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Splitting the relative assembly map, Nil-terms and involutions

Wolfgang Lück and Wolfgang Steimle

We show that the relative Farrell–Jones assembly map from the family of finite subgroups to the family of virtually cyclic subgroups for algebraic *K*-theory is split injective in the setting where the coefficients are additive categories with group action. This generalizes a result of Bartels for rings as coefficients. We give an explicit description of the relative term. This enables us to show that it vanishes rationally if we take coefficients in a regular ring. Moreover, it is, considered as a $\mathbb{Z}[\mathbb{Z}/2]$-module by the involution coming from taking dual modules, an extended module and in particular all its Tate cohomology groups vanish, provided that the infinite virtually cyclic subgroups of type I of G are orientable. The latter condition is for instance satisfied for torsionfree hyperbolic groups.

Introduction

0A. ***Motivation.*** The *K-theoretic Farrell–Jones conjecture* [1993, 1.6 on page 257 and 1.7 on page 262] for a group G and a ring R predicts that the *assembly map*

$$\operatorname{asmb}_n : H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_R) \to H_n^G(G/G; \boldsymbol{K}_R) = K_n(RG)$$

is an isomorphism for all $n \in \mathbb{Z}$. Here $\underline{\underline{E}}G = E_{\mathcal{VC}}(G)$ is the *classifying space for the family $\mathcal{VC}$ of virtually cyclic subgroups* and $H_n^G(-; \boldsymbol{K}_R^G)$ is the *G-homology theory* associated to a specific covariant functor $\boldsymbol{K}_R^G$ from the orbit category $\mathsf{Or}(G)$ to the category $\mathsf{Spectra}$ of spectra. It satisfies $H_n^G(G/H; \boldsymbol{K}_R^G) = \pi_n(\boldsymbol{K}^G(G/H)) = K_n(RH)$ for any subgroup $H \subseteq G$ and $n \in \mathbb{Z}$. The assembly map is induced by the projection $\underline{\underline{E}}G \to G/G$. More information about the Farrell–Jones conjecture and the classifying spaces for families can be found for instance in the survey articles [Lück 2005; Lück and Reich 2005].

Let $\underline{E}G = E_{\mathcal{F}\text{in}}(G)$ be the *classifying space for the family $\mathcal{F}in$ of finite subgroups*, sometimes also called the *classifying space for proper G-actions*. The G-map $\underline{E}G \to \underline{\underline{E}}G$, which is unique up to G-homotopy, induces a so-called *relative*

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assembly map

$$\overline{\mathrm{asmb}}_n : H_n^G(\underline{E}G;\, \boldsymbol{K}_R) \to H_n^G(\underline{\underline{E}}G;\, \boldsymbol{K}_R).$$

The main result of a paper by Bartels [2003, Theorem 1.3] says that $\overline{\mathrm{asmb}}_n$ is split injective for all $n \in \mathbb{Z}$.

In this paper we improve on this result in two different directions: First, we generalize from the context of rings R to the context of additive categories $\mathcal{A}$ with G-action. This improvement allows us to consider twisted group rings and involutions twisted by an orientation homomorphism $G \to \{\pm 1\}$; moreover one obtains better inheritance properties and gets fibered versions for free.

Secondly, we give an explicit description of the relative term in terms of so-called NK-spectra. This becomes relevant for instance in the study of the involution on the cokernel of the relative assembly map induced by an involution of $\mathcal{A}$. In more detail, we prove:

0B. ***Splitting the relative assembly map.*** Our main splitting result is:

Theorem 0.1 (splitting the K-theoretic assembly map from $\mathcal{F}\mathrm{in}$ to $\mathcal{VC}$). *Let G be a group and let $\mathcal{A}$ be an additive G-category. Let n be any integer.*

Then the relative K-theoretic assembly map

$$\overline{\mathrm{asmb}}_n : H_n^G(\underline{E}G;\, \boldsymbol{K}_{\mathcal{A}}^G) \to H_n^G(\underline{\underline{E}}G;\, \boldsymbol{K}_{\mathcal{A}}^G)$$

is split injective. In particular we obtain a natural splitting

$$H_n^G(\underline{\underline{E}}G;\, \boldsymbol{K}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{E}G;\, \boldsymbol{K}_{\mathcal{A}}^G) \oplus H_n^G(\underline{E}G \to \underline{\underline{E}}G;\, \boldsymbol{K}_{\mathcal{A}}^G).$$

Moreover, *there exists an* $\mathsf{Or}(G)$*-spectrum* $\boldsymbol{NK}_{\mathcal{A}}^G$ *and a natural isomorphism*

$$H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G);\, \boldsymbol{NK}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{E}G \to \underline{\underline{E}}G;\, \boldsymbol{K}_{\mathcal{A}}^G).$$

Here $E_{\mathcal{VC}_I}(G)$ denotes the classifying space for the family of virtually cyclic subgroups of type I; see Section 1. The proof will appear in Section 7. The point is that, instead of considering $\boldsymbol{K}_R^G$ for a ring R, we can treat the more general setup $\boldsymbol{K}_{\mathcal{A}}^G$ for an additive G-category $\mathcal{A}$, as explained in [Bartels and Lück 2010; Bartels and Reich 2007]. (One recovers the case of a ring R if one considers for $\mathcal{A}$ the category R-FGF of finitely generated free R-modules with the trivial G-action. Notice that we tacitly always apply idempotent completion to the additive categories before taking K-theory.) Whereas in [Bartels 2003, Theorem 1.3] just a splitting is constructed, we construct explicit $\mathsf{Or}(G)$-spectra $\boldsymbol{NK}_{\mathcal{A}}^G$ and identify the relative terms. This is crucial for the following results.

0C. ***Involutions and vanishing of Tate cohomology.*** We will prove in Section 8C:

Theorem 0.2 (the relative term is induced). *Let G be a group and let $\mathcal{A}$ be an additive G-category with involution. Suppose that the virtually cyclic subgroups of type I of G are orientable* (see Definition 8.5).

Then the $\mathbb{Z}/2$-module $H_n(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G)$ is isomorphic to $\mathbb{Z}[\mathbb{Z}/2] \otimes_{\mathbb{Z}} A$ for some $\mathbb{Z}$-module A.

In [Farrell et al. 2016] we consider the conclusion of Theorem 0.2 that the Tate cohomology groups $\widehat{H}^i(\mathbb{Z}/2, H_n(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G))$ vanish for all $i, n \in \mathbb{Z}$ if the virtually cyclic subgroups of type I of G are orientable. In general the Tate spectrum of the involution on the Whitehead spectrum plays a role in the study of automorphisms of manifolds (see [Weiss and Williams 2000, Section 4], for example).

0D. ***Rational vanishing of the relative term.***

Theorem 0.3. *Let G be a group and let R be a regular ring.*

Then the relative assembly map

$$\overline{\mathrm{asmb}}_n : H_n^G(\underline{E}G; \boldsymbol{K}_R^G) \to H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_R^G)$$

is rationally bijective for all $n \in \mathbb{Z}$.

If $R = \mathbb{Z}$ and $n \leq -1$, then, by [Farrell and Jones 1995], the relative assembly map $H_n^G(\underline{E}G; \boldsymbol{K}_{\mathbb{Z}}^G) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_{\mathbb{Z}}^G)$ is an isomorphism.

In Section 10, we briefly discuss further computations of the relative term $H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^G) \cong H_n^G(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G)$.

0E. ***A fibered case.*** In Section 11 we discuss a fibered situation which will be relevant for [Farrell et al. 2016] and can be handled by our general treatment for additive G-categories.

1. Virtually cyclic groups

A virtually cyclic group V is called *of type I* if it admits an epimorphism to the infinite cyclic group, and *of type II* if it admits an epimorphism onto the infinite dihedral group. The statements appearing in the next lemma are well-known; we insert a proof for the reader's convenience.

Lemma 1.1. *Let V be an infinite virtually cyclic group.*

(i) *V is either of type I or of type II.*

(ii) *The following assertions are equivalent*:

  (a) *V is of type I*;

  (b) *$H_1(V)$ is infinite*;

(c) $H_1(V)/\operatorname{tors}(V)$ *is infinite cyclic*;
(d) *the center of* V *is infinite.*

(iii) *There exists a unique maximal normal finite subgroup* $K_V \subseteq V$, *i.e.*, K_V *is a finite normal subgroup and every normal finite subgroup of* V *is contained in* K_V.

(iv) *Let* $Q_V := V/K_V$. *Then we obtain a canonical exact sequence*

$$1 \to K_V \xrightarrow{i_V} V \xrightarrow{p_V} Q_V \to 1.$$

Moreover, Q_V *is infinite cyclic if and only if* V *is of type I*, *and* Q_V *is isomorphic to the infinite dihedral group if and only if* V *is of type II.*

(v) *Let* $f : V \to Q$ *be any epimorphism onto the infinite cyclic group or onto the infinite dihedral group. Then the kernel of* f *agrees with* K_V.

(vi) *Let* $\phi : V \to W$ *be a homomorphism of infinite virtually cyclic groups with infinite image. Then* ϕ *maps* K_V *to* K_W *and we obtain the canonical commutative diagram with exact rows*

$$\begin{tikzcd} 1 \arrow[r] & K_V \arrow[r, "i_V"] \arrow[d, "\phi_K"] & V \arrow[r, "p_V"] \arrow[d, "\phi"] & Q_V \arrow[r] \arrow[d, "\phi_Q"] & 1 \\ 1 \arrow[r] & K_W \arrow[r, "i_W"] & W \arrow[r, "p_W"] & Q_W \arrow[r] & 1 \end{tikzcd}$$

with injective ϕ_Q.

Proof. (ii) If V is of type I, then we obtain epimorphisms

$$V \to H_1(V) \to H_1(V)/\operatorname{tors}(H_1(V)) \to \mathbb{Z}.$$

The kernel of $V \to \mathbb{Z}$ is finite, since for an exact sequence $1 \to \mathbb{Z} \xrightarrow{i} V \xrightarrow{q} H \to 1$ with finite H the composite of $V \to \mathbb{Z}$ with i is injective and hence the restriction of q to the kernel of $V \to \mathbb{Z}$ is injective. This implies that $H_1(V)$ is infinite and $H_1(V)/\operatorname{tors}(H_1(V))$ is infinite cyclic. If $H_1(V)/\operatorname{tors}(H_1(V))$ is infinite cyclic or if $H_1(V)$ is infinite, then $H_1(V)$ surjects onto $\mathbb{Z}$ and hence so does V. This shows (a) $\iff$ (b) $\iff$ (c).

Consider the exact sequence $1 \to \operatorname{cent}(V) \to V \to V/\operatorname{cent}(V) \to 1$, where $\operatorname{cent}(V)$ is the center of V. Suppose that $\operatorname{cent}(V)$ is infinite. Then $V/\operatorname{cent}(V)$ is finite and the Lyndon–Serre spectral sequence yields an isomorphism $\operatorname{cent}(V) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_1(V;\mathbb{Q})$. Hence $H_1(V)$ is infinite. This shows (d) $\Longrightarrow$ (b).

Suppose that V is of type I. Choose an exact sequence $1 \to K \to V \to \mathbb{Z} \to 1$ with finite K. Let $v \in V$ be an element which is mapped to a generator of $\mathbb{Z}$. Then conjugation with v induces an automorphism of K. Since K is finite, we can find a natural number k such that conjugation with v^k induces the identity on K. One easily checks that v^k belongs to the center of V and v has infinite order. This shows (a) $\Longrightarrow$ (d) and finishes the proof of assertion (ii).

(iii) If K_1 and K_2 are two finite normal subgroups of V, then

$$K_1 \cdot K_2 := \{v \in V \mid v = k_1 k_2 \text{ for some } k_1 \in K_1 \text{ and } k_2 \in K_2\}$$

is a finite normal subgroup of V. Hence we are left to show that V has only finitely many different finite normal subgroups.

To see this, choose an exact sequence $1 \to \mathbb{Z} \xrightarrow{i} V \xrightarrow{f} H \to 1$ for some finite group H. The map f induces a map from the finite normal subgroups of V to the normal subgroups of H; we will show that it is an injection. Let $t \in V$ be the image under i of some generator of $\mathbb{Z}$ and consider two finite normal subgroups K_1 and K_2 of V with $f(K_1) = f(K_2)$. Consider $k_1 \in K_1$. We can find $k_2 \in K_2$ and $n \in \mathbb{Z}$ with $k_2 = k_1 \cdot t^n$. Then t^n belongs to the finite normal subgroup $K_1 \cdot K_2$. This implies $n = 0$ and hence $k_1 = k_2$. This shows $K_1 \subseteq K_2$. By symmetry we get $K_1 = K_2$. Since H contains only finitely many subgroups, we conclude that there are only finitely many different finite normal subgroups in V. Now assertion (iii) follows.

(i) and (iv) Let V be an infinite virtually cyclic group. Then Q_V is an infinite virtually cyclic subgroup which does not contain a nontrivial finite normal subgroup. There exists an exact sequence $1 \to \mathbb{Z} \xrightarrow{i} Q_V \xrightarrow{f} H \to 1$ for some finite group H. There exists a subgroup of index at most two $H' \subseteq H$ such that the conjugation action of H on $\mathbb{Z}$ restricted to H' is trivial. Put $Q'_V = f^{-1}(H')$. Then the center of Q'_V contains $i(\mathbb{Z})$ and hence is infinite. By assertion (ii) we can find an exact sequence $1 \to K \to Q'_V \xrightarrow{f} \mathbb{Z} \to 1$ with finite K. The group Q'_V contains a unique maximal normal finite subgroup K' by assertion (iii). This implies that $K' \subseteq Q'_V$ is characteristic. Since Q'_V is a normal subgroup of Q_V, $K' \subseteq Q_V$ is a normal subgroup and therefore K' is trivial. Hence Q'_V contains no nontrivial finite normal subgroup. This implies that Q'_V is infinite cyclic. Since Q'_V is a normal subgroup of index 2 in Q_V and Q_V contains no nontrivial finite normal subgroup, Q_V is infinite cyclic or D_∞.

In particular we see that every infinite virtual cyclic group is of type I or of type II. It remains to show that an infinite virtually cyclic group V which is of type II cannot be of type I. If $1 \to K \to V \to D_\infty \to 1$ is an extension with finite K, then we obtain from the Lyndon–Serre spectral sequence an exact sequence $H_1(K) \otimes_{\mathbb{Z}Q} \mathbb{Z} \to H_1(V) \to H_1(D_\infty)$. Hence $H_1(V)$ is finite, since both $H_1(D_\infty)$ and $H_1(K)$ are finite. We conclude from assertion (ii) that V is not of type I. This finishes the proof of assertions (i) and (iv).

(v) Since V is virtually cyclic, the kernel of f is finite. Since Q does not contain a nontrivial finite normal subgroup, every normal finite subgroup of V is contained in the kernel of f. Hence $\ker(f)$ is the unique maximal finite normal subgroup of V.

(vi) Since K_W is finite and the image of ϕ is by assumption infinite, the composite $p_W \circ \phi : V \to Q_W$ has infinite image. Since Q_W is isomorphic to $\mathbb{Z}$ or D_∞, the same is true for the image of $p_W \circ \phi : V \to Q_W$. By assertion (v) the kernel of $p_W \circ \phi : V \to Q_W$ is K_V. Hence $\phi(K_V) \subseteq K_W$ and ϕ induces maps ϕ_K and ϕ_Q making the diagram appearing in assertion (vi) commutative. Since the image of $p_W \circ \phi : V \to Q_W$ is infinite, $\phi_Q(Q_V)$ is infinite. This implies that ϕ_Q is injective since both Q_V and Q_W are isomorphic to D_∞ or $\mathbb{Z}$. This finishes the proof of Lemma 1.1. $\square$

2. Some categories attached to homogeneous spaces

Let G be a group and let S be a G-set, for instance a homogeneous space G/H. Let $\mathcal{G}^G(S)$ be the associated *transport groupoid*. Objects are the elements in S. The set of morphisms from s_1 to s_2 consists of those elements $g \in G$ for which $gs_1 = s_2$. Composition is given by the group multiplication in G. Obviously $\mathcal{G}^G(S)$ is a connected groupoid if G acts transitively on S. A G-map $f : S \to T$ induces a functor $\mathcal{G}^G(f) : \mathcal{G}^G(S) \to \mathcal{G}^G(T)$ by sending an object $s \in S$ to $f(s)$ and a morphism $g : s_1 \to s_2$ to the morphism $g : f(s_1) \to f(s_2)$. We mention that for two objects s_1 and s_2 in $\mathcal{G}^G(S)$ the induced map $\mathrm{mor}_{\mathcal{G}^G(S)}(s_1, s_2) \to \mathrm{mor}_{\mathcal{G}^G(T)}(f(s_1), f(s_2))$ is injective.

A functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ of categories is called an *equivalence* if there exists a functor $F' : \mathcal{C}_1 \to \mathcal{C}_0$ with the property that $F' \circ F$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{C}_0}$ and $F \circ F'$ is naturally equivalent to the identity functor $\mathrm{id}_{\mathcal{C}_1}$. A functor F is a natural equivalence if and only if it is *essentially surjective* (i.e., it induces a bijection on the isomorphism classes of objects) and it is *full* and *faithful*, (i.e., for any two objects c, d in $\mathcal{C}_0$ the induced map $\mathrm{mor}_{\mathcal{C}_0}(c, d) \to \mathrm{mor}_{\mathcal{C}_1}(F(c), F(d))$ is bijective).

Given a monoid M, let $\widehat{M}$ be the category with precisely one object and M as the monoid of endomorphisms of this object. For any subgroup H of G, the inclusion

$$e(G/H) : \widehat{H} \to \mathcal{G}^G(G/H), \quad g \mapsto (eH \xrightarrow{g} eH)$$

(where $e \in G$ is the unit element), is an equivalence of categories, whose inverse sends $g : g_1H \to g_2H$ to $g_2^{-1}gg_1 \in G$.

Now fix an infinite virtually cyclic subgroup $V \subseteq G$ of type I. Then Q_V is an infinite cyclic group. Let $\mathrm{gen}(Q_V)$ be the set of generators. Given a generator $\sigma \in \mathrm{gen}(Q_V)$, define $Q_V[\sigma]$ to be the submonoid of Q_V consisting of elements of the form σ^n for $n \in \mathbb{Z}, n \geq 0$. Let $V[\sigma] \subseteq V$ be the submonoid given by $p_V^{-1}(Q_V[\sigma])$. Let $\mathcal{G}^G(G/V)[\sigma]$ be the subcategory of $\mathcal{G}^G(G/V)$ whose objects are the objects in

$\mathcal{G}^G(G/V)$ and whose morphisms $g : g_1V \to g_2V$ satisfy $g_2^{-1}gg_1 \in V[\sigma]$. Notice that $\mathcal{G}^G(G/V)[\sigma]$ is not a groupoid anymore, but any two objects are isomorphic. Let $\mathcal{G}^G(G/V)_K$ be the subcategory of $\mathcal{G}^G(G/V)$ whose objects are the objects in $\mathcal{G}^G(G/V)$ and whose morphisms $g : g_1V \to g_2V$ satisfy $g_2^{-1}gg_1 \in K_V$. Obviously $\mathcal{G}^G(G/V)_K$ is a connected groupoid and a subcategory of $\mathcal{G}^G(G/V)[\sigma]$.

We obtain the commutative diagram of categories

$$\begin{tikzcd} \mathcal{G}^G(G/V)[\sigma] \arrow[r, "j(G/V)[\sigma]"'] & \mathcal{G}^G(G/V) \\ \widehat{V[\sigma]} \arrow[u, "e(G/V)[\sigma]", "\cong"'] \arrow[r, "\widehat{j_V[\sigma]}"'] & \widehat{V} \arrow[u, "e(G/V)"', "\cong"] \end{tikzcd} \tag{2.1}$$

whose horizontal arrows are induced by the obvious inclusions and whose left vertical arrow is the restriction of $e(G/V)$ (and is also an equivalence of categories). The functor $e(G/V)$ also restricts to an equivalence of categories

$$e(G/V)_K : \widehat{K_V} \xrightarrow{\simeq} \mathcal{G}^G(G/V)_K. \tag{2.2}$$

Remark. The relation between the categories $\widehat{K_V}$, $\widehat{V[\sigma]}$ and $\widehat{V}$ and the categories $\mathcal{G}^G(G/V)_K$, $\mathcal{G}^G(G/V)[\sigma]$ and $\mathcal{G}^G(G/V)$ is analogous to the relation between the fundamental group of a path connected space and its fundamental groupoid.

Let $\overline{\sigma} \in V$ be any element which is mapped under the projection $p_V : V \to Q_V$ to the fixed generator σ. Right multiplication with $\overline{\sigma}$ induces a G-map $R_\sigma : G/K_V \to G/K_V,\ gK_V \mapsto g\overline{\sigma}K_V$. One easily checks that R_σ depends only on σ and is independent of the choice of $\overline{\sigma}$. Let $\mathrm{pr}_V : G/K_V \to G/V$ be the projection. We obtain the following commutative diagram:

$$\begin{tikzcd} \mathcal{G}^G(G/K_V) \arrow[rr, "R_\sigma"] \arrow[dr, "\mathcal{G}^G(\mathrm{pr}_V)"'] & & \mathcal{G}^G(G/K_V) \arrow[dl, "\mathcal{G}^G(\mathrm{pr}_V)"] \\ & \mathcal{G}^G(G/V) & \end{tikzcd} \tag{2.3}$$

3. Homotopy colimits of $\mathbb{Z}$-linear and additive categories

Homotopy colimits of additive categories have been defined for instance in [Bartels and Lück 2010, Section 5]. Here we review their definition and describe some properties, first in the setting of $\mathbb{Z}$-linear categories.

Recall that a $\mathbb{Z}$-linear category is a category where all Hom-sets are provided with the structure of abelian groups and composition is bilinear. Denote by $\mathbb{Z}$-$\mathcal{C}at$ the category whose objects are $\mathbb{Z}$-linear categories and whose morphisms are additive functors between them. Given a collection of $\mathbb{Z}$-linear categories $(\mathcal{A}_i)_{i\in I}$, their coproduct $\coprod_{i\in I}\mathcal{C}_i$ in $\mathbb{Z}$-$\mathcal{C}at$ exists and has the following explicit description:

Objects are pairs (i, X) where $i \in I$ and $X \in \mathcal{A}_i$. The abelian group of morphisms $(i, X) \to (j, Y)$ is nonzero only if $i = j$, in which case it is $\mathrm{mor}_{\mathcal{A}_i}(X, Y)$.

Let $\mathcal{C}$ be a small category. Given a contravariant functor $F : \mathcal{C} \to \mathbb{Z}\text{-}\mathcal{C}\mathfrak{at}$, its *homotopy colimit* (see [Thomason 1979], for instance)

$$\int_{\mathcal{C}} F \tag{3.1}$$

is the $\mathbb{Z}$-linear category obtained from the coproduct $\coprod_{c\in\mathcal{C}} F(c)$ by adjoining morphisms

$$T_f : (d, f^*X) \to (c, X)$$

for each $(c, X) \in \coprod_{c\in\mathcal{C}} F(c)$ and each morphism $f : d \to c$ in $\mathcal{C}$. (Here we write f^*X for $F(f)(X)$.) They are subject to the relations that $T_{\mathrm{id}} = \mathrm{id}$ and that all possible diagrams

$$\begin{tikzcd}
(e, g^*f^*X) \arrow[r, "T_g"] \arrow[dr, "T_{f\circ g}"'] & (d, f^*X) \arrow[d, "T_f"] \\
& (c, X)
\end{tikzcd}
\qquad
\begin{tikzcd}
(d, f^*X) \arrow[r, "T_f"] \arrow[d, "f^*u"] & (c, X) \arrow[d, "u"] \\
(d, f^*Y) \arrow[r, "T_f"] & (c, Y)
\end{tikzcd}$$

are to be commutative.

Hence, a morphism in $\int_{\mathcal{C}} F$ from (x, A) to (y, B) can be uniquely written as a sum

$$\sum_{f\in\mathrm{mor}_{\mathcal{C}}(x,y)} T_f \circ \phi_f, \tag{3.2}$$

where $\phi_f : A \to f^*B$ is a morphism in $F(x)$ and all but finitely many of the morphisms ϕ_f are zero. The composition of two such morphisms can be determined by the distributivity law and the rule

$$(T_f \circ \phi) \circ (T_g \circ \psi) = T_{f\circ g} \circ (g^*\phi \circ \psi),$$

which just follows from the fact that both upper squares are commutative.

Using this description, it follows that the homotopy colimit has the following universal property for additive functors $\int_{\mathcal{C}} F \to \mathcal{A}$ into some other $\mathbb{Z}$-linear category $\mathcal{A}$: Suppose that we are given additive functors $j_c : F(c) \to \mathcal{A}$ for each $c \in \mathcal{C}$ and morphisms $S_f : j_d(f^*X) \to j_c(X)$ for each $X \in F(c)$ and each $f : d \to c$ in $\mathcal{C}$. If $S_{\mathrm{id}} = \mathrm{id}$ and all possible diagrams

$$\begin{tikzcd}
j_c(g^*f^*X) \arrow[r, "S_g"] \arrow[dr, "S_{f\circ g}"'] & j_d(f^*X) \arrow[d, "S_f"] \\
& j_c(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
j_d(f^*X) \arrow[r, "S_f"] \arrow[d, "j_d(f^*u)"] & j_c(X) \arrow[d, "j_c(u)"] \\
j_d(f^*Y) \arrow[r, "S_f"] & j_c(Y)
\end{tikzcd}$$

are commutative, then this data specifies an additive functor $\int_{\mathcal{C}} F \to \mathcal{A}$ by sending T_f to S_f.

The homotopy colimit is functorial in F. Namely, if $S : F_0 \to F_1$ is a natural transformation of contravariant functors $\mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Cat}$, then it induces an additive functor

$$\int_{\mathcal{C}} S : \int_{\mathcal{C}} F_0 \to \int_{\mathcal{C}} F_1 \tag{3.3}$$

of $\mathbb{Z}$-linear categories. It is defined using the universal property by sending $F_0(c)$ to $F_1(c) \subset \int_{\mathcal{C}} F_1$ via S and "sending T_f to T_f". In more detail, the image of $T_f : (c, f^*(X)) \to (d, X)$ in $\int_{\mathcal{C}} F_0$ is given by $T_f : (c, f^*(S(X))) \to (d, S(X))$ in $\int_{\mathcal{C}} F_1$. Obviously we have, for $S_1 : F_0 \to F_1$ and $S_2 : F_1 \to F_2$,

$$\left(\int_{\mathcal{C}} S_2\right) \circ \left(\int_{\mathcal{C}} S_1\right) = \int_{\mathcal{C}} (S_2 \circ S_1), \tag{3.4}$$

$$\int_{\mathcal{C}} \mathrm{id}_F = \mathrm{id}_{\int_{\mathcal{C}} F}. \tag{3.5}$$

The construction is also functorial in $\mathcal{C}$. Namely, let $W : \mathcal{C}_1 \to \mathcal{C}_2$ be a covariant functor. Then we obtain a covariant functor

$$W_* : \int_{\mathcal{C}_1} F \circ W \to \int_{\mathcal{C}_2} F \tag{3.6}$$

of additive categories that is the identity on each $F(W(c))$ and "sends T_f to $T_{W(f)}$", again interpreted appropriately. For covariant functors $W_1 : \mathcal{C}_1 \to \mathcal{C}_2$, $W_2 : \mathcal{C}_2 \to \mathcal{C}_3$ and a contravariant functor $F : \mathcal{C}_3 \to \mathsf{Add}\text{-}\mathsf{Cat}$, we have

$$(W_2)_* \circ (W_1)_* = (W_2 \circ W_1)_*, \tag{3.7}$$

$$(\mathrm{id}_{\mathcal{C}})_* = \mathrm{id}_{\int_{\mathcal{C}} F}. \tag{3.8}$$

These two constructions are compatible. Namely, given a natural transformation $S : F_1 \to F_2$ of contravariant functors $\mathcal{C}_2 \to \mathbb{Z}\text{-}\mathsf{Cat}$ and a covariant functor $W : \mathcal{C}_1 \to \mathcal{C}_2$, we get

$$\left(\int_{\mathcal{C}_2} S\right) \circ W_* = W_* \circ \left(\int_{\mathcal{C}_1} (S \circ W)\right). \tag{3.9}$$

Lemma 3.10.

(i) *Let $W : \mathcal{D} \to \mathcal{C}$ be an equivalence of categories. Let $F : \mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Cat}$ be a contravariant functor. Then*

$$W_* : \int_{\mathcal{D}} F \circ W \to \int_{\mathcal{C}} F$$

is an equivalence of categories.

(ii) *Let $\mathcal{C}$ be a category and let $S : F_1 \to F_2$ be a transformation of contravariant functors $\mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Cat}$ such that, for every object c in $\mathcal{C}$, the functor $S(c) : F_0(c) \to F_1(c)$ is an equivalence of categories. Then*

$$\int_{\mathcal{C}} S : \int_{\mathcal{C}} F_1 \to \int_{\mathcal{C}} F_2$$

is an equivalence of categories.

The proof is an easy exercise. Note the general fact that, if $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between $\mathbb{Z}$-linear categories such that F is an equivalence between the underlying categories, then it follows automatically that there exists an additive inverse equivalence F' and two additive natural equivalences $F' \circ F \simeq \mathrm{id}_{\mathcal{C}}$ and $F \circ F' \simeq \mathrm{id}_{\mathcal{D}}$.

Notation 3.11. If $W : \mathcal{C}_1 \to \mathcal{C}$ is the inclusion of a subcategory, then the same is true for W_*. If no confusion is possible, we just write

$$\int_{\mathcal{C}_1} F := \int_{\mathcal{C}_1} F \circ W \subset \int_{\mathcal{C}} F.$$

Denote by $\mathsf{Add}\text{-}\mathsf{Cat}$ the category whose objects are additive categories and whose morphisms are given by additive functors between them. Notice that $\int_{\mathcal{C}} F$ is not necessarily an additive category even if all the $F(c)$ are — the direct sum $(c, X) \oplus (d, Y)$ need not exist. However, any isomorphism $f : c \to d$ in $\mathcal{C}$ induces an isomorphism $T_f : (c, f^*Y) \to (d, Y)$, so that

$$(c, X) \oplus (d, Y) \cong (c, X) \oplus (c, f^*Y) \cong (c, X \oplus f^*Y).$$

Hence, if in the index category all objects are isomorphic and all the $F(c)$ are additive, then $\int_{\mathcal{C}} F$ is an additive category. Since for additive categories $\mathcal{A}$, $\mathcal{B}$ we have

$$\mathrm{mor}_{\mathbb{Z}\text{-}\mathsf{Cat}}(A, B) = \mathrm{mor}_{\mathsf{Add}\text{-}\mathsf{Cat}}(A, B)$$

(in both cases the morphisms are just additive functors), the universal property for additive functors $\int_{\mathcal{C}} F \to \mathcal{A}$ into $\mathbb{Z}$-linear categories extends to a universal property for additive functors into additive categories.

In the general case of an arbitrary indexing category, the homotopy colimit in the setting of additive categories still exists. It is obtained by freely adjoining direct sums to the homotopy colimit for $\mathbb{Z}$-linear categories; the universal properties then hold in the setting of "additive categories with choice of direct sum". We will not discuss this in detail here since, in all the cases we will consider, the indexing category has the property that any two objects are isomorphic.

Notation 3.12. If the indexing category $\mathcal{C}$ has a single object and $F : \mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Cat}$ is a contravariant functor, then we will write X instead of $(*, X)$ for a typical element of the homotopy colimit. The structural morphisms in $\int_{\mathcal{C}} F$ thus take the simple form

$$T_f : f^*X \to X$$

for f a morphism (from the single object to itself) in $\mathcal{C}$.

4. The twisted Bass–Heller–Swan theorem for additive categories

Given an additive category $\mathcal{A}$, we denote by $\boldsymbol{K}(\mathcal{A})$ the nonconnective K-theory spectrum associated to it (after idempotent completion); see [Lück and Steimle 2014; Pedersen and Weibel 1989]. Thus we obtain a covariant functor

$$\boldsymbol{K} : \mathsf{Add\text{-}Cat} \to \mathsf{Spectra}. \tag{4.1}$$

Let $\mathcal{B}$ be an additive $\mathbb{Z}$-category, i.e., an additive category with a right action of the infinite cyclic group. Fix a generator σ of the infinite cyclic group $\mathbb{Z}$. Let $\Phi : \mathcal{B} \to \mathcal{B}$ be the automorphism of additive categories given by multiplication with σ. Of course, one can recover the $\mathbb{Z}$-action from Φ. Since $\widehat{\mathbb{Z}}$ has precisely one object, we can and will identify the set of objects of $\int_{\widehat{\mathbb{Z}}} \mathcal{B}$ and $\mathcal{B}$ in the sequel. Let $i_{\mathcal{B}} : \mathcal{B} \to \int_{\widehat{\mathbb{Z}}} \mathcal{B}$ be the inclusion into the homotopy colimit.

The structural morphisms $T_\sigma : \Phi(B) \to B$ of $\int_{\widehat{\mathbb{Z}}} \mathcal{B}$ assemble to a natural isomorphism $i_{\mathcal{B}} \circ \Phi \to i_{\mathcal{B}}$ in the following diagram:

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "\Phi"] \arrow[dr, "i_{\mathcal{B}}"'] & & \mathcal{B} \arrow[dl, "i_{\mathcal{B}}"] \\
& \int_{\widehat{\mathbb{Z}}} \mathcal{B} &
\end{tikzcd}$$

If we apply the nonconnective K-theory spectrum to it, we obtain a diagram of spectra which commutes up to preferred homotopy:

$$\begin{tikzcd}
\boldsymbol{K}(\mathcal{B}) \arrow[rr, "\boldsymbol{K}(\Phi)"] \arrow[dr, "\boldsymbol{K}(i_{\mathcal{B}})"'] & & \boldsymbol{K}(\mathcal{B}) \arrow[dl, "\boldsymbol{K}(i_{\mathcal{B}})"] \\
& \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right) &
\end{tikzcd}$$

It induces a map of spectra

$$\boldsymbol{a}_{\mathcal{B}} : T_{\boldsymbol{K}(\Phi)} \to \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right),$$

where $T_{\boldsymbol{K}(\Phi)}$ is the mapping torus of the map of spectra $\boldsymbol{K}(\Phi) : \boldsymbol{K}(\mathcal{B}) \to \boldsymbol{K}(\mathcal{B})$, which is defined as the pushout

$$\begin{tikzcd}
\boldsymbol{K}(\mathcal{B}) \wedge \{0,1\}_+ = \boldsymbol{K}(\mathcal{B}) \vee \boldsymbol{K}(\mathcal{B}) \arrow[r, "\boldsymbol{K}(\Phi)\vee \mathrm{id}_{\boldsymbol{K}(\mathcal{B})}"] \arrow[d] & \boldsymbol{K}(\mathcal{B}) \arrow[d] \\
\boldsymbol{K}(\mathcal{B}) \wedge [0,1]_+ \arrow[r] & T_{\boldsymbol{K}(\Phi)}
\end{tikzcd}$$

Let $\mathbb{Z}[\sigma]$ be the submonoid $\{\sigma^n \mid n \in \mathbb{Z},\ n \geq 0\}$ generated by σ. Let $j[\sigma] : \mathbb{Z}[\sigma] \to \mathbb{Z}$ be the inclusion. Let $i_{\mathcal{B}}[\sigma] : \mathcal{B} \to \int_{\widehat{\mathbb{Z}]\sigma]}} \mathcal{B}$ be the inclusion induced by $i_{\mathcal{B}}$. Define a functor of additive categories

$$\mathrm{ev}_{\mathcal{B}}[\sigma] : \int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B} \to \mathcal{B}$$

extending the identity on $\mathcal{B}$ by sending a morphism T_{σ^n} to 0 for $n > 0$. (Of course, $\sigma^0 = \mathrm{id}$ must go to the identity.) We obtain the following diagram of spectra:

$$\begin{tikzcd}
\boldsymbol{K}(\mathcal{B}) \arrow[r, "\boldsymbol{K}(i_{\mathcal{B}}[\sigma])"] \arrow[rr, bend right, "\mathrm{id}"'] & \boldsymbol{K}\big(\int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B}\big) \arrow[r, "\boldsymbol{K}(\mathrm{ev}_{\mathcal{B}}[\sigma])"] & \boldsymbol{K}(\mathcal{B})
\end{tikzcd}$$

Define $\boldsymbol{NK}(\mathcal{B}, \sigma)$ as the homotopy fiber of the map $\boldsymbol{K}(\mathrm{ev}_{\mathcal{B}}[\sigma]) : \boldsymbol{K}\big(\int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B}\big) \to \boldsymbol{K}(\mathcal{B})$. Let $\boldsymbol{b}_{\mathcal{B}}[\sigma]$ denote the composite

$$\boldsymbol{b}_{\mathcal{B}}[\sigma] : \boldsymbol{NK}(\mathcal{B}, \sigma) \to \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}[\sigma]}} \mathcal{B}\right) \to \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right)$$

of the canonical map with the inclusion. Let $\mathrm{gen}(\mathbb{Z})$ be the set of generators of the infinite cyclic group $\mathbb{Z}$. Put

$$\boldsymbol{NK}(\mathcal{B}) := \bigvee_{\sigma \in \mathrm{gen}(\mathbb{Z})} \boldsymbol{NK}(\mathcal{B}, \sigma)$$

and define

$$\boldsymbol{b}_{\mathcal{B}} := \bigvee_{\sigma \in \mathrm{gen}(\mathbb{Z})} \boldsymbol{b}_{\mathcal{B}}[\sigma] : \bigvee_{\sigma \in \mathrm{gen}(\mathbb{Z})} \boldsymbol{NK}(\mathcal{B}, \sigma) \to \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right).$$

The proof of the following result can be found in [Lück and Steimle 2016]. The case where the $\mathbb{Z}$-action on $\mathcal{B}$ is trivial and one considers only K-groups in dimensions $n \leq 1$ has already been treated by [Ranicki 1992, Chapters 10 and 11]. If R is a ring with an automorphism and one takes $\mathcal{B}$ to be the category R-FGF of finitely generated free R-modules with the induced $\mathbb{Z}$-action, Theorem 4.2 boils down for higher algebraic K-theory to the twisted Bass–Heller–Swan decomposition of [Grayson 1988, Theorems 2.1 and 2.3].

Theorem 4.2 (twisted Bass–Heller–Swan decomposition for additive categories). *The map of spectra*

$$\boldsymbol{a}_{\mathcal{B}} \vee \boldsymbol{b}_{\mathcal{B}} : T_{\boldsymbol{K}(\Phi)} \vee \boldsymbol{NK}(\mathcal{B}) \xrightarrow{\simeq} \boldsymbol{K}\left(\int_{\widehat{\mathbb{Z}}} \mathcal{B}\right)$$

is a weak equivalence of spectra.

5. Some additive categories associated to an additive G-category

Let G be a group. Let $\mathcal{A}$ be an additive G-category, i.e., an additive category with a right G-operation by isomorphisms of additive categories. We can consider $\mathcal{A}$ as a contravariant functor $\widehat{G} \to \mathsf{Add\text{-}Cat}$. Fix a homogeneous G-space G/H. Let $\mathrm{pr}_{G/H} : \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/G) = \widehat{G}$ be the projection induced by the canonical G-map $G/H \to G/G$. Then we obtain a covariant functor $\mathcal{G}^G(G/H) \to \mathsf{Add\text{-}Cat}$ by sending G/H to $\mathcal{A} \circ \mathrm{pr}_G$. Let $\int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H}$ be the additive category given by the homotopy colimit (defined in (3.1)) of this functor. A G-map $f : G/H \to G/K$ induces a functor $\mathcal{G}^G(f) : \mathcal{G}^G(G/H) \to \mathcal{G}^G(G/K)$ which is compatible with the projections to $\widehat{G}$. Hence it induces a functor of additive categories — see (3.6) —

$$\mathcal{G}^G(f)_* : \int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H} \to \int_{\mathcal{G}^G(G/K)} \mathcal{A} \circ \mathrm{pr}_{G/K} .$$

Thus we obtain a covariant functor

$$\mathsf{Or}(G) \to \mathsf{Add\text{-}Cat}, \quad G/H \mapsto \int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H} . \tag{5.1}$$

Remark 5.2. Applying Lemma 3.10(i) to the equivalence of categories $e(G/H) : \widehat{H} \to \mathcal{G}^G(G/H)$, we see that the functor (5.1), at G/H, takes the value $\int_{\widehat{H}} \mathcal{A}$, where $\mathcal{A}$ carries the restricted H-action. The more complicated description is however needed for the functoriality.

Notation 5.3. For the sake of brevity, we will just write $\mathcal{A}$ for any composite $\mathcal{A} \circ \mathrm{pr}_{G/H}$ if no confusion is possible. In this notation, (5.1) takes the form

$$G/H \mapsto \int_{\mathcal{G}^G(G/H)} \mathcal{A}.$$

Let $V \subseteq G$ be an infinite virtually cyclic subgroup of type I. In the sequel we abbreviate $K = K_V$ and $Q = Q_V$. Let $\mathrm{pr}_K : \mathcal{G}^G(G/V)_K \to \widehat{K}$ be the functor which sends a morphism $g : g_1 V \to g_2 V$ to $g_2^{-1} g g_1 \in K$.

Fixing a generator σ of the infinite cyclic group Q, the inclusions $\mathcal{G}^G(G/V)_K \subset \mathcal{G}^G(G/V)[\sigma] \subset \mathcal{G}^G(G/V)$ induce inclusions

$$\int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \subset \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \subset \int_{\mathcal{G}^G(G/V)} \mathcal{A}. \tag{5.4}$$

Actually, the category into the middle retracts onto the smaller one. To see this, define a retraction

$$\operatorname{ev}(G/V)[\sigma]_K : \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \to \int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \tag{5.5}$$

as follows: It is the identity on every copy of the additive category $\mathcal{A}$ inside the homotopy colimit. Let $T_g : (g_1 V, g^* A) \to (g_2 V, A)$ be a structural morphism in the homotopy colimit, where $g : g_1 V \to g_2 V$ in $\mathcal{G}^G(G/V)[\sigma]$ is a morphism in $\mathcal{G}^G(G/V)[\sigma]$ (that is, g is an element of G satisfying $g_2^{-1} g g_1 \in V[\sigma]$). If

$$g_2^{-1} g g_1 \in K \subset V[\sigma],$$

then g is by definition a morphism in $\mathcal{G}^G(G/V)_K \subset \mathcal{G}^G(G/V)[\sigma]$ and we may let

$$\operatorname{ev}(G/V)[\sigma]_K(T_g) = T_g.$$

Otherwise we send the morphism T_g to 0. This is well-defined, since for two elements $h_1, h_2 \in V[\sigma]$ we have $h_1 h_2 \in K$ if and only if both $h_1 \in K$ and $h_2 \in K$ hold.

Similarly the inclusion $\int_{\widehat{K}} \mathcal{A} \subset \int_{\widehat{V[\sigma]}} \mathcal{A}$ is split by a retraction

$$\operatorname{ev}_V[\sigma] : \int_{\widehat{V[\sigma]}} \mathcal{A} \to \int_{\widehat{K}} \mathcal{A}$$

defined as follows: On the copy of $\mathcal{A}$ inside $\int_{\widehat{V[\sigma]}} \mathcal{A}$, the functor is defined to be the identity. A structural morphism $T_g : g^* A \to A$ is sent to itself if $g \in K$, and to zero otherwise. One easily checks that the following diagram commutes (where the unlabelled arrows are inclusions) and has equivalences of additive categories as vertical maps:

$$\begin{tikzcd}
\int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \arrow[r, "\operatorname{ev}(G/V)[\sigma]"] & \int_{\mathcal{G}^G(G/V)_K} \mathcal{A} \\
\int_{\widehat{K}} \mathcal{A} \arrow[r] \arrow[u, "(e(G/V)_K)_*", "\wr"'] \arrow[rr, bend right, "\mathrm{id}"'] & \int_{\widehat{V[\sigma]}} \mathcal{A} \arrow[r, "\operatorname{ev}_V[\sigma]"] \arrow[u, "e(G/V)[\sigma]_*", "\wr"'] & \int_{\widehat{K}} \mathcal{A} \arrow[u, "(e(G/V)_K)_*", "\wr"']
\end{tikzcd} \tag{5.6}$$

We obtain from (2.1) and Lemma 3.10(i) the following commutative diagram of additive categories with equivalences of additive categories as vertical maps:

$$\begin{tikzcd}
\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \arrow[r] & \int_{\mathcal{G}^G(G/V)} \mathcal{A} \\
\int_{\widehat{V[\sigma]}} \mathcal{A} \arrow[r] \arrow[u, "e(G/V)[\sigma]_*", "\wr"'] & \int_{\widehat{V}} \mathcal{A} \arrow[u, "\wr", "e(G/V)_*"']
\end{tikzcd} \tag{5.7}$$

(where again the unlabelled arrows are the inclusions).

Now we abbreviate $\mathcal{B} = \int_{\widehat{K}} \mathcal{A}$. Next we define a right Q-action on $\mathcal{B}$ which will depend on a choice of an element $\overline{\sigma} \in V$ such that $p_V : V \to Q$ sends $\overline{\sigma}$ to σ. Such an element induces a section of the projection $G \to Q$ by which any action of G induces an action of Q. In short, the action of Q on $\mathcal{B}$ is given by the action of G onto $\mathcal{A} \subset \mathcal{B}$, and by the conjugation action of Q on the indexing category $\widehat{K}$. In more detail, the action of $\sigma \in Q$ is specified by the automorphism

$$\Phi : \int_{\widehat{K}} \mathcal{A} \to \int_{\widehat{K}} \mathcal{A}$$

defined as follows: A morphism $\varphi : A \to B$ in $\mathcal{A}$ is sent to $\overline{\sigma}^*\varphi : \overline{\sigma}^*A \to \overline{\sigma}^*B$, and a structural morphism $T_g : g^*A \to A$ is sent to the morphism

$$T_{\overline{\sigma}^{-1}g\overline{\sigma}} : \overline{\sigma}^*g^*A = (\overline{\sigma}^{-1}g\overline{\sigma})^*\overline{\sigma}^*A \to \overline{\sigma}^*A.$$

With this notation we obtain an additive functor

$$\Psi : \int_{\widehat{Q}} \mathcal{B} \to \int_{\widehat{V}} \mathcal{A}$$

defined to extend the inclusion $\mathcal{B} = \int_{\widehat{K}} \mathcal{A} \to \int_{\widehat{V}} \mathcal{A}$ and such that a structural morphism $T_\sigma : \Phi(A) \to A$ is sent to $T_{\overline{\sigma}} : \Phi(A) = \overline{\sigma}^*A \to A$.

In more detail, a morphism in $\int_{\widehat{Q}} \mathcal{B}$ can be uniquely written as a finite sum

$$\sum_{n\in\mathbb{Z}} T_{\overline{\sigma}^n} \circ \left(\sum_{k\in K} T_k \circ \phi_{k,n} \right) = \sum_{n,k} T_{\overline{\sigma}^n\cdot k} \circ \phi_{k,n}.$$

Since any element in V is uniquely a product $\overline{\sigma}^n \cdot k$ with $k \in K$, the functor Ψ is fully faithful. As it is the identity on objects, Ψ is an isomorphism of categories. It also restricts to an isomorphism of categories

$$\Psi[\sigma] : \int_{\widehat{Q[\sigma]}} \mathcal{B} \to \int_{\widehat{V[\sigma]}} \mathcal{A}.$$

Define a functor

$$\mathrm{ev}_{\mathcal{B}}[\sigma] : \int_{\widehat{Q[\sigma]}} \mathcal{B} \to \mathcal{B}$$

as follows. It is the identity functor on $\mathcal{B}$, and a nonidentity structural morphism $T_q : q^*B \to B$ is sent to 0. One easily checks using (5.6) and (5.7) that the following diagram of additive categories commutes (with unlabelled arrows given by inclusions) and that all vertical arrows are equivalences of additive categories:

$$
\begin{tikzcd}
\int_{\mathcal{G}^G(G/V)_K} \mathcal{A} & \int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A} \arrow[l, "\mathrm{ev}(G/V)[\sigma]_K"'] \arrow[r] & \int_{\mathcal{G}^G(G/V)} \mathcal{A} \\
\int_{\widehat{K}} \mathcal{A} \arrow[u, "(e(G/V)_K)_*", "\wr"'] & \int_{\widehat{V[\sigma]}} \mathcal{A} \arrow[l, "\mathrm{ev}_V[\sigma]"'] \arrow[r] \arrow[u, "e(G/V)[\sigma]_*", "\wr"'] & \int_{\widehat{V}} \mathcal{A} \arrow[u, "e(G/V)_*", "\wr"'] \\
\mathcal{B} \arrow[u, "\mathrm{id}", "\cong"'] & \int_{\widehat{Q[\sigma]}} \mathcal{B} \arrow[l, "\mathrm{ev}_{\mathcal{B}}[\sigma]"'] \arrow[r] \arrow[u, "\Psi[\sigma]", "\cong"'] & \int_{\widehat{Q}} \mathcal{B} \arrow[u, "\Psi", "\cong"']
\end{tikzcd}
\tag{5.8}
$$

Recall from Section 2 that $q_V : G/K \to G/V$ is the projection and that R_σ is the automorphism of $\int_{\mathcal{G}^G(G/K)} \mathcal{A}$ induced by right multiplication with $\overline{\sigma}$.

We have the following (not necessarily commutative) diagram of additive categories, all of whose vertical arrows are equivalences of additive categories, and the unlabelled arrows are the inclusions:

$$
\begin{tikzcd}
\int_{\mathcal{G}^G(G/K)} \mathcal{A} \arrow[rrrr, "R_\sigma"] \arrow[drr, "\mathcal{G}^G(\mathrm{pr}_V)_*"'] & & & & \int_{\mathcal{G}^G(G/K)} \mathcal{A} \arrow[dll, "\mathcal{G}^G(\mathrm{pr}_V)_*"] \\
& & \int_{\mathcal{G}^G(G/V)} \mathcal{A} & & \\
\int_{\widehat{K}} \mathcal{A} \arrow[uu, "(e(G/V)_K)_*", "\wr"'] \arrow[rrrr, "\Phi"] \arrow[drr] & & & & \int_{\widehat{K}} \mathcal{A} \arrow[uu, "(e(G/V)_K)_*"', "\wr"] \arrow[dll] \\
& & \int_{\widehat{V}} \mathcal{A} \arrow[uu, "e(G/V)_*", "\wr"'] & & \\
\mathcal{B} \arrow[uu, "\mathrm{id}", "\cong"'] \arrow[rrrr, "\Phi"] \arrow[drr, "i_{\mathcal{B}}"'] & & & & \mathcal{B} \arrow[uu, "\mathrm{id}"', "\cong"] \arrow[dll, "i_{\mathcal{B}}"] \\
& & \int_{\widehat{Q}} \mathcal{B} \arrow[uu, "\Psi", "\wr"'] & &
\end{tikzcd}
\tag{5.9}
$$

The lowest triangle commutes up to a preferred natural isomorphism $T : i_{\mathcal{B}} \circ \Phi \xrightarrow{\cong} i_{\mathcal{B}}$, which is part of the structural data of the homotopy colimit. We equip the middle triangle with the natural isomorphism $\Psi \circ T$. Explicitly it is just given by the structural morphisms $T_{\overline{\sigma}} : \overline{\sigma}^* A \to A$.

The three squares ranging from the middle to the lower level commute and the two natural equivalences above are compatible with these squares. The top triangle commutes. The back upper square commutes up to a preferred natural isomorphism

$S : (e(G/V)_K)_* \circ \Phi \xrightarrow{\cong} R_\sigma \circ (e(G/V)_K)_*$. It assigns to an object $A \in \mathcal{A}$, which is the same as an object in $\int_{\widehat{K}} \mathcal{A}$, the structural isomorphism

$$S(A) := T_{\overline{\sigma}} : (eK, \overline{\sigma}^* A) \to (\overline{\sigma} K, A).$$

The other two squares joining the upper to the middle level commute. From the explicit description of the natural isomorphisms it becomes apparent that the preferred natural isomorphism for the middle triangle defined above and the preferred natural isomorphism for the upper back square are compatible, in the sense that $e(G/V)[\sigma]_* \circ \Psi \circ T = \mathcal{G}^G(\mathrm{pr}_V)_* \circ S$.

6. Some K-theory-spectra over the orbit category

In this section we introduce various K-theory spectra. For a detailed introduction to spaces, spectra and modules over a category and some constructions of K-theory spectra, we refer to [Davis and Lück 1998].

Given an additive G-category $\mathcal{A}$, we obtain a covariant $\mathsf{Or}(G)$-spectrum

$$\boldsymbol{K}^G_{\mathcal{A}} : \mathsf{Or}(G) \to \mathsf{Spectra}, \quad G/H \mapsto \boldsymbol{K}\Big(\int_{\mathcal{G}^G(G/H)} \mathcal{A} \circ \mathrm{pr}_{G/H}\Big), \tag{6.1}$$

by the composite of the two functors (4.1) and (5.1). It is naturally equivalent to the covariant $\mathsf{Or}(G)$-spectrum, which is written in the same way and constructed in [Bartels and Reich 2007, Definition 3.1].

We again adopt Notation 5.3, abbreviating an expression such as $\mathcal{A} \circ \mathrm{pr}_{G/H}$ just by $\mathcal{A}$. Given a virtually cyclic subgroup $V \subseteq G$, we obtain the following map of spectra induced by the functors $j(G/V)[\sigma]_*$ of (5.4) and $\mathrm{ev}(G/V)[\sigma]$ of (5.5):

$$\boldsymbol{K}\Big(\int_{\mathcal{G}^G(G/V)_K} \mathcal{A}\Big) \xleftarrow{\boldsymbol{K}(\mathrm{ev}(G/V)[\sigma])} \boldsymbol{K}\Big(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\Big) \xrightarrow{\boldsymbol{K}(j(G/V)[\sigma]_*)} \boldsymbol{K}\Big(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\Big).$$

Notation 6.2. Let $\boldsymbol{NK}(G/V; \mathcal{A}, \sigma)$ be the spectrum given by the homotopy fiber of $\boldsymbol{K}\big(\mathrm{ev}(G/V)[\sigma]_*\big) : \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\big) \to \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K} \mathcal{A}\big)$.

Let $\boldsymbol{l} : \boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \to \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\big)$ be the canonical map of spectra. Define the map of spectra

$$\boldsymbol{j}(G/V; \mathcal{A}, \sigma) : \boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \to \boldsymbol{K}\Big(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\Big)$$

to be the composite $\boldsymbol{K}(j(G/V)[\sigma]_*) \circ \boldsymbol{l}$.

Consider a G-map $f : G/V \to G/W$, where V and W are virtually cyclic groups of type I. It induces a functor $\mathcal{G}^G(f) : \mathcal{G}^G(G/V) \to \mathcal{G}^G(G/W)$.

It also induces a bijection

$$\mathrm{gen}(f) : \mathrm{gen}(Q_V) \to \mathrm{gen}(Q_W) \tag{6.3}$$

as follows. Fix an element $g \in G$ such that $f(eV) = gW$. Then $g^{-1}Vg \subseteq W$. The injective group homomorphism $c(g) : V \to W,\ v \mapsto g^{-1}vg$, induces an injective group homomorphism $Q_{c(g)} : Q_V \to Q_W$ by Lemma 1.1(vi). For $\sigma \in \operatorname{gen}(Q_V)$ let $\operatorname{gen}(f)(\sigma) \in \operatorname{gen}(Q_W)$ be uniquely determined by the property that $Q_{c(g)}(\sigma) = \operatorname{gen}(f)(\sigma)^n$ for some $n \geq 1$. One easily checks that this is independent of the choice of $g \in G$ with $f(eV) = gW$ since, for $w \in W$, the conjugation homomorphism $c(w) : W \to W$ induces the identity on Q_W. Using Lemma 1.1(vi) it follows that $\mathcal{G}^G(f) : \mathcal{G}^G(G/V) \to \mathcal{G}^G(G/W)$ induces functors

$$\begin{aligned} \mathcal{G}^G(f)[\sigma] : \mathcal{G}^G(G/V)[\sigma] &\to \mathcal{G}^G(G/W)[\operatorname{gen}(f)(\sigma)], \\ \mathcal{G}^G(f)_K : \mathcal{G}^G(G/V)_K &\to \mathcal{G}^G(G/W)_K. \end{aligned}$$

Hence we obtain a commutative diagram of maps of spectra

$$\begin{tikzcd}
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}((\mathcal{G}^G(f)_K)_*)"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)_K} \mathcal{A}\big) \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}(\mathcal{G}^G(f)[\sigma]_*)"] \arrow[u, "\boldsymbol{K}(\operatorname{ev}(G/V)[\sigma])"] \arrow[d, "\boldsymbol{K}(j(G/V)[\sigma]_*)"'] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)[\operatorname{gen}(f)(\sigma)]} \mathcal{A}\big) \arrow[u, "\boldsymbol{K}(\operatorname{ev}(G/W)[\operatorname{gen}(f)(\sigma)])"'] \arrow[d, "\boldsymbol{K}(j(G/W)[\operatorname{gen}(f)(\sigma)]_*)"] \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}(\mathcal{G}^G(f)_*)"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)} \mathcal{A}\big)
\end{tikzcd}$$

Thus we obtain a map of spectra

$$\boldsymbol{NK}(f; \mathcal{A}, \sigma) : \boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \to \boldsymbol{NK}(G/W; \mathcal{A}, \operatorname{gen}(f)(\sigma))$$

such that the following diagram of spectra commutes:

$$\begin{tikzcd}
\boldsymbol{NK}(G/V; \mathcal{A}, \sigma) \arrow[r, "\boldsymbol{NK}(f;\mathcal{A},\sigma)"] \arrow[d, "\boldsymbol{j}(G/V;\mathcal{A},\sigma)"'] & \boldsymbol{NK}(G/W; \mathcal{A}, \operatorname{gen}(f)(\sigma)) \arrow[d, "\boldsymbol{j}(G/W;\mathcal{A},\operatorname{gen}(f)(\sigma))"] \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)} \mathcal{A}\big) \arrow[r, "\boldsymbol{K}(\mathcal{G}^G(f)_*)"'] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/W)} \mathcal{A}\big)
\end{tikzcd}$$

Let $\mathcal{VC}_I$ be the family of subgroups of G which consists of all finite groups and all virtually cyclic subgroups of type I. Let $\mathsf{Or}_{\mathcal{VC}_I}(G)$ be the full subcategory of the orbit category $\mathsf{Or}(G)$ consisting of objects G/V for which V belongs to $\mathcal{VC}_I$. Define a functor

$$\boldsymbol{NK}^G_{\mathcal{A}} : \mathsf{Or}_{\mathcal{VC}_I}(G) \to \mathsf{Spectra}$$

as follows: It sends G/H for a finite subgroup H to the trivial spectrum and G/V for a virtually cyclic subgroup V of type I to $\bigvee_{\sigma \in \operatorname{gen}(Q_V)} \boldsymbol{NK}(G/V; \mathcal{A}, \sigma)$. Consider a map $f : G/V \to G/W$. If V or W is finite, it is sent to the trivial map.

Suppose that both V and W are infinite virtually cyclic subgroups of type I. Then it is sent to the wedge of the two maps

$$\boldsymbol{NK}(f;\mathcal{A},\sigma_1):\boldsymbol{NK}(G/V;\mathcal{A},\sigma_1)\to\boldsymbol{NK}(G/W;\mathcal{A},\mathrm{gen}(f)(\sigma_1)),$$
$$\boldsymbol{NK}(f;\mathcal{A},\sigma_2):\boldsymbol{NK}(G/V;\mathcal{A},\sigma_2)\to\boldsymbol{NK}(G/W;\mathcal{A},\mathrm{gen}(f)(\sigma_2)),$$

for $\mathrm{gen}(Q_V)=\{\sigma_1,\sigma_2\}$.

The restriction of the covariant $\mathsf{Or}(G)$-spectrum $\boldsymbol{K}^G_{\mathcal{A}}:\mathsf{Or}(G)\to\mathsf{Spectra}$ to $\mathsf{Or}_{\mathcal{VC}_I}(G)$ will be denoted by the same symbol

$$\boldsymbol{K}^G_{\mathcal{A}}:\mathsf{Or}_{\mathcal{VC}_I}(G)\to\mathsf{Spectra}.$$

The wedge of the maps $\boldsymbol{j}(G/V;\mathcal{A},\sigma_1)$ and $\boldsymbol{j}(G/V;\mathcal{A},\sigma_2)$ for V a virtually cyclic subgroup of G of type I yields a map of spectra $\boldsymbol{NK}^G_{\mathcal{A}}(G/V)\to\boldsymbol{K}^G_{\mathcal{A}}(G/V)$. Thus we obtain a transformation of functors from $\mathsf{Or}_{\mathcal{VC}_I}(G)$ to $\mathsf{Spectra}$,

$$\boldsymbol{b}^G_{\mathcal{A}}:\boldsymbol{NK}^G_{\mathcal{A}}\to\boldsymbol{K}^G_{\mathcal{A}}. \tag{6.4}$$

7. Splitting the relative assembly map and identifying the relative term

Let X be a G-space. It defines a contravariant $\mathsf{Or}(G)$-space $O^G(X)$, i.e., a contravariant functor from $\mathsf{Or}(G)$ to the category of spaces, by sending G/H to the H-fixed point set $\mathrm{map}_G(G/H,X)=X^H$. Let $O^G(X)_+$ be the pointed $\mathsf{Or}(G)$-space, where $O^G(X)_+(G/H)$ is obtained from $O^G(X)(G/H)$ by adding an extra base point. If $f:X\to Y$ is a G-map, we obtain a natural transformation $O^G(f)_+:O^G(X)_+\to O^G(Y)_+$.

Let $\boldsymbol{E}$ be a covariant $\mathsf{Or}(G)$-spectrum, i.e., a covariant functor from $\mathsf{Or}(G)$ to the category of spectra. Fix a G-space Z. Define the covariant $\mathsf{Or}(G)$-spectrum

$$\boldsymbol{E}_Z:\mathsf{Or}(G)\to\mathsf{Spectra}$$

as follows. It sends an object G/H to the spectrum $O^G(G/H\times Z)_+\wedge_{\mathsf{Or}(G)}\boldsymbol{E}$, where $\wedge_{\mathsf{Or}(G)}$ is the wedge product of a pointed space and a spectrum over a category (see [Davis and Lück 1998, Section 1], where $\wedge_{\mathsf{Or}(G)}$ is denoted by $\otimes_{\mathsf{Or}(G)}$). The obvious identification of $O^G(G/H)_+(?)\wedge_{\mathsf{Or}(G)}\boldsymbol{E}(?)$ with $\boldsymbol{E}(G/H)$ and the projection $G/H\times Z\to G/H$ yields a natural transformation of covariant functors $\mathsf{Or}(G)\to\mathsf{Spectra}$,

$$\boldsymbol{a}:\boldsymbol{E}_Z\to\boldsymbol{E}. \tag{7.1}$$

Lemma 7.2. *Given a G-space X, there exists an isomorphism of spectra*

$$\boldsymbol{u}^G(X):O^G(X\times Z)_+\wedge_{\mathsf{Or}(G)}\boldsymbol{E}\xrightarrow{\cong}O^G(X)_+\wedge_{\mathsf{Or}(G)}\boldsymbol{E}_Z,$$

which is natural in X and Z.

Proof. The smash product $\wedge_{\mathsf{Or}(G)}$ is associative, i.e., there is a natural isomorphism of spectra

$$\begin{aligned}\big(O^G(X)_+(?_1)\wedge_{\mathsf{Or}(G)} O^G(?_2\times Z)_+(?_1)\big)\wedge_{\mathsf{Or}(G)} \boldsymbol{E}(?_2)\\ \xrightarrow{\cong} O^G(X)_+(?_1)\wedge_{\mathsf{Or}(G)}\big(O^G(?_2\times Z)_+(?_1)\wedge_{\mathsf{Or}(G)} \boldsymbol{E}(?_2)\big).\end{aligned}$$

There is a natural isomorphism of covariant $\mathsf{Or}(G)$-spaces

$$O^G(X\times Z)_+ \xrightarrow{\cong} O^G(X)_+(?)\wedge_{\mathsf{Or}(G)} O^G(?\times Z)_+$$

which, evaluated at G/H, sends $\alpha: G/H\to X\times Z$ to $(\mathrm{pr}_1\circ\alpha)\wedge(\mathrm{id}_{G/H}\times(\mathrm{pr}_2\circ\alpha))$ if pr_i is the projection onto the i-th factor of $X\times Z$. The inverse evaluated at G/H sends $(\beta_1: G/K\to X)\wedge(\beta_2: G/H\to G/K\times Z)$ to $(\beta_1\times\mathrm{id}_Z)\circ\beta_2$. The composite of these two isomorphisms yield the desired isomorphism $\boldsymbol{u}^G(X)$. □

If $\mathcal{F}$ is a family of subgroups of the group G, e.g., $\mathcal{VC}_I$ or the family $\mathcal{F}\mathrm{in}$ of finite subgroups, then we denote by $E_{\mathcal{F}}(G)$ the classifying space of $\mathcal{F}$. (For a survey on these spaces we refer for instance to [Lück 2005].) Let $\underline{E}G$ denote the classifying space for proper G-actions, or in other words, a model for $E_{\mathcal{F}\mathrm{in}}(G)$. If we restrict a covariant $\mathsf{Or}(G)$ spectrum $\boldsymbol{E}$ to $\mathsf{Or}_{\mathcal{VC}_I}(G)$, we will denote it by the same symbol $\boldsymbol{E}$ and analogously for $O^G(X)$.

Lemma 7.3. *Let $\mathcal{F}$ be a family of subgroups. Let X be a G-CW-complex whose isotropy groups belong to $\mathcal{F}$. Let $\mathbf{E}$ be a covariant $\mathsf{Or}(G)$-spectrum. Then there is a natural homeomorphism of spectra*

$$O^G(X)_+\wedge_{\mathsf{Or}_{\mathcal{F}}(G)}\boldsymbol{E}\xrightarrow{\cong} O^G(X)_+\wedge_{\mathsf{Or}(G)}\boldsymbol{E}.$$

Proof. Let $I:\mathsf{Or}_{\mathcal{F}}(G)\to\mathsf{Or}(G)$ be the inclusion. The claim follows from the adjunction of the induction I_* and restriction I^* — see [Davis and Lück 1998, Lemma 1.9] — and the fact that for the $\mathsf{Or}(G)$-space $O^G(X)$ the canonical map $I_*I^*O^G(X)\to O^G(X)$ is a homeomorphism of $\mathsf{Or}(G)$-spaces. □

In the sequel we will abbreviate $\boldsymbol{E}_{\underline{E}G}$ by $\underline{\boldsymbol{E}}$.

Lemma 7.4. *Let $\boldsymbol{E}$ be a covariant $\mathsf{Or}(G)$-spectrum. Let $f:\underline{E}G\to E_{\mathcal{VC}_I}(G)$ be a G-map.* (*It is unique up to G-homotopy.*) *Then there is an up-to-homotopy commutative diagram of spectra whose upper horizontal map is a weak equivalence*

$$\begin{tikzcd}[column sep=tiny] O^G(E_{\mathcal{VC}_I}(G))\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\underline{\boldsymbol{E}} \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{id}\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\boldsymbol{a}"'] & & O^G(\underline{E}G)\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\boldsymbol{E} \arrow[dl, "O^G(f)\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\mathrm{id}"] \\ & O^G(E_{\mathcal{VC}_I}(G))\wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)}\boldsymbol{E} & \end{tikzcd}$$

Proof. From Lemma 7.2 we obtain a commutative diagram with an isomorphism as horizontal map

$$\begin{tikzcd} O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \underline{\boldsymbol{E}} \arrow[rr, "\cong"] \arrow[dr, "\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{a}"'] & & O^G(E_{\mathcal{VC}_I}(G) \times \underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} \arrow[dl, "O^G(\mathrm{pr}_1) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \mathrm{id}"] \\ & O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} & \end{tikzcd}$$

where $\mathrm{pr}_1 : E_{\mathcal{VC}_I}(G) \times \underline{E}G \to E_{\mathcal{VC}_I}(G)$ is the obvious projection. The projection $\mathrm{pr}_2 : E_{\mathcal{VC}_I}(G) \times \underline{E}G \to \underline{E}G$ is a G-homotopy equivalence and its composite with $f : \underline{E}G \to E_{\mathcal{VC}_I}(G)$ is G-homotopic to pr_1. Hence the following diagram of spectra commutes up to G-homotopy and has a weak equivalence as horizontal map:

$$\begin{tikzcd} O^G(E_{\mathcal{VC}_I}(G) \times \underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} \arrow[rr, "\simeq", "O^G(\mathrm{pr}_2) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \mathrm{id}"'] \arrow[dr, "O^G(\mathrm{pr}_1) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \mathrm{id}"'] & & O^G(\underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} \arrow[dl, "O^G(f) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \mathrm{id}"] \\ & O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{E} & \end{tikzcd}$$

Putting these two diagrams together finishes the proof of Lemma 7.4 □

If $\boldsymbol{E}$ is the functor $\boldsymbol{K}^G_{\mathcal{A}}$ defined in (6.1) and $Z = \underline{E}G$, we will write $\underline{\boldsymbol{K}}^G_{\mathcal{A}}$ for $\underline{\boldsymbol{E}} = \boldsymbol{E}_{\underline{E}G}$.

Lemma 7.5. *Let H be a finite group or an infinite virtually cyclic group of type I. Then the map of spectra* (see (6.4) and (7.1))

$$\boldsymbol{a}(G/H) \vee \boldsymbol{b}(G/H) : \underline{\boldsymbol{K}}^G_{\mathcal{A}}(G/H) \vee \boldsymbol{NK}^G_{\mathcal{A}}(G/H) \to \boldsymbol{K}^G_{\mathcal{A}}(G/H)$$

is a weak equivalence.

Proof. Given an infinite cyclic subgroup $V \subseteq G$ of type I, we next construct an up-to-homotopy commutative diagram (on the next page) of spectra whose vertical arrows are all weak homotopy equivalences for $K = K_V$ and $Q = Q_V$. Let $i_V : V \to G$ be the inclusion and $p_V : V \to Q_V := V/K_V$ be the projection.

We first explain the vertical arrows, starting at the top. The first one is the identity by definition. The second one comes from the G-homeomorphism $G/V \times \underline{E}G \xrightarrow{\cong} (i_V)_*(i_V)^*\underline{E}G = G \times_V \underline{E}G$ sending (gV, x) to $(g, g^{-1}x)$. The third one comes from the adjunction of the induction $(i_V)_*$ and restriction i_V^*; see [Davis and Lück 1998, Lemma 1.9]. The fourth one comes from the fact that $p_V^* EQ$ and $i_V^* \underline{E}G$ are both models for $\underline{E}V$ and hence are V-homotopy equivalent. The fifth one comes from the adjunction of the restriction p_V^* with the coinduction $(p_V)_!$; see [Davis and Lück 1998, Lemma 1.9]. The sixth one comes from the fact that EQ is a free Q-CW-complex and Lemma 7.3 applied to the family consisting of one subgroup, namely the trivial subgroup. The seventh one comes from the identification

$(p_V)_!(i_V)^*\boldsymbol{K}_{\mathcal{A}}^G(Q_V/1) = (i_V)^*\boldsymbol{K}_{\mathcal{A}}^G(V/K) = \boldsymbol{K}_{\mathcal{A}}^G(G/K)$. The last one comes from the obvious homeomorphism if we use for EQ_V the standard model with $\mathbb{R}$ as the underlying $Q_V{=}\mathbb{Z}$-space. The arrow $\boldsymbol{a}'(G/V)$ is induced by the upper triangle in (5.9), which commutes (strictly). One easily checks that the diagram commutes:

$$
\begin{tikzcd}
\underline{\boldsymbol{K}}_{\mathcal{A}}^G(G/V) \arrow[d, "(1)"', "\mathrm{id}"] \arrow[ddddr, bend left, "\boldsymbol{a}(G/V)"] & \\
O^G(G/V \times \underline{E}G)_+ \wedge_{\mathsf{Or}(G)} \boldsymbol{K}_{\mathcal{A}}^G \arrow[d, "(2)"', "\cong"] & \\
O^G((i_V)_*(i_V)^*\underline{E}G)_+ \wedge_{\mathsf{Or}(G)} \boldsymbol{K}_{\mathcal{A}}^G \arrow[d, "(3)"', "\cong"] & \\
O^V((i_V)^*\underline{E}G)_+ \wedge_{\mathsf{Or}(V)} (i_V)^*\boldsymbol{K}_{\mathcal{A}}^G \arrow[d, "(4)"', "\simeq"] & \\
O^V((p_V)^*EQ_V)_+ \wedge_{\mathsf{Or}(V)} (i_V)^*\boldsymbol{K}_{\mathcal{A}}^G \arrow[d, "(5)"', "\cong"] & \boldsymbol{K}_{\mathcal{A}}^G(G/V) \\
O^{Q_V}(EQ_V)_+ \wedge_{\mathsf{Or}(Q_V)} (p_V)_!(i_V)^*\boldsymbol{K}_{\mathcal{A}}^G \arrow[d, "(6)"', "\cong"] & \\
(EQ_V)_+ \wedge_{Q_V} (p_V)_!(i_V)^*\boldsymbol{K}_{\mathcal{A}}^G(Q_V/1) \arrow[d, "(7)"', "\cong"] & \\
(EQ_V)_+ \wedge_{Q_V} \boldsymbol{K}_{\mathcal{A}}^G(G/K) \arrow[d, "(8)"', "\cong"] & \\
T_{\boldsymbol{K}(R_\sigma)\colon \boldsymbol{K}_{\mathcal{A}}^G(G/K) \to \boldsymbol{K}_{\mathcal{A}}^G(G/K)} \arrow[uuuur, bend right, "\boldsymbol{a}'(G/V)"'] &
\end{tikzcd}
$$

Here is a short explanation of the diagram above. The map $\boldsymbol{a}(G/V)$ is basically given by the projection $G/V \times \underline{\mathrm{E}}G \to G/V$. Following the equivalences (1) through (5), this corresponds to projecting EQ_V to a point. On the domain of the equivalence (8), this corresponds to projecting EQ_V to a point and taking the inclusion-induced map $\boldsymbol{K}_{\mathcal{A}}^G(G/K) \to \boldsymbol{K}_{\mathcal{A}}^G(G/V)$ on the other factor. But this is precisely the definition of the map $\boldsymbol{a}'(G/V)$.

From the diagram (5.9) (including the preferred equivalences and the fact that a natural isomorphism of functors induces a preferred homotopy after applying the

K-theory spectrum) we obtain the following diagram of spectra, which commutes up to homotopy and has weak homotopy equivalences as vertical arrows:

$$\begin{tikzcd}
T_{\boldsymbol{K}(R_\sigma):\boldsymbol{K}_{\mathcal{A}}^G(G/K)\to \boldsymbol{K}_{\mathcal{A}}^G(G/K)} \arrow[r, "\boldsymbol{a}'(G/V)"] & \boldsymbol{K}_{\mathcal{A}}^G(G/V) \\
T_{\boldsymbol{K}(\Phi):\boldsymbol{K}\left(\int_{\widehat{K}}\mathcal{A}\right)\to \boldsymbol{K}\left(\int_{\widehat{K}}\mathcal{A}\right)} \arrow[u, "\simeq"] \arrow[r, "\boldsymbol{a}''(G/V)"] & \boldsymbol{K}\left(\int_{\widehat{V}}\mathcal{A}\right) \arrow[u, "\simeq"] \\
T_{\boldsymbol{K}(\phi):\boldsymbol{K}(\mathcal{B})\to \boldsymbol{K}(\mathcal{B})} \arrow[u, "\mathrm{id}"] \arrow[r, "\boldsymbol{a}_{\mathcal{B}}"] & \boldsymbol{K}\left(\int_{\widehat{Q}}\mathcal{B}\right) \arrow[u, "\cong"]
\end{tikzcd}$$

We obtain from the diagram (5.8) the following commutative diagram of spectra with weak homotopy equivalences as vertical arrows:

$$\begin{tikzcd}
\boldsymbol{NK}_{\mathcal{A}}^G(G/V) \arrow[r, "\boldsymbol{b}(G/V)"] & \boldsymbol{K}_{\mathcal{A}}^G(G/V) \\
\boldsymbol{NK}(\mathcal{B}) \arrow[u, "\simeq"] \arrow[r, "\boldsymbol{b}_{\mathcal{B}}"] & \boldsymbol{K}\left(\int_{\widehat{Q_V}}\mathcal{B}\right) \arrow[u, "\simeq"]
\end{tikzcd}$$

We conclude from the three diagrams of spectra above that

$$\boldsymbol{a}(G/V)\vee \boldsymbol{b}(G/V) : \underline{\boldsymbol{K}}_{\mathcal{A}}^G(G/V)\vee \boldsymbol{NK}_{\mathcal{A}}^G(G/V)\to \boldsymbol{K}_{\mathcal{A}}^G(G/V)$$

is a weak homotopy of spectra if and only if

$$\boldsymbol{a}_{\mathcal{B}}\vee \boldsymbol{b}_{\mathcal{B}} : T_{\boldsymbol{K}(\phi):\boldsymbol{K}(\mathcal{B})\to \boldsymbol{K}(\mathcal{B})}\vee \boldsymbol{NK}(\mathcal{B})\to \boldsymbol{K}\left(\int_{\widehat{Q_V}}\mathcal{B}\right)$$

is a weak homotopy equivalence. Since this is just the assertion of Theorem 4.2, the claim of Lemma 7.5 follows in the case where H is an infinite virtually cyclic group of type I.

It remains to consider the case where H is finite. Then $\boldsymbol{NK}_{\mathcal{A}}^G(G/V)$ is, by definition, the trivial spectrum. Hence it remains to show for a finite subgroup H of G that $\boldsymbol{a}(G/H) : \underline{\boldsymbol{K}}_{\mathcal{A}}^G(G/H)\to \boldsymbol{K}_{\mathcal{A}}^G(G/H)$ is a weak homotopy equivalence. This follows from the fact that the projection $G/H\times \underline{E}G\to G/H$ is a G-homotopy equivalence for finite H. □

Recall that any covariant $\mathsf{Or}(G)$-spectrum $\boldsymbol{E}$ determines a G-homology theory $H_*^G(-;\boldsymbol{E})$ satisfying $H_n^G(G/H;\boldsymbol{E})=\pi_n(\boldsymbol{E}(G/H))$, namely (see [Davis and Lück 1998]) put

$$H_*^G(X;\boldsymbol{E}) := \pi_*(O^G(X)\wedge_{\mathsf{Or}(G)}\boldsymbol{E}). \tag{7.6}$$

In the sequel we often follow the convention in the literature to abbreviate $\underline{\underline{E}}G := E_{\mathcal{VC}}(G)$ for the family $\mathcal{VC}$ of virtually cyclic subgroups. Recall that for two families of subgroups $\mathcal{F}_1$ and $\mathcal{F}_2$ with $\mathcal{F}_1 \subseteq \mathcal{F}_2$ there is, up to G-homotopy, one G-map $f : E_{\mathcal{F}_1}(G) \to E_{\mathcal{F}_2}(G)$. We will define $H_n(E_{\mathcal{F}_1}(G) \to E_{\mathcal{F}_2}(G); \boldsymbol{K}_R^G) := H_n(\mathrm{cyl}(f), E_{\mathcal{F}_1}(G); \boldsymbol{K}_R^G)$, where $(\mathrm{cyl}(f), E_{\mathcal{F}_1}(G))$ is the G-pair coming from the mapping cylinder of f.

Notice that $\boldsymbol{NK}_{\mathcal{A}}^G$ is defined only over $\mathrm{Or}_{VCyc_I}(G)$. It can be extended to a spectrum over $\mathrm{Or}(G)$ by applying the coinduction functor — see [Davis and Lück 1998, Definition 1.8] — associated to the inclusion $\mathrm{Or}_{\mathcal{VC}_I}(G) \to \mathrm{Or}(G)$, so that the G-homology theory $H_n^G(-; \boldsymbol{NK}_{\mathcal{A}}^G)$ makes sense for all pairs of G-CW-complexes (X, A). Moreover, $H_n^G(X; \boldsymbol{NK}_{\mathcal{A}}^G)$ can be identified with $\pi_n(O^G(X) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G)$ for all G-CW-complexes X.

The remainder of this section is devoted to the proof of Theorem 0.1. Its proof will need the following result, taken from [Davis et al. 2011, Remark 1.6]:

Theorem 7.7 (passage from $\mathcal{VC}_I$ to $\mathcal{VC}$ in K-theory). *The relative assembly map*

$$H_n^G(E_{\mathcal{VC}_I}(G); \boldsymbol{K}_{\mathcal{A}}^G) \xrightarrow{\cong} H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}^G)$$

is bijective for all $n \in \mathbb{Z}$.

Hence, in the proof of Theorem 0.1 we only have to deal with the passage from $\mathcal{F}\mathrm{in}$ to $\mathcal{VC}_I$.

Proof of Theorem 0.1. From Lemma 7.5 and [Davis and Lück 1998, Lemma 4.6], we obtain a weak equivalence of spectra

$$\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} (\boldsymbol{a} \vee \boldsymbol{b}) : O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} (\underline{\boldsymbol{K}}_{\mathcal{A}}^G \vee \boldsymbol{NK}_{\mathcal{A}}^G) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G.$$

Hence we obtain a weak equivalence of spectra

$$(\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{a}) \vee (\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{b}) : \\ \big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \underline{\boldsymbol{K}}_{\mathcal{A}}^G\big) \vee \big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G\big) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G.$$

If we combine this with Lemma 7.4 we obtain a weak equivalence of spectra

$$(f \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \mathrm{id}) \vee (\mathrm{id} \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{b}) : \\ (O^G(\underline{E}G) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G) \vee (O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}_{\mathcal{A}}^G) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathrm{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{K}_{\mathcal{A}}^G.$$

Using Lemma 7.3 this yields a natural weak equivalence of spectra

$$(f \wedge_{\mathsf{Or}(G)} \mathrm{id}) \vee \boldsymbol{b}' : (O^G(\underline{E}G) \wedge_{\mathsf{Or}(G)} \boldsymbol{K}^G_{\mathcal{A}}) \vee \big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}^G_{\mathcal{A}}\big) \\ \to O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathsf{Or}(G)} \boldsymbol{K}^G_{\mathcal{A}},$$

where $\boldsymbol{b}'$ comes from $\mathrm{id} \wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{b}$. If we take homotopy groups, we obtain for every $n \in \mathbb{Z}$ an isomorphism

$$H^G_n(f; \boldsymbol{K}^G_{\mathcal{A}}) \oplus \pi_n(\boldsymbol{b}') : H^G_n(\underline{E}G; \boldsymbol{K}^G_{\mathcal{A}}) \oplus \pi_n\big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}^G_{\mathcal{A}}\big) \\ \xrightarrow{\cong} H_n(E_{\mathcal{VC}_I}(G); \boldsymbol{K}^G_{\mathcal{A}}).$$

We have already explained above that $H^G_n(E_{\mathcal{VC}_I}(G); \boldsymbol{NK}^G_{\mathcal{A}})$ can be identified with $\pi_n\big(O^G(E_{\mathcal{VC}_I}(G)) \wedge_{\mathsf{Or}_{\mathcal{VC}_I}(G)} \boldsymbol{NK}^G_{\mathcal{A}}\big)$. Since, by construction, $\boldsymbol{NK}^G_{\mathcal{A}}(G/H)$ is the trivial spectrum for finite H and all isotropy groups of $\underline{E}G$ are finite, we conclude $H^G_n(\underline{E}G; \boldsymbol{NK}^G_{\mathcal{A}}) = 0$ for all $n \in \mathbb{Z}$ from Lemma 7.3. We derive from the long exact sequence of $f : \underline{E}(G) \to E_{\mathcal{VC}_I}(G)$ that the canonical map

$$H^G_n(E_{\mathcal{VC}_I}(G); \boldsymbol{NK}^G_{\mathcal{A}}) \xrightarrow{\cong} H^G_n(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}^G_{\mathcal{A}})$$

is bijective for all $n \in \mathbb{Z}$. Hence we obtain for all $n \in \mathbb{Z}$ a natural isomorphism

$$H^G_n(f; \boldsymbol{K}^G_{\mathcal{A}}) \oplus b_n : H^G_n(\underline{E}G; \boldsymbol{K}^G_{\mathcal{A}}) \oplus H^G_n(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}^G_{\mathcal{A}}) \\ \xrightarrow{\cong} H_n(E_{\mathcal{VC}_I}(G); \boldsymbol{K}^G_{\mathcal{A}}).$$

From the long exact homology sequence associated to $f : \underline{E}G \to E_{\mathcal{VC}_I}(G)$, we conclude that the map

$$H^G_n(f; \boldsymbol{K}^G_{\mathcal{A}}) : H^G_n(\underline{E}G; \boldsymbol{K}^G_{\mathcal{A}}) \to H^G_n(E_{\mathcal{VC}_I}(G); \boldsymbol{K}^G_{\mathcal{A}})$$

is split injective, there is a natural splitting

$$H^G_n(E_{\mathcal{VC}_I}(G); \boldsymbol{K}^G_{\mathcal{A}}) \xrightarrow{\cong} H^G_n(\underline{E}G; \boldsymbol{K}^G_{\mathcal{A}}) \oplus H_n(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}^G_{\mathcal{A}}),$$

and there exists a natural isomorphism, which is induced by the natural transformation $\boldsymbol{b} : \boldsymbol{NK}^G_{\mathcal{A}} \to \boldsymbol{K}^G_{\mathcal{A}}$ of spectra over $\mathsf{Or}_{\mathcal{VC}_I}(G)$,

$$H^G_n(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}^G_{\mathcal{A}}) \xrightarrow{\cong} H^G_n(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}^G_{\mathcal{A}}).$$

Now Theorem 0.1 follows from Theorem 7.7. □

8. Involutions and vanishing of Tate cohomology

8A. ***Involutions on K-theory spectra.*** Let $\mathcal{A} = (\mathcal{A}, I)$ be *an additive G-category with involution*, i.e., an additive G-category $\mathcal{A}$ together with a contravariant functor

$I\colon \mathcal{A}\to\mathcal{A}$ satisfying $I\circ I=\mathrm{id}_{\mathcal{A}}$ and $I\circ R_g=R_g\circ I$ for all $g\in G$. Examples coming from twisted group rings, or more generally crossed product rings equipped with involutions twisted by orientation homomorphisms, are discussed in [Bartels and Lück 2010, Section 8].

In the sequel for a category $\mathcal{C}$ we denote its *opposite* category by $\mathcal{C}^{\mathrm{op}}$. It has the same objects as $\mathcal{C}$. A morphism in $\mathcal{C}^{\mathrm{op}}$ from x to y is a morphism $y\to x$ in $\mathcal{C}$. Obviously we can and will identify $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}=\mathcal{C}$.

Next we define a covariant functor

$$I(G/H)\colon \int_{\mathcal{G}^G(G/H)}\mathcal{A}\to\Big(\int_{\mathcal{G}^G(G/H)}\mathcal{A}\Big)^{\mathrm{op}}. \tag{8.1}$$

It is defined to extend the involution

$$\coprod_{x\in\mathcal{G}^G(G/H)} I\colon \coprod_{x\in\mathcal{G}^G(G/H)}\mathcal{A}\to\Big(\coprod_{x\in\mathcal{G}^G(G/H)}\mathcal{A}\Big)^{\mathrm{op}}$$

and to send a structural morphism $T_g\colon (g_1H, A\cdot g)\to(g_2H, A)$ to the morphism $T_{g^{-1}}\colon (g_2H, I(A))\to(g_1H, I(A)\cdot g)$. One easily checks $I(G/H)\circ I(G/H)=\mathrm{id}$.

Notice that there is a canonical identification $\boldsymbol{K}(\mathcal{B}^{\mathrm{op}})=\boldsymbol{K}(\mathcal{B})$ for every additive category $\mathcal{B}$. Hence $I(G/H)$ induces a map of spectra

$$\boldsymbol{i}(G/H)=\boldsymbol{K}(I(G/H))\colon \boldsymbol{K}\Big(\int_{\mathcal{G}^G(G/H)}\mathcal{A}\Big)\to\boldsymbol{K}\Big(\int_{\mathcal{G}^G(G/H)}\mathcal{A}\Big)$$

such that $\boldsymbol{i}(G/H)\circ\boldsymbol{i}(G/H)=\mathrm{id}$. Let $\mathbb{Z}/2\text{-}\mathsf{Spectra}$ be the category of spectra with a (strict) $\mathbb{Z}/2$-operation. Thus the functor $\boldsymbol{K}_R^G$ becomes a functor

$$\boldsymbol{K}_R^G\colon \mathsf{Or}(G)\to\mathbb{Z}/2\text{-}\mathsf{Spectra}. \tag{8.2}$$

Consider an infinite virtually cyclic subgroup $V\subseteq G$ and a fixed generator $\sigma\in Q_V$. The functor $I(G/V)$ of (8.1) induces functors

$$I(G/H)[\sigma]\colon \int_{\mathcal{G}^G(G/H)[\sigma]}\mathcal{A}\to\Big(\int_{\mathcal{G}^G(G/H)[\sigma^{-1}]}\mathcal{A}\Big)^{\mathrm{op}},$$
$$I(G/H)_K\colon \int_{\mathcal{G}^G(G/H)_K}\mathcal{A}\to\Big(\int_{\mathcal{G}^G(G/H)_K}\mathcal{A}\Big)^{\mathrm{op}}.$$

Since $\mathrm{ev}(G/V)[\sigma^{-1}]_*\circ I(G/V)[\sigma]=I(G/V)_K\circ\mathrm{ev}(G/V)[\sigma]$ and

$$j(G/V)[\sigma^{-1}]_*\circ I(G/V)[\sigma]=I(G/V)\circ j(G/V)[\sigma]_*,$$

we obtain a commutative diagram of spectra

$$\begin{tikzcd}
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K}\mathcal{A}\big) \arrow[r, "\boldsymbol{K}(I(G/V)_K)"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)_K}\mathcal{A}\big) \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma]}\mathcal{A}\big) \arrow[r, "\boldsymbol{K}(I(G/V))[\sigma]"] \arrow[u, "\boldsymbol{K}(\mathrm{ev}(G/V)[\sigma]_*)"] \arrow[d, "\boldsymbol{K}(j(G/V)[\sigma]_*)"'] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)[\sigma^{-1}]}\mathcal{A}\big) \arrow[u, "\boldsymbol{K}(\mathrm{ev}(G/V)[\sigma^{-1}]_*)"'] \arrow[d, "\boldsymbol{K}(j(G/V)[\sigma^{-1}]_*)"] \\
\boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)}\mathcal{A}\big) \arrow[r, "\boldsymbol{K}(I(G/V))"] & \boldsymbol{K}\big(\int_{\mathcal{G}^G(G/V)}\mathcal{A}\big)
\end{tikzcd}$$

Since $I(G/H)[\sigma^{-1}]\circ I(G/H)[\sigma]=\mathrm{id}$ and $I(G/H)_K\circ I(G/H)_K=\mathrm{id}$, we obtain a $\mathbb{Z}/2$-operation on $\boldsymbol{NK}^G_{\mathcal{A}}$ and hence a functor

$$\boldsymbol{NK}^G_{\mathcal{A}}:\mathsf{Or}(G)\to\mathbb{Z}/2\text{-}\mathsf{Spectra}, \tag{8.3}$$

and we conclude:

Lemma 8.4. *The transformation $\boldsymbol{b}:\boldsymbol{NK}^G_{\mathcal{A}}\to\boldsymbol{K}^G_{\mathcal{A}}$ of $\mathsf{Or}_{\mathcal{VC}_I}(G)$-spectra is compatible with the $\mathbb{Z}/2$-actions.*

8B. *Orientable virtually cyclic subgroups of type I.*

Definition 8.5 (orientable virtually cyclic subgroups of type I). Given a group G, we say that *the infinite virtually cyclic subgroups of type I of G are orientable* if there is, for every virtually cyclic subgroup V of type I, a choice σ_V of a generator of the infinite cyclic group Q_V with the following property: whenever V and V' are infinite virtually cyclic subgroups of type I, and $f:V\to V'$ is an inclusion or a conjugation by some element of G, then the map $Q_f:Q_V\to Q_W$ sends σ_V to a positive multiple of σ_W. Such a choice of elements $\{\sigma_V\mid V\in\mathcal{VC}_I\}$ is called an *orientation.*

Lemma 8.6. *Suppose that the virtually cyclic subgroups of type I of G are orientable. Then all infinite virtually cyclic subgroups of G are of type I, and the fundamental group $\mathbb{Z}\rtimes\mathbb{Z}$ of the Klein bottle is not a subgroup of G.*

Proof. Suppose that G contains an infinite virtually cyclic subgroup V of type II. Then Q_V is the infinite dihedral group. Its commutator $[Q_V,Q_V]$ is infinite cyclic. Let W be the preimage of the commutator $[Q_V,Q_V]$ under the canonical projection $p_V:V\to Q_V$. There exists an element $y\in Q_V$ such that conjugation by y induces $-\mathrm{id}$ on $[Q_V,Q_V]$. Obviously W is an infinite virtually cyclic group of type I, and the restriction of p_V to W is the canonical map $p_W:W\to Q_W=[Q_V,Q_V]$. Choose an element $x\in V$ with $p_V(x)=y$. Conjugation by x induces an automorphism of W which induces $-\mathrm{id}$ on Q_W. Hence the virtually cyclic subgroups of type I of G are not orientable.

The statement about the Klein bottle is obvious. ∎

For the notions of a CAT(0)-group and of a hyperbolic group we refer for instance to [Bridson and Haefliger 1999; Ghys and de la Harpe 1990; Gromov 1987]. The fundamental group of a closed Riemannian manifold is hyperbolic if the sectional curvature is strictly negative, and is a CAT(0)-group if the sectional curvature is nonpositive.

Lemma 8.7. *Let G be a hyperbolic group. Then the infinite virtually cyclic subgroups of type I of G are orientable if and only if all infinite virtually cyclic subgroups of G are of type I.*

Proof. The "only if" statement follows from Lemma 8.6. To prove the "if" statement, assume that all infinite virtually cyclic subgroups of G are of type I.

By [Lück and Weiermann 2012, Example 3.6], every hyperbolic group satisfies the condition $(NM_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$, i.e., every infinite virtually cyclic subgroup V is contained in a unique maximal one $V_{\max}$ and the normalizer of $V_{\max}$ satisfies $NV_{\max} = V_{\max}$. Let $\mathcal{M}$ be a complete system of representatives of the conjugacy classes of maximal infinite virtually cyclic subgroups. Since by assumption $V \in \mathcal{M}$ is of type I, we can fix a generator $\sigma_V \in Q_V$ for each $V \in \mathcal{M}$.

Consider any infinite virtually cyclic subgroup W of G type I. Choose $g \in G$ and $V \in \mathcal{M}$ such that $gWg^{-1} \subseteq V$. Then conjugation with g induces an injection $Q_{c(g)} : Q_W \to Q_V$ by Lemma 1.1(vi). We equip W with the generator $\sigma_W \in Q_W$ for which there exists an integer $n \geq 1$ with $Q_{c(g)}(\sigma_W) = (\sigma_V)^n$. This is independent of the choice of g and V: for every $g \in G$ and $V \in \mathcal{M}$ with $|gVg^{-1} \cap V| = \infty$, the condition $(NM_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$ implies that g belongs to V and conjugation with an element $g \in V$ induces the identity on Q_V. □

Lemma 8.8. *Let G be a* CAT(0)*-group. Then the infinite virtually cyclic subgroups of type I of G are orientable if and only if all infinite virtually cyclic subgroups of G are of type I and the fundamental group $\mathbb{Z} \rtimes \mathbb{Z}$ of the Klein bottle is not a subgroup of G.*

Proof. Because of Lemma 8.6 it suffices to construct for a CAT(0)-group an orientation for its infinite virtually cyclic subgroups of type I, provided that all infinite virtually cyclic subgroups of G are of type I and the fundamental group $\mathbb{Z} \rtimes \mathbb{Z}$ of the Klein bottle is not a subgroup of G.

Consider on the set of infinite virtually cyclic subgroups of type I of G the relation $\sim$, where we put $V_1 \sim V_2$ if and only if there exists an element $g \in G$ with $|gV_1g^{-1} \cap V_2| = \infty$. This is an equivalence relation since, for any infinite virtually cyclic group V and elements $v_1, v_2 \in V$ of infinite order, we can find integers n_1, n_2 with $v_1^{n_1} = v_2^{n_2}$, $n_1 \neq 0$ and $n_2 \neq 0$. Choose a complete system of representatives $\mathcal{S}$ for the classes under $\sim$. For each element $V \in \mathcal{S}$ we choose an orientation $\sigma_V \in Q_V$.

Given any infinite virtually cyclic subgroup $W \subseteq G$ of type I we define a preferred generator $\sigma_W \in Q_W$ as follows: Choose $g \in G$ and $V \in \mathcal{S}$ with $|gWg^{-1} \cap V| = \infty$. Let $i_1 : gWg^{-1} \cap V \to W$ be the injection sending v to $g^{-1}vg$ and $i_2 : gWg^{-1} \cap V \to V$ be the inclusion. By Lemma 1.1(vi) we obtain injections of infinite cyclic groups $Q_{i_1} : Q_{gWg^{-1}\cap V} \to Q_W$ and $Q_{i_2} : Q_{gWg^{-1}\cap V} \to Q_V$. Equip Q_W with the generator σ_W for which there exist integers $n_1,\ n_2 \geq 1$ and $\sigma \in Q_{gWg^{-1}\cap V}$ with $Q_{i_1}(\sigma) = (\sigma_W)^{n_1}$ and $Q_{i_2}(\sigma) = (\sigma_V)^{n_2}$.

We have to show that this is well-defined. Obviously it is independent of the choice of σ, n_1 and n_2. It remains to show that the choice of g does not matter. For this purpose we have to consider the special case $W = V$ and have to show that the new generator σ_W agrees with the given one σ_V. We conclude from [Lück 2009, Lemma 4.2] and the argument about the validity of condition (C) appearing in the proof of [Lück 2009, Theorem 1.1(ii)] that there exists an infinite cyclic subgroup $C \subseteq gVg^{-1} \cap V$ such that g belongs to the normalizer $N_G C$. It suffices to show that conjugation with g induces the identity on C. Let $H \subseteq G$ be the subgroup generated by g and C. We obtain a short exact sequence $1 \to C \to H \xrightarrow{\mathrm{pr}} H/C \to 1$, where H/C is the cyclic subgroup generated by $\mathrm{pr}(g)$. Suppose that H/C is finite. Then H is an infinite virtually cyclic subgroup of G which must, by assumption, be of type I. Since the center of H must be infinite by Lemma 1.1(ii) and hence the intersection of the center of H with C is infinite cyclic, the conjugation action of g on C must be trivial. Suppose that H/C is infinite. Then H is the fundamental group of the Klein bottle if the conjugation action of g on C is nontrivial. Since the fundamental group of the Klein bottle is not a subgroup of G by assumption, the conjugation action of g on C is trivial also in this case. □

8C. *Proof of Theorem 0.2.* Let $\mathsf{Or}_{\mathcal{VC}_I \setminus \mathcal{F}\mathrm{in}}(G)$ be the full subcategory of the orbit category $\mathsf{Or}(G)$ consisting of those objects G/V for which V is an infinite virtually cyclic subgroup of type I. We obtain a functor

$$\mathrm{gen}(Q_?) : \mathsf{Or}_{\mathcal{VC}_I \setminus \mathcal{F}\mathrm{in}}(G) \to \mathbb{Z}/2\text{-}\mathsf{Sets}$$

sending G/V to $\mathrm{gen}(Q_V)$, and a G-map $f : G/V \to G/W$ to $\mathrm{gen}(f)$ as defined in (6.3). The $\mathbb{Z}/2$-action on $\mathrm{gen}(Q_V)$ is given by taking the inverse of a generator. The condition that the virtually cyclic subgroups of type I of G are orientable (see Definition 8.5) is equivalent to the condition that the functor $\mathrm{gen}(Q_?)$ is isomorphic to the constant functor sending G/V to $\mathbb{Z}/2$. A choice of an orientation corresponds to a choice of such an isomorphism.

Proof of Theorem 0.2. Because of Theorem 0.1 and Lemma 8.4 it suffices to show that the $\mathbb{Z}[\mathbb{Z}/2]$-module $H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}^G_{\mathcal{A}})$ is isomorphic to $\mathbb{Z}[\mathbb{Z}/2] \otimes_{\mathbb{Z}} A$ for some $\mathbb{Z}$-module A.

Fix an orientation $\{\sigma_V \mid V \in \mathcal{VC}_I\}$ in the sense of Definition 8.5. We have the $\mathrm{Or}_{\mathcal{VC}_I}(G)$-spectrum

$$\boldsymbol{NK}_R^{G'} : \mathrm{Or}_{\mathcal{VC}_I}(G) \to \mathsf{Spectra},$$

which sends G/V to the trivial spectrum if V is finite and to $\boldsymbol{NK}(G/V; \mathcal{A}, \sigma_V)$ if V is infinite virtually cyclic of type I. This is well-defined by the orientability assumption. Now there is an obvious natural isomorphism of functors from $\mathrm{Or}_{\mathcal{VC}_I}(G)$ to the category of $\mathbb{Z}/2$-spectra

$$\boldsymbol{NK}_{\mathcal{A}}^{G'} \wedge (\mathbb{Z}/2)_+ \xrightarrow{\cong} \boldsymbol{NK}_{\mathcal{A}}^{G},$$

which is a weak equivalence of $\mathrm{Or}_{\mathcal{VC}_I}(G)$-spectra. It induces a $\mathbb{Z}[\mathbb{Z}/2]$-isomorphism

$$H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^{G'}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_{\mathcal{A}}^{G}).$$

This finishes the proof of Theorem 0.2. $\square$

9. Rational vanishing of the relative term

This section is devoted to the proof of Theorem 0.3.

Consider the following diagram of groups, where the vertical maps are inclusions of subgroups of finite index and the horizontal arrows are automorphisms:

$$\begin{array}{ccc} H & \xrightarrow{\phi} & H \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ K & \xrightarrow{\psi} & K \end{array}$$

We obtain a commutative diagram

$$\begin{array}{ccccc} K_n(RH_\phi[t]) & \xrightarrow{i[t]_*} & K_n(RK_\psi[t]) & \xrightarrow{i[t]^*} & K_n(RH_\phi[t]) \\ \downarrow{\scriptstyle (\mathrm{ev}_H)_*} & & \downarrow{\scriptstyle (\mathrm{ev}_K)_*} & & \downarrow{\scriptstyle (\mathrm{ev}_H)_*} \\ K_n(RH) & \xrightarrow{i_*} & K_n(RK) & \xrightarrow{i^*} & K_n(RH) \end{array} \tag{9.1}$$

as follows: i_* and i^* are the maps induced by induction and restriction with the ring homomorphism $Ri : RH \to RK$; $i[t]_*$ and $i[t]^*$ are the maps induced by induction and restriction with the ring homomorphism $Ri[t] : RH_\phi[t] \to RK_\psi[t]$; $\mathrm{ev}_H : RH_\phi[t] \to RH$ and $\mathrm{ev}_K : RK_\psi[t] \to RK$ are the ring homomorphisms given by putting $t = 0$.

The left square is obviously well-defined and commutative. The right square is well-defined since the restriction of RK to RH by Ri is a finitely generated free RH-module and the restriction of $RK_\psi[t]$ to $RH_\phi[t]$ by $Ri[t]$ is a finitely generated free RH_ϕ-module by the following argument.

Put $l := [K : H]$. Choose a subset $\{k_1, k_2, \dots, k_l\}$ of K such that K/H can be written as $\{k_1 H, k_2 H, \dots, k_l H\}$. The map

$$\alpha : \bigoplus_{i=1}^{l} RH \xrightarrow{\cong} i^*RK, \quad (x_1, x_2, \dots, x_l) \mapsto \sum_{i=1}^{l} x_i \cdot k_i,$$

is an homomorphism of RH-modules and the map

$$\beta : \bigoplus_{i=1}^{l} RH_\phi[t] \xrightarrow{\cong} i[t]^*RK_\psi[t], \quad (y_1, y_2, \dots, y_l) \mapsto \sum_{i=1}^{l} y_i \cdot k_i,$$

is a homomorphism of $RH_\phi[t]$-modules. Obviously α is bijective. The map β is bijective since for any integer m we get $K/H = \{\psi^m(k_1)H, \psi^m(k_2)H, \dots, \psi^m(k_i)H\}$.

To show that the right square commutes we have to define for every finitely generated projective $RK_\psi[t]$-module P a natural RH-isomorphism

$$T(P) : (\mathrm{ev}_H)_* i[t]^* P \xrightarrow{\cong} i^*(\mathrm{ev}_K)_* P.$$

First we define $T(P)$. By the adjunction of induction and restriction it suffices to construct a natural map $T'(P) : i_*(\mathrm{ev}_H)_* i[t]^* P \to (\mathrm{ev}_K)_* P$. Since $i \circ \mathrm{ev}_H = \mathrm{ev}_K \circ i[t]$ we have to construct a natural map $T''(P) : i[t]_* i[t]^* P \to P$, since then we define $T'(P)$ to be $(\mathrm{ev}_K)_*(T''(P))$. Now define $T''(P)$ to be the adjoint of the identity $\mathrm{id} : i[t]^* P \to i[t]^* P$. Explicitly $T(P)$ sends an element $h \otimes x$ in $(\mathrm{ev}_H)_* i[t]^* P = RH \otimes_{\mathrm{ev}_H} i[t]^* P$ to the element $i(h) \otimes x$ in $i^*(\mathrm{ev}_K)_* P = RK \otimes_{\mathrm{ev}_K} P$.

Obviously $T(P)$ is natural in P and compatible with direct sums. Hence, in order to show that $T(P)$ is bijective for all finitely generated projective $RK_\psi[t]$-modules P, it suffices to do that for $P = RK_\psi[t]$. Now the claim follows since the following diagram of RH-modules commutes:

$$\begin{tikzcd}
RH \otimes_{\mathrm{ev}_H} i[t]^* RK_\psi[t] \arrow[r, "T(RK_\psi[t])"] & i^*(RK \otimes_{\mathrm{ev}_K} RK_\psi[t]) \arrow[d, "\cong"] \\
RH \otimes_{\mathrm{ev}_H} \left(\bigoplus_{i=1}^{l} RH_\phi[t]\right) \arrow[u, "\mathrm{id} \otimes_{\mathrm{ev}_H} \beta", "\cong"'] & i^*RK \\
\bigoplus_{i=1}^{l} RH \otimes_{\mathrm{ev}_H} RH_\phi[t] \arrow[u, "\cong"'] \arrow[r, "\cong"] & \bigoplus_{i=1}^{l} RH \arrow[u, "\alpha"']
\end{tikzcd}$$

where the isomorphisms α and β have been defined above and all other arrows marked with $\cong$ are the obvious isomorphisms. Recall that $NK_n(RH, R\phi)$ is by definition the kernel of $(\mathrm{ev}_H)_* : K_n(RH_\phi[t]) \to K_n(RH)$ and the analogous statement holds for $NK_n(RK, R\psi)$.

The diagram (9.1) induces homomorphisms

$$\begin{aligned} i_* &: NK_n(RH, R\phi) \to NK_n(RK, R\psi), \\ i^* &: NK_n(RK, R\psi) \to NK_n(RH, R\phi). \end{aligned}$$

Since both composites

$$K_n(RH_\phi[t]) \xrightarrow{i[t]^* \circ i[t]_*} K_n(RH_\phi[t]) \quad \text{and} \quad K_n(RH) \xrightarrow{i^* \circ i_*} K_n(RH)$$

are multiplication with l, we conclude:

Lemma 9.2. *The composite $i^* \circ i_* : NK_n(RH, R\phi) \to NK_n(RH, R\phi)$ is multiplication with l for all $n \in \mathbb{Z}$.*

Lemma 9.3. *Let $\phi : K \to K$ be an inner automorphism of the group K. Then there is, for all $n \in \mathbb{Z}$, an isomorphism*

$$NK_n(RK, R\phi) \xrightarrow{\cong} NK_n(RK).$$

Proof. Let k be an element such that ϕ is given by conjugation with k. We obtain a ring isomorphism

$$\eta : RK_{R\phi}[t] \xrightarrow{\cong} RK[t], \qquad \sum_i \lambda_i t^i \mapsto \lambda_i k^i t^i.$$

Let $\mathrm{ev}_{RK,\phi} : RK_\phi[t] \to RK$ and $\mathrm{ev}_{RK} : RK[t] \to RK$ be the ring homomorphisms given by putting $t = 0$. Then we obtain a commutative diagram with isomorphisms as vertical arrows

$$\begin{tikzcd} K_n(RK_{R\phi}[t]) \arrow[r, "\cong", "\eta"'] \arrow[d, "\mathrm{ev}_{RK,\phi}"] & K_n(RK[t]) \arrow[d, "\mathrm{ev}_{RK}"] \\ K_n(RK) \arrow[r, "\cong", "\mathrm{id}"'] & K_n(RK) \end{tikzcd}$$

It induces the desired isomorphism $NK_n(RK, R\phi) \xrightarrow{\cong} NK_n(RK)$. □

Remark. As the referee has pointed out, this results holds more generally (with identical proof) for the twisted Bass group $NF(S, \phi)$ of any functor F from rings to abelian groups and any inner ring automorphism $\phi : S \to S$.

Theorem 9.4. *Let R be a regular ring. Let K be a finite group of order r and let $\phi : K \xrightarrow{\cong} K$ be an automorphism of order s. Then $NK_n(RK, R\phi)[1/rs] = 0$ for all $n \in \mathbb{Z}$. In particular, $NK_n(RK, R\phi) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for all $n \in \mathbb{Z}$.*

Proof. Let t be a generator of the cyclic group $\mathbb{Z}/s$ of order s. Consider the semidirect product $K \rtimes_\phi \mathbb{Z}/s$. Let $i : K \to K \rtimes_\phi \mathbb{Z}/s$ be the canonical inclusion.

Let ψ be the inner automorphism of $K \rtimes_\phi \mathbb{Z}/s$ given by conjugation with t. Then $[K \rtimes_\phi \mathbb{Z}/s : K] = s$ and the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "i"'] & K \arrow[d, "i"] \\ K \rtimes_\phi \mathbb{Z}/s \arrow[r, "\psi"'] & K \rtimes_\phi \mathbb{Z}/s \end{tikzcd}$$

Lemmas 9.2 and 9.3 yield maps $i_* : NK_n(RK, \phi) \to NK_n(R[K \rtimes_\phi \mathbb{Z}/s])$ and $i^* : NK_n(R[K \rtimes_\phi \mathbb{Z}/s]) \to NK_n(RK, \phi)$ such that $i^* \circ i_* = s \cdot \mathrm{id}$. This implies that $NK_n(RK, \phi)[1/s]$ is a direct summand in $NK_n(R[K \rtimes_\phi \mathbb{Z}/s])[1/s]$. Since R is regular by assumption and hence $NK_n(R)$ vanishes for all $n \in \mathbb{Z}$, we conclude from [Hambleton and Lück 2012, Theorem A] that

$$NK_n(R[K \rtimes_\phi \mathbb{Z}/s])[1/rs] = 0.$$

(For $R = \mathbb{Z}$ and some related rings, this has already been proved by Weibel [1981, (6.5), p. 490].) This implies $NK_n(RK, \phi)[1/rs] = 0$. □

Theorem 9.4 has already been proved for $R = \mathbb{Z}$ in [Grunewald 2008, Theorem 5.11].

Now we are ready to give the proof of Theorem 0.3.

Proof of Theorem 0.3. Because of Theorem 0.1 it suffices to prove, for all $n \in \mathbb{Z}$,

$$H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_R^G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \{0\}.$$

There is a spectral sequence converging to $H_{p+q}^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{NK}_R^G)$ whose E^2-term is the Bredon homology

$$E^2_{p,q} = H_p^{\mathbb{Z}\mathrm{Or}_{\mathcal{VC}_I}(G)}(\underline{E}G \to E_{\mathcal{VC}_I}(G); \pi_q(\boldsymbol{NK}_R^G))$$

with coefficients in the covariant functor from $\mathrm{Or}_{\mathcal{VC}_I}(G)$ to the category of $\mathbb{Z}$-modules coming from composing $\boldsymbol{NK}_R^G : \mathrm{Or}_{\mathcal{VC}_I}(G) \to \mathsf{Spectra}$ with the functor taking the q-homotopy group; see [Davis and Lück 1998, Theorems 4.7 and 7.4]. Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, it suffices to show, for all $V \in \mathcal{VC}_I$,

$$\pi_q(\boldsymbol{NK}_R^G(G/V)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0.$$

If V is finite, $\boldsymbol{NK}_R^G(G/V)$ is by construction the trivial spectrum and the claim is obviously true. If V is a virtually cyclic group of type I, then we conclude from the diagram (5.6) that

$$\pi_n(\boldsymbol{NK}_R^G(G/V)) \cong NK_n(RK_V, R\phi) \oplus NK_n(RK_V, R\phi^{-1}).$$

Now the claim follows from Theorem 9.4. □

10. On the computation of the relative term

In this section we give some further information about the computation of the relative term $H_n^G(\underline{E}G \to \underline{\underline{E}}G; \boldsymbol{K}_R^G) \cong H_n^G(\underline{E}G \to E_{\mathcal{VC}}(G); \boldsymbol{NK}_R^G)$.

Lück and Weiermann [2012] give a systematic analysis of how the space $E_{\mathcal{VC}_I}(G)$ is obtained from $\underline{E}G$. We say that G satisfies the condition $(M_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$ if any virtually cyclic subgroup of type I is contained in a unique maximal infinite cyclic subgroup of type I. We say that G satisfies the condition $(NM_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$ if it satisfies $(M_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$ and, for any maximal virtually cyclic subgroup V of type I, its normalizer N_GV agrees with V. Every word hyperbolic group satisfies $(NM_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$; see [Lück and Weiermann 2012, Example 3.6].

Suppose that G satisfies $(M_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$. Let $\mathcal{M}$ be a complete system of representatives V of the conjugacy classes of maximal virtually cyclic subgroups of type I. Then we conclude from [Lück and Weiermann 2012, Corollary 2.8] that there exists a G-pushout of G-CW-complexes with inclusions as horizontal maps

$$\begin{tikzcd}
\coprod_{V\in\mathcal{M}} G\times_{N_GV}\underline{E}N_GV \arrow[r, "i"] \arrow[d, "\coprod_{V\in\mathcal{M}}\mathrm{id}_G\times f_V"] & \underline{E}G \arrow[d, "f"] \\
\coprod_{V\in\mathcal{M}} G\times_{N_GV} E_{\mathcal{VC}_I}(N_GV) \arrow[r] & E_{\mathcal{VC}_I}(G)
\end{tikzcd}$$

This yields for all $n\in\mathbb{Z}$ an isomorphism, using the induction structure in the sense of [Lück 2002, Section 1],

$$\bigoplus_{V\in\mathcal{M}} H_n^{N_GV}(\underline{E}N_GV \to E_{\mathcal{VC}_I}(N_GV); \boldsymbol{K}_R^{N_GV}) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_R^G).$$

Combining this with Theorem 0.1 yields the isomorphism

$$\bigoplus_{V\in\mathcal{M}} H_n^{N_GV}(\underline{E}N_GV \to E_{\mathcal{VC}_I}(N_GV); \boldsymbol{NK}_R^{N_GV}) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_R^G).$$

Suppose now that G satisfies $(NM_{\mathcal{F}\mathrm{in}\subseteq\mathcal{VC}_I})$ and recall that $NK_R^G(V/H)=0$ for finite H, by definition. Then the isomorphism above reduces to the isomorphism

$$\bigoplus_{V\in\mathcal{M}} \pi_n(\boldsymbol{NK}_R^V(V/V)) \xrightarrow{\cong} H_n^G(\underline{E}G \to E_{\mathcal{VC}_I}(G); \boldsymbol{K}_R^G),$$

and $\pi_n(\boldsymbol{NK}_R^V(V/V))$ is the Nil-term $NK_n(RK_V, R\phi)\oplus NK_n(RK_V; R\phi^{-1})$ appearing in the twisted version of the Bass–Heller–Swan decomposition of RV (see [Grayson 1988, Theorems 2.1 and 2.3]) if we write $V\cong K_V\rtimes_\phi\mathbb{Z}$.

11. Fibered version

We illustrate in this section, by an example which will be crucial in [Farrell et al. 2016], that we do get information from our setting also in a fibered situation.

Let $p : X \to B$ be a map of path connected spaces. We will assume that it is π_1-surjective, i.e., induces an epimorphism on fundamental groups. Suppose that B admits a universal covering $q : \tilde{B} \to B$.

Choose base points $x_0 \in X$, $b_0 \in B$ and $\tilde{b}_0 \in \tilde{B}$ satisfying $p(x_0) = b_0 = q(\tilde{b}_0)$. We will abbreviate $\Gamma = \pi_1(X, x_0)$ and $G = \pi_1(B, b_0)$. Recall that we have a free right proper G-action on $\tilde{B}$ and q induces a homeomorphism $\tilde{B}/G \xrightarrow{\cong} B$. For a subgroup $H \subseteq G$ denote by $q(G/H) : \tilde{B} \times_G G/H = \tilde{B}/H \to B$ the obvious covering induced by q. The pullback construction yields a commutative square of spaces

$$\begin{tikzcd} X(G/H) \arrow[r, "\bar{q}(G/H)"] \arrow[d, "\bar{p}(G/H)"'] & X \arrow[d, "p"] \\ \tilde{B} \times_G G/H \arrow[r, "q(G/H)"'] & B \end{tikzcd}$$

where $\bar{q}(G/H)$ is again a covering. This yields covariant functors from the orbit category of G to the category of topological spaces,

$$\begin{aligned} \underline{B} &: \mathsf{Or}(G) \to \mathsf{Spaces}, \quad G/H \mapsto \tilde{B} \times_G G/H, \\ \underline{X} &: \mathsf{Or}(G) \to \mathsf{Spaces}, \quad G/H \mapsto X(G/H). \end{aligned}$$

The assumption that p is π_1-surjective ensures that $X(G/H)$ is path connected for all $H \subseteq G$.

By composition with the fundamental groupoid functor we obtain a functor

$$\underline{\Pi(X)} : \mathsf{Or}(G) \to \mathsf{Groupoids}, \quad G/H \mapsto \Pi(X(G/H)).$$

Let R-FGF be the additive category whose set of objects is $\{R^n \mid n = 0, 1, 2, \dots\}$ and whose morphisms are R-linear maps. In the sequel it will always be equipped with the trivial G- or Γ-action or considered as constant functor $\mathcal{G} \to \mathsf{Add\text{-}Cat}$. Consider the functor

$$\xi : \mathsf{Groupoids} \to \mathsf{Spectra}, \quad \mathcal{G} \mapsto \boldsymbol{K}\Big(\int_{\mathcal{G}} R\text{-}\mathsf{FGF}\Big).$$

The composite of the last two functors yields a functor

$$\boldsymbol{K}(p) := \xi \circ \underline{\Pi(X)} : \mathsf{Or}(G) \to \mathsf{Spectra}.$$

Associated to this functor there is — see [Davis and Lück 1998] — a G-homology theory $H_*^G(-; \boldsymbol{K}(p)) := \pi_n(O^G(-) \wedge_{\mathsf{Or}(G)} \boldsymbol{K}(p))$. We will be interested in the associated assembly map induced by the projection $\underline{\underline{E}}G \to G/G$,

$$H_n^G(\underline{\underline{E}}G; \boldsymbol{K}(p)) \to H_n^G(G/G; \boldsymbol{K}(p)) \cong K_n(R\Gamma). \tag{11.1}$$

The goal of this section is to identify this assembly map with the assembly map

$$H_n^G(\underline{\underline{E}}G; \boldsymbol{K}_{\mathcal{A}}) \to H_n^G(G/G; \boldsymbol{K}_{\mathcal{A}}) = K_n(R\Gamma)$$

for a suitable additive category with G-action $\mathcal{A}$. Thus the results of this paper apply also in the fibered setup.

Consider the functor

$$\underline{\mathcal{G}^\Gamma} : \mathtt{Or}(G) \to \mathtt{Groupoids}, \quad G/H \mapsto \mathcal{G}^\Gamma(G/H),$$

where we consider G/H as a Γ-set by restriction along the group homomorphism $\Gamma \to G$ induced by p.

Lemma 11.2. *There is a natural equivalence*

$$T : \underline{\mathcal{G}^\Gamma} \to \underline{\Pi(X)}$$

of covariant functors $\mathtt{Or}(G) \to \mathtt{Groupoids}$.

Proof. Given an object G/H in $\mathtt{Or}(G)$, we have to specify an equivalence of groupoids $T(G/H) : \mathcal{G}^\Gamma(G/H) \to \Pi(X(G/H))$. For an object in $\mathcal{G}^\Gamma(G/H)$ which is given by an element $wH \in G/H$, define $T(wH)$ to be the point in $X(G/H)$ which is determined by $(\tilde{b}_0, wH) \in \tilde{B} \times_G G/H$ and $x_0 \in X$. This makes sense since $q(G/H)((\tilde{b}_0, wH)) = b_0 = q(x_0)$.

Let $\gamma : w_0H \to w_1H$ be a morphism in $\mathcal{G}^\Gamma(G/H)$. Choose a loop u_X in X at $x_0 \in X$ which represents γ. Let u_B be the loop $p \circ u_X$ in B at $b_0 \in B$. There is precisely one path $u_{\tilde{B}}$ in $\tilde{B}$ which starts at $\tilde{b}_0$ and satisfies $q \circ u_{\tilde{B}} = u_B$. Let $[u_B] \in G$ be the class of u_B, or, equivalently, the image of γ under $\pi_1(p, x_0) : \Gamma \to G$. By definition of the right G-action on $\tilde{B}$ we have $\tilde{b}_0 \cdot [u_B] = u_B(1)$. Define a path $u_{\tilde{B}/H}$ in $\tilde{B} \times_G G/H$ from $(\tilde{b}_0, w_0H)$ to $(\tilde{b}_0, w_1H)$ by $t \mapsto (u_B(t), w_0H)$. This is indeed a path ending at $(\tilde{b}_0, w_1H)$ since $(\tilde{b}_0 \cdot [u_B], w_0H) = (\tilde{b}_0, [u_B] \cdot w_0H) = (\tilde{b}_0, w_1H)$ holds in $\tilde{B} \times_G G/H$. Obviously the composite of $u_{\tilde{B}/H}$ with $q(G/H) : \tilde{B} \times_G G/H \to B$ is u_B. Hence $u_{\tilde{B}/H}$ and u_X determine a path in $X(G/H)$ from $T(w_0H) \to T(w_1H)$ and hence a morphism $T(w_0H) \to T(w_1H)$ in $\Pi(X(G/H))$. One easily checks that the homotopy class (relative to the endpoints) of u depends only on γ. Thus we obtain the desired functor $T(G/H) : \mathcal{G}^\Gamma(G/H) \to \Pi(X(G/H))$. One easily checks that they fit together, so that we obtain a natural transformation $T : \underline{\mathcal{G}^\Gamma} \to \underline{\Pi(X)}$.

At a homogeneous space G/H, the value of $\underline{\mathcal{G}^\Gamma}$ is a groupoid equivalent to the group $\pi_1(p, x_0)^{-1}(H)$, while the value of $\underline{\Pi(X)}$ is a groupoid equivalent to the fundamental group of $X(G/H)$. Up to this equivalence, the functor T, at G/H, is the standard identification of these two groupoids. Hence T is a natural equivalence. □

We obtain a covariant functor

$$\boldsymbol{K}(p)' : \mathsf{Or}(G) \to \mathsf{Spectra}, \quad G/H \mapsto \boldsymbol{K}\left(\int_{\mathcal{G}^\Gamma(G/H)} R\text{-}\mathsf{FGF}\right).$$

Lemma 11.2 implies that the following diagram commutes, where the vertical arrow is the isomorphism induced by T:

$$\begin{tikzcd}
H_n^G(\underline{\underline{E}}G; \boldsymbol{K}(p)) \arrow[dr, "H_n^G(\mathrm{pr};\boldsymbol{K}(p))"] & \\
& H_n^G(G/G; \boldsymbol{K}(p)) = K_n(R\Gamma) \\
H_n^G(\underline{\underline{E}}G; \boldsymbol{K}(p)') \arrow[uu, "T_*"', "\cong"] \arrow[ur, "H_n^G(\mathrm{pr};\boldsymbol{K}(p)')"'] &
\end{tikzcd}$$

Now the functor $\boldsymbol{K}(p)'$ is, up to natural equivalence, of the form $\boldsymbol{K}_{\mathcal{A}}^G$ for some additive G-category, namely for $\mathcal{A} = \mathrm{ind}_{q:\Gamma\to G}\, R\text{-}\mathsf{FGF}$; see [Bartels and Lück 2010, (11.5) and Lemma 11.6]. We conclude:

Lemma 11.3. *The assembly map (11.1) is an isomorphism for all $n \in \mathbb{Z}$ if the K-theoretic Farrell–Jones conjecture for additive categories holds for G.*

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WOLFGANG LÜCK: wolfgang.lueck@him.uni-bonn.de
Mathematisches Institut, Universität Bonn, Endenicher Allee 60, D-53123 Bonn, Germany

WOLFGANG STEIMLE: steimle@math.uni-leipzig.de
Mathematisches Institut, Universität Leipzig, Postfach 10 09 20, D-04009 Leipzig, Germany

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Birational motives
I: Pure birational motives

Bruno Kahn and Ramdorai Sujatha

We define a category of pure birational motives over a field, depending on the choice of an adequate equivalence relation on algebraic cycles. It is obtained by "killing" the Lefschetz motive in the corresponding category of effective motives. For rational equivalence, it encompasses Bloch's decomposition of the diagonal. We study the induced Chow–Künneth decompositions in this category, and establish relationships with Rost's cycle modules and the Albanese functor for smooth projective varieties.

Introduction

In the preprint [Kahn and Sujatha 2002], we toyed with birational ideas in three areas of algebraic geometry: plain varieties, pure motives in the sense of Grothendieck, and triangulated motives in the sense of Voevodsky. These three themes are finally treated separately in revised versions. The first one is the object of [Kahn

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and Sujatha 2015a]; the second one is the object of the present paper; the third one is the object of [Kahn and Sujatha 2015b].

We work over a field F. Recall that we introduced in [Kahn and Sujatha 2015a] two "birational" categories. The first, $\mathbf{place}(F)$, has for objects the function fields over F and for morphisms the F-places. The second one is the Gabriel–Zisman localisation of the category $\mathbf{Sm}(F)$ of smooth F-varieties obtained by inverting birational morphisms [Gabriel and Zisman 1967, Chapter 1]; we denote this category by $S_b^{-1}\,\mathbf{Sm}(F)$.

We may also invert stable birational morphisms: those which are dominant and induce a purely transcendental extension of function fields, and invert the corresponding morphisms in $\mathbf{place}(F)$. We denote the sets of such morphisms by S_r.

In order to simplify the exposition, let us assume that F is of characteristic 0. Then the main results of [Kahn and Sujatha 2015a] and its predecessor [Kahn and Sujatha 2007] can be summarised in a diagram

$$\begin{tikzcd}
\mathbf{place}(F)^{\mathrm{op}} \arrow[r] \arrow[d] & S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \arrow[r, "\sim"] \arrow[d, "\wr"'] & S_b^{-1}\,\mathbf{Sm}(F) \arrow[d, "\wr"'] \\
S_r^{-1}\,\mathbf{place}(F)^{\mathrm{op}} \arrow[r] & S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \arrow[r, "\sim"] & S_r^{-1}\,\mathbf{Sm}(F)
\end{tikzcd}$$

where $\mathbf{Sm}^{\mathrm{proj}}(F)$ is the full subcategory of smooth projective varieties and the symbols $\sim$ denote equivalences of categories; see [Kahn and Sujatha 2007, Proposition 8.5] and [Kahn and Sujatha 2015a, Theorems 1.7.2 and 4.2.4].

Moreover, if X is smooth and Y is smooth proper, then $\mathrm{Hom}(X, Y) = Y(F(X))/R$ in $S_b^{-1}\,\mathbf{Sm}(F)$, where R is R-equivalence [ibid., Theorem 6.6.3].

In this paper, we consider the effect of inverting birational morphisms in categories of *effective pure motives*. For simplicity, let us still assume $\operatorname{char} F = 0$, and consider only the category of effective Chow motives $\mathbf{Chow}^{\mathrm{eff}}(F)$, defined by using algebraic cycles modulo rational equivalence. The graph functor then induces a commutative square (compare (5.1.1))

$$\begin{tikzcd}
S_b^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \arrow[r] \arrow[d, "\wr"'] & S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F) \arrow[d] \\
S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \arrow[r] & S_r^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F)
\end{tikzcd}$$

One can expect that the right vertical functor is an equivalence of categories, and indeed this is not difficult to prove (Corollary 2.2.5(b)). But we have two other descriptions of this category of "birational motives":

- The functor $\mathbf{Chow}^{\mathrm{eff}}(F) \to S_b^{-1}\,\mathbf{Chow}^{\mathrm{eff}}(F)$ is full, and its kernel is the ideal $\mathcal{L}_{\mathrm{rat}}$ of morphisms which factor through some object of the form $M \otimes \mathbb{L}$, where $\mathbb{L}$ is the *Lefschetz motive* [ibid].

- If X, Y are smooth projective varieties, then $\mathcal{L}_{\text{rat}}(h(X), h(Y))$ coincides with the group of Chow correspondences represented by algebraic cycles on $X \times Y$ whose irreducible components are not dominant over X (Theorem 2.4.2).

As a consequence, the group of morphisms from $h(X)$ to $h(Y)$ in $S_b^{-1}\,\mathbf{Chow}^{\text{eff}}(F)$ is isomorphic to $CH_0(Y_{F(X)})$. Given the similar description of Hom sets in

$$S_b^{-1}\,\mathbf{Sm}^{\text{proj}}(F)$$

recalled above, this places the classical map

$$Y(F(X))/R \to CH_0(Y_{F(X)})$$

in a categorical context.

Note that, by [Kahn and Sujatha 2015a, Theorem 8.5.1(b)], if $X \simeq \operatorname{Spec} F$ in $S_b^{-1}\,\mathbf{Sm}$ then X must be rationally connected; on the other hand, there are surfaces of general type with trivial birational motive, see Remarks 3.1.5(1) and (3). So the birational motive of a smooth projective variety detects much less geometry than its class in $S_b^{-1}\,\mathbf{Sm}$, but on the other hand it is much more computable.

This paper is organised as follows. In Section 1 we review pure motives. In Section 2 we study pure birational motives, in greater generality than outlined in this introduction. In particular, many results are valid for other adequate equivalence relations than rational equivalence, see Section 2.3; moreover, most results extend to characteristic p if p is invertible in the ring of coefficients, by using the de Jong–Gabber alteration theorem [Illusie and Temkin 2014]; see Theorem 2.4.2.

Section 3 consists of examples. We study varieties whose birational motive is trivial, in the line of the remarks above. We also study the Chow–Künneth decomposition in the category of birational motives, special attention being devoted to the case of complete intersections.

Let $\mathbf{Chow}^{\text{o}}(F)$ denote the pseudoabelian envelope of $S_b^{-1}\,\mathbf{Chow}^{\text{eff}}(F)$. In Section 4, we examine two questions: the existence of a right adjoint to the projection functor $\mathbf{Chow}^{\text{eff}}(F) \to \mathbf{Chow}^{\text{o}}(F)$ (and similarly for more general adequate equivalences), and whether pseudoabelian completion is really necessary. It turns out that the answer to the first question is negative (Theorems 4.3.2 and 4.3.3; this is related to the nontriviality of the Griffiths group for some 3-folds) and the answer to the second question is positive with rational coefficients under a nilpotence conjecture (Conjecture 3.3.1). We can get an unconditional positive answer to the second question if we restrict to a suitable type of motives (Proposition 4.4.1 and Example 4.4.2).

In Section 5, we define a functor $S_r^{-1}\,\mathbf{field}(F)^{\text{op}} \to S_r^{-1}\,\mathbf{Chow}^{\text{eff}}(F, \mathbb{Q})$ in characteristic p, using de Jong's theorem again. Here $\mathbf{field}(F)$ denotes the subcategory of $\mathbf{place}(F)$ with the same objects but morphisms restricted to field extensions (Proposition 5.1.4).

We end this paper by relating the previous constructions to more classical objects. In Section 6 we relate birational motives to cycle cohomology [Rost 1996], expanding a bit on previous results by Rost and Merkurjev [2001; 2008]. In Section 7, we define a tensor additive category $\mathbf{AbS}(F)$ of *locally abelian schemes*, whose objects are those F-group schemes that are extensions of a lattice (i.e., locally isomorphic for the étale topology to a free finitely generated abelian group) by an abelian variety. We then show in Section 8 that the classical construction of the Albanese variety of a smooth projective variety extends to a tensor functor

$$\mathrm{Alb} : \mathbf{Chow}^{\mathrm{o}}(F) \to \mathbf{AbS}(F),$$

which becomes full and essentially surjective after tensoring morphisms with $\mathbb{Q}$ (Proposition 8.2.1). So, one could say that $\mathbf{AbS}(F)$ is the *representable part* of $\mathbf{Chow}^{\mathrm{o}}(F)$. We also show that, after tensoring with $\mathbb{Q}$, Alb has a right adjoint which identifies $\mathbf{AbS}(F) \otimes \mathbb{Q}$ with the thick subcategory of $\mathbf{Chow}^{\mathrm{o}}(F) \otimes \mathbb{Q}$ generated by motives of varieties of dimension ≤ 1.

Some results of the preliminary version [Kahn and Sujatha 2002] of this work were used in other papers, namely [Kahn et al. 2007; Kahn 2009], and we occasionally refer to these papers to ease the exposition. Here is a correspondence guide between the results from [Kahn and Sujatha 2002] used in these papers and those in the present version:

- In [Kahn 2009], Lemma 7.2 uses [Kahn and Sujatha 2002, Lemmas 5.3 and 5.4], which correspond to Proposition 2.3.5 and Theorem 2.4.2 of the present paper. The reader will verify that the proofs of Proposition 2.3.5 and Theorem 2.4.2 are the same as those of [Kahn and Sujatha 2002, Lemmas 5.3 and 5.4], mutatis mutandis, and do not use any result from [Kahn 2009].
- In [Kahn et al. 2007], Lemma 7.5.3 uses the same references; the same comment as above applies. Moreover, Proposition 9.5 of [Kahn and Sujatha 2002] is used on pp. 174–175 of [Kahn et al. 2007]; this result is now Theorem 8.2.4. Again, its proof is identical to the one in the preliminary version and does not use results from [Kahn et al. 2007].

The idea of considering birational Chow correspondences, which yield here a category in which $\mathrm{Hom}([X], [Y]) = CH_0(Y_{F(X)})$ for two smooth projective varieties X, Y, goes back to S. Bloch's method of "decomposition of the diagonal" in [Bloch 2010, Appendix to Lecture 1] (see also [Bloch and Srinivas 1983]). He attributes the idea of considering the generic point of a smooth projective variety X as a 0-cycle over its function field to Colliot-Thélène; here, this corresponds to the identity endomorphism of $h^{\mathrm{o}}(X) \in \mathbf{Chow}^{\mathrm{o}}(F)$. We realised the connection with Bloch's ideas after reading H. Esnault's article [2003], and this led to another proof of her theorem by the present birational techniques in [Kahn 2009]. M. Rost has considered this category independently [Merkurjev 2001]; this was pointed out to us by N. Karpenko.

1. Review of pure motives

In this section, we recall the definition of categories of pure motives in a way which is suited to our needs. A slight variance to the usual exposition is the notion of *adequate pair*, which is a little more precise than the notion of adequate equivalence relation (it explicitly takes the coefficients into account).

We adopt the covariant convention, for future comparison with Voevodsky's triangulated categories of motives: here, the functor which sends a smooth projective variety to its motive is covariant. For a dictionary between the covariant and contravariant conventions, the reader may refer to [Kahn et al. 2007, Lemma 7.1.2].

1.1. ***Adequate pairs.*** We give ourselves

- a commutative ring of coefficients A;
- an adequate equivalence relation $\sim$ on algebraic cycles with coefficients in A [Samuel 1960].

We refer to $(A,\sim)$ as an *adequate pair*. Classical examples for $\sim$ are rat (rational equivalence), alg (algebraic equivalence), num (numerical equivalence), $\sim_H$ (homological equivalence relative to a fixed Weil cohomology theory H). A less classical example is Voevodsky's smash-nilpotence tnil [1995]; see [André and Kahn 2002, Example 7.4.3] (a cycle α is smash-nilpotent if $\alpha^{\otimes n}\sim_{\mathrm{rat}} 0$ for some $n>0$). We then have a notion of domination $(A,\sim)\geq(A,\sim')$ if $\sim$ is finer than $\sim'$ (i.e., the groups of cycles modulo $\sim$ surjects onto the one for $\sim'$). It is well known that $(A,\mathrm{rat})\geq(A,\sim)$ for any $\sim$ (see [Fulton 1984, Example 1.7.5]), and that $(A,\sim)\geq(A,\mathrm{num}_A)$ if A is a field.

Since the issue of coefficients is sometimes confusing, the following remarks may be helpful. Given a pair $(A,\sim)$ and a commutative A-algebra B, we get a new pair $B\otimes_A(A,\sim)$ by tensoring algebraic cycles with B: for example, $(A,\sim)=A\otimes_{\mathbb{Z}}(\mathbb{Z},\sim)$ for $\sim=\mathrm{rat}$, alg or tnil by definition. On the other hand, given a pair $(B,\sim)$ and a ring homomorphism $A\to B$ we get a "restriction of scalars" pair $(A,\sim_{|A})$ by considering cycles with coefficients in A which become ~ 0 after tensoring with B: for example, if H is a Weil cohomology theory with coefficients in K, this applies to any ring homomorphism $A\to K$. Obviously

$$B\otimes_A(A,\sim_{|A})\geq(B,\sim),$$

but this need not be an equality in general.

In the case of numerical equivalence (a cycle with coefficients in A is numerically equivalent to 0 if the degree of its intersection with any cycle of complementary dimension in good position is 0), we have $B \otimes_A (A, \mathrm{num}_A) \geq (B, \mathrm{num}_B)$, with equality if B is flat over A.

Given a pair $(A, \sim)$, to any smooth projective F-variety X and integer $n \geq 0$ we may associate its group of cycles of codimension n with coefficients in A modulo $\sim$, which will be denoted by $\mathcal{Z}^n_\sim(X, A)$. If X has pure dimension d, we also denote this group by $\mathcal{Z}^\sim_{d-n}(X, A)$.

1.2. ***Smooth projective varieties, connected and nonconnected.*** In [Kahn and Sujatha 2015a] we were only considering (connected) varieties over F. Classically, pure motives are defined using not necessarily connected smooth projective varieties. One could base the treatment on connected smooth varieties, but this would introduce problems with the tensor product, since a product of connected varieties need not be connected in general (e.g., if neither of them is geometrically connected). Thus we prefer to use here:

Definition 1.2.1. We write $\mathbf{Sm}_\amalg(F)$ for the category of smooth separated schemes of finite type over F. For $\% \in \{\mathrm{prop}, \mathrm{qp}, \mathrm{proj}\}$, we write $\mathbf{Sm}^\%_\amalg(F)$ for the full subcategory of $\mathbf{Sm}_\amalg(F)$ consisting of proper, quasiprojective or projective varieties.

Unlike their counterparts considered in [Kahn and Sujatha 2015a], these categories enjoy finite products and coproducts.

The following lemma is clear.

Lemma 1.2.2. *The categories considered in Definition 1.2.1 are the "finite coproduct envelopes" of those considered in* [Kahn and Sujatha 2015a], *in the sense of* [Kahn and Sujatha 2007, Proposition 6.1].

1.3. ***Review of correspondences.*** We associate to two smooth projective varieties X, Y the group $\mathcal{Z}^{\dim Y}_\sim(X \times Y, A)$ of correspondences from X to Y relative to $(A, \sim)$. The composition of correspondences is defined as follows:[1] if X, Y, Z are smooth projective and $(\alpha, \beta) \in \mathcal{Z}^{\dim Y}_\sim(X \times Y, A) \times \mathcal{Z}^{\dim Z}_\sim(Y \times Z, A)$, then

$$\beta \circ \alpha = (p_{XZ})_*(p^*_{XY}\alpha \cdot p^*_{YZ}\beta),$$

where p_{XY}, p_{YZ} and p_{XZ} denote the partial projections from $X \times Y \times Z$ onto two-fold factors.

We then get an A-linear tensor (i.e., symmetric monoidal) category $\mathbf{Cor}_\sim(F, A)$. The graph map defines a *covariant* functor

$$\mathbf{Sm}^{\mathrm{proj}}_\amalg(F) \to \mathbf{Cor}_\sim(F, A), \quad X \mapsto [X], \tag{1.3.1}$$

[1] We follow here the convention of Voevodsky [2000]. It is also the one used by Fulton [1984, Section 16]. See [Kahn et al. 2007, Lemma 7.1.2].

so that $[X \amalg Y] = [X] \oplus [Y]$, and $[X \times Y] = [X] \otimes [Y]$ for the tensor structure. The unit object is $\mathbb{1} = [\mathrm{Spec}\, F]$.

If $f : X \to Y$ is a morphism of smooth varieties, let Γ_f denote its graph and $[\Gamma_f]$ denote the class of Γ_f in $\mathcal{Z}^{\dim Y}_{\sim}(X \times Y)$. We write f_* for the correspondence $[\Gamma_f] : [X] \to [Y]$ (the image of f under the functor (1.3.1)). Note that if $f : X \to Y$ and $g : Y \to Z$ are two morphisms of smooth projective varieties, then the cycles $\Gamma_f \times Z$ and $X \times \Gamma_g$ on $X \times Y \times Z$ intersect properly, so that $g_* \circ f_*$ is well defined as a cycle and not just as an equivalence class of cycles; the equation $g_* \circ f_* = (g \circ f)_*$ is an equality of cycles. (This is a very special case of the composition of finite correspondences; see [Mazza et al. 2006, Lemma 1.7].)

1.4. ***The correspondence attached to a rational map.*** We first define rational maps between not necessarily connected smooth varieties X, Y in the obvious way: it is a morphism from a suitable *dense* open subset of X to Y. Like morphisms, rational maps split as disjoint unions of "connected" rational maps. A rational map f is *dominant* if all its connected components are dominant and if the image of f meets all connected components of Y.

Let $f : X \dashrightarrow Y$ be a rational map between two smooth projective varieties X, Y. To f we associated in [Kahn and Sujatha 2015a, Section 6.3] a morphism in the category $S_b^{-1}\,\mathbf{Sm}$. In the case of Chow motives, we can do better: define the correspondence $f_* : [X] \to [Y]$ in $\mathbf{Cor}_{\sim}(F, A)$ as the closure of the graph of f inside $X \times Y$. The formula $g_* \circ f_* = (g \circ f)_*$ need not be valid in general, even if $g \circ f$ is defined (but see Proposition 2.3.8 below). Yet we have:

Lemma 1.4.1. *Let $X \overset{f}{\dashrightarrow} Y \xrightarrow{g} Z$ be a diagram of smooth projective varieties, where f is a rational map and g is a morphism. Then we have an equality of cycles*

$$g_* \circ f_* = (g \circ f)_*$$

in $\mathcal{Z}^{\dim Z}(X \times Z)$.

Proof. Let U be an open subset of X on which f, hence also $g \circ f$, is defined. As explained in Section 1.3, we have an equality of reduced closed subschemes

$$\Gamma_{g \circ f} = p_{UZ}(\Gamma_f \times Z \cap X \times \Gamma_g).$$

Since Y is proper, $p_{UZ}(\Gamma_f \times Z \cap X \times \Gamma_g)$ is dense in $p_{XZ}(\overline{\Gamma}_f \times Z \cap X \times \Gamma_g) = g_* \circ f_*$, hence the conclusion. □

1.5. ***Effective pure motives.*** We now define as usual the category of effective pure motives $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$ relative to $(A, \sim)$ as the pseudoabelian envelope of $\mathbf{Cor}_{\sim}(F, A)$. We denote the composition of (1.3.1) with the pseudoabelianisation functor by $h_{\sim}$. If $\sim = \mathrm{rat}$, we usually abbreviate $h_{\sim}$ to h.

In $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$ we have

- $h_\sim(\operatorname{Spec} F) = \mathbb{1}$ (the unit object for the tensor structure);
- $h_\sim(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$, where $\mathbb{L}$ is the *Lefschetz motive*.

If $n \geq 0$, we write $M(n)$ for the motive $M \otimes \mathbb{L}^{\otimes n}$ (beware that the "standard" notation is $M(-n)$!)

We then have the formula, for two smooth projective X, Y and integers $p, q \geq 0$,

$$\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)(h_\sim(X)(p), h_\sim(Y)(q)) = \mathcal{Z}^{\dim Y+q-p}_\sim(X \times Y). \tag{1.5.1}$$

In particular, the endofunctor $-\otimes \mathbb{L}$ of $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ is fully faithful.

If $f : X \to Y$ is a morphism, then the correspondence $[{}^t\Gamma_f] \in \mathcal{Z}^{\dim Y}(Y \times X)$ obtained by the "switch" defines a morphism $f^* : h_\sim(Y)(\dim X) \to h_\sim(X)(\dim Y)$, i.e., from $h_\sim(Y)$ to $h_\sim(X)(\dim Y - \dim X)$ or from $h_\sim(Y)(\dim X - \dim Y)$ to $h_\sim(X)$ according to the sign of $\dim X - \dim Y$. In particular, if f has relative dimension 0 then f^* maps $h_\sim(Y)$ to $h_\sim(X)$. We similarly define f^* for a rational map f.

We recall the well-known lemma:

Lemma 1.5.2. *Suppose that f is generically finite of degree d. Then $f_* \circ f^* = d1_Y$.*

Proof. It suffices to prove this for the action on cycles, and then the lemma follows by Manin's identity principle [Scholl 1994, Section 2]. Let $\alpha \in \mathcal{Z}^*_\sim(Y, A)$. By the projection formula,

$$f_* f^*(\alpha) = \alpha \cdot f_*(1).$$

But $f_*(1) \in \mathcal{Z}^0_\sim(Y, A)$ may be computed after restriction to any open subset U of X, and for U small enough it is clear that $f_*(1) = d$. □

1.6. ***Pure motives.*** The category $\mathbf{Mot}_\sim(F, A)$ is now obtained from $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ by inverting the endofunctor $-\otimes \mathbb{L}$, i.e., adjoining a $\otimes$-quasi-inverse $\mathbb{T}$ of $\mathbb{L}$ (the Tate motive) to $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$. The resulting category is rigid and the functor $\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A) \to \mathbf{Mot}_\sim(F, A)$ is fully faithful; we refer to [Scholl 1994] for details. We still write $h_\sim(X)$ for the image of $h_\sim(X)$ in $\mathbf{Mot}_\sim(F, A)$.

1.7. ***Pure motives and purely inseparable extensions.*** This subsection will be needed for the proof of Remarks 2.3.10 below. It shows that extending scalars along a purely inseparable extension is harmless as long as the exponential characteristic is inverted.

Lemma 1.7.1. *Let $f : X \to Y$ be a finite, flat and radicial morphism* [Grothendieck and Dieudonné 1971, Définition 3.7.2] *between smooth projective F-varieties. Let $(A, \sim)$ be an adequate pair, with p invertible in A (where p is the exponential characteristic of F).*

(a) *$f_* : \mathcal{Z}^\sim_*(X, A) \to \mathcal{Z}^\sim_*(Y, A)$ is an isomorphism.*

(b) *$f_* : h(X) \to h(Y)$ is an isomorphism in $\mathbf{Cor}_\sim(F, A)$.*

Proof. Let p^n be the generic degree of f. We have $f_* f^* = f^* f_* = p^n$ (on the level of algebraic cycles), hence (a). Part (b) follows by Manin's identity principle (Yoneda lemma). □

Proposition 1.7.2. *Let K/F be a purely inseparable extension. Then, for any adequate pair $(A, \sim)$ as in Lemma 1.7.1, the extension of scalars functors*

$$\mathbf{Cor}_\sim(F, A) \to \mathbf{Cor}_\sim(K, A),$$
$$\mathbf{Mot}_\sim^{\text{eff}}(F, A) \to \mathbf{Mot}_\sim^{\text{eff}}(K, A),$$
$$\mathbf{Mot}_\sim(F, A) \to \mathbf{Mot}_\sim(K, A),$$

are equivalences of categories.

Proof. It suffices to show this for the first functor. Let X, Y be two smooth projective F-varieties. Then, for any finite subextension L/F of K/F, the morphism $(X \times_F Y)_L \to X \times_F Y$ is finite, flat and radicial; by Lemma 1.7.1(a) and a limit argument, this implies that the functor is fully faithful. For its essential surjectivity, we steal an idea from [Lang 1959, Chapter VIII, Section 1, proof of Theorem 2]. Let X be a smooth projective K-variety. Then X is defined over a finite subextension L/F of K/F. Let $p^n = [L : F]$, and let Φ_L be the absolute Frobenius of L. The relative Frobenius morphism (an L-morphism)

$$X \to (\Phi_L^n)^* X$$

is finite, flat[2] and radicial; by Lemma 1.7.1(b), $h(X) \to h((\Phi_L^n)^* X)$ is an isomorphism in $\mathbf{Cor}_\sim(L, A)$, hence also in $\mathbf{Cor}_\sim(K, A)$. Since $\Phi_L^n : \operatorname{Spec} L \to \operatorname{Spec} L$ factors through $\operatorname{Spec} F$, $(\Phi_L^n)^* X$ is defined over F, proving that the functor is essentially surjective. □

1.8. ***Image motives.*** In the study of projective homogeneous varieties, several people (starting with Vishik) have been led to introduce the following:

Definition 1.8.1. Let X be a smooth projective variety. We write

$$\bar{\mathcal{Z}}^*_\sim(X, A) = \operatorname{Im}(\mathcal{Z}^*_\sim(X, A) \to \mathcal{Z}^*_\sim(X_{F_s}, A)),$$

where F_s is a separable closure of F.

Using correspondences based on these groups, we define $\overline{\mathbf{Mot}}_\sim(F, A)$, etc. This is mainly interesting when $A = \mathbb{Z}$ or $\mathbb{Z}/p$: for $A = \mathbb{Q}$ the extension of scalars map is injective (by a transfer argument).

[2] To see this, one may use the fact that X is locally isomorphic to $\mathbb{A}^n$ for the étale topology.

2. Pure birational motives

2.1. ***First approach: localisation.*** The first idea to define a notion of pure birational motives is to localise $\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$ with respect to stable birational morphisms as in [Kahn and Sujatha 2015a], hence getting a functor

$$S_r^{-1}\,\mathbf{Sm}_{\amalg}^{\mathrm{proj}}(F) \to S_r^{-1}\,\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A).$$

This idea turns out to be the good one in all important cases, but to see this we first need some preliminary work. We start by reviewing the sets of morphisms used in [Kahn and Sujatha 2015a, Section 1.7]:

- S_b: birational morphisms;
- S_h: projections of the form $X \times (\mathbb{P}^1)^n \to X$;
- S_r: stably birational morphisms, where $s \in S_r$ if and only if s is dominant and gives a purely transcendental function field extension;

to which we adjoin

- S_b^w: compositions of blow-ups with smooth centres;
- $S_r^w = S_b^w \cup S_h$.

These morphisms, defined for connected varieties in [Kahn and Sujatha 2015a], extend trivially to the categories of Definition 1.2.1 as explained in [Kahn and Sujatha 2007, Corollary 6.3]. More precisely, if S is a set of morphisms of $\mathbf{Sm}(F)$, we define $S^{\amalg} \subset \mathbf{Sm}_{\amalg}(F)$ as the set of those morphisms which are dominant and whose connected components are all in S. For simplicity, we shall write S rather than $S^{\amalg}$ in the sequel.

By Lemma 1.2.2 and [Kahn and Sujatha 2007, Theorem 6.4], the localisation results of [Kahn and Sujatha 2007; 2015a] extend to the category $\mathbf{Sm}_{\amalg}(F)$ and, moreover, the functors

$$S^{-1}\,\mathbf{Sm}(F) \to S^{-1}\,\mathbf{Sm}_{\amalg}(F)$$

identify the right-hand side with the "finite coproduct envelope" of the left-hand side. Similarly for their analogues with decorations $\mathbf{Sm}^{\%}$.

We shall view the above morphisms as correspondences via the graph functor. We introduce two more sets which are convenient here:

Definition 2.1.1. We write $\tilde{S}_b$ and $\tilde{S}_r$ for the sets of dominant rational maps which induce, respectively, an isomorphism of function fields and a purely transcendental extension. We let these rational maps act on pure motives via their graphs, as in Section 1.4.

Thus we have a diagram of inclusions of morphisms on $\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$:

$$\begin{array}{ccccc} S_b^w & \subset & S_b^w \cup S_h & = & S_r^w \\ \cap & & \cap & & \cap \\ S_b & \subset & S_b \cup S_h & \subset & S_r \\ \cap & & \cap & & \cap \\ \tilde{S}_b & \subset & \tilde{S}_b \cup S_h & \subset & \tilde{S}_r \end{array} \tag{2.1.2}$$

Let us immediately notice:

Proposition 2.1.3. *Let S be one of the systems of morphisms in (2.1.2). Then the category $S^{-1}\,\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$ is an A-linear category provided with a tensor structure, compatible with the corresponding structures of $\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$ via the localisation functor.*

Proof. This follows from Theorem A.3.4, Proposition A.1.2 and the fact that elements of S are stable under disjoint unions and products. □

2.2. *Second approach: the Lefschetz ideal.*

Definition 2.2.1. We denote by $\mathcal{L}_\sim$ the ideal of $\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$ consisting of those morphisms which factor through some object of the form $P(1)$; this is the *Lefschetz ideal*. It is a monoidal ideal (i.e., it is closed with respect to composition and tensor products on the left and on the right).

Remark 2.2.2. In any additive category $\mathcal{A}$ there is a notion of product of two ideals $\mathcal{I}, \mathcal{J}$:

$$\mathcal{I} \circ \mathcal{J} = \langle f \circ g \mid f \in \mathcal{I}, g \in \mathcal{J} \rangle.$$

If $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ and $\mathcal{J} = \{f \mid f \text{ factors through some } A \in \mathcal{B}\}$, then $\mathcal{J}$ is idempotent because it is generated by idempotent morphisms, namely the identity maps of the objects of $\mathcal{B}$. In $\mathcal{A} = \mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$, this applies to $\mathcal{L}_\sim$.

On the other hand, in a tensor additive category $\mathcal{A}$ there is also the tensor product of two ideals $\mathcal{I}, \mathcal{J}$: for $A, B \in \mathcal{A}$,

$$(\mathcal{I} \otimes \mathcal{J})(A, B) = \langle \mathcal{A}(E \otimes F, B) \circ (\mathcal{I}(C, E) \otimes \mathcal{J}(D, F)) \circ \mathcal{A}(A, C \otimes D) \rangle,$$

where C, D, E, F run through all objects of $\mathcal{A}$. Coming back to $\mathcal{A} = \mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$, we have $\mathcal{L}_\sim \otimes \mathcal{L}_\sim = \mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)(2) \neq \mathcal{L}_\sim \circ \mathcal{L}_\sim = \mathcal{L}_\sim$. This is in sharp contrast with the case where $\mathcal{A}$ is rigid [André and Kahn 2002, Lemme 6.15].

Proposition 2.2.3. (a) *The localisation functor*

$$\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A) \to (S_b^w)^{-1}\,\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$$

factors through $\mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)/\mathcal{L}_\sim$.

(b) *The functors*
$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim} \to (S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A) \to (S_r^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$
are both isomorphisms of categories.

(c) *The functor*
$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim} \to S_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$
is full.

(d) *For any $s \in \tilde{S}_r$, s_* becomes invertible in $\tilde{S}_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$.*

Proof. (a) By Proposition 2.1.3, it is sufficient to show that $\mathbb{L} \mapsto 0$ in
$$(S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A).$$
Here as in the proof of (b) we shall use the following formula of Manin [1968, Section 9, Corollary, p. 463]: if $p : \tilde{X} \to X$ is a blow-up with smooth centre $Z \subset X$ of codimension n, then
$$h^{\mathrm{eff}}_{\sim}(\tilde{X}) \simeq h^{\mathrm{eff}}_{\sim}(X) \oplus \bigoplus_{i=1}^{n-1} h^{\mathrm{eff}}_{\sim}(Z) \otimes \mathbb{L}^{\otimes i}, \tag{2.2.4}$$
where projecting the right-hand side onto $h^{\mathrm{eff}}_{\sim}(X)$ we get p_*.

In (2.2.4), take $X = \mathbb{P}^2$ and for $\tilde{X}$ the blow-up of X at (say) $Z = \{(1:0:0)\}$. Since p is invertible in $(S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$, we get $\mathbb{L} = 0$ in this category as requested.

(b) It suffices to show that morphisms of S_r^w become invertible in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim}$, which immediately follows from (2.2.4) and the easier projective line formula.

(c) It suffices to show that members of S_b have right inverses in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$; this follows from Lemma 1.5.2.

(d) Let $g : X \dashrightarrow Y$ be an element of $\tilde{S}_r$. Then X is birational to $Y \times (\mathbb{P}^1)^n$ for some $n \geq 0$, and if $f : X \dashrightarrow Y \times (\mathbb{P}^1)^n$ is the corresponding birational map, its composition with the first projection π is g. By Lemma 1.4.1, it suffices to show that π_* is invertible in $\tilde{S}_b^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$, which follows from (b). □

Corollary 2.2.5. *Let $M = \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$.*

(a) *Diagram (2.1.2) induces a commutative diagram of categories and functors*
$$\begin{tikzcd}
M/\mathcal{L}_{\sim} \arrow[r, "\sim"] & (S_b^w)^{-1}M \arrow[r, "\sim"] \arrow[d, "\text{full}"'] & (S_b^w \cup S_h)^{-1}M \arrow[r, "\sim"] \arrow[d, "\text{full}"'] & (S_r^w)^{-1}M \arrow[d] \\
& S_b^{-1}M \arrow[r, "\sim"] \arrow[d] & (S_b \cup S_h)^{-1}M \arrow[r] \arrow[d] & S_r^{-1}M \arrow[d] \\
& \tilde{S}_b^{-1}M \arrow[r, "\sim"] & (\tilde{S}_b \cup S_h)^{-1}M \arrow[r, "\sim"] & \tilde{S}_r^{-1}M
\end{tikzcd} \tag{2.2.6}$$

where the functors with a sign $\sim$ are isomorphisms of categories and the indicated functors are full.

(b) *If* char $F = 0$, *all functors are isomorphisms of categories.*

Proof. (a) follows from Proposition 2.2.3; (b) follows from Hironaka's resolution of singularities (see [Kahn and Sujatha 2015a, Lemma 1.7.1]). □

Remark 2.2.7. Tracking isomorphisms in diagram (2.2.6), one sees that without assuming resolution of singularities we get a priori 4 different categories of "pure birational motives". If $p : \tilde{X} \to X$ is a birational morphism, then at least $h_\sim(X)$ is a direct summand of $h_\sim(\tilde{X})$ by Lemma 1.5.2. However it is not clear how to prove that the other summand is divisible by $\mathbb{L}$ without using resolution. We shall get by for special pairs $(A, \sim)$ in Theorem 2.4.2 below, using the alteration theorem of de Jong and Gabber.

We now introduce:

Definition 2.2.8. The category of *pure birational motives* is

$$\mathbf{Mot}^{\mathrm{b}}_\sim(F, A) = \big(\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)/\mathcal{L}_\sim\big)^\natural.$$

We also set

$$\mathbf{Chow}^{\mathrm{eff}}(F, A) = \mathbf{Mot}^{\mathrm{eff}}_{\mathrm{rat}}(F, A),$$
$$\mathbf{Chow}^{\mathrm{b}}(F, A) = \mathbf{Mot}^{\mathrm{b}}_{\mathrm{rat}}(F, A).$$

When $A = \mathbb{Z}$, we abbreviate this notation to $\mathbf{Chow}^{\mathrm{eff}}(F)$ and $\mathbf{Chow}^{\mathrm{b}}(F)$.

We note:

Proposition 2.2.9. *Taking pseudoabelian envelopes, the first functor in Corollary 2.2.5(a) induces an isomorphism of categories*

$$\mathbf{Mot}^{\mathrm{b}}_\sim(F, A) \xrightarrow{\sim} \big((S^w_b)^{-1}\,\mathbf{Cor}_\sim(F, A)\big)^\natural.$$

In particular, the functor $(S^w_b)^{-1}\,\mathbf{Cor}_\sim(F, A) \to (S^w_b)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_\sim(F, A)$ *is fully faithful and the functor* $\mathbf{Cor}_\sim(F, A) \to S_b^{-1}\,\mathbf{Cor}_\sim(F, A)$ *is full.*

Proof. All follows from Lemma A.4.1, except for the last statement, which follows from Proposition 2.2.3(c). □

In Section 4, we shall examine to what extent it is really necessary to adjoin idempotents in Definition 2.2.8.

2.3. ***Third approach: extendible pairs.*** To go further, we need to restrict the adequate equivalence relation we are using:

Definition 2.3.1. An adequate pair $(A, \sim)$ is *extendible* if

- $\sim$ is defined on cycles over arbitrary quasiprojective F-varieties;

- it is preserved by inverse image under flat morphisms and direct image under proper morphisms;
- if X is smooth projective, Z is a closed subset of X and $U = X - Z$, then the sequence

$$\mathcal{Z}_n^{\sim}(Z, A) \to \mathcal{Z}_n^{\sim}(X, A) \to \mathcal{Z}_n^{\sim}(U, A) \to 0 \tag{2.3.2}$$

  is exact.

Note that in (2.3.2), surjectivity always holds because this is already true on the level of cycles. So the issue is exactness at $\mathcal{Z}_n^{\sim}(X, A)$.

Examples 2.3.3. (a) Rational equivalence (with any coefficients) is extendible.

(b) Algebraic equivalence (with any coefficients) is extendible; see [Fulton 1984, Example 10.3.4].

(c) The status of homological equivalence is very interesting:

(1) Under the standard conjecture that homological and numerical equivalences agree, homological equivalence with respect to a "classical" Weil cohomology theory is extendible if char $F = 0$ [Corti and Hanamura 2000, Proposition 6.7]. The proof involves resolution of singularities and the weight spectral sequences for Borel–Moore Hodge homology, their degeneration at E_2 and the semisimplicity of numerical motives [Jannsen 1992]. Presumably the same arguments work in characteristic p by using de Jong's alteration theorem [1996] instead of Hironaka's resolution of singularities; we thank Yves André for pointing this out. See [Voisin 2013, Proposition 1.6] for a more precise statement and a different proof.

(2) It seems that the Corti–Hanamura argument implies unconditionally that André's motivated cycles [1996] verify the axioms of an extendible pair.

(3) For Betti cohomology with integral coefficients or l-adic cohomology with $\mathbb{Z}_l$ coefficients, homological equivalence is not extendible. (Counterexample: $F = \mathbb{C}$, $n = 1$, Z a general surface of degree ≥ 4 in $\mathbb{P}^3$; this example goes back to Kollár [1992, p. 134].) This is closely related to the failure of the Hodge or Tate conjecture integrally for Z (see [Soulé and Voisin 2005, Section 2]).

(4) Hodge cycles with coefficients in $\mathbb{Q}$ verify the axioms of an extendible pair: similarly to (1), the proof involves resolving the singularities of Z in (2.3.2) and using the semisimplicity of polarisable pure Hodge structures. See also [Jannsen 1994]. We are indebted to Claire Voisin for explaining these last two points.

(5) Taking Tate cycles for l-adic cohomology, the same argument works if we assume the semisimplicity of Galois action on the cohomology of smooth projective varieties.

Lemma 2.3.4. *If $(A, \sim)$ verifies the first two conditions of Definition 2.3.1, then $(A, \mathrm{rat}) \geq (A, \sim)$ (also over arbitrary quasiprojective varieties).*

Proof. Again, this follows from [Fulton 1984, Example 1.7.5]. □

Proposition 2.3.5. *Let $(A, \sim)$ be an extendible pair. For two smooth projective varieties X, Y, let $\mathcal{I}_\sim(X, Y)$ be the subgroup of $\mathcal{Z}_\sim^{\dim Y}(X \times Y, A)$ consisting of those classes vanishing in $\mathcal{Z}_\sim^{\dim Y}(U \times Y, A)$ for some open subset U of X. Then $\mathcal{I}_\sim$ is a monoidal ideal in $\mathbf{Cor}_\sim(F, A)$.*

Proof. Note that by Lemma 2.3.4 and the third condition of Definition 2.3.1, the map $\mathcal{I}_{\mathrm{rat}}(X, Y) \to \mathcal{I}_\sim(X, Y)$ is surjective for any X, Y; this reduces us to the case $\sim = \mathrm{rat}$. We further reduce immediately to $A = \mathbb{Z}$.

Let X, Y, Z be three smooth projective varieties. If U is an open subset of X, it is clear that the usual formula defines a composition of correspondences

$$CH^{\dim Y}(U \times Y) \times CH^{\dim Z}(Y \times Z) \to CH^{\dim Z}(U \times Z)$$

and that this composition commutes with restriction to smaller and smaller open subsets. Passing to the limit on U, we get a composition

$$CH^{\dim Y}(Y_{F(X)}) \times CH^{\dim Z}(Y \times Z) \to CH^{\dim Z}(Z_{F(X)})$$

or

$$CH_0(Y_{F(X)}) \times CH^{\dim Z}(Y \times Z) \to CH_0(Z_{F(X)}).$$

Here we used the fact that (codimensional) Chow groups commute with filtering inverse limits of schemes; see [Bloch 2010].

We now need to prove that this pairing factors through

$$CH_0(Y_{F(X)}) \times CH^{\dim Z}(V \times Z)$$

for any open subset V of Y. One checks that it is induced by the standard action of correspondences in $CH^{\dim Z}(Y_{F(X)} \times_{F(X)} Z_{F(X)})$ on groups of 0-cycles. Hence it is sufficient to show that the standard action of correspondences factors as indicated, and up to changing the base field we may replace $F(X)$ by F.

We now show that the pairing

$$CH_0(Y) \times CH^{\dim Z}(Y \times Z) \to CH_0(Z)$$

factors as indicated. The proof is a variant of Fulton's proof [1984, Example 16.1.11] of the Colliot-Thélène–Coray theorem [1979] that CH_0 is a birational invariant of smooth projective varieties. Let M be a proper closed subset of Y and let $i : M \to Y$ be the corresponding closed immersion. We have to prove that for any $\alpha \in CH_0(Y)$ and $\beta \in CH_{\dim Y}(M \times Z)$,

$$(i \times 1_Z)_*(\beta)(\alpha) := (p_2)_*((i \times 1_Z)_*\beta \cdot p_1^*\alpha) = 0,$$

where p_1 and p_2 are respectively the first and second projections on $Y \times Z$.

We shall actually prove that $(i \times 1_Z)_*\beta \cdot p_1^*\alpha = 0$. For this, we may assume that α is represented by a closed point $y \in Y$ and β by some integral variety $W \subseteq M \times Z$. Then $(i \times 1_Z)_*\beta \cdot p_1^*\alpha$ has support in $(i \times 1_Z)(W) \cap (\{y\} \times Z) \subset (M \times Z) \cap (\{y\} \times Z)$. If $y \notin M$, this subset is empty and we are done. Otherwise, up to rational equivalence, we may replace y by a 0-cycle disjoint from M (see [Roberts 1972]), and we are back to the previous case.

This shows that $\mathcal{I}_\sim$ is an ideal of $\mathbf{Cor}_\sim(F, A)$. The fact that it is a monoidal ideal is essentially obvious. □

Definition 2.3.6. For an extendible pair $(A, \sim)$, we abbreviate $\mathbf{Cor}_\sim(F, A)/\mathcal{I}_\sim$ (resp. $(\mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)/\mathcal{I}_\sim)^\natural$) into $\mathbf{Cor}_\sim^{\mathrm{o}}(F, A)$ (resp. $\mathbf{Mot}_\sim^{\mathrm{o}}(F, A)$). (Here o stands for "open".) We write $h_\sim^{\mathrm{o}}(X)$ for the image of $h_\sim(X)$ in $\mathbf{Mot}_\sim^{\mathrm{o}}(F, A)$. We also set $\mathbf{Chow}^{\mathrm{o}}(F, A) = \mathbf{Mot}_{\mathrm{rat}}^{\mathrm{o}}(F, A)$ and $\mathbf{Chow}^{\mathrm{o}}(F) = \mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z})$.

For future reference, let us record here the value of the Hom groups in the most important case, that of rational equivalence (see also Remark 2.3.10(2) below):

Lemma 2.3.7. *We have*

$$\mathbf{Cor}_{\mathrm{rat}}^{\mathrm{o}}(F, A)([X], [Y]) = CH_0(Y_{F(X)}) \otimes A.$$

Proposition 2.3.8. *In* $\mathbf{Cor}_\sim^{\mathrm{o}}(F, A)$:

(a) $(g \circ f)_* = g_* \circ f_*$ *for any composable rational maps* $X \overset{f}{\dashrightarrow} Y \overset{g}{\dashrightarrow} Z$.

(b) [Fulton 1984, Example 16.1.11] $f^* f_* = 1_X$ *and* $f_* f^* = 1_Y$ *for any birational map* $f : X \dashrightarrow Y$.

(c) *Morphisms of* $\tilde{S}_r$ (see Definition 2.1.1) *are invertible.*

Proof. (a) Let F be the fundamental set of f, G be the fundamental set of g, $U = X - F$, $V = Y - G$. By assumption, $f(U) \cap V \neq \varnothing$, hence $W = f^{-1}(V)$ is a nonempty open subset of U, on which $g \circ f$ is a morphism.

Let us abuse notation and still write f for the morphism f_U, etc. Then, by definition,

$$g_* \circ f_* = (p_{XZ})_*((\overline{\Gamma}_f \times Z) \cap (X \times \overline{\Gamma}_g))$$

(note that the two intersected cycles are in good position). This cycle clearly contains $(g \circ f)_* = \overline{\Gamma}_{g \circ f}$ as a closed subset. One sees immediately that the restriction of $g_* \circ f_*$ and $(g \circ f)_*$ to $W \times Z$ are equal.

(b) This is proven in the same way (or is a special case of (a)).

(c) Let $g : X \dashrightarrow Y$ be an element of $\tilde{S}_r$. Then X is birational to $Y \times (\mathbb{P}^1)^n$ for some $n \geq 0$, and if $f : X \dashrightarrow Y \times (\mathbb{P}^1)^n$ is a birational map, its composition with the first projection π is g. By (a) and (b), it suffices to show that π_* is invertible in

$\mathbf{Cor}_\sim(F, A)/\mathcal{I}_\sim$. For this we may reduce to $n = 1$ and even to $Y = \operatorname{Spec} F$ since $\mathcal{I}_\sim$ is a monoidal ideal. Let $s : \operatorname{Spec} F \to \mathbb{P}^1$ be the ∞ section; it suffices to show that $(s \circ \pi)_* = 1_{\mathbb{P}^1}$. But the cycle $(s \circ \pi)_* - 1_{\mathbb{P}^1}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is linearly equivalent to $\infty \times \mathbb{P}^1$ (this is the idempotent defining the Lefschetz motive), and the latter cycle vanishes when restricted to $\mathbb{A}^1 \times \mathbb{P}^1$. □

We shall also need the following lemma in the proof of Proposition 5.1.4(c).

Lemma 2.3.9. *Let L/K be an extension of function fields over F, with $K = F(X)$ and $L = F(Y)$ for X, Y two smooth projective F-varieties. Let $\varphi : Y \dashrightarrow X$ be the rational map corresponding to the inclusion $K \hookrightarrow L$. Let Z be another smooth projective F-variety. Then the map*

$$\mathbf{Chow}^{\mathrm{o}}(F, A)(h^{\mathrm{o}}(X), h^{\mathrm{o}}(Z)) \to \mathbf{Chow}^{\mathrm{o}}(F, A)(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(Z))$$

given by composition with $\varphi_* : h^{\mathrm{o}}(Y) \to h^{\mathrm{o}}(X)$ (see Section 1.4) *coincides via Lemma 2.3.7 with the base-change map $CH_0(Z_K) \otimes A \to CH_0(Z_L) \otimes A$.*

Proof. Let $V \subseteq Y$ and $U \subseteq X$ be open subsets such that φ is defined on V and $\varphi(V) \subseteq U$. Up to shrinking U, we may assume that φ is flat [Grothendieck and Dieudonné 1966, Théorème 11.1.1]. As in the proof of Proposition 2.3.5, the composition of correspondences induces a pairing

$$CH^{\dim X}(V \times U) \times CH^{\dim Z}(U \times Z) \to CH^{\dim Z}(V \times Z),$$

and the action of $\varphi_* \in CH^{\dim X}(V \times U)$ on $\alpha \in CH^{\dim Z}(U \times Z)$ is given by the flat pull-back of cycles. Therefore, φ_* induces in the limit the flat pull-back of 0-cycles from $CH_0(Z_K)$ to $CH_0(Z_L)$. □

Remarks 2.3.10. (1) Propositions 2.3.5 and 2.3.8(a) were independently observed by Markus Rost in the case $\sim=$rat [Merkurjev 2001, Proposition 3.1 and Lemma 3.3]. We are indebted to Karpenko for pointing this out and for referring us to Merkurjev's preprint.

(2) In $\mathbf{Cor}^{\mathrm{o}}_\sim(F, A)$, morphisms are by definition given by the formula

$$\mathbf{Cor}^{\mathrm{o}}_\sim(F, A)([X], [Y]) = \varinjlim_{U \subseteq X} \mathcal{Z}^{\dim Y}_\sim(U \times Y, A).$$

The latter group maps onto $\mathcal{Z}_0^\sim(Y_{F(X)}, A)$. If $\sim =$ rat, this map is an isomorphism (see Lemma 2.3.7). For other equivalence relations, this is far from being the case: for example, if $\sim =$ alg, F is algebraically closed, X, Y are two curves and, say, $A = \mathbb{Z}$, then

$$\begin{aligned}\mathcal{Z}^1_{\mathrm{alg}}(X \times Y, \mathbb{Z}) &= \mathrm{NS}(X \times Y) = \mathrm{NS}(X) \oplus \mathrm{NS}(Y) \oplus \operatorname{Hom}(J_X, J_Y)\\ &= \mathbb{Z} \oplus \mathbb{Z} \oplus \operatorname{Hom}(J_X, J_Y),\end{aligned}$$

where NS is the Néron–Severi group, and J_X and J_Y are the Jacobians of X and Y. On the other hand,

$$\mathcal{Z}_0^{\mathrm{alg}}(Y_{F(X)}, \mathbb{Z}) = \mathrm{NS}(Y_{F(X)}) = \mathbb{Z}.$$

When we remove a point from X, we kill the factor $\mathrm{NS}(X) = \mathbb{Z}$. But any two points of X are algebraically equivalent, so removing further points does not modify the group any further. Hence

$$\varinjlim_{U \subseteq X} \mathcal{Z}_{\mathrm{alg}}^{\dim Y}(U \times Y, \mathbb{Z}) = \mathbb{Z} \oplus \mathrm{Hom}(J_X, J_Y).$$

We thank Colliot-Thélène for helping clarify this matter.

2.4. ***The main theorem.*** We now extend the ideal $\mathcal{I}_\sim$ from

$$\mathbf{Cor}_\sim(F, A) \quad \text{to} \quad \mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)$$

in the usual way (see [André and Kahn 2002, Lemme 1.3.10]), without changing notation. By Propositions 2.2.3(a) and 2.3.8, we get a composite functor

$$\mathbf{Mot}_\sim^{\mathrm{b}}(F, A) \to (\tilde{S}_r^{-1}\, \mathbf{Mot}_\sim^{\mathrm{eff}}(F, A))^\natural \to \mathbf{Mot}_\sim^{\mathrm{o}}(F, A) \tag{2.4.1}$$

for any extendible pair $(A, \sim)$. Since both categories are (idempotent completions of) full images of $\mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)$, this functor is automatically full. We are going to show that it is an equivalence of categories in some important cases.

Theorem 2.4.2. *Let $(A, \sim)$ be an extendible pair. Suppose that the exponential characteristic p of F is invertible in A. Then the functor (2.4.1) is an isomorphism of categories.*

Proof.[3] We have to show that $\mathcal{I}_\sim(M, N) \subseteq \mathcal{L}_\sim(M, N)$ for any $M, N \in \mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)$. Proposition 1.7.2 reduces us to the case where F is *perfect*. Clearly we may assume $M = h_\sim(X)$, $N = h_\sim(Y)$ for two smooth projective varieties X, Y.

Let $f \in \mathcal{I}_\sim(h_\sim(X), h_\sim(Y))$. By the third condition in Definition 2.3.1, the cycle class $f \in \mathcal{Z}^\sim_{\dim X}(X \times Y, A)$ is of the form $(i \times 1_Y)_* g$ for some closed immersion $i : Z \to X$, where $g \in \mathcal{Z}^\sim_{\dim X}(Z \times Y, A)$. Let $\tilde{g}$ be a cycle representing g. Write $\tilde{g} = \sum_k a_k g_k$, with $a_k \in A$ and g_k irreducible. Then $(i \times 1_Y)_*(g_k) \in \mathcal{I}_\sim(h_\sim(X), h_\sim(Y))$. This reduces us to the case where g is represented by an irreducible cycle $\tilde{g}$.

Choose Z minimal among the closed subsets of X such that $\tilde{g}$ is supported on $Z \times Y$. In particular, Z is irreducible.

Consider Z with its reduced structure. Let l be a prime number different from p; by Gabber's refinement of de Jong's theorem [Illusie and Temkin 2014, Théorème X.2.1], we may choose a proper, generically finite morphism $\pi_l : \widetilde{Z}_l \to Z$ where

[3] We thank N. Fakhruddin for his help, which removes the recourse to Chow's moving lemma in [Kahn and Sujatha 2002].

$\widetilde{Z}_l$ is smooth projective (irreducible) and π_l is an alteration of generic degree d_l prime to l. (Recall that an alteration is a proper, generically finite morphism.)

By the minimality of Z, the support of $\tilde{g}$ has nonempty intersection $\tilde{g}_1$ with $V \times Y$, where $V = Z - (Z_{\mathrm{sing}} \cup T)$ with Z_{sing} the singular locus of Z and T the closed subset over which π_l is not finite. Let $\pi_V : \pi_l^{-1}(V) \to V$ be the map induced by π_l; note that π_V is flat since V and $\pi_l^{-1}(V)$ are smooth. We then have an equality of cycles

$$d_l \tilde{g}_1 = (\pi_V \times 1_Y)_*(\pi_V \times 1_Y)^* \tilde{g}_1.$$

Let γ_l be the closure of $(\pi_V \times 1_Y)^* \tilde{g}_1$ in $\widetilde{Z}_l$.[4] We get an equality of cycles (the support of $(\pi_V \times 1_Y)_*(\pi_V \times 1_Y)^* \tilde{g}_1$ is dense in that of $(\pi_l \times 1_Y)_* \gamma_l$):

$$d_l \tilde{g} = (\pi_l \times 1_Y)_* \gamma_l.$$

Let $d = \gcd_l(d_l)$, which is a power of p; then $d = \gcd(d_{l_1}, \dots, d_{l_r})$ for some finite set of primes $\{l_1, \dots, l_r\}$. For simplicity, write $Z_{l_i} = Z_i$, $\pi_{l_i} = \pi_i$ and $\gamma_{l_i} = \gamma_i$.

Let $h_i = d^{-1}[\gamma_i] \in \mathcal{Z}^{\sim}_{\dim X}(\widetilde{Z}_i \times Y, A)$. Choose $a_1, \dots, a_r \in \mathbb{Z}$ such that $d = \sum_i a_i d_i$, so that

$$f = \sum_i a_i ((i \circ \pi_i) \times 1_Y)_* h_i.$$

Then the correspondence $f \in \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F)(h_\sim(X), h_\sim(Y))$ factors as

$$h_\sim(X) \xrightarrow{(i\circ\pi)^*} h_\sim\Big(\coprod \widetilde{Z}_i\Big)(\dim X - \dim Z) \xrightarrow{(h_i)} h_\sim(Y)$$

(see (1.5.1)), which concludes the proof. □

Corollary 2.4.3. *Under the assumptions of Theorem 2.4.2, all the categories of diagram (2.2.6) are isomorphic to* $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{I}_\sim$.

Proof. By Proposition 2.2.3(b) and (d) we already know that the categories

$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_\sim, \quad (S_b^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A) \quad \text{and} \quad (S_r^w)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$

are isomorphic and that

$$(\tilde{S}_b)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A) \quad \text{and} \quad (\tilde{S}_r)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$

are isomorphic. We also know that the functor

$$\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_\sim \to (S_b)^{-1}\,\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$$

is full (Proposition 2.2.3(c)); by Theorem 2.4.2, this implies that it is an isomorphism. To conclude the proof, it is sufficient to show that any morphism of $\tilde{S}_r$, hence of S_r, has a right inverse in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_\sim$ (see (2.2.6)). Since $\tilde{S}_r$ is

[4] More correctly, the cycle associated to the schematic closure of $(\pi_V \times 1_Y)^{-1}(\tilde{g}_1)$ in $\widetilde{Z}_l$: take the topological closure of each component of $(\pi_V \times 1_Y)^* \tilde{g}_1$ and keep the same multiplicities.

generated by $\tilde{S}_b$ and projections of the form $X \times \mathbb{P}^1 \to X$ (see the proof of Proposition 2.2.3(d)) and since this is obvious for these projections, we are left to prove it for elements $f : X \dashrightarrow Y$ of $\tilde{S}_b$. But we have $f_* f^* = 1_X$ in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{I}_{\sim}$ by Proposition 2.3.8(b), hence in $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)/\mathcal{L}_{\sim}$ by Theorem 2.4.2. □

2.5. ***Birational image motives.*** Based on the categories of Section 1.8, we define categories $\overline{\mathbf{Mot}}^{\mathrm{b}}_{\sim}(F, A)$. If $\sim$ is extendible and p is invertible in A, the analogue of Theorem 2.4.2 holds, with the same proof.

2.6. ***Recapitulation, comments and notation.*** In Definition 2.2.8, we associated to any admissible pair $(A, \sim)$ a category of birational motives $\mathbf{Mot}^{\mathrm{b}}_{\sim}(F, A)$. If $(A, \sim)$ is extendible (Definition 2.3.1), we introduced in Definition 2.3.6 another category $\mathbf{Mot}^{\mathrm{o}}_{\sim}(F, A)$ plus a full functor $\mathbf{Mot}^{\mathrm{b}}_{\sim}(F, A) \to \mathbf{Mot}^{\mathrm{o}}_{\sim}(F, A)$. We showed in Theorem 2.4.2 that this functor is an isomorphism of categories when the exponential characteristic p is invertible in A; in particular, this is true for any A in characteristic 0. This gives a great flexibility in computing Hom groups, as in some cases one can use their "algebraic" description in terms of killing the Lefschetz motive, and in other cases their "geometric" description as Chow groups of 0-cycles if $\sim$ is rational equivalence.

In the sequel, we commit the abuse of notation which consists of writing $\mathbf{Mot}^{\mathrm{o}}_{\sim}$ for $\mathbf{Mot}^{\mathrm{b}}_{\sim}$ even when we don't know if the pair $(A, \sim)$ is extendible (notably, when $\sim$ is numerical equivalence). We do this because we feel that keeping the distinction would create more confusion than this choice.

3. Examples

We give some examples and computations of birational motives.

3.1. ***Varieties with trivial birational motive.*** These were initially studied by Bloch and Srinivas [1983] over a universal domain. The reader should compare the following to [Kahn and Sujatha 2015a, Theorem 8.5.1]; see also [Totaro 2014, Theorem 2.1].

Proposition 3.1.1. *Let A be a connected commutative ring, and let X be a smooth projective F-variety. Then the following conditions are equivalent:*

(i) *For any smooth projective F-variety Y, $CH_0(X_{F(Y)}) \otimes A \xrightarrow{\sim} A$ (by the degree map).*

(ii) $CH_0(X_{F(X)}) \otimes A \xrightarrow{\sim} A$.

(iii) *The class of the generic point η_X in $CH_0(X_{F(X)}) \otimes A$ belongs to*

$$\operatorname{Im}(CH_0(X) \otimes A \to CH_0(X_{F(X)}) \otimes A).$$

(iv) $h^{\mathrm{o}}(X) = \mathbb{1}$ *in* $\mathbf{Chow}^{\mathrm{o}}(F, A)$.

(v) (*For* $A=\mathbb{Z}$:) $M_0(F) \xrightarrow{\sim} A^0(X, M_0)$ *for any cycle module* M.

If p *is invertible in* A, *they are also equivalent to*:

(vi) *For any extension* K/F, $CH_0(X_K)\otimes A \xrightarrow{\sim} A$.

If F *is a universal domain and* $A \supseteq \mathbb{Q}$, *they are also equivalent to*:

(vii) $CH_0(X)\otimes A \xrightarrow{\sim} A$.

(viii) $CH_0(X) \xrightarrow{\sim} \mathbb{Z}$.

(Parts of this proposition are standard; see, e.g., [Auel et al. 2013, Lemma 1.3].)

Proof. (i) ⇒ (ii) ⇒ (iii) is obvious. By Lemma 2.3.7, the map of (iii) can be translated into

$$\mathbf{Chow}^{\mathrm{o}}(F, A)(\mathbb{1}, h^{\mathrm{o}}(X)) \to \mathbf{Chow}^{\mathrm{o}}(F, A)(h^{\mathrm{o}}(X), h^{\mathrm{o}}(X))$$

via the projection $h^{\mathrm{o}}(X) \to h^{\mathrm{o}}(\operatorname{Spec} k) = \mathbb{1}$. Since η_X represents the identity endomorphism of $h^{\mathrm{o}}(X)$, (iii) means that the latter factors through $\mathbb{1}$. Since $\operatorname{End}(\mathbb{1}) = A$, the resulting idempotent endomorphism of $\mathbb{1}$ must be 0 or 1; so $h^{\mathrm{o}}(X) = 0$ or $\mathbb{1}$, but the first case is impossible as it would imply that $\eta_X = 0$, while $\deg(\eta_X) = 1$. So (iii) ⇒ (iv). Using Lemma 2.3.7 again, we get (iv) ⇒ (i).

(vi) ⇒ (i) is obvious; to prove the converse, we reduce to F perfect by using Proposition 1.7.2, and then to K/F finitely generated by a limit argument. Then K is the function field of some smooth F-variety. We argue as in the proof of Theorem 2.4.2: using [Illusie and Temkin 2014, Théorème X.2.1], we can find finite extensions L_i/K such that $L_i = F(Y_i)$ for Y_i smooth projective, such that the gcd of the $[L_i : K]$ is a power of p. Then $(CH_0(X_K)\otimes A)_{\deg=0}$ is a direct summand of $\bigoplus_i (CH_0(X_{L_i})\otimes A)_{\deg=0} = 0$ by a transfer argument, hence (vi).

(iv) ⇒ (v) ⇒ (iii): see Section 6.

It remains to prove (iii) ⇐ (vii) ⇒ (viii) when F is a universal domain, since (viii) ⇒ (vii) is obvious. The implication (vii) ⇒ (iii) is the classical Bloch–Srinivas argument [1983, Proposition 1]: X is defined over a subfield $F' \subset F$ finitely generated over the prime field; for clarity, write X' for this F'-model. Now $F'(X')$ embeds into F over F'. Since

$$\operatorname{Ker}\bigl(CH_0(X'_{F'(X')}) \to CH_0(X'_F) = CH_0(X)\bigr)$$

is torsion by a transfer argument, (vii) implies that $CH_0(X'_{F'(X')})\otimes A \xrightarrow{\sim} A$. Thus $\eta_{X'}$ is A-rationally equivalent to a closed point of X', hence (iii). If (vii) is true, then $\operatorname{Alb}(X)(F)\otimes A = 0$, where $\operatorname{Alb}(X)$ is the Albanese variety of X; this implies $\operatorname{Alb}(X) = 0$. But Roĭtman's theorem [1980b] then implies that $CH_0(X)_{\mathrm{tors}} = 0$, whence (viii). □

Corollary 3.1.2. *Conditions* (i)–(v) *of Proposition 3.1.1 are stable under products of varieties*; *so are* (vi), (vii) *and* (viii) *under the stated conditions on* A *and* F.

Proof. Indeed, this is obviously the case for condition (iv). □

Remarks 3.1.3. (1) Condition (v) of Proposition 3.1.1 can be extended to any A if we consider cycle modules with coefficients in A.

(2) Except for (iv), Corollary 3.1.2 can also be proven without reference to birational motives when $A \supseteq \mathbb{Q}$, using that the product map

$$(CH_0(X) \otimes A) \otimes (CH_0(Y) \otimes A) \to CH_0(X \times Y) \otimes A$$

is then surjective for any smooth projective X, Y: reduce to F algebraically closed by a transfer argument, when this even holds integrally.

We now give some examples. In part (3) of the following proposition, the Betti numbers $b^i(X) = \dim H^i(X)$ refer to a "classical" Weil cohomology H: Betti or de Rham in characteristic 0, crystalline in characteristic > 0, l-adic in characteristic $\neq l$. It is known that $b^i(X)$ does not depend on the choice of such a Weil cohomology.

Proposition 3.1.4. (1) *If X is retract rational, then $h^o(X) = \mathbb{1}$ in* $\mathbf{Chow}^o(F, \mathbb{Z})$.

(2) *If X is rationally chain connected, then $h^o(X) = \mathbb{1}$ in* $\mathbf{Chow}^o(F, \mathbb{Q})$.

(3) *If $h^o(X) = \mathbb{1}$ in $\mathbf{Chow}^o(F, \mathbb{Q})$, then $b^1(X) = 0$ and $b^2(X) = \rho(X)$ (the Picard number).*

(4) *If $\dim X = 2$, the converse of (3) is true if and only if X verifies Bloch's conjecture on 0-cycles.*

Proof. (1) This follows from [Kahn and Sujatha 2015a, Proposition 8.6.2] and the functor (5.1.1) below. (One could also give a direct proof.)

(2) Let $\overline{F(X)}$ be an algebraic closure of $F(X)$; then $X(\overline{F(X)})/R = *$. Since the group of 0-cycles on $X_{\overline{F(X)}}$ is generated by $X(\overline{F(X)})$, this in turn implies that $CH_0(X_{\overline{F(X)}}) \xrightarrow{\sim} \mathbb{Z}$, which implies by a transfer argument that

$$CH_0(X_{F(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}.$$

(3) Since the hypothesis and conclusion do not change by extension of F, we may assume that F is a universal domain. We use Theorem 2.4.2: in $\mathbf{Chow}^{\text{eff}} = \mathbf{Chow}^{\text{eff}}(F, \mathbb{Q})$ we get a decomposition

$$h(X) = \mathbb{1} \oplus M \otimes \mathbb{L}$$

for some $M \in \mathbf{Chow}^{\text{eff}}$. Applying the cycle class map, we get a commutative diagram

$$\begin{tikzcd} CH^1(X) \otimes K \arrow[r, equal] \arrow[d, "\mathrm{cl}^1_X"'] & CH^0(M) \otimes K \arrow[d, "\mathrm{cl}^0_M"'] \\ H^2(X) \arrow[r, equal] & H^0(M) \end{tikzcd}$$

Here K is the field of coefficients of H and, as usual, $CH^i(M) := \mathbf{Chow}^{\mathrm{eff}}(M, \mathbb{L}^i)$ (giving back the rational Chow groups of smooth projective varieties) and cl is the cycle class map; for simplicity, we neglect Tate twists on cohomology. But cl^0_M is an isomorphism, as one sees by writing M as a direct summand of $h(Y)$ for some smooth projective Y; therefore cl^1_X is an isomorphism as well. Since this map factors through the Néron–Severi group $\mathrm{NS}(X) \otimes K$, this implies $\mathrm{Pic}^0(X) = 0$ (hence $b^1(X) = 0$), and $b^2(X) = \rho(X)$ as requested.

(4) The conditions in the conclusion of (3) imply $\mathrm{Alb}(X) = 0$ and (under Bloch's conjecture) $T(X_K) = 0$ for any extension K/F, where T is the Albanese kernel; the conclusion now follows from condition (i) of Proposition 3.1.1. □

Remarks 3.1.5. (1) As noted in [Kahn 2009, Example 7.3], an Enriques surface verifies the conditions of Proposition 3.1.1 (for 2 invertible in A); this can be recovered from Proposition 3.1.4(4) in a rather silly way. On the other hand, Inose and Mizukami's [1979] and Voisin's [1992] proofs of the Bloch conjecture for some quotients of hypersurfaces by finite groups give examples of surfaces of general type having trivial birational motive (with $\mathbb{Q}$-coefficients), which shows once again how motivic information is in some sense orthogonal to geometric information related to the Kodaira dimension. For a more refined example, see remark (3) below.

(2) Applying the reasoning in the proof of Proposition 3.1.4(3) to CH^2 and CH_1, one recovers some of the representability results of [Bloch and Srinivas 1983] in a different way. (The situation considered by Bloch and Srinivas is more general, and in the present terms amounts to the following: assume that, in $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Q})$, $h^{\mathrm{o}}(X)$ is isomorphic to a direct summand of $h^{\mathrm{o}}(Y)$ for some smooth projective variety Y of dimension $n \leq 3$.)

(3) Let X be a smooth projective variety such that $h^{\mathrm{o}}(X) = \mathbb{1}$ in $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Q})$. For simplicity, assume that X has a rational point x. By condition (iii) of Proposition 3.1.1, there is an integer $N > 0$ such that $N(\eta_X - x) = 0$ in $CH_0(X_{F(X)})$. Then in $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z})$, we have

$$h^{\mathrm{o}}(X) = \mathbb{1} \oplus M \quad \text{with } N 1_M = 0.$$

Indeed, x defines an idempotent endomorphism of $h^{\mathrm{o}}(X)$ which splits off the summand $\mathbb{1}$, and $\eta_X - x$ is the complementary idempotent. It follows that

$$N CH_0(X_K)_0 = 0$$

for any extension K/F and (for instance) that

$$N\,\mathrm{Coker}(M_n(K) \to A^0(X_K, M_n)) = N\,\mathrm{Ker}(A_0(X_K, M_n) \to M_n(K)) = 0$$

for any cycle module M and any $K \supseteq F$ (see Section 6): compare [Auel et al. 2013, Theorem 1.4].

If N is minimal, then $N > 1$ is an obstruction to having

$$h^{\mathrm{o}}(X) = \mathbb{1} \quad \text{in } \mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z});$$

this obstruction has been studied recently in [Auel et al. 2013; Voisin 2014; 2015]. Using the cycle module $M_n(K) = H^n(K, \mathbb{Q}/\mathbb{Z}(n-1))$ for $n = 1$, one finds that N is divisible by the exponent e of $H^1_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}/\mathbb{Z})$. One can show that $N = e$ if F is algebraically closed and X is a surface [Kahn 2016]; for $e = 1$, this was proven by Voisin [2014, Proposition 2.2] and by Auel, Colliot-Thélène and Parimala [Auel et al. 2013, Corollary 1.10]. For example, $N = 2$ for an Enriques surface and $N = 1$ for Barlow's surface [1985a; 1985b] (of general type), showing that its motive is $\mathbb{1}$ in $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z})$. (See the recent survey paper [Bauer et al. 2011] for more examples of surfaces of general type with $p_g = 0$.)

3.2. ***Quadrics.*** Suppose $\operatorname{char} F \neq 2$ and let X be a smooth projective quadric over F. By a theorem of Swan [1989] and Karpenko [1990], the degree map

$$\deg : CH_0(X) \to \mathbb{Z}$$

is injective, with image $\mathbb{Z}$ if X has a rational point and $2\mathbb{Z}$ otherwise. This implies:

Proposition 3.2.1. *Let X, Y be two smooth projective over F. Suppose that Y is a quadric. Then, in $\mathbf{Chow}^{\mathrm{o}}(F)$, we have*

$$\operatorname{Hom}(h^{\mathrm{o}}(X), h^{\mathrm{o}}(Y)) = \begin{cases} \mathbb{Z} & \text{if } Y_{F(X)} \text{ is isotropic}, \\ 2\mathbb{Z} & \text{otherwise}, \end{cases}$$

where we have used the degree map $\deg : CH_0(Y_{F(X)}) \to \mathbb{Z}$. *Similarly, in*

$$\overline{\mathbf{Chow}}^{\mathrm{o}}(F, \mathbb{Z}/2)$$

(see Section 2.5), *we have*

$$\operatorname{Hom}(h^{\mathrm{o}}(X), h^{\mathrm{o}}(Y)) = \begin{cases} \mathbb{Z}/2 & \text{if } Y_{F(X)} \text{ is isotropic}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2.2. Much work has been done recently on torsion in CH_0 of projective homogeneous varieties: we may quote [Chernousov et al. 2005; Krashen 2010; Petrov et al. 2008; Chernousov and Merkurjev 2006]. There are many examples of projective homogeneous varieties other than quadrics for which $CH_0(Y)$ is torsion-free; by [Chernousov and Merkurjev 2006, Corollary 4.3], this is always the case if Y is isotropic. This allows one to extend the second part of Proposition 3.2.1 to arbitrary projective homogeneous Y (with suitable coefficients). On the other hand, there are examples of anisotropic Y such that $CH_0(Y)_{\text{tors}} \neq 0$ [Krashen 2010, Proposition 1.1; Chernousov and Merkurjev 2006, Section 18], so the first part of Proposition 3.2.1 does not extend in full generality.

3.3. *The nilpotence conjecture.*

Conjecture 3.3.1. *For any two adequate pairs* $(A, \sim)$, $(A, \sim')$ *with* $A \supseteq \mathbb{Q}$ *and* $\sim \geq \sim'$, *and any* $M \in \mathbf{Mot}_{\sim}(F, A)$, $\operatorname{Ker}(\operatorname{End}(M) \to \operatorname{End}(M_{\sim'}))$ *is nilpotent.* (*We say that the kernel of* $\mathbf{Mot}_{\sim}(F, A) \to \mathbf{Mot}_{\sim'}(F, A)$ *is* locally nilpotent.)

Since rat is the finest, and num is the coarsest, adequate equivalence relation, this conjecture is clearly equivalent to the same statement for $\sim\, = \mathrm{rat}$ and $\sim'\, = \mathrm{num}$, but it may be convenient to consider it for selected adequate equivalence relations. For example:

Proposition 3.3.2. (a) *Conjecture 3.3.1 is true for* $M \in \mathbf{Mot}_{\sim}^{\mathrm{eff}}(F, A)$ (*and any* $\sim' \leq \sim$) *provided* M *is finite-dimensional in the sense of Kimura and O'Sullivan* [Kimura 2005, Definition 3.7]. *In particular*, *it is true if* M *is of abelian type*, *i.e.,* M *is a direct summand of* $h_{\sim}(A_K)$ *for* A *an abelian* F*-variety and* K *an étale* F*-algebra.*

(b) *If* $\sim\, = \mathrm{hom}$, $\sim'\, = \mathrm{num}$, *the condition of* (*a*) *is equivalent to the sign conjecture*:

> *If* H *is the Weil cohomology theory defining* hom, *the projector of* $\operatorname{End} H(M)$ *projecting* $H(M) = H^+(M) \oplus H^-(M)$ *onto its summand* $H^+(M)$ *is algebraic.*

In particular, *it is true if* M *satisfies the standard conjecture* C (*algebraicity of the Künneth projectors*).

(c) *Conjecture 3.3.1 is true in the following cases*:

(i) $\sim\, = \mathrm{rat}$, $\sim'\, = \mathrm{tnil}$.

(ii) $\sim\, = \mathrm{rat}$, $\sim'\, = \mathrm{alg}$.

Proof. (a) This is a theorem of Kimura and O'Sullivan; see [Kimura 2005, Proposition 7.5; André and Kahn 2002, Proposition 9.1.14]. The second assertion follows from Kimura's results; see [Kahn et al. 2007, Example 7.6.3(4)].

(b) See [André and Kahn 2002, Theorem 9.2.1(c)].

(c) (i) follows from the Voevodsky–Kimura lemma that smash-nilpotent correspondences are nilpotent; see [Voevodsky 1995, Lemma 2.7; Kimura 2005, Proposition 2.16; André and Kahn 2002, Lemma 7.4.2(ii)]. (ii) follows from (i) and Voevodsky's theorem [1995, Corollary 3.2] that $\mathrm{alg} \geq \mathrm{tnil}$. □

Let us recall some conjectures which imply Conjecture 3.3.1:

Proposition 3.3.3. (a) *Conjecture 3.3.1 is implied by Voevodsky's conjecture* [1995, Conjecture 4.2] *that smash-nilpotence equivalence equals numerical equivalence.*

(b) *It is also implied by the sign conjecture plus the Bloch–Beĭlinson–Murre conjecture* [Jannsen 1994; Murre 1993].

Proof. (a) This follows from Proposition 3.3.2(c)(i).

(b) Recall that the Bloch–Beĭlinson conjecture is equivalent to Murre's conjecture [1993] by [Jannsen 1994, Theorem 5.2]. Now the formulation of the former conjecture [Jannsen 1994, Conjecture 2.1] implies the existence of an increasing chain of equivalence relations $(\sim_\nu)_{1\le\nu\le\infty}$ such that

- $\sim_1 = \text{hom}$;
- if α, β are composable Chow correspondences such that $\alpha\sim_\mu 0$ and $\beta\sim_\nu 0$, then $\beta\circ\alpha\sim_{\mu+\nu} 0$;
- for any smooth projective variety X, there is $\nu=\nu(X)$ such that $A_{\sim_\nu}(X\times X)=A_{\text{rat}}(X\times X)$.

There properties, together with the sign conjecture, imply Conjecture 3.3.1 by Proposition 3.3.2(b). □

Remark 3.3.4. In fact, one has more precise but slightly weaker implications: the Bloch–Beĭlinson–Murre conjecture + "hom = num" conjecture $\Longrightarrow$ Voevodsky's conjecture $\Longrightarrow$ the Kimura–O'Sullivan conjecture [any Chow motive is finite-dimensional] $\Longrightarrow$ Conjecture 3.3.1; see the synoptic table at the end of Chapter 12 in [André 2004].

For the first implication, see [André 2004, Théorème 11.5.3.1]. For the second one, see [André 2004, Théorème 12.1.6.6]. The third one is in Proposition 3.3.2(a).

Definition 3.3.5. Let $M\in\mathbf{Mot}_\sim(F,A)$. For $n\in\mathbb{Z}$, we write $\nu(M)\ge n$ if $M\otimes\mathbb{L}^{\otimes -n}$ is effective.[5]

Proposition 3.3.6. *Suppose $A\supseteq\mathbb{Q}$ and the nilpotence conjecture holds for $\sim\,\ge\,\sim'$. Then*:

(a) *The functor $\mathbf{Mot}_\sim(F,A)\to\mathbf{Mot}_{\sim'}(F,A)$ is conservative, and for*

$$M\in\mathbf{Mot}_\sim(F,A)$$

any set of orthogonal idempotents in the endomorphism ring of $M_{\sim'}$ lifts.

(b) *If $M\in\mathbf{Mot}_\sim(F,A)$ and $M_{\sim'}$ is effective, then M is effective.*

(c) *If $M\in\mathbf{Mot}_\sim(F,A)$ and $\nu(M_{\sim'})\ge n$, then $\nu(M)\ge n$.*

(d) [André 2004, Section 13.2.1] *The map $K_0(\mathbf{Mot}_\sim(F,A))\to K_0(\mathbf{Mot}_{\sim'}(F,A))$ is an isomorphism* (here, the K_0-groups are those of additive categories).

[5]By convention, we say here that a motive $N\in\mathbf{Mot}_\sim(F,A)$ is *effective* if it is isomorphic to a motive of $\mathbf{Mot}^{\text{eff}}_\sim(F,A)$.

Proof. (a) This is classical (see [Jannsen 1994, Lemma 5.4] for the second statement).

(b) By definition, $M_{\sim'}$ effective means that $M_{\sim'}$ is isomorphic to a direct summand of $h_{\sim'}(X)$ for some smooth projective X. By (a), one may lift the corresponding idempotent $e_{\sim'}$ to an idempotent endomorphism e of $h_{\sim}(X)$, and the isomorphism $M_{\sim'} \simeq (h_{\sim'}(X), e_{\sim'})$ to an isomorphism $M \simeq (h_{\sim}(X), e)$.

(c) This follows from (b) applied to $M \otimes \mathbb{L}^{\otimes -n}$.

(d) This follows from (a), since then the functor $\mathbf{Mot}_{\sim}(F, A) \to \mathbf{Mot}_{\sim'}(F, A)$ is conservative and essentially surjective. □

The importance of Conjecture 3.3.1 will appear again in the next subsection and in Section 4 (see Remark 4.3.4 and Proposition 4.4.1).

3.4. ***The Chow–Künneth decomposition.*** Here we take $(A, \sim) = (\mathbb{Q}, \mathrm{rat})$. Recall that Murre [1993] strengthened the standard conjecture C (algebraicity of the Künneth projectors) to the existence of a *Chow–Künneth decomposition*

$$h(X) \simeq \bigoplus_{i=0}^{2d} h_i(X)$$

in $\mathbf{Chow}(F, \mathbb{Q})$. (This is part of the Bloch–Beĭlinson–Murre conjecture appearing in Proposition 3.3.3(b)). By Proposition 3.3.6(a), the nilpotence conjecture together with the standard conjecture C imply the existence of Chow–Künneth decompositions.

Here are some cases where the existence of a Chow–Künneth decomposition is known independently of any conjecture:

(1) Varieties of dimension ≤ 2 [Murre 1990] (see also [Scholl 1994]). In fact, Murre constructs for any X a partial decomposition

$$h(X) \simeq h_0(X) \oplus h_1(X) \oplus h_{[2,2d-2]}(X) \oplus h_{2d-1}(X) \oplus h_{2d}(X).$$

(2) Abelian varieties [Shermenev 1974].

(3) Complete intersections in $\mathbb{P}^N$ (see the next subsection).

(4) If X and Y have a Chow–Künneth decomposition, then so does $X \times Y$.

Suppose that the nilpotence conjecture holds for $h(X) \in \mathbf{Chow}(F, \mathbb{Q})$ and that homological and numerical equivalences coincide on $X \times X$. The latter then implies the standard conjecture C for X [Kleiman 1994], hence the existence of a Chow–Künneth decomposition by the remark above. In [Kahn et al. 2007, Theorem 14.7.3(iii)], it is proven:

Proposition 3.4.1. *Under these hypotheses, there exists a further decomposition for each* $i \in [0, 2d]$:

$$h_i(X) \simeq \bigoplus h_{i,j}(X)(j),$$

such that $h_{i,j}(X) = 0$ *for* $j \notin [0, [i/2]]$ *and, for each* j, $\nu(h_{i,j}^{\mathrm{hom}}(X)) = 0$ (see Definition 3.3.5). *Moreover, one has isomorphisms*

$$h_{2d-i,d-i+j}(X) \xrightarrow{\sim} h_{i,j}(X) \tag{3.4.2}$$

for $i \leq d$. *In particular,* $\nu(h_i(X)) > 0$ *for* $i > d$.

Let us justify the last assertion; the isomorphisms (3.4.2) imply that, when $i > d$, $h_{i,j}(X) = 0$ for $j < i - d$.

Since $\mathbf{Chow}^{\mathrm{eff}}(F, \mathbb{Q}) \to \mathbf{Chow}(F, \mathbb{Q})$ is fully faithful, all the above (refined) Chow–Künneth decompositions hold for the effective Chow motives

$$h(X) \in \mathbf{Chow}^{\mathrm{eff}}(F, \mathbb{Q}).$$

We deduce:

Corollary 3.4.3. *Under the nilpotence conjecture and the conjecture that homological and numerical equivalences coincide, for any smooth projective variety* X *the image of its Chow–Künneth decomposition in* $\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Q})$ *is of the form*

$$h^{\mathrm{o}}(X) \simeq \bigoplus_{i=0}^{d} h_i^{\mathrm{o}}(X).$$

Moreover, with the notation of Proposition 3.4.1, one has

$$h_i^{\mathrm{o}}(X) \simeq h_{i,0}^{\mathrm{o}}(X) \quad \text{for } i \leq d.$$

Examples where this conclusion is true unconditionally follow faithfully the examples where the Chow–Künneth decomposition is unconditionally known:

Proposition 3.4.4. *The conclusion of Corollary 3.4.3 holds in the following cases*:

(1) *Varieties of dimension* ≤ 2.

(2) *Abelian varieties.*

(3) *Complete intersections in* $\mathbb{P}^N$.

(4) *If* X *and* Y *have a Chow–Künneth decomposition and verify this conclusion, then so does* $X \times Y$.

Proof. In cases (1) and (2), the conclusion holds because one has “Lefschetz isomorphisms” $h_{2d-i}(X) \xrightarrow{\sim} h_i(X)(d-i)$ for $i > d$. For curves, it is trivial, for surfaces they are constructed in [Murre 1990] (see [Scholl 1994, Theorem 4.4(ii)]; the isomorphism is constructed for $i = 0, 1$ and any X), and for abelian varieties they are constructed in [Shermenev 1974]. For (3), see the next subsection. Finally, (4) is clear. □

In the case of a surface, Kahn et al. [2007] construct a refined Chow–Künneth decomposition

$$h(X) = h_0(X) \oplus h_1(X) \oplus \mathrm{NS}_X(1) \oplus t_2(X) \oplus h_3(X) \oplus h_4(X),$$

where NS_X is the Artin motive corresponding to the Galois representation defined by $\mathrm{NS}(\overline{X}) \otimes \mathbb{Q}$, and $t_2(X)$ is the *transcendental part of* $h(X)$. (In the notation of Proposition 3.4.1, $h_{2,0}(X) = t_2(X)$ and $h_{2,1}(X) = \mathrm{NS}_X$.) This translates on the birational motive of X as

$$h^{\mathrm{o}}(X) = h_0^{\mathrm{o}}(X) \oplus h_1^{\mathrm{o}}(X) \oplus t_2^{\mathrm{o}}(X).$$

3.5. ***Motives of complete intersections.*** These computations will be used in Section 4. Here we take $A \supseteq \mathbb{Q}$.

For convenience, we take the notation of [Deligne 1973]; so let $X \subset \mathbb{P}^r$ be a smooth complete intersection of multidegree $\underline{a} = (a_1, \dots, a_d)$, and let

$$n = r - d = \dim X.$$

Then the cohomology of X coincides with the cohomology of $\mathbb{P}^r$ except in middle dimension [Deligne 1973], and in particular it is fully algebraic except in middle dimension. This allows us to easily write down a Chow–Künneth decomposition for $h(X)$ in the sense of [Murre 1993] (see also [Esnault et al. 1997, Corollary 5.3]):

(1) (Murre) For each $i \neq n/2$, let $c^i \in \mathcal{Z}^i(X)$ be an algebraic cycle whose cohomology class generates $H^{2i}(X)$ (here H is some Weil cohomology). Then the Chow–Künneth projector π_{2i} is given by $c^i \times c^{n-i}$. We take $\pi_j = 0$ for j odd $\neq n$, and $\pi_n := \Delta_X - \sum_{j\neq n} \pi_j$.

(2) Consider the inclusion $i : X \hookrightarrow \mathbb{P}^r$. This yields morphisms of motives

$$h(\mathbb{P}^r)(-d) \xrightarrow{i^*} h(X) \xrightarrow{i_*} h(\mathbb{P}^r).$$

Given the decomposition $h(\mathbb{P}^r) \simeq \bigoplus_{j=0}^r \mathbb{L}^j$, this yields for each $j \in [0, n]$ morphisms

$$\mathbb{L}^j \xrightarrow{i_j^*} h(X) \xrightarrow{i_*^j} \mathbb{L}^j$$

with composition $a = \prod a_i$. Then $(1/a) i_j^* i_*^j$ defines the $2i$-th Chow–Künneth projector of X (π_{2i} in (1)), except if $2i = n$. Let $\pi_n^{\mathrm{prim}} := 1_{h(X)} - \sum_{i=0}^n (1/a) i_j^* i_*^j$ and let the image $p_n(X)$ of the projector π_n^{prim} be *the primitive part* of $h_n(X)$.

Note that the Chow–Künneth projectors of (1) and (2) are actually equal. Let us record here the corresponding (refined) Chow–Künneth decomposition:

$$h(X) \simeq \mathbb{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n \oplus p_n(X). \tag{3.5.1}$$

Lemma 3.5.2. (a) *Homological and numerical equivalences agree on all* (*rational*) *Chow groups of X provided n is odd or* (*if* char $F = 0$) *the Hodge realisation of $p_n(X)$ does not contain any direct summand isomorphic to $\mathbb{L}^{n/2}$.*

(b) *Suppose* (a) *is satisfied. Then for any adequate pair* $(\sim, A)$ *with $A \supseteq \mathbb{Q}$ and any $j \in [0, n]$, we have*

$$\mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)) = \operatorname{Ker}(A_j^\sim(X, A) \to A_j^{\mathrm{num}}(X, A)).$$

Proof. We have

$$\begin{aligned} A_j^\sim(X, A) &= \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, h(X)) \\ &= \bigoplus_{i=0}^{n} \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, \mathbb{L}^i) \oplus \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)) \\ &= \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, \mathbb{L}^j) \oplus \mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)). \end{aligned}$$

For $\sim\, =$ hom, we have $\mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, p_n(X)) = 0$ by weight reasons for $2j \neq n$ and under the hypothesis of (a) for $2j = n$ (note that the Hodge realisation of $p_n(X)$ is semisimple, as a polarisable Hodge structure). Hence the same is true for any $\sim$ finer than hom, in particular $\sim\, =$ num. This proves (a). Moreover, $\mathbf{Mot}_\sim(F, A)(\mathbb{L}^j, \mathbb{L}^j) = A$ for any choice of $\sim$. Hence (b). □

Equation (3.5.1) shows that the birational motive of X reduces to $\mathbb{1} \oplus p_n^\sim(X)^{\mathrm{o}}$. In fact, it is possible to be much more precise:

Proposition 3.5.3. *Let $\underline{a} = (a_1, \dots, a_d)$ be the multidegree of $X \subset \mathbb{P}^r$.*

(a) *If $a_1 + \cdots + a_d \leq r$, then $h^{\mathrm{o}}_{\mathrm{rat}}(X) = \mathbb{1}$.*

(b) *If $a_1 + \cdots + a_d > r$, then $h^{\mathrm{o}}_{\mathrm{num}}(X) \neq \mathbb{1}$* (*equivalently, $p_n^{\mathrm{num}}(X)^{\mathrm{o}} \neq 0$*) *provided* char $F = 0$ *or X is generic.*

Proof. (a) Under the hypothesis, we conclude from Roĭtman's theorem [1980a] that $CH_0(X_K) \otimes \mathbb{Q} = \mathbb{Q}$ for any extension K/F.[6] Assertion (a) then follows from Proposition 3.1.1.

(b) It suffices to prove the statement for homological equivalence, since the kernel of $\mathbf{Mot}_{\mathrm{hom}}(F,\mathbb{Q})(h(X), h(X)) \to \mathbf{Mot}_{\mathrm{num}}(F,\mathbb{Q})(h(X), h(X))$ is a nilpotent ideal (see Propositions 3.3.2(b) and 3.3.6(a)).

If char $F = 0$, we may use Hodge cohomology and Deligne's theorem [1973, Théorème 2.5(ii), p. 54]. Namely, with the notation of [loc. cit.], the condition

[6] Of course we could also invoke Proposition 3.1.4(2) since X is Fano, hence rationally chain connected, but this theorem of Campana [1992] and Kollár, Miyaoka and Mori [Kollár et al. 1992] was proven much later than Roĭtman's work.

$p_n^{\mathrm{hom}}(X)^{\mathrm{o}} = 0$ implies $h_0^{0,n}(\underline{a}) = 0$, which is equivalent by Deligne's theorem to

$$0 \le \left[\frac{n+d-\sum a_i}{\sup(a_i)}\right],$$

that is, $\sum a_i \le n+d = r$.

If char $F > 0$ and X is generic, we may use Katz's theorem [1973, p. 382, Theorem 4.1]. □

Remarks 3.5.4. (1) Katz also has a result [1973, Theorem 4.2] concerning a generic hyperplane section of a given complete intersection.

(2) It seems possible to remove the genericity assumption in positive characteristic by lifting the coefficients of the equations defining X to characteristic 0. We have not worked out the details.

4. On adjoints and idempotents

We now want to examine two related questions:

(1) Does the projection functor $\mathbf{Mot}_\sim^{\mathrm{eff}}(F, A) \to \mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)/\mathcal{L}_\sim$ have a right adjoint? This question was raised by Luca Barbieri-Viale and is closely related to a conjecture of Voevodsky [1992, Conjecture 0.0.11].

(2) Is the category $\mathbf{Mot}_\sim^{\mathrm{eff}}(F, A)/\mathcal{L}_\sim$ pseudoabelian, i.e., is it superfluous to take the pseudoabelian envelope in Definition 2.2.8?

The answer to both questions is "yes" for $\sim\, =$ num and $A \supseteq \mathbb{Q}$, as an easy consequence of Jannsen's semisimplicity theorem for numerical motives [1992]. In fact:

Proposition 4.0.1 [Kahn 2009, Proposition 7.7]. (a) *The projection functor*

$$\pi : \mathbf{Mot}_{\mathrm{num}}^{\mathrm{eff}} \to \mathbf{Mot}_{\mathrm{num}}^{\mathrm{o}}$$

is essentially surjective.

(b) *π has a section i which is also a left and right adjoint.*

(c) *The category $\mathbf{Mot}_{\mathrm{num}}^{\mathrm{eff}}$ is the coproduct of $\mathbf{Mot}_{\mathrm{num}}^{\mathrm{eff}} \otimes \mathbb{L}$ and $i(\mathbf{Mot}_{\mathrm{num}}^{\mathrm{o}})$, i.e., any object of $\mathbf{Mot}_{\mathrm{num}}^{\mathrm{eff}}$ can be uniquely written as a direct sum of objects of these two subcategories.*

In the sequel, we want to examine these questions for a general adequate pair; see Theorems 4.3.2 and 4.3.3 for (1) and Proposition 4.4.1 for (2). This requires some preparation.

4.1. ***A lemma on base change.*** Let $P : \mathcal{A} \to \mathcal{B}$ be a functor. Recall that one says that "its" right adjoint is *defined at* $B \in \mathcal{B}$ if the functor

$$\mathcal{A} \ni A \mapsto \mathcal{B}(PA, B)$$

is representable. We write $P^\sharp B$ for a representing object (unique up to unique isomorphism).

Let

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "Q"'] \\ \mathcal{C} \arrow[r, "\psi"] & \mathcal{D} \end{tikzcd}$$

be a naturally commutative diagram of pseudoabelian additive categories, and let $A \in \mathcal{A}$.

Suppose that "the" right adjoint $P^\sharp$ of P is defined at $PA \in \mathcal{C}$ and that the right adjoint $Q^\sharp$ of Q is defined at $\psi PA \simeq Q\varphi A$. We then have two corresponding unit maps (adjoint to the identities of PA and $Q\varphi A$)

$$\varepsilon_P : A \to P^\sharp PA, \quad \varepsilon_Q : \varphi A \to Q^\sharp Q\varphi A.$$

Lemma 4.1.1. *Suppose that ε_Q is an isomorphism. Then $\varphi\varepsilon_P$ has a retraction. If moreover φ is full and* $\mathrm{Ker}(\mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathcal{B}}(\varphi A))$ *is a nil ideal, then ε_P has a retraction.*

Proof. Let $\eta_P : PP^\sharp PA \to PA$ be the counit map of the adjunction at PA (adjoint to the identity of $P^\sharp PA$), and let $u : Q\varphi A \xrightarrow{\sim} \psi PA$ and $v : Q\varphi P^\sharp PA \xrightarrow{\sim} \psi PP^\sharp PA$ be the natural isomorphisms from $Q\varphi$ to ψP evaluated respectively at A and $P^\sharp PA$. We then have a composition

$$Q\varphi P^\sharp PA \xrightarrow{v} \psi PP^\sharp PA \xrightarrow{\psi\eta_P} \psi PA,$$

which yields by adjunction a "base change morphism"

$$\varphi P^\sharp PA \xrightarrow{b} Q^\sharp\psi PA.$$

Inspection shows that the diagram

$$\begin{tikzcd} \varphi A \arrow[r, "\varphi\varepsilon_P"] \arrow[d, "\varepsilon_Q"'] & \varphi P^\sharp PA \arrow[d, "b"'] \\ Q^\sharp Q\varphi A \arrow[r, "Q^\sharp u"] & Q^\sharp\psi PA \end{tikzcd}$$

commutes. The first claim follows, and the second claim follows from the first. $\square$

4.2. ***Right adjoints.*** We come back to question (1), posed at the beginning of this section. In [Kahn et al. 2007, Remark 14.8.7; Kahn 2009, Remark 7.8(3)], it was announced that one can show the nonexistence of the right adjoint for $\sim = \mathrm{rat}$, using the results of [Huber 2008, Appendix]. The proof turns out not to be exactly along these lines, but is closely related; see Lemma 4.2.1 and Theorems 4.3.2 and 4.3.3.

Let us abbreviate the notation to $\mathbf{Mot}^{\mathrm{eff}} = \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, A)$, $\mathbf{Mot}^{\mathrm{o}} = \mathbf{Mot}^{\mathrm{o}}_{\sim}(F, A)$. Let $P : \mathbf{Mot}^{\mathrm{eff}} \to \mathbf{Mot}^{\mathrm{o}}$ denote the projection functor, and let $P^\sharp$ denote its (a priori partially defined) right adjoint. Let $\mathcal{L}^\perp$ be the full subcategory of $\mathbf{Mot}^{\mathrm{eff}}$ consisting of those M such that $\mathrm{Hom}(N(1), M) = 0$ for all $N \in \mathbf{Mot}^{\mathrm{eff}}$. Recall from [Kahn et al. 2007, Proposition 7.8.1] that

- if $P^\sharp$ is defined at M, then $P^\sharp M \in \mathcal{L}^\perp$;
- the full subcategory $\mathbf{Mot}^\sharp$ of $\mathbf{Mot}^{\mathrm{o}}$ where $P^\sharp$ is defined equals $P(\mathcal{L}^\perp)$;
- $P^\sharp$ and the restriction of P to $\mathcal{L}^\perp$ define quasi-inverse equivalences of categories between $\mathcal{L}^\perp$ and $\mathbf{Mot}^\sharp$.

The right adjoint $P^\sharp$ is defined at birational motives of varieties of dimension ≤ 2 for any adequate pair $(A, \sim)$ such that $A \supseteq \mathbb{Q}$ by [Kahn et al. 2007, Corollary 7.8.6]. (The proof there is given for $(A, \sim) = (\mathbb{Q}, \mathrm{rat})$, but the argument works in general.) Recall that

$$P^\sharp h^{\mathrm{o}}(C) = \mathbb{1} \oplus h_1(C), \qquad P^\sharp h^{\mathrm{o}}(S) = \mathbb{1} \oplus h_1(S) \oplus t_2(S)$$

with the notation at the end of Section 3.4, where C is a curve and S is a surface.

The following lemma gives a sufficient condition for the nonexistence of $P^\sharp PM$ for an effective motive M.

Lemma 4.2.1. *Let* $(\mathbb{Q}, \sim)$ *be an adequate pair, and let* $M \in \mathbf{Mot}^{\mathrm{eff}}_{\sim}(F, \mathbb{Q})$. *Assume that*

(i) $M_{\mathrm{num}} \in \mathbf{Mot}^{\mathrm{eff}}_{\mathrm{num}}(F, \mathbb{Q})$ *does not contain any direct summand divisible by* $\mathbb{L}$;

(ii) $\mathrm{Ker}(\mathrm{End}(M) \to \mathrm{End}(M_{\mathrm{num}}))$ *is a nilideal*;

(iii) *there exists* $r > 0$ *such that* $\mathrm{Hom}(\mathbb{L}^r, M) \neq 0$.

Then $P^\sharp PM$ *does not exist.*

Proof. Suppose that $P^\sharp$ is defined at PM. Consider the unit map

$$\varepsilon_\sim : M \to P^\sharp PM. \tag{4.2.2}$$

For $\sim = \mathrm{num}$, $P^\sharp_{\mathrm{num}} P_{\mathrm{num}} M_{\mathrm{num}}$ exists by Proposition 4.0.1. Moreover, part (c) of this proposition shows that, under condition (i) of the lemma, $\varepsilon_{\mathrm{num}}$ is an isomorphism. By Lemma 4.1.1, the image of $\varepsilon_\sim$ modulo numerical equivalence then has a retraction, and so does $\varepsilon_\sim$ itself under condition (ii). If this is the case, $M \in \mathcal{L}^\perp$, and in particular, $\mathrm{Hom}(\mathbb{L}^r, M) = 0$ for all $r > 0$, contradiction. $\square$

4.3. ***Counterexamples.*** To give examples where the conditions of Lemma 4.2.1 are satisfied, we appeal as in [Huber 2008] to the nontriviality of the Griffiths group.

We start with an example which a priori only works for a specific adequate equivalence, because the proof is simpler. Unlike in [Huber 2008], we don't need the full force of Clemens' theorem [1983, Theorem 0.2], but merely the previous results of Griffiths [1969].

Definition 4.3.1 ("Abel–Jacobi equivalence"). Let $k = \mathbb{C}$. For X smooth projective, $\mathcal{Z}^j_{\mathrm{AJ}}(X, \mathbb{Q})$ is the image of $CH^j(X) \otimes \mathbb{Q}$ in Deligne–Beĭlinson cohomology via the (Deligne–Beĭlinson) cycle class map [Esnault and Viehweg 1988]. This defines an adequate equivalence relation.

Theorem 4.3.2. *Let $F = \mathbb{C}$ and $\sim = \mathrm{AJ}$. Then*:

(a) *Condition* (ii) *of Lemma 4.2.1 is satisfied for any pure motive M. Let X be a generic hypersurface of degree a in $\mathbb{P}^{n+1}$.*

(b) *Condition* (i) *of Lemma 4.2.1 is satisfied for $M = p_n(X)$* (see (3.5.1)) *provided X is not a quadric, a cubic surface or an even-dimensional intersection of two quadrics, and $a \geq n+1$.*

(c) *If $n = 2m-1$ is odd and $a \geq 2+3/(m-1)$, then condition* (iii) *of Lemma 4.2.1 is satisfied for $r = m-1$.*

(d) *$P^\sharp$ is not defined at $h^{\mathrm{o}}(X)$ in the following cases: n is odd and*
   (i) *if $n = 3$ then $a \geq 5$;*
   (ii) *if $n > 3$ then $a \geq n+1$.*

Proof. We see that (a) holds because $\mathrm{Ker}(\mathrm{End}_{\mathrm{AJ}}(M) \to \mathrm{End}_{\mathrm{hom}}(M))$ has square 0 [Esnault and Viehweg 1988, Proposition 7.10][7] and $\mathrm{Ker}(\mathrm{End}_{\mathrm{hom}}(M) \to \mathrm{End}_{\mathrm{num}}(M))$ is nilpotent.

(b) By [Peters and Steenbrink 2003, Example 5 and Corollary 18], the Hodge realisation $P_n(X)$ of $p_n(X)$ is an absolutely simple pure Hodge structure; this, together with Proposition 3.5.3(b), is amply sufficient to imply condition (i) of Lemma 4.2.1.

(c) By [Griffiths 1969, Corollaries 13.2 and 14.2],

$$\mathrm{Ker}(A^{\sim}_{m-1}(X, \mathbb{Q}) \to A^{\mathrm{num}}_{m-1}(X, \mathbb{Q})) \neq 0.$$

But by Lemma 3.5.2, this group is $\mathrm{Hom}(\mathbb{L}^{m-1}, p_n(X))$.

[7] A more functorial justification is: (1) Deligne–Beĭlinson cohomology can be computed as absolute Hodge cohomology as in [Beilinson 1986]; (2) the category of polarisable $\mathbb{Q}$-mixed Hodge structures has Ext-dimension 1.

(d) Note that, by the refined Chow–Künneth decomposition (3.5.1), $P^\sharp$ is defined at $Ph(X)$ if and only if it is defined at $Pp_n(X)$. The conclusion now follows from Lemma 4.2.1 and from collecting the results of (a), (b) and (c). □

To get a counterexample with rational equivalence, we appeal to a result of Nori [1989]. We thank Srinivas for pointing out this reference.

Theorem 4.3.3. *Let X be a generic abelian threefold over $k=\mathbb{C}$. If $\sim\ \geq$ alg, then $P^\sharp$ is not defined at $h^o_\sim(X)$.*

Proof. The proof is similar to that of Theorem 4.3.2, except that the motive of an abelian variety is more complicated than that of a hypersurface. We only sketch the argument (details will appear elsewhere).

It is enough to show that $P^\sharp$ is not defined at $h^o_{3,0}(X)$, where $h_{3,0}(X)$ is as in Proposition 3.4.1 (here we use that the nilpotence conjecture is true for motives of abelian varieties, see Proposition 3.3.2(a)). We check the conditions of Lemma 4.2.1 for $M=h_{3,0}(X)$. Item (i) is true by definition; and (ii) is true by Proposition 3.3.2(a). For (iii), one can show that computing the decomposition

$$A_1^\sim(X)=\mathbf{Mot}^{\mathrm{eff}}_\sim(\mathbb{L},h(X))\simeq\bigoplus_{i=0}^{6}\bigoplus_{j=0}^{[i/2]}\mathbf{Mot}^{\mathrm{eff}}_\sim(\mathbb{L},h_{i,j}(X)(j))$$

yields a surjection

$$\mathbf{Mot}^{\mathrm{eff}}_\sim(\mathbb{L},h_{3,0}(X))\twoheadrightarrow\mathrm{Griff}_1(X)$$

for $\sim\ \geq$ alg, where $\mathrm{Griff}_1(X)=\mathrm{Ker}(A_1^{\mathrm{alg}}(X)\to A_1^{\mathrm{num}}(X))$ is the Griffiths group of X. By Nori's theorem [1989], $\mathrm{Griff}_1(X)\neq 0$, and the proof is complete. □

Remark 4.3.4. It is easy to get examples of any dimension ≥ 4 by multiplying the example of Theorem 4.3.3 with $\mathbb{P}^n$.

4.4. ***Idempotents.*** We now address question (2) from the beginning of this section.

Proposition 4.4.1. *Let $(A,\sim)$ be an adequate pair with $A\supseteq\mathbb{Q}$, and let $\mathcal{M}$ be a full subcategory of $\mathbf{Mot}^{\mathrm{eff}}_\sim(F,A)$ closed under direct summands. If Conjecture 3.3.1 holds for the objects of $\mathcal{M}$, then the category $\mathcal{M}/\mathcal{L}_\sim$ is pseudoabelian.*

Proof. Let $\mathcal{M}_{\mathrm{num}}$ denote the pseudoabelian envelope of the image of $\mathcal{M}$ in

$$\mathbf{Mot}^{\mathrm{eff}}_{\mathrm{num}}(F,A).$$

We have a commutative diagram of categories:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "P"] \arrow[d, "\pi"'] & \mathcal{M}/\mathcal{L}_\sim \arrow[d, "\bar\pi"] \\
\mathcal{M}_{\mathrm{num}} \arrow[r, "P_{\mathrm{num}}"] & \mathcal{M}_{\mathrm{num}}/\mathcal{L}_{\mathrm{num}}
\end{tikzcd}$$

Under the hypothesis, π is essentially surjective (one can lift idempotents). Hence $\bar{\pi}$ is essentially surjective as well. Since P is essentially surjective and π, P_{num} are full, $\bar{\pi}$ is full, and its kernel is locally nilpotent as a quotient of the kernel of π (fullness of P). Thus $\bar{\pi}$ is full, essentially surjective and conservative.

Since $\mathbf{Mot}^{\mathrm{eff}}_{\mathrm{num}}(F, A)$ is abelian semisimple, $\mathcal{M}_{\mathrm{num}}$ is also abelian semisimple, hence so is $\mathcal{M}_{\mathrm{num}}/\mathcal{L}_{\mathrm{num}}$ which is in particular pseudoabelian.

Let now $M \in \mathcal{M}/\mathcal{L}_\sim$, and let $p = p^2 \in \mathrm{End}(M)$. Write $M_{\mathrm{num}} \simeq M_1 \oplus M_2$, where $M_1 = \mathrm{Im}\, p_{\mathrm{num}}$ and $M_2 = \mathrm{Ker}\, p_{\mathrm{num}}$. By essential surjectivity, we may lift M_1 and M_2 to objects $\widetilde{M}_1, \widetilde{M}_2 \in \mathcal{M}/\mathcal{L}_\sim$.

By fullness, we may lift the isomorphism $M_1 \oplus M_2 \xrightarrow{\sim} M_{\mathrm{num}}$ to a morphism $\widetilde{M}_1 \oplus \widetilde{M}_2 \to M$ in $\mathcal{M}/\mathcal{L}_\sim$, and this lift is an isomorphism by conservativity. This concludes the proof. □

Example 4.4.2. Proposition 4.4.1 applies taking for $\mathcal{M}$ the category of motives of abelian type (direct summands of the tensor product of an Artin motive and the motive of an abelian variety), since such motives are finite-dimensional [Kimura 2005].

The situation when A does not contain $\mathbb{Q}$, for example $A = \mathbb{Z}$, is unclear.

5. Birational motives and birational categories

In this section, we relate the categories studied in [Kahn and Sujatha 2015a] with the categories of pure birational motives introduced here.

5.1. From (2.4.1), we get a composite functor:

$$S_r^{-1}\, \mathbf{Sm}^{\mathrm{proj}}(F) \to S_r^{-1}\, \mathbf{Chow}^{\mathrm{eff}}(F) \to \mathbf{Chow}^{\mathrm{o}}(F). \tag{5.1.1}$$

The morphisms in the first category can be described by means of R-equivalence classes [Kahn and Sujatha 2015a, Theorem 6.6.3, Corollary 6.6.4 and Remark 6.6.5]; by Lemma 2.3.7, those in the last category can be described by means of Chow groups of 0-cycles. One checks easily that the action of the composite functor on Hom sets is just the map which sends R-equivalence classes of rational points to 0-cycles modulo rational equivalence. This puts this map within a functorial setting.

Let us now recall further results from [Kahn and Sujatha 2015a]. Let $\mathbf{place}(F)$ denote the category of finitely generated extensions of F, with F-places as morphisms. In [Kahn and Sujatha 2015a, (4.3)], we constructed a functor

$$\mathbf{place}_*(F)^{\mathrm{op}} \to S_b^{-1}\, \mathbf{Sm}^{\mathrm{prop}}(F),$$

hence a functor

$$S_r^{-1}\, \mathbf{place}_*(F)^{\mathrm{op}} \to S_r^{-1}\, \mathbf{Sm}^{\mathrm{prop}}(F),$$

where $\mathbf{place}_*(F)$ denotes the full subcategory of $\mathbf{place}(F)$ defined by those K/F which have a cofinal set of smooth proper models, and $S_r \subset Ar(\mathbf{place}(F))$ denotes the set of purely transcendental extensions. The same arguments as in [loc. cit.] give an analogous functor

$$S_r^{-1}\,\mathbf{place}_\sharp(F)^{\mathrm{op}} \to S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F), \tag{5.1.2}$$

where $\mathbf{place}_\sharp(F)$ has the same definition as $\mathbf{place}_*(F)$, replacing "smooth proper" by "smooth projective". Composing (5.1.2) with (5.1.1), we get a functor

$$S_r^{-1}\,\mathbf{place}_\sharp(F)^{\mathrm{op}} \to \mathbf{Chow}^{\mathrm{o}}(F). \tag{5.1.3}$$

We can describe the image under this functor of a place $\lambda : K \rightsquigarrow L$ in $CH_0(X_L)$, where X is a smooth projective model of K: it is just the class of the centre of λ. Hence the image of (5.1.3) on morphisms consists of the classes of L-rational points. This answers a question of Déglise.

In characteristic 0, $\mathbf{place}_\sharp(F) = \mathbf{place}(F)$ by resolution of singularities and $S_r^{-1}\,\mathbf{Sm}^{\mathrm{proj}}(F) \xrightarrow{\sim} S_r^{-1}\,\mathbf{Sm}(F)$ by [Kahn and Sujatha 2007, Proposition 8.5]. In characteristic p, we would ideally like to get functors

$$S_r^{-1}\,\mathbf{place}(F)^{\mathrm{op}} \to \mathbf{Chow}^{\mathrm{o}}(F), \quad S_r^{-1}\,\mathbf{Sm}(F) \to \mathbf{Chow}^{\mathrm{o}}(F)$$

extending (5.1.1) and (5.1.3). Constructing the first functor looks technically difficult: we shall content ourselves with extending [Kahn 2009, Remark 7.4] to all finitely generated fields K/F, by using an adjunction result from [Kahn 2015]; this will not be used in the rest of the paper. The second functor is constructed in [Kahn and Sujatha 2015b, Corollary 2.4.2].

Proposition 5.1.4. *Let p be the exponential characteristic of F.*

(a) *There is a unique functor (up to unique isomorphism)*

$$h^{\mathrm{o}} : S_r^{-1}\,\mathbf{field}(F)^{\mathrm{op}} \to \mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z}[1/p])$$

such that, for any $K \in \mathbf{field}(F)$ and any $Y \in \mathbf{Sm}^{\mathrm{proj}}(F)$, one has

$$\mathbf{Chow}^{\mathrm{o}}(F, \mathbb{Z}[1/p])(h^{\mathrm{o}}(K), h^{\mathrm{o}}(Y)) \simeq CH_0(Y_K) \otimes \mathbb{Z}[1/p]. \tag{5.1.5}$$

This functor transforms purely inseparable extensions into isomorphisms.

(b) *If $K \subseteq L$, the map $h^{\mathrm{o}}(L) \to h^{\mathrm{o}}(K)$ has a section.*

(c) *We have $h^{\mathrm{o}}(K) = h^{\mathrm{o}}(X)$ if $K = F(X)$ for a smooth projective variety X. Moreover, if $K = F(X)$, $L = F(Y)$ with X, Y smooth projective, and if $f : K \to L$ corresponds to a rational map $\varphi : Y \dashrightarrow X$, then $h^{\mathrm{o}}(f)$ is given by the graph of φ.*

Proof. (a) Note that the isomorphism (5.1.5) determines $h^{\mathrm{o}}(K)$ up to unique isomorphism, by Yoneda's lemma. By Lemma 2.3.7 applied over K, this isomorphism may be rewritten as

$$\mathbf{Chow}^{\mathrm{o}}(F,\mathbb{Z}[1/p])(h^{\mathrm{o}}(K),h^{\mathrm{o}}(Y))\simeq\mathbf{Chow}^{\mathrm{o}}(K,\mathbb{Z}[1/p])(\mathbb{1}_K,h^{\mathrm{o}}(Y_K)),$$

where $\mathbb{1}_K=h^{\mathrm{o}}(\operatorname{Spec}K)$ is the unit object of $\mathbf{Chow}^{\mathrm{o}}(K,\mathbb{Z}[1/p])$.

By [Kahn 2015, Theorem 6.5], the base-change functor

$$\mathbf{Chow}^{\mathrm{o}}(F,\mathbb{Z}[1/p])\to\mathbf{Chow}^{\mathrm{o}}(K,\mathbb{Z}[1/p])$$

has a left adjoint $l_{K/F}$. Therefore we may define $h^{\mathrm{o}}(K)=l_{K/F}(\mathbb{1}_K)$.

Suppose $F\to K\xrightarrow{f}L$ are successive finitely generated extensions. Since the base-change of $\mathbb{1}_K$ is $\mathbb{1}_L$, the identity map $\mathbb{1}_L\to\mathbb{1}_L$ gives by adjunction a map

$$l_{L/K}\mathbb{1}_L\to\mathbb{1}_K,$$

hence a map

$$h^{\mathrm{o}}(f):h^{\mathrm{o}}(L)=r_{L/F}(\mathbb{1}_L)\to r_{K/F}(\mathbb{1}_K)=h^{\mathrm{o}}(K).$$

We just used the transitivity of adjoints; using it a second time on a 3-layer extension shows that we have indeed defined a functor $\mathbf{field}(F)^{\mathrm{op}}\to\mathbf{Chow}^{\mathrm{o}}(F,\mathbb{Z}[1/p])$.

Suppose that $L=K(t)$. Then $l_{L/K}(\mathbb{1}_L)=h^{\mathrm{o}}(\mathbb{P}^1)=\mathbb{1}_K$, hence $h^{\mathrm{o}}(f)$ is an isomorphism. This shows that our functor induces a functor

$$h^{\mathrm{o}}:S_r^{-1}\,\mathbf{field}(F)^{\mathrm{op}}\to\mathbf{Chow}^{\mathrm{o}}(F,\mathbb{Z}[1/p]),$$

as required.

Suppose now that $K\xrightarrow{f}L$ is a finite and purely inseparable extension of finitely generated fields over F. If X is a smooth projective K-variety, then the map $CH_0(X)\otimes\mathbb{Z}[1/p]\to CH_0(X_L)\otimes\mathbb{Z}[1/p]$ is an isomorphism by Lemma 1.7.1; this shows that $l_{L/K}(\mathbb{1}_L)=\mathbb{1}_K$, hence that $h^{\mathrm{o}}(f)$ is invertible.

(b) The proof is the same as in [Kahn 2009, Remark 7.4]: Write L as a finite purely inseparable extension of a finite separable extension of a purely transcendental extension of K. Then (a) reduces us to the case where L/K is finite and separable. We may write $L=\operatorname{Spec}X$, where X is a 0-dimensional smooth projective K-variety, and $l_{L/K}(\mathbb{1}_L)=h^{\mathrm{o}}(X)$. The conclusion now follows from Lemma 1.5.2.

(c) If $K=F(X)$ for X smooth projective, then Lemma 2.3.7 and Yoneda's lemma show that $h^{\mathrm{o}}(K)\simeq h^{\mathrm{o}}(X)$. For the claim on morphisms, we are reduced (again by Yoneda's lemma) to determining the map

$$\mathbf{Chow}^{\mathrm{o}}(F,\mathbb{Z}[1/p])(h^{\mathrm{o}}(K),h^{\mathrm{o}}(Z))\xrightarrow{h^{\mathrm{o}}(f)^*}\mathbf{Chow}^{\mathrm{o}}(F,\mathbb{Z}[1/p])(h^{\mathrm{o}}(L),h^{\mathrm{o}}(Z))$$

for a smooth projective F-variety Z. By definition of $h^{\mathrm{o}}(f)$, an adjunction computation shows that this map may be rewritten as the map

$$\begin{aligned} CH_0(Z_K)\otimes\mathbb{Z}[1/p] &= \mathbf{Chow}^{\mathrm{o}}(K,\mathbb{Z}[1/p])(\mathbb{1}_K, h^{\mathrm{o}}(Z_K)) \\ &\to \mathbf{Chow}^{\mathrm{o}}(L,\mathbb{Z}[1/p])(\mathbb{1}_L, h^{\mathrm{o}}(Z_L)) = CH_0(Z_L)\otimes\mathbb{Z}[1/p] \end{aligned}$$

given by extension of scalars. The conclusion now follows from Lemma 2.3.9. □

6. Birational motives and cycle modules

Rost [1996] introduced the notion of cycle module and cycle cohomology; he proved [1996, Corollary 12.10] that for any cycle module M, $A^0(X, M)$ is a birational invariant of smooth projective varieties X. In [Merkurjev 2001, Corollary 3.5], he extended this to $A_0(X, M)$ by introducing the category $\mathbf{Chow}^{\mathrm{o}}(F)$ of Definition 2.3.6 (independently from this paper). In the first subsection, we essentially reproduce Section 3 of [Merkurjev 2001]; we don't claim any originality here, but hope this will be a service to the reader since this preprint remains unpublished. In the second subsection, we connect these results with more recent work of Merkurjev.

To lighten notation, we drop the reference to the base field F in the relevant categories.

6.1. ***The functors A^0 and A_0.*** Let $M = (M_n)_{n\in\mathbb{Z}}$ be a cycle module over F in the sense of [Rost 1996]; recall that this is a functor from **field** to graded abelian groups, provided with extra structure (transfers, residues, cup-products by units) subject to certain axioms. To a smooth variety $X \in \mathbf{Sm}$, one associates its *cycle cohomology with coefficients in M* [Rost 1996, Section 5],

$$A^p(X, M_n) = H\Big(\cdots \xrightarrow{\partial} \bigoplus_{x\in X^{(p)}} M_{n-p}(F(x)) \xrightarrow{\partial} \cdots\Big),$$

where the differentials ∂ are induced by the residue homomorphisms. We also have the homological notation

$$A_p(X, M_n) = H\Big(\cdots \xrightarrow{\partial} \bigoplus_{x\in X_{(p)}} M_{n+p}(F(x)) \xrightarrow{\partial} \cdots\Big),$$

so that $A_p(X, M_n) = A^{d-p}(X, M_{d+n})$ if X is purely of dimension d.

Proposition 6.1.1. (a) *Let X, Y be two smooth projective varieties and let*

$$\alpha \in CH_{\dim X}(X \times Y)$$

be a Chow correspondence. Then α induces homomorphisms

$$\alpha^* : A^p(Y, M_n) \to A^p(X, M_n), \quad \alpha_* : A_p(X, M_n) \to A_p(Y, M_n),$$

which make $A^p(-, M_n)$ (*resp.* $A_p(-, M_n)$) *a contravariant* (*resp. covariant*) *functor on* $\mathbf{Chow}^{\text{eff}}$.

(b) *Suppose that* $\alpha \in \mathcal{I}_{\text{rat}}(X, Y)$, *where* $\mathcal{I}_{\text{rat}}$ *is as in Proposition 2.3.5. Then* $\alpha^* A^0(Y, M_n) = 0$ (*resp.* $\alpha_* A_0(X, M_n) = 0$).

Proof. (a) This follows easily from the functoriality of cycle cohomology [Rost 1996, Proposition 4.6, Sections 13 and 14]. Namely, we define α^* as the composition

$$A^p(Y, M_n) \xrightarrow{p_Y^*} A^p(X \times Y, M_n) \xrightarrow{\cup\alpha} A^{p+\dim Y}(X \times Y, M_{n+\dim Y}) \xrightarrow{p_{X*}} A^p(X, M_n), \quad (6.1.2)$$

where $\cup\alpha$ is cup-product with α as in [Rost 1996, Section 14], and α_* similarly. Checking the identities $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ and $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ is a routine matter, using the compatibility of cup-product with pull-backs and the projection formula [ibid].

(b) We may assume X irreducible; let $Z \subset X$ be a proper closed subset such that α is supported on $Z \times Y$, and let $U = X - Z$. We consider the cases of α^* and α_* separately.

In the first case, we observe that (6.1.2) also makes sense for X smooth (not necessarily projective) and that $A^0(X, M_n) \to A^0(U, M_n)$ is injective (both groups being subsets of $M_n(F(X))$). Therefore it suffices to see that (6.1.2) is 0 when X is replaced by U, which is obvious since $\alpha_{|CH_{\dim X}(U \times Y)} = 0$.

In the second case, we generalise the argument in the proof of Proposition 2.3.5: if $x \in X_{(0)}$, it suffices to show that the composition

$$M_n(F(x)) \xrightarrow{i_{x*}} A_0(X, M_n) = A^{\dim X}(X, M_{n+\dim X}) \xrightarrow{p_Y^*} A^{\dim X}(X \times Y, M_{n+\dim X}) \xrightarrow{\cup\alpha} A^{\dim X+\dim Y}(X \times Y, M_{n+\dim X+\dim Y}) \xrightarrow{p_{Y*}} A^{\dim Y}(Y, M_{n+\dim Y}) = A_0(Y, M_n)$$

is 0. If $q_Y : x \times Y \to x$ is the first projection, we have

$$p_Y^* i_{x*} = (i_x \times 1_Y)_* q_Y^*$$

[Rost 1996, Proposition 4.1(3)]. For $a \in M_n(F(x))$, we now have

$$p_Y^* i_{x*} a \cup \alpha = (i_x \times 1_Y)_* q_Y^* a \cup \alpha = (i_x \times 1_Y)_* (q_Y^* a \cup (i_x \times 1_Y)^* \alpha)$$

by the projection formula [Rost 1996, Section 14.5]. As in the proof of Proposition 2.3.5 we reduce to the case where $x \notin Z$, and then $(i_x \times 1_Y)^* \alpha = 0$. $\square$

From Proposition 6.1.1(b), we immediately deduce:

Corollary 6.1.3. (a) *For any cycle module M and any $n \in \mathbb{Z}$, the assignment*

$$\mathbf{Sm}^{\mathrm{proj}} \ni X \mapsto A^0(X, M_n) \quad (\textit{resp.}\ A_0(X, M_n))$$

extends to a contravariant (*resp. a covariant*) *additive functor*

$$A^0(-, M_n) : \mathbf{Chow}^{\mathrm{o}} \to \mathbf{Ab} \qquad (\textit{resp.}\ A_0(-, M_n)).$$

(b) *Let $X \in \mathbf{Sm}^{\mathrm{proj}}$ be such that $h^{\mathrm{o}}(X) \simeq \mathbb{1} \in \mathbf{Chow}^{\mathrm{o}}(F)$. Then the maps*

$$M_n(F) \to A^0(X, M_n), \quad A_0(X, M_n) \to M_n(F)$$

induced by the structural map $\pi_X : X \to \operatorname{Spec} F$ are isomorphisms for any cycle module M and any $n \in \mathbb{Z}$.

This proves the implication (iv) $\Longrightarrow$ (v) in Proposition 3.1.1.

6.2. ***Relationship with Merkurjev's work.*** For $A^0(X, M_n)$, Corollary 6.1.3(b) is part of a theorem of Merkurjev:

Proposition 6.2.1 [Merkurjev 2008, Theorem 2.11(3) $\Longrightarrow$ (1)]. *If $CH_0(X_E) \xrightarrow{\sim} \mathbb{Z}$ for any extension E/F, then $M_n(F) \xrightarrow{\sim} A^0(X, M_n)$ for all cycle modules M and all $n \in \mathbb{Z}$.*

Indeed, this condition is equivalent to $h^{\mathrm{o}}(X) \simeq \mathbb{1}$ in $\mathbf{Chow}^{\mathrm{o}}$ by (iv) $\Longleftrightarrow$ (i) in Proposition 3.1.1.

Merkurjev proves the converse implication. For this, he defines a cycle module K^X such that

$$K_n^X(E) = A_0(X_E, K_n)$$

for any extension E/F. Here, K is the cycle module given by Milnor K-theory. He shows:

Theorem 6.2.2 [Merkurjev 2008, Theorem 2.10]. *The functor*

$$\mathbf{CM} \to \mathbf{Ab}, \quad M \mapsto A^0(X, M_0),$$

from the category of cycle modules to abelian groups is corepresented by K^X.

See [Kahn 2011, Theorem 1.3] for a generalisation to nonproper X.

Let us give a proof of the converse to Proposition 6.2.1 via birational motives, using only the existence of K^X and thus completing the proof of Proposition 3.1.1. Let us say that a cycle module M is *connected* if $M_n = 0$ for $n < 0$; we note that

$$A^0(X, M_0) = M_0(F(X)) \quad \text{if } M \text{ is connected.} \tag{6.2.3}$$

As K^X is connected and $K_0^X(E) = CH_0(X_E)$, the condition

$$K_0^X(F) \xrightarrow{\sim} A^0(X, K_0^X)$$

translates as $CH_0(X) \xrightarrow{\sim} CH_0(X_{F(X)})$, which in turn implies condition (iii) in Proposition 3.1.1.

We are now going to use Theorem 6.2.2 to clarify the relationship between birational motives and cycle modules.

Theorem 6.2.4. *Let* $\mathrm{Mod}\text{–}\mathbf{Chow}^{\mathrm{o}}$ *be the category of additive contravariant functors from* $\mathbf{Chow}^{\mathrm{o}}$ *to* $\mathbf{Ab}$. *The functor*

$$A^0 : \mathbf{CM} \to \mathrm{Mod}\text{–}\mathbf{Chow}^{\mathrm{o}}$$

from Corollary 6.1.3(a) *has a fully faithful left adjoint* $\Lambda \mapsto K^\Lambda$; *the essential image of this left adjoint is contained in the full subcategory of connected cycle modules.*

Proof. We first observe that $X \mapsto K^X$ extends to a functor

$$\mathbf{Chow}^{\mathrm{o}} \to \mathbf{CM}$$

thanks to Corollary 6.1.3(a) (case of A_0). Let $\Lambda \in \mathrm{Mod}\text{–}\mathbf{Chow}^{\mathrm{o}}$. We define

$$K^\Lambda = \varinjlim_{y(X)\to\Lambda} K^X,$$

where $y : \mathbf{Chow}^{\mathrm{o}} \to \mathrm{Mod}\text{–}\mathbf{Chow}^{\mathrm{o}}$ is the additive Yoneda functor, and the colimit is taken on the comma category $y \downarrow \Lambda$ [Mac Lane 1998, Chapter II, Section 6]. Since K^X is connected for any smooth projective X, K^Λ is connected. For a cycle module M, the identity

$$\mathbf{CM}(K^\Lambda, M) \simeq \mathrm{Mod}\text{–}\mathbf{Chow}^{\mathrm{o}}(\Lambda, A^0(M))$$

follows from Theorem 6.2.2 and Yoneda's lemma, thus proving the existence of the left adjoint and the statement on its essential image.

It remains to show that $\Lambda \mapsto K^\Lambda$ is fully faithful or, equivalently, that the unit map

$$\Lambda \to A^0(K^\Lambda)$$

is an isomorphism for all Λ. Let $Y \in \mathbf{Sm}^{\mathrm{proj}}$; we need to show that

$$\Lambda(h^{\mathrm{o}}(Y)) \to A^0(Y, K_0^\Lambda) = K_0^\Lambda(F(Y))$$

is an isomorphism, where we just used (6.2.3). We compute:

$$\begin{aligned} K_0^\Lambda(F(Y)) &= \varinjlim_{y(X)\to\Lambda} K_0^X(F(Y)) = \varinjlim_{y(X)\to\Lambda} CH_0(X_{F(Y)}) \\ &= \varinjlim_{y(X)\to\Lambda} \mathbf{Chow}^{\mathrm{o}}(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(X)) \\ &= \varinjlim_{y(X)\to\Lambda} y(h^{\mathrm{o}}(X))(h^{\mathrm{o}}(Y)) = \Lambda(h^{\mathrm{o}}(Y)). \end{aligned}$$

$\square$

We describe the essential image of the functor $K^?$ in [Kahn and Sujatha 2015b, Theorem 5.1.2].

7. Locally abelian schemes

In this section, F is perfect. We drop it from the notation for relevant categories.

7.1. *The Albanese scheme of a smooth projective variety.*

Definition 7.1.1. (a) Let X be a smooth separated F-scheme (not necessarily of finite type). For each connected component X_i of X, let E_i be its field of constants, that is, the algebraic closure of F in $F(X_i)$. We define

$$\pi_0(X) = \coprod_i \operatorname{Spec} E_i.$$

There is a canonical F-morphism $X \to \pi_0(X)$; $\pi_0(X)$ is called the *scheme of constants* of X.

(b) If $\dim X = 0$ (equivalently $X \xrightarrow{\sim} \pi_0(X)$), we write $\mathbb{Z}[X]$ for the 0-dimensional group scheme representing the étale sheaf $f_*\mathbb{Z}$, where $f : X \to \operatorname{Spec} F$ is the structural morphism.

Definition 7.1.2. (a) For an F-group scheme G, we denote by G^0 the kernel of the canonical map $G \to \pi_0(G)$ of Definition 7.1.1; this is the *neutral component* of G.

(b) An F-group scheme G is called a *lattice* if $G^0 = \{1\}$ and the geometric fibre of $\pi_0(G) = G$ is a free finitely generated abelian group.

Definition 7.1.3 [Ramachandran 2001]. (a) Recall that a *semiabelian variety* is an extension of an abelian variety by a torus. We denote by **SAb** the category of semiabelian F-varieties, and by **Ab** the full subcategory of abelian varieties.

(b) We denote by **SAbS** the full subcategory of the category of commutative F-group schemes consisting of those objects $\mathcal{A}$ such that

- $\pi_0(\mathcal{A})$ is a lattice;
- $\mathcal{A}^0$ is a semiabelian variety.

Objects of **SAbS** will be called *locally semiabelian F-schemes*.

(c) We denote by **AbS** the full subcategory of **SAbS** consisting of those $\mathcal{A}$ such that $\mathcal{A}^0$ is an abelian variety. Its objects are called *locally abelian F-schemes*.

Note that **SAbS** is a Serre subcategory of the abelian category of commutative F-group schemes locally of finite type (see [Demazure and Grothendieck 2011, Exp. VI, Proposition 5.4.1 and Théorème 5.4.2]); in particular it is abelian, and **AbS** is idempotent-closed in **SAbS**, hence pseudoabelian.

For any smooth F-variety X, let $\mathcal{A}_{X/F} = \mathcal{A}_X$ be the Albanese scheme of X over F [Ramachandran 2001]: it is an object of **SAbS** and there is a canonical morphism

$$\varphi_X : X \to \mathcal{A}_X, \tag{7.1.4}$$

which is universal for morphisms from X to objects of **SAbS**. There is an exact sequence of group schemes

$$0 \to \mathcal{A}_X^0 \to \mathcal{A}_X \to \mathbb{Z}[\pi_0(X)] \to 0,$$

where $\mathcal{A}_X^0$ is the Albanese variety of X (a semiabelian variety) and $\pi_0(X)$ has been defined above.

The aim of this section is to endow **SAbS** and **AbS** with symmetric monoidal structures, and to relate the latter one to birational motives (see Propositions 7.2.7 and 8.2.1).

Let us recall from [Ramachandran 2001] a description of $\mathcal{A}_X$. Let $\mathbb{Z}[X]$ be the "free" presheaf on F-schemes defined by $\mathbb{Z}[X](Y) = \mathbb{Z}[X(Y)]$ and $\mathcal{Z}_{X/F} = \mathcal{Z}_X$ the associated sheaf on the big fppf site of Spec F. Then $\mathcal{A}_X$ is the universal representable quotient of $\mathcal{Z}_X$. In other words, there is a homomorphism

$$\mathcal{Z}_X \to \mathcal{A}_X,$$

where $\mathcal{A}_X$ is considered as a representable sheaf, which is universal for homomorphisms from $\mathcal{Z}_X$ to sheaves of abelian groups representable by a locally semiabelian F-scheme.

Let us also denote by P_X the universal torsor under $\mathcal{A}_X^0$ constructed by Serre [1958/1959]. There is a map $X \xrightarrow{\tilde{\varphi}_X} P_X$, which is universal for maps from X to torsors under semiabelian varieties. The torsor P_X and the group scheme $\mathcal{A}_X$ have the same class in $\operatorname{Ext}^1_{(\mathrm{Sch}/F)_{\text{ét}}}(\pi_0(\mathcal{A}_X), \mathcal{A}_X^0) = H^1_{\text{ét}}(\pi_0(X), \mathcal{A}_X^0)$ (here we identify $\mathcal{A}_X^0$ with the corresponding representable étale sheaf over the big étale site of Spec F). A beautiful concrete description of this correspondence is given in [Ramachandran 2001, Section 1.2]. The map $\tilde{\varphi}_X$ induces an isomorphism

$$\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{P_X}.$$

We repeat some properties of $\mathcal{A}_X$ as taken from [Ramachandran 2001, Proposition 1.6 and Corollary 1.12] and add one.

Proposition 7.1.5. (a) *$\mathcal{A}_X$ is covariant in X.*

(b) *Let K/F be an extension. Then the natural map*

$$\mathcal{A}_{X_K/K} \to \mathcal{A}_{X/F} \otimes_F K$$

stemming from the universal property is an isomorphism.

(c) *If* $X = Y \amalg Z$, *then the natural map* $\mathcal{A}_{Y/F} \oplus \mathcal{A}_{Z/F} \to \mathcal{A}_{X/F}$ *is an isomorphism.*

(d) *Let* E/F *be a finite extension. For any* E*-scheme* S, *let* $S_{(F)}$ *denote the* (*ordinary*) *restriction of scalars of* S, *i.e.*, *we view* S *as an* F*-scheme. Then there is a natural isomorphism for* X *smooth*

$$R_{E/F}\mathcal{A}_{X/E} \xrightarrow{\sim} \mathcal{A}_{X_{(F)}/F},$$

where $R_{E/F}$ *denotes Weil's restriction of scalars.*

Proof. The only thing which is not in [Ramachandran 2001] is (d). We shall construct the isomorphism by descent from (c), using (b).

Let $f : \operatorname{Spec} E \to \operatorname{Spec} F$ be the structural morphism. Recall that, for any abelian sheaf $\mathcal{G}$ on $(\mathrm{Sch}/E)_{\text{ét}}$, the trace map defines an isomorphism [Milne 1980, Chapter V, Lemma 1.12]

$$f_*\mathcal{G} \xrightarrow{\sim} f_!\mathcal{G},$$

where $f_!$ (resp. f_*) is the left (resp. right) adjoint of the restriction functor f^*. This isomorphism is natural in $\mathcal{G}$.

This being said, the additive version of Yoneda's lemma immediately yields

$$f_!\mathcal{Z}_{X/E} = \mathcal{Z}_{X_{(F)}/F},$$

hence a composition of homomorphisms of sheaves

$$f_*\mathcal{Z}_{X/E} \xrightarrow{\sim} \mathcal{Z}_{X_{(F)}/F} \to \mathrm{Shv}(\mathcal{A}_{X_{(F)}/F}), \tag{7.1.6}$$

where, for clarity, $\mathrm{Shv}(\mathcal{A}_{X_{(F)}/F})$ denotes the sheaf associated to the group scheme $\mathcal{A}_{X_{(F)}/F}$. We also have a chain of homomorphisms

$$f_*\mathcal{Z}_{X/E} \to f_*\,\mathrm{Shv}(\mathcal{A}_{X/E}) \xrightarrow{\sim} \mathrm{Shv}(R_{E/F}\mathcal{A}_{X/E}), \tag{7.1.7}$$

where the last isomorphism is formal. If we can prove that (7.1.6) factors through (7.1.7) into an isomorphism, we are done by Yoneda.

In order to do this, we may assume via (b) that F is algebraically closed, hence that f is completely split. Then the claim follows from (c). □

We record here similar properties for the torsor $P_X = P_{X/F}$ (proofs are similar):

Proposition 7.1.8. (a) $X \mapsto P_X$ *is a functor.*

(b) *Let* K/F *be an extension. Then the natural map* $P_{X_K/K} \to P_{X/F} \otimes_F K$ *stemming from the universal property is an isomorphism.*

(c) *If* $X = Y \amalg Z$, *then there is an isomorphism* $P_{Y/F} \times P_{Z/F} \xrightarrow{\sim} P_{X/F}$ *which is natural in* (Y, Z).

(d) *Let* E/F *be a finite extension. Then there is a natural isomorphism*

$$P_{X_{(F)}/F} \to R_{E/F}P_{X/E}.$$

□

(In (c), the map stems from the fact that coproducts correspond to scheme-theoretic products in an appropriate category of torsors.)

7.2. ***The tensor category of locally semiabelian schemes.*** Recall the Yoneda full embedding $\mathrm{Shv} : \mathbf{SAbS} \to \mathrm{Ab}((\mathrm{Sch}/F)_{\text{ét}})$, where the latter is the category of sheaves of abelian groups over the big étale site of Spec F.

Lemma 7.2.1. (a) *If a sheaf $\mathcal{F} \in \mathrm{Ab}((\mathrm{Sch}/F)_{\text{ét}})$ is an extension of a lattice L by a semiabelian variety A, it is represented by an object of* **SAbS**.

(b) *Let A be a semiabelian variety and L a lattice. Then the étale sheaf $B = A \otimes L$ is represented by a semiabelian variety.*

Proof. (a) If L is constant, then the choice of a basis of L determines a section of the projection $\mathcal{F} \to \mathrm{Shv}(L)$, hence an isomorphism $\mathcal{F} \simeq \mathrm{Shv}(A) \oplus \mathrm{Shv}(L)$. Then $\mathcal{F}$ is represented by $\coprod_{l \in L} A$. In general, L becomes constant on some finite extension E/F, hence $\mathcal{F}_E$ is representable. By full faithfulness, the descent data of $\mathcal{F}_E$ are morphisms of schemes; then we may apply [Serre 1988, Corollary V.4.2(a) or (b)].

(b) Same method as in (a). □

Example 7.2.2. If $L = \mathbb{Z}[\mathrm{Spec}\, E]$, where E is an étale F-algebra, then $A \otimes L = R_{E/F} A_E$.

We shall also need:

Lemma 7.2.3. *Let F be a field, G_1, G_2, G_3 be three semiabelian F-varieties, and let $\varphi : G_1 \times G_2 \to G_3$ be an F-morphism. Assume that $\varphi(g_1, 0) = \varphi(0, g_2) = 0$ identically. Then $\varphi = 0$.*

Proof. By [Kahn 2014, Lemma 3], φ is a homomorphism and the conclusion is obvious. □

Let $\mathcal{A}, \mathcal{B} \in \mathbf{SAbS}$. Viewing them as étale sheaves, we may consider their tensor product $\mathcal{A} \otimes_{\mathrm{shv}} \mathcal{B}$. This tensor product contains the subsheaf $\mathcal{A}^0 \otimes_{\mathrm{shv}} \mathcal{B}^0$, which is clearly not representable. We define

$$\mathcal{A} \otimes_{\mathrm{rep}} \mathcal{B} = \mathcal{A} \otimes_{\mathrm{shv}} \mathcal{B} / \mathcal{A}^0 \otimes_{\mathrm{shv}} \mathcal{B}^0.$$

Proposition 7.2.4. (a) *$\mathcal{A} \otimes_{\mathrm{rep}} \mathcal{B}$ is representable by an object of* **SAbS**.

(b) *For $X, Y \in$* **Sm**, *the natural map*

$$\mathcal{Z}_X \otimes_{\mathrm{shv}} \mathcal{Z}_Y = \mathcal{Z}_{X \times Y} \to \mathcal{A}_{X \times Y}$$

factors into an isomorphism

$$\mathcal{A}_X \otimes_{\mathrm{rep}} \mathcal{A}_Y \xrightarrow{\sim} \mathcal{A}_{X \times Y}.$$

(This corrects [Ramachandran 2001, Corollary 1.12(vi)].)

Proof. (a) We have a short exact sequence

$$0 \to \mathcal{A}^0 \otimes \pi_0(\mathcal{B}) \oplus \mathcal{B}^0 \otimes \pi_0(\mathcal{A}) \to \mathcal{A} \otimes_{\mathrm{rep}} \mathcal{B} \to \pi_0(\mathcal{A}) \otimes \pi_0(\mathcal{B}) \to 0.$$

By Lemma 7.2.1(b), the left-hand side is representable by a semiabelian variety, and the right-hand side is clearly a lattice. We conclude by Lemma 7.2.1(a).

(b) It is enough to show that this holds over the algebraic closure of F. Using Proposition 7.1.5(c) (and the similar statement for $\mathcal{Z}$), we may assume that X and Y are connected. We shall show more generally that, for any locally semiabelian scheme $\mathcal{B}$ and any map $X \times Y \to \mathcal{B}$, the induced sheaf-theoretic map

$$\mathcal{Z}_X \otimes_{\mathrm{shv}} \mathcal{Z}_Y \to \mathcal{B} \tag{7.2.5}$$

factors through $\mathcal{A}_X \otimes_{\mathrm{rep}} \mathcal{A}_Y$. By (a), this will show that the latter has the universal property of $\mathcal{A}_{X\times Y}$.

For $n \in \mathbb{Z}$, we denote by $\mathcal{Z}_X^n$ or $\mathcal{A}_X^n$ the inverse image of n under the augmentation map $\mathcal{Z}_X \to \mathbb{Z}$ or $\mathcal{A}_X \to \mathbb{Z}$ stemming from the structural morphism $X \to \operatorname{Spec} F$. It is a subsheaf of $\mathcal{Z}_X$ or $\mathcal{A}_X$, and $\mathcal{A}_X^n$ is clearly representable (by a variety $\bar{F}$-isomorphic to the semiabelian variety $\mathcal{A}_X^0$). We shall also identify varieties with representable sheaves; this should create no confusion in view of Yoneda's lemma.

We first show that (7.2.5) factors through $\mathcal{A}_X \otimes_{\mathrm{shv}} \mathcal{A}_Y$. It suffices to show that the composition

$$\mathcal{Z}_X \times Y \to \mathcal{Z}_X \otimes_{\mathrm{shv}} \mathcal{Z}_Y \to \mathcal{B}$$

factors through $\mathcal{A}_X \times Y$, and to conclude by symmetry. But $X \times Y$ is connected, so its image in $\mathcal{B}$ falls in some connected component $\mathcal{B}^t$ of $\mathcal{B}$, which is a torsor under $\mathcal{B}^0$; applying the "Variation en fonction d'un paramètre" statement in [Serre 1958/1959, p. 10-05], we see that it extends to a morphism $\mathcal{A}_X^1 \times Y \to \mathcal{B}^t$. Including $\mathcal{B}^t$ into $\mathcal{B}$, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{A}_X^1 \times Y \arrow[r] & \mathcal{B} \\
\mathcal{Z}_X^1 \times Y \arrow[u] \arrow[r] & \mathcal{Z}_X \times Y \arrow[u]
\end{tikzcd}$$

Let $\mathcal{K} = \operatorname{Ker}(\mathcal{Z}_X \to \mathcal{A}_X) = \operatorname{Ker}(\mathcal{Z}_X^0 \to \mathcal{A}_X^0)$. The above diagram shows that the diagram

$$\begin{tikzcd}
\mathcal{K} \times \mathcal{Z}_X^1 \times Y \arrow[r, "a"] \arrow[d, "c"'] & \mathcal{Z}_X^1 \times Y \arrow[d, "d"] \\
\mathcal{Z}_X^1 \times Y \arrow[r, "b"] & \mathcal{B}
\end{tikzcd}$$

commutes, where a is given by the action of $\mathcal{K}$ on $\mathcal{Z}^1_X$ by left translation and c is given by $(k, z, y) \mapsto (z, y)$. Since b is a homomorphism in the first variable, this implies the desired factorisation.

We now show that the composition

$$\mathcal{A}^0_X \otimes_{\mathrm{shv}} \mathcal{A}^0_Y \to \mathcal{A}_X \otimes_{\mathrm{shv}} \mathcal{A}_Y \to \mathcal{B}$$

is 0. It is sufficient to show that the composition of this map with the inclusion $\mathcal{A}^0_X \times \mathcal{A}^0_Y \to \mathcal{A}^0_X \otimes \mathcal{A}^0_Y$ is 0. But $\mathcal{A}^0_X \times \mathcal{A}^0_Y$ is connected, hence its image falls in some connected component, in fact in $\mathcal{B}^0$. This map verifies the hypothesis of Lemma 7.2.3, hence it is 0. □

As a variant, we have:

Proposition 7.2.6. *We have an isomorphism*

$$P_{X\times Y} \xrightarrow{\sim} R_{\pi_0(X)/F}(P_Y \times_F \pi_0(X)) \times R_{\pi_0(Y)/F}(P_X \times_F \pi_0(Y)).$$

Since we are not going to use this, we leave the easy proof to the reader.

Proposition 7.2.4(a) endows **SAbS** with a symmetric monoidal structure compatible with its additive structure, hence also its full subcategory **AbS**. From now on we concentrate on this latter category.

Proposition 7.2.7. *The category* **AbS** *is symmetric monoidal* (*for* $\otimes_{\mathrm{rep}}$) *and pseudo-abelian. Its Kelly radical* $\mathcal{R}$ *is monoidal and has square* 0*. After tensoring with* $\mathbb{Q}$*,* **AbS** $/\mathcal{R}$ *becomes isomorphic to the semisimple category product of the category of abelian varieties up to isogenies and the category of* G_F*-*$\mathbb{Q}$*-lattices.*

Recall that the *Kelly radical* [1964] $\mathcal{R}$ of an additive category $\mathcal{A}$ is defined by

$$\mathcal{R}(A, B) = \{f \in \mathcal{A}(A, B) \mid 1_A - gf \text{ is invertible for all } g \in \mathcal{A}(B, A)\}$$

and that it is a (two-sided) ideal of $\mathcal{A}$.

Proof. For the first claim, we just observe that kernels exist in the category of commutative F-group schemes, and that a direct summand of an abelian variety (resp. of a lattice) is an abelian variety (resp. a lattice). For the second claim, consider the functor

$$T : \mathbf{AbS} \to \mathbf{Ab} \times \mathbf{Lat}, \quad \mathcal{A} \mapsto (\mathcal{A}^0, \pi_0(\mathcal{A})),$$

where **Ab** and **Lat** are respectively the category of abelian varieties and the category of lattices over F (viewed, for example, as full subcategories of the category of étale sheaves over Sm/F). This functor is obviously essentially surjective. After tensoring with $\mathbb{Q}$, it becomes full, because any extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

is rationally split. Now the collection of sets

$$\mathcal{I}(\mathcal{A}, \mathcal{B}) = \{f : \mathcal{A} \to \mathcal{B} \mid T(f) = 0\}$$

defines an ideal $\mathcal{I}$ of **AbS**. If $f \in \mathcal{I}(\mathcal{A}, \mathcal{B})$, then f induces a map

$$\bar{f} : \pi_0(\mathcal{A}) \to \mathcal{B}^0,$$

and this gives a description of $\mathcal{I}$. From this description, it follows immediately that $\mathcal{I}^2 = 0$. In particular, $\mathcal{I} \subseteq \mathcal{R}$.

If we tensor with $\mathbb{Q}$, then **Ab** $\times$ **Lat** becomes semisimple; since **AbS** $/\mathcal{I} \otimes \mathbb{Q}$ is semisimple and $\mathcal{I} \otimes \mathbb{Q}$ is nilpotent, it follows that $\mathcal{I} \otimes \mathbb{Q} = \mathcal{R} \otimes \mathbb{Q}$. In other words, $\mathcal{R}/\mathcal{I}$ is torsion.

Let $f \in \mathcal{R}(\mathcal{A}, \mathcal{B})$. There exists $n > 0$ such that $nf(\mathcal{A}^0) = 0$. But $f(\mathcal{A}^0)$ is an abelian subvariety of $\mathcal{B}^0$, hence $f(\mathcal{A}^0) = 0$ and $f \in \mathcal{I}(\mathcal{A}, \mathcal{B})$. So $\mathcal{R} = \mathcal{I}$.

If we endow the category **Ab** $\times$ **Lat** with the tensor structure

$$(A, L) \otimes (B, M) = (A \otimes M \oplus B \otimes L, L \otimes M),$$

then T becomes a monoidal functor, which shows that $\mathcal{R} = \mathcal{I}$ is monoidal. This completes the proof of Proposition 7.2.7. □

Remarks 7.2.8. (a) The morphisms in **AbS** are best represented in matrix form:

$$\operatorname{Hom}(\mathcal{A}, \mathcal{B}) = \begin{pmatrix} \operatorname{Hom}(\mathcal{A}_0, \mathcal{B}_0) & \operatorname{Hom}(\pi_0(\mathcal{A}), \mathcal{B}_0) \\ 0 & \operatorname{Hom}(\pi_0(\mathcal{A}), \pi_0(\mathcal{B})) \end{pmatrix}$$

(note that $\operatorname{Hom}(\mathcal{A}_0, \pi_0(\mathcal{B})) = 0$). This clarifies the arguments in the proof of Proposition 7.2.7 somewhat.

(b) The Hom groups of **Ab** $\times$ **Lat** are finitely generated $\mathbb{Z}$-modules. It follows from the proof of Proposition 7.2.7 that, for $\mathcal{A}, \mathcal{B} \in$ **AbS**, $T(\operatorname{Hom}(\mathcal{A}, \mathcal{B}))$ has finite index in $\operatorname{Hom}(T(\mathcal{A}), T(\mathcal{B}))$. In particular, for any $\mathcal{A} \in$ **AbS**, $\operatorname{End}(\mathcal{A})$ is an extension of an order in a semisimple $\mathbb{Q}$-algebra by an ideal of square 0.

(c) The functor T has the explicit section

$$(A, L) \mapsto A \oplus L.$$

This section is symmetric monoidal.

8. Chow birational motives and locally abelian schemes

8.1. *The Albanese map.* For any smooth projective variety X, there is a canonical map

$$CH_0(X) \xrightarrow{\mathrm{Alb}_X^F} \mathcal{A}_X(F). \tag{8.1.1}$$

Recall the construction of Alb_X: The map φ_X of (7.1.4) defines for any extension E/F a map $X(E) \to \mathcal{A}_X(E)$, still denoted by φ_X. When E/F is finite, viewing $\mathcal{A}_X$ as an étale sheaf, we have a trace map $\mathrm{Tr}_{E/F} : \mathcal{A}_X(E) \to \mathcal{A}_X(F)$. Then Alb_X maps the class of a closed point $x \in X$ with residue field E to $\mathrm{Tr}_{E/F}\ \varphi_X(x)$.

The map Alb_X is injective for $\dim X = 1$ and surjective if F is algebraically closed. For a curve, this map corresponds to the isomorphism $\mathrm{Pic}_X \simeq \mathcal{A}_X$, where Pic_X is the Picard scheme of X; we then also have $\mathcal{A}_X^0 \simeq J_X$, where J_X is the Jacobian variety of X.

The functoriality of $\mathcal{A}$ shows that there is a chain of isomorphisms

$$\Phi_{X,Y} : \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \xrightarrow{\sim} \mathrm{Mor}(X, \mathcal{A}_Y) \xrightarrow{\sim} \mathcal{A}_Y(F(X)) \tag{8.1.2}$$

(the latter by Weil's theorem on extension of morphisms to abelian varieties [Milne 1986, Theorem 3.1]), hence a canonical map

$$CH_0(Y_{F(X)}) \xrightarrow{\mathrm{Alb}_{X,Y}} \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y), \tag{8.1.3}$$

which generalises (8.1.1); more precisely, we have

$$\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y} = \mathrm{Alb}_Y^{F(X)}. \tag{8.1.4}$$

On the other hand, there is an exact sequence

$$\begin{aligned} 0 \to \mathcal{A}_Y(\pi_0(X)) = \mathrm{Hom}(\mathbb{Z}[\pi_0(X)], \mathcal{A}_Y) \to \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \\ \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y) \to \mathrm{Ext}^1(\mathbb{Z}[\pi_0(X)], \mathcal{A}_Y) = H^1(\pi_0(X), \mathcal{A}_Y), \end{aligned}$$

and the map $\mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y)$ is an isomorphism. From this and (8.1.3) we get a zero sequence

$$0 \to CH_0(Y) \to CH_0(Y_{F(X)}) \to \mathrm{Hom}(\mathcal{A}_X^0, \mathcal{A}_Y^0) \to 0. \tag{8.1.5}$$

Lemma 8.1.6. *Let Y, Z be two smooth projective varieties and $\beta \in CH_0(Z_{F(Y)})$. Then the following diagram commutes*:

$$\begin{tikzcd} CH_0(Y) \arrow[r, "\beta_*"] \arrow[d, "\mathrm{Alb}_Y^F"'] & CH_0(Z) \arrow[d, "\mathrm{Alb}_Z^F"'] \\ \mathcal{A}_Y(F) \arrow[r, "\mathrm{Alb}_{Y,Z}(\beta)_*"] & \mathcal{A}_Z(F) \end{tikzcd}$$

Proof. Without loss of generality, we may assume that β is given by an integral subscheme W in $Y \times Z$. Then the composite $f = p_Y i_W$ is a proper surjective generically finite morphism, where p_Y denotes the projection and i_W is the inclusion of W in $Y \times Z$.

Let V be an affine dense open subset of Y such that $f_{|f^{-1}(V)}$ is finite. Any element of $CH_0(Y)$ may be represented by a zero-cycle with support in V (see

[Roberts 1972]), so it is enough to check the commutativity of the diagram on zero-cycles on Y of the form y, where $y \in V_{(0)}$. For such a y, we have $\beta_* y = p_*(f^{-1}(y))$, where $p = p_Z i_W$.

On the other hand, the composition $\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V}$ may be described as follows: Let d be the degree of $f_{|f^{-1}(V)}$, $f^{-1}(V)^{[d]}$ the d-fold symmetric power of $f^{-1}(V)$ and $f^* : V \to f^{-1}(V)^{[d]}$ the map $x \mapsto f^{-1}(x)$. Then

$$\mathrm{Alb}_{Y,Z}(\beta)_* \circ (\mathrm{Alb}_Y^F)_{|V} = \Sigma_d \circ (\varphi_Z)^{[d]} \circ p_*^{[d]} \circ f^*,$$

where $\Sigma_d : \mathcal{A}_Z^{[d]} \to \mathcal{A}_Z$ is the summation map. The commutativity of the diagram is now clear. □

8.2. *The Albanese functor.*

Proposition 8.2.1. *The assignment $X \mapsto \mathcal{A}_X$ defines, via (8.1.3), a symmetric monoidal additive functor*

$$\mathrm{Alb} : \mathbf{Chow}^{\mathrm{o}} \to \mathbf{AbS},$$

which becomes full and essentially surjective after tensoring with $\mathbb{Q}$.

Proof. Since **AbS** is pseudoabelian, it suffices to construct the functor on $\mathbf{Cor}^{\mathrm{o}}$. Let $\alpha \in CH_0(Y_{F(X)})$ and $\beta \in CH_0(Z_{F(Y)})$. We want to show that

$$\mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta) \circ \mathrm{Alb}_{X,Y}(\alpha).$$

But β induces a map

$$\beta_* : CH_0(Y_{F(X)}) \to CH_0(Z_{F(X)}),$$

and we have the equality $\beta_*\alpha = \beta \circ \alpha$ (see the proof of Proposition 2.3.5). Hence, applying Lemma 8.1.6, in which we replace F by $F(X)$, we get

$$\mathrm{Alb}_Z^{F(X)}(\beta \circ \alpha) = \mathrm{Alb}_Z^{F(X)}(\beta_*\alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\mathrm{Alb}_Y^{F(X)}(\alpha)).$$

Applying now (8.1.4), we get

$$\Phi_{X,Z} \circ \mathrm{Alb}_{X,Z}(\beta \circ \alpha) = \mathrm{Alb}_{Y,Z}(\beta)_*(\Phi_{X,Y} \circ \mathrm{Alb}_{X,Y}(\alpha)).$$

On the other hand, the diagram

$$\begin{tikzcd}
\mathcal{A}_Y(F(X)) \arrow[r, "\mathrm{Alb}_{Y,Z}(\beta)_*"] & \mathcal{A}_Z(F(X)) \\
\mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \arrow[u, "\Phi_{X,Y}", "\wr"'] \arrow[r, "\mathrm{Alb}_{Y,Z}(\beta)_*"] & \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \arrow[u, "\Phi_{X,Z}", "\wr"']
\end{tikzcd}$$

obviously commutes, which concludes the proof that Alb is a functor.

Compatibility with the monoidal structures follows from Proposition 7.2.4(b). It remains to show the assertions on fullness and essential surjectivity.

Fullness: For any Y, the map $\mathrm{Alb}_Y^F \otimes\mathbb{Q}$ is surjective. This follows from the case where F is algebraically closed (in which case Alb_Y^F itself is surjective) by a transfer argument. Replacing the ground field F by $F(X)$ for some other X, we get that $\mathrm{Alb}_{X,Y} \otimes\mathbb{Q}$ is surjective. This shows that the restriction of $\mathrm{Alb}\otimes\mathbb{Q}$ to $\mathbf{Cor}^{\mathrm{o}}\otimes\mathbb{Q}$ is full; but the pseudoabelianisation of a full functor is evidently full (a direct summand of a surjective homomorphism of abelian groups is surjective).

Essential surjectivity: We first note that, after tensoring with $\mathbb{Q}$, the extension

$$0 \to \mathcal{A}^0 \to \mathcal{A} \to \pi_0(\mathcal{A}) \to 0$$

becomes split for any $\mathcal{A} \in \mathbf{AbS}$. Indeed the extension class belongs to

$$\mathrm{Ext}^1_F(\pi_0(\mathcal{A}), \mathcal{A}^0);$$

this group sits in an exact sequence (coming from an Ext spectral sequence)

$$0 \to H^1(F, \mathrm{Hom}_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})) \to \mathrm{Ext}^1_F(\pi_0(\mathcal{A}), \mathcal{A}^0) \\ \to H^0\big(F, \mathrm{Ext}^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})\big).$$

Since the restriction $\pi_0(\mathcal{A})_{|\bar F}$ is a constant sheaf of free finitely generated abelian groups, the group $\mathrm{Ext}^1_{\bar F}(\pi_0(\mathcal{A})_{|\bar F}, \mathcal{A}^0_{|\bar F})$ is 0, while the left group is torsion as a Galois cohomology group. It is now sufficient to show separately that L and A are in the essential image of $\mathrm{Alb}\otimes\mathbb{Q}$, where L (resp. A) is a lattice (resp. an abelian variety).

A lattice L corresponds to a continuous integral representation ρ of G_F. But it is well known that $\rho\otimes\mathbb{Q}$ is of the form $\theta\otimes\mathbb{Q}$, where θ is a direct summand of a permutation representation of G_F. If E is the corresponding étale algebra, we therefore have an isomorphism of L with a direct summand of $(\mathrm{Alb}\otimes\mathbb{Q})(E)$.

Given an abelian variety A, we simply note that

$$A = \mathrm{Alb}(\tilde h(A)),$$

where $\tilde h(A)$ is the reduced motive of A, that is, $h(A) = \mathbb{1}\oplus\tilde h(A)$, where the splitting is given by the rational point $0 \in A(F)$. □

Remark 8.2.2. Let $\mathcal{R}$ be the Kelly radical of $\mathbf{AbS}$ (see Proposition 7.2.7). If F is a finitely generated field, the groups $\mathcal{R}(\mathcal{A}, \mathcal{B})$ are finitely generated by the Mordell–Weil–Néron theorem. To see this, note that if L is a lattice and A an abelian variety, then

$$\mathrm{Hom}(L, A) \xrightarrow{\sim} \mathrm{Hom}(L_{|\bar F}, A_{|\bar F})^{G_F}$$

and that the right term may be rewritten as $B(F)$, where $B = L^*\otimes A$ (compare Lemma 7.2.1). Hence the Hom groups in $\mathbf{AbS}$ are finitely generated as well. In this case, Proposition 8.2.1 implies that, for any $M, N \in \mathbf{Chow}^{\mathrm{o}}$, *the image of the map* $\mathrm{Alb}_{M,N}$ *has finite index in the group* $\mathrm{Hom}(\mathrm{Alb}(M), \mathrm{Alb}(N))$.

Lemma 8.2.3. *Suppose that Y is a curve. Then the map (8.1.3) fits into an exact sequence*

$$0 \to CH_0(Y_{F(X)}) \xrightarrow{\mathrm{Alb}_{X,Y}} \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_Y) \to Br(F(X)) \to Br(F(X \times Y)),$$

where Br denotes the Brauer group. In particular, (8.1.3) $\otimes \mathbb{Q}$ *is an isomorphism.*

Proof. First assume that X is a point; then (8.1.3) reduces to (8.1.1). Suppose first that F is separably closed. Then (8.1.1) is bijective (see comments at the beginning of this section). In the general case, let F_s be a separable closure of F, and $G = \mathrm{Gal}(F_s/F)$. Since $\mathcal{A}_Y$ is a sheaf for the étale topology, we get a commutative diagram

$$\begin{tikzcd} CH_0(Y_s)^G \arrow[r, "\mathrm{Alb}_Y^{F_s}", "\sim"'] & \mathcal{A}_Y(F_s)^G \\ CH_0(Y) \arrow[u] \arrow[r, "\mathrm{Alb}_Y^F"] & \mathcal{A}_Y(F) \arrow[u, "\wr"] \end{tikzcd}$$

where $Y_s = Y \times_F F_s$ and the top horizontal and right vertical maps are bijective. The lemma then follows from the classical exact sequence

$$0 \to CH_0(Y) \to CH_0(Y_s)^G \to Br(F) \to Br(F(Y)).$$

The case where X is not necessarily a point now follows from this special case and the construction of (8.1.3). □

Theorem 8.2.4. *Let $\mathbf{Chow}^{\mathrm{o}}_{\leq 1}$ denote the thick subcategory of $\mathbf{Chow}^{\mathrm{o}}$ generated by motives of varieties of dimension ≤ 1, and let $\iota : \mathbf{Chow}^{\mathrm{o}}_{\leq 1} \to \mathbf{Chow}^{\mathrm{o}}$ be the canonical inclusion. Then*:

(a) *After tensoring morphisms with $\mathbb{Q}$, $\mathrm{Alb} \circ \iota : \mathbf{Chow}^{\mathrm{o}}_{\leq 1} \to \mathbf{AbS}$ becomes an equivalence of categories.*

(b) *Let j be a quasi-inverse. Then $\iota \circ j$ is right adjoint to* Alb.

Proof. (a) The full faithfulness follows from Lemma 8.2.3. For the essential surjectivity, we may reduce as in the proof of Proposition 8.2.1 to proving that lattices and abelian varieties are in the essential image. For lattices, this is proven in Proposition 8.2.1. For an abelian variety A, use the fact that A is isogenous to a quotient of the Jacobian of a curve, and Poincaré's complete reducibility theorem.

(b) Let $(M, \mathcal{A}) \in \mathbf{Chow}^{\mathrm{o}}_{\leq 1}(F, \mathbb{Q}) \times \mathbf{AbS}(F, \mathbb{Q})$. To produce a natural isomorphism $\mathbf{Chow}^{\mathrm{o}}_{\leq 1}(F, \mathbb{Q})(M, \iota j(\mathcal{A})) \simeq \mathbf{AbS}(F)(\mathrm{Alb}(M), \mathcal{A}) \otimes \mathbb{Q}$, it is sufficient by (a) to handle the case $M = h^{\mathrm{o}}(X)$, $\mathcal{A} = \mathcal{A}_Y$ for some smooth projective curves X, Y. Then the isomorphism follows from (8.1.2) and Lemma 8.2.3. □

Remarks 8.2.5. (a) Of course the functor $\iota \circ j$ is not a tensor functor (since its image is not closed under tensor product).

(b) In particular, the inclusion functor ι has the left adjoint $j \circ \mathrm{Alb}$. This is a birational version of Murre's results [1990; 1993, Section 2.1] for effective Chow motives; see also [Scholl 1994, Section 4]. Beware however that we have taken the opposite to usual convention for the variance of Chow motives (our functor $X \mapsto h(X)$ is covariant rather than contravariant), so the direction of arrows has to be reversed with respect to Murre's work.

Appendix: Complements on localisation of categories

A.1. *Localisation of symmetric monoidal categories.*

Lemma A.1.1. (a) *Localisation commutes with products of categories for sets of morphisms containing all identities.*[8]

(b) *Let $T_0, T_1 : \mathcal{C} \rightrightarrows \mathcal{D}$ be two functors and $f : T_0 \Rightarrow T_1$ a natural transformation. Let S, S' be collections of morphisms in $\mathcal{C}$ and $\mathcal{D}$ such that $T_i(S) \subseteq S'$, so that T_0 and T_1 pass to localisation. Then f remains a natural transformation between the localised functors.*

Proof. (a) Let S_i be a collection of morphisms in $\mathcal{C}_i$ for $i = 1, 2$, such that S_i contains the identities of all objects of $\mathcal{C}_i$. Then $S_1 \times S_2$ is generated by S_1 and S_2 in the sense that the equality

$$(s_1, s_2) = (s_1, 1) \circ (1, s_2)$$

holds in $S_1 \times S_2$ for any pair (s_1, s_2). The conclusion easily follows (see [Maltsiniotis 2005, Lemme 2.1.7]).

(b) This is true because f commuted with the members of S, hence it now commutes with their inverses. □

Proposition A.1.2. *Let $\mathcal{C}$ be a category with a product $\bullet : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and let S be a collection of morphisms in $\mathcal{C}$ containing all identities. Assume that $S \bullet S \subseteq S$. Then*:

(a) *There is a unique product $S^{-1}\mathcal{C} \times S^{-1}\mathcal{C} \to S^{-1}\mathcal{C}$ such that the localisation functor $P_S : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with the two products.*

(b) *If $\bullet$ is monoidal* (*resp. braided*, *symmetric*, *unital*), *the induced product on $S^{-1}\mathcal{C}$ enjoys the same properties and P_S is monoidal* (*resp. braided*, *symmetric*, *unital*).

Proof. Item (a) follows from Lemma A.1.1(a), and (b) from Lemma A.1.1(b). □

[8] We thank M. Bondarko for pointing out the importance of the identities.

A.2. ***Semiadditive categories.*** This subsection is a reformulation of [Mac Lane 1998, Chapter VIII, Section 2]; see also [Mac Lane 1950, Section 18 and beginning of Section 19].

Lemma A.2.1. (a) *For a category $\mathcal{A}$, the following conditions are equivalent*:

(i) *$\mathcal{A}$ has a* 0 *object* (*initial and final*), *binary products and coproducts*, *and for any $A, B \in \mathcal{A}$, the map*

$$A \amalg B \to A \times B$$

given on A by $(1_A, 0)$ and on B by $(0, 1_B)$ is an isomorphism.

(ii) *$\mathcal{A}$ has finite products*, *and for any $A, B \in \mathcal{A}$, $\mathcal{A}(A, B)$ has a structure of a commutative monoid*, *and composition is distributive with respect to these monoid laws.*

(iii) *Same as* (ii), *replacing product by coproduct.*

We then say that $\mathcal{A}$ is a semiadditive category *and write $A \oplus B$ for the product or coproduct of two objects A, B.*

(b) *If $\mathcal{A}$ is a semiadditive category*, *the law $(A, B) \mapsto A \oplus B$ endows $\mathcal{A}$ with a canonical unital symmetric monoidal structure.*

Proof. (a) By duality, we only need to show (i) $\iff$ (ii). (i) $\implies$ (ii) follows from [Mac Lane 1998, Chapter VIII, Section 2, Example 4(a)]; recall that for two morphisms $f, g : A \to B$ in $\mathcal{A}$, Mac Lane defines their sum $f + g$ as the composition

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[dd, "f+g"'] & A \times A \arrow[dr, "f\times g"] & \\ & & B \times B \\ B & B \amalg B \arrow[l, "\nabla_B"'] \arrow[ur, "\sim"'] & \end{tikzcd}$$

where Δ_A is the diagonal and ∇_B is the codiagonal.

As for (ii) $\implies$ (i), it is implicit in the proof of [Mac Lane 1998, Chapter VIII, Section 2, Theorem 2]. Indeed, Mac Lane defines a biproduct of two objects $A, B \in \mathcal{A}$ as a diagram

$$A \underset{i_1}{\overset{p_1}{\leftrightarrows}} C \underset{i_2}{\overset{p_2}{\leftrightarrows}} B$$

satisfying $p_1 i_1 = 1_A$, $p_2 i_2 = 1_B$ and $i_1 p_1 + i_2 p_2 = 1_C$. Let us say that such a diagram is a *biproduct** if the further identities $p_1 i_2 = 0$ and $p_2 i_1 = 0$ hold. Then, Mac Lane proves that a biproduct* is a product and that a product is a biproduct*. Dually, a biproduct* is the same as a coproduct, hence binary products and coproducts are canonically isomorphic, and one checks from his proof that the isomorphism is given by the map of (i).

(Let us clarify that Mac Lane proves that a biproduct is a biproduct* if the addition law on morphisms has the cancellation property; but we don't use this part of his proof.)

(b) This is obvious: already finite products or coproducts define a canonical symmetric monoidal structure. □

Define a *semiadditive functor* between two semiadditive categories $\mathcal{A}, \mathcal{B}$ as a functor $F : \mathcal{A} \to \mathcal{B}$ which preserves addition of morphisms. Note that any semiadditive functor preserves $\oplus$, by the characterisation of biproducts via equations (see proof of Lemma A.2.1(a)).

A.3. *Localisation of R-linear categories.*

Theorem A.3.1. *Let $\mathcal{A}$ be a semiadditive category and S a family of morphisms of $\mathcal{A}$, containing all identities and stable under $\oplus$. Then $S^{-1}\mathcal{A}$ and the localisation functor $P_S : \mathcal{A} \to S^{-1}\mathcal{A}$ are semiadditive.*

Proof. We use the characterisation (i) of semiadditive categories in Lemma A.2.1; by [Maltsiniotis 2005, Lemme 1.3.6 and Proposition 2.1.8], P_S preserves products and coproducts, and transforms the isomorphisms $A \amalg B \xrightarrow{\sim} A \times B$ into isomorphisms. □

To "catch" additive categories (as opposed to semiadditive categories), we could do as in [Mac Lane 1950] and postulate the existence of an endomorphism -1_A for each object A. We prefer to do this more generally by dealing with R-linear categories, where R is an arbitrary ring (an R-linear category is simply a semiadditive R-category).

More precisely, let $\mathcal{A}$ be an R-linear category. Then in particular:

- $\mathcal{A}$ is a semiadditive category.
- It enjoys an action of the multiplicative monoid underlying R, i.e., there is a homomorphism of monoids $R \to \mathrm{End}(Id_A)$, where $\mathrm{End}(Id_A)$ is the monoid of natural transformations of the identity functor of A.
- For $\lambda \in R$ and $A \in \mathcal{A}$, let λ_A denote the corresponding endomorphism of A. Then we have identities

$$(\lambda + \mu)_A = \lambda_A + \mu_A. \tag{A.3.2}$$

Conversely, the following lemma is straightforward.

Lemma A.3.3. *Let $\mathcal{A}$ be a semiadditive category provided with an action of R verifying (A.3.2). Then $\mathcal{A}$ is an R-linear category.*

From this lemma, it follows:

Theorem A.3.4. *Theorem A.3.1 extends to R-linear categories.*

A.4. ***Localisation and pseudoabelian envelope.***

Lemma A.4.1. *Let $\mathcal{A}$ an additive category and S a family of morphisms in $\mathcal{A}$, stable under direct sums. Let $\mathcal{A} \to \mathcal{A}^\natural$ denote the pseudoabelian envelope of $\mathcal{A}$, and let us denote by $S^\natural$ the set of direct summands of members of S in $\mathcal{A}^\natural$. Then the natural functors*

$$(S^{-1}\mathcal{A})^\natural \to (S^{-1}(\mathcal{A}^\natural))^\natural \to ((S^\natural)^{-1}(\mathcal{A}^\natural))^\natural$$

are equivalences of categories.

Proof. All categories are universal for additive functors T from $\mathcal{A}$ to a pseudo-abelian category such that $T(S)$ is invertible. □

A.5. ***Localisation and group completion.***

Lemma A.5.1. *Let $\mathcal{A}$ be a semiadditive category. There exists an additive category $\mathcal{A}^+$ and a semiadditive functor $\iota : \mathcal{A} \to \mathcal{A}^+$ with the following 2-universal property: any semiadditive functor from $\mathcal{A}$ to an additive category factors through ι up to a unique natural isomorphism.*

A model of $\mathcal{A}^+$ may be given as follows: the objects of $\mathcal{A}^+$ are those of $\mathcal{A}$; if $A, B \in \mathcal{A}$, then $\mathcal{A}^+(A, B)$ is the group completion of the commutative monoid $\mathcal{A}(A, B)$.

The category $\mathcal{A}^+$ is called the group completion *of $\mathcal{A}$.*

The proof is straightforward and omitted.

Proposition A.5.2. *Let $\mathcal{A}$ be a semiadditive category, and let S be a family of morphisms in $\mathcal{A}$, containing the identities and stable under direct sums. Keep writing S for the image of S in the group completion $\mathcal{A}^+$. Then the functor*

$$S^{-1}\iota : S^{-1}\mathcal{A} \to S^{-1}(\mathcal{A}^+)$$

induces an equivalence of categories

$$\tilde{\iota} : (S^{-1}\mathcal{A})^+ \xrightarrow{\sim} S^{-1}(\mathcal{A}^+).$$

Here we use the structure of semiadditive category on $S^{-1}\mathcal{A}$ given in Theorem A.3.1.

Proof. The existence of $\tilde{\iota}$ follows from the universal property of group completion. A quasi-inverse to $\tilde{\iota}$ is obtained by group-completing the functor $\mathcal{A} \to S^{-1}\mathcal{A}$ (which is semiadditive by Theorem A.3.1), and then extending the resulting functor to $S^{-1}(\mathcal{A}^+)$. □

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BRUNO KAHN: bruno.kahn@imj-prg.fr
Institut de Mathématiques de Jussieu-Paris Rive Gauche, 4 place Jussieu, Case 247, 75252 Paris Cedex 05, France

RAMDORAI SUJATHA: sujatha@math.ubc.ca
Mathematics Department, University of British Columbia, Vancouver BC V6T 1Z2, Canada

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On the *K*-theory of linear groups

Daniel Kasprowski

We prove that for a finitely generated linear group over a field of positive characteristic the family of quotients by finite subgroups has finite asymptotic dimension. We use this to show that the K-theoretic assembly map for the family of finite subgroups is split injective for every finitely generated linear group G over a commutative ring with unit under the assumption that G admits a finite-dimensional model for the classifying space for the family of finite subgroups. Furthermore, we prove that this is the case if and only if an upper bound on the rank of the solvable subgroups of G exists.

1. Introduction

For every group G and every ring A there is a functor $\mathbb{K}_A : \operatorname{Or} G \to \mathfrak{Spectra}$ from the orbit category of G to the category of spectra sending G/H to (a spectrum weakly equivalent to) the K-theory spectrum $\mathbb{K}(A[H])$ for every subgroup $H \leq G$. For any such functor $F : \operatorname{Or} G \to \mathfrak{Spectra}$, a G-homology theory $\mathbb{F}$ can be constructed via

$$\mathbb{F}(X) := \operatorname{Map}_G(\_, X_+) \wedge_{\operatorname{Or} G} F;$$

see [Davis and Lück 1998]. We will write $H_n^G(X; F) := \pi_n \mathbb{F}(X)$ for its homotopy groups. The assembly map for the family of finite subgroups is the map

$$H_n^G(\underline{E}G; \mathbb{K}_A) \to H_n^G(\mathrm{pt}; \mathbb{K}_A) \cong K_n(A[G])$$

induced by the map $\underline{E}G \to \mathrm{pt}$. Here $\underline{E}G$ denotes the classifying space for the family of finite subgroups; see [Lück 2000]. The assembly map is a helpful tool for relating the K-theory of the group ring $A[G]$ to the K-theory of the group rings over the finite subgroups $H \leq G$. It can more generally be defined for any additive G-category instead of A; see [Bartels and Reich 2007]. Note that additive categories will always be small and that K-theory will always mean the nonconnective K-theory constructed by Pedersen and Weibel [1985].

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Theorem 1.1. *Let R be a commutative ring with unit and let $G \le \mathrm{GL}_n(R)$ be finitely generated. If G admits a finite-dimensional model for the classifying space $\underline{E}G$, then the assembly map*

$$H_n^G(\underline{E}G; \mathbb{K}_{\mathcal{A}}) \to K_n(\mathcal{A}[G])$$

is split injective for every additive G-category $\mathcal{A}$.

If $\mathcal{A}$ is an additive G-category with involution such that, for every virtually nilpotent subgroup $A \le G$, there exists $i_0 \in \mathbb{N}$ such that for $i \ge i_0$ we have $K_{-i}(\mathcal{A}[A]) = 0$, then the L-theoretic assembly map

$$H_n^G(\underline{E}G; \mathbb{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(\mathcal{A}[G])$$

is split injective.

Theorem 1.1 implies the (generalized integral) Novikov conjecture for these groups by [Kasprowski 2015b, Section 6], since virtually nilpotent groups satisfy the Farrell–Jones conjecture by [Wegner 2015]. The (rational) Novikov conjecture for these groups is already known, by Guentner, Higson and Weinberger [Guentner et al. 2005], where it is shown that the Baum–Connes assembly map is split injective for linear groups.

We will use inheritance properties to reduce the proof of the theorem to the case where the ring R has trivial nilradical and show that in this case the family $\{F \backslash G\}_{F \in \mathcal{F}in}$ has finite decomposition complexity, where $\mathcal{F}in$ denotes the family of finite subgroups of G. Then the theorem follows from the main theorem of [Kasprowski 2014]. For convenience, the necessary results of [Kasprowski 2014] are recalled in the Appendix.

By a result of Alperin and Shalen [1982], a finitely generated subgroup G of $\mathrm{GL}_n(F)$, where F is a field of characteristic zero, has finite virtual cohomological dimension if and only if there is a bound on the Hirsch rank of the unipotent subgroups of G. This in particular implies that it has a finite-dimensional model for the classifying space $\underline{E}G$. In positive characteristic, a finitely generated subgroup $G \le \mathrm{GL}_n(F)$ always admits a finite-dimensional model for $\underline{E}G$, by [Degrijse and Petrosyan 2015, Corollary 5]. In Section 5 we prove the following generalization:

Proposition 1.2. *Let R be a commutative ring with unit and let $G \le \mathrm{GL}_n(R)$ be finitely generated. Then G admits a finite-dimensional model for $\underline{E}G$ if and only if there exists $N \in \mathbb{N}$ such that $l(A) \le N$ for every solvable subgroup $A \le G$, where $l(A)$ denotes the Hirsch rank of A.*

Let G be a solvable group and $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots G_{n-1} \trianglelefteq G_n = G$ a normal series with abelian factors. The *Hirsch rank* (or *Hirsch length*) $l(G)$ of G is

$$l(G) = \sum_{i=1}^{n} \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} (A_i / A_{i-1}).$$

2. Finite decomposition complexity

Let X be a metric space. A decomposition $X = \bigcup_{i\in I} U_i$ is called *r-disjoint*, if $d(U_i, U_j) > r$ for all $i \neq j \in I$. We then denote the decomposition by

$$X = \bigcup^{r\text{-disj.}} U_i.$$

A *metric family* is a set of metric spaces. A metric family $\{X_i\}_{i\in I}$ has *finite asymptotic dimension uniformly* if there exists an $n \in \mathbb{N}$ such that for every $r > 0$ and $i \in I$ there exist decompositions

$$X_i = \bigcup_{k=0}^{n} U_i^k \quad \text{and} \quad U_i^k = \bigcup_{j\in J_{i,k}}^{r\text{-disj.}} U_{i,j}^k$$

such that $\sup_{i,j,k} U_{i,j}^k < \infty$.

Guentner, Tessera and Yu [Guentner et al. 2013] introduced the following generalization of finite asymptotic dimension:

Definition 2.1. Let $r > 0$. A metric family $\mathcal{X} = \{X_\alpha\}_{\alpha\in A}$ *r-decomposes over a class of metric families* $\mathfrak{D}$ if for every $\alpha \in A$ there exists a decomposition $X_\alpha = U_\alpha^r \cup V_\alpha^r$ and r-disjoint decompositions

$$U_\alpha^r = \bigcup_{i\in I(r,\alpha)}^{r\text{-disj.}} U_{\alpha,i}^r \quad \text{and} \quad V_\alpha^r = \bigcup_{j\in J(r,\alpha)}^{r\text{-disj.}} V_{\alpha,j}^r$$

such that the families $\{U_{\alpha,i}^r\}_{\alpha\in A,\, i\in I(r,\alpha)}$ and $\{V_{\alpha,j}^r\}_{\alpha\in A,\, j\in J(r,\alpha)}$ lie in $\mathfrak{D}$. A metric family $\mathcal{X}$ *decomposes over* $\mathfrak{D}$ if it r-decomposes over $\mathfrak{D}$ for all $r > 0$.

Let $\mathfrak{B}$ denote the class of *bounded families*, i.e., $\mathcal{X} \in \mathfrak{B}$ if there exists $R > 0$ such that $\operatorname{diam} X < R$ for all $X \in \mathcal{X}$. We set $\mathfrak{D}_0 = \mathfrak{B}$. For a successor ordinal $\gamma + 1$ we define $\mathfrak{D}_{\gamma+1}$ to be the class of all metric families which decompose over $\mathfrak{D}_\gamma$. For a limit ordinal λ we define

$$\mathfrak{D}_\lambda = \bigcup_{\gamma<\lambda} \mathfrak{D}_\gamma.$$

A metric family $\mathcal{X}$ has *finite decomposition complexity* (*FDC*) if $\mathcal{X} \in \mathfrak{D}_\gamma$ for some ordinal γ.

A metric space X has FDC if the family $\{X\}$ consisting only of X has FDC. A group G has FDC if it has FDC with any (and thus every) proper left-invariant metric.

A subfamily $\mathcal{Z}$ of a metric family $\mathcal{Y}$ is a metric family $\mathcal{Z}$ such that for each $Z \in \mathcal{Z}$ there exists $Y \in \mathcal{Z}$ with $Y \subseteq X$.

A map $F : \mathcal{X} \to \mathcal{Y}$ between metric families $\mathcal{X}$ and $\mathcal{Y}$ is a set of maps from elements of $\mathcal{X}$ to elements of $\mathcal{Y}$ such that every $X \in \mathcal{X}$ is the domain of at least one $f \in F$. The inverse image $F^{-1}(\mathcal{Z})$ of a subfamily $\mathcal{Z}$ of $\mathcal{Y}$ is the metric family $\{f^{-1}(Z) \mid Z \in \mathcal{Z},\ f \in F\}$. A map $F : \mathcal{X} \to \mathcal{Y}$ is called *uniformly expansive* if there exists a nondecreasing function $\rho : [0, \infty) \to [0, \infty)$ such that for every $f : X \to Y$ in F and every $x, y \in X$ we have

$$d(f(x), f(y)) \le \rho(d(x, y)).$$

We will use the following three results about FDC:

Theorem 2.2 [Guentner et al. 2013, Fibering theorem 3.1.4]. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families and let $F : \mathcal{X} \to \mathcal{Y}$ be uniformly expansive. Assume $\mathcal{Y}$ has FDC and that for every bounded subfamily $\mathcal{Z}$ of $\mathcal{Y}$ the inverse image $F^{-1}(\mathcal{Z})$ has FDC. Then $\mathcal{X}$ also has FDC.*

Theorem 2.3 [Guentner et al. 2013, Theorem 4.1]. *A metric space X with finite asymptotic dimension has FDC.*

While the above theorem is stated only for metric spaces it also holds for metric families which have finite asymptotic dimension uniformly.

Theorem 2.4 [Guentner et al. 2013, Theorem 3.1.7]. *Let X be a metric space, expressed as a union of finitely many subspaces $X = \bigcup_{i=0}^{n} X_i$. If the metric family $\{X_i\}_{i=0,\dots,n}$ has FDC, so does X.*

This theorem again holds for metric families instead of metric spaces, i.e., a metric family $\left\{\bigcup_{i=0}^{n} X_{ij}\right\}_{j\in J}$ has FDC if and only if the family $\{X_{ij}\}_{j\in J,\, i=0,\dots,n}$ has FDC. We will also need the following two results about finite asymptotic dimension:

Lemma 2.5. *Let $P : \mathcal{X} \to \mathcal{Y}$ be a family of maps such that for some $k > 0$ each $p \in P$ is k-Lipschitz. Suppose that $\mathcal{Y}$ has finite asymptotic dimension uniformly and that for each $R > 0$ the family*

$$\left\{p^{-1}(B_R(y)) \mid X \in \mathcal{X},\ Y \in \mathcal{Y},\ y \in Y,\ (p : X \to Y) \in P\right\}$$

has finite asymptotic dimension uniformly. Then $\mathcal{X}$ has finite asymptotic dimension uniformly.

Lemma 2.6. *Let $\mathcal{X} = \{U_\alpha \cup V_\alpha\}_{\alpha\in A}$ be a metric family. Then*

$$\operatorname{asdim} \mathcal{X} = \max\left\{\operatorname{asdim}\{U_\alpha\}_\alpha, \operatorname{asdim}\{V_\alpha\}_{\alpha\in A}\right\}.$$

These results are [Roe 2003, Lemma 9.16 and Proposition 9.13], respectively, for metric families instead of metric spaces. The proofs are the same.

In the next section it will be more convenient to work with pseudometrics instead of metrics, i.e., allowing $d(x, y) = 0$ for $x \neq y$. Finite asymptotic dimension and FDC are defined in the same way for pseudometrics. If d is a pseudometric on X, then we can define a metric d' on X by setting $d'(x, y) := \max\{1, d(x, y)\}$

for $x \neq y$. The metric d' is proper (resp. left-invariant) if and only if d is. It has finite asymptotic dimension (resp. FDC) if and only if d does. Therefore, to show that a group has finite asymptotic dimension or FDC, it suffices to show this for G equipped with a left-invariant proper pseudometric.

Notation 2.7. We write $\mathcal{F}in_G$ for the set of finite subgroups of a group G. For a subgroup H of G, by $\{F\backslash G\}_{F\in\mathcal{F}in_H}$ we will mean the family of quotients of G by all finite subgroups of H. When H is the group of which we take the quotients, we will drop the subscript on $\mathcal{F}in$, that is, $\{F\backslash G\}_{F\in\mathcal{F}in} = \{F\backslash G\}_{F\in\mathcal{F}in_G}$.

3. Linear groups over fields of positive characteristic

In this section K will always denote a field of positive characteristic. Every finitely generated subgroup G of $\mathrm{GL}_n(K)$ has finite asymptotic dimension, by [Guentner et al. 2012, Theorem 3.1]. Here we want to show that the family $\{F\backslash G\}_{F\in\mathcal{F}in}$ has finite asymptotic dimension uniformly. We begin by recalling the argument from [Guentner et al. 2012].

A *length function* on a group G is a function $l : G \to [0,\infty)$ such that, for all $g, h \in G$,

(1) $l(e) = 0$,

(2) $l(g) = l(g^{-1})$, and

(3) $l(gh) \le l(g) + l(h)$.

We do not require that l be proper, nor that $l(g) = 0$ if and only if $g = e$. By setting $d(g,h) := l(g^{-1}h)$ we obtain a pseudometric.

A *discrete norm* on a field K is a map $\gamma : K \to [0,\infty)$ satisfying that for all $x, y \in K$ we have

(1) $\gamma(x) = 0$ if and only if $x = 0$,

(2) $\gamma(xy) = \gamma(x)\gamma(y)$,

(3) $\gamma(x+y) \le \max\{\gamma(x), \gamma(y)\}$,

and that the range of γ on $K \setminus \{0\}$ is a discrete subgroup of the multiplicative group $(0,\infty)$.

Following [Guentner et al. 2005], we obtain for every discrete norm γ on K a length function l_γ on $\mathrm{GL}_n(K)$ by

$$l_\gamma(g) = \log \max_{i,j}\{\gamma(g_{ij}), \gamma(g^{ij})\},$$

where g_{ij} and g^{ij} are the matrix coefficients of g and g^{-1}, respectively. By [Guentner et al. 2013, Propostion 5.2.4] the group $\mathrm{GL}_n(K)$ equipped with the pseudometric $d(g,h) = l_\gamma(g^{-1}h)$ has finite asymptotic dimension for every discrete norm γ. Let us review the proof.

The subset $\mathcal{O} := \{x \in K \mid \gamma(x) \leq 1\}$ is a subring of K called the *ring of integers* and $\mathfrak{m} := \{x \in K \mid \gamma(x) < 1\}$ is a principal ideal in $\mathcal{O}$. Let π be a fixed generator of $\mathfrak{m}$ and let D denote the subgroup of diagonal matrices with powers of π on the diagonal. Let U denote the unipotent upper triangular matrices. By [Guentner et al. 2013, Lemma 5.2.5] the group U has asymptotic dimension zero. We have $D \cong \mathbb{Z}^n$ and the restriction of l_γ to D is given by

$$l_\gamma(a) := \max_i |k_i| \log \gamma(\pi^{-1}),$$

where a is the diagonal matrix with entries π^{k_i} on the diagonal. The group D therefore is quasi-isometric to $\mathbb{Z}^n$ with the standard metric and has asymptotic dimension n. The group $T := DU$ is again a subgroup of $\mathrm{GL}_n(K)$ and $U \leq T$ is normal. Considering the extension $1 \to U \to T \to D \to 1$, we see that T has finite asymptotic dimension.

Let H be the subgroup of those $g \in \mathrm{GL}_n(F)$ for which the entries of g and g^{-1} are in $\mathcal{O}$. Then $\mathrm{GL}_n(K) = TH$ by [Guentner et al. 2005, Lemma 5]. For $h \in H$ let h_{ij} and h^{ij} denote the matrix coefficients of h and h^{-1}, respectively. By definition, $\gamma(h_{ij}), \gamma(h^{ij}) \leq 1$ and thus

$$0 \leq l_\gamma(h) = \log \max_{ij}\{\gamma(h_{ij}), \gamma(h^{ij})\} \leq 0.$$

This implies that the inclusion $T \to \mathrm{GL}_n(K)$ is isometric and metrically onto, i.e., for every $g \in \mathrm{GL}_n(K)$ there exists a $t \in T$ with $d(g, t) = 0$. Hence, $\mathrm{GL}_n(K)$ has finite asymptotic dimension with respect to the pseudometric d.

Lemma 3.1. *For every discrete norm the family $\{F\backslash \mathrm{GL}_n(K)\}_F$, where F ranges over all finite subgroups of U, has finite asymptotic dimension uniformly with respect to the associated pseudometric.*

Proof. Let F be a finite subgroup of U. Then we can consider the map

$$F\backslash T \xrightarrow{\rho_F} D.$$

We want to apply Lemma 2.5 to the family $\{\rho_F : F\backslash T \xrightarrow{\rho} D\}_{F\in\mathcal{F}in_U}$, For this we have to show that for every $R > 0$ the family $\{\rho_F^{-1}(B_R(d))\}_{d\in D,\, F\in\mathcal{F}in_U}$ has finite asymptotic dimension uniformly. The preimage $\rho_F^{-1}(d) = \{Fud \mid u \in U\}$ of a point $d \in D$ is isometric to $(F)^d\backslash U$, by mapping Fud to $d^{-1}Fdd^{-1}ud$, where $(F)^d := \{d^{-1}fd \mid f \in F\}$. Therefore, the preimage of $B_R(d)$ for any $R > 0$ is a finite union of spaces isometric to spaces of the form $(F)^{d'}\backslash U$ with $d' \in D$. The number of spaces appearing in this union only depends on R and not on d (or F). Thus, by Lemma 2.6,

$$\mathrm{asdim}\{\rho_F^{-1}(B_R(d))\}_{d\in D,\, F\in\mathcal{F}in_U} = \mathrm{asdim}\{F\backslash U\}_{F\in\mathcal{F}in}.$$

Since the inclusion $F\backslash T \to F\backslash \mathrm{GL}_n(K)$ is isometric and metrically onto, to prove the lemma it remains to show that the family $\{F\backslash U\}_{F\in\mathcal{F}in}$ has asymptotic dimension zero uniformly.

Let $R > 0$ be given and let $\mathcal{S}$ denote the partition of U into r-connected components, i.e., two elements $u, u' \in U$ lie inside the same $S \in \mathcal{S}$ if and only if there exists a sequence $u_0, \dots, u_n$ with $u = u_0$, $u' = u_n$ and $d(u_{i-1}, u_i) \le R$ for all $i = 1, \dots, n$. Since U has asymptotic dimension zero we have that $r := \sup_{S\in\mathcal{S}} \operatorname{diam} S < \infty$. Since the left action of F on U is isometric, if $fu = u'$ for some $f \in F$ and $u, u' \in U$, then f maps the r-connected component of u bijectively onto the r-connected component of u'. This implies that every r-connected component of $F\backslash U$ is a quotient of an r-connected component of U and in particular has diameter at most r. Therefore, the family $\{F\backslash U\}_{F\in\mathcal{F}in}$ has asymptotic dimension zero uniformly, as claimed. □

Proposition 3.2. *Let $G \le \mathrm{GL}_n(K)$ be a finitely generated subgroup. Then for every discrete norm γ the family $\{F\backslash G\}_{F\in\mathcal{F}in}$ has finite asymptotic dimension uniformly with respect to the associated pseudometric.*

Proof. By the main theorem of [Alperin 1987] there exists a normal subgroup $G' \le G$ with index $[G : G'] =: N < \infty$ such that every finite subgroup of G' is unipotent. Therefore, every finite subgroup $F \le G'$ is conjugate in $\mathrm{GL}_n(K)$ to a finite subgroup $F' \le U$. Let $g = th$ with $t \in T$, $h \in H$ be such that $g^{-1}F'g = F$. Since U is normal in T, we have that $t^{-1}F't \le U$ and we can assume $g \in H$ and in particular $l_\gamma(g) = 0$. This implies that conjugation by g is an isometry and induces an isometry between $F'\backslash \mathrm{GL}_n(K)$ and $F\backslash \mathrm{GL}_n(K)$. By Lemma 3.1 the family $\{F'\backslash \mathrm{GL}_n(K)\}_{F'\in\mathcal{F}in_U}$ has finite asymptotic dimension uniformly and, by the above isometry, the family $\{F\backslash \mathrm{GL}_n(K)\}_{F\in\mathcal{F}in_{G'}}$ therefore also has finite asymptotic dimension uniformly. This also holds for the subfamily $\{F\backslash G\}_{F\in\mathcal{F}in_{G'}}$. Since $[G : G'] = N$, every finite subgroup $\tilde{F}$ of G has a normal subgroup F of index at most N lying in G'. The quotient group $F\backslash\tilde{F}$ acts isometrically on $F\backslash G$. Thus, projecting the covers that give finite asymptotic dimension for $\{F\backslash G\}_{F\in\mathcal{F}in_{G'}}$ down to the quotient $\{\tilde{F}\backslash G\}_{\tilde{F}\in\mathcal{F}in}$ shows that this family still has finite asymptotic dimension uniformly. □

Theorem 3.3. *Let $G \le \mathrm{GL}_n(K)$ be a finitely generated subgroup. There exists a proper, left-invariant metric on G such that the family $\{F\backslash G\}_{F\in\mathcal{F}in}$ has finite asymptotic dimension uniformly.*

Proof. The subring of K generated by the matrix entries of a finite generating set for G is a finitely generated domain A with $G \le \mathrm{GL}_n(A)$ and we may replace K by the (finitely generated) fraction field of A; thus, we can assume that K is a finitely generated field of positive characteristic. By [Guentner et al. 2012, Proposition 3.4], for every finitely generated subring A of K there exists a finite set N_A of discrete norms such that for every $k \in \mathbb{N}$ the set

$$B_A(k) = \{a \in A \mid \gamma(a) \le e^k \text{ for all } \gamma \in N_A\}$$

is finite. Let A again be the subring generated by the matrix entries of a finite generating set for G and $N_A = \{\gamma_1, \dots, \gamma_q\}$ be the finite set of discrete norms, as above. Consider the length function $l := l_{\gamma_1} + \dots + l_{\gamma_q}$. The pseudometric on G defined by $d(g, g') := l(g^{-1}g')$ now is proper and left-invariant, and the diagonal embedding

$$(G, d) \to (\mathrm{GL}_n(K), d_{\gamma_1}) \times \dots \times (\mathrm{GL}_n(K), d_{\gamma_q})$$

is isometric when the product is given the sum metric. It suffices to show that the family

$$\left\{F\backslash((G, d_{\gamma_1}) \times \dots \times (G, d_{\gamma_q}))\right\}_{F \in \mathcal{F}in_G}$$

has finite asymptotic dimension uniformly. Now let $F \le G$ be finite and consider the projection

$$F\backslash(G \times \dots \times G) \xrightarrow{p} F\backslash G \times \dots \times F\backslash G$$

using the same metrics as above. The image has finite asymptotic dimension uniformly in F by Proposition 3.2, and using Lemma 2.5 it suffices to show that the preimage of $B_R(Fg_1) \times \dots \times B_R(Fg_n)$ under p has finite asymptotic dimension uniformly. The preimage is a finite union of metric spaces of the form $F\backslash(Fg'_1 \times Fg'_n)$ and the number of the spaces appearing in the union only depends on R, not on F or $g_1, \dots, g_n$. By the main theorem of [Alperin 1987] there exists a normal subgroup $G' \trianglelefteq \mathrm{GL}_n(A)$ with index $[\mathrm{GL}_n(A) : G'] =: N < \infty$ such that every finite subgroup of G' is unipotent. In particular, we have a normal unipotent subgroup $F' := G' \cap F$ of F of index at most N. The space $Fg'_1 \times \dots \times Fg'_n$ is a union of at most N subspaces isometric to $F'g'_1 \times \dots \times F'g'_n$, and as in the proof of Proposition 3.2 there exists an isometry of these to $F_1 \times \dots \times F_n$ with $F_i \le U$. By Lemma 2.6 this shows that $Fg'_1 \times \dots \times Fg'_n$ has asymptotic dimension zero uniformly in F. As in the proof of Lemma 3.1, we see that $F\backslash(Fg'_1 \times \dots \times Fg'_n)$ also has asymptotic dimension zero. This completes the proof of Theorem 3.3. □

Remark 3.4. Note that the family $\{F\backslash G\}_{F \in \mathcal{F}in}$ has finite asymptotic dimension uniformly for some proper, left-invariant (pseudo)metric on G if and only if it has finite asymptotic dimension for every proper, left-invariant metric on G.

4. Linear groups over commutative rings with unit

Lemma 4.1. *Let H_1 and H_2 be groups such that $\{F\backslash H_i\}_{F \in \mathcal{F}in}$ has FDC for $i = 1, 2$. Then $\{F\backslash(H_1 \times H_2)\}_{F \in \mathcal{F}in}$ has FDC.*

Proof. Let proper, left-invariant metrics d_i on H_i be given and consider $H_1 \times H_2$ with the metric $d_1 + d_2$. Let $p_i : H_1 \times H_2 \to H_i$ denote the projection. Consider the uniformly expansive map

$$\{F\backslash(H_1 \times H_2)\}_{F\in\mathcal{F}in} \to \big\{(p_1(F) \times p_2(F))\backslash(H_1 \times H_2)\big\}_{F\in\mathcal{F}in}.$$

Then the range has FDC by assumption and by the fibering theorem [Guentner et al. 2013, Theorem 3.1.4] it suffices to show that the family

$$\big\{F\backslash(p_1(F) \times p_2(F))(B_R(h_1) \times B_R(h_2))\big\}_{h_i\in H_i,\ F\in\mathcal{F}in_{H_1\times H_2}}$$

has FDC for every $R > 0$. Every space in this family is a union of $|B_R(h_1)\times B_R(h_2)|$ many spaces of the form $F\backslash(p_1(F)\times p_2(F))(h, h')$. The number $|B_R(h_1)\times B_R(h_2)|$ only depends on R, not on h_1 and h_2, and every space $F\backslash(p_1(F) \times p_2(F))(h, h')$ is isometric to $(F)^{(h,h')}\backslash(p_1(F)\times p_2(F))^{(h,h')}$, where $(F)^{(h,h')}$ is $(h, h')^{-1}F(h, h')$ and similarly for $(p_1(F)\times p_2(F))^{(h,h')}$. By Theorem 2.4 it suffices to show that the family $\{F\backslash F'\}_{F\le F'}$ has FDC, where $F \le F'$ ranges over all pairs of finite subgroups of $H_1 \times H_2$. Let $\mathcal{S}_R$ denote the family of finite subgroups of $H_1 \times H_2$ generated by elements from $B_R(e)$ and let $s_R := \sup_{S\in\mathcal{S}_R} \operatorname{diam} S$. Let $F \le H_1\times H_2$ be finite. Then for every $R > 0$ the group F is the r-disjoint union of the cosets of $\langle F\cap B_R(e)\rangle$ and each of these has diameter at most s_R. We see that the family of finite subgroups of $H_1 \times H_2$ has asymptotic dimension zero uniformly. This implies that the above family $\{F\backslash F'\}_{F\le F'}$ also has asymptotic dimension zero uniformly, since every r-connected component of $F\backslash F'$ is a quotient of an r-connected component of F' and thus has uniformly bounded diameter. □

Lemma 4.2 [Guentner et al. 2013, Lemma 5.2.3]. *Let R be a finitely generated commutative ring with unit and let $\mathfrak{n}$ be the nilpotent radical of R,*

$$\mathfrak{n} = \{r \in R \mid r^n = 0 \textit{ for some } n\}.$$

The quotient ring $S = R/\mathfrak{n}$ contains a finite number of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ such that the diagonal map

$$S \to S/\mathfrak{p}_1 \oplus \cdots \oplus S/\mathfrak{p}_k$$

embeds S into a finite direct sum of domains.

Theorem 4.3. *Let R be a commutative ring with unit and trivial nilradical and let G be a finitely generated subgroup of* $\mathrm{GL}(n, R)$*. Then $\{F\backslash G\}_{F\in\mathcal{F}in}$ has FDC.*

Proof. Because G is finitely generated we can assume that R is finitely generated as well. Since the nilradical of R is trivial, we have $R = S$ in the notation of the previous lemma and there is an embedding

$$\mathrm{GL}_n(S)\hookrightarrow \mathrm{GL}_n(S/\mathfrak{p}_1)\times\cdots\times\mathrm{GL}_n(S/\mathfrak{p}_k)\hookrightarrow \mathrm{GL}_n(Q(S/\mathfrak{p}_1))\times\cdots\times\mathrm{GL}_n(Q(S/\mathfrak{p}_k)),$$

where $Q(S/\mathfrak{p}_i)$ is the quotient field of $S/\mathfrak{p}_i$. Let G_i be the image of the group G in $\mathrm{GL}_n(Q(S/\mathfrak{p}_i))$. If $S/\mathfrak{p}_i$ has positive characteristic, the family $\{F\backslash G_i\}_{F\in\mathcal{F}in}$ has FDC by Theorem 3.3. If $S/\mathfrak{p}_i$ has characteristic zero, then G_i is virtually torsion-free by Selberg's lemma and thus $\{F\backslash G_i\}_{F\in\mathcal{F}in}$ has FDC by [Kasprowski 2015a, Theorem 4.10]. Now Lemma 4.1 implies that the family $\{F\backslash G\}_{F\in\mathcal{F}in}$ also has FDC. □

Proof of Theorem 1.1. This follows directly from Theorem 4.3 and [Kasprowski 2014, Theorems 3.2.2 and 3.3.1] if R has trivial nilradical. Note that these theorems are stronger than the similar [Kasprowski 2015a, Theorems A and 9.1], where an upper bound on the order of the finite subgroups is needed. For convenience we show in the Appendix how the results from [Kasprowski 2015a] can be used to prove the theorems from [Kasprowski 2014].

If the nilradical $\mathfrak{n}$ of R is nontrivial, we have an exact sequence

$$1 \to (1+M_n(\mathfrak{n}))\cap G \to G \to H \to 1,$$

where H denotes the image of G in $\mathrm{GL}_n(R/\mathfrak{n})$. Now the K-theoretic assembly map for H is split injective and $(1+M_n(\mathfrak{n}))\cap G$ is nilpotent. Therefore, the preimage of every virtually cyclic subgroup of H is virtually solvable and satisfies the Farrell–Jones conjecture, by [Wegner 2015]. By [Kasprowski 2015b, Proposition 4.1] this implies that the K-theoretic assembly map for G is split injective as well. The L-theory version of the theorem follows in the same way from the results in [Kasprowski 2015b, Section 6]. □

5. Dimension of the classifying space

In this section we want to prove Proposition 1.2. We will need the following result about classifying spaces. The proof is the same as the proof of [Lück 2000, Theorem 3.1].

Theorem 5.1. *Let $1 \to K \to G \xrightarrow{\pi} Q \to 1$ be an exact sequence of groups. Assume that Q admits a finite-dimensional model for $\underline{E}Q$ and that there exists $N \in \mathbb{N}$ such that for every finite subgroup $F \in Q$ the preimage admits a model for $\underline{E}\pi^{-1}(F)$ of dimension at most N. Then there exists a finite-dimensional model for $\underline{E}G$.*

Proof of Proposition 1.2. For a group G let $\underline{cd}G$ be the shortest length of a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module and let $\underline{hd}G$ be the shortest length of a flat resolution of $\mathbb{Z}$ of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module. Let $\underline{gd}G$ denote the minimal dimension of a model for $\underline{E}G$. For a countable group G by [Nucinkis 2004, Theorem 4.1] we have

$$\underline{hd}G \le \underline{cd}G \le \underline{hd}G + 1.$$

Furthermore,

$$\underline{cd}G \le \underline{gd}G \le \max\{\underline{cd}G, 3\},$$

where the first inequality follows from taking the cellular chain complex of $\underline{E}G$ as a resolution and the second inequality follows from [Lück 1989, Theorem 13.19]. By [Flores and Nucinkis 2007, Theorem 1], for a solvable group with finite Hirsch length $l(G)$ it holds that $l(G) = \underline{hd}G$. Note that Flores and Nucinkis use Hillman's definition of the Hirsch rank for an elementary amenable group. It can be shown by a simple transfinite induction that for solvable groups this agrees with the definition given in the introduction. Furthermore, every solvable group with infinite Hirsch length has a subgroup with arbitrary large Hirsch length. In particular, the existence of a finite-dimensional model X for $\underline{E}G$ directly implies that the Hirsch rank of the solvable subgroups of G is bounded by $\dim X$. It remains to prove the other direction.

Let R be a fixed commutative ring with unit and let $G \leq \mathrm{GL}_n(R)$ be finitely generated with $N \in \mathbb{N}$ an upper bound on the Hirsch rank of the solvable subgroups of G. Since G is finitely generated, we can assume that R is also finitely generated and let $\mathfrak{n}$, S and $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be as in Lemma 4.2. Furthermore, let H denote the image of G in $\mathrm{GL}_n(S)$ and $p : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ the projection. Let A be a finitely generated abelian subgroup of H. Then $p^{-1}(A)$ is solvable. This implies that the rank of the finitely generated abelian subgroups of H is also bounded by N.

First let us show that H admits a finite-dimensional model for $\underline{E}H$. By Lemma 4.2 H embeds into $\mathrm{GL}_n(S/\mathfrak{p}_1) \times \cdots \times \mathrm{GL}_n(S/\mathfrak{p}_k)$ and, since H is finitely generated, we can assume that all the domains $S/\mathfrak{p}_i$ are as well. Order them in such a way that $S/\mathfrak{p}_1, \dots, S/\mathfrak{p}_q$ are of positive characteristic and $S/\mathfrak{p}_{q+1}, \dots, S/\mathfrak{p}_k$ are of characteristic zero. Then $\mathrm{GL}_n(S/\mathfrak{p}_{q+1}) \times \cdots \times \mathrm{GL}_n(S/\mathfrak{p}_k)$ embeds into $\mathrm{GL}_{n(k-q)}(\mathbb{C})$. Let π denote the projection of H to $\mathrm{GL}_n(S/\mathfrak{p}_1) \times \cdots \times \mathrm{GL}_n(S/\mathfrak{p}_q)$ and let π_i denote the projection of H to $\mathrm{GL}_n(S/\mathfrak{p}_i)$ for $i = 1, \dots, q$; then $\pi_i(H)$ admits a finite-dimensional model E_i for $\underline{E}\pi_i(H)$, by [Degrijse and Petrosyan 2015, Corollary 5], and thus $E_1 \times \cdots \times E_q$ is a finite-dimensional model for $\underline{E}\pi(H)$. By Theorem 5.1 it remains to show that for every finite subgroup $F \in \pi(H)$ the preimage $\pi^{-1}(F)$ admits a finite-dimensional model with dimension bounded uniformly in F. Let ρ denote the projection from H to $\mathrm{GL}_{n(k-q)}(\mathbb{C})$. Then $\rho(H)$ is virtually torsion-free, by Selberg's lemma [1960]. The group $\rho(\ker \pi)$ is isomorphic to $\ker \pi$ and thus N is a bound on the rank of its finitely generated abelian subgroups. Furthermore, $\rho(\ker \pi)$ has finite index in $\rho(\pi^{-1}(F))$ for every finite subgroup $F \leq \pi(H)$. Thus, the rank of the finitely generated abelian subgroups of $\rho(\pi^{-1}(F))$ is also bounded by N. By [Kasprowski 2015b, Proposition 3.1] this implies that the rank of the finitely generated unipotent subgroups of $\rho(\pi^{-1}(F))$ is bounded by $\frac{1}{2}N(N+1)$. This implies that $\rho(\pi^{-1}(F))$ has finite virtual cohomological dimension bounded uniformly in F; see [Alperin and Shalen 1982, Remark after Theorem 3.3]. The order of the finite subgroups in $\rho(\pi^{-1}(F))$ is bounded uniformly in F since they are all contained inside the virtually torsion-free group $\rho(H)$. By Theorem 1.10 of [Lück 2000] this implies that there exist finite-dimensional models for $\underline{E}\rho(\pi^{-1}(F))$ with dimension bounded uniformly in F and, since $\rho : \pi^{-1}(F) \to \rho(\pi^{-1}(F))$ has finite kernel, they are also models for $\underline{E}\pi^{-1}(F)$. This completes the proof that H admits a finite-dimensional model for $\underline{E}H$.

For every finite subgroup $F \leq H$, its preimage A in G is virtually nilpotent and thus elementary amenable, and the Hirsch rank of A is bounded by N. By the inequalities from the beginning of the proof this implies that there is a model for $\underline{E}A$ of dimension at most $N+2$. Using Theorem 5.1 again we conclude that there exists a finite-dimensional model for $\underline{E}G$. □

Appendix

In this appendix we want to prove the following:

Theorem A.1 [Kasprowski 2014, Theorem 3.2.2]. *Let G be a discrete group such that $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC and let $\mathcal{A}$ be a small additive G-category. Assume that there is a finite-dimensional G-CW model for the classifying space for proper G-actions $\underline{E}G$. Then the assembly map in algebraic K-theory*

$$H_*^G(\underline{E}G; \mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$$

is a split injection.

The analogous result in L-theory [Kasprowski 2014, Theorem 3.3.1] follows in the same way from the results of [Kasprowski 2015a]. We will use the notation introduced in [Kasprowski 2015a]. Note that, in the appendix, metrics are allowed to take on the value ∞. We will need the following equivariant version of FDC.

Definition A.2. An *equivariant metric family* is a family $\{(X_\alpha, G_\alpha)\}_{\alpha\in A}$, where G_α is a group and X_α is a metric G_α-space.

Definition A.3. An equivariant metric family $\mathcal{X} = \{(X_\alpha, G_\alpha)\}_{\alpha\in A}$ *decomposes over a class of equivariant metric families* $\mathfrak{D}$ if for every $r > 0$ and every $\alpha \in A$ there exists a decomposition $X_\alpha = U_\alpha^r \cup V_\alpha^r$ into G_α-invariant subspaces and r-disjoint decompositions

$$U_\alpha^r = \bigcup_{i\in I(r,\alpha)}^{r\text{-disj.}} U_{\alpha,i}^r \quad\text{and}\quad V_\alpha^r = \bigcup_{j\in J(r,\alpha)}^{r\text{-disj.}} V_{\alpha,j}^r$$

such that G_α acts on $I(r,\alpha)$ and $J(r,\alpha)$ and, for every $g \in G_\alpha$, we have $gU_{\alpha,i}^r = U_{\alpha,gi}^r$ and $gV_{\alpha,j}^r = V_{\alpha,gj}^r$. Furthermore, the families

$$\biggl\{\Bigl(\coprod_{i\in I(r,\alpha)} U_{\alpha,i}^r, G_\alpha\Bigr)\biggr\}_{\alpha\in A} \quad\text{and}\quad \biggl\{\Bigl(\coprod_{j\in J(r,\alpha)} V_{\alpha,j}^r, G_\alpha\Bigr)\biggr\}_{\alpha\in A}$$

have to lie in $\mathfrak{D}$.

Notice that the underlying sets of U^r_α and $\coprod_{i\in I(r,\alpha)} U^r_{\alpha,i}$ are canonically isomorphic and in this sense the G_α-action on $\coprod_{i\in I(r,\alpha)} U^r_{\alpha,i}$ is the same as the action on U^r_α, only the metric has changed.

Definition A.4. An equivariant metric family $\mathcal{X}$ is called *semibounded* if there exists $R > 0$ such that for all $(X, G) \in \mathcal{X}$ and $x, y \in X$ we have $d(x, y) < R$ or $d(x, y) = \infty$.

Let $e\mathfrak{B}$ denote the class of semibounded equivariant families. We set $e\mathfrak{D}_0 = e\mathfrak{B}$ and, given a successor ordinal $\gamma + 1$, we define $e\mathfrak{D}_{\gamma+1}$ to be the class of all equivariant metric families which decompose over $e\mathcal{D}_\gamma$. For a limit ordinal λ we define

$$e\mathfrak{D}_\lambda = \bigcup_{\gamma<\lambda} e\mathfrak{D}_\gamma.$$

An equivariant metric family $\mathcal{X}$ has *finite decomposition complexity* (*FDC*) if $\mathcal{X}$ lies in $e\mathfrak{D}_\gamma$ for some ordinal γ.

Note that the equivariant metric family $\{(X_\alpha, \{e\})\}_{\alpha\in A}$ has FDC if and only if the metric family $\{X_\alpha\}_{\alpha\in A}$ has FDC.

A metric family $\{X_\alpha\}_{\alpha\in A}$ has uniformly bounded geometry if for every $R > 0$ there exists $N \in \mathbb{N}$ such that, for every $\alpha \in A$ and $U \subseteq X_\alpha$ with $\operatorname{diam}(U) \le R$, the set U contains at most N elements.

The following is a generalization of Ramras, Tessera and Yu [Ramras et al. 2014, Theorem 6.4]. The proof is analogous to the proof of theirs and can be found in [Kasprowski 2014]. The additive G-category $\mathcal{A}_G(X)$ is defined in [Kasprowski 2015a, Definition 5.1] and $\mathcal{A}_G^G(X)$ denotes the fixed-point category. For a definition of the bounded product see [Kasprowski 2015a, Definition 5.11].

Theorem A.5. *Let $\mathcal{X} = \{(X_\alpha, G_\alpha)\}_{\alpha\in A}$ be an equivariant family with FDC, and let the family $\{X_\alpha\}_{\alpha\in A}$ have bounded geometry uniformly. Then*

$$\operatorname*{colim}_s K_n\Bigl(\prod_{\alpha\in A}^{bd} \mathcal{A}_{G_\alpha}^{G_\alpha}(P_s X_\alpha)\Bigr) = 0$$

for all $n \in \mathbb{Z}$, where the colimit is taken over the maps induced by the inclusion of the respective Rips complexes $P_s X_\alpha$.

Furthermore, recall the following:

Theorem A.6 [Kasprowski 2015a, Theorem 7.6]. *Let G be a discrete group admitting a finite-dimensional model for $\underline{E}G$ and let X be a simplicial G-CW complex with a proper G-invariant metric. Assume that, for every G-set J with finite stabilizers,*

$$\operatorname*{colim}_K K_n\Bigl(\prod_{j\in J}^{bd} \mathcal{A}_G(GK)\Bigr)^G = 0,$$

where the colimit is taken over all finite subcomplexes $K \subseteq X$. Then the assembly map
$$H_*^G(X;\mathbb{K}_{\mathcal{A}}) \to K_*(\mathcal{A}[G])$$
is a split injection.

Proposition A.7 [Kasprowski 2014, Proposition 3.2.1]. *Let G be a group such that the metric family $\{H\backslash G\}_{H\in\mathcal{F}in}$ has FDC. Then the equivariant metric family $\{(G,H)\}_{H\in\mathcal{F}in}$ has FDC as well.*

Proof. Let $\{(X_i,G_i)\}_{i\in I}$ be an equivariant metric family with $G_i \le G$ a finite subgroup and assume $X_i \subseteq \coprod_{A_i} G$ is a G_i-invariant subspace, where A_i is a G_i-set. We prove by induction on the decomposition complexity that the family $\{(X_i,G_i)\}_{i\in I}$ lies in $e\mathfrak{D}_{\gamma+1}$ if $\{G_i\backslash X_i\}_{i\in I}\in\mathfrak{D}_\gamma$. For the start of the induction let $\{G_i\backslash X_i\}_{i\in I}$ be in $\mathfrak{D}_0=\mathfrak{B}$. Since $G_i\backslash X_i$ is bounded, there is $a_i\in A_i$ with $X_i\subseteq\coprod_{G_ia_i}G$. Then there exist $R>0$ and $Y_i\subseteq G=\coprod_{\{a_i\}}G\subseteq\coprod_{A_i}G$ with $\operatorname{diam} Y_i<R$ for all $i\in I$ such that $X_i=G_iY_i\subseteq\coprod_{A_i}G$. Let $G_i'\subseteq G_i$ be the stabilizer of a_i. Then
$$X_i\cong\coprod_{[g]\in G_i/G_i'} gG_i'Y_i \quad\text{with}\quad G_i'Y_i\subseteq G.$$

Let $r>0$ be given and define $\mathcal{S}_r:=\{H\in\mathcal{F}in \mid H=\langle S\rangle,\ S\subseteq B_{2R+r}(e)\}$ and $k:=\max_{H\in\mathcal{S}}|H|$. Let $g_i\in Y_i$ be a fixed base point. Let $H_i\le G_i'$ be the subgroup generated by those $g\in G_i'$ with $d(Y_i,gY_i)<r$. For these g we have $d(e,g_i^{-1}gg_i)<2R+r$. Therefore, $g_i^{-1}H_ig_i\in\mathcal{S}_r$ and $|H_i|\le k$. We have the decomposition
$$X_i=\bigcup_{[g]\in G_i/H_i} gH_iY_i.$$

This decomposition is r-disjoint, since $d(ghy,g'h'y')<r$ with $g,g'\in G_i$, $h,h'\in H_i$ and $y,\ y'\in Y_i$ implies that $d(Y_i,h^{-1}g^{-1}g'h'Y_i)<r$ and so, by definition, the element $h^{-1}g^{-1}g'h'$ lies in H_i, which is equivalent to $gH_i=g'H_i$.

By definition of H_i each $h\in H_i$ can be written as $h=g_1\cdots g_l$ with $l\le|H|\le k$ and g_j such that $d(Y_i,g_jY_i)<r$. For every $y,\ y'\in Y_i$, by left-invariance and the triangle inequality we obtain
$$\begin{aligned} d(y,hy') &\le d(y,g_1y')+d(g_1y',g_1g_2y')+\cdots+d(g_1\cdots g_{l-1}y',hy')\\ &= d(y,g_1y')+d(y',g_2y')+\cdots+d(y',g_ly')<lr. \end{aligned}$$

Therefore $\operatorname{diam} gH_iY_i=\operatorname{diam} H_iY_i<kr$. Thus, $\{(X_i,G_i)\}_{i\in I}$ is r-decomposable over $e\mathfrak{D}_0=e\mathfrak{B}$ for every $r>0$ and lies in $e\mathfrak{D}_1$.

If $\{G_i\backslash X_i\}_{i\in I}$ lies in $\mathfrak{D}_{\gamma+1}$, then it decomposes over $\mathfrak{D}_\gamma$ and the preimages under the projection $X_i\to G_i\backslash X_i$ give a decomposition of $\{(X_i,G_i)\}$ over $e\mathfrak{D}_{\gamma+1}$ by the induction hypothesis. Here G_i acts trivially on the index set of the decomposition. The induction step for limit ordinals is trivial. ∎

Proof of Theorem A.1. By [Kasprowski 2015a, Proposition 1.5], G admits a finite-dimensional model X for $\underline{E}G$ with a left-invariant proper metric. By Theorem A.6 we have to show that

$$\operatorname*{colim}_K K_n\biggl(\prod_{j\in J}^{bd}\mathcal{A}_G(GK)\biggr)^G=0,$$

where the colimit is taken over all finite subcomplexes $K\subseteq X$. Since the category $\bigl(\prod_{j\in J}^{bd}\mathcal{A}_G(GK)\bigr)^G$ is equivalent to $\prod_{Gj\in G\backslash J}^{bd}\mathcal{A}_G^{G_j}(P_sG)$, where G_j is the stabilizer of $j\in J$, this is equivalent to showing that, for every family of finite subgroups $\{G_i\}_{i\in I}$ over some index set I,

$$\operatorname*{colim}_K K_n\biggl(\prod_{i\in I}^{bd}\mathcal{A}_G^{G_i}(GK)\biggr)=0.$$

By [Kasprowski 2015a, Lemma 1.8 and Proposition 6.3], for every finite subcomplex $K\subseteq X$ there exists $K'\subseteq X$ finite and $s>0$ with maps $GK\to P_s(G)\to GK'$ such that the composition is metrically homotopic to the identity. In particular, the composition induces the identity in the K-theory of the associated controlled categories. Thus it remains to show

$$\operatorname*{colim}_s K_n\biggl(\prod_{i\in I}^{bd}\mathcal{A}_G^{G_i}(P_sG)\biggr)=0.$$

Since $\{(G,H)\}_{H\in\mathcal{F}in}$ has FDC by Proposition A.7 and the category $\mathcal{A}_G^{G_i}(P_sG)$ is equivalent to $\mathcal{A}_{G_i}^{G_i}(P_sG)$, this follows from Theorem A.5. □

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DANIEL KASPROWSKI: kasprowski@mpim-bonn.mpg.de
Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

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Standard norm varieties for Milnor symbols mod p

Dinh Huu Nguyen

We prove that the standard norm varieties for Milnor symbols mod p of length n are birationally isomorphic to Pfister quadrics when $p = 2$, to Severi–Brauer varieties when $p > 2$ and $n = 2$, and to varieties defined by reduced norms of cyclic algebras when $p > 2$ and $n = 3$. In the case $p = 2$ and the case $p > 2$ and $n = 2$, the results imply that the standard norm varieties for two equal Milnor symbols mod p are birationally isomorphic, and we conjecture this in general.

1. Introduction

The norm residue theorem relates the Milnor K-theory mod p of a field k with the étale cohomology of k with coefficients in the twists of μ_p. More precisely, it states that for each prime $p \neq \mathrm{char}(k)$ and each weight $n \geq 0$ there exists an isomorphism

$$K_n^M(k)/p \cong H^n_{\text{ét}}(k, \mu_p^n)$$

In 1996, V. Voevodsky [2003] proved the special case of $p = 2$, known as the Milnor conjecture. He later [2011] proved the general case of the norm residue theorem, also known as the Bloch–Kato conjecture. His proof used a splitting variety with certain properties for a given Milnor symbol $\{a_1, \dots, a_n\}$ in $K_n^M(k)/p$. One construction for such splitting varieties was provided by M. Rost in [Haesemeyer and Weibel 2009, Section 3]. Another construction for these varieties was suggested by Voevodsky in [Suslin and Joukhovitski 2006, Section 2]. The entire theorem has been written in book form by C. Haesemeyer and C. Weibel [2016].

In Section 2, we summarize Voevodsky's construction. It uses symmetric powers and produces what are called standard norm varieties.

In Section 3, we show in Theorem 3.7 that the standard norm varieties are birationally isomorphic to Pfister quadrics defined by subforms of Pfister forms when $p = 2$. Then we combine this result with the chain P-equivalence theorem

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by R. Elman and T. Y. Lam [1972, Main Theorem 3.2] and properties of quadratic forms to prove that the standard norm varieties for two equal symbols are birationally isomorphic in Corollary 3.13.

In Section 4, we use Galois descent to show in Theorem 4.1 that the standard norm varieties are birationally isomorphic to Severi–Brauer varieties when $p > 2$ and $n = 2$ and get the similar Corollary 4.2.

In Section 5, we use Galois descent to show in Theorem 5.1 that the standard norm varieties are birationally isomorphic to varieties defined by reduced norms of cyclic algebras when $p > 2$ and $n = 3$. N. Karpenko and A. Merkurjev [2013] use this result and induction to prove A-triviality for standard norm varieties.

Given the above two corollaries, we make the following conjecture:

Conjecture 1.1. *The standard norm varieties for* $\{a_1, \dots, a_n\}$ *and* $\{b_1, \dots, b_n\}$ *are birationally isomorphic if* $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$ *in* $K_n^M(k)/p$ *for all* p *and* n.

2. Symmetric powers

A general reference for Milnor K-theory is [Milnor 1970]. Throughout this paper, p is a prime and k is a base field of characteristic 0 containing the p-th roots of unity. Associated to each nontrivial Milnor symbol $\{a_1, \dots, a_n\}$ in $K_n^M(k)/p$ are the following notions:

Definition 2.1. A field extension L/k is called a *splitting field* for $\{a_1, \dots, a_n\}$ if $\{a_1, \dots, a_n\} = 0$ in $K_n^M(L)/p$.

Definition 2.2. A smooth variety X is called a *splitting variety* for $\{a_1, \dots, a_n\}$ if its function field $k(X)$ is a splitting field for $\{a_1, \dots, a_n\}$. In addition, it is called a *generic splitting variety* for $\{a_1, \dots, a_n\}$ if any splitting field L for $\{a_1, \dots, a_n\}$ has a point in X, i.e., if there exists a morphism $\mathrm{Spec}(L) \to X$ over k.

Such generic splitting varieties are known to exist for all n when $p = 2$ and only for $n \leq 3$ when $p > 2$. However, if L'/L if a finite extension of degree prime to p and L' splits $\{a_1, \dots, a_n\}$, then L also splits $\{a_1, \dots, a_n\}$ (using transfer and norm maps). Therefore we can relax our last definition.

Definition 2.3. A smooth variety X is called a *p-generic splitting variety* for $\{a_1, \dots, a_n\}$ if it is a splitting variety for $\{a_1, \dots, a_n\}$ and, for any splitting field L for $\{a_1, \dots, a_n\}$, there exists an extension L'/L of degree prime to p with a point in X. In addition, it is called a *norm variety* for $\{a_1, \dots, a_n\}$ if it is projective and geometrically irreducible of dimension $p^{n-1} - 1$.

Example 2.4. When $n = 1$, a norm variety for $\{a_1\}$ is $\mathrm{Spec}(L)$, where $L = k(\sqrt[p]{a_1}\,)$. When $n = 2$, a norm variety for $\{a_1, a_2\}$ is the Severi–Brauer variety $\mathrm{SB}(A)$ associated to the cyclic algebra $A = (a_1, a_2, \zeta_p)_k$.

We now describe a standard way to produce these norm varieties for all n, which are called *standard norm varieties*.

Let X be a smooth, quasiprojective, geometrically irreducible variety. The symmetric group S_p acts on the product X^p and induces the quotient variety $S^p(X)$. This quotient variety is geometrically irreducible and normal. Note that S_p acts freely on $X^p \backslash \triangle$ and $U := (X^p \backslash \triangle)/S_p$ is an open subset in $S^p(X)$, where $\triangle$ is the union of all diagonals in X^p.

For every normal and irreducible scheme Y, the set of morphisms $\mathrm{Hom}(Y, S^p(X))$ can be identified with the set of all effective cycles $Z \subset X \times Y$ such that each component of Z is finite surjective over Y, and that the degree of Z over Y is p. In particular, the identity map $S^p(X) \xrightarrow{\mathrm{id}} S^p(X)$ corresponds to the incidence cycle $Z \subset X \times S^p(X)$. In fact, Z is a closed subscheme: it is the image of the closed embedding $X \times S^{p-1}(X) \hookrightarrow X \times S^p(X)$, $(x, y) \mapsto (x, x+y)$. Compose this with projection onto the second factor and we get a map

$$p : X \times S^{p-1}(X) \to X \times S^p(X) \to S^p(X).$$

It is finite surjective of degree p. Thus we get a diagram

$$\begin{tikzcd}
X \times S^{p-1}(X) \arrow[d, "p"'] & p^{-1}(U) \arrow[l] \arrow[d, "p|_{p^{-1}(U)}"'] & X^p \backslash \triangle \arrow[l, "/S_{p-1}"'] \arrow[dl, "\backslash S_p"'] \\
S^p(X) & U \arrow[l] &
\end{tikzcd}$$

We see that both maps from $X^p \backslash \triangle$ are Galois étale coverings, $p|_{p^{-1}(U)}$ is a finite étale map of degree p, and U is smooth. Furthermore $p_* f(\mathcal{O}_{X \times S^{p-1}(X)})$ is a coherent $\mathcal{O}_{S^p(X)}$-algebra and the sheaf $\mathcal{A} := p_*(\mathcal{O}_{X \times S^{p-1}(X)}|_{p^{-1}(U)})$ is a locally free $\mathcal{O}_U$-algebra of rank p. This latter sheaf corresponds to the vector bundle $V := \mathrm{Spec}(S^\bullet \mathcal{A}^\nu)$ of rank p over U. Here $\mathcal{A}^\nu$ denotes the dual of $\mathcal{A}$ and $S^\bullet \mathcal{A}^\nu$ denotes its symmetric algebra. There is a well-defined norm map $\mathcal{A} \xrightarrow{N} \mathcal{O}_U$. Locally N is a homogeneous polynomial of degree p, that is, $N \in S^p(\mathcal{A}^\nu)$.

A norm variety $X(a_1, \ldots, a_n)$ for $\{a_1, \ldots, a_n\}$ is then constructed by induction. For $n = 2$, we take $X = X(a_1, a_2)$ in the preceding construction to be the Severi–Brauer variety $\mathrm{SB}(A)$ associated to the cyclic algebra $A = (a_1, a_2, \zeta_p)_k$. Suppose we have constructed a norm variety $X(a_1, \ldots, a_{n-1})$ for $\{a_1, \ldots, a_{n-1}\}$. Again let that be X and let $W \subset V$ be the hypersurface defined by the equation $N - a_n = 0$. By construction, W has dimension $p^{n-1} - 1$. By [Suslin and Joukhovitski 2006, Lemma 2.1] it is smooth over U (hence smooth) and geometrically irreducible. By resolution of singularities we can embed W as an open subvariety of a new smooth, projective, geometrically irreducible variety X' of the same dimension. Together

[Suslin and Joukhovitski 2006, Lemma 2.3 and Proposition 2.4] and its subsequent argument show this X' is a p-generic splitting variety for $\{a_1, \dots, a_n\}$. Hence X' is the norm variety that we seek. Note that its construction depends solely on the tuple $(a_1, \dots, a_n)$.

Remark 2.5. The inductive construction could in fact start with $n = 1$. We describe explicitly what happens at this stage. Take $X = X(a_1) = \mathrm{Spec}(L)$, where $L = k(\sqrt[p]{a_1})$. If $\bar{k}$ is the separable closure of k then $\overline{X} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ has p points; call them $1, 2, \dots, p-1, p$. From there,

$$\begin{aligned}
\overline{X}^p &= \{\text{points on the diagonals}\} \\
&\qquad \sqcup \{(n_1, n_2, \dots, n_p) \mid 1 \le n_i \le p \text{ and } n_i \ne n_j \text{ for all } i, j\}, \\
S^p(\overline{X}) = \overline{X}^p/S_p &= \{\text{classes of points on the diagonals}\} \sqcup \{\overline{(1, 2, \dots, p)}\}, \\
\overline{X}^p \backslash \triangle &= \{(n_1, n_2, \dots, n_p) \mid 1 \le n_i \le p \text{ and } n_i \ne n_j \text{ for all } i, j\}, \\
(\overline{X}^p \backslash \triangle)/S_p &= \{\overline{(1, 2, \dots, p)}\}.
\end{aligned}$$

The above square thus looks like this:

$$\begin{tikzcd}
\overline{X} \times S^{p-1}(\overline{X}) \arrow[d, "p"'] & p^{-1}(U) = \{(n, \overline{(2, 3, \dots, p)}) \mid 1 \le n \le p\} \arrow[l] \arrow[d, "p|_{p^{-1}(U)}"'] \\
S^p(\overline{X}) & U = \{\overline{(1, 2, \dots, p)}\} \arrow[l]
\end{tikzcd}$$

Over k it looks like this:

$$\begin{tikzcd}
X \times S^{p-1}(X) \arrow[d, "p"'] & p^{-1}(U) \cong \mathrm{Spec}(L) \arrow[l] \arrow[d, "p|_{p^{-1}(U)}"'] \\
S^p(X) & U \cong \mathrm{Spec}(k) \arrow[l]
\end{tikzcd}$$

We will use this in Theorems 3.7 and 4.1.

Remark 2.6. Since our problem only concerns birational isomorphism, we can always replace our varieties with birationally isomorphic ones when it suits our purpose but does not change our result. Or we can consider what happens with the generic fiber. For example, in Theorem 3.7 we consider the residue field of the generic fiber of the map p in our construction without mentioning V, W and X'.

3. When $p = 2$, all n

When $p = 2$, we show that the standard norm varieties are birationally isomorphic to Pfister quadrics associated to Pfister forms. This result together with the chain

P-equivalence theorem and properties of quadratic forms will allow us to compare the standard norm varieties for two equal symbols.

For a quadratic form φ, let A_φ denote its symmetric matrix and $D_k(\varphi) \subseteq k$ denote the set of its values. Also, for an n-tuple $(a_1, \ldots, a_n)$, let φ_n denote the n-fold Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle = \prod_{i=1}^n \langle 1, -a_i\rangle$. Furthermore, we associate to φ_n the subform $\psi_n = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$ and denote the quadric defined by ψ_n as $Z(\psi_n)$, known as a Pfister quadric. Below are a few more definitions. A general reference for quadratic forms is [Lam 2005].

Definition 3.1. Two quadratic forms φ and φ' are said to be equivalent, written $\varphi \cong \varphi'$, if there exists a matrix $C \in \mathrm{GL}(k)$ such that $A_{\varphi'} = CA_\varphi C^t$.

Definition 3.2. Two Pfister forms $\varphi = \langle\langle a_1, \ldots, a_n\rangle\rangle$ and $\varphi' = \langle\langle a'_1, \ldots, a'_n\rangle\rangle$ are said to be simply P-equivalent if there exist two indices i and j such that $\langle\langle a_i, a_j\rangle\rangle \cong \langle\langle a'_i, a'_j\rangle\rangle$ and $a_k = a'_k$ for $k \neq i,\ j$. More generally, they are said to be chain P-equivalent, written $\varphi \approx \varphi'$, if there exists a sequence $\varphi_0, \varphi_1, \ldots, \varphi_{m-1}, \varphi_m$ of Pfister forms such that $\varphi = \varphi_0,\ \varphi' = \varphi_m$ and φ_i is simply P-equivalent to φ_{i+1} for $0 \leq i \leq m-1$.

Clearly $\varphi \approx \varphi'$ implies $\varphi \cong \varphi'$. The converse statement was proven by Elman and Lam [1972] and is called the chain P-equivalence theorem. We recall the statement here, for use in Proposition 3.10.

Theorem 3.3 (chain P-equivalence theorem). *Let φ and φ' be n-fold Pfister forms. Then $\varphi \cong \varphi'$ if and only if $\varphi \approx \varphi'$.*

Definition 3.4. Two quadratic forms φ and φ' are said to be birationally equivalent if the quadrics they define are birationally isomorphic, i.e., if the function fields $k(Z(\varphi))$ and $k(Z(\varphi'))$ are isomorphic.

We begin with a lemma about two equivalent Pfister forms and the matrix that connects them.

Lemma 3.5. *If φ_{n-1} and $\varphi_n = \langle 1, -b\rangle\varphi_{n-1}$ are Pfister forms with matrices $A_{\varphi_{n-1}}$ and A_{φ_n}, and $c = \varphi_n(x_1, \ldots, x_{2^n})$, then $\varphi_n \cong \langle c\rangle\varphi_n$ via a matrix*

$$C_n \in \mathrm{GL}_{2^n}(k(x_1, \ldots, x_{2^n}))$$

— that is, $C_n A_{\varphi_n} C_n^t = cA_{\varphi_n}$ — which satisfies two properties:

(1) *$C_n^{-1} = C_n/c$, hence $(C_n^t)^{-1} = C_n^t/c$ as well.*

(2) *The first row and first column of C_n are $(x_1 \cdots x_{2^n})$ and $A_{\varphi_n}(x_1 \cdots x_{2^n})^t$.*

Proof. We induce on n. For $n = 1$ and $c = x_1^2 - ax_2^2$, we have $\varphi_1 \cong c\varphi_1$ via

$$C_1 = \begin{pmatrix} x_1 & x_2 \\ -ax_2 & -x_1 \end{pmatrix},$$

which satisfies (1) and (2).

Next, write

$$A_{\varphi_n} = \begin{pmatrix} A_{\varphi_{n-1}} & 0 \\ 0 & -bA_{\varphi_{n-1}} \end{pmatrix};$$

then $c = \varphi_n(x_1, \ldots, x_{2^n}) = xA_{\varphi_n}x^t = s - bt \in D_k(\varphi_n)$, where $s = \varphi_{n-1}(x_1, \ldots, x_{2^{n-1}})$ and $t = \varphi_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})$ are in $D_k(\varphi_{n-1})$. By induction, $\varphi_{n-1} \cong \langle s\rangle\varphi_{n-1}$ via a matrix $C \in \mathrm{GL}_{2^{n-1}}(k(x_1, \ldots, x_{2^{n-1}}))$, that is, $CA\varphi_{n-1}C^t = sA_{\varphi_{n-1}}$, which satisfies:

(1) $C^{-1} = C/s$, hence $(C^t)^{-1} = C^t/s$.

(2) The first row and first column of C are $(x_1 \cdots x_{2^{n-1}})$ and $A_{\varphi_{n-1}}(x_1 \cdots x_{2^{n-1}})^t$.

Similarly, $\varphi_{n-1} \cong \langle t\rangle\varphi_{n-1}$ via $C' \in \mathrm{GL}_{2^{n-1}}(F(x_{2^{n-1}+1}, \ldots, x_{2^n}))$ with the same properties. From this, we have:

(i) $\varphi_n \cong \langle s\rangle\varphi_{n-1} \perp \langle -b\rangle\langle t\rangle\varphi_{n-1} = \langle s, -bt\rangle\varphi_{n-1}$ with

$$\begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix}\begin{pmatrix} A_{\varphi_{n-1}} & 0 \\ 0 & -bA_{\varphi_{n-1}} \end{pmatrix}\begin{pmatrix} C^t & 0 \\ 0 & C'^t \end{pmatrix} = \begin{pmatrix} sA_{\varphi_{n-1}} & 0 \\ 0 & -btA_{\varphi_{n-1}} \end{pmatrix}.$$

(ii) $\langle s, -bt\rangle\varphi_{n-1} \cong \langle c, -cbst\rangle\varphi_{n-1}$ with

$$\begin{pmatrix} I & I \\ btI & sI \end{pmatrix}\begin{pmatrix} sA_{\varphi_{n-1}} & 0 \\ 0 & -btA_{\varphi_{n-1}} \end{pmatrix}\begin{pmatrix} I & btI \\ I & sI \end{pmatrix} = \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstA_{\varphi_{n-1}} \end{pmatrix}.$$

(iii) Let $D = (CC')^{-1} = C'^{-1}C^{-1} = C'C/ts$; then

$$\langle c, -cbst\rangle\varphi_{n-1} \cong \langle c, -cb\rangle\varphi_{n-1} = \langle c\rangle\varphi_n$$

with

$$\begin{aligned}
\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}&\begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstA_{\varphi_{n-1}} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D^t \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstDA_{\varphi_{n-1}} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D^t \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstDA_{\varphi_{n-1}}D^t \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbstA_{\varphi_{n-1}}/st \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -cbA_{\varphi_{n-1}} \end{pmatrix}.
\end{aligned}$$

(iv) Putting (i), (ii) and (iii) together, we get $\varphi_n \cong \langle s\rangle\varphi_{n-1} \perp \langle -b\rangle\langle t\rangle\varphi_{n-1} = \langle s, -bt\rangle\varphi_{n-1} \cong \langle c, -cbst\rangle\varphi_{n-1} \cong \langle c, -cb\rangle\varphi_{n-1} = \langle c\rangle\varphi_n$ via C_n', where

$$
\begin{aligned}
C_n' &= \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & I \\ btI & sI \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix} \\
&= \begin{pmatrix} I & I \\ btD & sD \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix} \\
&= \begin{pmatrix} C & C' \\ btC'^{-1}C^{-1}C & sC'^{-1}C^{-1}C' \end{pmatrix} \\
&= \begin{pmatrix} C & C' \\ bC' & C'CC'/t \end{pmatrix}.
\end{aligned}
$$

Finally, let

$$
C_n = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} C_n' = \begin{pmatrix} C & C' \\ -bC' & -C'CC'/t \end{pmatrix};
$$

then its inverse C_n^{-1} equals C_n/c, its first row and first column are $(x_1 \cdots x_{2^n})$ and $A_{\varphi_n}(x_1 \cdots x_{2^n})^t$, and

$$
C_n A_{\varphi_n} C_n^t = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} C_n' A_{\varphi_n} C_n'^t \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = cA_{\varphi_n}.
$$

The last equality can be verified directly:

$$
\begin{aligned}
&C_n A_{\varphi_n} C_n^t \\
&= \begin{pmatrix} C & C' \\ -bC' & -C'CC'/t \end{pmatrix}\begin{pmatrix} A_{\varphi_{n-1}} & 0 \\ 0 & -bA_{\varphi_{n-1}} \end{pmatrix}\begin{pmatrix} C^t & -bC'^t \\ C'^t & -C'^tC^tC'^t/t \end{pmatrix} \\
&= \begin{pmatrix} CA_{\varphi_{n-1}} & -bC'A_{\varphi_{n-1}} \\ -bC'A_{\varphi_{n-1}} & (b/t)C'CC'A_{\varphi_{n-1}} \end{pmatrix}\begin{pmatrix} C^t & -bC'^t \\ C'^t & -C'^tC^tC'^t/t \end{pmatrix} \\
&= \begin{pmatrix} CA_{\varphi_{n-1}}C^t - bC'A_{\varphi_{n-1}}C'^t & -bCA_{\varphi_{n-1}}C'^t + (b/t)C'A_{\varphi_{n-1}}C'^tC^tC'^t \\ -bC'A_{\varphi_{n-1}}C^t + (b/t)C'CC'A_{\varphi_{n-1}}C'^t & b^2C'A_{\varphi_{n-1}}C'^t - (b/t^2)C'CC'A_{\varphi_{n-1}}C'^tC^tC'^t \end{pmatrix} \\
&= \begin{pmatrix} sA_{\varphi_{n-1}} - btA_{\varphi_{n-1}} & -bCA_{\varphi_{n-1}}C'^t + bA_{\varphi_{n-1}}C^tC'^t \\ -bC'A_{\varphi_{n-1}}C^t + bC'CA_{\varphi_{n-1}} & b^2tA_{\varphi_{n-1}} - bsA_{\varphi_{n-1}} \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & -bCA_{\varphi_{n-1}}C'^t + (b/s)CA_{\varphi_{n-1}}C^tC^tC'^t \\ -bC'A_{\varphi_{n-1}}C^t + (b/s)C'CCA_{\varphi_{n-1}}C^t & -bcA_{\varphi_{n-1}} \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & -bCA_{\varphi_{n-1}}C'^t + bCA_{\varphi_{n-1}}C'^t \\ -bC'A_{\varphi_{n-1}}C^t + bC'A_{\varphi_{n-1}}C^t & -bcA_{\varphi_{n-1}} \end{pmatrix} \\
&= \begin{pmatrix} cA_{\varphi_{n-1}} & 0 \\ 0 & -bcA_{\varphi_{n-1}} \end{pmatrix} \\
&= cA_{\varphi_n}.
\end{aligned}
$$

This concludes the proof. $\square$

The next lemma is needed to show that the residue field in Theorem 3.7 stays the same.

Lemma 3.6. *The $n \times n$ matrix*

$$M = \begin{pmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{pmatrix}$$

has characteristic polynomial $\mathrm{char}_M(t) = t^{n-1}(t - a_1b_1 - a_2b_2 - \cdots - a_nb_n)$.

Proof. We consider what M does to the standard basis:

$$\begin{aligned} k^n &\xrightarrow{M} k^n, \\ (1, 0, \dots, 0) &\longmapsto b_1(a_1, \dots, a_n), \\ (0, 1, \dots, 0) &\longmapsto b_2(a_1, \dots, a_n), \\ &\vdots \\ (0, 0, \dots, 1) &\longmapsto b_n(a_1, \dots, a_n). \end{aligned}$$

Thus M sends $(a_1, \dots, a_n)$ to $\alpha(a_1, \dots, a_n)$, where $\alpha = a_1b_1 + a_2b_2 + \cdots + a_nb_n$. Letting $v_1 = (a_1, \dots, a_n)$, we choose a new basis $\{v_1, \dots, v_n\}$ for k^n such that $\ker(M) = \langle v_2, \dots, v_n\rangle$ and again look at what M does as a linear map:

$$\begin{aligned} k^n &\xrightarrow{M} k^n, \\ v_1 &\longmapsto (\alpha, 0, \dots, 0), \\ v_2 &\longmapsto (0, \dots, 0), \\ &\vdots \\ v_n &\longmapsto (0, \dots, 0). \end{aligned}$$

Therefore M has canonical form

$$\begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and $\det(tI - M) = \mathrm{char}_M(t) = t^n - \alpha t^{n-1} = t^{n-1}(t - a_1b_1 - a_2b_2 - \cdots - a_nb_n)$. $\square$

We are now ready to turn the standard norm varieties into Pfister quadrics defined by subforms of Pfister forms.

Theorem 3.7. *The standard norm variety $X(a_1, \dots, a_n)$ for $\{a_1, \dots, a_n\}$ is birationally isomorphic to the Pfister quadric $Z(\psi_n) \subset \mathbb{P}_k^{2^{n-1}}$ defined by the subform $\psi_n = \langle\langle a_1, \dots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle$ of the Pfister form $\varphi_n = \langle\langle a_1, \dots, a_n\rangle\rangle$.*

Proof. We induce on n. First we verify the case $n = 2$. As described in Remark 2.5, we begin our symmetric power construction with $X(a_1) = \mathrm{Spec}(L)$, where $L = k(\sqrt{a_1})$ and get

$$\begin{tikzcd} \mathrm{Spec}(L) \times \mathrm{Spec}(L) \arrow[d, "p"'] & p^{-1}(U) \cong \mathrm{Spec}(L) \arrow[l] \arrow[d, "p|_{p^{-1}(U)}"'] \\ S^2(\mathrm{Spec}(L)) & U \cong \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

Hence $X(a_1, a_2) = Z(N_{L/k} - a_2) = Z(x_1^2 - a_1 x_2^2 - a_2)$, the hypersurface defined by the equation $N_{L/k} - a_2 = x_1^2 - a_1 x_2^2 - a_2 = 0$. Projectivization then gives $X(a_1, a_2) = Z(x_1^2 - a_1 x_2^2 - a_2 x_3^2) = Z(\psi_2) \subset \mathbb{P}_k^2$, as required.

By induction, $X(a_1, \dots, a_{n+1}) \approx Z(\psi_{n+1})$. Write $\psi = \psi_{n+1} = \varphi_n \perp \langle -a_{n+1} \rangle = \langle 1 \rangle \perp \varphi' \perp \langle -a_{n+1} \rangle \cong \langle 1, -a_{n+1} \rangle \perp \varphi'$, where φ' is the pure subform of φ. By construction, we get

$$(X_{n+1} \times X_{n+1}) \setminus \Delta \longrightarrow ((X_{n+1} \times X_{n+1}) \setminus \Delta)/_{S_2} \longrightarrow \mathrm{Gr}(2, \mathbb{A}_k^{2^n+1}).$$

Let $U = \langle u, v \rangle = \langle (1, 0, x_2, \dots, x_{2^n}), (0, 1, y_2, \dots, y_{2^n}) \rangle$ be the generic plane in $\mathbb{A}_k^{2^n+1}$ and moreover let $\{u, v\}$ be a basis for U. Over this basis, the restriction $\psi_{k(x_i, y_i)}|_U$ has matrix form

$$\begin{pmatrix} \psi(u) & b(u, v) \\ b(u, v) & \psi(v) \end{pmatrix},$$

where

$$U \times U \xrightarrow{b} k, \quad (u', v') \mapsto \tfrac{1}{2}(\psi(u' + v') - \psi(u') - \psi(v')),$$

is the symmetric bilinear form associated to $\psi_{k(x_i, y_i)}|_U$.

The generic fiber is then the point $(r, s) \in U$ such that

$$\psi(r, s) = \psi(u, u) r^2 + 2b(u, v) rs + \psi(v, v) s^2 = 0,$$

with residue field

$$qf\Big(k(x_i, y_i)\Big[\frac{r}{s}\Big] \Big/ \Big(\psi(u, u)\Big(\frac{r}{s}\Big)^2 + 2b(u, v)\frac{r}{s} + \psi(v, v)\Big)\Big) = k(x_i, y_j)(\sqrt{-\theta}\,),$$

where

$$\begin{aligned} \theta &= \psi(u)\psi(v) - b(u, v)^2 \\ &= (1 + \varphi'(x_2, \dots, x_{2^n}))(-a_{n+1} + \varphi'(y_2, \dots, y_{2^n})) - b(u, v)^2 \\ &= (\varphi(1, x_2, \dots, x_{2^n}))(-a_{n+1} + \varphi'(y_2, \dots, y_{2^n})) - b(u, v)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi(1, x_2, \dots, x_{2^n})\varphi(0, y_2, \dots, y_{2^n}) - b(u, v)^2. \end{aligned}$$

If we write $\varphi = \langle 1, c_2, \dots, c_{2^n}\rangle$ then, by Lemma 3.5, there exists a matrix

$$C_n = \begin{pmatrix} 1 & x_2 & \cdots & x_{2^n} \\ c_2x_2 & \ddots & & \\ \vdots & & \ddots & \\ c_{2^n}x_{2^n} & & & \ddots \end{pmatrix}$$

such that $\varphi(1, x_2, \dots, x_{2^n})\varphi(0, y_2, \dots, y_{2^n}) = \varphi((0, y_2, \dots, y_{2^n})C_n)$. So

$$\begin{aligned}\theta &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi((0, y_2, \dots, y_{2^n})C_n) - b(u, v)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi((0, y_2, \dots, y_{2^n})A_\varphi(1, x_2, \dots, x_{2^n})^t, z_2, \dots, z_{2^n}) \\ &\qquad\qquad - ((y_2, \dots, y_{2^n})A_{\varphi'}(x_2, \dots, x_{2^n})^t)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi((y_2, \dots, y_{2^n})A_{\varphi'}(x_2, \dots, x_{2^n})^t, z_2, \dots, z_{2^n}) \\ &\qquad\qquad - ((y_2, \dots, y_{2^n})A_{\varphi'}(x_2, \dots, x_{2^n})^t)^2 \\ &= (-a_{n+1})\varphi(1, x_2, \dots, x_{2^n}) + \varphi'(z_2, \dots, z_{2^n}).\end{aligned}$$

Above, we let $(z_1, z_2, \dots, z_{2^n}) = (0, y_2, \dots, y_{2^n})C_n$, so that $(z_2, \dots, z_{2^n}) = (y_2, \dots, y_{2^n})M$, where M is C_n without its first row and first column. Since $C_n^2 = \varphi(1, x_2, \dots, x_{2^n})I$, it follows that $M^2 = \varphi(1, x_2, \dots, x_n)I - (c_ix_ix_j)$ for $2 \le i, j \le 2^n$. By Lemma 3.6,

$$\det(M^2) = \varphi(1, x_2, \dots, x_{2^n})^{2^n-2}.$$

Thus $\det(M) = \varphi(1, x_2, \dots, x_{2^n})^{2^{n-1}-1}$ and $M \in \mathrm{GL}_{2^n-1}(F(x_2, \dots, x_{2^n}))$. So the residue field stays the same:

$$F(x_i, y_j)(\sqrt{-\theta}\,) = F(x_i, z_j)(\sqrt{-\theta}\,)$$

It has quadratic norm

$$\begin{aligned}N(m + n\sqrt{-\theta}\,) &= m^2 - a_{n+1}\varphi(1, x_2, \dots, x_{2^n})n^2 + \varphi'(z_2, \dots, z_{2^n})n^2 \\ &= \varphi(m, nz_2, \dots, nz_{2^n}) - a_{n+1}\varphi(n, nx_2, \dots, nx_{2^n}) \\ &= \langle 1, -a_{n+1}\rangle\varphi(m, nz_2, \dots, nz_{2^n}, n, nx_2, \dots, nx_{2^n}) \\ &= \varphi_{n+1}(t_1, \dots, t_{2^{n+1}}).\end{aligned}$$

Therefore our projectivized $X(a_1, \dots, a_{n+2}) = Z(N - a_{n+2}t^2_{2^{n+1}+1})$ is birationally isomorphic to $Z(\varphi_{n+1} \perp \langle -a_{n+2}\rangle) = Z(\psi_{n+2}) \subset \mathbb{P}_k^{2^{n+1}}$, as wanted. □

Next, we show that interchanging a_i and a_j or multiplying a_i by any nonzero norm $N_{k(\sqrt{a_j})/k}(u)$ in the symbol $\{a_1, \dots, a_n\}$ does not change its standard norm variety. For this, we need two more lemmas about Pfister neighbors; the first one we will use toward our Corollary 3.13 and the second one we will use toward our Example 3.14.

Lemma 3.8. *If $\varphi = \langle\langle a_1, \dots, a_n\rangle\rangle$ is an anisotropic Pfister form then the two forms $\varphi \perp \langle -b\varphi\rangle \perp \langle -c\rangle$ and $\varphi \perp \langle -c\varphi\rangle \perp \langle -b\rangle$ are birationally equivalent.*

Proof. We connect the quadrics defined by these two forms by a sequence of birationally isomorphic ones. Let (x, y, z) be the generic zero for the form $\varphi \perp \langle -b\varphi\rangle \perp \langle -c\rangle$; then

$$\varphi(x) - b\varphi(y) - cz^2 = 0.$$

Since φ is Pfister and $\varphi(y) \in D_{k(y)}(\varphi)$, it follows $\varphi \cong \varphi(y)\varphi$ over $k(y)$. That means there exists a matrix $C \in \mathrm{GL}(k(y))$ such that $\varphi(x) = \varphi(y)\varphi(Cx)$. Let $x' = Cx$; then $k(x, y, z) = k(x', y, z)$ and

$$\varphi(y)\varphi(x') - b\varphi(y) - cz^2 = 0,$$

hence

$$\varphi(x') - b - c\frac{z^2}{\varphi(y)} = 0.$$

Now let $y' = y/\varphi(y)$; then $k(x, y, z) = k(x', y', z)$ and

$$\varphi(x') - b - cz^2\varphi(y') = 0,$$

hence

$$\frac{\varphi(x')}{z^2} - \frac{b}{z^2} - c\varphi(y') = 0.$$

Finally, let $x'' = x'/z$ and $z' = 1/z$; then (x'', y', z') is a generic zero for $\varphi \perp \langle -c\varphi\rangle \perp \langle -b\rangle$, $k(x, y, z) = k(x'', y', z')$ and

$$\varphi(x'') - c\varphi(y') - bz'^2 = 0.$$

Therefore, the two forms $\varphi \perp \langle -b\varphi\rangle \perp \langle -c\rangle$ and $\varphi \perp \langle -c\varphi\rangle \perp \langle -b\rangle$ are birationally equivalent. □

Lemma 3.9. *If $\varphi = \langle\langle a_1, \dots, a_n\rangle\rangle$ is an anisotropic Pfister form then the two forms $\varphi \perp \langle -b\rangle$ and $\varphi \perp \langle -b\varphi(x_0)\rangle$ with $\varphi(x_0) \neq 0$ are birationally equivalent. In particular, $\varphi \perp \langle -b\rangle \approx \varphi \perp \langle -bN_{k(\sqrt{a_i})/k}(u)\rangle$ for any nonzero norm $N_{k(\sqrt{a_i})/k}(u)$.*

Proof. We use the same approach as in Lemma 3.8. Let (x, y) be a generic zero for the form $\varphi \perp \langle -b\varphi(x_0)\rangle$; then

$$\varphi(x) - b\varphi(x_0)y^2 = 0,$$

hence

$$\varphi(x_0)\varphi(x) - b\varphi(x_0)^2y^2 = 0.$$

Again $\varphi \cong \varphi(x_0)\varphi$ over k, i.e., there exists a matrix $C \in \mathrm{GL}(k)$ such that $\varphi(Cx) = \varphi(x_0)\varphi(x)$. Let $x' = Cx$ and $y' = \varphi(x_0)y$; then (x', y') is a generic zero for $\varphi \perp \langle -b\rangle$, $k(x, y) = k(x', y')$ and

$$\varphi(x') - by'^2 = 0$$

Therefore, the two forms $\varphi \perp \langle -b\rangle$ and $\varphi \perp \langle -b\varphi(x_0)\rangle$ with $\varphi(x_0) \neq 0$ are birationally equivalent. The last statement follows when we choose x_0 such that $\varphi(x_0) = N_{k(\sqrt{a_i})/k}(u)$. □

Proposition 3.10. *If two Pfister forms φ and φ' are equivalent then their associated subforms ψ and ψ' are birationally equivalent.*

Proof. By the chain P-equivalence theorem, $\varphi \approxeq \varphi'$. So there exists a sequence of Pfister forms $\varphi_0, \varphi_1, \ldots, \varphi_t, \ldots, \varphi_{m-1}, \varphi_m$ such that $\varphi = \varphi_0$, $\varphi' = \varphi_m$ and φ_t is simply P-equivalent to φ_{t+1} for $0 \le t \le m-1$. Write $\varphi_t = \langle\langle a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n\rangle\rangle$ and $\varphi_{t+1} = \langle\langle a_1, \ldots, a'_i, \ldots, a'_j, \ldots, a_n\rangle\rangle$, where $\langle\langle a_i, a_j\rangle\rangle \cong \langle\langle a'_i, a'_j\rangle\rangle$. If $i = j$ then there is nothing to do. Otherwise, we consider each case separately:

(1) If $j \neq n$ then

$$\begin{aligned}\psi_t &= \langle\langle a_1, \ldots, a_i, \ldots, a_j, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle\\ &\cong \langle\langle a_1, \ldots, a'_i, \ldots, a'_j, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_n\rangle\\ &= \psi_{t+1}.\end{aligned}$$

(2) If $j = n$ and $i \neq n-1$ then, by Lemma 3.8,

$$\begin{aligned}\psi_t &= \langle\langle a_1, \ldots, a_i, \ldots, a_{n-1}\rangle\rangle \perp \langle -a_j\rangle\\ &\approx \langle\langle a_1, \ldots, a_i, \ldots, a_j\rangle\rangle \perp \langle -a_{n-1}\rangle\\ &\cong \langle\langle a_1, \ldots, a'_i, \ldots, a'_j\rangle\rangle \perp \langle -a_{n-1}\rangle\\ &\approx \langle\langle a_1, \ldots, a'_i, \ldots, a_{n-1}\rangle\rangle \perp \langle -a'_j\rangle\\ &= \psi_{t+1}.\end{aligned}$$

(3) If $j = n$ and $i = n-1$ then, again by Lemma 3.8,

$$\begin{aligned}\psi_t &= \langle\langle a_1, \ldots, a_{n-2}, a_i\rangle\rangle \perp \langle -a_j\rangle\\ &\cong \langle\langle a_1, \ldots, a_i, a_{n-2}\rangle\rangle \perp \langle -a_j\rangle\\ &\approx \langle\langle a_1, \ldots, a_i, a_j\rangle\rangle \perp \langle -a_{n-2}\rangle\\ &\cong \langle\langle a_1, \ldots, a'_i, a'_j\rangle\rangle \perp \langle -a_{n-2}\rangle\\ &\approx \langle\langle a_1, \ldots, a'_i, a_{n-2}\rangle\rangle \perp \langle -a'_j\rangle\\ &\cong \langle\langle a_1, \ldots, a_{n-2}, a'_i\rangle\rangle \perp \langle -a'_j\rangle\\ &= \psi_{t+1}.\end{aligned}$$

Hence $\psi_t \approx \psi_{t+1}$ for all t, and $\psi \approx \psi'$. □

Remark 3.11. Let φ be a Pfister form of dimension greater than or equal to 2, $c \in k^\times$, and φ_1 a nonzero subform of φ. In [Ahmad and Ohm 1995], H. Ahmad called (φ, c, φ_1) a Pfister triple, $\varphi \perp \langle c\rangle$ the base form, $\varphi \perp c\varphi_1$ the form defined by the triple, $\varphi \perp c\varphi$ the associated Pfister form, and any form similar to such $\varphi \perp c\varphi_1$ a special Pfister neighbor. In this setting the forms in Lemma 3.8 and the forms in Lemma 3.9 are pairwise special Pfister neighbors of the same dimensions and have the same associated Pfister forms $\varphi \otimes \langle\langle b, c\rangle\rangle$ and $\varphi \otimes \langle\langle b\rangle\rangle$, respectively. The lemmas then follow from his more general [Ahmad and Ohm 1995, Theorem 1.6].

Remark 3.12. One sees that Lemmas 3.8 and 3.9 hold for any strongly multiplicative form φ as defined in [Lam 2005]. The work lies with anisotropic Pfister forms. The remaining strongly multiplicative forms are isotropic, hence their function fields are rational and both lemmas become trivial.

Proposition 3.10 enables us to compare the standard norm varieties for two equal symbols.

Corollary 3.13. *The standard norm varieties $X(a_1, \dots, a_n)$ and $X(b_1, \dots, b_n)$ for $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are birationally isomorphic if $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$ in $K_n^M(k)/2$.*

Proof. By [Elman et al. 2008, Theorem 6.20], the two Pfister forms $\varphi = \langle\langle a_1, \dots, a_n\rangle\rangle$ and $\varphi' = \langle\langle b_1, \dots, b_n\rangle\rangle$ are equivalent. Proposition 3.10 now implies their associated subforms ψ and ψ' are birationally equivalent. By Theorem 3.7, $X(a_1, \dots, a_n)$ and $X(b_1, \dots, b_n)$ are birationally isomorphic. $\square$

Example 3.14. For any nonzero norm $N_{k(\sqrt{a_i})/k}(u)$, we know

$$\{a_1, \dots, a_i, \dots, a_j, \dots, a_n\} = \{a_1, \dots, a_i, \dots, a_j N_{k(\sqrt{a_i})/k}(u), \dots, a_n\}$$

in $K_n^M(k)/2$. By Corollary 3.13, their standard norm varieties are birationally isomorphic. Or we can use Theorem 3.7 and Lemma 3.9, bypassing the chain P-equivalence theorem to see this as well.

4. When $p > 2$ and $n = 2$

When $p > 2$ and $n = 2$ we show that the standard norm varieties are birationally isomorphic to Severi–Brauer varieties.

Theorem 4.1. *The standard norm variety $X(a, b)$ for $\{a, b\}$ is birationally isomorphic to the Severi–Brauer variety* $\mathrm{SB}(A)$ *associated to the cyclic algebra $A = (a, b, \zeta_p)_k$.*

Proof. Again, if we start the symmetric power construction with $X(a) = \mathrm{Spec}(L)$, where $L = k(\sqrt[p]{a})$, then $X(a, b) = Z(N_{L/k} - b)$ by Remark 2.5. We consider what happens in a split case, where $A_L \cong M_p(L)$ and $\mathrm{SB}(A_L) \cong \mathbb{P}_L^{p-1}$. Furthermore, if $G = \mathrm{Gal}(L/k) = \langle\sigma\rangle$ of order p then over L, the norm $N_{L/k}(x)$ splits in to a product $\prod_{i=0}^{p-1} \sigma^i(x)$ for every $x \in L$. Define

$$U_L = \{I \subset M_p(L)\},$$

where

$$I = \left\{ \begin{pmatrix} \alpha_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{p-1} & 0 & \cdots & 0 \end{pmatrix} M \;\middle|\; \alpha_i \neq 0 \text{ for all } i \text{ and } M \in M_p(L) \right\};$$

then U_L is an open subset in $\mathrm{SB}(A_L)$ and we have a diagram

$$\begin{tikzcd} Z(N_{L/k}-b)_L \arrow[r, "f_L"] \arrow[d, "/G"] & U_L \arrow[r, "\text{open}"] \arrow[d, "/G"] & \mathrm{SB}(A_L) \arrow[d, "/G"] \\ Z(N_{L/k}-b) \arrow[r, "f"] & U \arrow[r, "\text{open}"] & \mathrm{SB}(A) \end{tikzcd}$$

where f_L can be described as

$$\begin{aligned} Z(N_{L/k}-b)_L &\xrightarrow{f_L} U_L, \\ (x, \sigma(x), \ldots, \sigma^{p-1}(x)) &\longmapsto (x : x\sigma(x) : \ldots : x\sigma(x)\cdots\sigma^{p-1}(x)), \end{aligned}$$

if we abuse notation and write points in $\mathrm{SB}(A_L)$ in projective coordinates. We verify that f_L is G-equivariant:

$$\begin{aligned} &f_L(\sigma \cdot (x, \sigma(x), \ldots, \sigma^{p-2}(x), \sigma^{p-1}(x))) \\ &\qquad = f(\sigma(x), \sigma^2(x), \ldots, \sigma^{p-1}(x), \sigma^p(x)) \\ &\qquad = (\sigma(x) : \sigma(x)\sigma^2(x) : \ldots : \sigma(x)\cdots\sigma^{p-1}(x) : \sigma(x)\sigma^2(x)\cdots\sigma^p(x)) \\ &\qquad = (\sigma(x) : \sigma(x)\sigma^2(x) : \ldots : \sigma(x)\cdots\sigma^{p-1}(x) : b), \end{aligned}$$

while

$$\begin{aligned} &\sigma \cdot f_L(x, \sigma(x), \ldots, \sigma^{p-1}(x)) \\ &\qquad = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ b & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ x\sigma(x) \\ \vdots \\ x\sigma(x)\cdots\sigma^{p-2}(x) \\ x\sigma(x)\cdots\sigma^{p-1}(x) \end{pmatrix} \\ &\qquad = (x\sigma(x) : x\sigma(x)\sigma^2(x) : \ldots : x\sigma(x)\cdots\sigma^{p-1}(x) : bx) \\ &\qquad = (\sigma(x) : \sigma(x)\sigma^2(x) : \ldots : \sigma(x)\cdots\sigma^{p-1}(x) : b). \end{aligned}$$

In function fields, we have an isomorphism of the same name f_L from $L(U_L) = L(t_1/t_0, \dots, t_p/t_0)$ to $L(Z(N_{L/k} - b)_L) = L(x, \sigma(x), \dots, \sigma^{p-1}(x))$,

$$L\left(\frac{t_1}{t_0}, \dots, \frac{t_p}{t_0}\right) \xrightarrow{f_L} L(x, \sigma(x), \dots, \sigma^{p-1}(x)), \quad \frac{t_i}{t_0} \mapsto x\sigma(x)\cdots\sigma^{i-1}(x),$$

where $i = 1, \dots, p$ and $t_p/t_0 = b$ with inverse

$$L(x, \sigma(x), \dots, \sigma^{p-1}(x)) \xrightarrow{f_L^{-1}} L\left(\frac{t_1}{t_0}, \dots, \frac{t_p}{t_0}\right), \quad \sigma^{i-1}(x) \mapsto \frac{t_i}{t_{i-1}}.$$

We verify that f_L respects the G-action:

$$\begin{aligned} f_L\left(\sigma \cdot \frac{t_i}{t_0}\right) &= f_L\left(\frac{t_{i+1}}{t_1}\right) \\ &= f_L\left(\left(\frac{t_{i+1}}{t_0}\right)\left(\frac{t_1}{t_0}\right)^{-1}\right) \\ &= x\sigma(x)\cdots\sigma^i(x)x^{-1} \\ &= \sigma(x)\cdots\sigma^i(x), \end{aligned}$$

while

$$\sigma \cdot f_L\left(\frac{t_i}{t_0}\right) = \sigma \cdot (x\sigma(x)\cdots\sigma^{i-1}(x)) = \sigma(x)\cdots\sigma^i(x).$$

Therefore $Z(N_{L/k} - b)_L$ is birationally isomorphic to U_L. So $Z(N_{L/k} - b)$ is birationally isomorphic to U, hence to $\mathrm{SB}(A)$. □

This theorem enables us to compare the standard norm varieties for two equal symbols.

Corollary 4.2. *The standard norm varieties $X(a_1, a_2)$ and $X(b_1, b_2)$ for $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are birationally isomorphic if $\{a_1, a_2\} = \{b_1, b_2\}$ in $K_2^M(k)/p$.*

Proof. By the norm residue homomorphism $K_2^M(k)/p \to \mathrm{Br}_p(k)$, the classes of $(a_1, a_2, \zeta_p)_k$ and $(b_1, b_2, \zeta_p)_k$ are equal in the subgroup $\mathrm{Br}_p(k)$ of elements of exponent p in the Brauer group $\mathrm{Br}(k)$. Since they have the same dimension, $(a_1, a_2, \zeta_p)_k$ and $(b_1, b_2, \zeta_p)_k$ are isomorphic as algebras. Hence

$$\mathrm{SB}((a_1, a_2, \zeta_p)_k) \cong \mathrm{SB}((a_1, a_2, \zeta_p)_k).$$

It follows from the theorem that $X(a_1, a_2) \approx X(b_1, b_2)$. □

5. When $p > 2$ and $n = 3$

When $p > 2$ and $n = 3$, we show that the standard norm varieties are birationally isomorphic to varieties defined by reduced norms of cyclic algebras.

Theorem 5.1. *The standard norm variety $X(a, b, c)$ for $\{a, b, c\}$ is birationally isomorphic to $Z(\mathrm{Nrd}_{A/k} - c)$, where $A = (a, b, \zeta_p)_k$.*

Proof. We consider what happens in a split case. Let $L = k(\sqrt[p]{a}\,)$ and use $\mathrm{SB}(A)$ as the standard norm variety $X(a, b)$ for $\{a, b\}$. Once again, $A_L \cong M_p(L)$ and $\mathrm{SB}(A_L) \cong \mathbb{P}_L^{p-1}$. Our symmetric power construction looks like the front square over k and the back square over L:

$$\begin{tikzcd}
 & \mathrm{SB}(A_L) \times S^{p-1}(\mathrm{SB}(A_L)) \arrow[dl, "/G"'] \arrow[dd, "p_L"] & & p^{-1}(U_L) \arrow[ll] \arrow[dl, "/G"'] \arrow[dd, "p_L|"] & \\
\mathrm{SB}(A) \times \mathrm{SB}^{p-1}(A) \arrow[dd, "p"'] & & p^{-1}(U) \arrow[ll] \arrow[dd, "p|"'] & & \\
 & S^p(\mathrm{SB}(A_L)) \arrow[dl, "/G"'] & & U_L \arrow[ll] \arrow[dl, "/G"'] & V_L \arrow[l, "\pi_L"'] \arrow[dl, "/G"'] \\
S^p(\mathrm{SB}(A)) & & U \arrow[ll] & V \arrow[l, "\pi"'] &
\end{tikzcd}$$

Now let X_L denote the variety of all étale subalgebras of degree p in $\mathrm{End}_L(L^p)$. If each subalgebra $D \in X_L$ is generated by a matrix λ, where $\lambda = (\lambda_1, \ldots, \lambda_p)$ is its diagonal form, then S_p acts trivially on X_L by permuting the diagonal entries. So we have an S_p-equivariant map

$$U_L \xrightarrow{f_L} X_L, \quad (u_1, \ldots, u_p) \mapsto D,$$

where D is the étale subalgebra whose eigenspaces are the lines $u_1, \ldots, u_p$, with inverse $f_L^{-1} : D \mapsto (u_1, \ldots, u_p)$. This map fits into the following commutative diagram:

$$\begin{tikzcd}
U_L \arrow[r, "f_L"] \arrow[d, "/G"] & X_L \arrow[d, "/G"] \\
U \arrow[r, "f"] & X
\end{tikzcd}$$

and we get vector bundles over the last diagram,

$$\begin{tikzcd}
& U_L \arrow[dl, "/G"'] \arrow[dd, "f_L"] & & V_L \arrow[ll, "\pi_L"'] \arrow[dl] \arrow[dd, "f_L^*"] \\
U \arrow[dd, "f"] & & V \arrow[ll, "\pi"', crossing over] & \\
& X_L \arrow[dl, "/G"'] & & V_{X_L} \arrow[ll, "\pi_{X_L}"'] \arrow[dl] \\
X & & V_X \arrow[ll, "\pi_X"'] \arrow[from=uu, "f^*", crossing over] &
\end{tikzcd}$$

For each $(u_1, \ldots, u_p) \in U_L$, the preimage $p_L^{-1}((u_1, \ldots, u_p))$ consists of p points $y_1, \ldots, y_p$, where each y_i is of the form $(u_i, (u_1, \ldots, \check{u}_i, \ldots, u_p))$. So $\pi_L^{-1}((u_1, \ldots, u_p)) = \{((u_1, \ldots, u_p), x_1, \ldots, x_p) \mid x_i \in L(y_i)\}$. Correspondingly, $\pi_{X_L}^{-1}(D) = \{(D, d) \mid d \in D\}$. Both are algebras of rank p over L. We can describe the back face of the cube pointwise:

$$\begin{tikzcd}
((u_1, \ldots, u_p), x_1, \ldots, x_p) \arrow[r, "f_L^*"] \arrow[d, "\pi_L"'] & \left(D, \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_p \end{pmatrix}\right) \arrow[d, "\pi_{X_L}"] \\
(u_1, \ldots, u_p) \arrow[r, "f_L"] & D
\end{tikzcd}$$

Note that if $q(t) = a_1 t + \cdots + a_p t^p$ and $d = q(\lambda) \in D$ with eigenvalues $q(\lambda_i)$ then $f_L^{*^{-1}}(D, d) = ((u_1, \ldots, u_p), q(\lambda_1), \ldots, q(\lambda_p))$.

Therefore, in V_L and V_{X_L} we have two birationally isomorphic subvarieties $Z(N - c)_L$ and $Z(\mathrm{Nrd}_{A_L/L} - c)$, since

$$\begin{aligned}
Z(N - c)_L &= \{((u_1, \ldots, u_p), x_1, \ldots, x_p) \mid x_1 \cdots x_p = c\} \\
&\cong \{(D, d) \mid D \subset A_L \text{ étale of rank } p \text{ and } d \in D \text{ with } N_{D/L}(d) = c\} \\
&= \{(D, d) \mid D \subset A_L \text{ étale of rank } p \text{ and } d \in D \text{ with } \mathrm{Nrd}_{A_L/L}(d) = c\} \\
&\cong \{d \in A_L \mid \langle d \rangle \subset A_L \text{ étale of rank } p \text{ and } \mathrm{Nrd}_{A_L/L}(d) = c\} \\
&\qquad\qquad (\text{via } (D, d) \mapsto d)
\end{aligned}$$

$$
\begin{aligned}
&= \{d \in A_L \mid \mathrm{Nrd}_{A_L/L}(d) = c\} \\
&\qquad\qquad \cap \{d \in A_L \mid \text{its minimal polynomial } m_d(t) \text{ is of degree } p\} \\
&= \{d \in A_L \mid \mathrm{Nrd}_{A_L/L}(d) = c\} \\
&\qquad\qquad\qquad \cap \{d \in A_L \mid x_i \neq x_j \text{ for all of its eigenvalues } x_i, x_j\} \\
&\approx \{d \in A_L \mid \mathrm{Nrd}_{A_L/L}(d) = c\} \\
&= Z(\mathrm{Nrd}_{A_L/L} - c).
\end{aligned}
$$

Note that the intersection above is nonempty — it contains, for example, the diagonal matrix $(c/\zeta_p^{(p-1)/2}, \zeta_p, \ldots, \zeta_p^{p-1})$ — and the second set is open. Hence our standard norm variety $X(a, b, c) = Z(N - c)$ is birationally isomorphic to $Z(\mathrm{Nrd}_{A/k} - c)$ over k. □

Knowing that $X(a, b, c)$ is birationally isomorphic to $Z(\mathrm{Nrd}_{A/k} - c)$, where $A = (a, b, \zeta_p)_k$, may allow us to compare $X(a, b, c)$ and $X(a', b', c')$ when $\{a, b, c\} = \{a', b', c'\}$ in $K_3^M(k)/p$. If we know $Z(\mathrm{Nrd}_{A/k} - c) \approx Z(\mathrm{Nrd}_{A'/k} - c')$, where $A' = (a', b', \zeta_p)_k$, then we can draw the same corollary for $p > 2$ and $n = 3$ as we did for $p = 2$ in Corollary 3.13 and for $p > 2$ and $n = 2$ in Corollary 4.2.

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DINH HUU NGUYEN: dinhuun@gmail.com
Department of Mathematics, UCLA, Los Angeles, CA 90095, United States

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