4.2).

This is an analog of a similar result by the author and Panin [Garkusha and Panin 2014a] computing a fibrant replacement of the suspension  $\mathbb{P}^1$ -spectrum  $\Sigma_{\mathbb{P}^1}^\infty X_+$  of a smooth algebraic variety  $X$ . The main reason that computation of a fibrant replacement for  $\Sigma_{\mathbb{P}^1}^\infty X_+$  is possible is the existence of framed correspondences of [Voevodsky 2001] on homotopy groups of (motivically fibrant)  $\mathbb{P}^1$ -spectra. In turn, the main reason why the computation of a fibrant replacement for  $\Sigma^\infty_r A$  is possible is that algebras have universal extensions.

Let  $\text{SH}_{S^1}(\mathfrak{A})$  denote the homotopy category of  $\text{Sp}(\mathfrak{A})$ .  $\text{SH}_{S^1}(\mathfrak{A})$  plays the same role as the stable homotopy category of motivic  $S^1$ -spectra  $\text{SH}_{S^1}(F)$  over a field  $F$ . It is a compactly generated triangulated category with compact generators  $\{\Sigma^\infty_r A[n]\}_{A \in \mathfrak{A}, n \in \mathbb{Z}}$ . One of the important consequences of the above computation is that we are able to give an *explicit* description of the Hom-groups

$$\text{SH}_{S^1}(\mathfrak{A})(\Sigma^\infty_r B[n], \Sigma^\infty_r A).$$

Precisely, there is an isomorphism of abelian groups (see Corollary 4.3)

$$\text{SH}_{S^1}(\mathfrak{A})(\Sigma^\infty_r B[n], \Sigma^\infty_r A) \cong \mathbb{K}_n(A, B), \quad A, B \in \mathfrak{A}, n \in \mathbb{Z}.$$

It is important to note that the groups  $\mathbb{K}_n(A, B)$  have an *explicit* description in terms of nonunital algebra homomorphisms (see [Garkusha 2014, Section 7] for details).

We also show in Theorem 4.4 that the full subcategory  $\mathcal{S}$  of  $\mathrm{SH}_{S^1}(\mathfrak{A})$  spanned by the compact generators  $\{\Sigma^\infty rA[n]\}_{A \in \mathfrak{A}, n \in \mathbb{Z}}$  is triangulated and there is a contravariant equivalence of triangulated categories

$$D(\mathfrak{A}, \mathfrak{F}) \xrightarrow{\sim} \mathcal{S}$$

with  $\mathfrak{A} \rightarrow D(\mathfrak{A}, \mathfrak{F})$  the universal unstable excisive homotopy invariant homology theory in the sense of [Garkusha 2013] with respect to the class of  $k$ -split surjective algebra homomorphisms  $\mathfrak{F}$ . This equivalence is an extension of the contravariant functor  $A \in \mathfrak{A} \mapsto \Sigma^\infty rA \in \mathrm{SH}_{S^1}(\mathfrak{A})$  to  $D(\mathfrak{A}, \mathfrak{F})$ . Thus  $D(\mathfrak{A}, \mathfrak{F})$  is recovered from  $\mathrm{SH}_{S^1}(\mathfrak{A})$ . It also follows that the *small* triangulated category  $D(\mathfrak{A}, \mathfrak{F})^{\mathrm{op}}$  lives inside the “big” ambient triangulated category  $\mathrm{SH}_{S^1}(\mathfrak{A})$ . This is reminiscent of Voevodsky’s theorem [2000] saying that there is a full embedding of the *small* triangulated category  $DM_{gm}^{eff}(F)$  of effective geometrical motives into the “big” triangulated category  $DM^{eff}(F)$  of motivic complexes of Nisnevich sheaves with transfers.

Next, we introduce matrices into the game. Namely, if we localise  $\mathrm{SH}_{S^1}(\mathfrak{A})$  with respect to the family of compact objects

$$\{\mathrm{cone}(\Sigma^\infty r(M_n A) \rightarrow \Sigma^\infty rA)\}_{n>0},$$

we shall get a compactly generated triangulated category  $\mathrm{SH}_{S^1}^{\mathrm{mor}}(\mathfrak{A})$  with compact generators  $\{\Sigma^\infty rA[n]\}_{A \in \mathfrak{A}, n \in \mathbb{Z}}$ . It is in fact the homotopy category of a model category  $\mathrm{Sp}_{\mathrm{mor}}(\mathfrak{A})$ , which is the same category as  $\mathrm{Sp}(\mathfrak{A})$  but with a new model structure. We construct in a similar way a compactly generated triangulated category  $\mathrm{SH}_{S^1}^\infty(\mathfrak{A})$ , obtained from  $\mathrm{SH}_{S^1}(\mathfrak{A})$  by localisation with respect to the family of compact objects

$$\{\mathrm{cone}(\Sigma^\infty r(M_\infty A) \rightarrow \Sigma^\infty rA)\},$$

where  $M_\infty A = \bigcup_n M_n A$ . It is also the homotopy category of a model category  $\mathrm{Sp}_\infty(\mathfrak{A})$ , which is the same category as  $\mathrm{Sp}(\mathfrak{A})$  but with a new model structure.

We prove in Theorems 5.1 and 6.1 that for any algebra  $A \in \mathfrak{A}$  there are natural weak equivalences of spectra

$$\Sigma^\infty rA \longrightarrow \mathbb{K}^{\mathrm{mor}}(A, -) \quad \text{and} \quad \Sigma^\infty rA \longrightarrow \mathbb{K}^{\mathrm{st}}(A, -)$$

in  $\mathrm{Sp}_{\mathrm{mor}}(\mathfrak{A})$  and  $\mathrm{Sp}_\infty(\mathfrak{A})$ , respectively. Also, for all  $A, B \in \mathfrak{A}$  and  $n \in \mathbb{Z}$  there are isomorphisms of abelian groups

$$\mathrm{SH}_{S^1}^{\mathrm{mor}}(\mathfrak{A})(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n^{\mathrm{mor}}(A, B)$$

and

$$\mathrm{SH}_{S^1}^\infty(\mathfrak{R})(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n^{st}(A, B),$$

respectively. Furthermore, the full subcategories  $\mathcal{S}_{\mathrm{mor}}$  and  $\mathcal{S}_\infty$  of  $\mathrm{SH}_{S^1}^{\mathrm{mor}}(\mathfrak{R})$  and  $\mathrm{SH}_{S^1}^\infty(\mathfrak{R})$  spanned by the compact generators  $\{\Sigma^\infty rA[n]\}_{A \in \mathfrak{R}, n \in \mathbb{Z}}$  are triangulated and there are contravariant equivalences of triangulated categories

$$D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F}) \xrightarrow{\sim} \mathcal{S}_{\mathrm{mor}} \quad \text{and} \quad D_{st}(\mathfrak{R}, \mathfrak{F}) \xrightarrow{\sim} \mathcal{S}_\infty.$$

Here  $\mathfrak{R} \rightarrow D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F})$  (respectively  $\mathfrak{R} \rightarrow D_{st}(\mathfrak{R}, \mathfrak{F})$ ) is the universal Morita stable (respectively stable) excisive homotopy invariant homology theory in the sense of [Garkusha 2013]. Thus  $D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F})$  and  $D_{st}(\mathfrak{R}, \mathfrak{F})$  are recovered from  $\mathrm{SH}_{S^1}^{\mathrm{mor}}(\mathfrak{R})$  and  $\mathrm{SH}_{S^1}^{st}(\mathfrak{R})$ , respectively. It also follows that the *small* triangulated categories  $D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F})^{\mathrm{op}}$ ,  $D_{st}(\mathfrak{R}, \mathfrak{F})^{\mathrm{op}}$  live inside the ambient triangulated categories  $\mathrm{SH}_{S^1}^{\mathrm{mor}}(\mathfrak{R})$  and  $\mathrm{SH}_{S^1}^{st}(\mathfrak{R})$ .

We next introduce a symmetric monoidal compactly generated triangulated category of  $K$ -motives  $DK(\mathfrak{R})$  together with a canonical contravariant functor

$$M_K : \mathfrak{R} \rightarrow DK(\mathfrak{R}).$$

The category  $DK(\mathfrak{R})$  is an analog of the triangulated category of  $K$ -motives for algebraic varieties introduced in [Garkusha and Panin 2012; 2014b].

For any algebra  $A \in \mathfrak{R}$  its  $K$ -motive is, by definition, the object  $M_K(A)$  of  $DK(\mathfrak{R})$ . We have that

$$M_K(A) \otimes M_K(B) \cong M_K(A \otimes B)$$

for all  $A, B \in \mathfrak{R}$  (see Proposition 7.1).

We prove in Theorem 7.2 that for any two algebras  $A, B \in \mathfrak{R}$  and any integer  $n$  there is a natural isomorphism

$$DK(\mathfrak{R})(M_K(B)[n], M_K(A)) \cong \mathbb{K}_n^{st}(A, B).$$

Moreover, if  $\mathcal{T}$  is the full subcategory of  $DK(\mathfrak{R})$  spanned by  $K$ -motives of algebras  $\{M_K(A)\}_{A \in \mathfrak{R}}$  then  $\mathcal{T}$  is triangulated and there is an equivalence of triangulated categories

$$D_{st}(\mathfrak{R}, \mathfrak{F}) \rightarrow \mathcal{T}^{\mathrm{op}}$$

sending an algebra  $A \in \mathfrak{R}$  to its  $K$ -motive  $M_K(A)$  (see Theorem 7.2). It is also proved in Corollary 7.3 that for any algebra  $A \in \mathfrak{R}$  and any integer  $n$  one has a natural isomorphism

$$DK(\mathfrak{R})(M_K(A)[n], M_K(k)) \cong KH_n(A),$$

where the right hand side is the  $n$ -th homotopy  $K$ -theory group in the sense of Weibel [1989]. This result is reminiscent of a similar result for  $K$ -motives of algebraic varieties in the sense of [Garkusha and Panin 2012; 2014b] identifying the  $K$ -motive of the point with algebraic  $K$ -theory.

Cortiñas and Thom [2007] constructed a universal excisive homotopy invariant and  $M_\infty$ -invariant homology theory on all  $k$ -algebras

$$j : \text{Alg}_k \rightarrow kk.$$

The triangulated category  $kk$  is an analog of Cuntz’s triangulated category  $kk^{\text{lca}}$  whose objects are the locally convex algebras [Cuntz 1997; 2005; Cuntz and Thom 2006].

We show in Theorem 7.4 that, if we denote by  $kk(\mathfrak{R})$  the full subcategory of  $kk$  spanned by the objects from  $\mathfrak{R}$  and assume that the cone ring  $\Gamma k$  in the sense of [Karoubi and Villamayor 1969] is in  $\mathfrak{R}$ , then there is a natural triangulated equivalence

$$kk(\mathfrak{R}) \xrightarrow{\sim} \mathcal{T}^{\text{op}}$$

sending  $A \in kk(\mathfrak{R})$  to its  $K$ -motive  $M_K(A)$ . Thus we can identify  $kk(\mathfrak{R})$  with the  $K$ -motives of algebras. It also follows that the *small* triangulated category  $kk(\mathfrak{R})^{\text{op}}$  lives inside the ambient triangulated category  $DK(\mathfrak{R})$ .

One of the equivalent approaches to stable motivic homotopy theory in the sense of Morel and Voevodsky [1999] is the theory of  $(S^1, \mathbb{G}_m)$ -bispectra. The role of  $\mathbb{G}_m$  in our context is played by the representable functor  $\mathbb{G} := r(\sigma)$ , where  $\sigma = (t - 1)k[t^{\pm 1}]$ . We develop the motivic theory of  $(S^1, \mathbb{G})$ -bispectra. As usual they form a model category which we denote by  $\text{Sp}_{\mathbb{G}}(\mathfrak{R})$ . The homotopy category  $\text{SH}_{S^1, \mathbb{G}}(\mathfrak{R})$  of  $\text{Sp}_{\mathbb{G}}(\mathfrak{R})$  plays the same role as the stable motivic homotopy category  $\text{SH}(F)$  over a field  $F$ . We construct an explicit fibrant  $(S^1, \mathbb{G})$ -bispectrum  $\Theta_{\mathbb{G}}^\infty \mathbb{K}\mathbb{G}(A, -)$ , obtained from fibrant  $S^1$ -spectra  $\mathbb{K}(\sigma^n A, -)$ ,  $n \geq 0$ , by stabilisation in the  $\sigma$ -direction.

The main computational result for bispectra, stated in Theorem 8.1, says that  $\Theta_{\mathbb{G}}^\infty \mathbb{K}\mathbb{G}(A, -)$  is a fibrant replacement of the suspension bispectrum associated with an algebra  $A$ . Namely, there is a natural weak equivalence of bispectra in  $\text{Sp}_{\mathbb{G}}(\mathfrak{R})$

$$\Sigma_{\mathbb{G}}^\infty \Sigma^\infty rA \rightarrow \Theta_{\mathbb{G}}^\infty \mathbb{K}\mathbb{G}(A, -),$$

where  $\Sigma_{\mathbb{G}}^\infty \Sigma^\infty rA$  is the suspension bispectrum of  $rA$ .

Let  $k$  be the field of complex numbers  $\mathbb{C}$  and let  $\mathcal{K}^\sigma(A, -)$  be the  $(0,0)$ -space of the bispectrum  $\Theta_{\mathbb{G}}^\infty \mathbb{K}\mathbb{G}(A, -)$ . We raise a question whether there is a category  $\mathfrak{R}$  of commutative  $\mathbb{C}$ -algebras such that the fibrant simplicial set  $\mathcal{K}^\sigma(\mathbb{C}, \mathbb{C})$  has the homotopy type of  $\Omega^\infty \Sigma^\infty S^0$ . The question is justified by a recent result of Levine [2014] saying that over an algebraically closed field  $F$  of characteristic zero the

homotopy groups of weight zero of the motivic sphere spectrum evaluated at  $F$  are isomorphic to the stable homotopy groups of the classical sphere spectrum. The role of the motivic sphere spectrum in our context is played by the bispectrum  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} r\mathbb{C}$ .

We finish the paper by proving that the triangulated category  $DK(\mathfrak{R})$  of  $K$ -motives is fully faithfully embedded into the homotopy category of  $(S^1, \mathbb{G})$ -bispectra. We also construct an explicit fibrant  $(S^1, \mathbb{G})$ -bispectrum  $\mathbb{K}\mathbb{G}^{st}(A, -)$  consisting of fibrant  $S^1$ -spectra  $\mathbb{K}^{st}(\sigma^n A, -)$ ,  $n \geq 0$ . For this we prove the ‘‘cancellation theorem’’ for the spectra  $\mathbb{K}^{st}(\sigma^n A, -)$  (see Theorem 9.5). It is reminiscent of the cancellation theorem proved by Voevodsky [2010a] for motivic cohomology. The same theorem was proved for  $K$ -theory of algebraic varieties in [Garkusha and Panin 2015].

We show in Theorem 9.7 that  $\mathbb{K}\mathbb{G}^{st}(A, -)$  is  $(2, 1)$ -periodic and represents stable algebraic Kasparov  $K$ -theory (cf. [Voevodsky 1998, Theorems 6.8 and 6.9]). Precisely, for any algebras  $A, B \in \mathfrak{R}$  and any integers  $p, q$  there is an isomorphism

$$\pi_{p,q}(\mathbb{K}\mathbb{G}^{st}(A, B)) \cong \mathbb{K}_{p-2q}^{st}(A, B).$$

As a consequence, one has that for any algebra  $B \in \mathfrak{R}$  and any integers  $p, q$  there is an isomorphism

$$\pi_{p,q}(\mathbb{K}\mathbb{G}^{st}(k, B)) \cong KH_{p-2q}(B).$$

Thus the bispectrum  $\mathbb{K}\mathbb{G}^{st}(k, B)$  yields an explicit model for homotopy  $K$ -theory.

We should stress that the term ‘‘motivic’’ is used in the paper only for the reason that the  $k[t]$ -homotopy theory of algebras shares many properties with Morel and Voevodsky’s motivic homotopy theory of smooth schemes [1999] (see remarks on page 288 as well). If there is a likelihood of confusion with other motivic theories of commutative or noncommutative objects, the reader can just omit the term ‘‘motivic’’ everywhere.

In general, we shall not be very explicit about set-theoretical foundations, and we shall tacitly assume we are working in some fixed universe  $\mathbb{U}$  of sets. Members of  $\mathbb{U}$  are then called *small sets*, whereas a collection of members of  $\mathbb{U}$  which does not itself belong to  $\mathbb{U}$  will be referred to as a *large set* or a *proper class*. If there is no likelihood of confusion, we replace  $\otimes_k$  by  $\otimes$ .

## 2. Preliminaries

In this section we collect basic facts about admissible categories of algebras and triangulated categories associated with them. We mostly follow [Garkusha 2007; 2013].

**2.1. Algebraic homotopy.** Following [Gersten 1971b] a category  $\mathfrak{R}$  of  $k$ -algebras without unit is *admissible* if it is a full subcategory of  $\text{Alg}_k$  and



- (1) if  $R$  is in  $\mathfrak{R}$  and  $I$  is a (two-sided) ideal of  $R$  then  $I$  and  $R/I$  are in  $\mathfrak{R}$ ;
- (2) if  $R$  is in  $\mathfrak{R}$ , then so is  $R[x]$ , the polynomial algebra, in one variable;
- (3) given a cartesian square

$$\begin{array}{ccc} D & \xrightarrow{\rho} & A \\ \sigma \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

in  $\text{Alg}_k$  with  $A, B, C$  in  $\mathfrak{R}$ , then  $D$  is in  $\mathfrak{R}$ .

One may abbreviate (1)–(3) by saying that  $\mathfrak{R}$  is closed under operations of taking ideals, homomorphic images, polynomial extensions in a finite number of variables, and fibre products. For instance, the *category of commutative  $k$ -algebras*  $\text{CAlg}_k$  is admissible.

Observe that every  $k$ -module  $M$  can be regarded as a nonunital  $k$ -algebra with trivial multiplication:  $m_1 \cdot m_2 = 0$  for all  $m_1, m_2 \in M$ . Then  $\text{Mod } k$  is an admissible category of commutative  $k$ -algebras.

If  $R$  is an algebra then the polynomial algebra  $R[x]$  admits two homomorphisms onto  $R$

$$R[x] \begin{array}{c} \xrightarrow{\partial_x^0} \\ \xrightarrow{\partial_x^1} \end{array} R ,$$

where

$$\partial_x^i|_R = 1_R, \quad \partial_x^i(x) = i, \quad i = 0, 1.$$

Of course,  $\partial_x^1(x) = 1$  has to be understood in the sense that  $\Sigma r_n x^n \mapsto \Sigma r_n$ .

**Definition.** Two homomorphisms  $f_0, f_1 : S \rightarrow R$  are *elementary homotopic*, written  $f_0 \sim f_1$ , if there exists a homomorphism

$$f : S \rightarrow R[x]$$

such that  $\partial_x^0 f = f_0$  and  $\partial_x^1 f = f_1$ . A map  $f : S \rightarrow R$  is called an *elementary homotopy equivalence* if there is a map  $g : R \rightarrow S$  such that  $fg$  and  $gf$  are elementary homotopic to  $\text{id}_R$  and  $\text{id}_S$  respectively.

For example, let  $A$  be a  $\mathbb{Z}_{n \geq 0}$ -graded algebra, then the inclusion  $A_0 \rightarrow A$  is an elementary homotopy equivalence. The homotopy inverse is given by the projection  $A \rightarrow A_0$ . Indeed, the map  $A \rightarrow A[x]$  sending a homogeneous element  $a_n \in A_n$  to  $a_n x^n$  is a homotopy between the composite  $A \rightarrow A_0 \rightarrow A$  and the identity  $\text{id}_A$ .

The relation “elementary homotopic” is reflexive and symmetric [Gersten 1971b, p. 62]. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol “ $\simeq$ ”). Following notation of [Gersten 1971a], the set of equivalence classes of morphisms  $R \rightarrow S$  is written  $[R, S]$ .

**Lemma 2.1** [Gersten 1971a]. *Given morphisms in  $\text{Alg}_k$*

$$R \xrightarrow{f} S \begin{array}{c} \xrightarrow{g} \\ \xleftarrow{g'} \end{array} T \xrightarrow{h} U$$

*such that  $g \simeq g'$ , then  $gf \simeq g'f$  and  $hg \simeq hg'$ .*

Thus homotopy behaves well with respect to composition and we have category  $\text{Hotalg}$ , the *homotopy category of  $k$ -algebras*, whose objects are  $k$ -algebras and such that  $\text{Hotalg}(R, S) = [R, S]$ . The homotopy category of an admissible category of algebras  $\mathfrak{A}$  will be denoted by  $\mathcal{H}(\mathfrak{A})$ . Call a homomorphism  $s : A \rightarrow B$  an *I-weak equivalence* if its image in  $\mathcal{H}(\mathfrak{A})$  is an isomorphism. Observe that *I-weak equivalences* are those homomorphisms which have homotopy inverses.

A diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in  $\text{Alg}_k$  is a short exact sequence if  $f$  is injective,  $g$  is surjective, and the image of  $f$  is equal to the kernel of  $g$ .

**Definition.** An algebra  $R$  is *contractible* if  $0 \sim 1$ ; that is, if there is a homomorphism  $f : R \rightarrow R[x]$  such that  $\partial_x^0 f = 0$  and  $\partial_x^1 f = 1_R$ .

For example, every square zero algebra  $A \in \text{Alg}_k$  is contractible by means of the homotopy  $A \rightarrow A[x]$ ,  $a \in A \mapsto ax \in A[x]$ . In other words, every  $k$ -module, regarded as a  $k$ -algebra with trivial multiplication, is contractible.

Following [Karoubi and Villamayor 1969] we define  $ER$ , the *path algebra* on  $R$ , as the kernel of  $\partial_x^0 : R[x] \rightarrow R$ , so

$$ER \rightarrow R[x] \xrightarrow{\partial_x^0} R$$

is a short exact sequence in  $\text{Alg}_k$ . Also  $\partial_x^1 : R[x] \rightarrow R$  induces a surjection  $\partial_x^1 : ER \rightarrow R$  and we define the *loop algebra*  $\Omega R$  of  $R$  to be its kernel, so we have a short exact sequence in  $\text{Alg}_k$

$$\Omega R \rightarrow ER \xrightarrow{\partial_x^1} R.$$

We call it the *loop extension* of  $R$ . Clearly,  $\Omega R$  is the intersection of the kernels of  $\partial_x^0$  and  $\partial_x^1$ . By [Gersten 1971b, Lemma 3.3]  $ER$  is contractible for any algebra  $R$ .

### 2.2. Categories of fibrant objects.

**Definition.** Let  $\mathcal{A}$  be a category with finite products and a final object  $e$ . Assume that  $\mathcal{A}$  has two distinguished classes of maps, called *weak equivalences* and *fibrations*. A map is called a *trivial fibration* if it is both a weak equivalence and a

fibration. We define a *path space* for an object  $B$  to be an object  $B^I$  together with maps

$$B \xrightarrow{s} B^I \xrightarrow{(d_0, d_1)} B \times B,$$

where  $s$  is a weak equivalence,  $(d_0, d_1)$  is a fibration, and the composite is the diagonal map.

Following [Brown 1973], we call  $\mathcal{A}$  a *category of fibrant objects* or a *Brown category* if the following axioms are satisfied.

- (A) Let  $f$  and  $g$  be maps such that  $gf$  is defined. If two of  $f$ ,  $g$ ,  $gf$  are weak equivalences then so is the third. Any isomorphism is a weak equivalence.
- (B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.
- (C) Given a diagram

$$A \xrightarrow{u} C \xleftarrow{v} B,$$

with  $v$  a fibration (respectively a trivial fibration), the pullback  $A \times_C B$  exists and the map  $A \times_C B \rightarrow A$  is a fibration (respectively a trivial fibration).

- (D) For any object  $B$  in  $\mathcal{A}$  there exists at least one path space  $B^I$  (not necessarily functorial in  $B$ ).
- (E) For any object  $B$  the map  $B \rightarrow e$  is a fibration.

**2.3. The triangulated category  $D(\mathfrak{R}, \mathfrak{F})$ .** In what follows we denote by  $\mathfrak{F}$  the class of  $k$ -split surjective algebra homomorphisms. We shall also refer to  $\mathfrak{F}$  as *fibrations*.

Let  $\mathfrak{W}$  be a class of weak equivalences in an admissible category of algebras  $\mathfrak{R}$  containing homomorphisms  $A \rightarrow A[t]$ ,  $A \in \mathfrak{R}$ , such that the triple  $(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  is a Brown category.

**Definition.** The *left derived category*  $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  of  $\mathfrak{R}$  with respect to  $(\mathfrak{F}, \mathfrak{W})$  is the category obtained from  $\mathfrak{R}$  by inverting the weak equivalences.

By [Garkusha 2013] the family of weak equivalences in the category  $\mathcal{H}\mathfrak{R}$  admits a calculus of right fractions. The left derived category  $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  (possibly “large”) is obtained from  $\mathcal{H}\mathfrak{R}$  by inverting the weak equivalences. The left derived category  $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  is left triangulated (see [Garkusha 2007; 2013] for details) with  $\Omega$  a loop functor on it.

There is a general method of stabilising  $\Omega$  (see Heller [Heller 1968]) and producing a triangulated (possibly “large”) category  $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  from the left triangulated structure on  $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ .

An object of  $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  is a pair  $(A, m)$  with  $A \in D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  and  $m \in \mathbb{Z}$ . If  $m, n \in \mathbb{Z}$  then we consider the directed set  $I_{m,n} = \{k \in \mathbb{Z} \mid m, n \leq k\}$ . The morphisms

between  $(A, m)$  and  $(B, n) \in D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  are defined by

$$D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})[(A, m), (B, n)] := \operatorname{colim}_{k \in I_{m,n}} D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})(\Omega^{k-m}(A), \Omega^{k-n}(B)).$$

Morphisms of  $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  are composed in the obvious fashion. We define the *loop* automorphism on  $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  by  $\Omega(A, m) := (A, m - 1)$ . There is a natural functor  $S : D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}) \rightarrow D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  defined by  $A \mapsto (A, 0)$ .

$D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  is an additive category [Garkusha 2007; 2013]. We define a triangulation  $\mathcal{T}r(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  of the pair  $(D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}), \Omega)$  as follows. A sequence

$$\Omega(A, l) \rightarrow (C, n) \rightarrow (B, m) \rightarrow (A, l)$$

belongs to  $\mathcal{T}r(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  if there is an even integer  $k$  and a left triangle of representatives  $\Omega(\Omega^{k-l}(A)) \rightarrow \Omega^{k-n}(C) \rightarrow \Omega^{k-m}(B) \rightarrow \Omega^{k-l}(A)$  in  $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ . Then the functor  $S$  takes left triangles in  $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  to triangles in  $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$ . By [Garkusha 2007; 2013]  $\mathcal{T}r(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  is a triangulation of  $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W})$  in the classical sense of [Verdier 1996].

By an  $\mathfrak{F}$ -*extension* or just *extension* in  $\mathfrak{R}$  we mean a short exact sequence of algebras

$$(E) : A \rightarrow B \xrightarrow{\alpha} C$$

such that  $\alpha \in \mathfrak{F}$ . Let  $\mathcal{E}$  be the class of all  $\mathfrak{F}$ -extensions in  $\mathfrak{R}$ .

**Definition.** Following [Cortiñas and Thom 2007] a  $(\mathfrak{F}$ -)excisive homology theory on  $\mathfrak{R}$  with values in a triangulated category  $(\mathcal{T}, \Omega)$  consists of a functor  $X : \mathfrak{R} \rightarrow \mathcal{T}$ , together with a collection  $\{\partial_E : E \in \mathcal{E}\}$  of maps  $\partial_E^X = \partial_E \in \mathcal{T}(\Omega X(C), X(A))$ . The maps  $\partial_E$  are to satisfy the following requirements.

- (1) For all  $E \in \mathcal{E}$  as above,

$$\Omega X(C) \xrightarrow{\partial_E} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

is a distinguished triangle in  $\mathcal{T}$ .

- (2) If

$$(E) : \begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ (E') : & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

is a map of extensions, then the following diagram commutes

$$\begin{array}{ccc}
 \Omega X(C) & \xrightarrow{\partial_E} & X(A) \\
 \Omega X(\gamma) \downarrow & & \downarrow X(\alpha) \\
 \Omega X(C') & \xrightarrow{\partial_{E'}} & X(A)
 \end{array}$$

We say that the functor  $X : \mathfrak{R} \rightarrow \mathcal{T}$  is *homotopy invariant* if it maps homotopic homomorphisms to equal maps, or equivalently, if for every  $A \in \text{Alg}_k$ ,  $X$  maps the inclusion  $A \subset A[t]$  to an isomorphism.

Denote by  $\mathfrak{W}_\Delta$  the class of homomorphisms  $f$  such that  $X(f)$  is an isomorphism for any excisive, homotopy invariant homology theory  $X : \mathfrak{R} \rightarrow \mathcal{T}$ . We shall refer to the maps from  $\mathfrak{W}_\Delta$  as *stable weak equivalences*. The triple  $(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}_\Delta)$  is a Brown category. In what follows we shall write  $D^-(\mathfrak{R}, \mathfrak{F})$  and  $D(\mathfrak{R}, \mathfrak{F})$  to denote  $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}_\Delta)$  and  $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}_\Delta)$  respectively, dropping  $\mathfrak{W}_\Delta$  from the notation.

By [Garkusha 2013] the canonical functor

$$\mathfrak{R} \rightarrow D(\mathfrak{R}, \mathfrak{F})$$

is the universal excisive, homotopy invariant homology theory on  $\mathfrak{R}$ .

### 3. Homotopy theory of algebras

Let  $\mathfrak{R}$  be a *small* admissible category of algebras. We shall work with various model category structures for the category of simplicial functors on  $\mathfrak{R}$ . We mostly adhere to [Garkusha 2007; 2014].

**3.1. The categories of pointed simplicial functors  $U_\bullet \mathfrak{R}$ .** Throughout this paper we work with a model category  $U_\bullet \mathfrak{R}$ . To define it, we first enrich  $\mathfrak{R}$  over pointed simplicial sets  $\mathbb{S}_\bullet$ . Given an algebra  $A \in \mathfrak{R}$ , denote by  $rA$  the representable functor  $\text{Hom}_{\mathfrak{R}}(A, -)$ . Let  $\mathfrak{R}_\bullet$  have the same objects as  $\mathfrak{R}$  and have pointed simplicial sets of morphisms being the  $rA(B) = \text{Hom}_{\mathfrak{R}}(A, B)$  pointed at zero. Denote by  $U_\bullet \mathfrak{R}$  the category of  $\mathbb{S}_\bullet$ -enriched functors from  $\mathfrak{R}_\bullet$  to  $\mathbb{S}_\bullet$ . One easily checks that  $U_\bullet \mathfrak{R}$  can be regarded as the category of covariant pointed simplicial functors  $X : \mathfrak{R} \rightarrow \mathbb{S}_\bullet$  such that  $X(0) = *$ .

By [Dundas et al. 2003, Theorem 4.2] we define the projective model structure on  $U_\bullet \mathfrak{R}$ . This is a proper, simplicial, cellular model category with weak equivalences and fibrations being defined object-wise, and cofibrations being those maps having the left lifting property with respect to trivial fibrations.

The class of projective cofibrations for  $U_\bullet \mathfrak{R}$  is generated by the set

$$I_{U_\bullet \mathfrak{R}} = \{rA \wedge (\partial \Delta^n \subset \Delta^n)_+\}^{n \geq 0},$$

indexed by  $A \in \mathfrak{R}$ . Likewise, the class of acyclic projective cofibrations is generated by

$$J_{U_\bullet \mathfrak{R}} = \{rA \wedge (\Delta_n^k \subset \Delta^n)_+ \}_{0 \leq k \leq n}^{n > 0}.$$

Given  $\mathcal{X}, \mathcal{Y} \in U_\bullet \mathfrak{R}$  the pointed function complex  $\text{Map}_\bullet(\mathcal{X}, \mathcal{Y})$  is defined as

$$\text{Map}_\bullet(\mathcal{X}, \mathcal{Y})_n = \text{Hom}_{U_\bullet \mathfrak{R}}(\mathcal{X} \wedge \Delta_+^n, \mathcal{Y}), \quad n \geq 0.$$

By [Dundas et al. 2003, Lemma 2.1] there is a natural isomorphism of pointed simplicial sets

$$\text{Map}_\bullet(rA, \mathcal{X}) \cong \mathcal{X}(A)$$

for all  $A \in \mathfrak{R}$  and  $\mathcal{X} \in U_\bullet \mathfrak{R}$ .

Recall that the model category  $U\mathfrak{R}$  of functors from  $\mathfrak{R}$  to unpointed simplicial sets  $\mathbb{S}$  is defined in a similar fashion (see [Garkusha 2007]). Since we mostly work with spectra in this paper, the category of spectra associated with  $U_\bullet \mathfrak{R}$  is technically more convenient than the category of spectra associated with  $U\mathfrak{R}$ .

**3.2. The model categories  $U_\bullet \mathfrak{R}_I, U_\bullet \mathfrak{R}_J, U_\bullet \mathfrak{R}_{I,J}$ .** Let

$$I = \{i = i_A : r(A[t]) \rightarrow r(A) \mid A \in \mathfrak{R}\},$$

where each  $i_A$  is induced by the natural homomorphism  $i : A \rightarrow A[t]$ . Recall that a functor  $F : \mathfrak{R} \rightarrow \mathbb{S}/\text{Spectra}$  is *homotopy invariant* if  $F(A) \rightarrow F(A[t])$  is a weak equivalence for all  $A \in \mathfrak{R}$ . Consider the projective model structure on  $U_\bullet \mathfrak{R}$ . We shall refer to the  $I$ -local equivalences as (projective)  $I$ -weak equivalences. Denote by  $U_\bullet \mathfrak{R}_I$  the model category obtained from  $U_\bullet \mathfrak{R}$  by Bousfield localisation with respect to the family  $I$ . Notice that any objectwise fibrant homotopy invariant functor  $F \in U_\bullet \mathfrak{R}$  is an  $I$ -local object, hence fibrant in  $U_\bullet \mathfrak{R}_I$ .

Let us introduce the class of excisive functors on  $\mathfrak{R}$ . They look like flasque presheaves on a site defined by a cd-structure in the sense of [Voevodsky 2010b, Section 3].

**Definition.** A simplicial functor  $\mathcal{X} \in U_\bullet \mathfrak{R}$  is called *excisive* with respect to  $\mathfrak{F}$  if for any cartesian square in  $\mathfrak{R}$

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & C \end{array}$$

with  $f$  a fibration (we call such squares *distinguished*), the square of simplicial sets

$$\begin{array}{ccc} \mathcal{X}(D) & \longrightarrow & \mathcal{X}(A) \\ \downarrow & & \downarrow \\ \mathcal{X}(B) & \longrightarrow & \mathcal{X}(C) \end{array}$$

is a homotopy pullback square. It immediately follows from the definition that every excisive object takes  $\mathfrak{F}$ -extensions in  $\mathfrak{A}$  to homotopy fibre sequences of simplicial sets.

Let  $\alpha$  denote a distinguished square in  $\mathfrak{A}$  as shown:

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

Let us apply the simplicial mapping cylinder construction  $\text{cyl}$  to  $\alpha$  and form the pushouts:

$$\begin{array}{ccccc} rC & \longrightarrow & \text{cyl}(rC \rightarrow rA) & \longrightarrow & rA \\ \downarrow & & \downarrow & & \downarrow \\ rB & \longrightarrow & \text{cyl}(rC \rightarrow rA) \sqcup_{rC} rB & \longrightarrow & rD \end{array}$$

Note that  $rC \rightarrow \text{cyl}(rC \rightarrow rA)$  is a projective cofibration between (projective) cofibrant objects of  $U_*\mathfrak{A}$ . Thus  $s(\alpha) = \text{cyl}(rC \rightarrow rA) \sqcup_{rC} rB$  is (projective) cofibrant [Hovey 1999, 1.1.11]. For the same reasons, applying the simplicial mapping cylinder to  $s(\alpha) \rightarrow rD$  and setting  $t(\alpha) = \text{cyl}(s(\alpha) \rightarrow rD)$  we get a projective cofibration

$$\text{cyl}(\alpha): s(\alpha) \longrightarrow t(\alpha).$$

Let  $J_{U_*\mathfrak{A}}^{\text{cyl}(\alpha)}$  consists of all pushout product maps

$$s(\alpha) \wedge \Delta_+^n \sqcup_{s(\alpha) \wedge \partial \Delta_+^n} t(\alpha) \wedge \partial \Delta_+^n \longrightarrow t(\alpha) \wedge \Delta_+^n,$$

and let  $J = J_{U_*\mathfrak{A}} \cup J_{U_*\mathfrak{A}}^{\text{cyl}(\alpha)}$ . Denote by  $U_*\mathfrak{A}_J$  the model category obtained from  $U_*\mathfrak{A}$  by Bousfield localisation with respect to the family  $J$ . It is directly verified that  $\mathcal{X} \in U_*\mathfrak{A}$  is  $J$ -local if and only if it has the right lifting property with respect to  $J$ . Also,  $\mathcal{X}$  is  $J$ -local if and only if it is objectwise fibrant and excisive [Garkusha 2007, Lemma 4.3].

Finally, let us introduce the model category  $U_*\mathfrak{A}_{I,J}$ . It is, by definition, the Bousfield localisation of  $U_*\mathfrak{A}$  with respect to  $I \cup J$ . The weak equivalences (trivial cofibrations) of  $U_*\mathfrak{A}_{I,J}$  will be referred to as (projective)  $(I, J)$ -weak equivalences

((projective)  $(I, J)$ -trivial cofibrations). By [Garkusha 2007, Lemma 4.5] a functor  $\mathcal{X} \in U_*\mathfrak{R}$  is  $(I, J)$ -local if and only if it is objectwise fibrant, homotopy invariant and excisive.

**Remark.** The model category  $U_*\mathfrak{R}_{I,J}$  can also be regarded as a kind of unstable motivic model category associated with  $\mathfrak{R}$ . Indeed, the construction of  $U_*\mathfrak{R}_{I,J}$  is similar to Morel–Voevodsky’s unstable motivic theory for smooth schemes  $\text{Sm}/F$  over a field  $F$  [Morel and Voevodsky 1999]. If we replace  $I$  by

$$I' = \{X \times \mathbb{A}^1 \xrightarrow{pr} X \mid X \in \text{Sm}/F\},$$

and the family of distinguished squares by the family of elementary Nisnevich squares and get the corresponding family  $J'$  associated to it, then one of the equivalent models for Morel–Voevodsky’s unstable motivic theory is obtained by Bousfield localisation of simplicial presheaves with respect to  $I' \cup J'$ .

For this reason,  $U_*\mathfrak{R}_{I,J}$  can also be called the category of (pointed) *motivic spaces*, where each algebra  $A$  is identified with the pointed motivic space  $rA$ . One can also refer to  $(I, J)$ -weak equivalences as *motivic weak equivalences*.

**3.3. Monoidal structure on  $U_*\mathfrak{R}$ .** In this section we mostly follow [Østvær 2010, Section 2.1]. Suppose  $\mathfrak{R}$  is tensor closed, that is  $k \in \mathfrak{R}$  and  $A \otimes B \in \mathfrak{R}$  for all  $A, B \in \mathfrak{R}$ . We introduce the monoidal product  $\mathcal{X} \otimes \mathcal{Y}$  of  $\mathcal{X}$  and  $\mathcal{Y}$  in  $U_*\mathfrak{R}$  by the formulas

$$\mathcal{X} \otimes \mathcal{Y}(A) = \text{colim}_{A_1 \otimes A_2 \rightarrow A} \mathcal{X}(A_1) \wedge \mathcal{Y}(A_2).$$

The colimit is indexed on the category with objects  $\alpha: A_1 \otimes A_2 \rightarrow A$  and maps the pairs of maps  $(\varphi, \psi): (A_1, A_2) \rightarrow (A'_1, A'_2)$  such that  $\alpha'(\psi \otimes \varphi) = \alpha$ . By functoriality of colimits it follows that  $\mathcal{X} \otimes \mathcal{Y}$  is in  $U_*\mathfrak{R}$ .

The tensor product can also be defined by the formula

$$\mathcal{X} \otimes \mathcal{Y}(A) = \int^{A_1, A_2 \in \mathfrak{R}} (\mathcal{X}(A_1) \wedge \mathcal{Y}(A_2)) \wedge \text{Hom}_{\mathfrak{R}}(A_1 \otimes A_2, A).$$

This formula is obtained from a theorem of Day [1970], which also asserts that the triple  $(U_*\mathfrak{R}, \otimes, r(k))$  forms a closed symmetric monoidal category.

The internal Hom functor, right adjoint to  $\mathcal{X} \otimes -$ , is given by

$$\underline{\text{Hom}}(\mathcal{X}, \mathcal{Y})(A) = \int_{B \in \mathfrak{R}} \text{Map}_*(\mathcal{X}(B), \mathcal{Y}(A \otimes B)),$$

where  $\text{Map}_*$  stands for the function complex in  $\mathbb{S}_*$ .

So there exist natural isomorphisms

$$\underline{\text{Hom}}(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z}) \cong \underline{\text{Hom}}(\mathcal{X}, \underline{\text{Hom}}(\mathcal{Y}, \mathcal{Z}))$$



and

$$\underline{\text{Hom}}(r(k), \mathcal{X}) \cong \mathcal{X}.$$

Concerning smash products of representable functors, one has a natural isomorphism

$$rA \otimes rB \cong r(A \otimes B), \quad A, B \in \mathfrak{R}.$$

Note as well that, for pointed simplicial sets  $K$  and  $L$ , one has  $K \otimes L = K \wedge L$ .

We recall a pointed simplicial set tensor and cotensor structure on  $U_{\bullet}\mathfrak{R}$ . If  $\mathcal{X}$  and  $\mathcal{Y}$  are in  $U_{\bullet}\mathfrak{R}$  and  $K$  is a pointed simplicial set, the tensor  $\mathcal{X} \otimes K$  is given by

$$\mathcal{X} \otimes K(A) = \mathcal{X}(A) \wedge K$$

and the cotensor  $\mathcal{Y}^K$  is given in terms of the ordinary function complex:

$$\mathcal{Y}^K(A) = \text{Map}_{\bullet}(K, \mathcal{Y}(A)).$$

The function complex  $\text{Map}_{\bullet}(\mathcal{X}, \mathcal{Y})$  of  $\mathcal{X}$  and  $\mathcal{Y}$  is defined by setting

$$\text{Map}_{\bullet}(\mathcal{X}, \mathcal{Y})_n = \text{Hom}_{U_{\bullet}\mathfrak{R}}(\mathcal{X} \otimes \Delta_+^n, \mathcal{Y}).$$

By the Yoneda lemma there exists a natural isomorphism of pointed simplicial sets

$$\text{Map}_{\bullet}(rA, \mathcal{Y}) \cong \mathcal{Y}(A).$$

Using these definitions  $U_{\bullet}\mathfrak{R}$  is enriched in pointed simplicial sets  $\mathbb{S}_{\bullet}$ . Moreover, there are natural isomorphisms of pointed simplicial sets

$$\text{Map}_{\bullet}(\mathcal{X} \otimes K, \mathcal{Y}) \cong \text{Map}_{\bullet}(K, \text{Map}_{\bullet}(\mathcal{X}, \mathcal{Y})) \cong \text{Map}_{\bullet}(\mathcal{X}, \mathcal{Y}^K).$$

It is also useful to note that

$$\underline{\text{Hom}}(\mathcal{X}, \mathcal{Y})(A) = \text{Map}_{\bullet}(\mathcal{X}, \mathcal{Y}(A \otimes -)) \quad \text{and} \quad \underline{\text{Hom}}(rB, \mathcal{Y}) = \mathcal{Y}(- \otimes B).$$

It can be shown similarly to [Østvær 2010, Lemma 3.10; Propositions 3.43 and 3.89] that the model categories  $U_{\bullet}\mathfrak{R}$ ,  $U_{\bullet}\mathfrak{R}_I$ ,  $U_{\bullet}\mathfrak{R}_J$ ,  $U_{\bullet}\mathfrak{R}_{I,J}$  are monoidal.

#### 4. Unstable algebraic Kasparov K-theory

Let  $\mathcal{U}$  be an arbitrary category and let  $\mathfrak{R}$  be an admissible category of  $k$ -algebras. Suppose that there are functors  $F : \mathfrak{R} \rightarrow \mathcal{U}$  and  $\tilde{T} : \mathcal{U} \rightarrow \mathfrak{R}$  such that  $\tilde{T}$  is left adjoint to  $F$ . We denote  $\tilde{T}FA$ , for  $A \in \mathfrak{R}$ , by  $TA$  and the counit map  $\tilde{T}FA \rightarrow A$  by  $\eta_A$ . If  $X \in \text{Ob } \mathcal{U}$  then the unit map  $X \rightarrow F\tilde{T}X$  is denoted by  $i_X$ . We note that the composition

$$FA \xrightarrow{i_{FA}} F\tilde{T}FA \xrightarrow{F\eta_A} FA$$

equals  $1_{FA}$  for every  $A \in \mathfrak{R}$ , and hence  $F\eta_A$  splits in  $\mathcal{U}$ . We call an admissible category of  $k$ -algebras  $T$ -closed if  $TA \in \mathfrak{R}$  for all  $A \in \mathfrak{R}$ .

**Lemma 4.1.** *Suppose  $\mathcal{U}$  is either a full subcategory of the category of sets or a full subcategory of the category of  $k$ -modules. Suppose as well that  $F : \mathfrak{R} \rightarrow \mathcal{U}$  is the forgetful functor. Then for every  $A \in \mathfrak{R}$  the algebra  $TA$  is contractible, i.e., there is a contraction  $\tau : TA \rightarrow TA[x]$  such that  $\partial_x^0 \tau = 0, \partial_x^1 \tau = 1$ . Moreover, the contraction is functorial in  $A$ .*

*Proof.* Consider a map  $u : FTA \rightarrow FTA[x]$  sending an element  $b \in FTA$  to  $bx \in FTA[x]$ . By assumption,  $u$  is a morphism of  $\mathcal{U}$ . The desired contraction  $\tau$  is uniquely determined by the map  $u \circ i_{FA} : FA \rightarrow FTA[x]$ . By using elementary properties of adjoint functors, one can show that  $\partial_x^0 \tau = 0$  and  $\partial_x^1 \tau = 1$ .  $\square$

Throughout this paper, whenever we deal with a  $T$ -closed admissible category of  $k$ -algebras  $\mathfrak{R}$  we assume to be fixed an underlying category  $\mathcal{U}$ , which is a full subcategory of  $\text{Mod } k$ .

**Examples.** (1) Let  $\mathfrak{R} = \text{Alg}_k$ . Given an algebra  $A$ , consider the algebraic tensor algebra

$$TA = A \oplus A \otimes A \oplus A^{\otimes 3} \oplus \dots,$$

with the usual product given by concatenation of tensors. In Cuntz’s treatment of bivariant  $K$ -theory [Cuntz 1997; 2005; Cuntz and Thom 2006], tensor algebras play a prominent role.

There is a canonical  $k$ -linear map  $A \rightarrow TA$  mapping  $A$  into the first direct summand. Every  $k$ -linear map  $s : A \rightarrow B$  into an algebra  $B$  induces a homomorphism  $\gamma_s : TA \rightarrow B$  defined by

$$\gamma_s(x_1 \otimes \dots \otimes x_n) = s(x_1)s(x_2) \dots s(x_n).$$

Plainly  $\mathfrak{R}$  is  $T$ -closed.

(2) If  $\mathfrak{R} = \text{CAlg}_k$ , then

$$T(A) = \text{Sym}(A) = \bigoplus_{n \geq 1} S^n A,$$

the symmetric algebra of  $A$ , and  $\mathfrak{R}$  is  $T$ -closed. Here

$$S^n A = A^{\otimes n} / \langle a_1 \otimes \dots \otimes a_n - a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(n)} \rangle \quad \text{for } \sigma \in \Sigma_n.$$

We have a natural extension of algebras

$$0 \longrightarrow JA \xrightarrow{\iota_A} TA \xrightarrow{\eta_A} A \longrightarrow 0.$$

Here  $JA$  is defined as  $\text{Ker } \eta_A$ . Clearly,  $JA$  is functorial in  $A$ .

Given a small  $T$ -closed admissible category of  $k$ -algebras  $\mathfrak{R}$ , we denote by  $\text{Sp}(\mathfrak{R})$  the category of  $S^1$ -spectra in the sense of [Hovey 2001] associated with the model category  $U.\mathfrak{R}_{I,J}$ . Recall that a spectrum consists of sequences  $\mathcal{E} = (\mathcal{E}_n)_{n \geq 0}$  of

pointed simplicial functors in  $U_*\mathfrak{R}$  equipped with structure maps  $\sigma_n^{\mathcal{E}} : \Sigma \mathcal{E}_n \rightarrow \mathcal{E}_{n+1}$ , where  $\Sigma = - \wedge S^1$  is the suspension functor. A map  $f : \mathcal{E} \rightarrow \mathcal{F}$  of spectra consists of compatible maps  $f_n : \mathcal{E}_n \rightarrow \mathcal{F}_n$  in the sense that the diagrams

$$\begin{array}{ccc} \Sigma \mathcal{E}_n & \xrightarrow{\sigma_n^{\mathcal{E}}} & \mathcal{E}_{n+1} \\ \Sigma f_n \downarrow & & \downarrow f_{n+1} \\ \Sigma \mathcal{F}_n & \xrightarrow{\sigma_n^{\mathcal{F}}} & \mathcal{F}_{n+1} \end{array}$$

commute for all  $n \geq 0$ . The category  $\text{Sp}(\mathfrak{R})$  is endowed with the stable model structure (see [Hovey 2001] for details).

Given an algebra  $A \in \mathfrak{R}$ , we denote by  $\Sigma^\infty rA$  the suspension spectrum associated with the functor  $rA$  pointed at zero. By definition,  $(\Sigma^\infty rA)_n = rA \wedge S^n$  with obvious structure maps.

In order to define one of the main spectra of the paper  $\mathcal{R}(A)$  associated to an algebra  $A \in \mathfrak{R}$ , we have to recall some definitions from [Garkusha 2014].

For any  $B \in \mathfrak{R}$  we define a simplicial algebra

$$B^\Delta : [n] \mapsto B^{\Delta^n} := B[t_0, \dots, t_n] / \left\langle 1 - \sum_i t_i \right\rangle \quad (\cong B[t_1, \dots, t_n]).$$

Given a map of posets  $\alpha : [m] \rightarrow [n]$ , the map  $\alpha^* : B^{\Delta^n} \rightarrow B^{\Delta^m}$  is defined by  $\alpha^*(t_j) = \sum_{\alpha(i)=j} t_i$ . We have that  $B^\Delta \cong B \otimes k^\Delta$  and  $B^\Delta$  is pointed at zero.

For any pointed simplicial set  $X \in \mathbb{S}_*$ , we denote by  $B^\Delta(X)$  the simplicial algebra  $\text{Map}_*(X, B^\Delta)$ . The simplicial algebra associated to any unpointed simplicial set and any simplicial algebra is defined in a similar way. By  $\mathbb{B}^\Delta(X)$  we shall mean the pointed simplicial ind-algebra

$$B^\Delta(X) \rightarrow B^\Delta(\text{sd}^1 X) \rightarrow B^\Delta(\text{sd}^2 X) \rightarrow \dots$$

In particular, one defines the ‘‘path space’’ simplicial ind-algebra  $P\mathbb{B}^\Delta$ . We shall also write  $\mathbb{B}^\Delta(\Omega^n)$  to denote  $\mathbb{B}^\Delta(S^n)$ , where  $S^n = S^1 \wedge \dots \wedge S^1$  is the simplicial  $n$ -sphere. For any  $A \in \mathfrak{R}$  we denote by  $\text{Hom}_{\text{Alg}_k^{\text{ind}}}(A, \mathbb{B}^\Delta(\Omega^n))$  the colimit of the sequence in  $\mathbb{S}_*$

$$\text{Hom}_{\text{Alg}_k}(A, B^\Delta(S^n)) \rightarrow \text{Hom}_{\text{Alg}_k}(A, B^\Delta(\text{sd}^1 S^n)) \rightarrow \text{Hom}_{\text{Alg}_k}(A, B^\Delta(\text{sd}^2 S^n)) \rightarrow \dots$$

The natural simplicial map  $d_1 : P\mathbb{B}^\Delta(\Omega^n) \rightarrow \mathbb{B}^\Delta(\Omega^n)$  has a natural  $k$ -linear splitting described below. Let  $\mathbf{t} \in Pk^\Delta(\Delta^1 \times \dots \times \Delta^1)_0$  stand for the composite map

$$\text{sd}^m(\Delta^1 \times \dots \times \Delta^1) \xrightarrow{pr} \text{sd}^m \Delta^1 \rightarrow \Delta^1 \xrightarrow{t} k^\Delta,$$

where  $pr$  is the projection onto the  $(n + 1)$ -th direct factor  $\Delta^1$  and  $t = t_0 \in k^{\Delta^1}$ . The element  $\mathbf{t}$  can be regarded as a 1-simplex of the unital ind-algebra

$$\mathbb{k}^\Delta(\Delta^1 \times \dots \times \Delta^1)$$

such that  $\partial_0(\mathbf{t}) = 0$  and  $\partial_1(\mathbf{t}) = 1$ . Let  $\iota : \mathbb{B}^\Delta(\Omega^n) \rightarrow (\mathbb{B}^\Delta(\Omega^n))^{\Delta^1}$  be the natural inclusion. Multiplication with  $\mathbf{t}$  determines a  $k$ -linear map

$$(\mathbb{B}^\Delta(\Omega^n))^{\Delta^1} \xrightarrow{\mathbf{t}} P\mathbb{B}^\Delta(\Omega^n).$$

Now the desired  $k$ -linear splitting  $\mathbb{B}^\Delta(\Omega^n) \xrightarrow{v} P\mathbb{B}^\Delta(\Omega^n)$  of simplicial ind-modules is defined as

$$v := \mathbf{t} \cdot \iota.$$

If we consider  $\mathbb{B}^\Delta(\Omega^n)$  as a  $(\mathbb{Z}_{\geq 0} \times \Delta)$ -diagram in  $\mathfrak{R}$ , then there is a commutative diagram of extensions for  $(\mathbb{Z}_{\geq 0} \times \Delta)$ -diagrams

$$\begin{array}{ccccc} J\mathbb{B}^\Delta(\Omega^n) & \longrightarrow & T\mathbb{B}^\Delta(\Omega^n) & \longrightarrow & \mathbb{B}^\Delta(\Omega^n) \\ \xi_v \downarrow & & \downarrow & & \parallel \\ \mathbb{B}^\Delta(\Omega^{n+1}) & \longrightarrow & P\mathbb{B}^\Delta(\Omega^n) & \xrightarrow{d_1} & \mathbb{B}^\Delta(\Omega^n) \end{array}$$

where the map  $\xi_v$  is uniquely determined by the  $k$ -linear splitting  $v$ . For every element  $f \in \text{Hom}_{\text{Alg}_k^{\text{ind}}}(J^n A, \mathbb{B}^\Delta(\Omega^n))$  one sets:

$$\zeta(f) := \xi_v \circ J(f) \in \text{Hom}_{\text{Alg}_k^{\text{ind}}}(J^{n+1} A, \mathbb{B}^\Delta(\Omega^{n+1})).$$

The spectrum  $\mathcal{R}(A)$  is defined at every  $B \in \mathfrak{R}$  as the sequence of spaces pointed at zero

$$\text{Hom}_{\text{Alg}_k^{\text{ind}}}(A, \mathbb{B}^\Delta), \text{Hom}_{\text{Alg}_k^{\text{ind}}}(JA, \mathbb{B}^\Delta), \text{Hom}_{\text{Alg}_k^{\text{ind}}}(J^2 A, \mathbb{B}^\Delta), \dots$$

By [Garkusha 2014, Section 2] each  $\mathcal{R}(A)_n(B)$  is a fibrant simplicial set and

$$\Omega^k \mathcal{R}(A)_0(B) = \text{Hom}_{\text{Alg}_k^{\text{ind}}}(A, \mathbb{B}^\Delta(\Omega^k)).$$

Each structure map  $\sigma_n : \mathcal{R}(A)_n \wedge S^1 \rightarrow \mathcal{R}(A)_{n+1}$  is defined at  $B$  as adjoint to the map  $\zeta : \text{Hom}_{\text{Alg}_k^{\text{ind}}}(J^n A, \mathbb{B}^\Delta) \rightarrow \text{Hom}_{\text{Alg}_k^{\text{ind}}}(J^{n+1} A, \mathbb{B}^\Delta(\Omega))$ .

For every  $A \in \mathfrak{R}$  there is a natural map in  $\text{Sp}(\mathfrak{R})$

$$i : \Sigma^\infty rA \rightarrow \mathcal{R}(A)$$

functorial in  $A$ .

**Definition** [Garkusha 2014]. (1) Given two  $k$ -algebras  $A, B \in \mathfrak{R}$ , the *unstable algebraic Kasparov K-theory space*  $\mathcal{K}(A, B)$  is the fibrant space

$$\text{colim}_n \text{Hom}_{\text{Alg}_k^{\text{ind}}}(J^n A, \mathbb{B}^\Delta(\Omega^n)),$$

where the colimit maps are given by  $\xi_\nu$ -s and  $JA$  is as defined on page 290. Its homotopy groups will be denoted by  $\mathcal{K}_n(A, B)$ ,  $n \geq 0$ . The simplicial functor  $\mathcal{K}(A, -)$  is fibrant in  $U_*(\mathfrak{R})_{I,J}$  by [Garkusha 2014, Section 4]. Also, there is a natural isomorphism of simplicial sets

$$\mathcal{K}(A, B) \cong \Omega \mathcal{K}(JA, B).$$

In particular,  $\mathcal{K}(A, B)$  is an infinite loop space with  $\mathcal{K}(A, B)$  which simplicially isomorphic to  $\Omega^n \mathcal{K}(J^n A, B)$  (see [Garkusha 2014, Theorem 5.1]).

(2) The *unstable algebraic Kasparov KK-theory spectrum* of  $(A, B)$  consists of the sequence of spaces

$$\mathcal{K}(A, B), \mathcal{K}(JA, B), \mathcal{K}(J^2 A, B), \dots,$$

together with the natural isomorphisms  $\mathcal{K}(J^n A, B) \cong \Omega \mathcal{K}(J^{n+1} A, B)$ . It forms an  $\Omega$ -spectrum which we also denote by  $\mathbb{K}(A, B)$ . Its homotopy groups will be denoted by  $\mathbb{K}_n(A, B)$ ,  $n \in \mathbb{Z}$ . Observe that  $\mathbb{K}_n(A, B) \cong \mathcal{K}_n(A, B)$  for any  $n \geq 0$  and  $\mathbb{K}_n(A, B) \cong \mathcal{K}_0(J^{-n} A, B)$  for any  $n < 0$ .

There is a natural map of spectra

$$j : \mathcal{R}(A) \rightarrow \mathbb{K}(A, -).$$

By [Garkusha 2014, Section 6] this is a stable equivalence and  $\mathbb{K}(A, -)$  is a fibrant object of  $\text{Sp}(\mathfrak{R})$ . In fact, for any algebra  $B \in \mathfrak{R}$  the map

$$j : \mathcal{R}(A)(B) \rightarrow \mathbb{K}(A, B)$$

is a stable equivalence of ordinary spectra.

The following theorem is crucial in our analysis. It states that  $\mathbb{K}(A, -)$  is a fibrant replacement of  $\Sigma^\infty rA$  in  $\text{Sp}(\mathfrak{R})$ .

**Theorem 4.2.** *Given  $A \in \mathfrak{R}$  the map  $i : \Sigma^\infty rA \rightarrow \mathcal{R}(A)$  is a level  $(I, J)$ -weak equivalence, and therefore the composite map*

$$\Sigma^\infty rA \xrightarrow{i} \mathcal{R}(A) \xrightarrow{j} \mathbb{K}(A, -)$$

*is a stable equivalence in  $\text{Sp}(\mathfrak{R})$ , functorial in  $A$ .*

*Proof.* Recall that for any functor  $F$  from rings to simplicial sets,  $\text{Sing}(F)$  is defined at each ring  $R$  as the diagonal of the bisimplicial set  $F(R[\Delta])$ . The map

$$i_0 : (\Sigma^\infty rA)_0 \rightarrow \mathcal{R}(A)_0$$

equals  $rA \rightarrow \text{Ex}^\infty \circ \text{Sing}(rA)$ , which is an  $I$ -weak equivalence by [Garkusha 2007, Corollary 3.8]. Let us show that

$$i_1 : rA \wedge S^1 \rightarrow \mathcal{R}(A)_1 = \text{Ex}^\infty \circ \text{Sing}(r(JA))$$

is an  $(I, J)$ -weak equivalence. It is fully determined by the element  $\rho_A : JA \rightarrow \Omega A$ , which is a zero simplex of  $\Omega(\text{Ex}^\infty \circ \text{Sing}(r(JA)))(A)$ , coming from the adjunction isomorphism

$$\text{Map}_\bullet(rA \wedge S^1, \text{Ex}^\infty \circ \text{Sing}(r(JA))) \cong \Omega(\text{Ex}^\infty \circ \text{Sing}(r(JA)))(A).$$

Let  $(I, 0)$  denote  $\Delta[1]$  pointed at 0. Consider a commutative diagram of cofibrant objects in  $U_\bullet \mathfrak{N}$

$$\begin{array}{ccccc} rA & \xrightarrow{v} & rA \wedge (I, 0) & \twoheadrightarrow & rA \wedge S^1 \\ \eta_A^* \downarrow & & \downarrow & & \parallel \\ r(TA) & \twoheadrightarrow & \mathcal{X} & \xrightarrow{\alpha} & rA \wedge S^1 \end{array}$$

where the left square is pushout, the left map is induced by the canonical homomorphism  $\eta_A : TA \rightarrow A$  and  $v$  is induced by the natural inclusion  $d^0 : \Delta[0] \rightarrow \Delta[1]$ . Lemma 4.1 implies  $r(TA)$  is weakly equivalent to zero in  $U_\bullet \mathfrak{N}_I$ . It follows that  $\alpha$  is an  $I$ -weak equivalence.

By the universal property of pullback diagrams there is a unique morphism  $\sigma : \mathcal{X} \rightarrow r(JA)$  whose restriction to  $r(TA)$  equals  $\iota_A^*$ , where  $\iota_A = \text{Ker } \eta_A$ , which makes the diagram

$$\begin{array}{ccccc} rA \wedge (I, 0) & \longrightarrow & \mathcal{X} & & \\ \nearrow & \downarrow & \nearrow & & \downarrow \sigma \\ rA & \longrightarrow & r(TA) & & \\ \downarrow & \downarrow & \downarrow & & \\ 1 & \nearrow & pt & \longrightarrow & r(JA) \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ rA & \longrightarrow & r(TA) & & \\ & & \eta_A^* & & \end{array}$$

commutative. Since the upper and the lower squares are homotopy pushouts in  $U_\bullet \mathfrak{N}_{I,J}$  and  $rA \wedge (I, 0)$  is weakly equivalent to zero, it follows from [Hirschhorn 2003, Proposition 13.5.10] that  $\sigma$  is an  $(I, J)$ -weak equivalence. Therefore the composite map, we shall denote it by  $\rho$ ,

$$\mathcal{X} \xrightarrow{\sigma} r(JA) \rightarrow \mathcal{R}(A)_1$$

is an  $(I, J)$ -weak equivalence, where the right map is the natural  $I$ -weak equivalence.

Let  $\mathcal{R}(A)_1[x] \in U_\bullet \mathfrak{N}$  be a simplicial functor defined as

$$\mathcal{R}(A)_1[x](B) = \text{Hom}_{\text{Alg}_k^{\text{ind}}}(JA, \mathbb{B}^\Delta[x]) = \text{Ex}^\infty \circ \text{Hom}_{\text{Alg}_k}(JA, B[x]^\Delta), \quad B \in \mathfrak{N}.$$

There is a natural map  $s : \mathcal{R}(A)_1 \rightarrow \mathcal{R}(A)_1[x]$ , induced by the monomorphism  $B \rightarrow B[x]$  at each  $B$ . It follows from [Garkusha 2007, Proposition 3.2] that this

map is a weak equivalence in  $U.\mathfrak{R}$ . The evaluation homomorphisms

$$\partial_x^0, \partial_x^1 : B[x] \rightarrow B$$

induce a map  $(\partial_x^0, \partial_x^1) : \mathcal{R}(A)_1[x] \rightarrow \mathcal{R}(A)_1 \times \mathcal{R}(A)_1$ , whose composition with  $s$  is the diagonal map  $\mathcal{R}(A)_1 \rightarrow \mathcal{R}(A)_1 \times \mathcal{R}(A)_1$ . We see that  $\mathcal{R}(A)_1[x]$  is a path object for the projectively fibrant object  $\mathcal{R}(A)_1$ .

If we constructed a homotopy  $H : \mathcal{X} \rightarrow \mathcal{R}(A)_1[x]$  such that  $\partial_x^0 H = i_1 \alpha$  and  $\partial_x^1 H = \rho$  it would follow that  $i_1 \alpha$ , being homotopic to the  $(I, J)$ -weak equivalence  $\rho$ , is an  $(I, J)$ -weak equivalence. Since also  $\alpha$  is an  $(I, J)$ -weak equivalence, then so would be  $i_1$ .

The desired map  $H$  is uniquely determined by maps  $h_1 : r(TA) \rightarrow \mathcal{R}(A)_1[x]$  and  $h_2 : rA \wedge (I, 0) \rightarrow \mathcal{R}(A)_1[x]$  such that  $h_1 \eta_A^* = h_2 v$  is defined as follows. The map  $h_1$  is uniquely determined by the homomorphism  $JA \rightarrow TA[x]$  which is the composition of  $\iota_A$  and the contraction homomorphism  $\tau : TA \rightarrow TA[x]$ , functorial in  $A$ , that exists by Lemma 4.1. The map  $h_2$  is uniquely determined by the one-simplex  $JA \rightarrow A[\Delta^1][x]$  of  $\text{Ex}^\infty \circ \text{Hom}_{\text{Alg}_k}(JA, A[x]^\Delta)$  which is the composition of

$$\rho_A : JA \rightarrow \Omega A = (t^2 - t)A[t] \subset A[\Delta^1]$$

and the homomorphism  $\omega : A[\Delta^1] \rightarrow A[\Delta^1][x]$  sending the variable  $t$  to

$$1 - (1 - t)(1 - x).$$

Thus we have shown that

$$i_1 : rA \wedge S^1 \rightarrow \mathcal{R}(A)_1$$

is an  $(I, J)$ -weak equivalence. It follows that the composite map

$$rA \wedge S^1 \xrightarrow{i_0 \wedge S^1} \mathcal{R}(A)_0 \wedge S^1 \xrightarrow{\sigma_0} \mathcal{R}(A)_1,$$

which is equal to  $i_1$ , is an  $(I, J)$ -weak equivalence. Hence  $\sigma_0$  is an  $(I, J)$ -weak equivalence, because  $i_0 \wedge S^1$  is an  $I$ -weak equivalence. More generally, one gets that every structure map

$$\mathcal{R}(A)_n \wedge S^1 \xrightarrow{\sigma_n} \mathcal{R}(A)_{n+1}$$

is an  $(I, J)$ -weak equivalence.

By induction, assume that  $i_n : rA \wedge S^n \rightarrow \mathcal{R}(A)_n$  is an  $(I, J)$ -weak equivalence. Then  $i_n \wedge S^1$  is an  $(I, J)$ -weak equivalence, and hence so is  $i_{n+1} = \sigma_n \circ (i_n \wedge S^1)$ .  $\square$

Denote by  $\text{SH}_{S^1}(\mathfrak{R})$  the stable homotopy category of  $\text{Sp}(\mathfrak{R})$ . Since the endofunctor  $-\wedge S^1$  is an equivalence on  $\text{SH}_{S^1}(\mathfrak{R})$  by [Hovey 2001], it follows from [Hovey 1999, Chapter 7] that  $\text{SH}_{S^1}(\mathfrak{R})$  is a triangulated category. Moreover, it is compactly generated with compact generators  $\{(\Sigma^\infty rA)[n]\}_{A \in \mathfrak{R}, n \in \mathbb{Z}}$ .

**Corollary 4.3.**  $\{\Sigma^\infty r A[n]\}_{A \in \mathfrak{A}, n \in \mathbb{Z}}$  is a family of compact generators for  $\mathrm{SH}_{S^1}(\mathfrak{A})$ . Moreover, there is a natural isomorphism

$$\mathrm{SH}_{S^1}(\mathfrak{A})(\Sigma^\infty r B[n], \Sigma^\infty r A) \cong \mathbb{K}_n(A, B)$$

for all  $A, B \in \mathfrak{A}$  and  $n \in \mathbb{Z}$ .

Denote by  $\mathcal{S}$  the full subcategory of  $\mathrm{SH}_{S^1}(\mathfrak{A})$  whose objects are

$$\{\Sigma^\infty r A[n]\}_{A \in \mathfrak{A}, n \in \mathbb{Z}}.$$

The next statement gives another description of the triangulated category  $D(\mathfrak{A}, \mathfrak{F})$ .

**Theorem 4.4.** *The category  $\mathcal{S}$  is triangulated. Moreover, there is a contravariant equivalence of triangulated categories*

$$T : D(\mathfrak{A}, \mathfrak{F}) \rightarrow \mathcal{S}.$$

*Proof.* By [Garkusha 2013] the natural functor

$$j : \mathfrak{A} \rightarrow D(\mathfrak{A}, \mathfrak{F})$$

is a universal excisive homotopy invariant homology theory. Consider the homology theory

$$t : \mathfrak{A} \rightarrow \mathrm{SH}_{S^1}(\mathfrak{A})^{\mathrm{op}}$$

that takes an algebra  $A \in \mathfrak{A}$  to  $\Sigma^\infty r A$ . It is homotopy invariant and excisive, hence there is a unique triangulated functor

$$T : D(\mathfrak{A}, \mathfrak{F}) \rightarrow \mathrm{SH}_{S^1}(\mathfrak{A})^{\mathrm{op}},$$

such that  $t = T \circ j$ . If we apply  $T$  to the loop extension

$$\Omega A \rightarrow EA \rightarrow A,$$

we get an isomorphism

$$T(\Omega A) \cong \Sigma^\infty r A[1],$$

which is functorial in  $A$ .

It follows from Comparison Theorem B of [Garkusha 2014] and Corollary 4.3 that  $T$  is full and faithful. Every object of  $\mathcal{S}$  is plainly equivalent to the image of an object in  $D(\mathfrak{A}, \mathfrak{F})$ . □

**Remark.** Suppose  $I$  is an infinite index set and  $\{B_i\}_{i \in I}$  is a family of algebras from  $\mathfrak{A}$  such that the algebra  $B = \bigoplus_I B_i$  is in  $\mathfrak{A}$ . Then  $\Sigma^\infty r B$  is a compact object of  $\mathrm{SH}_{S^1}(\mathfrak{A})$ , but  $\bigoplus_I \Sigma^\infty r(B_i)$  may not be compact. Furthermore, suppose  $B = \bigoplus_I B_i$  is also a direct sum object of the  $B_i$ -s in the triangulated category  $D(\mathfrak{A}, \mathfrak{F})$ . Then  $\mathrm{Hom}_{D(\mathfrak{A}, \mathfrak{F})}(B, \bigoplus_I C_i) \neq \bigoplus_I \mathrm{Hom}_{D(\mathfrak{A}, \mathfrak{F})}(B, C_i)$  in general, where  $\{C_i\}_{i \in I}$  is a family of algebras from  $\mathfrak{A}$  such that the algebra  $\bigoplus_I C_i$  is in  $\mathfrak{A}$ .



For instance, consider the triangulated category  $KK$  of [Kasparov 1980], with which  $D(\mathfrak{R}, \mathfrak{F})$  shares many properties. It follows from [Rosenberg and Schochet 1987, Theorem 1.12] that  $KK$  has countable coproducts given by  $A = \bigoplus_I A_i$ , where  $I$  is a countable set. However, the functor  $KK(A, -)$  does not respect countable coproducts by [Rosenberg and Schochet 1987, Remark 7.12].

Recall from [Garkusha 2014] that we can vary  $\mathfrak{R}$  in the following sense. If  $\mathfrak{R}'$  is another  $T$ -closed admissible category of algebras containing  $\mathfrak{R}$ , then  $D(\mathfrak{R}, \mathfrak{F})$  is a full subcategory of  $D(\mathfrak{R}', \mathfrak{F})$ .

### 5. Morita stable algebraic Kasparov $K$ -theory

If  $A$  is an algebra and  $n > 0$  is a positive integer, then there is a natural inclusion  $\iota : A \rightarrow M_n A$  of algebras, sending  $A$  to the upper left corner of  $M_n A$ . Throughout this section  $\mathfrak{R}$  is a small  $T$ -closed admissible category of  $k$ -algebras with  $M_n A \in \mathfrak{R}$  for every  $A \in \mathfrak{R}$  and  $n \geq 1$ .

Denote by  $U_{\bullet} \mathfrak{R}_{I,J}^{\text{mor}}$  the model category obtained from  $U_{\bullet} \mathfrak{R}_{I,J}$  by Bousfield localisation with respect to the family of maps of cofibrant objects

$$\{r(M_n A) \rightarrow rA \mid A \in \mathfrak{R}, n > 0\}.$$

Let  $\text{Sp}_{\text{mor}}(\mathfrak{R})$  be the stable model category of  $S^1$ -spectra associated with  $U_{\bullet} \mathfrak{R}_{I,J}^{\text{mor}}$ . Observe that it is also obtained from  $\text{Sp}(\mathfrak{R})$  by Bousfield localisation with respect to the family of maps of cofibrant objects in  $\text{Sp}(\mathfrak{R})$

$$\{F_s(r(M_n A)) \rightarrow F_s(rA) \mid A \in \mathfrak{R}, n > 0, s \geq 0\}.$$

Here  $F_s : U_{\bullet} \mathfrak{R}_{I,J}^{\text{mor}} \rightarrow \text{Sp}_{\text{mor}}(\mathfrak{R})$  is the canonical functor adjoint to the evaluation functor  $E v_s : \text{Sp}_{\text{mor}}(\mathfrak{R}) \rightarrow U_{\bullet} \mathfrak{R}_{I,J}^{\text{mor}}$ .

**Definition** [Garkusha 2014]. (1) The *Morita stable algebraic Kasparov  $K$ -theory space* of two algebras  $A, B \in \mathfrak{R}$  is the space

$$\mathcal{K}^{\text{mor}}(A, B) = \text{colim}(\mathcal{K}(A, B) \rightarrow \mathcal{K}(A, M_2 k \otimes B) \rightarrow \mathcal{K}(A, M_3 k \otimes B) \rightarrow \dots).$$

Its homotopy groups will be denoted by  $\mathcal{K}_n^{\text{mor}}(A, B), n \geq 0$ .

- (2) A functor  $X : \mathfrak{R} \rightarrow \mathbb{S}/(\text{Spectra})$  is *Morita invariant* if each morphism  $X(A) \rightarrow X(M_n A), A \in \mathfrak{R}, n > 0$ , is a weak equivalence.
- (3) An excisive, homotopy invariant homology theory  $X : \mathfrak{R} \rightarrow \mathcal{T}$  is *Morita invariant* if each morphism  $X(A) \rightarrow X(M_n A), A \in \mathfrak{R}, n > 0$ , is an isomorphism.
- (4) The *Morita stable algebraic Kasparov  $K$ -theory spectrum* of  $A, B \in \mathfrak{R}$  is the  $\Omega$ -spectrum

$$\mathbb{K}^{\text{mor}}(A, B) = (\mathcal{K}^{\text{mor}}(A, B), \mathcal{K}^{\text{mor}}(JA, B), \mathcal{K}^{\text{mor}}(J^2 A, B), \dots).$$

Denote by  $\mathrm{SH}_{\mathcal{S}^1}^{\mathrm{mor}}(\mathfrak{A})$  the (stable) homotopy category of  $\mathrm{Sp}_{\mathrm{mor}}(\mathfrak{A})$ . It is a compactly generated triangulated category with compact generators  $\{\Sigma^\infty r A[n]\}_{A \in \mathfrak{A}, n \in \mathbb{Z}}$ . Let  $\mathcal{S}_{\mathrm{mor}}$  be the full subcategory of  $\mathrm{SH}_{\mathcal{S}^1}^{\mathrm{mor}}(\mathfrak{A})$  whose objects are  $\{\Sigma^\infty r A[n]\}_{A \in \mathfrak{A}, n \in \mathbb{Z}}$ .

Recall the definition of the triangulated category  $D_{\mathrm{mor}}(\mathfrak{A}, \mathfrak{F})$  from [Garkusha 2013]. Its objects are those of  $\mathfrak{A}$  and the set of morphisms between two algebras  $A, B \in \mathfrak{A}$  is defined as the colimit of the sequence of abelian groups

$$D(\mathfrak{A}, \mathfrak{F})(A, B) \rightarrow D(\mathfrak{A}, \mathfrak{F})(A, M_2 B) \rightarrow D(\mathfrak{A}, \mathfrak{F})(A, M_3 B) \rightarrow \dots$$

There is a canonical functor  $\mathfrak{A} \rightarrow D_{\mathrm{mor}}(\mathfrak{A}, \mathfrak{F})$ . It is a universal excisive, homotopy invariant and Morita invariant homology theory on  $\mathfrak{A}$ .

**Theorem 5.1.** *Given  $A \in \mathfrak{A}$  the composite map*

$$\Sigma^\infty r A \xrightarrow{i} \mathcal{R}(A) \xrightarrow{j} \mathbb{K}(A, -) \rightarrow \mathbb{K}^{\mathrm{mor}}(A, -) \tag{5.1}$$

*is a stable equivalence in  $\mathrm{Sp}_{\mathrm{mor}}(\mathfrak{A})$ , functorial in  $A$ . In particular, there is a natural isomorphism*

$$\mathrm{SH}_{\mathcal{S}^1}^{\mathrm{mor}}(\mathfrak{A})(\Sigma^\infty r B[n], \Sigma^\infty r A) \cong \mathbb{K}_n^{\mathrm{mor}}(A, B)$$

*for all  $A, B \in \mathfrak{A}$  and  $n \in \mathbb{Z}$ . Furthermore, the category  $\mathcal{S}_{\mathrm{mor}}$  is triangulated and there is a contravariant equivalence of triangulated categories*

$$T : D_{\mathrm{mor}}(\mathfrak{A}, \mathfrak{F}) \rightarrow \mathcal{S}_{\mathrm{mor}}.$$

*Proof.* Let  $\mathcal{S}^c$  and  $\mathcal{S}_{\mathrm{mor}}^c$  be the categories of compact objects in  $\mathrm{SH}_{\mathcal{S}^1}(\mathfrak{A})$  and  $\mathrm{SH}_{\mathcal{S}^1}^{\mathrm{mor}}(\mathfrak{A})$  respectively. Denote by  $\mathcal{R}$  the full triangulated subcategory of  $\mathcal{S}$  generated by objects

$$\{\mathrm{cone}(\Sigma^\infty r(M_n A) \rightarrow \Sigma^\infty r A)[k] \mid A \in \mathfrak{A}, n > 0, k \in \mathbb{Z}\}.$$

Let  $\mathcal{R}^c$  be the thick closure of  $\mathcal{R}$  in  $\mathrm{SH}_{\mathcal{S}^1}(\mathfrak{A})$ . It follows from [Neeman 1996, Theorem 2.1] that the natural functor

$$\mathcal{S}^c / \mathcal{R}^c \rightarrow \mathcal{S}_{\mathrm{mor}}^c$$

is full and faithful and  $\mathcal{S}_{\mathrm{mor}}^c$  is the thick closure of  $\mathcal{S}^c / \mathcal{R}^c$ .

We claim that the natural functor

$$\mathcal{S} / \mathcal{R} \rightarrow \mathcal{S}^c / \mathcal{R}^c \tag{5.2}$$

is full and faithful. For this consider a map  $\alpha : X \rightarrow Y$  in  $\mathcal{S}^c$  such that its cone  $Z$  is in  $\mathcal{R}^c$  and  $Y \in \mathcal{S}$ . We can find  $Z' \in \mathcal{R}^c$  such that  $Z \oplus Z'$  is isomorphic to an

object  $W \in \mathcal{R}$ . Construct a commutative diagram in  $\mathcal{S}^c$

$$\begin{array}{ccccccc}
 U & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & \Sigma U \\
 s \downarrow & & \parallel & & p \downarrow & & \downarrow \\
 X & \xrightarrow{\alpha} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X
 \end{array}$$

where  $p$  is the natural projection. We see that  $\alpha s$  is such that its cone  $W$  belongs to  $\mathcal{R}$ . Standard facts for Gabriel–Zisman localisation theory imply (5.2) is a fully faithful embedding. It also follows that

$$\mathcal{S}_{\text{mor}} = \mathcal{S} / \mathcal{R}.$$

We want to compute Hom sets in  $\mathcal{S} / \mathcal{R}$ . For this observe first that there is a contravariant equivalence of triangulated categories

$$\tau : D(\mathfrak{R}, \mathfrak{F}) / \mathcal{U} \rightarrow \mathcal{S}_{\text{mor}},$$

where  $\mathcal{U}$  is the smallest full triangulated subcategory of  $D(\mathfrak{R}, \mathfrak{F})$  containing

$$\{\text{cone}(A \xrightarrow{\iota} M_n A) \mid A \in \mathfrak{R}, n > 0\}.$$

This follows from Theorem 4.4.

By construction, every excisive homotopy invariant Morita invariant homology theory  $\mathfrak{R} \rightarrow \mathcal{T}$  factors through  $D(\mathfrak{R}, \mathfrak{F}) / \mathcal{U}$ . Since  $\mathfrak{R} \rightarrow D_{\text{mor}}(\mathfrak{R}, \mathfrak{F})$  is a universal excisive homotopy invariant Morita invariant homology theory [Garkusha 2013], we see that there exists a triangle equivalence of triangulated categories

$$D_{\text{mor}}(\mathfrak{R}, \mathfrak{F}) \simeq D(\mathfrak{R}, \mathfrak{F}) / \mathcal{U}.$$

So there is a natural contravariant triangle equivalence of triangulated categories

$$T : D_{\text{mor}}(\mathfrak{R}, \mathfrak{F}) \rightarrow \mathcal{S}_{\text{mor}}.$$

Using this and [Garkusha 2014, Theorem 9.8], there is a natural isomorphism

$$\mathcal{S}_{\text{mor}}(\Sigma^\infty r B[n], \Sigma^\infty r A) \cong \mathbb{K}_n^{\text{mor}}(A, B)$$

for all  $A, B \in \mathfrak{R}$  and  $n \in \mathbb{Z}$ . The fact that (5.1) is a stable equivalence in  $\text{Sp}_{\text{mor}}(\mathfrak{R})$  is now obvious. □

### 6. Stable algebraic Kasparov K-theory

If  $A$  is an algebra set  $M_\infty A = \cup_n M_n A$ . There is a natural inclusion  $\iota : A \rightarrow M_\infty A$  of algebras, sending  $A$  to the upper left corner of  $M_\infty A$ . Throughout the section  $\mathfrak{R}$  is a small  $T$ -closed admissible category of  $k$ -algebras with  $M_\infty(A) \in \mathfrak{R}$  for all  $A \in \mathfrak{R}$ .

Denote by  $U_*\mathfrak{R}_{I,J}^\infty$  the model category obtained from  $U_*\mathfrak{R}_{I,J}$  by Bousfield localisation with respect to the family of maps of cofibrant objects

$$\{r(M_\infty A) \rightarrow rA \mid A \in \mathfrak{R}\}.$$

Let  $\mathrm{Sp}_\infty(\mathfrak{R})$  be the stable model category of  $S^1$ -spectra associated with  $U_*\mathfrak{R}_{I,J}^\infty$ . Observe that it is also obtained from  $\mathrm{Sp}(\mathfrak{R})$  by Bousfield localisation with respect to the family of maps of cofibrant objects in  $\mathrm{Sp}(\mathfrak{R})$

$$\{F_s(r(M_\infty A)) \rightarrow F_s(rA) \mid A \in \mathfrak{R}, s \geq 0\}.$$

**Definition** [Garkusha 2014]. (1) The *stable algebraic Kasparov K-theory space* of two algebras  $A, B \in \mathfrak{R}$  is the space

$$\mathcal{K}^{st}(A, B) = \mathrm{colim}(\mathcal{K}(A, B) \rightarrow \mathcal{K}(A, M_\infty k \otimes B) \rightarrow \mathcal{K}(A, M_\infty k \otimes M_\infty k \otimes B) \rightarrow \dots).$$

Its homotopy groups will be denoted by  $\mathcal{K}_n^{st}(A, B)$ ,  $n \geq 0$ .

- (2) A functor  $X: \mathfrak{R} \rightarrow \mathbb{S}/(\mathrm{Spectra})$  is *stable* or  *$M_\infty$ -invariant* if  $X(A) \rightarrow X(M_\infty A)$  is a weak equivalence for all  $A \in \mathfrak{R}$ .
- (3) An excisive, homotopy invariant homology theory  $X: \mathfrak{R} \rightarrow \mathcal{T}$  is *stable* or  *$M_\infty$ -invariant* if  $X(A) \rightarrow X(M_\infty A)$  is an isomorphism for all  $A \in \mathfrak{R}$ .
- (4) The *stable algebraic Kasparov K-theory spectrum* for  $A, B \in \mathfrak{R}$  is the  $\Omega$ -spectrum

$$\mathbb{K}^{st}(A, B) = (\mathcal{K}^{st}(A, B), \mathcal{K}^{st}(JA, B), \mathcal{K}^{st}(J^2 A, B), \dots).$$

Denote by  $\mathrm{SH}_{\mathrm{St}}^\infty(\mathfrak{R})$  the (stable) homotopy category of  $\mathrm{Sp}_\infty(\mathfrak{R})$ . It is a compactly generated triangulated category with compact generators  $\{\Sigma^\infty rA[n]\}_{A \in \mathfrak{R}, n \in \mathbb{Z}}$ . Let  $\mathcal{S}_\infty$  be the full subcategory of  $\mathrm{SH}_{\mathrm{St}}^\infty(\mathfrak{R})$  whose objects are  $\{\Sigma^\infty rA[n]\}_{A \in \mathfrak{R}, n \in \mathbb{Z}}$ .

Recall from [Garkusha 2013] the definition of the triangulated category  $D_{st}(\mathfrak{R}, \mathfrak{F})$ . Its objects are those of  $\mathfrak{R}$  and the set of morphisms between two algebras  $A, B \in \mathfrak{R}$  is defined as the colimit of the sequence of abelian groups

$$\begin{aligned} D(\mathfrak{R}, \mathfrak{F})(A, B) &\rightarrow D(\mathfrak{R}, \mathfrak{F})(A, M_\infty k \otimes_k B) \\ &\rightarrow D(\mathfrak{R}, \mathfrak{F})(A, M_\infty k \otimes_k M_\infty k \otimes_k B) \rightarrow \dots \end{aligned}$$

There is a canonical functor  $\mathfrak{R} \rightarrow D_{st}(\mathfrak{R}, \mathfrak{F})$ . It is the universal excisive, homotopy invariant and stable homology theory on  $\mathfrak{R}$ .

The proof of the next result literally repeats that of Theorem 5.1 if we replace the algebras  $M_n A$  with  $M_\infty A$  and the categories  $\mathcal{S}_{\mathrm{mor}}$  and  $D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F})$  with  $\mathcal{S}_\infty$  and  $D_{st}(\mathfrak{R}, \mathfrak{F})$  respectively.

**Theorem 6.1.** *Given  $A \in \mathfrak{R}$ , the composite map*

$$\Sigma^\infty rA \xrightarrow{i} \mathcal{R}(A) \xrightarrow{j} \mathbb{K}(A, -) \rightarrow \mathbb{K}^{st}(A, -)$$

*is a stable equivalence in  $\mathrm{Sp}_\infty(\mathfrak{R})$ , functorial in  $A$ . In particular, there is a natural isomorphism*

$$\mathrm{SH}_{st}^\infty(\mathfrak{R})(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n^{st}(A, B)$$

*for all  $A, B \in \mathfrak{R}$  and  $n \in \mathbb{Z}$ . Furthermore, the category  $\mathcal{S}_\infty$  is triangulated and there is a contravariant equivalence of triangulated categories*

$$T : D_{st}(\mathfrak{R}, \mathfrak{F}) \rightarrow \mathcal{S}_\infty.$$

Let  $\Gamma A$ , for  $A \in \mathrm{Alg}_k$ , be the algebra of  $\mathbb{N} \times \mathbb{N}$ -matrices which satisfy the following two properties.

- (i) The set  $\{a_{ij} \mid i, j \in \mathbb{N}\}$  is finite.
- (ii) There exists a natural number  $N \in \mathbb{N}$  such that each row and each column has at most  $N$  nonzero entries.

$M_\infty A \subset \Gamma A$  is an ideal. We put

$$\Sigma A = \Gamma A / M_\infty A.$$

We note that  $\Gamma A, \Sigma A$  are the cone and suspension rings of  $A$  considered by Karoubi and Villamayor [1969, p. 269], where a different but equivalent definition is given. By [Cortiñas and Thom 2007] there are natural ring isomorphisms

$$\Gamma A \cong \Gamma k \otimes A, \quad \Sigma A \cong \Sigma k \otimes A.$$

We call the short exact sequence

$$M_\infty A \twoheadrightarrow \Gamma A \twoheadrightarrow \Sigma A$$

the *cone extension*. By [Cortiñas and Thom 2007]  $\Gamma A \twoheadrightarrow \Sigma A$  is a split surjection of  $k$ -modules.

Let  $\tau$  be the  $k$ -algebra which is unital and free on two generators  $\alpha$  and  $\beta$  satisfying the relation  $\alpha\beta = 1$ . By [Cortiñas and Thom 2007, Lemma 4.10.1] the kernel of the natural map

$$\tau \rightarrow k[t^{\pm 1}]$$

is isomorphic to  $M_\infty k$ . We set  $\tau_0 = \tau \oplus_{k[t^{\pm 1}]} \sigma$ .

Let  $A$  be a  $k$ -algebra. We get an extension

$$M_\infty A \longrightarrow \tau A \longrightarrow A[t^{\pm 1}],$$

and an analogous extension

$$M_\infty A \longrightarrow \tau_0 A \longrightarrow \sigma A.$$

**Definition.** We say that an admissible category of  $k$ -algebras  $\mathfrak{R}$  is  $\tau_0$ -closed (respectively  $\Gamma$ -closed) if  $\tau_0 A \in \mathfrak{R}$  (respectively  $\Gamma A \in \mathfrak{R}$ ) for all  $A \in \mathfrak{R}$ .

Cuntz [1997; 2005; Cuntz and Thom 2006] constructed a triangulated category  $kk^{\text{lca}}$  whose objects are the locally convex algebras. Later Cortiñas and Thom [2007] constructed in a similar fashion a triangulated category  $kk$  whose objects are all  $k$ -algebras  $\text{Alg}_k$ . If we suppose that  $\mathfrak{R}$  is also  $\Gamma$ -closed, then one can define a full triangulated subcategory  $kk(\mathfrak{R})$  of  $kk$  whose objects are those of  $\mathfrak{R}$ .

It can be shown similar to [Garkusha 2007, Theorem 7.4] or [Garkusha 2013, Corollary 9.4] that there is an equivalence of triangulated categories

$$D_{st}(\mathfrak{R}, \mathfrak{F}) \xrightarrow{\sim} kk(\mathfrak{R}).$$

An important computational result of Cortiñas and Thom [2007] states that there is an isomorphism of graded abelian groups

$$\bigoplus_{n \in \mathbb{Z}} kk(\mathfrak{R})(k, \Omega^n A) \cong \bigoplus_{n \in \mathbb{Z}} KH_n(A),$$

where the right hand side is the homotopy  $K$ -theory of  $A \in \mathfrak{R}$  in the sense of [Weibel 1989].

Summarising the above arguments together with Theorem 6.1 we obtain the following:

**Theorem 6.2.** *Suppose  $\mathfrak{R}$  is  $\Gamma$ -closed. Then there is a contravariant equivalence of triangulated categories*

$$kk(\mathfrak{R}) \rightarrow \mathcal{S}_\infty.$$

*Moreover, there is a natural isomorphism*

$$\text{SH}_{\mathbb{S}^1}^\infty(\mathfrak{R})(\Sigma^\infty r A[n], \Sigma^\infty r(k)) \cong KH_n(A)$$

*for any  $A \in \mathfrak{R}$  and any integer  $n$ .*

### 7. $K$ -motives of algebras

Throughout the section we assume that  $\mathfrak{R}$  is a small tensor closed and  $T$ -closed admissible category of  $k$ -algebras with  $M_\infty(k) \in \mathfrak{R}$ . It follows that

$$M_\infty A := A \otimes M_\infty(k) \in \mathfrak{R}$$

for all  $A \in \mathfrak{R}$ .

In this section we define and study the triangulated category of  $K$ -motives. It shares many properties with the category of  $K$ -motives for algebraic varieties constructed in [Garkusha and Panin 2012; 2014b]

Since  $\mathfrak{R}$  is tensor closed, it follows that  $U_*\mathfrak{R}_{I,J}^\infty$  is a monoidal model category. Let  $\mathrm{Sp}_\infty^\Sigma(\mathfrak{R})$  be the monoidal category of symmetric spectra in the sense of [Hovey 2001] associated to  $U_*\mathfrak{R}_{I,J}^\infty$ .

**Definition.** The *category of  $K$ -motives*  $DK(\mathfrak{R})$  is the stable homotopy category of  $\mathrm{Sp}_\infty^\Sigma(\mathfrak{R})$ . The  *$K$ -motive*  $M_K(A)$  of an algebra  $A \in \mathfrak{R}$  is the image of  $A$  in  $DK(\mathfrak{R})$ , that is  $M_K(A) = \Sigma^\infty rA$ . Thus one has a canonical contravariant functor

$$M_K : \mathfrak{R} \rightarrow DK(\mathfrak{R})$$

sending algebras to their  $K$ -motives.

The following proposition follows from standard facts for monoidal model categories.

**Proposition 7.1.**  *$DK(\mathfrak{R})$  is a symmetric monoidal compactly generated triangulated category with compact generators  $\{M_K(A)\}_{A \in \mathfrak{R}}$ . For any two algebras  $A, B \in \mathfrak{R}$  one has a natural isomorphism*

$$M_K(A) \otimes M_K(B) \cong M_K(A \otimes B).$$

Furthermore, any extension of algebras in  $\mathfrak{R}$

$$(E) : \quad A \rightarrow B \rightarrow C$$

induces a triangle in  $DK(\mathfrak{R})$

$$M_K(E) : \quad M_K(C) \rightarrow M_K(B) \rightarrow M_K(A) \xrightarrow{+} .$$

There is a pair of adjoint functors

$$V : \mathrm{Sp}_\infty(\mathfrak{R}) \rightleftarrows \mathrm{Sp}_\infty^\Sigma(\mathfrak{R}) : U,$$

where  $U$  is the right Quillen forgetful functor. These form a Quillen equivalence. In particular, the induced functors

$$V : \mathrm{SH}_{\mathcal{S}^1}^\infty(\mathfrak{R}) \rightleftarrows DK(\mathfrak{R}) : U$$

are equivalences of triangulated categories. It follows from Proposition 7.1 that  $\mathrm{SH}_{\mathcal{S}^1}^\infty(\mathfrak{R})$  is a symmetric monoidal category and

$$\Sigma^\infty rA \otimes \Sigma^\infty rB \cong \Sigma^\infty r(A \otimes B)$$

for all  $A, B \in \mathfrak{R}$ . Moreover,

$$V(\Sigma^\infty rA) \cong M_K(A)$$

for all  $A \in \mathfrak{R}$ .

Summarising the above arguments together with Theorem 6.1 we get the following:

**Theorem 7.2.** *For any two algebras  $A, B \in \mathfrak{R}$  and any integer  $n$  one has a natural isomorphism of abelian groups*

$$DK(\mathfrak{R})(M_K(B)[n], M_K(A)) \cong \mathbb{K}_n^{st}(A, B).$$

*The full subcategory  $\mathcal{T}$  of  $DK(\mathfrak{R})$  spanned by  $K$ -motives of algebras  $\{M_K(A)\}_{A \in \mathfrak{R}}$  is triangulated and there is an equivalence of triangulated categories*

$$D_{st}(\mathfrak{R}, \mathfrak{F}) \rightarrow \mathcal{T}^{\text{op}}$$

*sending an algebra  $A \in \mathfrak{R}$  to its  $K$ -motive  $M_K(A)$ .*

The next result is reminiscent of a similar result for  $K$ -motives of algebraic varieties in the sense of [Garkusha and Panin 2012; 2014b] identifying the  $K$ -motive of the point with algebraic  $K$ -theory.

**Corollary 7.3.** *Suppose  $\mathfrak{R}$  is  $\Gamma$ -closed. Then for any algebra  $A$  and any integer  $n$  one has a natural isomorphism of abelian groups*

$$DK(\mathfrak{R})(M_K(A)[n], M_K(k)) \cong KH_n(A),$$

*where the right hand side is the  $n$ -th homotopy  $K$ -theory group in the sense of [Weibel 1989].*

*Proof.* This follows from [Garkusha 2013, Theorem 10.6] and the preceding theorem.  $\square$

We finish the section by showing that the category  $kk(\mathfrak{R})$  of [Cortiñas and Thom 2007] can be identified with the  $K$ -motives of algebras.

**Theorem 7.4.** *Suppose  $\mathfrak{R}$  is  $\Gamma$ -closed. Then there is a natural equivalence of triangulated categories*

$$kk(\mathfrak{R}) \xrightarrow{\sim} \mathcal{T}^{\text{op}}$$

*sending an algebra  $A \in \mathfrak{R}$  to its  $K$ -motive  $M_K(A)$ .*

*Proof.* This follows from Theorem 7.2 and the fact that  $D_{st}(\mathfrak{R}, \mathfrak{F})$  and  $kk(\mathfrak{R})$  are triangle equivalent (see [Garkusha 2007, Theorem 7.4] or [Garkusha 2013, Corollary 9.4]).  $\square$

The latter theorem shows in particular that  $kk(\mathfrak{R})$  is embedded into the compactly generated triangulated category of  $K$ -motives  $DK(\mathfrak{R})$  and generates it.



### 8. The $\mathbb{G}$ -stable theory

The stable motivic homotopy theory over a field is the homotopy theory of  $T$ -spectra, where  $T = S^1 \wedge \mathbb{G}_m$  (see [Voevodsky 1998; Jardine 2000]). There are various equivalent definitions of the theory, one of which is given in terms of  $(S^1, \mathbb{G}_m)$ -bispectra. In our context the role of the motivic space  $\mathbb{G}_m$  is played by  $\sigma = (t - 1)k[t^{\pm 1}]$ . Its simplicial functor  $r(\sigma)$  is denoted by  $\mathbb{G}$ . In this section we define the stable category of  $(S^1, \mathbb{G})$ -bispectra and construct an explicit fibrant replacement of the  $(S^1, \mathbb{G})$ -bispectrum  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA$  of an algebra  $A$ . One can also define a Quillen equivalent category of  $T$ -spectra, where  $T = S^1 \wedge \mathbb{G}$ , and compute an explicit fibrant replacement for the  $T$ -spectrum of an algebra. However we prefer to work with  $(S^1, \mathbb{G})$ -bispectra rather than  $T$ -spectra in order to study  $K$ -motives of algebras in terms of associated  $(S^1, \mathbb{G})$ -bispectra (see the next section).

Throughout the section we assume that  $\mathfrak{R}$  is a small tensor closed and  $T$ -closed admissible category of  $k$ -algebras. We have that  $\sigma A := A \otimes \sigma \in \mathfrak{R}$  for all  $A \in \mathfrak{R}$ .

Recall that  $U_*\mathfrak{R}_{I,J}$  is a monoidal model category. It follows from [Hovey 2001, Section 6.3] that  $\mathrm{Sp}(\mathfrak{R})$  is a  $U_*\mathfrak{R}_{I,J}$ -model category. In particular

$$- \otimes \mathbb{G} : \mathrm{Sp}(\mathfrak{R}) \rightarrow \mathrm{Sp}(\mathfrak{R})$$

is a left Quillen endofunctor.

By definition, a  $(S^1, \mathbb{G})$ -bispectrum or bispectrum  $\mathcal{E}$  is given by a sequence  $(E_0, E_1, \dots)$ , where each  $E_j$  is a  $S^1$ -spectrum of  $\mathrm{Sp}(\mathfrak{R})$ , together with bonding morphisms  $\varepsilon_n : E_n \wedge \mathbb{G} \rightarrow E_{n+1}$ . Maps are sequences of maps in  $\mathrm{Sp}(\mathfrak{R})$  respecting the bonding morphisms. We denote the category of bispectra by  $\mathrm{Sp}_{\mathbb{G}}(\mathfrak{R})$ . It can be regarded as the category of  $\mathbb{G}$ -spectra on  $\mathrm{Sp}(\mathfrak{R})$  in the sense of [Hovey 2001].

$\mathrm{Sp}_{\mathbb{G}}(\mathfrak{R})$  is equipped with the stable  $U_*\mathfrak{R}_{I,J}$ -model structure in which weak equivalences are defined by means of bigraded homotopy groups. The bispectrum object  $\mathcal{E}$  determines a sequence of maps of  $S^1$ -spectra

$$E_0 \xrightarrow{\tilde{\varepsilon}_0} \Omega_{\mathbb{G}} E_1 \xrightarrow{\Omega_{\mathbb{G}}(\tilde{\varepsilon}_1)} \Omega_{\mathbb{G}}^2 E_2 \rightarrow \dots,$$

where  $\Omega_{\mathbb{G}}$  is the functor  $\underline{\mathrm{Hom}}(\mathbb{G}, -)$  and  $\tilde{\varepsilon}_n$ -s are adjoint to the structure maps of  $\mathcal{E}$ . We define  $\pi_{p,q} \mathcal{E}$  in  $A$ -sections as the colimit

$$\begin{aligned} \mathrm{colim}_l (\mathrm{Hom}_{\mathrm{SH}_{S^1}(\mathfrak{R})}(S^{p-q}, \Omega_{\mathbb{G}}^{q+l} J E_l(A)) \\ \rightarrow \mathrm{Hom}_{\mathrm{SH}_{S^1}(\mathfrak{R})}(S^{p-q}, \Omega_{\mathbb{G}}^{q+l+1} J E_{l+1}(A)) \rightarrow \dots) \end{aligned}$$

once  $\mathcal{E}$  has been replaced up to levelwise equivalence by a levelwise fibrant object  $J\mathcal{E}$  so that the “loop” constructions make sense. We also call  $\pi_{*,q} \mathcal{E}$  the homotopy groups of weight  $q$ .

By definition, a map of bispectra is a weak equivalence in  $\mathrm{Sp}_{\mathbb{G}}(\mathfrak{A})$  if it induces an isomorphism on bigraded homotopy groups. We denote the homotopy category of  $\mathrm{Sp}_{\mathbb{G}}(\mathfrak{A})$  by  $\mathrm{SH}_{S^1, \mathbb{G}}(\mathfrak{A})$ . It is a compactly generated triangulated category.

To define the main  $(S^1, \mathbb{G})$ -bispectrum of this section, denoted by  $\mathbb{K}\mathbb{G}(A, -)$ , we should first establish some facts for algebra homomorphisms.

Suppose  $A, C \in \mathfrak{A}$ , then one has a commutative diagram

$$\begin{array}{ccccc} J(A \otimes C) & \xrightarrow{\quad} & T(A \otimes C) & \xrightarrow{\eta_{A \otimes C}} & A \otimes C \\ \gamma_{A,C} \downarrow & & \downarrow & & \parallel \\ JA \otimes C & \xrightarrow{\quad} & T(A) \otimes C & \xrightarrow{\eta_{A \otimes C}} & A \otimes C \end{array}$$

in which  $\gamma_{A,C}$  is uniquely determined by the split monomorphism

$$i_A \otimes C : A \otimes C \rightarrow T(A) \otimes C.$$

One sets  $\gamma_{A,C}^0 := 1_{A \otimes C}$ . We construct inductively

$$\gamma_{A,C}^n : J^n(A \otimes C) \rightarrow J^n(A) \otimes C, \quad n \geq 1.$$

Namely,  $\gamma_{A,C}^{n+1}$  is the composite

$$J^{n+1}(A \otimes C) \xrightarrow{J(\gamma_{A,C}^n)} J(J^n(A) \otimes C) \xrightarrow{\gamma_{J^n(A), C}^n} J^{n+1}(A) \otimes C.$$

Given  $n \geq 0$ , we define a map

$$t_n = t_n^{A,C} : \mathcal{K}(J^n A, -) \rightarrow \mathcal{K}(J^n(A \otimes C), - \otimes C) = \underline{\mathrm{Hom}}(rC, \mathcal{K}(J^n(A \otimes C), -))$$

as follows. Let  $B \in \mathfrak{A}$  and  $(\alpha : J^{n+m} A \rightarrow \mathbb{B}(\Omega^m)) \in \mathcal{K}(J^n A, B)$ . We set

$$t_n(\alpha) \in \mathcal{K}(J^n(A \otimes C), B \otimes C)$$

to be the composite

$$J^{n+m}(A \otimes C) \xrightarrow{\gamma_{A,C}^{n+m}} J^{n+m}(A) \otimes C \xrightarrow{\alpha \otimes C} \mathbb{B}^\Delta(\Omega^m) \otimes C \xrightarrow{\tau} (\mathbb{B} \otimes C)^\Delta(\Omega^m).$$

Here  $\tau$  is a canonical isomorphism (see [Cortiñas and Thom 2007, Proposition 3.1.3]) and  $(\mathbb{B} \otimes C)^\Delta$  stands for the simplicial ind-algebra

$$[m, \ell] \mapsto \mathrm{Hom}_{\mathbb{S}}(\mathrm{sd}^m \Delta^\ell, (B \otimes C)^\Delta) = (B \otimes C)^{\mathrm{sd}^m \Delta^\ell} \cong k^{\mathrm{sd}^m \Delta^\ell} \otimes (B \otimes C).$$

One has to verify that  $t_n$  is consistent with maps

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+m} A, \mathbb{B}^\Delta(\Omega^m)) \xrightarrow{\zeta} \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+m+1} A, \mathbb{B}^\Delta(\Omega^{m+1})).$$

More precisely, we must show that the map

$$J^{n+m+1}(A \otimes C) \xrightarrow{J(\gamma_{A,C}^{n+m})} J(J^{n+m} A \otimes C) \xrightarrow{J(\alpha \otimes 1)} J(\mathbb{B}^\Delta(\Omega^m) \otimes C) \xrightarrow{J\tau} J((\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^m)) \xrightarrow{\xi_\nu} (\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^{m+1})$$

is equal to the map

$$J^{n+m+1}(A \otimes C) \xrightarrow{\gamma_{A,C}^{n+m+1}} J^{n+m+1} A \otimes C \xrightarrow{J\alpha \otimes 1} J(\mathbb{B}^\Delta(\Omega^m)) \otimes C \xrightarrow{\xi_\nu \otimes 1} \mathbb{B}^\Delta(\Omega^{m+1}) \otimes C \xrightarrow{\tau} (\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^{m+1}).$$

The desired property follows from commutativity of the diagram (we use [Garkusha 2014, Lemma 3.4] here)

$$\begin{array}{ccccccc} J^{n+m+1}(A \otimes C) & & & & & & \\ \downarrow J(\gamma_{A,C}^{n+m}) & & & & & & \\ J(J^{n+m} A \otimes C) & \xrightarrow{\gamma_{J^{n+m} A, C}} & J^{n+m+1} A \otimes C & \xrightarrow{\quad} & T J^{n+m} A \otimes C & \twoheadrightarrow & J^{n+m} A \otimes C \\ \downarrow J(\alpha \otimes 1) & & \downarrow J(\alpha \otimes 1) & & \downarrow & & \downarrow \alpha \otimes 1 \\ J(\mathbb{B}^\Delta(\Omega^m) \otimes C) & \longrightarrow & J(\mathbb{B}^\Delta(\Omega^m)) \otimes C & \xrightarrow{\quad} & T(\mathbb{B}^\Delta(\Omega^m)) \otimes C & \longrightarrow & \mathbb{B}^\Delta(\Omega^m) \otimes C \\ \parallel & & \downarrow \xi_\nu \otimes 1 & & \downarrow & & \parallel \\ J(\mathbb{B}^\Delta(\Omega^m) \otimes C) & \longrightarrow & \mathbb{B}^\Delta(\Omega^{m+1}) \otimes C & \xrightarrow{\quad} & P(\mathbb{B}^\Delta(\Omega^m)) \otimes C & \twoheadrightarrow & \mathbb{B}^\Delta(\Omega^m) \otimes C \\ \downarrow J\tau & & \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ J((\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^m)) & \xrightarrow{\xi_\nu} & (\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^{m+1}) & \xrightarrow{\quad} & P(\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^m) & \twoheadrightarrow & (\mathbb{B} \otimes \mathbb{C})^\Delta(\Omega^m) \end{array}$$

We see that  $t_n$  is well defined. We claim that the collection of maps  $(t_n)_n$  defines a map of  $S^1$ -spectra

$$t : \mathbb{K}(A, B) \rightarrow \mathbb{K}(A \otimes C, B \otimes C).$$

We have to check that for each  $n \geq 0$  the diagram

$$\begin{array}{ccc} \mathcal{H}(J^n A, B) & \xrightarrow{\cong} & \Omega \mathcal{H}(J^{n+1} A, B) \\ \downarrow t_n & & \downarrow \Omega t_{n+1} \\ \mathcal{H}(J^n(A \otimes C), B \otimes C) & \xrightarrow{\cong} & \Omega \mathcal{H}(J^{n+1}(A \otimes C), B \otimes C) \end{array}$$

is commutative. But this directly follows from the definition of the horizontal maps (see [Garkusha 2014, Theorem 5.1]) and arguments above made for the  $t_n$ .

If we replace  $C$  by  $\sigma$  we get that the array

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 \mathbb{K}(\sigma^2 A, B) : & \mathcal{H}(\sigma^2 A, B) & & \mathcal{H}(J\sigma^2 A, B) & & \mathcal{H}(J^2\sigma^2 A, B) & \dots \\
 \mathbb{K}(\sigma A, B) : & \mathcal{H}(\sigma A, B) & & \mathcal{H}(J\sigma A, B) & & \mathcal{H}(J^2\sigma A, B) & \dots \\
 \mathbb{K}(A, B) : & \mathcal{H}(A, B) & & \mathcal{H}(JA, B) & & \mathcal{H}(J^2 A, B) & \dots
 \end{array}$$

together with structure maps

$$\mathbb{K}(\sigma^n A, -) \otimes \mathbb{G} \rightarrow \mathbb{K}(\sigma^{n+1} A, -)$$

defined as adjoint maps to

$$t : \mathbb{K}(\sigma^n A, -) \rightarrow \underline{\text{Hom}}(\mathbb{G}, \mathbb{K}(\sigma^{n+1} A, -))$$

forms a  $(S^1, \mathbb{G})$ -bispectrum, which we denote by  $\mathbb{K}\mathbb{G}(A, -)$ .

There is a natural map of  $(S^1, \mathbb{G})$ -bispectra

$$\Gamma : \Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA \rightarrow \mathbb{K}\mathbb{G}(A, -),$$

where  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA$  is the  $(S^1, \mathbb{G})$ -bispectrum represented by the array

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 \Sigma^{\infty} rA \otimes \mathbb{G}^2 : & rA \otimes \mathbb{G}^2 (\cong r(\sigma^2 A)) & & (rA \wedge S^1) \otimes \mathbb{G}^2 (\cong r(\sigma^2 A) \wedge S^1) & & & \dots \\
 \Sigma^{\infty} rA \otimes \mathbb{G} : & rA \otimes \mathbb{G} (\cong r(\sigma A)) & & (rA \wedge S^1) \otimes \mathbb{G} (\cong r(\sigma A) \wedge S^1) & & & \dots \\
 \Sigma^{\infty} rA : & rA & & rA \wedge S^1 & & & \dots
 \end{array}$$

with obvious structure maps.

By Theorem 4.2 each map

$$\Gamma_n : \Sigma^{\infty} rA \otimes \mathbb{G}^n \rightarrow \mathbb{K}\mathbb{G}(A, -)_n = \mathbb{K}(\sigma^n A, -)$$

is a stable weak equivalence in  $\text{Sp}(\mathfrak{N})$ . By [Garkusha 2014] each  $\mathbb{K}(\sigma^n A, -)$  is a fibrant object in  $\text{Sp}(\mathfrak{N})$ . For each  $n \geq 0$  we set

$$\begin{aligned}
 & \Theta_{\mathbb{G}}^{\infty} \mathbb{K}\mathbb{G}(A, -)_n \\
 & = \text{colim}(\mathbb{K}(\sigma^n A, -) \xrightarrow{t_0} \mathbb{K}(\sigma^{n+1} A, - \otimes \sigma) \xrightarrow{\Omega_{\mathbb{G}}(t_1)} \mathbb{K}(\sigma^{n+2} A, - \otimes \sigma^2) \rightarrow \dots).
 \end{aligned}$$

By specialising a collection of results in [Hovey 2001, Section 4] to our setting we have that  $\Theta_{\mathbb{G}}^{\infty} \mathbb{K}\mathbb{G}(A, -)$  is a fibrant bispectrum and the natural map

$$j : \mathbb{K}\mathbb{G}(A, -) \rightarrow \Theta_{\mathbb{G}}^{\infty} \mathbb{K}\mathbb{G}(A, -)$$

is a weak equivalence in  $\text{Sp}_{\mathbb{G}}(\mathfrak{N})$ .

We have thus shown that  $\Theta_{\mathbb{G}}^{\infty} \mathbb{K}\mathbb{G}(A, -)$  is an explicit fibrant replacement for the bispectrum  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA$  of the algebra  $A$ . Denote by  $\mathcal{K}^{\sigma}(A, B)$  the  $(0, 0)$ -space of the bispectrum  $\Theta_{\mathbb{G}}^{\infty} \mathbb{K}\mathbb{G}(A, B)$ . It is, by construction, the colimit

$$\text{colim}_n \mathcal{K}(\sigma^n A, \sigma^n B).$$

Its homotopy groups will be denoted by  $\mathcal{K}_n^{\sigma}(A, B)$ ,  $n \geq 0$ .

**Theorem 8.1.** *Let  $A$  be an algebra in  $\mathfrak{A}$ ; then the composite map*

$$j \circ \Gamma : \Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA \rightarrow \Theta_{\mathbb{G}}^{\infty} \mathbb{K}\mathbb{G}(A, -)$$

*is a fibrant replacement of  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA$ . In particular,*

$$\text{SH}_{S^1, \mathbb{G}}(\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rB, \Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA) = \mathcal{K}_0^{\sigma}(A, B)$$

*for all  $B \in \mathfrak{A}$ .*

**Remark.** Let  $\text{SH}(F)$  be the motivic stable homotopy category over a field  $F$ . The category  $\text{SH}_{S^1, \mathbb{G}}(\mathfrak{A})$  shares many properties with  $\text{SH}(F)$ . The author and Panin [Garkusha and Panin 2014a] have recently computed a fibrant replacement of  $\Sigma_{s,t}^{\infty} X_+$ ,  $X \in \text{Sm}/F$ , by developing the machinery of framed motives. The machinery is based on the theory of framed correspondences developed by Voevodsky [2001]. In turn, the computation of Theorem 8.1 is possible thanks to the existence of universal extensions of algebras.

Let  $F$  be an algebraically closed field of characteristic zero with an embedding  $F \hookrightarrow \mathbb{C}$  and let  $\text{SH}$  be the stable homotopy category of ordinary spectra. Let  $c : \text{SH} \rightarrow \text{SH}(F)$  be the functor induced by sending a space to the constant presheaf of spaces on  $\text{Sm}/F$ . Levine [2014] has recently shown that  $c$  is fully faithful, a fact implied by his result that the Betti realisation functor in the sense of [Ayoub 2010]

$$\text{Re}_B : \text{SH}(F) \rightarrow \text{SH}$$

gives an isomorphism

$$\text{Re}_{B*} : \pi_{n,0} \mathcal{S}_F(F) \rightarrow \pi_n(\mathcal{S})$$

for all  $n \in \mathbb{Z}$ . Here  $\mathcal{S}_F$  is the motivic sphere spectrum in  $\text{SH}(F)$  and  $\mathcal{S}$  is the classical sphere spectrum in  $\text{SH}$ . These results use recent developments for the spectral sequence associated with the slice filtration of the motivic sphere  $\mathcal{S}_F$ .

All this justifies raising the following questions.

**Questions.** (1) Is there an admissible category of commutative algebras  $\mathfrak{A}$  over the field of complex numbers  $\mathbb{C}$  such that the natural functor

$$c : \text{SH} \rightarrow \text{SH}_{S^1, \mathbb{G}}(\mathfrak{A}),$$

induced by the functor  $\mathcal{S} \rightarrow U\mathfrak{R}$  sending a simplicial set to the constant simplicial functor on  $\mathfrak{R}$ , is fully faithful?

(2) Let  $\mathfrak{R}$  be an admissible category of commutative  $\mathbb{C}$ -algebras and let  $\mathcal{S}_{\mathbb{C}}$  be the bispectrum  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} r\mathbb{C}$ . Is it true that the homotopy groups of weight zero  $\pi_{n,0} \mathcal{S}_{\mathbb{C}}(\mathbb{C}) = \mathcal{K}_n^{\sigma}(\mathbb{C}, \mathbb{C})$ ,  $n \geq 0$ , are isomorphic to the stable homotopy groups  $\pi_n(\mathcal{S})$  of the classical sphere spectrum?

We should also mention that one can define  $(S^1, \mathbb{G})$ -bispectra by starting at the monoidal category of symmetric spectra  $\mathrm{Sp}^{\Sigma}(\mathfrak{R})$  associated with the monoidal category  $U_{\bullet}(\mathfrak{R})_{I,J}$  and then stabilising the left Quillen functor

$$- \otimes \mathbb{G} : \mathrm{Sp}^{\Sigma}(\mathfrak{R}) \rightarrow \mathrm{Sp}^{\Sigma}(\mathfrak{R}).$$

One produces a model category  $\mathrm{Sp}_{\mathbb{G}}^{\Sigma}(\mathfrak{R})$  of (usual, nonsymmetric)  $\mathbb{G}$ -spectra in  $\mathrm{Sp}^{\Sigma}(\mathfrak{R})$ . Using Hovey’s notation [2001], one has, by definition,

$$\mathrm{Sp}_{\mathbb{G}}^{\Sigma}(\mathfrak{R}) = \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}^{\Sigma}(\mathfrak{R}), - \otimes \mathbb{G}).$$

There is a Quillen equivalence

$$V : \mathrm{Sp}(\mathfrak{R}) \rightleftarrows \mathrm{Sp}^{\Sigma}(\mathfrak{R}) : U$$

as well as a Quillen equivalence

$$V : \mathrm{Sp}_{\mathbb{G}}(\mathfrak{R}) \rightleftarrows \mathrm{Sp}_{\mathbb{G}}^{\Sigma}(\mathfrak{R}) : U,$$

where  $U$  is the forgetful functor (see [Hovey 2001, Section 5.7]).

If we denote by  $\mathrm{SH}_{S^1}^{\Sigma}(\mathfrak{R})$  and  $\mathrm{SH}_{S^1, \mathbb{G}}^{\Sigma}(\mathfrak{R})$  the homotopy categories of  $\mathrm{Sp}^{\Sigma}(\mathfrak{R})$  and  $\mathrm{Sp}_{\mathbb{G}}^{\Sigma}(\mathfrak{R})$  respectively, then one has equivalences of categories

$$V : \mathrm{SH}_{S^1}(\mathfrak{R}) \rightleftarrows \mathrm{SH}_{S^1}^{\Sigma}(\mathfrak{R}) : U \quad \text{and} \quad V : \mathrm{SH}_{S^1, \mathbb{G}}(\mathfrak{R}) \rightleftarrows \mathrm{SH}_{S^1, \mathbb{G}}^{\Sigma}(\mathfrak{R}) : U.$$

We refer the interested reader to [Hovey 2001; Jardine 2000] for further details.

### 9. $K$ -motives and $(S^1, \mathbb{G})$ -bispectra

We prove in this section that the triangulated category of  $K$ -motives is fully faithfully embedded into the stable homotopy category of  $(S^1, \mathbb{G})$ -bispectra  $\mathrm{SH}_{S^1, \mathbb{G}}(\mathfrak{R})$ . In particular, the triangulated category  $kk(\mathfrak{R})$  of [Cortiñas and Thom 2007] is fully faithfully embedded into  $\mathrm{SH}_{S^1, \mathbb{G}}(\mathfrak{R})$  by means of a contravariant functor. As an application we construct an explicit fibrant  $(S^1, \mathbb{G})$ -bispectrum representing homotopy  $K$ -theory in the sense of [Weibel 1989].

Throughout this section we assume that  $\mathfrak{R}$  is a small tensor closed,  $T$ -,  $\Gamma$ - and  $\tau_0$ -closed admissible category of  $k$ -algebras. It follows that  $\sigma A, \Sigma A, M_{\infty} A \in \mathfrak{R}$  for all  $A \in \mathfrak{R}$ .

Let  $\mathrm{Sp}_{\infty, \mathbb{G}}^{\Sigma}(\mathfrak{R})$  denote the model category of (usual, nonsymmetric)  $\mathbb{G}$ -spectra in  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$ . Using Hovey's notation [2001],  $\mathrm{Sp}_{\infty, \mathbb{G}}^{\Sigma}(\mathfrak{R}) = \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R}), - \otimes \mathbb{G})$ .

**Proposition 9.1.** *The functor*

$$- \otimes \mathbb{G} : \mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R}) \rightarrow \mathrm{Sp}_{\infty, \mathbb{G}}^{\Sigma}(\mathfrak{R})$$

*and the canonical functor*

$$F_{0, \mathbb{G}} = \Sigma_{\mathbb{G}}^{\infty} : \mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R}) \rightarrow \mathrm{Sp}_{\infty, \mathbb{G}}^{\Sigma}(\mathfrak{R})$$

*are left Quillen equivalences.*

*Proof.* We first observe that  $- \otimes \mathbb{G}$  is a left Quillen equivalence on  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$  if and only if so is  $- \otimes \Sigma^{\infty} \mathbb{G}$ . By [Cortiñas and Thom 2007, Section 4] there is an extension

$$M_{\infty} k \mapsto \tau_0 \twoheadrightarrow \sigma.$$

It follows from [Cortiñas and Thom 2007, Lemma 7.3.2] that  $\Sigma^{\infty}(r(\tau_0)) = 0$  in  $DK(\mathfrak{R})$ , and hence  $\Sigma^{\infty}(r(\tau_0))$  is weakly equivalent to zero in  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$ .

The extension above yields therefore a zigzag of weak equivalences between cofibrant objects in  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$  from  $\Sigma^{\infty}(r(M_{\infty} k))$  to  $\Sigma^{\infty} \mathbb{G} \wedge S^1$ . Since  $\Sigma^{\infty}(r(M_{\infty} k))$  is weakly equivalent to the monoidal unit  $\Sigma^{\infty}(r(k))$ , we see that  $\Sigma^{\infty}(r(k))$  is zigzag weakly equivalent to  $(\Sigma^{\infty} \mathbb{G}) \wedge S^1$  in the category of cofibrant objects in  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$ .

Since  $\Sigma^{\infty}(r(k))$  is a monoidal unit in  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$ , then  $- \otimes \Sigma^{\infty}(r(k))$  is a left Quillen equivalence on  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$ , and hence so is  $- \otimes ((\Sigma^{\infty} \mathbb{G}) \wedge S^1)$ . But  $- \wedge S^1$  is a left Quillen equivalence on  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$ . Therefore  $- \otimes \Sigma^{\infty} \mathbb{G}$  is a left Quillen equivalence by [Hovey 1999, Corollary 1.3.15].

The fact that the canonical functor

$$F_{0, \mathbb{G}} : \mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R}) \rightarrow \mathrm{Sp}_{\infty, \mathbb{G}}^{\Sigma}(\mathfrak{R})$$

is a left Quillen equivalence now follows from [Hovey 2001, Section 5.1]. □

Denote the homotopy category of  $\mathrm{Sp}_{\infty, \mathbb{G}}^{\Sigma}(\mathfrak{R})$  by  $\mathrm{SH}_{S^1, \mathbb{G}}^{\Sigma, \infty}(\mathfrak{R})$ .

**Corollary 9.2.** *The canonical functor*

$$F_{0, \mathbb{G}} = \Sigma_{\mathbb{G}}^{\infty} : DK(\mathfrak{R}) \rightarrow \mathrm{SH}_{S^1, \mathbb{G}}^{\Sigma, \infty}(\mathfrak{R})$$

*is an equivalence of triangulated categories.*

Recall that  $\mathrm{Sp}_{\infty}^{\Sigma}(\mathfrak{R})$  is the Bousfield localisation of  $\mathrm{Sp}^{\Sigma}(\mathfrak{R})$  with respect to

$$\{F_s(r(M_{\infty} A)) \rightarrow F_s(rA) \mid A \in \mathfrak{R}, s \geq 0\}.$$

It follows that the induced triangulated functor is fully faithful

$$DK(\mathfrak{R}) \rightarrow \mathrm{SH}_{S^1}^\Sigma(\mathfrak{R}).$$

In a similar fashion,  $\mathrm{Sp}_{\infty, \mathbb{G}}^\Sigma(\mathfrak{R})$  can be obtained from  $\mathrm{Sp}_{\mathbb{G}}^\Sigma(\mathfrak{R})$  by Bousfield localisation with respect to

$$\{F_{k, \mathbb{G}}(F_s(r(M_\infty A))) \rightarrow F_{k, \mathbb{G}}(F_s(rA)) \mid A \in \mathfrak{R}, k, s \geq 0\}.$$

We summarise all of this together with Proposition 9.1 as follows.

**Theorem 9.3.** *There is an adjoint pair of triangulated functors*

$$\Phi : \mathrm{SH}_{S^1, \mathbb{G}}^\Sigma(\mathfrak{R}) \rightleftarrows DK(\mathfrak{R}) : \Psi$$

such that  $\Psi$  is fully faithful. Moreover,  $\mathcal{T} = \mathrm{Ker} \Phi$  is the localising subcategory of  $\mathrm{SH}_{S^1, \mathbb{G}}^\Sigma(\mathfrak{R})$  generated by the compact objects

$$\{\mathrm{cone}(F_{k, \mathbb{G}}(F_s(r(M_\infty A))) \rightarrow F_{k, \mathbb{G}}(F_s(rA))) \mid A \in \mathfrak{R}\}$$

and  $DK(\mathfrak{R})$  is triangle equivalent to  $\mathrm{SH}_{S^1, \mathbb{G}}^\Sigma(\mathfrak{R})/\mathcal{T}$ .

**Corollary 9.4.** *There is a contravariant fully faithful triangulated functor*

$$kk(\mathfrak{R}) \rightarrow \mathrm{SH}_{S^1, \mathbb{G}}(\mathfrak{R}).$$

*Proof.* This follows from Theorems 7.4 and 9.3. □

Let  $\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R})$  denote the model category of  $\mathbb{G}$ -spectra in  $\mathrm{Sp}_\infty(\mathfrak{R})$ . Using Hovey's notation [2001], we have

$$\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R}) = \mathrm{Sp}^{\mathbb{N}}(\mathrm{Sp}_\infty(\mathfrak{R}), - \otimes \mathbb{G}).$$

As above, there is a Quillen equivalence

$$V : \mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R}) \rightleftarrows \mathrm{Sp}_{\infty, \mathbb{G}}^\Sigma(\mathfrak{R}) : U,$$

where  $U$  is the forgetful functor. It induces an equivalence of triangulated categories

$$V : \mathrm{SH}_{S^1, \mathbb{G}}^\infty(\mathfrak{R}) \rightleftarrows \mathrm{SH}_{S^1, \mathbb{G}}^{\Sigma, \infty}(\mathfrak{R}) : U,$$

where the left hand side is the homotopy category of  $\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R})$ .

Given  $A \in \mathfrak{R}$ , consider a  $(S^1, \mathbb{G})$ -bispectrum  $\mathbb{K}\mathbb{G}^{st}(A, -)$  which we define at each  $B \in \mathfrak{R}$  as

$$\mathrm{colim}_n(\mathbb{K}\mathbb{G}(A, B) \rightarrow \mathbb{K}\mathbb{G}(A, M_\infty k \otimes B) \rightarrow \mathbb{K}\mathbb{G}(A, M_\infty^2 k \otimes B) \rightarrow \dots).$$



It can also be presented as the array

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 \mathbb{K}^{st}(\sigma^2 A, B) : & \mathcal{H}^{st}(\sigma^2 A, B) & \mathcal{H}^{st}(J\sigma^2 A, B) & \mathcal{H}^{st}(J^2\sigma^2 A, B) & \dots & & \\
 \mathbb{K}^{st}(\sigma A, B) : & \mathcal{H}^{st}(\sigma A, B) & \mathcal{H}^{st}(J\sigma A, B) & \mathcal{H}^{st}(J^2\sigma A, B) & \dots & & \\
 \mathbb{K}^{st}(A, B) : & \mathcal{H}^{st}(A, B) & \mathcal{H}^{st}(JA, B) & \mathcal{H}^{st}(J^2A, B) & \dots & & 
 \end{array}$$

It follows from Theorem 6.1 that the canonical map of bispectra

$$\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA \rightarrow \mathbb{K}\mathbb{G}^{st}(A, -)$$

is a level weak equivalence in  $\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R})$ . In fact we can say more. We shall show below that  $\mathbb{K}\mathbb{G}^{st}(A, -)$  is a fibrant bispectrum and this arrow is a fibrant replacement of  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA$  in  $\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R})$ . To this end we have to prove the cancellation theorem for the  $S^1$ -spectrum  $\mathbb{K}^{st}(A, -)$ . The cancellation theorem for  $K$ -theory of algebraic varieties was proved in [Garkusha and Panin 2015]. It is also reminiscent of the cancellation theorem for motivic cohomology proved by Voevodsky [2010a].

**Theorem 9.5** (cancellation for  $K$ -theory). *Each structure map of the bispectrum  $\mathbb{K}\mathbb{G}^{st}(A, -)$*

$$\mathbb{K}^{st}(\sigma^n A, -) \rightarrow \Omega_{\mathbb{G}} \mathbb{K}^{st}(\sigma^{n+1} A, -), \quad n \geq 0,$$

*is a weak equivalence of fibrant  $S^1$ -spectra.*

*Proof.* It follows from Proposition 9.1 that the functor

$$- \otimes_{\mathbb{G}} : \mathrm{Sp}_{\infty}(\mathfrak{R}) \rightarrow \mathrm{Sp}_{\infty}(\mathfrak{R})$$

is a left Quillen equivalence. It remains to apply Theorem 6.1. □

**Corollary 9.6.** *For any  $A \in \mathfrak{R}$  the bispectrum  $\mathbb{K}\mathbb{G}^{st}(A, -)$  is fibrant in  $\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R})$ . Moreover, the canonical map of bispectra*

$$\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA \rightarrow \mathbb{K}\mathbb{G}^{st}(A, -)$$

*is a fibrant resolution for  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA$  in  $\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R})$ .*

The following result says that the bispectrum  $\mathbb{K}\mathbb{G}^{st}(A, -)$  is  $(2, 1)$ -periodic and represents stable algebraic Kasparov  $K$ -theory (cf. [Voevodsky 1998, Theorems 6.8 and 6.9]).

**Theorem 9.7.** *For any algebras  $A, B \in \mathfrak{R}$  and any integers  $p, q$  there is an isomorphism of abelian groups*

$$\begin{aligned}
 & \pi_{p,q}(\mathbb{K}\mathbb{G}^{st}(A, B)) \\
 & \cong \mathrm{Hom}_{\mathrm{SH}_{S^1, \mathbb{G}}(\mathfrak{R})}(\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rB \otimes S^{p-q} \otimes \mathbb{G}^q, \mathbb{K}\mathbb{G}^{st}(A, -)) \cong \mathbb{K}_{p-2q}^{st}(A, B).
 \end{aligned}$$

In particular,

$$\pi_{p,q}(\mathbb{K}\mathbb{G}^{st}(A, B)) \cong \pi_{p+2,q+1}(\mathbb{K}\mathbb{G}^{st}(A, B)).$$

*Proof.* By Corollary 9.6 the bispectrum  $\mathbb{K}\mathbb{G}^{st}(A, -)$  is a fibrant replacement for  $\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA$  in  $\mathrm{Sp}_{\infty, \mathbb{G}}(\mathfrak{R})$ . Therefore,

$$\pi_{p,q}(\mathbb{K}\mathbb{G}^{st}(A, B)) \cong \mathrm{Hom}_{\mathrm{SH}_{S^1, \mathbb{G}}^{\infty}(\mathfrak{R})}(\Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rB \otimes S^{p-q} \otimes \mathbb{G}^q, \Sigma_{\mathbb{G}}^{\infty} \Sigma^{\infty} rA).$$

Corollary 9.2 implies that the right hand side is isomorphic to

$$DK(\mathfrak{R})(M_K(B) \otimes S^{p-q} \otimes \mathbb{G}^q, M_K(A)).$$

On the other hand,

$$\begin{aligned} DK(\mathfrak{R})(M_K(B) \otimes S^{p-q} \otimes \mathbb{G}^q, M_K(A)) \\ \cong DK(\mathfrak{R})(M_K(B) \otimes S^{p-2q} \otimes S^q \otimes \mathbb{G}^q, M_K(A)). \end{aligned}$$

The proof of Proposition 9.1 implies  $\Sigma^{\infty}(S^1 \otimes \mathbb{G})$  is isomorphic to the monoidal unit. Therefore,

$$DK(\mathfrak{R})(M_K(B) \otimes S^{p-2q} \otimes S^q \otimes \mathbb{G}^q, M_K(A)) \cong DK(\mathfrak{R})(M_K(B)[p-2q], M_K(A)).$$

Our statement now follows from Theorem 7.2.  $\square$

The next statement says that the bispectrum  $\mathbb{K}\mathbb{G}^{st}(k, B)$  gives a model for homotopy  $K$ -theory in the sense of [Weibel 1989] (compare [Voevodsky 1998, Theorem 6.9]).

**Corollary 9.8.** *For any algebra  $B \in \mathfrak{R}$  and any integers  $p, q$  there is an isomorphism*

$$\pi_{p,q}(\mathbb{K}\mathbb{G}^{st}(k, B)) \cong KH_{p-2q}(B).$$

*Proof.* This follows from the preceding theorem and [Garkusha 2014, 9.11].  $\square$

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