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Birational motives
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We define a category of pure birational motives over a field, depending on the choice of an adequate equivalence relation on algebraic cycles. It is obtained by "killing" the Lefschetz motive in the corresponding category of effective motives. For rational equivalence, it encompasses Bloch's decomposition of the diagonal. We study the induced Chow–Künneth decompositions in this category, and establish relationships with Rost's cycle modules and the Albanese functor for smooth projective varieties.

Introduction

In the preprint [Kahn and Sujatha 2002], we toyed with birational ideas in three areas of algebraic geometry: plain varieties, pure motives in the sense of Grothendieck, and triangulated motives in the sense of Voevodsky. These three themes are finally treated separately in revised versions. The first one is the object of [Kahn

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and Sujatha 2015a]; the second one is the object of the present paper; the third one is the object of [Kahn and Sujatha 2015b].

We work over a field F. Recall that we introduced in [Kahn and Sujatha 2015a] two "birational" categories. The first, $\mathbf{place}(F)$, has for objects the function fields over F and for morphisms the F-places. The second one is the Gabriel–Zisman localisation of the category $\mathbf{Sm}(F)$ of smooth F-varieties obtained by inverting birational morphisms [Gabriel and Zisman 1967, Chapter 1]; we denote this category by $S_b^{-1}\mathbf{Sm}(F)$.

We may also invert stable birational morphisms: those which are dominant and induce a purely transcendental extension of function fields, and invert the corresponding morphisms in $\mathbf{place}(F)$. We denote the sets of such morphisms by S_r.

In order to simplify the exposition, let us assume that F is of characteristic 0. Then the main results of [Kahn and Sujatha 2015a] and its predecessor [Kahn and Sujatha 2007] can be summarised in a diagram

$$\begin{array}{ccccc} \mathbf{place}(F)^{\mathrm{op}} & \longrightarrow & S_b^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) & \xrightarrow{\sim} & S_b^{-1}\mathbf{Sm}(F) \\ \downarrow & & \wr\downarrow & & \wr\downarrow \\ S_r^{-1}\mathbf{place}(F)^{\mathrm{op}} & \to & S_r^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) & \xrightarrow{\sim} & S_r^{-1}\mathbf{Sm}(F) \end{array}$$

where $\mathbf{Sm}^{\mathrm{proj}}(F)$ is the full subcategory of smooth projective varieties and the symbols $\sim$ denote equivalences of categories; see [Kahn and Sujatha 2007, Proposition 8.5] and [Kahn and Sujatha 2015a, Theorems 1.7.2 and 4.2.4].

Moreover, if X is smooth and Y is smooth proper, then $\mathrm{Hom}(X, Y) = Y(F(X))/R$ in $S_b^{-1}\mathbf{Sm}(F)$, where R is R-equivalence [ibid., Theorem 6.6.3].

In this paper, we consider the effect of inverting birational morphisms in categories of *effective pure motives*. For simplicity, let us still assume $\operatorname{char} F = 0$, and consider only the category of effective Chow motives $\mathbf{Chow}^{\mathrm{eff}}(F)$, defined by using algebraic cycles modulo rational equivalence. The graph functor then induces a commutative square (compare (5.1.1))

$$\begin{array}{ccc} S_b^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) & \to & S_b^{-1}\mathbf{Chow}^{\mathrm{eff}}(F) \\ \wr\downarrow & & \downarrow \\ S_r^{-1}\mathbf{Sm}^{\mathrm{proj}}(F) & \to & S_r^{-1}\mathbf{Chow}^{\mathrm{eff}}(F) \end{array}$$

One can expect that the right vertical functor is an equivalence of categories, and indeed this is not difficult to prove (Corollary 2.2.5(b)). But we have two other descriptions of this category of "birational motives":

- The functor $\mathbf{Chow}^{\mathrm{eff}}(F) \to S_b^{-1}\mathbf{Chow}^{\mathrm{eff}}(F)$ is full, and its kernel is the ideal $\mathcal{L}_{\mathrm{rat}}$ of morphisms which factor through some object of the form $M \otimes \mathbb{L}$, where $\mathbb{L}$ is the *Lefschetz motive* [ibid].

- If X, Y are smooth projective varieties, then $\mathcal{L}_{\mathrm{rat}}(h(X), h(Y))$ coincides with the group of Chow correspondences represented by algebraic cycles on $X \times Y$ whose irreducible components are not dominant over X (Theorem 2.4.2).

As a consequence, the group of morphisms from $h(X)$ to $h(Y)$ in $S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F)$ is isomorphic to $CH_0(Y_{F(X)})$. Given the similar description of Hom sets in

$$S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F)$$

recalled above, this places the classical map

$$Y(F(X))/R \to CH_0(Y_{F(X)})$$

in a categorical context.

Note that, by [Kahn and Sujatha 2015a, Theorem 8.5.1(b)], if $X \simeq \operatorname{Spec} F$ in $S_b^{-1}\,\mathbf{Sm}$ then X must be rationally connected; on the other hand, there are surfaces of general type with trivial birational motive, see Remarks 3.1.5(1) and (3). So the birational motive of a smooth projective variety detects much less geometry than its class in $S_b^{-1}\,\mathbf{Sm}$, but on the other hand it is much more computable.

This paper is organised as follows. In Section 1 we review pure motives. In Section 2 we study pure birational motives, in greater generality than outlined in this introduction. In particular, many results are valid for other adequate equivalence relations than rational equivalence, see Section 2.3; moreover, most results extend to characteristic p if p is invertible in the ring of coefficients, by using the de Jong–Gabber alteration theorem [Illusie and Temkin 2014]; see Theorem 2.4.2.

Section 3 consists of examples. We study varieties whose birational motive is trivial, in the line of the remarks above. We also study the Chow–Künneth decomposition in the category of birational motives, special attention being devoted to the case of complete intersections.

Let $\mathbf{Chow}^{\mathrm{o}}(F)$ denote the pseudoabelian envelope of $S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F)$. In Section 4, we examine two questions: the existence of a right adjoint to the projection functor $\mathbf{Chow}^{\mathrm{eff}}(F) \to \mathbf{Chow}^{\mathrm{o}}(F)$ (and similarly for more general adequate equivalences), and whether pseudoabelian completion is really necessary. It turns out that the answer to the first question is negative (Theorems 4.3.2 and 4.3.3; this is related to the nontriviality of the Griffiths group for some 3-folds) and the answer to the second question is positive with rational coefficients under a nilpotence conjecture (Conjecture 3.3.1). We can get an unconditional positive answer to the second question if we restrict to a suitable type of motives (Proposition 4.4.1 and Example 4.4.2).

In Section 5, we define a functor $S_r^{-1}\,\mathbf{field}(F)^{\mathrm{op}} \to S_r^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F, \mathbb{Q})$ in characteristic p, using de Jong's theorem again. Here $\mathbf{field}(F)$ denotes the subcategory of $\mathbf{place}(F)$ with the same objects but morphisms restricted to field extensions (Proposition 5.1.4).

We end this paper by relating the previous constructions to more classical objects. In Section 6 we relate birational motives to cycle cohomology [Rost 1996], expanding a bit on previous results by Rost and Merkurjev [2001; 2008]. In Section 7, we define a tensor additive category $\mathbf{AbS}(F)$ of *locally abelian schemes*, whose objects are those F-group schemes that are extensions of a lattice (i.e., locally isomorphic for the étale topology to a free finitely generated abelian group) by an abelian variety. We then show in Section 8 that the classical construction of the Albanese variety of a smooth projective variety extends to a tensor functor

$$\mathrm{Alb} : \mathbf{Chow}^{\mathrm{o}}(F) \to \mathbf{AbS}(F),$$

which becomes full and essentially surjective after tensoring morphisms with $\mathbb{Q}$ (Proposition 8.2.1). So, one could say that $\mathbf{AbS}(F)$ is the *representable part* of $\mathbf{Chow}^{\mathrm{o}}(F)$. We also show that, after tensoring with $\mathbb{Q}$, Alb has a right adjoint which identifies $\mathbf{AbS}(F) \otimes \mathbb{Q}$ with the thick subcategory of $\mathbf{Chow}^{\mathrm{o}}(F) \otimes \mathbb{Q}$ generated by motives of varieties of dimension ≤ 1.

Some results of the preliminary version [Kahn and Sujatha 2002] of this work were used in other papers, namely [Kahn et al. 2007; Kahn 2009], and we occasionally refer to these papers to ease the exposition. Here is a correspondence guide between the results from [Kahn and Sujatha 2002] used in these papers and those in the present version:

- In [Kahn 2009], Lemma 7.2 uses [Kahn and Sujatha 2002, Lemmas 5.3 and 5.4], which correspond to Proposition 2.3.5 and Theorem 2.4.2 of the present paper. The reader will verify that the proofs of Proposition 2.3.5 and Theorem 2.4.2 are the same as those of [Kahn and Sujatha 2002, Lemmas 5.3 and 5.4], mutatis mutandis, and do not use any result from [Kahn 2009].
- In [Kahn et al. 2007], Lemma 7.5.3 uses the same references; the same comment as above applies. Moreover, Proposition 9.5 of [Kahn and Sujatha 2002] is used on pp. 174–175 of [Kahn et al. 2007]; this result is now Theorem 8.2.4. Again, its proof is identical to the one in the preliminary version and does not use results from [Kahn et al. 2007].

The idea of considering birational Chow correspondences, which yield here a category in which $\mathrm{Hom}([X],[Y]) = CH_0(Y_{F(X)})$ for two smooth projective varieties X, Y, goes back to S. Bloch's method of "decomposition of the diagonal" in [Bloch 2010, Appendix to Lecture 1] (see also [Bloch and Srinivas 1983]). He attributes the idea of considering the generic point of a smooth projective variety X as a 0-cycle over its function field to Colliot-Thélène; here, this corresponds to the identity endomorphism of $h^{\mathrm{o}}(X) \in \mathbf{Chow}^{\mathrm{o}}(F)$. We realised the connection with Bloch's ideas after reading H. Esnault's article [2003], and this led to another proof of her theorem by the present birational techniques in [Kahn 2009]. M. Rost has considered this category independently [Merkurjev 2001]; this was pointed out to us by N. Karpenko.

1. Review of pure motives

In this section, we recall the definition of categories of pure motives in a way which is suited to our needs. A slight variance to the usual exposition is the notion of *adequate pair*, which is a little more precise than the notion of adequate equivalence relation (it explicitly takes the coefficients into account).

We adopt the covariant convention, for future comparison with Voevodsky's triangulated categories of motives: here, the functor which sends a smooth projective variety to its motive is covariant. For a dictionary between the covariant and contravariant conventions, the reader may refer to [Kahn et al. 2007, Lemma 7.1.2].

1.1. ***Adequate pairs.*** We give ourselves

- a commutative ring of coefficients A;
- an adequate equivalence relation $\sim$ on algebraic cycles with coefficients in A [Samuel 1960].

We refer to $(A, \sim)$ as an *adequate pair*. Classical examples for $\sim$ are rat (rational equivalence), alg (algebraic equivalence), num (numerical equivalence), $\sim_H$ (homological equivalence relative to a fixed Weil cohomology theory H). A less classical example is Voevodsky's smash-nilpotence tnil [1995]; see [André and Kahn 2002, Example 7.4.3] (a cycle α is smash-nilpotent if $\alpha^{\otimes n} \sim_{\mathrm{rat}} 0$ for some $n > 0$). We then have a notion of domination $(A, \sim) \geq (A, \sim')$ if $\sim$ is finer than $\sim'$ (i.e., the groups of cycles modulo $\sim$ surjects onto the one for $\sim'$). It is well known that $(A, \mathrm{rat}) \geq (A, \sim)$ for any $\sim$ (see [Fulton 1984, Example 1.7.5]), and that $(A, \sim) \geq (A, \mathrm{num}_A)$ if A is a field.

Since the issue of coefficients is sometimes confusing, the following remarks may be helpful. Given a pair $(A, \sim)$ and a commutative A-algebra B, we get a new pair $B \otimes_A (A, \sim)$ by tensoring algebraic cycles with B: for example, $(A, \sim) = A \otimes_{\mathbb{Z}} (\mathbb{Z}, \sim)$ for $\sim =$ rat, alg or tnil by definition. On the other hand, given a pair $(B, \sim)$ and a ring homomorphism $A \to B$ we get a "restriction of scalars" pair $(A, \sim_{|A})$ by considering cycles with coefficients in A which become ~ 0 after tensoring with B: for example, if H is a Weil cohomology theory with coefficients in K, this applies to any ring homomorphism $A \to K$. Obviously

$$B \otimes_A (A, \sim_{|A}) \geq (B, \sim),$$

but this need not be an equality in general.

In the case of numerical equivalence (a cycle with coefficients in A is numerically equivalent to 0 if the degree of its intersection with any cycle of complementary dimension in good position is 0), we have $B \otimes_A (A, \mathrm{num}_A) \geq (B, \mathrm{num}_B)$, with equality if B is flat over A.

Given a pair $(A, \sim)$, to any smooth projective F-variety X and integer $n \geq 0$ we may associate its group of cycles of codimension n with coefficients in A modulo $\sim$, which will be denoted by $\mathcal{Z}^n_\sim(X, A)$. If X has pure dimension d, we also denote this group by $\mathcal{Z}^\sim_{d-n}(X, A)$.

1.2. ***Smooth projective varieties, connected and nonconnected.*** In [Kahn and Sujatha 2015a] we were only considering (connected) varieties over F. Classically, pure motives are defined using not necessarily connected smooth projective varieties. One could base the treatment on connected smooth varieties, but this would introduce problems with the tensor product, since a product of connected varieties need not be connected in general (e.g., if neither of them is geometrically connected). Thus we prefer to use here:

Definition 1.2.1. We write $\mathbf{Sm}_\amalg(F)$ for the category of smooth separated schemes of finite type over F. For $\% \in \{\mathrm{prop}, \mathrm{qp}, \mathrm{proj}\}$, we write $\mathbf{Sm}^\%_\amalg(F)$ for the full subcategory of $\mathbf{Sm}_\amalg(F)$ consisting of proper, quasiprojective or projective varieties.

Unlike their counterparts considered in [Kahn and Sujatha 2015a], these categories enjoy finite products and coproducts.

The following lemma is clear.

Lemma 1.2.2. *The categories considered in Definition 1.2.1 are the "finite coproduct envelopes" of those considered in* [Kahn and Sujatha 2015a], *in the sense of* [Kahn and Sujatha 2007, Proposition 6.1].

1.3. ***Review of correspondences.*** We associate to two smooth projective varieties X, Y the group $\mathcal{Z}^{\dim Y}_\sim(X \times Y, A)$ of correspondences from X to Y relative to $(A, \sim)$. The composition of correspondences is defined as follows:[1] if X, Y, Z are smooth projective and $(\alpha, \beta) \in \mathcal{Z}^{\dim Y}_\sim(X \times Y, A) \times \mathcal{Z}^{\dim Z}_\sim(Y \times Z, A)$, then

$$\beta \circ \alpha = (p_{XZ})_*(p^*_{XY}\alpha \cdot p^*_{YZ}\beta),$$

where p_{XY}, p_{YZ} and p_{XZ} denote the partial projections from $X \times Y \times Z$ onto two-fold factors.

We then get an A-linear tensor (i.e., symmetric monoidal) category $\mathbf{Cor}_\sim(F, A)$. The graph map defines a *covariant* functor

$$\mathbf{Sm}^{\mathrm{proj}}_\amalg(F) \to \mathbf{Cor}_\sim(F, A), \quad X \mapsto [X], \tag{1.3.1}$$

[1] We follow here the convention of Voevodsky [2000]. It is also the one used by Fulton [1984, Section 16]. See [Kahn et al. 2007, Lemma 7.1.2].

so that $[X \amalg Y] = [X] \oplus [Y]$, and $[X \times Y] = [X] \otimes [Y]$ for the tensor structure. The unit object is $\mathbb{1} = [\operatorname{Spec} F]$.

If $f : X \to Y$ is a morphism of smooth varieties, let Γ_f denote its graph and $[\Gamma_f]$ denote the class of Γ_f in $\mathcal{Z}^{\dim Y}_\sim(X \times Y)$. We write f_* for the correspondence $[\Gamma_f] : [X] \to [Y]$ (the image of f under the functor (1.3.1)). Note that if $f : X \to Y$ and $g : Y \to Z$ are two morphisms of smooth projective varieties, then the cycles $\Gamma_f \times Z$ and $X \times \Gamma_g$ on $X \times Y \times Z$ intersect properly, so that $g_* \circ f_*$ is well defined as a cycle and not just as an equivalence class of cycles; the equation $g_* \circ f_* = (g \circ f)_*$ is an equality of cycles. (This is a very special case of the composition of finite correspondences; see [Mazza et al. 2006, Lemma 1.7].)

1.4. ***The correspondence attached to a rational map.*** We first define rational maps between not necessarily connected smooth varieties X, Y in the obvious way: it is a morphism from a suitable *dense* open subset of X to Y. Like morphisms, rational maps split as disjoint unions of "connected" rational maps. A rational map f is *dominant* if all its connected components are dominant and if the image of f meets all connected components of Y.

Let $f : X \dashrightarrow Y$ be a rational map between two smooth projective varieties X, Y. To f we associated in [Kahn and Sujatha 2015a, Section 6.3] a morphism in the category $S_b^{-1}\,\mathbf{Sm}$. In the case of Chow motives, we can do better: define the correspondence $f_* : [X] \to [Y]$ in $\mathbf{Cor}_\sim(F, A)$ as the closure of the graph of f inside $X \times Y$. The formula $g_* \circ f_* = (g \circ f)_*$ need not be valid in general, even if $g \circ f$ is defined (but see Proposition 2.3.8 below). Yet we have:

Lemma 1.4.1. *Let $X \overset{f}{\dashrightarrow} Y \overset{g}{\to} Z$ be a diagram of smooth projective varieties, where f is a rational map and g is a morphism. Then we have an equality of cycles*

$$g_* \circ f_* = (g \circ f)_*$$

in $\mathcal{Z}^{\dim Z}(X \times Z)$.

Proof. Let U be an open subset of X on which f, hence also $g \circ f$, is defined. As explained in Section 1.3, we have an equality of reduced closed subschemes

$$\Gamma_{g \circ f} = p_{UZ}(\Gamma_f \times Z \cap X \times \Gamma_g).$$

Since Y is proper, $p_{UZ}(\Gamma_f \times Z \cap X \times \Gamma_g)$ is dense in $p_{XZ}(\overline{\Gamma}_f \times Z \cap X \times \Gamma_g) = g_* \circ f_*$, hence the conclusion. □

1.5. ***Effective pure motives.*** We now define as usual the category of effective pure motives $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ relative to $(A, \sim)$ as the pseudoabelian envelope of $\mathbf{Cor}_\sim(F, A)$. We denote the composition of (1.3.1) with the pseudoabelianisation functor by $h_\sim$. If $\sim = \mathrm{rat}$, we usually abbreviate $h_\sim$ to h.

In $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ we have

- $h_\sim(\operatorname{Spec} F) = \mathbb{1}$ (the unit object for the tensor structure);
- $h_\sim(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$, where $\mathbb{L}$ is the *Lefschetz motive*.

If $n \geq 0$, we write $M(n)$ for the motive $M \otimes \mathbb{L}^{\otimes n}$ (beware that the "standard" notation is $M(-n)$!)

We then have the formula, for two smooth projective X, Y and integers $p, q \geq 0$,

$$\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)(h_\sim(X)(p), h_\sim(Y)(q)) = \mathcal{Z}^{\dim Y + q - p}_\sim(X \times Y). \tag{1.5.1}$$

In particular, the endofunctor $- \otimes \mathbb{L}$ of $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ is fully faithful.

If $f : X \to Y$ is a morphism, then the correspondence $[{}^t\Gamma_f] \in \mathcal{Z}^{\dim Y}(Y \times X)$ obtained by the "switch" defines a morphism $f^* : h_\sim(Y)(\dim X) \to h_\sim(X)(\dim Y)$, i.e., from $h_\sim(Y)$ to $h_\sim(X)(\dim Y - \dim X)$ or from $h_\sim(Y)(\dim X - \dim Y)$ to $h_\sim(X)$ according to the sign of $\dim X - \dim Y$. In particular, if f has relative dimension 0 then f^* maps $h_\sim(Y)$ to $h_\sim(X)$. We similarly define f^* for a rational map f.

We recall the well-known lemma:

Lemma 1.5.2. *Suppose that f is generically finite of degree d. Then $f_* \circ f^* = d 1_Y$.*

Proof. It suffices to prove this for the action on cycles, and then the lemma follows by Manin's identity principle [Scholl 1994, Section 2]. Let $\alpha \in \mathcal{Z}^*_\sim(Y, A)$. By the projection formula,

$$f_* f^*(\alpha) = \alpha \cdot f_*(1).$$

But $f_*(1) \in \mathcal{Z}^0_\sim(Y, A)$ may be computed after restriction to any open subset U of X, and for U small enough it is clear that $f_*(1) = d$. □

1.6. *Pure motives.* The category $\mathbf{Mot}_\sim(F, A)$ is now obtained from $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ by inverting the endofunctor $- \otimes \mathbb{L}$, i.e., adjoining a $\otimes$-quasi-inverse $\mathbb{T}$ of $\mathbb{L}$ (the Tate motive) to $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$. The resulting category is rigid and the functor $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A) \to \mathbf{Mot}_\sim(F, A)$ is fully faithful; we refer to [Scholl 1994] for details. We still write $h_\sim(X)$ for the image of $h_\sim(X)$ in $\mathbf{Mot}_\sim(F, A)$.

1.7. *Pure motives and purely inseparable extensions.* This subsection will be needed for the proof of Remarks 2.3.10 below. It shows that extending scalars along a purely inseparable extension is harmless as long as the exponential characteristic is inverted.

Lemma 1.7.1. *Let $f : X \to Y$ be a finite, flat and radicial morphism* [Grothendieck and Dieudonné 1971, Définition 3.7.2] *between smooth projective F-varieties. Let $(A, \sim)$ be an adequate pair, with p invertible in A (where p is the exponential characteristic of F).*

(a) $f_* : \mathcal{Z}^\sim_*(X, A) \to \mathcal{Z}^\sim_*(Y, A)$ *is an isomorphism.*

(b) $f_* : h(X) \to h(Y)$ *is an isomorphism in* $\mathbf{Cor}_\sim(F, A)$.

Proof. Let p^n be the generic degree of f. We have $f_* f^* = f^* f_* = p^n$ (on the level of algebraic cycles), hence (a). Part (b) follows by Manin's identity principle (Yoneda lemma). □

Proposition 1.7.2. *Let K/F be a purely inseparable extension. Then, for any adequate pair $(A, \sim)$ as in Lemma 1.7.1, the extension of scalars functors*

$$\mathbf{Cor}_\sim(F, A) \to \mathbf{Cor}_\sim(K, A),$$
$$\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A) \to \mathbf{Mot}^{\mathrm{eff}}_\sim(K, A),$$
$$\mathbf{Mot}_\sim(F, A) \to \mathbf{Mot}_\sim(K, A),$$

are equivalences of categories.

Proof. It suffices to show this for the first functor. Let X, Y be two smooth projective F-varieties. Then, for any finite subextension L/F of K/F, the morphism $(X \times_F Y)_L \to X \times_F Y$ is finite, flat and radicial; by Lemma 1.7.1(a) and a limit argument, this implies that the functor is fully faithful. For its essential surjectivity, we steal an idea from [Lang 1959, Chapter VIII, Section 1, proof of Theorem 2]. Let X be a smooth projective K-variety. Then X is defined over a finite subextension L/F of K/F. Let $p^n = [L : F]$, and let Φ_L be the absolute Frobenius of L. The relative Frobenius morphism (an L-morphism)

$$X \to (\Phi_L^n)^* X$$

is finite, flat[2] and radicial; by Lemma 1.7.1(b), $h(X) \to h((\Phi_L^n)^* X)$ is an isomorphism in $\mathbf{Cor}_\sim(L, A)$, hence also in $\mathbf{Cor}_\sim(K, A)$. Since $\Phi_L^n : \operatorname{Spec} L \to \operatorname{Spec} L$ factors through $\operatorname{Spec} F$, $(\Phi_L^n)^* X$ is defined over F, proving that the functor is essentially surjective. □

1.8. *Image motives.* In the study of projective homogeneous varieties, several people (starting with Vishik) have been led to introduce the following:

Definition 1.8.1. Let X be a smooth projective variety. We write

$$\bar{\mathcal{Z}}^*_\sim(X, A) = \operatorname{Im}(\mathcal{Z}^*_\sim(X, A) \to \mathcal{Z}^*_\sim(X_{F_s}, A)),$$

where F_s is a separable closure of F.

Using correspondences based on these groups, we define $\overline{\mathbf{Mot}}_\sim(F, A)$, etc. This is mainly interesting when $A = \mathbb{Z}$ or $\mathbb{Z}/p$: for $A = \mathbb{Q}$ the extension of scalars map is injective (by a transfer argument).

[2]To see this, one may use the fact that X is locally isomorphic to $\mathbb{A}^n$ for the étale topology.

2. Pure birational motives

2.1. ***First approach: localisation.*** The first idea to define a notion of pure birational motives is to localise $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$ with respect to stable birational morphisms as in [Kahn and Sujatha 2015a], hence getting a functor

$$S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}_{\amalg}(F) \to S_r^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A).$$

This idea turns out to be the good one in all important cases, but to see this we first need some preliminary work. We start by reviewing the sets of morphisms used in [Kahn and Sujatha 2015a, Section 1.7]:

- S_b: birational morphisms;
- S_h: projections of the form $X \times (\mathbb{P}^1)^n \to X$;
- S_r: stably birational morphisms, where $s \in S_r$ if and only if s is dominant and gives a purely transcendental function field extension;

to which we adjoin

- S_b^w: compositions of blow-ups with smooth centres;
- $S_r^w = S_b^w \cup S_h$.

These morphisms, defined for connected varieties in [Kahn and Sujatha 2015a], extend trivially to the categories of Definition 1.2.1 as explained in [Kahn and Sujatha 2007, Corollary 6.3]. More precisely, if S is a set of morphisms of $\mathbf{Sm}(F)$, we define $S^{\amalg} \subset \mathbf{Sm}_{\amalg}(F)$ as the set of those morphisms which are dominant and whose connected components are all in S. For simplicity, we shall write S rather than $S^{\amalg}$ in the sequel.

By Lemma 1.2.2 and [Kahn and Sujatha 2007, Theorem 6.4], the localisation results of [Kahn and Sujatha 2007; 2015a] extend to the category $\mathbf{Sm}_{\amalg}(F)$ and, moreover, the functors

$$S^{-1}\,\mathbf{Sm}(F) \to S^{-1}\,\mathbf{Sm}_{\amalg}(F)$$

identify the right-hand side with the "finite coproduct envelope" of the left-hand side. Similarly for their analogues with decorations $\mathbf{Sm}^{\%}$.

We shall view the above morphisms as correspondences via the graph functor. We introduce two more sets which are convenient here:

Definition 2.1.1. We write $\tilde{S}_b$ and $\tilde{S}_r$ for the sets of dominant rational maps which induce, respectively, an isomorphism of function fields and a purely transcendental extension. We let these rational maps act on pure motives via their graphs, as in Section 1.4.

Thus we have a diagram of inclusions of morphisms on $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$:

$$\begin{array}{ccccc} S_b^w & \subset & S_b^w \cup S_h & = & S_r^w \\ \cap & & \cap & & \cap \\ S_b & \subset & S_b \cup S_h & \subset & S_r \\ \cap & & \cap & & \cap \\ \tilde{S}_b & \subset & \tilde{S}_b \cup S_h & \subset & \tilde{S}_r \end{array} \tag{2.1.2}$$

Let us immediately notice:

Proposition 2.1.3. *Let S be one of the systems of morphisms in* (2.1.2). *Then the category $S^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$ is an A-linear category provided with a tensor structure, compatible with the corresponding structures of $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$ via the localisation functor.*

Proof. This follows from Theorem A.3.4, Proposition A.1.2 and the fact that elements of S are stable under disjoint unions and products. □

2.2. ***Second approach: the Lefschetz ideal.***

Definition 2.2.1. We denote by $\mathcal{L}_{\sim}$ the ideal of $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$ consisting of those morphisms which factor through some object of the form $P(1)$; this is the *Lefschetz ideal*. It is a monoidal ideal (i.e., it is closed with respect to composition and tensor products on the left and on the right).

Remark 2.2.2. In any additive category $\mathcal{A}$ there is a notion of product of two ideals $\mathcal{I}, \mathcal{J}$:

$$\mathcal{I} \circ \mathcal{J} = \langle f \circ g \mid f \in \mathcal{I}, g \in \mathcal{J} \rangle.$$

If $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ and $\mathcal{J} = \{f \mid f \text{ factors through some } A \in \mathcal{B}\}$, then $\mathcal{J}$ is idempotent because it is generated by idempotent morphisms, namely the identity maps of the objects of $\mathcal{B}$. In $\mathcal{A} = \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$, this applies to $\mathcal{L}_{\sim}$.

On the other hand, in a tensor additive category $\mathcal{A}$ there is also the tensor product of two ideals $\mathcal{I}, \mathcal{J}$: for $A, B \in \mathcal{A}$,

$$(\mathcal{I} \otimes \mathcal{J})(A, B) = \langle \mathcal{A}(E \otimes F, B) \circ (\mathcal{I}(C, E) \otimes \mathcal{J}(D, F)) \circ \mathcal{A}(A, C \otimes D) \rangle,$$

where C, D, E, F run through all objects of $\mathcal{A}$. Coming back to $\mathcal{A} = \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$, we have $\mathcal{L}_{\sim} \otimes \mathcal{L}_{\sim} = \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)(2) \neq \mathcal{L}_{\sim} \circ \mathcal{L}_{\sim} = \mathcal{L}_{\sim}$. This is in sharp contrast with the case where $\mathcal{A}$ is rigid [André and Kahn 2002, Lemme 6.15].

Proposition 2.2.3. (a) *The localisation functor*

$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A) \to (S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$

factors through $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim}$.

(b) *The functors*

$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim} \to (S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A) \to (S_r^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$

are both isomorphisms of categories.

(c) *The functor*

$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim} \to S_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$

is full.

(d) *For any* $s \in \tilde{S}_r$, s_* *becomes invertible in* $\tilde{S}_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$.

Proof. (a) By Proposition 2.1.3, it is sufficient to show that $\mathbb{L} \mapsto 0$ in

$$(S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A).$$

Here as in the proof of (b) we shall use the following formula of Manin [1968, Section 9, Corollary, p. 463]: if $p : \tilde{X} \to X$ is a blow-up with smooth centre $Z \subset X$ of codimension n, then

$$h^{\mathrm{eff}}_{\sim}(\tilde{X}) \simeq h^{\mathrm{eff}}_{\sim}(X) \oplus \bigoplus_{i=1}^{n-1} h^{\mathrm{eff}}_{\sim}(Z) \otimes \mathbb{L}^{\otimes i}, \tag{2.2.4}$$

where projecting the right-hand side onto $h^{\mathrm{eff}}_{\sim}(X)$ we get p_*.

In (2.2.4), take $X = \mathbb{P}^2$ and for $\tilde{X}$ the blow-up of X at (say) $Z = \{(1:0:0)\}$. Since p is invertible in $(S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$, we get $\mathbb{L} = 0$ in this category as requested.

(b) It suffices to show that morphisms of S_r^w become invertible in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim}$, which immediately follows from (2.2.4) and the easier projective line formula.

(c) It suffices to show that members of S_b have right inverses in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$; this follows from Lemma 1.5.2.

(d) Let $g : X \dashrightarrow Y$ be an element of $\tilde{S}_r$. Then X is birational to $Y \times (\mathbb{P}^1)^n$ for some $n \geq 0$, and if $f : X \dashrightarrow Y \times (\mathbb{P}^1)^n$ is the corresponding birational map, its composition with the first projection π is g. By Lemma 1.4.1, it suffices to show that π_* is invertible in $\tilde{S}_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$, which follows from (b). □

Corollary 2.2.5. *Let* $M = \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$.

(a) *Diagram* (2.1.2) *induces a commutative diagram of categories and functors*

$$\begin{array}{ccccccc}
M/\mathcal{L}_{\sim} & \xrightarrow{\sim} & (S_b^w)^{-1}M & \xrightarrow{\sim} & (S_b^w \cup S_h)^{-1}M & \xrightarrow{\sim} & (S_r^w)^{-1}M \\
 & & \downarrow{\scriptstyle \text{full}} & & \downarrow{\scriptstyle \text{full}} & & \downarrow \\
 & & S_b^{-1}M & \xrightarrow{\sim} & (S_b \cup S_h)^{-1}M & \longrightarrow & S_r^{-1}M \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \tilde{S}_b^{-1}M & \xrightarrow{\sim} & (\tilde{S}_b \cup S_h)^{-1}M & \xrightarrow{\sim} & \tilde{S}_r^{-1}M
\end{array} \tag{2.2.6}$$

where the functors with a sign $\sim$ *are isomorphisms of categories and the indicated functors are full.*

(b) *If* $\operatorname{char} F = 0$, *all functors are isomorphisms of categories.*

Proof. (a) follows from Proposition 2.2.3; (b) follows from Hironaka's resolution of singularities (see [Kahn and Sujatha 2015a, Lemma 1.7.1]). □

Remark 2.2.7. Tracking isomorphisms in diagram (2.2.6), one sees that without assuming resolution of singularities we get a priori 4 different categories of "pure birational motives". If $p : \tilde{X} \to X$ is a birational morphism, then at least $h_\sim(X)$ is a direct summand of $h_\sim(\tilde{X})$ by Lemma 1.5.2. However it is not clear how to prove that the other summand is divisible by $\mathbb{L}$ without using resolution. We shall get by for special pairs $(A, \sim)$ in Theorem 2.4.2 below, using the alteration theorem of de Jong and Gabber.

We now introduce:

Definition 2.2.8. The category of *pure birational motives* is

$$\mathbf{Mot}^{\mathrm{b}}_\sim(F, A) = \big(\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)/\mathcal{L}_\sim\big)^\natural.$$

We also set

$$\mathbf{Chow}^{\mathrm{eff}}(F, A) = \mathbf{Mot}^{\mathrm{eff}}_{\mathrm{rat}}(F, A),$$
$$\mathbf{Chow}^{\mathrm{b}}(F, A) = \mathbf{Mot}^{\mathrm{b}}_{\mathrm{rat}}(F, A).$$

When $A = \mathbb{Z}$, we abbreviate this notation to $\mathbf{Chow}^{\mathrm{eff}}(F)$ and $\mathbf{Chow}^{\mathrm{b}}(F)$.

We note:

Proposition 2.2.9. *Taking pseudoabelian envelopes, the first functor in Corollary 2.2.5(a) induces an isomorphism of categories*

$$\mathbf{Mot}^{\mathrm{b}}_\sim(F, A) \xrightarrow{\sim} \big((S^w_b)^{-1}\,\mathbf{Cor}_\sim(F, A)\big)^\natural.$$

In particular, the functor $(S^w_b)^{-1}\,\mathbf{Cor}_\sim(F, A) \to (S^w_b)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ *is fully faithful and the functor* $\mathbf{Cor}_\sim(F, A) \to S_b^{-1}\,\mathbf{Cor}_\sim(F, A)$ *is full.*

Proof. All follows from Lemma A.4.1, except for the last statement, which follows from Proposition 2.2.3(c). □

In Section 4, we shall examine to what extent it is really necessary to adjoin idempotents in Definition 2.2.8.

2.3. *Third approach: extendible pairs.* To go further, we need to restrict the adequate equivalence relation we are using:

Definition 2.3.1. An adequate pair $(A, \sim)$ is *extendible* if

- $\sim$ is defined on cycles over arbitrary quasiprojective F-varieties;

- it is preserved by inverse image under flat morphisms and direct image under proper morphisms;
- if X is smooth projective, Z is a closed subset of X and $U = X - Z$, then the sequence

$$\mathcal{Z}_n^{\sim}(Z, A) \to \mathcal{Z}_n^{\sim}(X, A) \to \mathcal{Z}_n^{\sim}(U, A) \to 0 \tag{2.3.2}$$

is exact.

Note that in (2.3.2), surjectivity always holds because this is already true on the level of cycles. So the issue is exactness at $\mathcal{Z}_n^{\sim}(X, A)$.

Examples 2.3.3. (a) Rational equivalence (with any coefficients) is extendible.

(b) Algebraic equivalence (with any coefficients) is extendible; see [Fulton 1984, Example 10.3.4].

(c) The status of homological equivalence is very interesting:

(1) Under the standard conjecture that homological and numerical equivalences agree, homological equivalence with respect to a "classical" Weil cohomology theory is extendible if char $F = 0$ [Corti and Hanamura 2000, Proposition 6.7]. The proof involves resolution of singularities and the weight spectral sequences for Borel–Moore Hodge homology, their degeneration at E_2 and the semisimplicity of numerical motives [Jannsen 1992]. Presumably the same arguments work in characteristic p by using de Jong's alteration theorem [1996] instead of Hironaka's resolution of singularities; we thank Yves André for pointing this out. See [Voisin 2013, Proposition 1.6] for a more precise statement and a different proof.

(2) It seems that the Corti–Hanamura argument implies unconditionally that André's motivated cycles [1996] verify the axioms of an extendible pair.

(3) For Betti cohomology with integral coefficients or l-adic cohomology with $\mathbb{Z}_l$ coefficients, homological equivalence is not extendible. (Counterexample: $F = \mathbb{C}$, $n = 1$, Z a general surface of degree ≥ 4 in $\mathbb{P}^3$; this example goes back to Kollár [1992, p. 134].) This is closely related to the failure of the Hodge or Tate conjecture integrally for Z (see [Soulé and Voisin 2005, Section 2]).

(4) Hodge cycles with coefficients in $\mathbb{Q}$ verify the axioms of an extendible pair: similarly to (1), the proof involves resolving the singularities of Z in (2.3.2) and using the semisimplicity of polarisable pure Hodge structures. See also [Jannsen 1994]. We are indebted to Claire Voisin for explaining these last two points.

(5) Taking Tate cycles for l-adic cohomology, the same argument works if we assume the semisimplicity of Galois action on the cohomology of smooth projective varieties.

Lemma 2.3.4. *If* $(A, \sim)$ *verifies the first two conditions of Definition 2.3.1, then* $(A, \mathrm{rat}) \geq (A, \sim)$ (*also over arbitrary quasiprojective varieties*).

Proof. Again, this follows from [Fulton 1984, Example 1.7.5]. □

Proposition 2.3.5. *Let* $(A, \sim)$ *be an extendible pair. For two smooth projective varieties* X, Y, *let* $\mathcal{I}_\sim(X, Y)$ *be the subgroup of* $\mathcal{Z}_\sim^{\dim Y}(X \times Y, A)$ *consisting of those classes vanishing in* $\mathcal{Z}_\sim^{\dim Y}(U \times Y, A)$ *for some open subset* U *of* X. *Then* $\mathcal{I}_\sim$ *is a monoidal ideal in* $\mathbf{Cor}_\sim(F, A)$.

Proof. Note that by Lemma 2.3.4 and the third condition of Definition 2.3.1, the map $\mathcal{I}_{\mathrm{rat}}(X, Y) \to \mathcal{I}_\sim(X, Y)$ is surjective for any X, Y; this reduces us to the case $\sim = \mathrm{rat}$. We further reduce immediately to $A = \mathbb{Z}$.

Let X, Y, Z be three smooth projective varieties. If U is an open subset of X, it is clear that the usual formula defines a composition of correspondences

$$CH^{\dim Y}(U \times Y) \times CH^{\dim Z}(Y \times Z) \to CH^{\dim Z}(U \times Z)$$

and that this composition commutes with restriction to smaller and smaller open subsets. Passing to the limit on U, we get a composition

$$CH^{\dim Y}(Y_{F(X)}) \times CH^{\dim Z}(Y \times Z) \to CH^{\dim Z}(Z_{F(X)})$$

or

$$CH_0(Y_{F(X)}) \times CH^{\dim Z}(Y \times Z) \to CH_0(Z_{F(X)}).$$

Here we used the fact that (codimensional) Chow groups commute with filtering inverse limits of schemes; see [Bloch 2010].

We now need to prove that this pairing factors through

$$CH_0(Y_{F(X)}) \times CH^{\dim Z}(V \times Z)$$

for any open subset V of Y. One checks that it is induced by the standard action of correspondences in $CH^{\dim Z}(Y_{F(X)} \times_{F(X)} Z_{F(X)})$ on groups of 0-cycles. Hence it is sufficient to show that the standard action of correspondences factors as indicated, and up to changing the base field we may replace $F(X)$ by F.

We now show that the pairing

$$CH_0(Y) \times CH^{\dim Z}(Y \times Z) \to CH_0(Z)$$

factors as indicated. The proof is a variant of Fulton's proof [1984, Example 16.1.11] of the Colliot-Thélène–Coray theorem [1979] that CH_0 is a birational invariant of smooth projective varieties. Let M be a proper closed subset of Y and let $i : M \to Y$ be the corresponding closed immersion. We have to prove that for any $\alpha \in CH_0(Y)$ and $\beta \in CH_{\dim Y}(M \times Z)$,

$$(i \times 1_Z)_*(\beta)(\alpha) := (p_2)_*((i \times 1_Z)_*\beta \cdot p_1^*\alpha) = 0,$$

where p_1 and p_2 are respectively the first and second projections on $Y \times Z$.

We shall actually prove that $(i \times 1_Z)_*\beta \cdot p_1^*\alpha = 0$. For this, we may assume that α is represented by a closed point $y \in Y$ and β by some integral variety $W \subseteq M \times Z$. Then $(i \times 1_Z)_*\beta \cdot p_1^*\alpha$ has support in $(i \times 1_Z)(W) \cap (\{y\} \times Z) \subset (M \times Z) \cap (\{y\} \times Z)$. If $y \notin M$, this subset is empty and we are done. Otherwise, up to rational equivalence, we may replace y by a 0-cycle disjoint from M (see [Roberts 1972]), and we are back to the previous case.

This shows that $\mathcal{I}_\sim$ is an ideal of $\mathbf{Cor}_\sim(F, A)$. The fact that it is a monoidal ideal is essentially obvious. □

Definition 2.3.6. For an extendible pair $(A, \sim)$, we abbreviate $\mathbf{Cor}_\sim(F, A)/\mathcal{I}_\sim$ (resp. $(\mathbf{Mot}_\sim^{\text{eff}}(F, A)/\mathcal{I}_\sim)^\natural$) into $\mathbf{Cor}_\sim^{\text{o}}(F, A)$ (resp. $\mathbf{Mot}_\sim^{\text{o}}(F, A)$). (Here o stands for “open”.) We write $h_\sim^{\text{o}}(X)$ for the image of $h_\sim(X)$ in $\mathbf{Mot}_\sim^{\text{o}}(F, A)$. We also set $\mathbf{Chow}^{\text{o}}(F, A) = \mathbf{Mot}_{\text{rat}}^{\text{o}}(F, A)$ and $\mathbf{Chow}^{\text{o}}(F) = \mathbf{Chow}^{\text{o}}(F, \mathbb{Z})$.

For future reference, let us record here the value of the Hom groups in the most important case, that of rational equivalence (see also Remark 2.3.10(2) below):

Lemma 2.3.7. *We have*

$$\mathbf{Cor}_{\text{rat}}^{\text{o}}(F, A)([X], [Y]) = CH_0(Y_{F(X)}) \otimes A.$$

Proposition 2.3.8. *In* $\mathbf{Cor}_\sim^{\text{o}}(F, A)$:

(a) $(g \circ f)_* = g_* \circ f_*$ *for any composable rational maps* $X \overset{f}{\dashrightarrow} Y \overset{g}{\dashrightarrow} Z$.

(b) [Fulton 1984, Example 16.1.11] $f^* f_* = 1_X$ *and* $f_* f^* = 1_Y$ *for any birational map* $f : X \dashrightarrow Y$.

(c) *Morphisms of* $\tilde{S}_r$ (see Definition 2.1.1) *are invertible.*

Proof. (a) Let F be the fundamental set of f, G be the fundamental set of g, $U = X - F$, $V = Y - G$. By assumption, $f(U) \cap V \neq \varnothing$, hence $W = f^{-1}(V)$ is a nonempty open subset of U, on which $g \circ f$ is a morphism.

Let us abuse notation and still write f for the morphism f_U, etc. Then, by definition,

$$g_* \circ f_* = (p_{XZ})_*((\overline{\Gamma}_f \times Z) \cap (X \times \overline{\Gamma}_g))$$

(note that the two intersected cycles are in good position). This cycle clearly contains $(g \circ f)_* = \overline{\Gamma}_{g \circ f}$ as a closed subset. One sees immediately that the restriction of $g_* \circ f_*$ and $(g \circ f)_*$ to $W \times Z$ are equal.

(b) This is proven in the same way (or is a special case of (a)).

(c) Let $g : X \dashrightarrow Y$ be an element of $\tilde{S}_r$. Then X is birational to $Y \times (\mathbb{P}^1)^n$ for some $n \geq 0$, and if $f : X \dashrightarrow Y \times (\mathbb{P}^1)^n$ is a birational map, its composition with the first projection π is g. By (a) and (b), it suffices to show that π_* is invertible in

$\mathbf{Cor}_\sim(F, A)/\mathcal{I}_\sim$. For this we may reduce to $n = 1$ and even to $Y = \operatorname{Spec} F$ since $\mathcal{I}_\sim$ is a monoidal ideal. Let $s : \operatorname{Spec} F \to \mathbb{P}^1$ be the ∞ section; it suffices to show that $(s \circ \pi)_* = 1_{\mathbb{P}^1}$. But the cycle $(s \circ \pi)_* - 1_{\mathbb{P}^1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is linearly equivalent to $\infty \times \mathbb{P}^1$ (this is the idempotent defining the Lefschetz motive), and the latter cycle vanishes when restricted to $\mathbb{A}^1 \times \mathbb{P}^1$. □

We shall also need the following lemma in the proof of Proposition 5.1.4(c).

Lemma 2.3.9. *Let L/K be an extension of function fields over F, with $K = F(X)$ and $L = F(Y)$ for X, Y two smooth projective F-varieties. Let $\varphi : Y \dashrightarrow X$ be the rational map corresponding to the inclusion $K \hookrightarrow L$. Let Z be another smooth projective F-variety. Then the map*

$$\mathbf{Chow}^{\mathrm{o}}(F, A)(h^{\mathrm{o}}(X), h^{\mathrm{o}}(Z)) \to \mathbf{Chow}^{\mathrm{o}}(F, A)(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(Z))$$

given by composition with $\varphi_* : h^{\mathrm{o}}(Y) \to h^{\mathrm{o}}(X)$ (see Section 1.4) *coincides via Lemma 2.3.7 with the base-change map $CH_0(Z_K) \otimes A \to CH_0(Z_L) \otimes A$.*

Proof. Let $V \subseteq Y$ and $U \subseteq X$ be open subsets such that φ is defined on V and $\varphi(V) \subseteq U$. Up to shrinking U, we may assume that φ is flat [Grothendieck and Dieudonné 1966, Théorème 11.1.1]. As in the proof of Proposition 2.3.5, the composition of correspondences induces a pairing

$$CH^{\dim X}(V \times U) \times CH^{\dim Z}(U \times Z) \to CH^{\dim Z}(V \times Z),$$

and the action of $\varphi_* \in CH^{\dim X}(V \times U)$ on $\alpha \in CH^{\dim Z}(U \times Z)$ is given by the flat pull-back of cycles. Therefore, φ_* induces in the limit the flat pull-back of 0-cycles from $CH_0(Z_K)$ to $CH_0(Z_L)$. □

Remarks 2.3.10. (1) Propositions 2.3.5 and 2.3.8(a) were independently observed by Markus Rost in the case $\sim$=rat [Merkurjev 2001, Proposition 3.1 and Lemma 3.3]. We are indebted to Karpenko for pointing this out and for referring us to Merkurjev's preprint.

(2) In $\mathbf{Cor}^{\mathrm{o}}_\sim(F, A)$, morphisms are by definition given by the formula

$$\mathbf{Cor}^{\mathrm{o}}_\sim(F, A)([X], [Y]) = \varinjlim_{U \subseteq X} \mathcal{Z}^{\dim Y}_\sim(U \times Y, A).$$

The latter group maps onto $\mathcal{Z}_0^\sim(Y_{F(X)}, A)$. If $\sim$ = rat, this map is an isomorphism (see Lemma 2.3.7). For other equivalence relations, this is far from being the case: for example, if $\sim$ = alg, F is algebraically closed, X, Y are two curves and, say, $A = \mathbb{Z}$, then

$$\begin{aligned}\mathcal{Z}^1_{\mathrm{alg}}(X \times Y, \mathbb{Z}) &= \mathrm{NS}(X \times Y) = \mathrm{NS}(X) \oplus \mathrm{NS}(Y) \oplus \mathrm{Hom}(J_X, J_Y)\\ &= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Hom}(J_X, J_Y),\end{aligned}$$

where NS is the Néron–Severi group, and J_X and J_Y are the Jacobians of X and Y. On the other hand,

$$\mathcal{Z}_0^{\mathrm{alg}}(Y_{F(X)}, \mathbb{Z}) = \mathrm{NS}(Y_{F(X)}) = \mathbb{Z}.$$

When we remove a point from X, we kill the factor $\mathrm{NS}(X) = \mathbb{Z}$. But any two points of X are algebraically equivalent, so removing further points does not modify the group any further. Hence

$$\varinjlim_{U\subseteq X} \mathcal{Z}_{\mathrm{alg}}^{\dim Y}(U \times Y, \mathbb{Z}) = \mathbb{Z} \oplus \mathrm{Hom}(J_X, J_Y).$$

We thank Colliot-Thélène for helping clarify this matter.

2.4. ***The main theorem.*** We now extend the ideal $\mathcal{I}_\sim$ from

$$\mathbf{Cor}_\sim(F, A) \quad \text{to} \quad \mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)$$

in the usual way (see [André and Kahn 2002, Lemme 1.3.10]), without changing notation. By Propositions 2.2.3(a) and 2.3.8, we get a composite functor

$$\mathbf{Mot}_\sim^{\mathrm{b}}(F, A) \to (\tilde{S}_r^{-1}\,\mathbf{Mot}_\sim^{\mathrm{eff}}(F, A))^\natural \to \mathbf{Mot}_\sim^{\mathrm{o}}(F, A) \tag{2.4.1}$$

for any extendible pair $(A, \sim)$. Since both categories are (idempotent completions of) full images of $\mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)$, this functor is automatically full. We are going to show that it is an equivalence of categories in some important cases.

Theorem 2.4.2. *Let $(A, \sim)$ be an extendible pair. Suppose that the exponential characteristic p of F is invertible in A. Then the functor* (2.4.1) *is an isomorphism of categories.*

Proof.[3] We have to show that $\mathcal{I}_\sim(M, N) \subseteq \mathcal{L}_\sim(M, N)$ for any $M, N \in \mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)$. Proposition 1.7.2 reduces us to the case where F is *perfect*. Clearly we may assume $M = h_\sim(X)$, $N = h_\sim(Y)$ for two smooth projective varieties X, Y.

Let $f \in \mathcal{I}_\sim(h_\sim(X), h_\sim(Y))$. By the third condition in Definition 2.3.1, the cycle class $f \in \mathcal{Z}^\sim_{\dim X}(X \times Y, A)$ is of the form $(i \times 1_Y)_* g$ for some closed immersion $i : Z \to X$, where $g \in \mathcal{Z}^\sim_{\dim X}(Z \times Y, A)$. Let $\tilde{g}$ be a cycle representing g. Write $\tilde{g} = \sum_k a_k g_k$, with $a_k \in A$ and g_k irreducible. Then $(i \times 1_Y)_*(g_k) \in \mathcal{I}_\sim(h_\sim(X), h_\sim(Y))$. This reduces us to the case where g is represented by an irreducible cycle $\tilde{g}$.

Choose Z minimal among the closed subsets of X such that $\tilde{g}$ is supported on $Z \times Y$. In particular, Z is irreducible.

Consider Z with its reduced structure. Let l be a prime number different from p; by Gabber's refinement of de Jong's theorem [Illusie and Temkin 2014, Théorème X.2.1], we may choose a proper, generically finite morphism $\pi_l : \widetilde{Z}_l \to Z$ where

[3] We thank N. Fakhruddin for his help, which removes the recourse to Chow's moving lemma in [Kahn and Sujatha 2002].

$\widetilde{Z}_l$ is smooth projective (irreducible) and π_l is an alteration of generic degree d_l prime to l. (Recall that an alteration is a proper, generically finite morphism.)

By the minimality of Z, the support of $\tilde{g}$ has nonempty intersection $\tilde{g}_1$ with $V \times Y$, where $V = Z - (Z_{\rm sing} \cup T)$ with $Z_{\rm sing}$ the singular locus of Z and T the closed subset over which π_l is not finite. Let $\pi_V : \pi_l^{-1}(V) \to V$ be the map induced by π_l; note that π_V is flat since V and $\pi_l^{-1}(V)$ are smooth. We then have an equality of cycles

$$d_l \tilde{g}_1 = (\pi_V \times 1_Y)_* (\pi_V \times 1_Y)^* \tilde{g}_1.$$

Let γ_l be the closure of $(\pi_V \times 1_Y)^* \tilde{g}_1$ in $\widetilde{Z}_l$.[4] We get an equality of cycles (the support of $(\pi_V \times 1_Y)_* (\pi_V \times 1_Y)^* \tilde{g}_1$ is dense in that of $(\pi_l \times 1_Y)_* \gamma_l$):

$$d_l \tilde{g} = (\pi_l \times 1_Y)_* \gamma_l.$$

Let $d = \gcd_l(d_l)$, which is a power of p; then $d = \gcd(d_{l_1}, \dots, d_{l_r})$ for some finite set of primes $\{l_1, \dots, l_r\}$. For simplicity, write $Z_{l_i} = Z_i$, $\pi_{l_i} = \pi_i$ and $\gamma_{l_i} = \gamma_i$.

Let $h_i = d^{-1}[\gamma_i] \in \mathcal{Z}^{\sim}_{\dim X}(\widetilde{Z}_i \times Y, A)$. Choose $a_1, \dots, a_r \in \mathbb{Z}$ such that $d = \sum_i a_i d_i$, so that

$$f = \sum_i a_i ((i \circ \pi_i) \times 1_Y)_* h_i.$$

Then the correspondence $f \in \mathbf{Mot}^{\rm eff}_\sim(F)(h_\sim(X), h_\sim(Y))$ factors as

$$h_\sim(X) \xrightarrow{(i\circ\pi)^*} h_\sim\Big(\coprod \widetilde{Z}_i\Big)(\dim X - \dim Z) \xrightarrow{(h_i)} h_\sim(Y)$$

(see (1.5.1)), which concludes the proof. □

Corollary 2.4.3. *Under the assumptions of Theorem 2.4.2, all the categories of diagram* (2.2.6) *are isomorphic to* $\mathbf{Mot}^{\rm eff}_\sim(F, A)/\mathcal{I}_\sim$.

Proof. By Proposition 2.2.3(b) and (d) we already know that the categories

$$\mathbf{Mot}^{\rm eff}_\sim(F, A)/\mathcal{L}_\sim, \quad (S_b^w)^{-1} \mathbf{Mot}^{\rm eff}_\sim(F, A) \quad \text{and} \quad (S_r^w)^{-1} \mathbf{Mot}^{\rm eff}_\sim(F, A)$$

are isomorphic and that

$$(\tilde{S}_b)^{-1} \mathbf{Mot}^{\rm eff}_\sim(F, A) \quad \text{and} \quad (\tilde{S}_r)^{-1} \mathbf{Mot}^{\rm eff}_\sim(F, A)$$

are isomorphic. We also know that the functor

$$\mathbf{Mot}^{\rm eff}_\sim(F, A)/\mathcal{L}_\sim \to (S_b)^{-1} \mathbf{Mot}^{\rm eff}_\sim(F, A)$$

is full (Proposition 2.2.3(c)); by Theorem 2.4.2, this implies that it is an isomorphism. To conclude the proof, it is sufficient to show that any morphism of $\tilde{S}_r$, hence of S_r, has a right inverse in $\mathbf{Mot}^{\rm eff}_\sim(F, A)/\mathcal{L}_\sim$ (see (2.2.6)). Since $\tilde{S}_r$ is

[4] More correctly, the cycle associated to the schematic closure of $(\pi_V \times 1_Y)^{-1}(\tilde{g}_1)$ in $\widetilde{Z}_l$: take the topological closure of each component of $(\pi_V \times 1_Y)^* \tilde{g}_1$ and keep the same multiplicities.

generated by $\tilde{S}_b$ and projections of the form $X \times \mathbb{P}^1 \to X$ (see the proof of Proposition 2.2.3(d)) and since this is obvious for these projections, we are left to prove it for elements $f : X \dashrightarrow Y$ of $\tilde{S}_b$. But we have $f_* f^* = 1_X$ in $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)/\mathcal{I}_\sim$ by Proposition 2.3.8(b), hence in $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)/\mathcal{L}_\sim$ by Theorem 2.4.2. □

2.5. ***Birational image motives.*** Based on the categories of Section 1.8, we define categories $\overline{\mathbf{Mot}}^{\mathrm{b}}_\sim(F, A)$. If $\sim$ is extendible and p is invertible in A, the analogue of Theorem 2.4.2 holds, with the same proof.

2.6. ***Recapitulation, comments and notation.*** In Definition 2.2.8, we associated to any admissible pair $(A, \sim)$ a category of birational motives $\mathbf{Mot}^{\mathrm{b}}_\sim(F, A)$. If $(A, \sim)$ is extendible (Definition 2.3.1), we introduced in Definition 2.3.6 another category $\mathbf{Mot}^{\mathrm{o}}_\sim(F, A)$ plus a full functor $\mathbf{Mot}^{\mathrm{b}}_\sim(F, A) \to \mathbf{Mot}^{\mathrm{o}}_\sim(F, A)$. We showed in Theorem 2.4.2 that this functor is an isomorphism of categories when the exponential characteristic p is invertible in A; in particular, this is true for any A in characteristic 0. This gives a great flexibility in computing Hom groups, as in some cases one can use their "algebraic" description in terms of killing the Lefschetz motive, and in other cases their "geometric" description as Chow groups of 0-cycles if $\sim$ is rational equivalence.

In the sequel, we commit the abuse of notation which consists of writing $\mathbf{Mot}^{\mathrm{o}}_\sim$ for $\mathbf{Mot}^{\mathrm{b}}_\sim$ even when we don't know if the pair $(A, \sim)$ is extendible (notably, when $\sim$ is numerical equivalence). We do this because we feel that keeping the distinction would create more confusion than this choice.

3. Examples

We give some examples and computations of birational motives.

3.1. ***Varieties with trivial birational motive.*** These were initially studied by Bloch and Srinivas [1983] over a universal domain. The reader should compare the following to [Kahn and Sujatha 2015a, Theorem 8.5.1]; see also [Totaro 2014, Theorem 2.1].

Proposition 3.1.1. *Let A be a connected commutative ring, and let X be a smooth projective F-variety. Then the following conditions are equivalent*:

(i) *For any smooth projective F-variety Y, $CH_0(X_{F(Y)}) \otimes A \xrightarrow{\sim} A$ (by the degree map).*

(ii) $CH_0(X_{F(X)}) \otimes A \xrightarrow{\sim} A$.

(iii) *The class of the generic point η_X in $CH_0(X_{F(X)}) \otimes A$ belongs to*

$$\mathrm{Im}(CH_0(X) \otimes A \to CH_0(X_{F(X)}) \otimes A).$$

(iv) $h^{\mathrm{o}}(X) = \mathbb{1}$ *in* $\mathbf{Chow}^{\mathrm{o}}(F, A)$.

(v) *(For* $A = \mathbb{Z}$:*)* $M_0(F) \xrightarrow{\sim} A^0(X, M_0)$ *for any cycle module* M.

If p *is invertible in* A, *they are also equivalent to*:

(vi) *For any extension* K/F, $CH_0(X_K) \otimes A \xrightarrow{\sim} A$.

If F *is a universal domain and* $A \supseteq \mathbb{Q}$, *they are also equivalent to*:

(vii) $CH_0(X) \otimes A \xrightarrow{\sim} A$.

(viii) $CH_0(X) \xrightarrow{\sim} \mathbb{Z}$.

(Parts of this proposition are standard; see, e.g., [Auel et al. 2013, Lemma 1.3].)

Proof. (i) ⇒ (ii) ⇒ (iii) is obvious. By Lemma 2.3.7, the map of (iii) can be translated into

$$\mathbf{Chow}^{o}(F, A)(\mathbb{1}, h^{o}(X)) \to \mathbf{Chow}^{o}(F, A)(h^{o}(X), h^{o}(X))$$

via the projection $h^o(X) \to h^o(\operatorname{Spec} k) = \mathbb{1}$. Since η_X represents the identity endomorphism of $h^o(X)$, (iii) means that the latter factors through $\mathbb{1}$. Since $\operatorname{End}(\mathbb{1}) = A$, the resulting idempotent endomorphism of $\mathbb{1}$ must be 0 or 1; so $h^o(X) = 0$ or $\mathbb{1}$, but the first case is impossible as it would imply that $\eta_X = 0$, while $\deg(\eta_X) = 1$. So (iii) ⇒ (iv). Using Lemma 2.3.7 again, we get (iv) ⇒ (i).

(vi) ⇒ (i) is obvious; to prove the converse, we reduce to F perfect by using Proposition 1.7.2, and then to K/F finitely generated by a limit argument. Then K is the function field of some smooth F-variety. We argue as in the proof of Theorem 2.4.2: using [Illusie and Temkin 2014, Théorème X.2.1], we can find finite extensions L_i/K such that $L_i = F(Y_i)$ for Y_i smooth projective, such that the gcd of the $[L_i : K]$ is a power of p. Then $(CH_0(X_K) \otimes A)_{\deg=0}$ is a direct summand of $\bigoplus_i (CH_0(X_{L_i}) \otimes A)_{\deg=0} = 0$ by a transfer argument, hence (vi).

(iv) ⇒ (v) ⇒ (iii): see Section 6.

It remains to prove (iii) ⇐ (vii) ⇒ (viii) when F is a universal domain, since (viii) ⇒ (vii) is obvious. The implication (vii) ⇒ (iii) is the classical Bloch–Srinivas argument [1983, Proposition 1]: X is defined over a subfield $F' \subset F$ finitely generated over the prime field; for clarity, write X' for this F'-model. Now $F'(X')$ embeds into F over F'. Since

$$\operatorname{Ker}\big(CH_0(X'_{F'(X')}) \to CH_0(X'_F) = CH_0(X)\big)$$

is torsion by a transfer argument, (vii) implies that $CH_0(X'_{F'(X')}) \otimes A \xrightarrow{\sim} A$. Thus $\eta_{X'}$ is A-rationally equivalent to a closed point of X', hence (iii). If (vii) is true, then $\operatorname{Alb}(X)(F) \otimes A = 0$, where $\operatorname{Alb}(X)$ is the Albanese variety of X; this implies $\operatorname{Alb}(X) = 0$. But Roĭtman's theorem [1980b] then implies that $CH_0(X)_{\text{tors}} = 0$, whence (viii). □

Corollary 3.1.2. *Conditions* (i)–(v) *of Proposition 3.1.1 are stable under products of varieties*; *so are* (vi), (vii) *and* (viii) *under the stated conditions on* A *and* F.

Proof. Indeed, this is obviously the case for condition (iv). □

Remarks 3.1.3. (1) Condition (v) of Proposition 3.1.1 can be extended to any A if we consider cycle modules with coefficients in A.

(2) Except for (iv), Corollary 3.1.2 can also be proven without reference to birational motives when $A \supseteq \mathbb{Q}$, using that the product map

$$(CH_0(X) \otimes A) \otimes (CH_0(Y) \otimes A) \to CH_0(X \times Y) \otimes A$$

is then surjective for any smooth projective X, Y: reduce to F algebraically closed by a transfer argument, when this even holds integrally.

We now give some examples. In part (3) of the following proposition, the Betti numbers $b^i(X) = \dim H^i(X)$ refer to a "classical" Weil cohomology H: Betti or de Rham in characteristic 0, crystalline in characteristic > 0, l-adic in characteristic $\neq l$. It is known that $b^i(X)$ does not depend on the choice of such a Weil cohomology.

Proposition 3.1.4. (1) *If X is retract rational, then $h^o(X) = \mathbb{1}$ in* $\mathbf{Chow}^o(F, \mathbb{Z})$.

(2) *If X is rationally chain connected, then $h^o(X) = \mathbb{1}$ in* $\mathbf{Chow}^o(F, \mathbb{Q})$.

(3) *If $h^o(X) = \mathbb{1}$ in* $\mathbf{Chow}^o(F, \mathbb{Q})$, *then $b^1(X) = 0$ and $b^2(X) = \rho(X)$ (the Picard number).*

(4) *If* $\dim X = 2$, *the converse of* (3) *is true if and only if X verifies Bloch's conjecture on* 0*-cycles.*

Proof. (1) This follows from [Kahn and Sujatha 2015a, Proposition 8.6.2] and the functor (5.1.1) below. (One could also give a direct proof.)

(2) Let $\overline{F(X)}$ be an algebraic closure of $F(X)$; then $X(\overline{F(X)})/R = *$. Since the group of 0-cycles on $X_{\overline{F(X)}}$ is generated by $X(\overline{F(X)})$, this in turn implies that $CH_0(X_{\overline{F(X)}}) \xrightarrow{\sim} \mathbb{Z}$, which implies by a transfer argument that

$$CH_0(X_{F(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}.$$

(3) Since the hypothesis and conclusion do not change by extension of F, we may assume that F is a universal domain. We use Theorem 2.4.2: in $\mathbf{Chow}^{\text{eff}} = \mathbf{Chow}^{\text{eff}}(F, \mathbb{Q})$ we get a decomposition

$$h(X) = \mathbb{1} \oplus M \otimes \mathbb{L}$$

for some $M \in \mathbf{Chow}^{\text{eff}}$. Applying the cycle class map, we get a commutative diagram

$$\begin{array}{ccc} CH^1(X) \otimes K & = & CH^0(M) \otimes K \\ {\scriptstyle \mathrm{cl}^1_X}\downarrow & & {\scriptstyle \mathrm{cl}^0_M}\downarrow \\ H^2(X) & = & H^0(M) \end{array}$$

Here K is the field of coefficients of H and, as usual, $CH^i(M) := \mathbf{Chow}^{\mathrm{eff}}(M, \mathbb{L}^i)$ (giving back the rational Chow groups of smooth projective varieties) and cl is the cycle class map; for simplicity, we neglect Tate twists on cohomology. But cl_M^0 is an isomorphism, as one sees by writing M as a direct summand of $h(Y)$ for some smooth projective Y; therefore cl_X^1 is an isomorphism as well. Since this map factors through the Néron–Severi group $\mathrm{NS}(X) \otimes K$, this implies $\mathrm{Pic}^0(X) = 0$ (hence $b^1(X) = 0$), and $b^2(X) = \rho(X)$ as requested.

(4) The conditions in the conclusion of (3) imply $\mathrm{Alb}(X) = 0$ and (under Bloch's conjecture) $T(X_K) = 0$ for any extension K/F, where T is the Albanese kernel; the conclusion now follows from condition (i) of Proposition 3.1.1. □

Remarks 3.1.5. (1) As noted in [Kahn 2009, Example 7.3], an Enriques surface verifies the conditions of Proposition 3.1.1 (for 2 invertible in A); this can be recovered from Proposition 3.1.4(4) in a rather silly way. On the other hand, Inose and Mizukami's [1979] and Voisin's [1992] proofs of the Bloch conjecture for some quotients of hypersurfaces by finite groups give examples of surfaces of general type having trivial birational motive (with $\mathbb{Q}$-coefficients), which shows once again how motivic information is in some sense orthogonal to geometric information related to the Kodaira dimension. For a more refined example, see remark (3) below.

(2) Applying the reasoning in the proof of Proposition 3.1.4(3) to CH^2 and CH_1, one recovers some of the representability results of [Bloch and Srinivas 1983] in a different way. (The situation considered by Bloch and Srinivas is more general, and in the present terms amounts to the following: assume that, in $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Q})$, $h^{\mathrm{o}}(X)$ is isomorphic to a direct summand of $h^{\mathrm{o}}(Y)$ for some smooth projective variety Y of dimension $n \le 3$.)

(3) Let X be a smooth projective variety such that $h^{\mathrm{o}}(X) = \mathbb{1}$ in $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Q})$. For simplicity, assume that X has a rational point x. By condition (iii) of Proposition 3.1.1, there is an integer $N > 0$ such that $N(\eta_X - x) = 0$ in $CH_0(X_{F(X)})$. Then in $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z})$, we have

$$h^{\mathrm{o}}(X) = \mathbb{1} \oplus M \quad \text{with } N 1_M = 0.$$

Indeed, x defines an idempotent endomorphism of $h^{\mathrm{o}}(X)$ which splits off the summand $\mathbb{1}$, and $\eta_X - x$ is the complementary idempotent. It follows that

$$N CH_0(X_K)_0 = 0$$

for any extension K/F and (for instance) that

$$N \operatorname{Coker}(M_n(K) \to A^0(X_K, M_n)) = N \operatorname{Ker}(A_0(X_K, M_n) \to M_n(K)) = 0$$

for any cycle module M and any $K \supseteq F$ (see Section 6): compare [Auel et al. 2013, Theorem 1.4].

If N is minimal, then $N > 1$ is an obstruction to having

$$h^o(X) = \mathbb{1} \quad \text{in } \mathbf{Chow}^o(F, \mathbb{Z});$$

this obstruction has been studied recently in [Auel et al. 2013; Voisin 2014; 2015]. Using the cycle module $M_n(K) = H^n(K, \mathbb{Q}/\mathbb{Z}(n-1))$ for $n = 1$, one finds that N is divisible by the exponent e of $H^1_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}/\mathbb{Z})$. One can show that $N = e$ if F is algebraically closed and X is a surface [Kahn 2016]; for $e = 1$, this was proven by Voisin [2014, Proposition 2.2] and by Auel, Colliot-Thélène and Parimala [Auel et al. 2013, Corollary 1.10]. For example, $N = 2$ for an Enriques surface and $N = 1$ for Barlow's surface [1985a; 1985b] (of general type), showing that its motive is $\mathbb{1}$ in $\mathbf{Chow}^o(F, \mathbb{Z})$. (See the recent survey paper [Bauer et al. 2011] for more examples of surfaces of general type with $p_g = 0$.)

3.2. *Quadrics.* Suppose $\operatorname{char} F \neq 2$ and let X be a smooth projective quadric over F. By a theorem of Swan [1989] and Karpenko [1990], the degree map

$$\deg : CH_0(X) \to \mathbb{Z}$$

is injective, with image $\mathbb{Z}$ if X has a rational point and $2\mathbb{Z}$ otherwise. This implies:

Proposition 3.2.1. *Let X, Y be two smooth projective over F. Suppose that Y is a quadric. Then, in $\mathbf{Chow}^o(F)$, we have*

$$\operatorname{Hom}(h^o(X), h^o(Y)) = \begin{cases} \mathbb{Z} & \textit{if } Y_{F(X)} \textit{ is isotropic,} \\ 2\mathbb{Z} & \textit{otherwise,} \end{cases}$$

where we have used the degree map $\deg : CH_0(Y_{F(X)}) \to \mathbb{Z}$. *Similarly, in*

$$\overline{\mathbf{Chow}}^o(F, \mathbb{Z}/2)$$

(see Section 2.5), *we have*

$$\operatorname{Hom}(h^o(X), h^o(Y)) = \begin{cases} \mathbb{Z}/2 & \textit{if } Y_{F(X)} \textit{ is isotropic,} \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 3.2.2. Much work has been done recently on torsion in CH_0 of projective homogeneous varieties: we may quote [Chernousov et al. 2005; Krashen 2010; Petrov et al. 2008; Chernousov and Merkurjev 2006]. There are many examples of projective homogeneous varieties other than quadrics for which $CH_0(Y)$ is torsion-free; by [Chernousov and Merkurjev 2006, Corollary 4.3], this is always the case if Y is isotropic. This allows one to extend the second part of Proposition 3.2.1 to arbitrary projective homogeneous Y (with suitable coefficients). On the other hand, there are examples of anisotropic Y such that $CH_0(Y)_{\text{tors}} \neq 0$ [Krashen 2010, Proposition 1.1; Chernousov and Merkurjev 2006, Section 18], so the first part of Proposition 3.2.1 does not extend in full generality.

3.3. *The nilpotence conjecture.*

Conjecture 3.3.1. *For any two adequate pairs* $(A, \sim)$, $(A, \sim')$ *with* $A \supseteq \mathbb{Q}$ *and* $\sim \geq \sim'$, *and any* $M \in \mathbf{Mot}_\sim(F, A)$, $\mathrm{Ker}(\mathrm{End}(M) \to \mathrm{End}(M_{\sim'}))$ *is nilpotent.* (*We say that the kernel of* $\mathbf{Mot}_\sim(F, A) \to \mathbf{Mot}_{\sim'}(F, A)$ *is* locally nilpotent.)

Since rat is the finest, and num is the coarsest, adequate equivalence relation, this conjecture is clearly equivalent to the same statement for $\sim = \text{rat}$ and $\sim' = \text{num}$, but it may be convenient to consider it for selected adequate equivalence relations. For example:

Proposition 3.3.2. (a) *Conjecture 3.3.1 is true for* $M \in \mathbf{Mot}^{\text{eff}}_\sim(F, A)$ (*and any* $\sim' \leq \sim$) *provided* M *is finite-dimensional in the sense of Kimura and O'Sullivan* [Kimura 2005, Definition 3.7]. *In particular*, *it is true if* M *is of abelian type*, *i.e.,* M *is a direct summand of* $h_\sim(A_K)$ *for* A *an abelian* F*-variety and* K *an étale* F*-algebra.*

(b) *If* $\sim = \text{hom}$, $\sim' = \text{num}$, *the condition of* (*a*) *is equivalent to the sign conjecture*:

> *If* H *is the Weil cohomology theory defining* hom, *the projector of* End $H(M)$ *projecting* $H(M) = H^+(M) \oplus H^-(M)$ *onto its summand* $H^+(M)$ *is algebraic.*

In particular, *it is true if* M *satisfies the standard conjecture* C (*algebraicity of the Künneth projectors*).

(c) *Conjecture 3.3.1 is true in the following cases*:

(i) $\sim = \text{rat}$, $\sim' = \text{tnil}$.

(ii) $\sim = \text{rat}$, $\sim' = \text{alg}$.

Proof. (a) This is a theorem of Kimura and O'Sullivan; see [Kimura 2005, Proposition 7.5; André and Kahn 2002, Proposition 9.1.14]. The second assertion follows from Kimura's results; see [Kahn et al. 2007, Example 7.6.3(4)].

(b) See [André and Kahn 2002, Theorem 9.2.1(c)].

(c) (i) follows from the Voevodsky–Kimura lemma that smash-nilpotent correspondences are nilpotent; see [Voevodsky 1995, Lemma 2.7; Kimura 2005, Proposition 2.16; André and Kahn 2002, Lemma 7.4.2(ii)]. (ii) follows from (i) and Voevodsky's theorem [1995, Corollary 3.2] that alg $\geq$ tnil. □

Let us recall some conjectures which imply Conjecture 3.3.1:

Proposition 3.3.3. (a) *Conjecture 3.3.1 is implied by Voevodsky's conjecture* [1995, Conjecture 4.2] *that smash-nilpotence equivalence equals numerical equivalence.*

(b) *It is also implied by the sign conjecture plus the Bloch–Beĭlinson–Murre conjecture* [Jannsen 1994; Murre 1993].

Proof. (a) This follows from Proposition 3.3.2(c)(i).

(b) Recall that the Bloch–Beĭlinson conjecture is equivalent to Murre's conjecture [1993] by [Jannsen 1994, Theorem 5.2]. Now the formulation of the former conjecture [Jannsen 1994, Conjecture 2.1] implies the existence of an increasing chain of equivalence relations $(\sim_\nu)_{1\le\nu\le\infty}$ such that

- $\sim_1 = \mathrm{hom}$;
- if α, β are composable Chow correspondences such that $\alpha \sim_\mu 0$ and $\beta \sim_\nu 0$, then $\beta \circ \alpha \sim_{\mu+\nu} 0$;
- for any smooth projective variety X, there is $\nu = \nu(X)$ such that $A_{\sim_\nu}(X \times X) = A_{\mathrm{rat}}(X \times X)$.

There properties, together with the sign conjecture, imply Conjecture 3.3.1 by Proposition 3.3.2(b). □

Remark 3.3.4. In fact, one has more precise but slightly weaker implications: the Bloch–Beĭlinson–Murre conjecture + "hom = num" conjecture $\Longrightarrow$ Voevodsky's conjecture $\Longrightarrow$ the Kimura–O'Sullivan conjecture [any Chow motive is finite-dimensional] $\Longrightarrow$ Conjecture 3.3.1; see the synoptic table at the end of Chapter 12 in [André 2004].

For the first implication, see [André 2004, Théorème 11.5.3.1]. For the second one, see [André 2004, Théorème 12.1.6.6]. The third one is in Proposition 3.3.2(a).

Definition 3.3.5. Let $M \in \mathbf{Mot}_\sim(F, A)$. For $n \in \mathbb{Z}$, we write $\nu(M) \ge n$ if $M \otimes \mathbb{L}^{\otimes -n}$ is effective.[5]

Proposition 3.3.6. *Suppose $A \supseteq \mathbb{Q}$ and the nilpotence conjecture holds for $\sim \ge \sim'$. Then*:

(a) *The functor $\mathbf{Mot}_\sim(F, A) \to \mathbf{Mot}_{\sim'}(F, A)$ is conservative, and for*

$$M \in \mathbf{Mot}_\sim(F, A)$$

any set of orthogonal idempotents in the endomorphism ring of $M_{\sim'}$ lifts.

(b) *If $M \in \mathbf{Mot}_\sim(F, A)$ and $M_{\sim'}$ is effective, then M is effective.*

(c) *If $M \in \mathbf{Mot}_\sim(F, A)$ and $\nu(M_{\sim'}) \ge n$, then $\nu(M) \ge n$.*

(d) [André 2004, Section 13.2.1] *The map $K_0(\mathbf{Mot}_\sim(F, A)) \to K_0(\mathbf{Mot}_{\sim'}(F, A))$ is an isomorphism* (here, the K_0-groups are those of additive categories).

[5] By convention, we say here that a motive $N \in \mathbf{Mot}_\sim(F, A)$ is *effective* if it is isomorphic to a motive of $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$.

Proof. (a) This is classical (see [Jannsen 1994, Lemma 5.4] for the second statement).

(b) By definition, $M_{\sim'}$ effective means that $M_{\sim'}$ is isomorphic to a direct summand of $h_{\sim'}(X)$ for some smooth projective X. By (a), one may lift the corresponding idempotent $e_{\sim'}$ to an idempotent endomorphism e of $h_{\sim}(X)$, and the isomorphism $M_{\sim'} \simeq (h_{\sim'}(X), e_{\sim'})$ to an isomorphism $M \simeq (h_{\sim}(X), e)$.

(c) This follows from (b) applied to $M \otimes \mathbb{L}^{\otimes -n}$.

(d) This follows from (a), since then the functor $\mathbf{Mot}_{\sim}(F, A) \to \mathbf{Mot}_{\sim'}(F, A)$ is conservative and essentially surjective. □

The importance of Conjecture 3.3.1 will appear again in the next subsection and in Section 4 (see Remark 4.3.4 and Proposition 4.4.1).

3.4. ***The Chow–Künneth decomposition.*** Here we take $(A, \sim) = (\mathbb{Q}, \mathrm{rat})$. Recall that Murre [1993] strengthened the standard conjecture C (algebraicity of the Künneth projectors) to the existence of a *Chow–Künneth decomposition*

$$h(X) \simeq \bigoplus_{i=0}^{2d} h_i(X)$$

in $\mathbf{Chow}(F, \mathbb{Q})$. (This is part of the Bloch–Beĭlinson–Murre conjecture appearing in Proposition 3.3.3(b)). By Proposition 3.3.6(a), the nilpotence conjecture together with the standard conjecture C imply the existence of Chow–Künneth decompositions.

Here are some cases where the existence of a Chow–Künneth decomposition is known independently of any conjecture:

(1) Varieties of dimension ≤ 2 [Murre 1990] (see also [Scholl 1994]). In fact, Murre constructs for any X a partial decomposition

$$h(X) \simeq h_0(X) \oplus h_1(X) \oplus h_{[2,2d-2]}(X) \oplus h_{2d-1}(X) \oplus h_{2d}(X).$$

(2) Abelian varieties [Shermenev 1974].

(3) Complete intersections in $\mathbb{P}^N$ (see the next subsection).

(4) If X and Y have a Chow–Künneth decomposition, then so does $X \times Y$.

Suppose that the nilpotence conjecture holds for $h(X) \in \mathbf{Chow}(F, \mathbb{Q})$ and that homological and numerical equivalences coincide on $X \times X$. The latter then implies the standard conjecture C for X [Kleiman 1994], hence the existence of a Chow–Künneth decomposition by the remark above. In [Kahn et al. 2007, Theorem 14.7.3(iii)], it is proven:

Proposition 3.4.1. *Under these hypotheses, there exists a further decomposition for each* $i \in [0, 2d]$:

$$h_i(X) \simeq \bigoplus h_{i,j}(X)(j),$$

such that $h_{i,j}(X) = 0$ *for* $j \notin [0, [i/2]]$ *and, for each* j, $\nu(h^{\text{hom}}_{i,j}(X)) = 0$ (see Definition 3.3.5). *Moreover, one has isomorphisms*

$$h_{2d-i,d-i+j}(X) \xrightarrow{\sim} h_{i,j}(X) \tag{3.4.2}$$

for $i \le d$. *In particular,* $\nu(h_i(X)) > 0$ *for* $i > d$.

Let us justify the last assertion; the isomorphisms (3.4.2) imply that, when $i > d$, $h_{i,j}(X) = 0$ for $j < i - d$.

Since $\mathbf{Chow}^{\text{eff}}(F, \mathbb{Q}) \to \mathbf{Chow}(F, \mathbb{Q})$ is fully faithful, all the above (refined) Chow–Künneth decompositions hold for the effective Chow motives

$$h(X) \in \mathbf{Chow}^{\text{eff}}(F, \mathbb{Q}).$$

We deduce:

Corollary 3.4.3. *Under the nilpotence conjecture and the conjecture that homological and numerical equivalences coincide, for any smooth projective variety* X *the image of its Chow–Künneth decomposition in* $\mathbf{Chow}^{\text{o}}(F, \mathbb{Q})$ *is of the form*

$$h^{\text{o}}(X) \simeq \bigoplus_{i=0}^{d} h^{\text{o}}_i(X).$$

Moreover, with the notation of Proposition 3.4.1, one has

$$h^{\text{o}}_i(X) \simeq h^{\text{o}}_{i,0}(X) \quad \text{for } i \le d.$$

Examples where this conclusion is true unconditionally follow faithfully the examples where the Chow–Künneth decomposition is unconditionally known:

Proposition 3.4.4. *The conclusion of Corollary 3.4.3 holds in the following cases*:

(1) *Varieties of dimension* ≤ 2.

(2) *Abelian varieties.*

(3) *Complete intersections in* $\mathbb{P}^N$.

(4) *If* X *and* Y *have a Chow–Künneth decomposition and verify this conclusion, then so does* $X \times Y$.

Proof. In cases (1) and (2), the conclusion holds because one has "Lefschetz isomorphisms" $h_{2d-i}(X) \xrightarrow{\sim} h_i(X)(d-i)$ for $i > d$. For curves, it is trivial, for surfaces they are constructed in [Murre 1990] (see [Scholl 1994, Theorem 4.4(ii)]; the isomorphism is constructed for $i = 0, 1$ and any X), and for abelian varieties they are constructed in [Shermenev 1974]. For (3), see the next subsection. Finally, (4) is clear. □

In the case of a surface, Kahn et al. [2007] construct a refined Chow–Künneth decomposition

$$h(X) = h_0(X) \oplus h_1(X) \oplus \mathrm{NS}_X(1) \oplus t_2(X) \oplus h_3(X) \oplus h_4(X),$$

where NS_X is the Artin motive corresponding to the Galois representation defined by $\mathrm{NS}(\overline{X}) \otimes \mathbb{Q}$, and $t_2(X)$ is the *transcendental part of* $h(X)$. (In the notation of Proposition 3.4.1, $h_{2,0}(X) = t_2(X)$ and $h_{2,1}(X) = \mathrm{NS}_X$.) This translates on the birational motive of X as

$$h^{\mathrm{o}}(X) = h_0^{\mathrm{o}}(X) \oplus h_1^{\mathrm{o}}(X) \oplus t_2^{\mathrm{o}}(X).$$

3.5. ***Motives of complete intersections.*** These computations will be used in Section 4. Here we take $A \supseteq \mathbb{Q}$.

For convenience, we take the notation of [Deligne 1973]; so let $X \subset \mathbb{P}^r$ be a smooth complete intersection of multidegree $\underline{a} = (a_1, \dots, a_d)$, and let

$$n = r - d = \dim X.$$

Then the cohomology of X coincides with the cohomology of $\mathbb{P}^r$ except in middle dimension [Deligne 1973], and in particular it is fully algebraic except in middle dimension. This allows us to easily write down a Chow–Künneth decomposition for $h(X)$ in the sense of [Murre 1993] (see also [Esnault et al. 1997, Corollary 5.3]):

(1) (Murre) For each $i \neq n/2$, let $c^i \in \mathcal{Z}^i(X)$ be an algebraic cycle whose cohomology class generates $H^{2i}(X)$ (here H is some Weil cohomology). Then the Chow–Künneth projector π_{2i} is given by $c^i \times c^{n-i}$. We take $\pi_j = 0$ for j odd $\neq n$, and $\pi_n := \Delta_X - \sum_{j \neq n} \pi_j$.

(2) Consider the inclusion $i : X \hookrightarrow \mathbb{P}^r$. This yields morphisms of motives

$$h(\mathbb{P}^r)(-d) \xrightarrow{i^*} h(X) \xrightarrow{i_*} h(\mathbb{P}^r).$$

Given the decomposition $h(\mathbb{P}^r) \simeq \bigoplus_{j=0}^r \mathbb{L}^j$, this yields for each $j \in [0, n]$ morphisms

$$\mathbb{L}^j \xrightarrow{i_j^*} h(X) \xrightarrow{i_*^j} \mathbb{L}^j$$

with composition $a = \prod a_i$. Then $(1/a) i_j^* i_*^j$ defines the $2i$-th Chow–Künneth projector of X (π_{2i} in (1)), except if $2i = n$. Let $\pi_n^{\mathrm{prim}} := 1_{h(X)} - \sum_{i=0}^n (1/a) i_j^* i_*^j$ and let the image $p_n(X)$ of the projector π_n^{prim} be *the primitive part* of $h_n(X)$.

Note that the Chow–Künneth projectors of (1) and (2) are actually equal. Let us record here the corresponding (refined) Chow–Künneth decomposition:

$$h(X) \simeq \mathbb{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n \oplus p_n(X). \tag{3.5.1}$$

Lemma 3.5.2. (a) *Homological and numerical equivalences agree on all* (*rational*) *Chow groups of X provided n is odd or* (*if* char $F=0$) *the Hodge realisation of $p_n(X)$ does not contain any direct summand isomorphic to $\mathbb{L}^{n/2}$.*

(b) *Suppose* (a) *is satisfied. Then for any adequate pair* $(\sim, A)$ *with* $A\supseteq\mathbb{Q}$ *and any* $j\in[0,n]$, *we have*

$$\mathbf{Mot}_\sim(F,A)(\mathbb{L}^j, p_n(X)) = \operatorname{Ker}(A_j^\sim(X,A)\to A_j^{\mathrm{num}}(X,A)).$$

Proof. We have

$$\begin{aligned} A_j^\sim(X,A) &= \mathbf{Mot}_\sim(F,A)(\mathbb{L}^j, h(X))\\ &= \bigoplus_{i=0}^n \mathbf{Mot}_\sim(F,A)(\mathbb{L}^j,\mathbb{L}^i)\oplus \mathbf{Mot}_\sim(F,A)(\mathbb{L}^j,p_n(X))\\ &= \mathbf{Mot}_\sim(F,A)(\mathbb{L}^j,\mathbb{L}^j)\oplus \mathbf{Mot}_\sim(F,A)(\mathbb{L}^j,p_n(X)). \end{aligned}$$

For $\sim\,=$ hom, we have $\mathbf{Mot}_\sim(F,A)(\mathbb{L}^j,p_n(X))=0$ by weight reasons for $2j\neq n$ and under the hypothesis of (a) for $2j=n$ (note that the Hodge realisation of $p_n(X)$ is semisimple, as a polarisable Hodge structure). Hence the same is true for any $\sim$ finer than hom, in particular $\sim\,=$ num. This proves (a). Moreover, $\mathbf{Mot}_\sim(F,A)(\mathbb{L}^j,\mathbb{L}^j)=A$ for any choice of $\sim$. Hence (b). ☐

Equation (3.5.1) shows that the birational motive of X reduces to $\mathbb{1}\oplus p_n^\sim(X)^{\mathrm{o}}$. In fact, it is possible to be much more precise:

Proposition 3.5.3. *Let $\underline{a}=(a_1,\dots,a_d)$ be the multidegree of $X\subset\mathbb{P}^r$.*

(a) *If $a_1+\dots+a_d\le r$, then $h^{\mathrm{o}}_{\mathrm{rat}}(X)=\mathbb{1}$.*

(b) *If $a_1+\dots+a_d> r$, then $h^{\mathrm{o}}_{\mathrm{num}}(X)\neq\mathbb{1}$ (equivalently, $p_n^{\mathrm{num}}(X)^{\mathrm{o}}\neq 0$) provided* char $F=0$ *or X is generic.*

Proof. (a) Under the hypothesis, we conclude from Roĭtman's theorem [1980a] that $CH_0(X_K)\otimes\mathbb{Q}=\mathbb{Q}$ for any extension K/F.[6] Assertion (a) then follows from Proposition 3.1.1.

(b) It suffices to prove the statement for homological equivalence, since the kernel of $\mathbf{Mot}_{\mathrm{hom}}(F,\mathbb{Q})(h(X),h(X))\to\mathbf{Mot}_{\mathrm{num}}(F,\mathbb{Q})(h(X),h(X))$ is a nilpotent ideal (see Propositions 3.3.2(b) and 3.3.6(a)).

If char $F=0$, we may use Hodge cohomology and Deligne's theorem [1973, Théorème 2.5(ii), p. 54]. Namely, with the notation of [loc. cit.], the condition

[6]Of course we could also invoke Proposition 3.1.4(2) since X is Fano, hence rationally chain connected, but this theorem of Campana [1992] and Kollár, Miyaoka and Mori [Kollár et al. 1992] was proven much later than Roĭtman's work.

$p_n^{\text{hom}}(X)^{\text{o}} = 0$ implies $h_0^{0,n}(\underline{a}) = 0$, which is equivalent by Deligne's theorem to

$$0 \le \left[\frac{n+d-\sum a_i}{\sup(a_i)}\right],$$

that is, $\sum a_i \le n+d = r$.

If $\operatorname{char} F > 0$ and X is generic, we may use Katz's theorem [1973, p. 382, Theorem 4.1]. □

Remarks 3.5.4. (1) Katz also has a result [1973, Theorem 4.2] concerning a generic hyperplane section of a given complete intersection.

(2) It seems possible to remove the genericity assumption in positive characteristic by lifting the coefficients of the equations defining X to characteristic 0. We have not worked out the details.

4. On adjoints and idempotents

We now want to examine two related questions:

(1) Does the projection functor $\mathbf{Mot}_\sim^{\text{eff}}(F, A) \to \mathbf{Mot}_\sim^{\text{eff}}(F, A)/\mathcal{L}_\sim$ have a right adjoint? This question was raised by Luca Barbieri-Viale and is closely related to a conjecture of Voevodsky [1992, Conjecture 0.0.11].

(2) Is the category $\mathbf{Mot}_\sim^{\text{eff}}(F, A)/\mathcal{L}_\sim$ pseudoabelian, i.e., is it superfluous to take the pseudoabelian envelope in Definition 2.2.8?

The answer to both questions is "yes" for $\sim = \text{num}$ and $A \supseteq \mathbb{Q}$, as an easy consequence of Jannsen's semisimplicity theorem for numerical motives [1992]. In fact:

Proposition 4.0.1 [Kahn 2009, Proposition 7.7]. (a) *The projection functor*

$$\pi : \mathbf{Mot}_{\text{num}}^{\text{eff}} \to \mathbf{Mot}_{\text{num}}^{\text{o}}$$

is essentially surjective.

(b) *π has a section i which is also a left and right adjoint.*

(c) *The category $\mathbf{Mot}_{\text{num}}^{\text{eff}}$ is the coproduct of $\mathbf{Mot}_{\text{num}}^{\text{eff}} \otimes \mathbb{L}$ and $i(\mathbf{Mot}_{\text{num}}^{\text{o}})$, i.e., any object of $\mathbf{Mot}_{\text{num}}^{\text{eff}}$ can be uniquely written as a direct sum of objects of these two subcategories.*

In the sequel, we want to examine these questions for a general adequate pair; see Theorems 4.3.2 and 4.3.3 for (1) and Proposition 4.4.1 for (2). This requires some preparation.

4.1. ***A lemma on base change.*** Let $P : \mathcal{A} \to \mathcal{B}$ be a functor. Recall that one says that "its" right adjoint is *defined at* $B \in \mathcal{B}$ if the functor

$$\mathcal{A} \ni A \mapsto \mathcal{B}(PA, B)$$

is representable. We write $P^\sharp B$ for a representing object (unique up to unique isomorphism).

Let

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{C} & \xrightarrow{\psi} & \mathcal{D} \end{array}$$

be a naturally commutative diagram of pseudoabelian additive categories, and let $A \in \mathcal{A}$.

Suppose that "the" right adjoint $P^\sharp$ of P is defined at $PA \in \mathcal{C}$ and that the right adjoint $Q^\sharp$ of Q is defined at $\psi PA \simeq Q\varphi A$. We then have two corresponding unit maps (adjoint to the identities of PA and $Q\varphi A$)

$$\varepsilon_P : A \to P^\sharp PA, \quad \varepsilon_Q : \varphi A \to Q^\sharp Q\varphi A.$$

Lemma 4.1.1. *Suppose that ε_Q is an isomorphism. Then $\varphi\varepsilon_P$ has a retraction. If moreover φ is full and* $\operatorname{Ker}(\operatorname{End}_{\mathcal{A}}(A) \to \operatorname{End}_{\mathcal{B}}(\varphi A))$ *is a nil ideal, then ε_P has a retraction.*

Proof. Let $\eta_P : PP^\sharp PA \to PA$ be the counit map of the adjunction at PA (adjoint to the identity of $P^\sharp PA$), and let $u : Q\varphi A \xrightarrow{\sim} \psi PA$ and $v : Q\varphi P^\sharp PA \xrightarrow{\sim} \psi PP^\sharp PA$ be the natural isomorphisms from $Q\varphi$ to ψP evaluated respectively at A and $P^\sharp PA$. We then have a composition

$$Q\varphi P^\sharp PA \xrightarrow{v} \psi PP^\sharp PA \xrightarrow{\psi\eta_P} \psi PA,$$

which yields by adjunction a "base change morphism"

$$\varphi P^\sharp PA \xrightarrow{b} Q^\sharp\psi PA.$$

Inspection shows that the diagram

$$\begin{array}{ccc} \varphi A & \xrightarrow{\varphi\varepsilon_P} & \varphi P^\sharp PA \\ {\scriptstyle \varepsilon_Q}\downarrow & & \downarrow{\scriptstyle b} \\ Q^\sharp Q\varphi A & \xrightarrow{Q^\sharp u} & Q^\sharp\psi PA \end{array}$$

commutes. The first claim follows, and the second claim follows from the first. □

4.2. ***Right adjoints.*** We come back to question (1), posed at the beginning of this section. In [Kahn et al. 2007, Remark 14.8.7; Kahn 2009, Remark 7.8(3)], it was announced that one can show the nonexistence of the right adjoint for $\sim = \text{rat}$, using the results of [Huber 2008, Appendix]. The proof turns out not to be exactly along these lines, but is closely related; see Lemma 4.2.1 and Theorems 4.3.2 and 4.3.3.

Let us abbreviate the notation to $\mathbf{Mot}^{\text{eff}} = \mathbf{Mot}^{\text{eff}}_{\sim}(F, A)$, $\mathbf{Mot}^{\text{o}} = \mathbf{Mot}^{\text{o}}_{\sim}(F, A)$. Let $P : \mathbf{Mot}^{\text{eff}} \to \mathbf{Mot}^{\text{o}}$ denote the projection functor, and let $P^{\sharp}$ denote its (a priori partially defined) right adjoint. Let $\mathcal{L}^{\perp}$ be the full subcategory of $\mathbf{Mot}^{\text{eff}}$ consisting of those M such that $\text{Hom}(N(1), M) = 0$ for all $N \in \mathbf{Mot}^{\text{eff}}$. Recall from [Kahn et al. 2007, Proposition 7.8.1] that

- if $P^{\sharp}$ is defined at M, then $P^{\sharp}M \in \mathcal{L}^{\perp}$;
- the full subcategory $\mathbf{Mot}^{\sharp}$ of $\mathbf{Mot}^{\text{o}}$ where $P^{\sharp}$ is defined equals $P(\mathcal{L}^{\perp})$;
- $P^{\sharp}$ and the restriction of P to $\mathcal{L}^{\perp}$ define quasi-inverse equivalences of categories between $\mathcal{L}^{\perp}$ and $\mathbf{Mot}^{\sharp}$.

The right adjoint $P^{\sharp}$ is defined at birational motives of varieties of dimension ≤ 2 for any adequate pair $(A, \sim)$ such that $A \supseteq \mathbb{Q}$ by [Kahn et al. 2007, Corollary 7.8.6]. (The proof there is given for $(A, \sim) = (\mathbb{Q}, \text{rat})$, but the argument works in general.) Recall that

$$P^{\sharp}h^{\text{o}}(C) = \mathbb{1} \oplus h_1(C), \quad P^{\sharp}h^{\text{o}}(S) = \mathbb{1} \oplus h_1(S) \oplus t_2(S)$$

with the notation at the end of Section 3.4, where C is a curve and S is a surface.

The following lemma gives a sufficient condition for the nonexistence of $P^{\sharp}PM$ for an effective motive M.

Lemma 4.2.1. *Let* $(\mathbb{Q}, \sim)$ *be an adequate pair, and let* $M \in \mathbf{Mot}^{\text{eff}}_{\sim}(F, \mathbb{Q})$. *Assume that*

(i) $M_{\text{num}} \in \mathbf{Mot}^{\text{eff}}_{\text{num}}(F, \mathbb{Q})$ *does not contain any direct summand divisible by* $\mathbb{L}$;

(ii) $\text{Ker}(\text{End}(M) \to \text{End}(M_{\text{num}}))$ *is a nilideal*;

(iii) *there exists* $r > 0$ *such that* $\text{Hom}(\mathbb{L}^r, M) \neq 0$.

Then $P^{\sharp}PM$ *does not exist.*

Proof. Suppose that $P^{\sharp}$ is defined at PM. Consider the unit map

$$\varepsilon_{\sim} : M \to P^{\sharp}PM. \tag{4.2.2}$$

For $\sim = \text{num}$, $P^{\sharp}_{\text{num}}P_{\text{num}}M_{\text{num}}$ exists by Proposition 4.0.1. Moreover, part (c) of this proposition shows that, under condition (i) of the lemma, ε_{num} is an isomorphism. By Lemma 4.1.1, the image of $\varepsilon_{\sim}$ modulo numerical equivalence then has a retraction, and so does $\varepsilon_{\sim}$ itself under condition (ii). If this is the case, $M \in \mathcal{L}^{\perp}$, and in particular, $\text{Hom}(\mathbb{L}^r, M) = 0$ for all $r > 0$, contradiction. ∎

4.3. ***Counterexamples.*** To give examples where the conditions of Lemma 4.2.1 are satisfied, we appeal as in [Huber 2008] to the nontriviality of the Griffiths group.

We start with an example which a priori only works for a specific adequate equivalence, because the proof is simpler. Unlike in [Huber 2008], we don't need the full force of Clemens' theorem [1983, Theorem 0.2], but merely the previous results of Griffiths [1969].

Definition 4.3.1 ("Abel–Jacobi equivalence"). Let $k = \mathbb{C}$. For X smooth projective, $\mathcal{Z}^j_{\mathrm{AJ}}(X, \mathbb{Q})$ is the image of $CH^j(X) \otimes \mathbb{Q}$ in Deligne–Beĭlinson cohomology via the (Deligne–Beĭlinson) cycle class map [Esnault and Viehweg 1988]. This defines an adequate equivalence relation.

Theorem 4.3.2. *Let $F = \mathbb{C}$ and $\sim = \mathrm{AJ}$. Then*:

(a) *Condition* (ii) *of Lemma 4.2.1 is satisfied for any pure motive M. Let X be a generic hypersurface of degree a in $\mathbb{P}^{n+1}$.*

(b) *Condition* (i) *of Lemma 4.2.1 is satisfied for $M = p_n(X)$* (see (3.5.1)) *provided X is not a quadric, a cubic surface or an even-dimensional intersection of two quadrics, and $a \geq n+1$.*

(c) *If $n = 2m-1$ is odd and $a \geq 2+3/(m-1)$, then condition* (iii) *of Lemma 4.2.1 is satisfied for $r = m-1$.*

(d) *$P^\sharp$ is not defined at $h^o(X)$ in the following cases*: *n is odd and*

  (i) *if $n = 3$ then $a \geq 5$;*

  (ii) *if $n > 3$ then $a \geq n+1$.*

Proof. We see that (a) holds because $\mathrm{Ker}(\mathrm{End}_{\mathrm{AJ}}(M) \to \mathrm{End}_{\mathrm{hom}}(M))$ has square 0 [Esnault and Viehweg 1988, Proposition 7.10][7] and $\mathrm{Ker}(\mathrm{End}_{\mathrm{hom}}(M) \to \mathrm{End}_{\mathrm{num}}(M))$ is nilpotent.

(b) By [Peters and Steenbrink 2003, Example 5 and Corollary 18], the Hodge realisation $P_n(X)$ of $p_n(X)$ is an absolutely simple pure Hodge structure; this, together with Proposition 3.5.3(b), is amply sufficient to imply condition (i) of Lemma 4.2.1.

(c) By [Griffiths 1969, Corollaries 13.2 and 14.2],

$$\mathrm{Ker}(A^{\sim}_{m-1}(X, \mathbb{Q}) \to A^{\mathrm{num}}_{m-1}(X, \mathbb{Q})) \neq 0.$$

But by Lemma 3.5.2, this group is $\mathrm{Hom}(\mathbb{L}^{m-1}, p_n(X))$.

[7] A more functorial justification is: (1) Deligne–Beĭlinson cohomology can be computed as absolute Hodge cohomology as in [Beilinson 1986]; (2) the category of polarisable $\mathbb{Q}$-mixed Hodge structures has Ext-dimension 1.

(d) Note that, by the refined Chow–Künneth decomposition (3.5.1), $P^\sharp$ is defined at $Ph(X)$ if and only if it is defined at $Pp_n(X)$. The conclusion now follows from Lemma 4.2.1 and from collecting the results of (a), (b) and (c). □

To get a counterexample with rational equivalence, we appeal to a result of Nori [1989]. We thank Srinivas for pointing out this reference.

Theorem 4.3.3. *Let X be a generic abelian threefold over $k = \mathbb{C}$. If $\sim\, \geq$ alg, then $P^\sharp$ is not defined at $h^{\mathrm{o}}_\sim(X)$.*

Proof. The proof is similar to that of Theorem 4.3.2, except that the motive of an abelian variety is more complicated than that of a hypersurface. We only sketch the argument (details will appear elsewhere).

It is enough to show that $P^\sharp$ is not defined at $h^{\mathrm{o}}_{3,0}(X)$, where $h_{3,0}(X)$ is as in Proposition 3.4.1 (here we use that the nilpotence conjecture is true for motives of abelian varieties, see Proposition 3.3.2(a)). We check the conditions of Lemma 4.2.1 for $M = h_{3,0}(X)$. Item (i) is true by definition; and (ii) is true by Proposition 3.3.2(a). For (iii), one can show that computing the decomposition

$$A_1^\sim(X) = \mathbf{Mot}^{\mathrm{eff}}_\sim(\mathbb{L}, h(X)) \simeq \bigoplus_{i=0}^{6}\bigoplus_{j=0}^{[i/2]} \mathbf{Mot}^{\mathrm{eff}}_\sim(\mathbb{L}, h_{i,j}(X)(j))$$

yields a surjection

$$\mathbf{Mot}^{\mathrm{eff}}_\sim(\mathbb{L}, h_{3,0}(X)) \twoheadrightarrow \mathrm{Griff}_1(X)$$

for $\sim\, \geq$ alg, where $\mathrm{Griff}_1(X) = \mathrm{Ker}(A_1^{\mathrm{alg}}(X) \to A_1^{\mathrm{num}}(X))$ is the Griffiths group of X. By Nori's theorem [1989], $\mathrm{Griff}_1(X) \neq 0$, and the proof is complete. □

Remark 4.3.4. It is easy to get examples of any dimension ≥ 4 by multiplying the example of Theorem 4.3.3 with $\mathbb{P}^n$.

4.4. *Idempotents.* We now address question (2) from the beginning of this section.

Proposition 4.4.1. *Let $(A, \sim)$ be an adequate pair with $A \supseteq \mathbb{Q}$, and let $\mathcal{M}$ be a full subcategory of $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ closed under direct summands. If Conjecture 3.3.1 holds for the objects of $\mathcal{M}$, then the category $\mathcal{M}/\mathcal{L}_\sim$ is pseudoabelian.*

Proof. Let $\mathcal{M}_{\mathrm{num}}$ denote the pseudoabelian envelope of the image of $\mathcal{M}$ in

$$\mathbf{Mot}^{\mathrm{eff}}_{\mathrm{num}}(F, A).$$

We have a commutative diagram of categories:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{P} & \mathcal{M}/\mathcal{L}_\sim \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar\pi} \\ \mathcal{M}_{\mathrm{num}} & \xrightarrow{P_{\mathrm{num}}} & \mathcal{M}_{\mathrm{num}}/\mathcal{L}_{\mathrm{num}} \end{array}$$

Under the hypothesis, π is essentially surjective (one can lift idempotents). Hence $\bar{\pi}$ is essentially surjective as well. Since P is essentially surjective and π, P_{num} are full, $\bar{\pi}$ is full, and its kernel is locally nilpotent as a quotient of the kernel of π (fullness of P). Thus $\bar{\pi}$ is full, essentially surjective and conservative.

Since $\mathbf{Mot}^{\mathrm{eff}}_{\mathrm{num}}(F, A)$ is abelian semisimple, $\mathcal{M}_{\mathrm{num}}$ is also abelian semisimple, hence so is $\mathcal{M}_{\mathrm{num}}/\mathcal{L}_{\mathrm{num}}$ which is in particular pseudoabelian.

Let now $M \in \mathcal{M}/\mathcal{L}_\sim$, and let $p = p^2 \in \mathrm{End}(M)$. Write $M_{\mathrm{num}} \simeq M_1 \oplus M_2$, where $M_1 = \mathrm{Im}\, p_{\mathrm{num}}$ and $M_2 = \mathrm{Ker}\, p_{\mathrm{num}}$. By essential surjectivity, we may lift M_1 and M_2 to objects $\widetilde{M}_1, \widetilde{M}_2 \in \mathcal{M}/\mathcal{L}_\sim$.

By fullness, we may lift the isomorphism $M_1 \oplus M_2 \xrightarrow{\sim} M_{\mathrm{num}}$ to a morphism $\widetilde{M}_1 \oplus \widetilde{M}_2 \to M$ in $\mathcal{M}/\mathcal{L}_\sim$, and this lift is an isomorphism by conservativity. This concludes the proof. □

Example 4.4.2. Proposition 4.4.1 applies taking for $\mathcal{M}$ the category of motives of abelian type (direct summands of the tensor product of an Artin motive and the motive of an abelian variety), since such motives are finite-dimensional [Kimura 2005].

The situation when A does not contain $\mathbb{Q}$, for example $A = \mathbb{Z}$, is unclear.

5. Birational motives and birational categories

In this section, we relate the categories studied in [Kahn and Sujatha 2015a] with the categories of pure birational motives introduced here.

5.1. From (2.4.1), we get a composite functor:

$$S_r^{-1}\, \mathbf{Sm}^{\mathrm{proj}}(F) \to S_r^{-1}\, \mathbf{Chow}^{\mathrm{eff}}(F) \to \mathbf{Chow}^{\mathrm{o}}(F). \tag{5.1.1}$$

The morphisms in the first category can be described by means of R-equivalence classes [Kahn and Sujatha 2015a, Theorem 6.6.3, Corollary 6.6.4 and Remark 6.6.5]; by Lemma 2.3.7, those in the last category can be described by means of Chow groups of 0-cycles. One checks easily that the action of the composite functor on Hom sets is just the map which sends R-equivalence classes of rational points to 0-cycles modulo rational equivalence. This puts this map within a functorial setting.

Let us now recall further results from [Kahn and Sujatha 2015a]. Let $\mathbf{place}(F)$ denote the category of finitely generated extensions of F, with F-places as morphisms. In [Kahn and Sujatha 2015a, (4.3)], we constructed a functor

$$\mathbf{place}_*(F)^{\mathrm{op}} \to S_b^{-1}\, \mathbf{Sm}^{\mathrm{prop}}(F),$$

hence a functor

$$S_r^{-1}\, \mathbf{place}_*(F)^{\mathrm{op}} \to S_r^{-1}\, \mathbf{Sm}^{\mathrm{prop}}(F),$$

where $\mathbf{place}_*(F)$ denotes the full subcategory of $\mathbf{place}(F)$ defined by those K/F which have a cofinal set of smooth proper models, and $S_r \subset Ar(\mathbf{place}(F))$ denotes the set of purely transcendental extensions. The same arguments as in [loc. cit.] give an analogous functor

$$S_r^{-1}\,\mathbf{place}_\sharp(F)^{\mathrm{op}} \to S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F), \tag{5.1.2}$$

where $\mathbf{place}_\sharp(F)$ has the same definition as $\mathbf{place}_*(F)$, replacing "smooth proper" by "smooth projective". Composing (5.1.2) with (5.1.1), we get a functor

$$S_r^{-1}\,\mathbf{place}_\sharp(F)^{\mathrm{op}} \to \mathbf{Chow}^{\mathrm{o}}(F). \tag{5.1.3}$$

We can describe the image under this functor of a place $\lambda : K \rightsquigarrow L$ in $CH_0(X_L)$, where X is a smooth projective model of K: it is just the class of the centre of λ. Hence the image of (5.1.3) on morphisms consists of the classes of L-rational points. This answers a question of Déglise.

In characteristic 0, $\mathbf{place}_\sharp(F) = \mathbf{place}(F)$ by resolution of singularities and $S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \xrightarrow{\sim} S_r^{-1}\,\mathbf{Sm}(F)$ by [Kahn and Sujatha 2007, Proposition 8.5]. In characteristic p, we would ideally like to get functors

$$S_r^{-1}\,\mathbf{place}(F)^{\mathrm{op}} \to \mathbf{Chow}^{\mathrm{o}}(F), \quad S_r^{-1}\,\mathbf{Sm}(F) \to \mathbf{Chow}^{\mathrm{o}}(F)$$

extending (5.1.1) and (5.1.3). Constructing the first functor looks technically difficult: we shall content ourselves with extending [Kahn 2009, Remark 7.4] to all finitely generated fields K/F, by using an adjunction result from [Kahn 2015]; this will not be used in the rest of the paper. The second functor is constructed in [Kahn and Sujatha 2015b, Corollary 2.4.2].

Proposition 5.1.4. *Let p be the exponential characteristic of F.*

(a) *There is a unique functor (up to unique isomorphism)*

$$h^{\mathrm{o}} : S_r^{-1}\,\mathbf{field}(F)^{\mathrm{op}} \to \mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z}[1/p])$$

such that, for any $K \in \mathbf{field}(F)$ and any $Y \in \mathbf{Sm}^{\mathrm{proj}}(F)$, one has

$$\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z}[1/p])(h^{\mathrm{o}}(K), h^{\mathrm{o}}(Y)) \simeq CH_0(Y_K) \otimes \mathbb{Z}[1/p]. \tag{5.1.5}$$

This functor transforms purely inseparable extensions into isomorphisms.

(b) *If $K \subseteq L$, the map $h^{\mathrm{o}}(L) \to h^{\mathrm{o}}(K)$ has a section.*

(c) *We have $h^{\mathrm{o}}(K) = h^{\mathrm{o}}(X)$ if $K = F(X)$ for a smooth projective variety X. Moreover, if $K = F(X)$, $L = F(Y)$ with X, Y smooth projective, and if $f : K \to L$ corresponds to a rational map $\varphi : Y \dashrightarrow X$, then $h^{\mathrm{o}}(f)$ is given by the graph of φ.*

Proof. (a) Note that the isomorphism (5.1.5) determines $h^o(K)$ up to unique isomorphism, by Yoneda's lemma. By Lemma 2.3.7 applied over K, this isomorphism may be rewritten as

$$\mathbf{Chow}^o(F, \mathbb{Z}[1/p])(h^o(K), h^o(Y)) \simeq \mathbf{Chow}^o(K, \mathbb{Z}[1/p])(\mathbb{1}_K, h^o(Y_K)),$$

where $\mathbb{1}_K = h^o(\operatorname{Spec} K)$ is the unit object of $\mathbf{Chow}^o(K, \mathbb{Z}[1/p])$.

By [Kahn 2015, Theorem 6.5], the base-change functor

$$\mathbf{Chow}^o(F, \mathbb{Z}[1/p]) \to \mathbf{Chow}^o(K, \mathbb{Z}[1/p])$$

has a left adjoint $l_{K/F}$. Therefore we may define $h^o(K) = l_{K/F}(\mathbb{1}_K)$.

Suppose $F \to K \xrightarrow{f} L$ are successive finitely generated extensions. Since the base-change of $\mathbb{1}_K$ is $\mathbb{1}_L$, the identity map $\mathbb{1}_L \to \mathbb{1}_L$ gives by adjunction a map

$$l_{L/K}\mathbb{1}_L \to \mathbb{1}_K,$$

hence a map

$$h^o(f) : h^o(L) = r_{L/F}(\mathbb{1}_L) \to r_{K/F}(\mathbb{1}_K) = h^o(K).$$

We just used the transitivity of adjoints; using it a second time on a 3-layer extension shows that we have indeed defined a functor $\mathbf{field}(F)^{\mathrm{op}} \to \mathbf{Chow}^o(F, \mathbb{Z}[1/p])$.

Suppose that $L = K(t)$. Then $l_{L/K}(\mathbb{1}_L) = h^o(\mathbb{P}^1) = \mathbb{1}_K$, hence $h^o(f)$ is an isomorphism. This shows that our functor induces a functor

$$h^o : S_r^{-1}\,\mathbf{field}(F)^{\mathrm{op}} \to \mathbf{Chow}^o(F, \mathbb{Z}[1/p]),$$

as required.

Suppose now that $K \xrightarrow{f} L$ is a finite and purely inseparable extension of finitely generated fields over F. If X is a smooth projective K-variety, then the map $CH_0(X) \otimes \mathbb{Z}[1/p] \to CH_0(X_L) \otimes \mathbb{Z}[1/p]$ is an isomorphism by Lemma 1.7.1; this shows that $l_{L/K}(\mathbb{1}_L) = \mathbb{1}_K$, hence that $h^o(f)$ is invertible.

(b) The proof is the same as in [Kahn 2009, Remark 7.4]: Write L as a finite purely inseparable extension of a finite separable extension of a purely transcendental extension of K. Then (a) reduces us to the case where L/K is finite and separable. We may write $L = \operatorname{Spec} X$, where X is a 0-dimensional smooth projective K-variety, and $l_{L/K}(\mathbb{1}_L) = h^o(X)$. The conclusion now follows from Lemma 1.5.2.

(c) If $K = F(X)$ for X smooth projective, then Lemma 2.3.7 and Yoneda's lemma show that $h^o(K) \simeq h^o(X)$. For the claim on morphisms, we are reduced (again by Yoneda's lemma) to determining the map

$$\mathbf{Chow}^o(F, \mathbb{Z}[1/p])(h^o(K), h^o(Z)) \xrightarrow{h^o(f)^*} \mathbf{Chow}^o(F, \mathbb{Z}[1/p])(h^o(L), h^o(Z))$$

for a smooth projective F-variety Z. By definition of $h^o(f)$, an adjunction computation shows that this map may be rewritten as the map

$$CH_0(Z_K)\otimes\mathbb{Z}[1/p]=\mathbf{Chow}^o(K,\mathbb{Z}[1/p])(\mathbb{1}_K,h^o(Z_K))\\ \to \mathbf{Chow}^o(L,\mathbb{Z}[1/p])(\mathbb{1}_L,h^o(Z_L))=CH_0(Z_L)\otimes\mathbb{Z}[1/p]$$

given by extension of scalars. The conclusion now follows from Lemma 2.3.9. □

6. Birational motives and cycle modules

Rost [1996] introduced the notion of cycle module and cycle cohomology; he proved [1996, Corollary 12.10] that for any cycle module M, $A^0(X, M)$ is a birational invariant of smooth projective varieties X. In [Merkurjev 2001, Corollary 3.5], he extended this to $A_0(X, M)$ by introducing the category $\mathbf{Chow}^o(F)$ of Definition 2.3.6 (independently from this paper). In the first subsection, we essentially reproduce Section 3 of [Merkurjev 2001]; we don't claim any originality here, but hope this will be a service to the reader since this preprint remains unpublished. In the second subsection, we connect these results with more recent work of Merkurjev.

To lighten notation, we drop the reference to the base field F in the relevant categories.

6.1. ***The functors A^0 and A_0.*** Let $M=(M_n)_{n\in\mathbb{Z}}$ be a cycle module over F in the sense of [Rost 1996]; recall that this is a functor from **field** to graded abelian groups, provided with extra structure (transfers, residues, cup-products by units) subject to certain axioms. To a smooth variety $X\in\mathbf{Sm}$, one associates its *cycle cohomology with coefficients in M* [Rost 1996, Section 5],

$$A^p(X,M_n)=H\Bigl(\cdots\xrightarrow{\partial}\bigoplus_{x\in X^{(p)}}M_{n-p}(F(x))\xrightarrow{\partial}\cdots\Bigr),$$

where the differentials ∂ are induced by the residue homomorphisms. We also have the homological notation

$$A_p(X,M_n)=H\Bigl(\cdots\xrightarrow{\partial}\bigoplus_{x\in X_{(p)}}M_{n+p}(F(x))\xrightarrow{\partial}\cdots\Bigr),$$

so that $A_p(X,M_n)=A^{d-p}(X,M_{d+n})$ if X is purely of dimension d.

Proposition 6.1.1. (a) *Let X, Y be two smooth projective varieties and let*

$$\alpha\in CH_{\dim X}(X\times Y)$$

be a Chow correspondence. Then α induces homomorphisms

$$\alpha^*:A^p(Y,M_n)\to A^p(X,M_n),\quad \alpha_*:A_p(X,M_n)\to A_p(Y,M_n),$$

which make $A^p(-, M_n)$ (*resp.* $A_p(-, M_n)$) *a contravariant* (*resp. covariant*) *functor on* $\mathbf{Chow}^{\text{eff}}$.

(b) *Suppose that* $\alpha \in \mathcal{I}_{\text{rat}}(X, Y)$, *where* $\mathcal{I}_{\text{rat}}$ *is as in Proposition 2.3.5. Then* $\alpha^* A^0(Y, M_n) = 0$ (*resp.* $\alpha_* A_0(X, M_n) = 0$).

Proof. (a) This follows easily from the functoriality of cycle cohomology [Rost 1996, Proposition 4.6, Sections 13 and 14]. Namely, we define α^* as the composition

$$A^p(Y, M_n) \xrightarrow{p_Y^*} A^p(X \times Y, M_n) \xrightarrow{\cup\alpha} A^{p+\dim Y}(X \times Y, M_{n+\dim Y}) \xrightarrow{p_{X*}} A^p(X, M_n), \quad (6.1.2)$$

where $\cup\alpha$ is cup-product with α as in [Rost 1996, Section 14], and α_* similarly. Checking the identities $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ and $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ is a routine matter, using the compatibility of cup-product with pull-backs and the projection formula [ibid].

(b) We may assume X irreducible; let $Z \subset X$ be a proper closed subset such that α is supported on $Z \times Y$, and let $U = X - Z$. We consider the cases of α^* and α_* separately.

In the first case, we observe that (6.1.2) also makes sense for X smooth (not necessarily projective) and that $A^0(X, M_n) \to A^0(U, M_n)$ is injective (both groups being subsets of $M_n(F(X))$). Therefore it suffices to see that (6.1.2) is 0 when X is replaced by U, which is obvious since $\alpha_{|CH_{\dim X}(U \times Y)} = 0$.

In the second case, we generalise the argument in the proof of Proposition 2.3.5: if $x \in X_{(0)}$, it suffices to show that the composition

$$M_n(F(x)) \xrightarrow{i_{x*}} A_0(X, M_n) = A^{\dim X}(X, M_{n+\dim X}) \xrightarrow{p_Y^*} A^{\dim X}(X \times Y, M_{n+\dim X}) \xrightarrow{\cup\alpha} A^{\dim X+\dim Y}(X \times Y, M_{n+\dim X+\dim Y}) \xrightarrow{p_{Y*}} A^{\dim Y}(Y, M_{n+\dim Y}) = A_0(Y, M_n)$$

is 0. If $q_Y : x \times Y \to x$ is the first projection, we have

$$p_Y^* i_{x*} = (i_x \times 1_Y)_* q_Y^*$$

[Rost 1996, Proposition 4.1(3)]. For $a \in M_n(F(x))$, we now have

$$p_Y^* i_{x*} a \cup \alpha = (i_x \times 1_Y)_* q_Y^* a \cup \alpha = (i_x \times 1_Y)_* (q_Y^* a \cup (i_x \times 1_Y)^* \alpha)$$

by the projection formula [Rost 1996, Section 14.5]. As in the proof of Proposition 2.3.5 we reduce to the case where $x \notin Z$, and then $(i_x \times 1_Y)^* \alpha = 0$. □

From Proposition 6.1.1(b), we immediately deduce:

Corollary 6.1.3. (a) *For any cycle module M and any $n \in \mathbb{Z}$, the assignment*

$$\mathbf{Sm}^{\mathrm{proj}} \ni X \mapsto A^0(X, M_n) \quad (\textit{resp. } A_0(X, M_n))$$

extends to a contravariant (resp. a covariant) additive functor

$$A^0(-, M_n) : \mathbf{Chow}^{\mathrm{o}} \to \mathbf{Ab} \qquad (\textit{resp. } A_0(-, M_n)).$$

(b) *Let $X \in \mathbf{Sm}^{\mathrm{proj}}$ be such that $h^{\mathrm{o}}(X) \simeq \mathbb{1} \in \mathbf{Chow}^{\mathrm{o}}(F)$. Then the maps*

$$M_n(F) \to A^0(X, M_n), \quad A_0(X, M_n) \to M_n(F)$$

induced by the structural map $\pi_X : X \to \operatorname{Spec} F$ are isomorphisms for any cycle module M and any $n \in \mathbb{Z}$.

This proves the implication (iv) $\Longrightarrow$ (v) in Proposition 3.1.1.

6.2. ***Relationship with Merkurjev's work.*** For $A^0(X, M_n)$, Corollary 6.1.3(b) is part of a theorem of Merkurjev:

Proposition 6.2.1 [Merkurjev 2008, Theorem 2.11(3) $\Longrightarrow$ (1)]. *If $CH_0(X_E) \xrightarrow{\sim} \mathbb{Z}$ for any extension E/F, then $M_n(F) \xrightarrow{\sim} A^0(X, M_n)$ for all cycle modules M and all $n \in \mathbb{Z}$.*

Indeed, this condition is equivalent to $h^{\mathrm{o}}(X) \simeq \mathbb{1}$ in $\mathbf{Chow}^{\mathrm{o}}$ by (iv) $\Longleftrightarrow$ (i) in Proposition 3.1.1.

Merkurjev proves the converse implication. For this, he defines a cycle module K^X such that

$$K_n^X(E) = A_0(X_E, K_n)$$

for any extension E/F. Here, K is the cycle module given by Milnor K-theory. He shows:

Theorem 6.2.2 [Merkurjev 2008, Theorem 2.10]. *The functor*

$$\mathbf{CM} \to \mathbf{Ab}, \quad M \mapsto A^0(X, M_0),$$

from the category of cycle modules to abelian groups is corepresented by K^X.

See [Kahn 2011, Theorem 1.3] for a generalisation to nonproper X.

Let us give a proof of the converse to Proposition 6.2.1 via birational motives, using only the existence of K^X and thus completing the proof of Proposition 3.1.1. Let us say that a cycle module M is *connected* if $M_n = 0$ for $n < 0$; we note that

$$A^0(X, M_0) = M_0(F(X)) \quad \text{if } M \text{ is connected.} \tag{6.2.3}$$

As K^X is connected and $K_0^X(E) = CH_0(X_E)$, the condition

$$K_0^X(F) \xrightarrow{\sim} A^0(X, K_0^X)$$

translates as $CH_0(X) \xrightarrow{\sim} CH_0(X_{F(X)})$, which in turn implies condition (iii) in Proposition 3.1.1.

We are now going to use Theorem 6.2.2 to clarify the relationship between birational motives and cycle modules.

Theorem 6.2.4. *Let* Mod– $\mathbf{Chow}^{\mathrm{o}}$ *be the category of additive contravariant functors from* $\mathbf{Chow}^{\mathrm{o}}$ *to* $\mathbf{Ab}$. *The functor*

$$A^0 : \mathbf{CM} \to \text{Mod–}\,\mathbf{Chow}^{\mathrm{o}}$$

from Corollary 6.1.3(a) *has a fully faithful left adjoint* $\Lambda \mapsto K^\Lambda$; *the essential image of this left adjoint is contained in the full subcategory of connected cycle modules.*

Proof. We first observe that $X \mapsto K^X$ extends to a functor

$$\mathbf{Chow}^{\mathrm{o}} \to \mathbf{CM}$$

thanks to Corollary 6.1.3(a) (case of A_0). Let $\Lambda \in$ Mod– $\mathbf{Chow}^{\mathrm{o}}$. We define

$$K^\Lambda = \varinjlim_{y(X)\to\Lambda} K^X,$$

where $y : \mathbf{Chow}^{\mathrm{o}} \to$ Mod– $\mathbf{Chow}^{\mathrm{o}}$ is the additive Yoneda functor, and the colimit is taken on the comma category $y \downarrow \Lambda$ [Mac Lane 1998, Chapter II, Section 6]. Since K^X is connected for any smooth projective X, K^Λ is connected. For a cycle module M, the identity

$$\mathbf{CM}(K^\Lambda, M) \simeq \text{Mod–}\,\mathbf{Chow}^{\mathrm{o}}(\Lambda, A^0(M))$$

follows from Theorem 6.2.2 and Yoneda's lemma, thus proving the existence of the left adjoint and the statement on its essential image.

It remains to show that $\Lambda \mapsto K^\Lambda$ is fully faithful or, equivalently, that the unit map

$$\Lambda \to A^0(K^\Lambda)$$

is an isomorphism for all Λ. Let $Y \in \mathbf{Sm}^{\mathrm{proj}}$; we need to show that

$$\Lambda(h^{\mathrm{o}}(Y)) \to A^0(Y, K_0^\Lambda) = K_0^\Lambda(F(Y))$$

is an isomorphism, where we just used (6.2.3). We compute:

$$\begin{aligned} K_0^\Lambda(F(Y)) &= \varinjlim_{y(X)\to\Lambda} K_0^X(F(Y)) = \varinjlim_{y(X)\to\Lambda} CH_0(X_{F(Y)}) \\ &= \varinjlim_{y(X)\to\Lambda} \mathbf{Chow}^{\mathrm{o}}(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(X)) \\ &= \varinjlim_{y(X)\to\Lambda} y(h^{\mathrm{o}}(X))(h^{\mathrm{o}}(Y)) = \Lambda(h^{\mathrm{o}}(Y)). \end{aligned}$$

□

We describe the essential image of the functor $K^?$ in [Kahn and Sujatha 2015b, Theorem 5.1.2].

7. Locally abelian schemes

In this section, F is perfect. We drop it from the notation for relevant categories.

7.1. *The Albanese scheme of a smooth projective variety.*

Definition 7.1.1. (a) Let X be a smooth separated F-scheme (not necessarily of finite type). For each connected component X_i of X, let E_i be its field of constants, that is, the algebraic closure of F in $F(X_i)$. We define

$$\pi_0(X) = \coprod_i \operatorname{Spec} E_i.$$

There is a canonical F-morphism $X \to \pi_0(X)$; $\pi_0(X)$ is called the *scheme of constants* of X.

(b) If $\dim X = 0$ (equivalently $X \xrightarrow{\sim} \pi_0(X)$), we write $\mathbb{Z}[X]$ for the 0-dimensional group scheme representing the étale sheaf $f_*\mathbb{Z}$, where $f : X \to \operatorname{Spec} F$ is the structural morphism.

Definition 7.1.2. (a) For an F-group scheme G, we denote by G^0 the kernel of the canonical map $G \to \pi_0(G)$ of Definition 7.1.1; this is the *neutral component* of G.

(b) An F-group scheme G is called a *lattice* if $G^0 = \{1\}$ and the geometric fibre of $\pi_0(G) = G$ is a free finitely generated abelian group.

Definition 7.1.3 [Ramachandran 2001]. (a) Recall that a *semiabelian variety* is an extension of an abelian variety by a torus. We denote by **SAb** the category of semiabelian F-varieties, and by **Ab** the full subcategory of abelian varieties.

(b) We denote by **SAbS** the full subcategory of the category of commutative F-group schemes consisting of those objects $\mathcal{A}$ such that

- $\pi_0(\mathcal{A})$ is a lattice;
- $\mathcal{A}^0$ is a semiabelian variety.

Objects of **SAbS** will be called *locally semiabelian F-schemes*.

(c) We denote by **AbS** the full subcategory of **SAbS** consisting of those $\mathcal{A}$ such that $\mathcal{A}^0$ is an abelian variety. Its objects are called *locally abelian F-schemes*.

Note that **SAbS** is a Serre subcategory of the abelian category of commutative F-group schemes locally of finite type (see [Demazure and Grothendieck 2011, Exp. VI, Proposition 5.4.1 and Théorème 5.4.2]); in particular it is abelian, and **AbS** is idempotent-closed in **SAbS**, hence pseudoabelian.

For any smooth F-variety X, let $\mathcal{A}_{X/F} = \mathcal{A}_X$ be the Albanese scheme of X over F [Ramachandran 2001]: it is an object of **SAbS** and there is a canonical morphism

$$\varphi_X : X \to \mathcal{A}_X, \tag{7.1.4}$$

which is universal for morphisms from X to objects of **SAbS**. There is an exact sequence of group schemes

$$0 \to \mathcal{A}_X^0 \to \mathcal{A}_X \to \mathbb{Z}[\pi_0(X)] \to 0,$$

where $\mathcal{A}_X^0$ is the Albanese variety of X (a semiabelian variety) and $\pi_0(X)$ has been defined above.

The aim of this section is to endow **SAbS** and **AbS** with symmetric monoidal structures, and to relate the latter one to birational motives (see Propositions 7.2.7 and 8.2.1).

Let us recall from [Ramachandran 2001] a description of $\mathcal{A}_X$. Let $\mathbb{Z}[X]$ be the "free" presheaf on F-schemes defined by $\mathbb{Z}[X](Y) = \mathbb{Z}[X(Y)]$ and $\mathcal{Z}_{X/F} = \mathcal{Z}_X$ the associated sheaf on the big fppf site of $\operatorname{Spec} F$. Then $\mathcal{A}_X$ is the universal representable quotient of $\mathcal{Z}_X$. In other words, there is a homomorphism

$$\mathcal{Z}_X \to \mathcal{A}_X,$$

where $\mathcal{A}_X$ is considered as a representable sheaf, which is universal for homomorphisms from $\mathcal{Z}_X$ to sheaves of abelian groups representable by a locally semiabelian F-scheme.

Let us also denote by P_X the universal torsor under $\mathcal{A}_X^0$ constructed by Serre [1958/1959]. There is a map $X \xrightarrow{\tilde{\varphi}_X} P_X$, which is universal for maps from X to torsors under semiabelian varieties. The torsor P_X and the group scheme $\mathcal{A}_X$ have the same class in $\operatorname{Ext}^1_{(\mathrm{Sch}/F)_{\text{ét}}}(\pi_0(\mathcal{A}_X), \mathcal{A}_X^0) = H^1_{\text{ét}}(\pi_0(X), \mathcal{A}_X^0)$ (here we identify $\mathcal{A}_X^0$ with the corresponding representable étale sheaf over the big étale site of $\operatorname{Spec} F$). A beautiful concrete description of this correspondence is given in [Ramachandran 2001, Section 1.2]. The map $\tilde{\varphi}_X$ induces an isomorphism

$$\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{P_X}.$$

We repeat some properties of $\mathcal{A}_X$ as taken from [Ramachandran 2001, Proposition 1.6 and Corollary 1.12] and add one.

Proposition 7.1.5. (a) *$\mathcal{A}_X$ is covariant in X.*

(b) *Let K/F be an extension. Then the natural map*

$$\mathcal{A}_{X_K/K} \to \mathcal{A}_{X/F} \otimes_F K$$

stemming from the universal property is an isomorphism.

(c) *If* $X = Y \amalg Z$, *then the natural map* $\mathcal{A}_{Y/F} \oplus \mathcal{A}_{Z/F} \to \mathcal{A}_{X/F}$ *is an isomorphism.*

(d) *Let* E/F *be a finite extension. For any* E*-scheme* S, *let* $S_{(F)}$ *denote the* (*ordinary*) *restriction of scalars of* S, *i.e.*, *we view* S *as an* F*-scheme. Then there is a natural isomorphism for* X *smooth*

$$R_{E/F}\mathcal{A}_{X/E} \xrightarrow{\sim} \mathcal{A}_{X_{(F)}/F},$$

where $R_{E/F}$ *denotes Weil's restriction of scalars.*

Proof. The only thing which is not in [Ramachandran 2001] is (d). We shall construct the isomorphism by descent from (c), using (b).

Let $f : \operatorname{Spec} E \to \operatorname{Spec} F$ be the structural morphism. Recall that, for any abelian sheaf $\mathcal{G}$ on $(\mathrm{Sch}/E)_{\text{ét}}$, the trace map defines an isomorphism [Milne 1980, Chapter V, Lemma 1.12]

$$f_*\mathcal{G} \xrightarrow{\sim} f_!\mathcal{G},$$

where $f_!$ (resp. f_*) is the left (resp. right) adjoint of the restriction functor f^*. This isomorphism is natural in $\mathcal{G}$.

This being said, the additive version of Yoneda's lemma immediately yields

$$f_!\mathcal{Z}_{X/E} = \mathcal{Z}_{X_{(F)}/F},$$

hence a composition of homomorphisms of sheaves

$$f_*\mathcal{Z}_{X/E} \xrightarrow{\sim} \mathcal{Z}_{X_{(F)}/F} \to \mathrm{Shv}(\mathcal{A}_{X_{(F)}/F}), \tag{7.1.6}$$

where, for clarity, $\mathrm{Shv}(\mathcal{A}_{X_{(F)}/F})$ denotes the sheaf associated to the group scheme $\mathcal{A}_{X_{(F)}/F}$. We also have a chain of homomorphisms

$$f_*\mathcal{Z}_{X/E} \to f_*\,\mathrm{Shv}(\mathcal{A}_{X/E}) \xrightarrow{\sim} \mathrm{Shv}(R_{E/F}\mathcal{A}_{X/E}), \tag{7.1.7}$$

where the last isomorphism is formal. If we can prove that (7.1.6) factors through (7.1.7) into an isomorphism, we are done by Yoneda.

In order to do this, we may assume via (b) that F is algebraically closed, hence that f is completely split. Then the claim follows from (c). □

We record here similar properties for the torsor $P_X = P_{X/F}$ (proofs are similar):

Proposition 7.1.8. (a) $X \mapsto P_X$ *is a functor.*

(b) *Let* K/F *be an extension. Then the natural map* $P_{X_K/K} \to P_{X/F} \otimes_F K$ *stemming from the universal property is an isomorphism.*

(c) *If* $X = Y \amalg Z$, *then there is an isomorphism* $P_{Y/F} \times P_{Z/F} \xrightarrow{\sim} P_{X/F}$ *which is natural in* (Y, Z).

(d) *Let* E/F *be a finite extension. Then there is a natural isomorphism*

$$P_{X_{(F)}/F} \to R_{E/F}P_{X/E}.$$

□

(In (c), the map stems from the fact that coproducts correspond to scheme-theoretic products in an appropriate category of torsors.)

7.2. ***The tensor category of locally semiabelian schemes.*** Recall the Yoneda full embedding $\mathrm{Shv} : \mathbf{SAbS} \to \mathrm{Ab}((\mathrm{Sch}/F)_{\text{ét}})$, where the latter is the category of sheaves of abelian groups over the big étale site of Spec F.

Lemma 7.2.1. (a) *If a sheaf $\mathcal{F} \in \mathrm{Ab}((\mathrm{Sch}/F)_{\text{ét}})$ is an extension of a lattice L by a semiabelian variety A, it is represented by an object of* **SAbS**.

(b) *Let A be a semiabelian variety and L a lattice. Then the étale sheaf $B = A \otimes L$ is represented by a semiabelian variety.*

Proof. (a) If L is constant, then the choice of a basis of L determines a section of the projection $\mathcal{F} \to \mathrm{Shv}(L)$, hence an isomorphism $\mathcal{F} \simeq \mathrm{Shv}(A) \oplus \mathrm{Shv}(L)$. Then $\mathcal{F}$ is represented by $\coprod_{l \in L} A$. In general, L becomes constant on some finite extension E/F, hence $\mathcal{F}_E$ is representable. By full faithfulness, the descent data of $\mathcal{F}_E$ are morphisms of schemes; then we may apply [Serre 1988, Corollary V.4.2(a) or (b)].

(b) Same method as in (a). □

Example 7.2.2. If $L = \mathbb{Z}[\mathrm{Spec}\, E]$, where E is an étale F-algebra, then $A \otimes L = R_{E/F} A_E$.

We shall also need:

Lemma 7.2.3. *Let F be a field, G_1, G_2, G_3 be three semiabelian F-varieties, and let $\varphi : G_1 \times G_2 \to G_3$ be an F-morphism. Assume that $\varphi(g_1, 0) = \varphi(0, g_2) = 0$ identically. Then $\varphi = 0$.*

Proof. By [Kahn 2014, Lemma 3], φ is a homomorphism and the conclusion is obvious. □

Let $\mathcal{A}, \mathcal{B} \in \mathbf{SAbS}$. Viewing them as étale sheaves, we may consider their tensor product $\mathcal{A} \otimes_{\text{shv}} \mathcal{B}$. This tensor product contains the subsheaf $\mathcal{A}^0 \otimes_{\text{shv}} \mathcal{B}^0$, which is clearly not representable. We define

$$\mathcal{A} \otimes_{\text{rep}} \mathcal{B} = \mathcal{A} \otimes_{\text{shv}} \mathcal{B} / \mathcal{A}^0 \otimes_{\text{shv}} \mathcal{B}^0.$$

Proposition 7.2.4. (a) *$\mathcal{A} \otimes_{\text{rep}} \mathcal{B}$ is representable by an object of* **SAbS**.

(b) *For $X, Y \in$ **Sm**, the natural map*

$$\mathcal{Z}_X \otimes_{\text{shv}} \mathcal{Z}_Y = \mathcal{Z}_{X \times Y} \to \mathcal{A}_{X \times Y}$$

factors into an isomorphism

$$\mathcal{A}_X \otimes_{\text{rep}} \mathcal{A}_Y \xrightarrow{\sim} \mathcal{A}_{X \times Y}.$$

(This corrects [Ramachandran 2001, Corollary 1.12(vi)].)

Proof. (a) We have a short exact sequence

$$0 \to \mathcal{A}^0 \otimes \pi_0(\mathcal{B}) \oplus \mathcal{B}^0 \otimes \pi_0(\mathcal{A}) \to \mathcal{A} \otimes_{\mathrm{rep}} \mathcal{B} \to \pi_0(\mathcal{A}) \otimes \pi_0(\mathcal{B}) \to 0.$$

By Lemma 7.2.1(b), the left-hand side is representable by a semiabelian variety, and the right-hand side is clearly a lattice. We conclude by Lemma 7.2.1(a).

(b) It is enough to show that this holds over the algebraic closure of F. Using Proposition 7.1.5(c) (and the similar statement for $\mathcal{Z}$), we may assume that X and Y are connected. We shall show more generally that, for any locally semiabelian scheme $\mathcal{B}$ and any map $X \times Y \to \mathcal{B}$, the induced sheaf-theoretic map

$$\mathcal{Z}_X \otimes_{\mathrm{shv}} \mathcal{Z}_Y \to \mathcal{B} \tag{7.2.5}$$

factors through $\mathcal{A}_X \otimes_{\mathrm{rep}} \mathcal{A}_Y$. By (a), this will show that the latter has the universal property of $\mathcal{A}_{X \times Y}$.

For $n \in \mathbb{Z}$, we denote by $\mathcal{Z}_X^n$ or $\mathcal{A}_X^n$ the inverse image of n under the augmentation map $\mathcal{Z}_X \to \mathbb{Z}$ or $\mathcal{A}_X \to \mathbb{Z}$ stemming from the structural morphism $X \to \operatorname{Spec} F$. It is a subsheaf of $\mathcal{Z}_X$ or $\mathcal{A}_X$, and $\mathcal{A}_X^n$ is clearly representable (by a variety $\bar{F}$-isomorphic to the semiabelian variety $\mathcal{A}_X^0$). We shall also identify varieties with representable sheaves; this should create no confusion in view of Yoneda's lemma.

We first show that (7.2.5) factors through $\mathcal{A}_X \otimes_{\mathrm{shv}} \mathcal{A}_Y$. It suffices to show that the composition

$$\mathcal{Z}_X \times Y \to \mathcal{Z}_X \otimes_{\mathrm{shv}} \mathcal{Z}_Y \to \mathcal{B}$$

factors through $\mathcal{A}_X \times Y$, and to conclude by symmetry. But $X \times Y$ is connected, so its image in $\mathcal{B}$ falls in some connected component $\mathcal{B}^t$ of $\mathcal{B}$, which is a torsor under $\mathcal{B}^0$; applying the "Variation en fonction d'un paramètre" statement in [Serre 1958/1959, p. 10-05], we see that it extends to a morphism $\mathcal{A}_X^1 \times Y \to \mathcal{B}^t$. Including $\mathcal{B}^t$ into $\mathcal{B}$, we get a commutative diagram

$$\begin{array}{ccc}
\mathcal{A}_X^1 \times Y & \longrightarrow & \mathcal{B} \\
\uparrow & & \uparrow \\
\mathcal{Z}_X^1 \times Y & \longrightarrow & \mathcal{Z}_X \times Y
\end{array}$$

Let $\mathcal{K} = \operatorname{Ker}(\mathcal{Z}_X \to \mathcal{A}_X) = \operatorname{Ker}(\mathcal{Z}_X^0 \to \mathcal{A}_X^0)$. The above diagram shows that the diagram

$$\begin{array}{ccc}
\mathcal{K} \times \mathcal{Z}_X^1 \times Y & \xrightarrow{a} & \mathcal{Z}_X^1 \times Y \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\
\mathcal{Z}_X^1 \times Y & \xrightarrow{b} & \mathcal{B}
\end{array}$$

commutes, where a is given by the action of $\mathcal{K}$ on $\mathcal{Z}^1_X$ by left translation and c is given by $(k, z, y) \mapsto (z, y)$. Since b is a homomorphism in the first variable, this implies the desired factorisation.

We now show that the composition

$$\mathcal{A}^0_X \otimes_{\text{shv}} \mathcal{A}^0_Y \to \mathcal{A}_X \otimes_{\text{shv}} \mathcal{A}_Y \to \mathcal{B}$$

is 0. It is sufficient to show that the composition of this map with the inclusion $\mathcal{A}^0_X \times \mathcal{A}^0_Y \to \mathcal{A}^0_X \otimes \mathcal{A}^0_Y$ is 0. But $\mathcal{A}^0_X \times \mathcal{A}^0_Y$ is connected, hence its image falls in some connected component, in fact in $\mathcal{B}^0$. This map verifies the hypothesis of Lemma 7.2.3, hence it is 0. □

As a variant, we have:

Proposition 7.2.6. *We have an isomorphism*

$$P_{X\times Y} \xrightarrow{\sim} R_{\pi_0(X)/F}(P_Y \times_F \pi_0(X)) \times R_{\pi_0(Y)/F}(P_X \times_F \pi_0(Y)).$$

Since we are not going to use this, we leave the easy proof to the reader.

Proposition 7.2.4(a) endows **SAbS** with a symmetric monoidal structure compatible with its additive structure, hence also its full subcategory **AbS**. From now on we concentrate on this latter category.

Proposition 7.2.7. *The category* **AbS** *is symmetric monoidal* (*for* $\otimes_{\text{rep}}$) *and pseudo-abelian. Its Kelly radical* $\mathcal{R}$ *is monoidal and has square* 0*. After tensoring with* $\mathbb{Q}$, **AbS** $/\mathcal{R}$ *becomes isomorphic to the semisimple category product of the category of abelian varieties up to isogenies and the category of* G_F*-*$\mathbb{Q}$*-lattices.*

Recall that the *Kelly radical* [1964] $\mathcal{R}$ of an additive category $\mathcal{A}$ is defined by

$$\mathcal{R}(A, B) = \{f \in \mathcal{A}(A, B) \mid 1_A - gf \text{ is invertible for all } g \in \mathcal{A}(B, A)\}$$

and that it is a (two-sided) ideal of $\mathcal{A}$.

Proof. For the first claim, we just observe that kernels exist in the category of commutative F-group schemes, and that a direct summand of an abelian variety (resp. of a lattice) is an abelian variety (resp. a lattice). For the second claim, consider the functor

$$T : \mathbf{AbS} \to \mathbf{Ab} \times \mathbf{Lat}, \quad \mathcal{A} \mapsto (\mathcal{A}^0, \pi_0(\mathcal{A})),$$

where **Ab** and **Lat** are respectively the category of abelian varieties and the category of lattices over F (viewed, for example, as full subcategories of the category of étale sheaves over Sm/F). This functor is obviously essentially surjective. After tensoring with $\mathbb{Q}$, it becomes full, because any extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

is rationally split. Now the collection of sets

$$\mathcal{I}(\mathcal{A}, \mathcal{B}) = \{f : \mathcal{A} \to \mathcal{B} \mid T(f) = 0\}$$

defines an ideal $\mathcal{I}$ of **AbS**. If $f \in \mathcal{I}(\mathcal{A}, \mathcal{B})$, then f induces a map

$$\bar{f} : \pi_0(\mathcal{A}) \to \mathcal{B}^0,$$

and this gives a description of $\mathcal{I}$. From this description, it follows immediately that $\mathcal{I}^2 = 0$. In particular, $\mathcal{I} \subseteq \mathcal{R}$.

If we tensor with $\mathbb{Q}$, then **Ab** $\times$ **Lat** becomes semisimple; since **AbS** $/\mathcal{I} \otimes \mathbb{Q}$ is semisimple and $\mathcal{I} \otimes \mathbb{Q}$ is nilpotent, it follows that $\mathcal{I} \otimes \mathbb{Q} = \mathcal{R} \otimes \mathbb{Q}$. In other words, $\mathcal{R}/\mathcal{I}$ is torsion.

Let $f \in \mathcal{R}(\mathcal{A}, \mathcal{B})$. There exists $n > 0$ such that $nf(\mathcal{A}^0) = 0$. But $f(\mathcal{A}^0)$ is an abelian subvariety of $\mathcal{B}^0$, hence $f(\mathcal{A}^0) = 0$ and $f \in \mathcal{I}(\mathcal{A}, \mathcal{B})$. So $\mathcal{R} = \mathcal{I}$.

If we endow the category **Ab** $\times$ **Lat** with the tensor structure

$$(A, L) \otimes (B, M) = (A \otimes M \oplus B \otimes L, L \otimes M),$$

then T becomes a monoidal functor, which shows that $\mathcal{R} = \mathcal{I}$ is monoidal. This completes the proof of Proposition 7.2.7. □

Remarks 7.2.8. (a) The morphisms in **AbS** are best represented in matrix form:

$$\operatorname{Hom}(\mathcal{A}, \mathcal{B}) = \begin{pmatrix} \operatorname{Hom}(\mathcal{A}_0, \mathcal{B}_0) & \operatorname{Hom}(\pi_0(\mathcal{A}), \mathcal{B}_0) \\ 0 & \operatorname{Hom}(\pi_0(\mathcal{A}), \pi_0(\mathcal{B})) \end{pmatrix}$$

(note that $\operatorname{Hom}(\mathcal{A}_0, \pi_0(\mathcal{B})) = 0$). This clarifies the arguments in the proof of Proposition 7.2.7 somewhat.

(b) The Hom groups of **Ab** $\times$ **Lat** are finitely generated $\mathbb{Z}$-modules. It follows from the proof of Proposition 7.2.7 that, for $\mathcal{A}, \mathcal{B} \in$ **AbS**, $T(\operatorname{Hom}(\mathcal{A}, \mathcal{B}))$ has finite index in $\operatorname{Hom}(T(\mathcal{A}), T(\mathcal{B}))$. In particular, for any $\mathcal{A} \in$ **AbS**, $\operatorname{End}(\mathcal{A})$ is an extension of an order in a semisimple $\mathbb{Q}$-algebra by an ideal of square 0.

(c) The functor T has the explicit section

$$(A, L) \mapsto A \oplus L.$$

This section is symmetric monoidal.

8. Chow birational motives and locally abelian schemes

8.1. ***The Albanese map.*** For any smooth projective variety X, there is a canonical map

$$CH_0(X) \xrightarrow{\mathrm{Alb}_X^F} \mathcal{A}_X(F). \tag{8.1.1}$$

Recall the construction of Alb_X: The map φ_X of (7.1.4) defines for any extension E/F a map $X(E) \to \mathcal{A}_X(E)$, still denoted by φ_X. When E/F is finite, viewing $\mathcal{A}_X$ as an étale sheaf, we have a trace map $\mathrm{Tr}_{E/F} : \mathcal{A}_X(E) \to \mathcal{A}_X(F)$. Then Alb_X maps the class of a closed point $x \in X$ with residue field E to $\mathrm{Tr}_{E/F}\ \varphi_X(x)$.

The map Alb_X is injective for $\dim X = 1$ and surjective if F is algebraically closed. For a curve, this map corresponds to the isomorphism $\mathrm{Pic}_X \simeq \mathcal{A}_X$, where Pic_X is the Picard scheme of X; we then also have $\mathcal{A}_X^0 \simeq J_X$, where J_X is the Jacobian variety of X.

The functoriality of $\mathcal{A}$ shows that there is a chain of isomorphisms

$$\Phi_{X,Y} : \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \xrightarrow{\sim} \mathrm{Mor}(X, \mathcal{A}_Y) \xrightarrow{\sim} \mathcal{A}_Y(F(X)) \tag{8.1.2}$$

(the latter by Weil's theorem on extension of morphisms to abelian varieties [Milne 1986, Theorem 3.1]), hence a canonical map

$$CH_0(Y_{F(X)}) \xrightarrow{\mathrm{Alb}_{X,Y}} \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y), \tag{8.1.3}$$

which generalises (8.1.1); more precisely, we have

$$\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y} = \mathrm{Alb}_Y^{F(X)}. \tag{8.1.4}$$

On the other hand, there is an exact sequence

$$\begin{aligned} 0 \to \mathcal{A}_Y(\pi_0(X)) = \mathrm{Hom}(\mathbb{Z}[\pi_0(X)], \mathcal{A}_Y) \to \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \\ \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y) \to \mathrm{Ext}^1(\mathbb{Z}[\pi_0(X)], \mathcal{A}_Y) = H^1(\pi_0(X), \mathcal{A}_Y), \end{aligned}$$

and the map $\mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y)$ is an isomorphism. From this and (8.1.3) we get a zero sequence

$$0 \to CH_0(Y) \to CH_0(Y_{F(X)}) \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to 0. \tag{8.1.5}$$

Lemma 8.1.6. *Let Y, Z be two smooth projective varieties and $\beta \in CH_0(Z_{F(Y)})$. Then the following diagram commutes*:

$$\begin{array}{ccc} CH_0(Y) & \xrightarrow{\beta_*} & CH_0(Z) \\ {\scriptstyle \mathrm{Alb}_Y^F}\downarrow & & \downarrow{\scriptstyle \mathrm{Alb}_Z^F} \\ \mathcal{A}_Y(F) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathcal{A}_Z(F) \end{array}$$

Proof. Without loss of generality, we may assume that β is given by an integral subscheme W in $Y \times Z$. Then the composite $f = p_Y i_W$ is a proper surjective generically finite morphism, where p_Y denotes the projection and i_W is the inclusion of W in $Y \times Z$.

Let V be an affine dense open subset of Y such that $f_{|f^{-1}(V)}$ is finite. Any element of $CH_0(Y)$ may be represented by a zero-cycle with support in V (see

[Roberts 1972]), so it is enough to check the commutativity of the diagram on zero-cycles on Y of the form y, where $y \in V_{(0)}$. For such a y, we have $\beta_* y = p_*(f^{-1}(y))$, where $p = p_Z i_W$.

On the other hand, the composition $\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V}$ may be described as follows: Let d be the degree of $f_{|f^{-1}(V)}$, $f^{-1}(V)^{[d]}$ the d-fold symmetric power of $f^{-1}(V)$ and $f^* : V \to f^{-1}(V)^{[d]}$ the map $x \mapsto f^{-1}(x)$. Then

$$\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V} = \Sigma_d \circ (\varphi_Z)^{[d]} \circ p_*^{[d]} \circ f^*,$$

where $\Sigma_d : \mathcal{A}_Z^{[d]} \to \mathcal{A}_Z$ is the summation map. The commutativity of the diagram is now clear. □

8.2. *The Albanese functor.*

Proposition 8.2.1. *The assignment* $X \mapsto \mathcal{A}_X$ *defines, via* (8.1.3), *a symmetric monoidal additive functor*

$$\mathrm{Alb} : \mathbf{Chow}^{\mathrm{o}} \to \mathbf{AbS},$$

which becomes full and essentially surjective after tensoring with $\mathbb{Q}$.

Proof. Since **AbS** is pseudoabelian, it suffices to construct the functor on $\mathbf{Cor}^{\mathrm{o}}$. Let $\alpha \in CH_0(Y_{F(X)})$ and $\beta \in CH_0(Z_{F(Y)})$. We want to show that

$$\mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta) \circ \mathrm{Alb}_{X,Y}(\alpha).$$

But β induces a map

$$\beta_* : CH_0(Y_{F(X)}) \to CH_0(Z_{F(X)}),$$

and we have the equality $\beta_* \alpha = \beta \circ \alpha$ (see the proof of Proposition 2.3.5). Hence, applying Lemma 8.1.6, in which we replace F by $F(X)$, we get

$$\mathrm{Alb}_Z^{F(X)}(\beta \circ \alpha) = \mathrm{Alb}_Z^{F(X)}(\beta_* \alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\mathrm{Alb}_Y^{F(X)}(\alpha)).$$

Applying now (8.1.4), we get

$$\Phi_{X,Z} \circ \mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y}(\alpha)).$$

On the other hand, the diagram

$$\begin{array}{ccc}
\mathcal{A}_Y(F(X)) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathcal{A}_Z(F(X)) \\
\Phi_{X,Y} \uparrow \wr & & \Phi_{X,Z} \uparrow \wr \\
\mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) & \xrightarrow{\mathrm{Alb}_{Y,Z}(\beta)_*} & \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y)
\end{array}$$

obviously commutes, which concludes the proof that Alb is a functor.

Compatibility with the monoidal structures follows from Proposition 7.2.4(b). It remains to show the assertions on fullness and essential surjectivity.

Fullness: For any Y, the map $\mathrm{Alb}_Y^F \otimes \mathbb{Q}$ is surjective. This follows from the case where F is algebraically closed (in which case Alb_Y^F itself is surjective) by a transfer argument. Replacing the ground field F by $F(X)$ for some other X, we get that $\mathrm{Alb}_{X,Y} \otimes \mathbb{Q}$ is surjective. This shows that the restriction of $\mathrm{Alb} \otimes \mathbb{Q}$ to $\mathbf{Cor}^{\mathrm{o}} \otimes \mathbb{Q}$ is full; but the pseudoabelianisation of a full functor is evidently full (a direct summand of a surjective homomorphism of abelian groups is surjective).

Essential surjectivity: We first note that, after tensoring with $\mathbb{Q}$, the extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

becomes split for any $\mathcal{A} \in \mathbf{AbS}$. Indeed the extension class belongs to

$$\mathrm{Ext}^1_F(\pi_0(\mathcal{A}), \mathcal{A}^0);$$

this group sits in an exact sequence (coming from an Ext spectral sequence)

$$0 \to H^1(F, \mathrm{Hom}_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})) \to \mathrm{Ext}^1_F(\pi_0(\mathcal{A}), \mathcal{A}^0) \\ \to H^0\big(F, \mathrm{Ext}^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})\big).$$

Since the restriction $\pi_0(\mathcal{A})_{|\bar F}$ is a constant sheaf of free finitely generated abelian groups, the group $\mathrm{Ext}^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})$ is 0, while the left group is torsion as a Galois cohomology group. It is now sufficient to show separately that L and A are in the essential image of $\mathrm{Alb} \otimes \mathbb{Q}$, where L (resp. A) is a lattice (resp. an abelian variety).

A lattice L corresponds to a continuous integral representation ρ of G_F. But it is well known that $\rho \otimes \mathbb{Q}$ is of the form $\theta \otimes \mathbb{Q}$, where θ is a direct summand of a permutation representation of G_F. If E is the corresponding étale algebra, we therefore have an isomorphism of L with a direct summand of $(\mathrm{Alb} \otimes \mathbb{Q})(E)$.

Given an abelian variety A, we simply note that

$$A = \mathrm{Alb}(\tilde h(A)),$$

where $\tilde h(A)$ is the reduced motive of A, that is, $h(A) = \mathbb{1} \oplus \tilde h(A)$, where the splitting is given by the rational point $0 \in A(F)$. □

Remark 8.2.2. Let $\mathcal{R}$ be the Kelly radical of $\mathbf{AbS}$ (see Proposition 7.2.7). If F is a finitely generated field, the groups $\mathcal{R}(\mathcal{A}, \mathcal{B})$ are finitely generated by the Mordell–Weil–Néron theorem. To see this, note that if L is a lattice and A an abelian variety, then

$$\mathrm{Hom}(L, A) \xrightarrow{\sim} \mathrm{Hom}(L_{|\bar F}, A_{|\bar F})^{G_F}$$

and that the right term may be rewritten as $B(F)$, where $B = L^* \otimes A$ (compare Lemma 7.2.1). Hence the Hom groups in $\mathbf{AbS}$ are finitely generated as well. In this case, Proposition 8.2.1 implies that, for any $M, N \in \mathbf{Chow}^{o}$, *the image of the map* $\mathrm{Alb}_{M,N}$ *has finite index in the group* $\mathrm{Hom}(\mathrm{Alb}(M), \mathrm{Alb}(N))$.

Lemma 8.2.3. *Suppose that Y is a curve. Then the map* (8.1.3) *fits into an exact sequence*

$$0 \to CH_0(Y_{F(X)}) \xrightarrow{\mathrm{Alb}_{X,Y}} \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \to Br(F(X)) \to Br(F(X \times Y)),$$

where Br denotes the Brauer group. In particular, $(8.1.3) \otimes \mathbb{Q}$ *is an isomorphism.*

Proof. First assume that X is a point; then (8.1.3) reduces to (8.1.1). Suppose first that F is separably closed. Then (8.1.1) is bijective (see comments at the beginning of this section). In the general case, let F_s be a separable closure of F, and $G = \mathrm{Gal}(F_s/F)$. Since $\mathcal{A}_Y$ is a sheaf for the étale topology, we get a commutative diagram

$$\begin{array}{ccc} CH_0(Y_s)^G & \xrightarrow[\sim]{\mathrm{Alb}_Y^{F_s}} & \mathcal{A}_Y(F_s)^G \\ \uparrow & & \uparrow \wr \\ CH_0(Y) & \xrightarrow{\mathrm{Alb}_Y^{F}} & \mathcal{A}_Y(F) \end{array}$$

where $Y_s = Y \times_F F_s$ and the top horizontal and right vertical maps are bijective. The lemma then follows from the classical exact sequence

$$0 \to CH_0(Y) \to CH_0(Y_s)^G \to Br(F) \to Br(F(Y)).$$

The case where X is not necessarily a point now follows from this special case and the construction of (8.1.3). □

Theorem 8.2.4. *Let* $\mathbf{Chow}^{o}_{\leq 1}$ *denote the thick subcategory of* $\mathbf{Chow}^{o}$ *generated by motives of varieties of dimension* ≤ 1, *and let* $\iota : \mathbf{Chow}^{o}_{\leq 1} \to \mathbf{Chow}^{o}$ *be the canonical inclusion. Then*:

(a) *After tensoring morphisms with* $\mathbb{Q}$, $\mathrm{Alb} \circ \iota : \mathbf{Chow}^{o}_{\leq 1} \to \mathbf{AbS}$ *becomes an equivalence of categories.*

(b) *Let j be a quasi-inverse. Then $\iota \circ j$ is right adjoint to* Alb.

Proof. (a) The full faithfulness follows from Lemma 8.2.3. For the essential surjectivity, we may reduce as in the proof of Proposition 8.2.1 to proving that lattices and abelian varieties are in the essential image. For lattices, this is proven in Proposition 8.2.1. For an abelian variety A, use the fact that A is isogenous to a quotient of the Jacobian of a curve, and Poincaré's complete reducibility theorem.

(b) Let $(M, \mathcal{A}) \in \mathbf{Chow}^{o}_{\leq 1}(F, \mathbb{Q}) \times \mathbf{AbS}(F, \mathbb{Q})$. To produce a natural isomorphism $\mathbf{Chow}^{o}_{\leq 1}(F, \mathbb{Q})(M, \iota j(\mathcal{A})) \simeq \mathbf{AbS}(F)(\mathrm{Alb}(M), \mathcal{A}) \otimes \mathbb{Q}$, it is sufficient by (a) to handle the case $M = h^{o}(X)$, $\mathcal{A} = \mathcal{A}_Y$ for some smooth projective curves X, Y. Then the isomorphism follows from (8.1.2) and Lemma 8.2.3. □

Remarks 8.2.5. (a) Of course the functor $\iota \circ j$ is not a tensor functor (since its image is not closed under tensor product).

(b) In particular, the inclusion functor ι has the left adjoint $j \circ \mathrm{Alb}$. This is a birational version of Murre's results [1990; 1993, Section 2.1] for effective Chow motives; see also [Scholl 1994, Section 4]. Beware however that we have taken the opposite to usual convention for the variance of Chow motives (our functor $X \mapsto h(X)$ is covariant rather than contravariant), so the direction of arrows has to be reversed with respect to Murre's work.

Appendix: Complements on localisation of categories

A.1. *Localisation of symmetric monoidal categories.*

Lemma A.1.1. (a) *Localisation commutes with products of categories for sets of morphisms containing all identities.*[8]

(b) *Let $T_0, T_1 : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors and $f : T_0 \Rightarrow T_1$ a natural transformation. Let S, S' be collections of morphisms in $\mathcal{C}$ and $\mathcal{D}$ such that $T_i(S) \subseteq S'$, so that T_0 and T_1 pass to localisation. Then f remains a natural transformation between the localised functors.*

Proof. (a) Let S_i be a collection of morphisms in $\mathcal{C}_i$ for $i = 1, 2$, such that S_i contains the identities of all objects of $\mathcal{C}_i$. Then $S_1 \times S_2$ is generated by S_1 and S_2 in the sense that the equality

$$(s_1, s_2) = (s_1, 1) \circ (1, s_2)$$

holds in $S_1 \times S_2$ for any pair (s_1, s_2). The conclusion easily follows (see [Maltsiniotis 2005, Lemme 2.1.7]).

(b) This is true because f commuted with the members of S, hence it now commutes with their inverses. □

Proposition A.1.2. *Let $\mathcal{C}$ be a category with a product $\bullet : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and let S be a collection of morphisms in $\mathcal{C}$ containing all identities. Assume that $S \bullet S \subseteq S$. Then*:

(a) *There is a unique product $S^{-1}\mathcal{C} \times S^{-1}\mathcal{C} \to S^{-1}\mathcal{C}$ such that the localisation functor $P_S : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with the two products.*

(b) *If $\bullet$ is monoidal* (*resp. braided*, *symmetric*, *unital*), *the induced product on $S^{-1}\mathcal{C}$ enjoys the same properties and P_S is monoidal* (*resp. braided*, *symmetric*, *unital*).

Proof. Item (a) follows from Lemma A.1.1(a), and (b) from Lemma A.1.1(b). □

[8] We thank M. Bondarko for pointing out the importance of the identities.

A.2. ***Semiadditive categories.*** This subsection is a reformulation of [Mac Lane 1998, Chapter VIII, Section 2]; see also [Mac Lane 1950, Section 18 and beginning of Section 19].

Lemma A.2.1. (a) *For a category $\mathcal{A}$, the following conditions are equivalent*:

(i) *$\mathcal{A}$ has a* 0 *object* (*initial and final*), *binary products and coproducts*, *and for any $A, B \in \mathcal{A}$, the map*

$$A \amalg B \to A \times B$$

given on A by $(1_A, 0)$ and on B by $(0, 1_B)$ is an isomorphism.

(ii) *$\mathcal{A}$ has finite products*, *and for any $A, B \in \mathcal{A}$, $\mathcal{A}(A, B)$ has a structure of a commutative monoid*, *and composition is distributive with respect to these monoid laws.*

(iii) *Same as* (ii), *replacing product by coproduct.*

We then say that $\mathcal{A}$ is a semiadditive category *and write $A \oplus B$ for the product or coproduct of two objects A, B.*

(b) *If $\mathcal{A}$ is a semiadditive category*, *the law $(A, B) \mapsto A \oplus B$ endows $\mathcal{A}$ with a canonical unital symmetric monoidal structure.*

Proof. (a) By duality, we only need to show (i) $\iff$ (ii). (i) $\Longrightarrow$ (ii) follows from [Mac Lane 1998, Chapter VIII, Section 2, Example 4(a)]; recall that for two morphisms $f, g : A \to B$ in $\mathcal{A}$, Mac Lane defines their sum $f + g$ as the composition

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A \times A \\ {\scriptstyle f+g}\big\downarrow & & \searrow{\scriptstyle f\times g} \\ & & B \times B \\ & & \nearrow{\scriptstyle \sim} \\ B & \xleftarrow{\nabla_B} & B \amalg B \end{array}$$

where Δ_A is the diagonal and ∇_B is the codiagonal.

As for (ii) $\Longrightarrow$ (i), it is implicit in the proof of [Mac Lane 1998, Chapter VIII, Section 2, Theorem 2]. Indeed, Mac Lane defines a biproduct of two objects $A, B \in \mathcal{A}$ as a diagram

$$A \underset{i_1}{\overset{p_1}{\leftrightarrows}} C \underset{i_2}{\overset{p_2}{\leftrightarrows}} B$$

satisfying $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$ and $i_1 p_1 + i_2 p_2 = 1_C$. Let us say that such a diagram is a *biproduct** if the further identities $p_1 i_2 = 0$ and $p_2 i_1 = 0$ hold. Then, Mac Lane proves that a biproduct* is a product and that a product is a biproduct*. Dually, a biproduct* is the same as a coproduct, hence binary products and coproducts are canonically isomorphic, and one checks from his proof that the isomorphism is given by the map of (i).

(Let us clarify that Mac Lane proves that a biproduct is a biproduct* if the addition law on morphisms has the cancellation property; but we don't use this part of his proof.)

(b) This is obvious: already finite products or coproducts define a canonical symmetric monoidal structure. □

Define a *semiadditive functor* between two semiadditive categories $\mathcal{A}, \mathcal{B}$ as a functor $F : \mathcal{A} \to \mathcal{B}$ which preserves addition of morphisms. Note that any semiadditive functor preserves $\oplus$, by the characterisation of biproducts via equations (see proof of Lemma A.2.1(a)).

A.3. *Localisation of R-linear categories.*

Theorem A.3.1. *Let $\mathcal{A}$ be a semiadditive category and S a family of morphisms of $\mathcal{A}$, containing all identities and stable under $\oplus$. Then $S^{-1}\mathcal{A}$ and the localisation functor $P_S : \mathcal{A} \to S^{-1}\mathcal{A}$ are semiadditive.*

Proof. We use the characterisation (i) of semiadditive categories in Lemma A.2.1; by [Maltsiniotis 2005, Lemme 1.3.6 and Proposition 2.1.8], P_S preserves products and coproducts, and transforms the isomorphisms $A \amalg B \xrightarrow{\sim} A \times B$ into isomorphisms. □

To "catch" additive categories (as opposed to semiadditive categories), we could do as in [Mac Lane 1950] and postulate the existence of an endomorphism -1_A for each object A. We prefer to do this more generally by dealing with R-linear categories, where R is an arbitrary ring (an R-linear category is simply a semiadditive R-category).

More precisely, let $\mathcal{A}$ be an R-linear category. Then in particular:

- $\mathcal{A}$ is a semiadditive category.
- It enjoys an action of the multiplicative monoid underlying R, i.e., there is a homomorphism of monoids $R \to \mathrm{End}(Id_A)$, where $\mathrm{End}(Id_A)$ is the monoid of natural transformations of the identity functor of A.
- For $\lambda \in R$ and $A \in \mathcal{A}$, let λ_A denote the corresponding endomorphism of A. Then we have identities

$$(\lambda + \mu)_A = \lambda_A + \mu_A. \tag{A.3.2}$$

Conversely, the following lemma is straightforward.

Lemma A.3.3. *Let $\mathcal{A}$ be a semiadditive category provided with an action of R verifying* (A.3.2). *Then $\mathcal{A}$ is an R-linear category.*

From this lemma, it follows:

Theorem A.3.4. *Theorem A.3.1 extends to R-linear categories.*

A.4. *Localisation and pseudoabelian envelope.*

Lemma A.4.1. *Let $\mathcal{A}$ an additive category and S a family of morphisms in $\mathcal{A}$, stable under direct sums. Let $\mathcal{A} \to \mathcal{A}^\natural$ denote the pseudoabelian envelope of $\mathcal{A}$, and let us denote by $S^\natural$ the set of direct summands of members of S in $\mathcal{A}^\natural$. Then the natural functors*

$$(S^{-1}\mathcal{A})^\natural \to (S^{-1}(\mathcal{A}^\natural))^\natural \to ((S^\natural)^{-1}(\mathcal{A}^\natural))^\natural$$

are equivalences of categories.

Proof. All categories are universal for additive functors T from $\mathcal{A}$ to a pseudoabelian category such that $T(S)$ is invertible. □

A.5. *Localisation and group completion.*

Lemma A.5.1. *Let $\mathcal{A}$ be a semiadditive category. There exists an additive category $\mathcal{A}^+$ and a semiadditive functor $\iota : \mathcal{A} \to \mathcal{A}^+$ with the following 2-universal property: any semiadditive functor from $\mathcal{A}$ to an additive category factors through ι up to a unique natural isomorphism.*

A model of $\mathcal{A}^+$ may be given as follows: the objects of $\mathcal{A}^+$ are those of $\mathcal{A}$; if $A, B \in \mathcal{A}$, then $\mathcal{A}^+(A, B)$ is the group completion of the commutative monoid $\mathcal{A}(A, B)$.

The category $\mathcal{A}^+$ is called the group completion *of $\mathcal{A}$.*

The proof is straightforward and omitted.

Proposition A.5.2. *Let $\mathcal{A}$ be a semiadditive category, and let S be a family of morphisms in $\mathcal{A}$, containing the identities and stable under direct sums. Keep writing S for the image of S in the group completion $\mathcal{A}^+$. Then the functor*

$$S^{-1}\iota : S^{-1}\mathcal{A} \to S^{-1}(\mathcal{A}^+)$$

induces an equivalence of categories

$$\tilde{\iota} : (S^{-1}\mathcal{A})^+ \xrightarrow{\sim} S^{-1}(\mathcal{A}^+).$$

Here we use the structure of semiadditive category on $S^{-1}\mathcal{A}$ given in Theorem A.3.1.

Proof. The existence of $\tilde{\iota}$ follows from the universal property of group completion. A quasi-inverse to $\tilde{\iota}$ is obtained by group-completing the functor $\mathcal{A} \to S^{-1}\mathcal{A}$ (which is semiadditive by Theorem A.3.1), and then extending the resulting functor to $S^{-1}(\mathcal{A}^+)$. □

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References

[André 1996] Y. André, "Pour une théorie inconditionnelle des motifs", *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 5–49. MR Zbl

[André 2004] Y. André, *Une introduction aux motifs* (*motifs purs, motifs mixtes, périodes*), Panoramas et Synthèses **17**, Société Mathématique de France, Paris, 2004. MR Zbl

[André and Kahn 2002] Y. André and B. Kahn, "Nilpotence, radicaux et structures monoïdales", *Rend. Sem. Mat. Univ. Padova* **108** (2002), 107–291. MR Zbl

[Auel et al. 2013] A. Auel, J.-L. Colliot-Thélène, and R. Parimala, "Universal unramified cohomology of cubic fourfolds containing a plane", preprint, 2013. arXiv

[Barlow 1985a] R. Barlow, "Rational equivalence of zero cycles for some more surfaces with $p_g = 0$", *Invent. Math.* **79**:2 (1985), 303–308. MR Zbl

[Barlow 1985b] R. Barlow, "A simply connected surface of general type with $p_g = 0$", *Invent. Math.* **79**:2 (1985), 293–301. MR Zbl

[Bauer et al. 2011] I. Bauer, F. Catanese, and R. Pignatelli, "Surfaces of general type with geometric genus zero: a survey", pp. 1–48 in *Complex and differential geometry*, edited by W. Ebeling et al., Springer Proc. Math. **8**, Springer, Heidelberg, 2011. MR Zbl

[Beilinson 1986] A. A. Beilinson, "Notes on absolute Hodge cohomology", pp. 35–68 in *Applications of algebraic K-theory to algebraic geometry and number theory, I* (Boulder, CO, 1983), edited by S. J. Bloch et al., Contemp. Math. **55**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl

[Bloch 2010] S. Bloch, *Lectures on algebraic cycles*, 2nd ed., New Mathematical Monographs **16**, Cambridge University Press, 2010. MR Zbl

[Bloch and Srinivas 1983] S. Bloch and V. Srinivas, "Remarks on correspondences and algebraic cycles", *Amer. J. Math.* **105**:5 (1983), 1235–1253. MR Zbl

[Campana 1992] F. Campana, "Connexité rationnelle des variétés de Fano", *Ann. Sci. École Norm. Sup.* (4) **25**:5 (1992), 539–545. MR Zbl

[Chernousov and Merkurjev 2006] V. Chernousov and A. Merkurjev, "Connectedness of classes of fields and zero-cycles on projective homogeneous varieties", *Compos. Math.* **142**:6 (2006), 1522–1548. MR Zbl

[Chernousov et al. 2005] V. Chernousov, S. Gille, and A. Merkurjev, "Motivic decomposition of isotropic projective homogeneous varieties", *Duke Math. J.* **126**:1 (2005), 137–159. MR Zbl

[Clemens 1983] H. Clemens, "Homological equivalence, modulo algebraic equivalence, is not finitely generated", *Inst. Hautes Études Sci. Publ. Math.* **58** (1983), 19–38. MR Zbl

[Colliot-Thélène and Coray 1979] J.-L. Colliot-Thélène and D. Coray, "L'équivalence rationnelle sur les points fermés des surfaces rationnelles fibrées en coniques", *Compositio Math.* **39**:3 (1979), 301–332. MR Zbl

[Corti and Hanamura 2000] A. Corti and M. Hanamura, "Motivic decomposition and intersection Chow groups, I", *Duke Math. J.* **103**:3 (2000), 459–522. MR Zbl

[Deligne 1973] P. Deligne, “Cohomologie des intersections complètes”, exposé XI, pp. 39–61 in *Groupes de monodromie en géométrie algébrique, II: Exposés X–XXII* (Séminaire de Géométrie Algébrique du Bois Marie 1967–1969), edited by P. Deligne and N. Katz, Lecture Notes in Math. **340**, Springer, Berlin, 1973. MR Zbl

[Demazure and Grothendieck 2011] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome I: propriétés générales des schémas en groupes*, vol. 7, recomposée et annotée ed., Documents mathématiques, Paris, 2011.

[Esnault 2003] H. Esnault, “Varieties over a finite field with trivial Chow group of 0-cycles have a rational point”, *Invent. Math.* **151**:1 (2003), 187–191. MR Zbl

[Esnault and Viehweg 1988] H. Esnault and E. Viehweg, “Deligne–Beilinson cohomology”, pp. 43–91 in *Beilinson's conjectures on special values of L-functions*, edited by M. Rapoport et al., Perspect. Math. **4**, Academic Press, Boston, 1988. MR Zbl

[Esnault et al. 1997] H. Esnault, M. Levine, and E. Viehweg, “Chow groups of projective varieties of very small degree”, *Duke Math. J.* **87**:1 (1997), 29–58. MR Zbl

[Fulton 1984] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik (3) **2**, Springer, Berlin, 1984. MR Zbl

[Gabriel and Zisman 1967] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik **35**, Springer, New York, 1967. MR Zbl

[Griffiths 1969] P. A. Griffiths, “On the periods of certain rational integrals, II”, *Ann. of Math.* (2) **90** (1969), 496–541. MR Zbl

[Grothendieck and Dieudonné 1966] A. Grothendieck and J. A. Dieudonné, “Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, III”, *Inst. Hautes Études Sci. Publ. Math.* **28** (1966), 5–255. MR Zbl

[Grothendieck and Dieudonné 1971] A. Grothendieck and J. A. Dieudonné, *Eléments de géométrie algébrique, I*, Grundlehren der Math. Wissenschaften **166**, Springer, Berlin, 1971. MR Zbl

[Huber 2008] A. Huber, “Slice filtration on motives and the Hodge conjecture”, *Math. Nachr.* **281**:12 (2008), 1764–1776. MR

[Illusie and Temkin 2014] L. Illusie and M. Temkin, “Gabber's modification theorem (log smooth case)”, exposé X, pp. 167–212 in *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*, edited by L. Illusie et al., Astérisque **363–364**, Société Mathématique de France, Paris, 2014. MR Zbl

[Inose and Mizukami 1979] H. Inose and M. Mizukami, “Rational equivalence of 0-cycles on some surfaces of general type with $p_g = 0$”, *Math. Ann.* **244**:3 (1979), 205–217. MR Zbl

[Jannsen 1992] U. Jannsen, “Motives, numerical equivalence, and semi-simplicity”, *Invent. Math.* **107**:3 (1992), 447–452. MR Zbl

[Jannsen 1994] U. Jannsen, “Motivic sheaves and filtrations on Chow groups”, pp. 245–302 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[de Jong 1996] A. J. de Jong, “Smoothness, semi-stability and alterations”, *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 51–93. MR Zbl

[Kahn 2009] B. Kahn, “Zeta functions and motives”, *Pure Appl. Math. Q.* **5**:1 (2009), 507–570. MR Zbl

[Kahn 2011] B. Kahn, “Relatively unramified elements in cycle modules”, *J. K-Theory* **7**:3 (2011), 409–427. MR Zbl

[Kahn 2014] B. Kahn, "Algebraic tori as Nisnevich sheaves with transfers", *Ann. Fac. Sci. Toulouse Math.* (6) **23**:3 (2014), 699–715. MR Zbl

[Kahn 2015] B. Kahn, "Motifs et adjoints", preprint, 2015. arXiv

[Kahn 2016] B. Kahn, "Torsion order of smooth projective surfaces", preprint, 2016. arXiv

[Kahn and Sujatha 2002] B. Kahn and R. Sujatha, "Birational motives, I", preprint, 2002, available at http://www.math.uiuc.edu/K-theory/0596/birat11.pdf.

[Kahn and Sujatha 2007] B. Kahn and R. Sujatha, "A few localisation theorems", *Homology, Homotopy Appl.* **9**:2 (2007), 137–161. MR Zbl

[Kahn and Sujatha 2015a] B. Kahn and R. Sujatha, "Birational geometry and localisation of categories", *Doc. Math.* Extra vol. (2015), 277–334. MR

[Kahn and Sujatha 2015b] B. Kahn and R. Sujatha, "Birational motives, II: Triangulated birational motives", preprint, 2015. To appear in *Int. Math. Res. Not.* arXiv

[Kahn et al. 2007] B. Kahn, J. P. Murre, and C. Pedrini, "On the transcendental part of the motive of a surface", pp. 143–202 in *Algebraic cycles and motives, II*, edited by J. Nagel and C. Peters, London Math. Soc. Lecture Note Ser. **344**, Cambridge University Press, 2007. MR Zbl

[Karpenko 1990] N. A. Karpenko, "Algebro-geometric invariants of quadratic forms", *Algebra i Analiz* **2**:1 (1990), 141–162. MR Zbl

[Katz 1973] N. Katz, "Le niveau de la cohomologie des intersections complètes", exposé XXI, pp. 363–400 in *Groupes de monodromie en géométrie algébrique, II: Exposés X–XXII* (Séminaire de Géométrie Algébrique du Bois Marie 1967–1969), edited by P. Deligne and N. Katz, Lecture Notes in Math. **340**, Springer, Berlin, 1973. MR Zbl

[Kelly 1964] G. M. Kelly, "On the radical of a category", *J. Austral. Math. Soc.* **4** (1964), 299–307. MR Zbl

[Kimura 2005] S.-I. Kimura, "Chow groups are finite dimensional, in some sense", *Math. Ann.* **331**:1 (2005), 173–201. MR Zbl

[Kleiman 1994] S. L. Kleiman, "The standard conjectures", pp. 3–20 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Kollár 1992] J. Kollár, *Classification of irregular varieties: minimal models and abelian varieties*, edited by E. Ballico et al., Lecture Notes in Mathematics **1515**, Springer, Berlin, 1992. MR

[Kollár et al. 1992] J. Kollár, Y. Miyaoka, and S. Mori, "Rational connectedness and boundedness of Fano manifolds", *J. Differential Geom.* **36**:3 (1992), 765–779. MR Zbl

[Krashen 2010] D. Krashen, "Zero cycles on homogeneous varieties", *Adv. Math.* **223**:6 (2010), 2022–2048. MR Zbl

[Lang 1959] S. Lang, *Abelian varieties*, Interscience Tracts in Pure and Applied Mathematics **7**, Interscience, New York, 1959. MR Zbl

[Mac Lane 1950] S. Mac Lane, "Duality for groups", *Bull. Amer. Math. Soc.* **56**:6 (11 1950), 485–516. Zbl

[Mac Lane 1998] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics **5**, Springer, New York, 1998. MR Zbl

[Maltsiniotis 2005] G. Maltsiniotis, *La théorie de l'homotopie de Grothendieck*, Astérisque **301**, Société Mathématique de France, Paris, 2005. MR Zbl

[Manin 1968] Y. Manin, "Correspondences, motifs and monoidal transformations", *Math. USSR Sbornik* **6** (1968), 439–470. Zbl

[Mazza et al. 2006] C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs **2**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Merkurjev 2001] A. S. Merkurjev, "Rational correspondences", preprint, 2001, available at http://www.math.ucla.edu/~merkurev/papers/rat.pdf.

[Merkurjev 2008] A. Merkurjev, "Unramified elements in cycle modules", *J. Lond. Math. Soc.* (2) **78**:1 (2008), 51–64. MR Zbl

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton University Press, 1980. MR Zbl

[Milne 1986] J. S. Milne, "Abelian varieties", pp. 103–150 in *Arithmetic geometry* (Storrs, CT, 1984), edited by G. Cornell and J. H. Silverman, Springer, New York, 1986. MR Zbl

[Murre 1990] J. P. Murre, "On the motive of an algebraic surface", *J. Reine Angew. Math.* **409** (1990), 190–204. MR Zbl

[Murre 1993] J. P. Murre, "On a conjectural filtration on the Chow groups of an algebraic variety, I: The general conjectures and some examples", *Indag. Math.* (*N.S.*) **4**:2 (1993), 177–188. MR

[Nori 1989] M. V. Nori, "Cycles on the generic abelian threefold", *Proc. Indian Acad. Sci. Math. Sci.* **99**:3 (1989), 191–196. MR

[Peters and Steenbrink 2003] C. A. M. Peters and J. H. M. Steenbrink, "Monodromy of variations of Hodge structure", *Acta Appl. Math.* **75**:1-3 (2003), 183–194. MR Zbl

[Petrov et al. 2008] V. Petrov, N. Semenov, and K. Zainoulline, "Zero cycles on a twisted Cayley plane", *Canad. Math. Bull.* **51**:1 (2008), 114–124. MR Zbl

[Ramachandran 2001] N. Ramachandran, "Duality of Albanese and Picard 1-motives", *K-Theory* **22**:3 (2001), 271–301. MR Zbl

[Roberts 1972] J. Roberts, "Chow's moving lemma", pp. 89–96 in *Algebraic geometry: proceedings of the fifth Nordic Summer School in Mathematics* (Oslso, 1970), edited by F. Oort, Wolters-Noordhoff, Groningen, 1972. MR

[Roitman 1980a] A. A. Roitman, "Rational equivalence of zero-dimensional cycles", *Mat. Zametki* **28**:1 (1980), 85–90. In Russian. MR

[Roitman 1980b] A. A. Roitman, "The torsion of the group of 0-cycles modulo rational equivalence", *Ann. of Math.* (2) **111**:3 (1980), 553–569. MR Zbl

[Rost 1996] M. Rost, "Chow groups with coefficients", *Doc. Math.* **1**:16 (1996), 319–393. MR Zbl

[Samuel 1960] P. Samuel, "Relations d'équivalence en géométrie algébrique", pp. 470–487 in *Proceedings of the International Congress of Mathematicians, 1958*, edited by J. A. Todd, Cambridge University Press, 1960. MR

[Scholl 1994] A. J. Scholl, "Classical motives", pp. 163–187 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Serre 1958/1959] J.-P. Serre, "Morphismes universels et variétés d'Albanese", exposé X, pp. 1–22 in *Séminaire Claude Chevalley*, vol. 4, Secrétariat mathématique, Paris, 1958/1959. Reprinted in *Notes de séminaires* (*1950–1999*), Documents mathématiques, **1** (2001), 141–160. Zbl

[Serre 1988] J.-P. Serre, *Algebraic groups and class fields*, Graduate Texts in Mathematics **117**, Springer, New York, 1988. MR Zbl

[Shermenev 1974] A. M. Shermenev, "The motif of an Abelian variety", *Funkts. Anal. Prilozh.* **8**:1 (1974), 55–61. In Russian; translated in *Functional Anal. Appl.* **8**:1 (1974), 47–53. MR Zbl

[Soulé and Voisin 2005] C. Soulé and C. Voisin, "Torsion cohomology classes and algebraic cycles on complex projective manifolds", *Adv. Math.* **198**:1 (2005), 107–127. MR Zbl

[Swan 1989] R. G. Swan, "Zero cycles on quadric hypersurfaces", *Proc. Amer. Math. Soc.* **107**:1 (1989), 43–46. MR Zbl

[Totaro 2014] B. Totaro, "The motive of a classifying space", preprint, 2014. To appear in *Geom. Topol.* arXiv

[Voevodsky 1992] V. Voevodsky, letter to A. Beilinson, December 6, 1992, available at http://math.uiuc.edu/K-theory/0033/.

[Voevodsky 1995] V. Voevodsky, "A nilpotence theorem for cycles algebraically equivalent to zero", *Internat. Math. Res. Notices* **1995**:4 (1995), 187–198. MR Zbl

[Voevodsky 2000] V. Voevodsky, "Triangulated categories of motives over a field", pp. 188–238 in *Cycles, transfers, and motivic homology theories*, edited by V. Voevodsky et al., Ann. of Math. Stud. **143**, Princeton University Press, 2000. MR Zbl

[Voisin 1992] C. Voisin, "Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **19**:4 (1992), 473–492. MR Zbl

[Voisin 2013] C. Voisin, "The generalized Hodge and Bloch conjectures are equivalent for general complete intersections", *Ann. Sci. Éc. Norm. Supér.* (4) **46**:3 (2013), 449–475. MR Zbl

[Voisin 2014] C. Voisin, "On the universal CH_0 group of cubic hypersurfaces", preprint, 2014. arXiv

[Voisin 2015] C. Voisin, "Unirational threefolds with no universal codimension 2 cycle", *Invent. Math.* **201**:1 (2015), 207–237. MR Zbl

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