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$\mathbb{A}^1$-homotopy invariance of algebraic K-theory with coefficients and du Val singularities

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C. Weibel, and Thomason and Trobaugh, proved (under some assumptions) that algebraic K-theory with coefficients is $\mathbb{A}^1$-homotopy invariant. We generalize this result from schemes to the broad setting of dg categories. Along the way, we extend the Bass–Quillen fundamental theorem as well as Stienstra's foundational work on module structures over the big Witt ring to the setting of dg categories. Among other cases, the above $\mathbb{A}^1$-homotopy invariance result can now be applied to sheaves of (not necessarily commutative) dg algebras over stacks. As an application, we compute the algebraic K-theory with coefficients of dg cluster categories using solely the kernel and cokernel of the Coxeter matrix. This leads to a complete computation of the algebraic K-theory with coefficients of the du Val singularities parametrized by the simply laced Dynkin diagrams. As a byproduct, we obtain vanishing and divisibility properties of algebraic K-theory (without coefficients).

1. Introduction and statement of results

Let k be a base commutative ring, X a quasicompact, quasiseparated k-scheme, and l^ν a prime power. As proved by Weibel [1982, page 391; 1981, Theorem 5.2] and by Thomason and Trobaugh [1990, Theorems 9.5–9.6], we have the following result:

Theorem 1.1. (i) *When $1/l \in k$, the projection morphism $X[t] \to X$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(X; \mathbb{Z}/l^\nu) \to \mathbb{K}(X[t]; \mathbb{Z}/l^\nu)$.*

(ii) *When l is nilpotent in k, the projection morphism $X[t] \to X$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(X) \otimes \mathbb{Z}[1/l] \to \mathbb{K}(X[t]) \otimes \mathbb{Z}[1/l]$.*

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The proof of Theorem 1.1 is quite involved! The affine case, established by Weibel, makes use of a convergent right half-plane spectral sequence, of a universal coefficient sequence, of the Bass–Quillen fundamental theorem (see [Grayson 1976, page 236]), and more importantly of Stienstra's foundational work [1982, §8] on module structures over the big Witt ring. The extension to quasicompact, quasiseparated schemes, later established by Thomason and Trobaugh [1990, §9.1], is based on a powerful method known as "reduction to the affine case".

The first goal of this article is to generalize Theorem 1.1 from schemes to the broad setting of dg categories. Consult Sections 2–3 for applications and computations.

Statement of results. A *differential graded* (*dg*) *category* $\mathcal{A}$, over the base commutative ring k, is a category enriched over complexes of k-modules; see Section 4. Every (dg) k-algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes, since the category of perfect complexes $\mathrm{perf}(X)$ of every quasicompact, quasiseparated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; see [Keller 2006, §4.4]. Given a dg category $\mathcal{A}$, let us write $\mathcal{A}[t]$ for the tensor product $\mathcal{A} \otimes k[t]$. Our first main result is the following:

Theorem 1.2. (i) *When $1/l \in k$, the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu) \to \mathbb{K}(\mathcal{A}[t]; \mathbb{Z}/l^\nu)$.*

(ii) *When l is nilpotent in k, the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(\mathcal{A}) \otimes \mathbb{Z}[1/l] \to \mathbb{K}(\mathcal{A}[t]) \otimes \mathbb{Z}[1/l]$.*

For the proof of Theorem 1.2, we adapt the Bass–Quillen fundamental theorem, as well as Stienstra's foundational work on module structures over the big Witt ring, to the broad setting of dg categories; see Theorems 8.4 and 9.1, respectively. These results are of independent interest. Except in Theorem 9.1, we work more generally with a localizing invariant; see Definition 5.1.

2. Applications and computations

The second goal of this article is to explain how Theorem 1.2 leads naturally to several applications and computations.

Sheaves of dg algebras. Let X be a quasicompact, quasiseparated k-scheme and $\mathcal{S}$ a sheaf of (not necessarily commutative) dg $\mathcal{O}_X$-algebras. In addition to $\mathrm{perf}_{\mathrm{dg}}(X)$, we can consider the dg category $\mathrm{perf}_{\mathrm{dg}}(\mathcal{S})$ of perfect complexes of $\mathcal{S}$-modules; see [Tabuada and Van den Bergh 2015, §6]. By applying Theorem 1.2 to the dg category $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(\mathcal{S})$, we obtain the following generalization of Theorem 1.1:

Theorem 2.1. (i) *When $1/l \in k$, the projection morphism $\mathcal{S}[t] \to \mathcal{S}$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(\mathcal{S}; \mathbb{Z}/l^\nu) \to \mathbb{K}(\mathcal{S}[t]; \mathbb{Z}/l^\nu)$.*

(ii) *When l is nilpotent in k, the projection morphism $\mathcal{S}[t] \to \mathcal{S}$ gives rise to an homotopy equivalence of spectra* $\mathbb{K}(\mathcal{S}) \otimes \mathbb{Z}[1/l] \to \mathbb{K}(\mathcal{S}[t]) \otimes \mathbb{Z}[1/l]$.

Remark 2.2 (orbifolds and stacks). Given an orbifold, or more generally a stack $\mathcal{X}$, we can also consider the associated dg category $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X})$ of perfect complexes. Therefore, Theorem 2.1 holds more generally for every sheaf $\mathcal{S}$ of dg $\mathcal{O}_{\mathcal{X}}$-algebras.

DG orbit categories. Given a dg category $\mathcal{A}$ and a dg functor $F : \mathcal{A} \to \mathcal{A}$ which induces an equivalence of categories $\mathrm{H}^0(F) : \mathrm{H}^0(\mathcal{A}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{A})$, recall from [Keller 2005, §5.1] the construction of the associated dg orbit category $\mathcal{A}/F^{\mathbb{Z}}$. Thanks to Theorem 1.2, all the results established in [Tabuada 2015a] can now be applied to algebraic K-theory with coefficients. For example, Theorem 1.5 of [Tabuada 2015a] gives rise to the result:

Theorem 2.3. *When $1/l \in k$, we have a distinguished triangle of spectra*:

$$\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu) \xrightarrow{\mathbb{K}(F;\mathbb{Z}/l^\nu)-\mathrm{Id}} \mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu) \longrightarrow \mathbb{K}(\mathcal{A}/F^{\mathbb{Z}}; \mathbb{Z}/l^\nu) \longrightarrow \Sigma\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu).$$

When l is nilpotent in k, the same holds with $\mathbb{K}(-; \mathbb{Z}/l^\nu)$ replaced by $\mathbb{K}(-)\otimes\mathbb{Z}[1/l]$.

Remark 2.4 (fundamental isomorphism). When F is the identity dg functor, the dg orbit category $\mathcal{A}/F^{\mathbb{Z}}$ reduces to $\mathcal{A}[t, 1/t]$ and the above distinguished triangle splits. Thus, we obtain a fundamental isomorphism between $\mathbb{K}(\mathcal{A}[t, 1/t]; \mathbb{Z}/l^\nu)$ and the direct sum $\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu) \oplus \Sigma\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu)$. When l is nilpotent in k, the same holds with $\mathbb{K}(-; \mathbb{Z}/l^\nu)$ replaced by $\mathbb{K}(-) \otimes \mathbb{Z}[1/l]$.

DG cluster categories. Let k be an algebraically closed field, Q a finite quiver without oriented cycles, kQ the path k-algebra of Q, $\mathcal{D}^b(kQ)$ the bounded derived category of finitely generated right kQ-modules, and $\mathcal{D}^b_{\mathrm{dg}}(kQ)$ the canonical dg enhancement of $\mathcal{D}^b(kQ)$. Consider the dg functors

$$\tau^{-1}\Sigma^m : \mathcal{D}^b_{\mathrm{dg}}(kQ) \longrightarrow \mathcal{D}^b_{\mathrm{dg}}(kQ), \quad m \geq 0,$$

where τ is the Auslander–Reiten translation. Following Keller [2005, §7.2], the *dg (m)-cluster category $\mathcal{C}_Q^{(m)}$ of Q* is defined as the dg orbit category

$$\mathcal{D}^b_{\mathrm{dg}}(kQ)/(\tau^{-1}\Sigma^m)^{\mathbb{Z}}.$$

In the same vein, the *(m)-cluster category of Q* is defined as $\mathrm{H}^0(\mathcal{C}_Q^{(m)})$. These (dg) categories play, nowadays, a key role in the representation theory of finite-dimensional algebras; see Reiten's ICM address [2010]. As proved by Keller and Reiten [2008, §2], the (m)-cluster categories (with $m \geq 1$) can be conceptually characterized as those $(m+1)$-Calabi–Yau triangulated categories containing a cluster-tilting object whose endomorphism algebra has a quiver without oriented cycles.

As explained in [Tabuada 2015a, Corollary 2.11], in the particular case of dg cluster categories, Theorem 2.3 reduces to the following one:

Theorem 2.5. *When $l \neq \mathrm{char}(k)$, we have a distinguished triangle of spectra*

$$\bigoplus_{r=1}^{v} \mathbb{K}(k;\mathbb{Z}/l^{\nu}) \xrightarrow{(-1)^m\Phi_Q - \mathrm{Id}} \bigoplus_{r=1}^{v} \mathbb{K}(k;\mathbb{Z}/l^{\nu}) \to \mathbb{K}(\mathcal{C}_Q^{(m)};\mathbb{Z}/l^{\nu}) \to \bigoplus_{r=1}^{v} \Sigma\mathbb{K}(k;\mathbb{Z}/l^{\nu}),$$

where v stands for the number of vertices of Q and Φ_Q for the Coxeter matrix of Q. When $l = \mathrm{char}(k)$, the same holds with $\mathbb{K}(-;\mathbb{Z}/l^{\nu})$ replaced by $\mathbb{K}(-)\otimes\mathbb{Z}[1/l]$.

As proved by Suslin [1984, Corollary 3.13], we have $\mathbb{K}_i(k;\mathbb{Z}/l^{\nu}) \simeq \mathbb{Z}/l^{\nu}$ when $i \geq 0$ is even and $\mathbb{K}_i(k;\mathbb{Z}/l^{\nu}) = 0$ otherwise. Consequently, making use of the long exact sequence of algebraic K-theory groups with coefficients associated to the above distinguished triangle of spectra, we obtain the following result:

Corollary 2.6. *Consider the* (*matrix*) *homomorphism*

$$(-1)^m\Phi_Q - \mathrm{Id} : \bigoplus_{r=1}^{v} \mathbb{Z}/l^{\nu} \longrightarrow \bigoplus_{r=1}^{v} \mathbb{Z}/l^{\nu}. \tag{2.7}$$

When $l \neq \mathrm{char}(k)$, we have the following computation:

$$\mathbb{K}_i(\mathcal{C}_Q^{(m)};\mathbb{Z}/l^{\nu}) \simeq \begin{cases} \text{cokernel}\,(2.7) & \text{if } i \geq 0 \text{ even}, \\ \text{kernel}\,(2.7) & \text{if } i \geq 0 \text{ odd}, \\ 0 & \text{if } i < 0. \end{cases}$$

Corollary 2.6 provides a complete computation of the algebraic K-theory with coefficients of all dg orbit categories! Roughly speaking, all the information is encoded in the Coxeter matrix of the quiver. Note also that the kernel and cokernel of (2.7) have the same finite order. In particular, one is trivial if and only if the other one is trivial. Thanks to Corollary 2.6, this implies that the groups $\mathbb{K}_i(\mathcal{C}_Q^{(m)};\mathbb{Z}/l^{\nu})$, $i \geq 0$, are either all trivial or all nontrivial.

3. Du Val singularities

The third goal of this article is to explain how Corollary 2.6 provides us a complete computation of the algebraic K-theory with coefficients of the du Val singularities.

Let k be an algebraically closed field of characteristic zero. Recall that the *du Val* singularities[1] [1934a; 1934b; 1934c] are the isolated singularities of the singular affine hypersurfaces $R := k[x, y, z]/(f)$ parametrized by the simply laced Dynkin diagrams:

type	$A_n, n \geq 1$	$D_n, n \geq 4$	E_6	E_7	E_8
f	$x^{n+1} + yz$	$x^{n-1} + xy^2 + z^2$	$x^4 + y^3 + z^2$	$x^3y + y^3 + z^2$	$x^5 + y^3 + z^2$

[1] Also known as *rational double points* or *ADE singularities*.

Let $\underline{\mathrm{MCM}}(R)$ denote the stable category of maximal Cohen–Macaulay R-modules. Thanks to the work of Buchweitz [1986] and Orlov [2004; 2009], this category is also known as the category of singularities $\mathcal{D}^{\mathrm{sing}}(R)$ or equivalently as the category of matrix factorizations $\mathrm{MF}(k[x, y, z], f)$. Roughly speaking, $\underline{\mathrm{MCM}}(R)$ encodes the crucial information concerning the isolated singularity of the singular affine hypersurface R.

Let Q be a *Dynkin quiver*, i.e., a quiver whose underlying graph is a Dynkin diagram of type A, D, or E. As explained by Keller [2005, §7.3], $\underline{\mathrm{MCM}}(R)$ is equivalent to the category of finitely generated projective modules over the preprojective algebra $\Lambda(Q)$ and to the (0)-cluster category of Q. We conclude that the algebraic K-theory of the du Val singularities is given by the algebraic K-theory of the dg (0)-cluster categories $\mathcal{C}^{(0)}_{A_n}$, $\mathcal{C}^{(0)}_{D_n}$, $\mathcal{C}^{(0)}_{E_6}$, $\mathcal{C}^{(0)}_{E_7}$ and $\mathcal{C}^{(0)}_{E_8}$. In these cases, the homomorphisms (2.7) correspond to the following matrices (see [Auslander et al. 1995, pages 289–290]):

$$A_n\colon\ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n \qquad \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ -1 & -1 & \ddots & \ddots & \vdots \\ -1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & 0 & \cdots & 0 & -1 \end{bmatrix}_{n\times n}$$

$$D_n\colon\ \begin{tikzcd}[row sep=small] 1 \arrow[dr] & & & & \\ & 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n \\ 2 \arrow[ur] & & & & \end{tikzcd} \qquad \begin{bmatrix} -2 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -2 & 1 & 0 & \ddots & \ddots & \vdots \\ -1 & -1 & 0 & 1 & \ddots & \ddots & \vdots \\ -1 & -1 & 1 & -1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & -1 & 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n}$$

$$E_6\colon\ \begin{tikzcd}[row sep=small] & & 3 \arrow[d] & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \end{tikzcd} \qquad \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & -1 \end{bmatrix}$$

$$E_7\colon\ \begin{tikzcd}[row sep=small] & & 3 \arrow[d] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd} \qquad \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$E_8: \quad \begin{array}{ccccccccccccc} & & & & 3 & & & & & & & & \\ & & & & \downarrow & & & & & & & & \\ 1 & \to & 2 & \to & 4 & \to & 5 & \to & 6 & \to & 7 & \to & 8 \end{array} \qquad \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Thanks to Corollary 2.6, the computation of the algebraic K-theory with coefficients of the du Val singularities reduces then to the computation of the (co)kernels of the above explicit matrix homomorphisms! We now compute the type A_n and leave the remaining cases to the reader.

Theorem 3.1. *Let k be an algebraically closed field of characteristic zero and $n \geq 1$ a positive integer. Under these assumptions and notations, we have*

$$\mathbb{K}_i(\mathcal{C}^{(0)}_{A_n}; \mathbb{Z}/l^\nu) \simeq \begin{cases} \mathbb{Z}/\gcd(n+1, l^\nu) & \text{if } i \geq 0, \\ 0 & \text{if } i < 0. \end{cases}$$

Consequently, the group $\mathbb{K}_i(\mathcal{C}^{(0)}_{A_n}; \mathbb{Z}/l^\nu)$ is nontrivial if and only if $l \mid (n+1)$ and $i \geq 0$.

Intuitively, Theorem 3.1 shows that the algebraic K-theory with $\mathbb{Z}/l^\nu$-coefficients of the isolated singularity of the affine hypersurface $k[x, y, z]/(x^{n+1} + yz)$ measures the l-divisibility of the integer $n+1$. To the best of the author's knowledge, these computations are new in the literature. They lead to the following vanishing and divisibility properties of algebraic K-theory (without coefficients):

Corollary 3.2. (i) *For every $i \geq 0$, at least one of the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}^{(0)}_{A_n})$ and $\mathbb{K}_{i-1}(\mathcal{C}^{(0)}_{A_n})$ is nontrivial.*

(ii) *For every $l \nmid (n+1)$ the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}^{(0)}_{A_n})$, $i \in \mathbb{Z}$, are uniquely l-divisible, i.e., they are $\mathbb{Z}[1/l]$-modules.*

Roughly speaking, Corollary 3.2 shows that at least half of the groups $\mathbb{K}_i(\mathcal{C}^{(0)}_{A_n})$ are nontrivial and moreover that they are "large" from the divisibility viewpoint.

Proof. Consider the following universal coefficient sequences (see Section 5):

$$0 \to \mathbb{K}_i(\mathcal{C}^{(0)}_{A_n}) \otimes_{\mathbb{Z}} \mathbb{Z}/l \to \mathbb{K}_i(\mathcal{C}^{(0)}_{A_n}; \mathbb{Z}/l) \to \{l\text{-torsion in } \mathbb{K}_{i-1}(\mathcal{C}^{(0)}_{A_n})\} \to 0.$$

Let l be a prime factor of $n+1$. Thanks to Theorem 3.1, the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}^{(0)}_{A_n}; \mathbb{Z}/l)$, $i \geq 0$, are nontrivial. Therefore, (i) follows from the above short exact sequences. Let l be a prime number which does not divide $n+1$. Thanks to Theorem 3.1, the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}^{(0)}_{A_n}; \mathbb{Z}/l)$, $i \in \mathbb{Z}$, are trivial. Therefore, (ii) follows also from the above short exact sequences. $\square$

A cyclic quotient singularity. Let the cyclic group $\mathbb{Z}/3$ act on the power series ring $k[[x, y, z]]$ by multiplication with a primitive third root of unit. As proved by Keller and Reiten [2008, §2], the stable category of maximal Cohen–Macaulay modules $\underline{\mathrm{MCM}}(R)$ over the fixed point ring $R := k[[x, y, z]]^{\mathbb{Z}/3}$ is equivalent to the (1)-cluster category of the generalized Kronecker quiver $Q : 1 \Rrightarrow 2$. In this case the above homomorphism (2.7) is given by the matrix $\left[\begin{smallmatrix} -9 & 3 \\ -3 & 0 \end{smallmatrix}\right]$.

Proposition 3.3. *We have the following computation*:

$$\mathbb{K}_i(\mathcal{C}_Q^{(1)}; \mathbb{Z}/l^\nu) \simeq \begin{cases} \mathbb{Z}/3 \times \mathbb{Z}/3 & \text{if } i \geq 0 \text{ and } l = 3, \\ 0 & \text{otherwise}. \end{cases}$$

To the best of the author's knowledge, the above computation is new in the literature. Similarly to Corollary 3.2, for every $i \geq 0$ at least one of the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}_Q^{(1)})$ and $\mathbb{K}_{i-1}(\mathcal{C}_Q^{(1)})$ is nontrivial, and, for every prime number $l \neq 3$, the groups $\mathbb{K}_i(\mathcal{C}_Q^{(1)})$, $i \in \mathbb{Z}$, are uniquely l-divisible.

Remark 3.4. After the circulation of this manuscript, Christian Haesemeyer kindly informed the author that some related computations concerning the G-theory of a local ring of finite Cohen–Macaulay type have been performed by Viraj Navkal [2013].

4. Preliminaries

Throughout the article, k will be a base commutative ring. Unless stated differently, all tensor products will be taken over k.

Dg categories. Let $\mathcal{C}(k)$ be the category of cochain complexes of k-modules. A *differential graded* (*dg*) *category* $\mathcal{A}$ is a $\mathcal{C}(k)$-enriched category and a *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{C}(k)$-enriched functor; consult Keller's ICM survey [2006]. In what follows, $\mathrm{dgcat}(k)$ stands for the category of (small) dg categories and dg functors.

Let $\mathcal{A}$ be a dg category. The category $\mathrm{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and $\mathrm{H}^0(\mathcal{A})(x, y) := H^0\mathcal{A}(x, y)$. The dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of cochain complexes of k-modules. Let us write $\mathcal{C}(\mathcal{A})$ for the category of right $\mathcal{A}$-modules. As explained in [Keller 2006, §§3.1–3.2], the category $\mathcal{C}(\mathcal{A})$ carries a projective Quillen model structure in which the weak equivalences and fibrations are defined objectwise. The *derived category* $\mathcal{D}(\mathcal{A})$ *of* $\mathcal{A}$ is the associated homotopy category or, equivalently, the localization of $\mathcal{C}(\mathcal{A})$ with respect to the (objectwise) quasi-isomorphisms. The full triangulated subcategory of compact objects will be denoted by $\mathcal{D}_c(\mathcal{A})$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence of (triangulated) categories $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$; see [Keller 2006, §4.6]. As proved in [Tabuada 2005, Theorem 5.3], $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let $\mathrm{Hmo}(k)$ be the associated homotopy category.

The tensor product $\mathcal{A}\otimes\mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A}\otimes\mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y)\otimes\mathcal{B}(w,z)$. As explained in [Keller 2006, §2.3 and §4.3], this construction gives rise to symmetric monoidal categories $(\mathrm{dgcat}(k), -\otimes-, k)$ and $(\mathrm{Hmo}(k), -\otimes^{\mathbb{L}}-, k)$.

An *$\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $\mathrm{B} : \mathcal{A}\otimes^{\mathbb{L}}\mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ or, equivalently, a right $(\mathcal{A}^{\mathrm{op}}\otimes^{\mathbb{L}}\mathcal{B})$-module. A standard example is the $\mathcal{A}$-$\mathcal{B}$-bimodule

$$ {}_F\mathrm{B} : \mathcal{A}\otimes^{\mathbb{L}}\mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \quad (x,w)\mapsto \mathcal{B}(w,F(x)), \tag{4.1} $$

associated to a dg functor $F : \mathcal{A}\to\mathcal{B}$. Finally, let us denote by $\mathrm{rep}(\mathcal{A},\mathcal{B})$ the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes^{\mathbb{L}}\mathcal{B})$ consisting of those $\mathcal{A}$-$\mathcal{B}$-bimodules B such that $\mathrm{B}(x,-)\in\mathcal{D}_c(\mathcal{B})$ for every object $x\in\mathcal{A}$.

Exact categories. Let $\mathcal{E}$ be an exact category in the sense of [Quillen 1973, §2]. The following examples will be used in the sequel:

Example 4.2. Let A be a k-algebra. Recall from [Quillen 1973, §2] that the category $\mathrm{P}(A)$ of finitely generated projective right A-modules carries a canonical exact structure.

(i) Let $\mathrm{End}(A)$ be the category of endomorphisms in $\mathrm{P}(A)$. The objects are the pairs (M,f), with $M\in\mathrm{P}(A)$ and f an endomorphism of M. The morphisms $(M,f)\to(M',f')$ are the A-linear maps $h : M\to M'$ such that $hf = f'h$. Note that $\mathrm{End}(A)$ inherits naturally from $\mathrm{P}(A)$ an exact structure making the forgetful functor $\mathrm{End}(A)\to\mathrm{P}(A)$, $(M,f)\mapsto M$, exact.

(ii) Let $\mathrm{Nil}(A)$ be the category of nilpotent endomorphisms in $\mathrm{P}(A)$. By construction, $\mathrm{Nil}(A)$ is a full exact subcategory of $\mathrm{End}(A)$.

Following [Keller 2006, §4.4], the *bounded derived dg category $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$ of $\mathcal{E}$* is defined as Drinfeld's dg quotient $\mathcal{C}^b_{\mathrm{dg}}(\mathcal{E})/\mathcal{A}c^b_{\mathrm{dg}}(\mathcal{E})$ of the dg category of bounded cochain complexes over $\mathcal{E}$ by the full dg subcategory of acyclic complexes.

Notation 4.3. Let $\mathcal{E}$ be an exact category. In order to simplify the exposition, let us write $\mathcal{E}_{\mathbf{dg}}$ instead of $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$. By construction, we have $\mathrm{H}^0(\mathcal{E}_{\mathbf{dg}})\simeq\mathcal{D}^b(\mathcal{E})$. Note that when $\mathcal{E}=\mathrm{P}(A)$ we have a Morita equivalence between $\mathrm{P}(A)_{\mathbf{dg}}$ and A.

Every exact functor $\mathcal{E}\to\mathcal{E}'$ gives rise to a dg functor $\mathcal{E}_{\mathbf{dg}}\to\mathcal{E}'_{\mathbf{dg}}$. In the same vein, every multiexact functor $\mathcal{E}\times\cdots\times\mathcal{E}'\to\mathcal{E}''$ gives rise to a dg functor $\mathcal{E}_{\mathbf{dg}}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\mathcal{E}'_{\mathbf{dg}}\to\mathcal{E}''_{\mathbf{dg}}$.

Algebraic K-theory with coefficients. Let Spt be the homotopy category of spectra and $\mathbb{S}$ the sphere spectrum. Given a small dg category $\mathcal{A}$, its *nonconnective algebraic K-theory spectrum* $\mathbb{K}(\mathcal{A})$ is defined by applying Schlichting's construction [2006, §12.1] to the Frobenius pair associated to the category of those cofibrant right $\mathcal{A}$-modules which become compact in the derived category $\mathcal{D}(\mathcal{A})$. Let us denote by $\mathbb{K} : \mathrm{dgcat}(k) \to \mathrm{Spt}$ the associated functor. Given a prime power l^ν, the algebraic K-theory with $\mathbb{Z}/l^\nu$-coefficients is defined as[2]

$$\mathbb{K}(-; \mathbb{Z}/l^\nu) : \mathrm{dgcat}(k) \to \mathrm{Spt}, \quad \mathcal{A} \mapsto \mathbb{K}(\mathcal{A}) \wedge^{\mathbb{L}} \mathbb{S}/l^\nu, \tag{4.4}$$

where $\mathbb{S}/l^\nu$ stands for the mod l^ν Moore spectrum of $\mathbb{S}$. In the same vein, we have the functor $\mathbb{K}(-) \otimes \mathbb{Z}[1/l] : \mathrm{dgcat}(k) \to \mathrm{Spt}$ defined by the homotopy colimit

$$\mathbb{K}(\mathcal{A}) \otimes \mathbb{Z}[1/l] := \mathrm{hocolim}\big(\mathbb{K}(\mathcal{A}) \xrightarrow{\cdot l} \mathbb{K}(\mathcal{A}) \xrightarrow{\cdot l} \cdots\big).$$

When $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(X)$, with X a quasicompact, quasiseparated k-scheme, $\mathbb{K}(\mathcal{A})$ agrees with $\mathbb{K}(X)$; see [Keller 2006, §5.2; Schlichting 2006, §8]. Consequently, $\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu)$ agrees with $\mathbb{K}(X; \mathbb{Z}/l^\nu)$ and $\mathbb{K}(\mathcal{A}) \otimes \mathbb{Z}[1/l]$ agrees with $\mathbb{K}(X) \otimes \mathbb{Z}[1/l]$.

Bass's construction. Let $H : \mathrm{dgcat}(k) \to \mathrm{Ab}$ be a functor with values in the category of abelian groups. Following [Bass 1968, §XII], consider the sequence of functors $N^p H : \mathrm{dgcat}(k) \to \mathrm{Ab}$, $p \geq 0$, defined by $N^0 H(\mathcal{A}) := H(\mathcal{A})$ and

$$N^p H(\mathcal{A}) := \mathrm{kernel}\big(N^{p-1} H(\mathcal{A}[t]) \xrightarrow{\mathrm{id} \otimes (t=0)} N^{p-1} H(\mathcal{A})\big), \quad p \geq 1. \tag{4.5}$$

Note that the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to a splitting $N^{p-1} H(\mathcal{A}[t]) \simeq N^p H(\mathcal{A}) \oplus N^{p-1} H(\mathcal{A})$. Let $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ be the category of nonnegatively graded chain complexes of abelian groups. Following Bass, we also have the functor

$$N^\bullet H : \mathrm{dgcat}(k) \to \mathrm{Ch}_{\geq 0}(\mathbb{Z}), \quad \mathcal{A} \mapsto N^\bullet H(\mathcal{A}),$$

where the chain complex $N^\bullet H(\mathcal{A})$ is defined by $N^0 H(\mathcal{A}) := H(\mathcal{A})$ and, for $p \geq 1$,

$$N^p H(\mathcal{A}) := \bigcap_{i=1}^{p} \mathrm{kernel}\big(H(\mathcal{A}[t_1, \ldots, t_p]) \xrightarrow{\mathrm{id} \otimes (t_i=0)} H(\mathcal{A}[t_1, \ldots, \hat{t}_i, \ldots, t_p])\big),$$

$$N^p H(\mathcal{A}) \longrightarrow N^{p-1} H(\mathcal{A}), \quad t_i \mapsto \begin{cases} 1 - \sum_{i=2}^{p} t_i & \text{if } i = 1, \\ t_{i-1} & \text{if } i \neq 1. \end{cases}$$

Note that the above two definitions of $N^p H(\mathcal{A})$ are isomorphic. In what follows we will simply write $NH(\mathcal{A})$ instead of $N^1 H(\mathcal{A})$.

[2] Given any two prime numbers p and q, we have $\mathbb{S}/pq \simeq \mathbb{S}/p \oplus \mathbb{S}/q$ in Spt. Therefore, without loss of generality, we can (and will) work solely with one prime power l^ν.

5. Proof of Theorem 1.2

We will work often with the following general notion:

Definition 5.1. A functor $E : \mathrm{dgcat}(k) \to \mathrm{Spt}$ is called a *localizing invariant* if it inverts Morita equivalences and sends short exact sequences of dg categories (see [Keller 2006, §4.6]) to distinguished triangles of spectra

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \;\mapsto\; E(\mathcal{A}) \to E(\mathcal{B}) \to E(\mathcal{C}) \xrightarrow{\partial} \Sigma E(\mathcal{A}).$$

Thanks to the work of Blumberg and Mandell [2012], Keller [1998; 1999], Schlichting [2006], Thomason and Trobaugh [1990], and others, examples include not only nonconnective algebraic K-theory (with coefficients) but also Hochschild homology, cyclic homology, negative cyclic homology, periodic cyclic homology, topological Hochschild homology, topological cyclic homology, etc. Given an integer $q \in \mathbb{Z}$, the abelian group $\mathrm{Hom}_{\mathrm{Spt}}(\Sigma^q \mathbb{S}, E(\mathcal{A}))$ will be denoted by $E_q(\mathcal{A})$.

The proof of Theorem 1.2 is divided into four steps:

(I) Spectral sequence.

(II) Universal coefficient sequence.

(III) Fundamental theorem.

(IV) Module structure over the big Witt ring.

In order to simplify the exposition, we develop each one of these steps in a different section. Making use of Steps I–IV, we then conclude the proof of Theorem 1.2 in Section 10.

6. Step I: spectral sequence

Let E be a localizing invariant and $\Delta_n := k[t_0, \dots, t_n]/\big(\sum_{i=0}^n t_i - 1\big)$, $n \geq 0$, the simplicial k-algebra with faces and degeneracies given by the formulas

$$d_r(t_i) := \begin{cases} t_i & \text{if } i < r, \\ 0 & \text{if } i = r, \\ t_{i-1} & \text{if } i > r, \end{cases} \qquad \text{and} \qquad s_r(t_i) := \begin{cases} t_i & \text{if } i < r, \\ t_i + t_{i+1} & \text{if } i = r, \\ t_{i+1} & \text{if } i > r. \end{cases}$$

Out of this data, we can construct the $\mathbb{A}^1$-*homotopization* of E:

$$E^h : \mathrm{dgcat}(k) \to \mathrm{Spt}, \quad \mathcal{A} \mapsto \mathrm{hocolim}_n\, E(\mathcal{A} \otimes \Delta_n).$$

Note that E^h comes equipped with a natural 2-morphism $E \Rightarrow E^h$. As explained in [Tabuada 2015b, Proposition 5.2], E^h remains a localizing invariant and the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra $E^h(\mathcal{A}) \to E^h(\mathcal{A}[t])$.

Given an integer $q \in \mathbb{Z}$, consider the functor $E_q : \mathrm{dgcat}(k) \to \mathrm{Ab}$ and the associated nonnegatively graded chain complex of abelian groups

$$0 \longleftarrow E_q(\mathcal{A}) \xleftarrow{d_0-d_1} E_q(\mathcal{A}[t]) \longleftarrow \cdots \xleftarrow{(-1)^r \sum_r d_r} E_q(\mathcal{A}\otimes \Delta_n) \longleftarrow \cdots. \tag{6.1}$$

Under the isomorphisms

$$\Delta_n \xrightarrow{\sim} k[t_1, \dots, t_n], \quad t_i \mapsto \begin{cases} 1-\sum_{i=1}^n t_i & \text{if } i=0, \\ t_i & \text{if } i \neq 0, \end{cases}$$

the (Moore) normalization of (6.1) identifies with $N^\bullet E_q(\mathcal{A})$. Consequently, following [Quillen 1966], we obtain a standard convergent right half-plane spectral sequence $E^1_{pq} = N^p E_q(\mathcal{A}) \Rightarrow E^h_{p+q}(\mathcal{A})$. In the particular case of algebraic K-theory with coefficients, we have the convergent spectral sequence

$$E^1_{pq} = N^p \mathbb{K}_q(\mathcal{A}; \mathbb{Z}/l^\nu) \Rightarrow \mathbb{K}^h_{p+q}(\mathcal{A}; \mathbb{Z}/l^\nu). \tag{6.2}$$

Similarly, since $\pi_q(\mathbb{K}(\mathcal{A})\otimes \mathbb{Z}[1/l]) \simeq \mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]}$, we have the spectral sequence

$$E^1_{pq} = N^p \mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]} \Rightarrow \mathbb{K}^h_{p+q}(\mathcal{A})_{\mathbb{Z}[1/l]}. \tag{6.3}$$

Remark 6.4. The preceding constructions and spectral sequences have their roots in the work of Anderson [1973], in the definition of homotopy K-theory (see [Weibel 1989]), and in the work of Suslin and Voevodsky [1996].

7. Step II: universal coefficient sequence

Let E be a localizing invariant. Similarly to (4.4), consider the functor

$$E(-; \mathbb{Z}/l^\nu) : \mathrm{dgcat}(k) \to \mathrm{Spt}, \quad \mathcal{A} \mapsto E(\mathcal{A}) \wedge^{\mathbb{L}} \mathbb{S}/l^\nu.$$

For every dg category $\mathcal{A}$ we have a distinguished triangle of spectra

$$E(\mathcal{A}) \xrightarrow{\cdot l^\nu} E(\mathcal{A}) \longrightarrow E(\mathcal{A}; \mathbb{Z}/l^\nu) \xrightarrow{\partial} \Sigma E(\mathcal{A}). \tag{7.1}$$

Consequently, the associated long exact sequence (obtained by applying the functor $\mathrm{Hom}_{\mathrm{Spt}}(\mathbb{S}, -)$ to (7.1)) breaks up into short exact sequences

$$0 \to E_q(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \to E_q(\mathcal{A}; \mathbb{Z}/l^\nu) \to \{l^\nu\text{-torsion in } E_{q-1}(\mathcal{A})\} \to 0.$$

Note that since the above distinguished triangle of spectra (7.1) is functorial on $\mathcal{A}$, we have moreover the short exact sequences

$$0 \to N^p E_q(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \to N^p E_q(\mathcal{A}; \mathbb{Z}/l^\nu) \to \{l^\nu\text{-torsion in } N^p E_{q-1}(\mathcal{A})\} \to 0.$$

Remark 7.2. The preceding universal coefficient sequences are well known. In the case where $E = \mathbb{K}$, they were established by Thomason [1985, Appendix A].

8. Step III: fundamental theorem

Recall that we have the exact functors:

$$\mathrm{Nil}(k) \subset \mathrm{End}(k) \to \mathrm{P}(k), \qquad (M, f) \mapsto M, \tag{8.1}$$

$$\mathrm{P}(k) \to \mathrm{Nil}(k) \subset \mathrm{End}(k), \qquad M \mapsto (M, 0). \tag{8.2}$$

Let E be a localizing invariant and $\mathrm{Nil}(k)_{\mathbf{dg}}$, $\mathrm{P}(k)_{\mathbf{dg}}$ the dg categories introduced at Notation 4.3. Given a dg category $\mathcal{A}$ and an integer q, consider the abelian group

$$\mathrm{Nil}\, E_q(\mathcal{A}) := \mathrm{kernel}\big(E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \xrightarrow{\mathrm{id} \otimes (8.1)} E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}}) \simeq E_q(\mathcal{A})\big).$$

Note that since $(8.1) \circ (8.2) = \mathrm{id}$, the morphism

$$E(\mathcal{A}) \simeq E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}}) \xrightarrow{\mathrm{id} \otimes (8.2)} E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \tag{8.3}$$

gives rise to a splitting $E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \simeq \mathrm{Nil}\, E_q(\mathcal{A}) \oplus E_q(\mathcal{A})$.

Theorem 8.4 (fundamental theorem). *Given a localizing invariant E, we have $N E_{q+1}(\mathcal{A}) \simeq \mathrm{Nil}\, E_q(\mathcal{A})$.*

The remainder of this section is devoted to the proof of Theorem 8.4. Let $\mathbb{P}^1$ be the projective line over the base commutative ring k, with $i : \mathrm{Spec}(k[t]) \hookrightarrow \mathbb{P}^1$ and $j : \mathrm{Spec}(k[1/t]) \hookrightarrow \mathbb{P}^1$ the classical Zariski open cover.

Proposition 8.5. *We have a short exact sequence of dg categories*

$$0 \longrightarrow \mathrm{Nil}(k)_{\mathbf{dg}} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) \xrightarrow{\mathbb{L}j^*} \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[1/t])) \longrightarrow 0. \tag{8.6}$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)_Z \arrow[r] \arrow[d] & \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) \arrow[r, "\mathbb{L}j^*"] \arrow[d, "\mathbb{L}i^*"'] & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[1/t])) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'} \arrow[r] & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t])) \arrow[r] & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t, 1/t])) \arrow[r] & 0
\end{tikzcd}$$

where Z (resp. Z') stands for the complement of $\mathrm{Spec}(k[1/t])$ in $\mathbb{P}^1$ (resp. of $\mathrm{Spec}(k[t, 1/t])$ in $\mathrm{Spec}(k[t])$) and $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)_Z$ (resp. $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'}$) stands for the dg category of those perfect complexes of $\mathcal{O}_{\mathbb{P}^1}$-modules (resp. $\mathcal{O}_{\mathrm{Spec}(k[t])}$-modules) which are supported on Z (resp. on Z'). As proved by Thomason and Trobaugh [1990, Theorems 2.6.3 and 7.4], both rows are short exact sequences of dg categories and the left-hand side vertical dg functor is a Morita equivalence. It remains then only to show that $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'}$ is Morita equivalent to $\mathrm{Nil}(k)_{\mathbf{dg}}$.

Let us write $\mathbb{H}_{1,t}(k[t])$ for the exact category of finitely presented $k[t]$-modules of projective dimension ≤ 1 that are annihilated by some power t^n of t. Following

[Schlichting 2011, §§3.1.8–3.1.11], we have a short exact sequence of dg categories

$$0 \longrightarrow \mathbb{H}_{1,t}(k[t])_{\mathbf{dg}} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t])) \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t, 1/t])) \longrightarrow 0.$$

Making use of Keller's characterization [2006, Theorem 4.11(i)] of short exact sequences of dg categories, we conclude that $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'}$ is Morita equivalent to $\mathbb{H}_{1,t}(k[t])_{\mathbf{dg}}$. As proved by Grayson and Quillen [Grayson 1976, page 236], we have an equivalence of exact categories $\mathrm{Nil}(k) \to \mathbb{H}_{1,t}(k[t])$, $(M, f) \mapsto M_f$, where M_f stands for the $k[t]$-module M on which t acts as f. Consequently, we obtain an induced equivalence of dg categories $\mathbb{H}_{1,t}(k[t])_{\mathbf{dg}} \simeq \mathrm{Nil}(k)_{\mathbf{dg}}$. This concludes the proof of Proposition 8.5. □

As proved by Drinfeld [2004, Proposition 1.6.3], the functor

$$\mathcal{A} \otimes^{\mathbb{L}} - : \mathrm{Hmo}(k) \to \mathrm{Hmo}(k)$$

is well defined and preserves short exact sequences of dg categories. Consequently, (8.6) gives rise to the short exact sequence of dg categories

$$0 \longrightarrow \mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}} \longrightarrow \mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1 \xrightarrow{\mathrm{id} \otimes \mathbb{L} j^*} \mathcal{A}[1/t] \longrightarrow 0, \tag{8.7}$$

where $\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1$ stands for $\mathcal{A} \otimes^{\mathbb{L}} \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)$. By applying the functor E to (8.7), we obtain a distinguished triangle of spectra

$$E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \to E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1) \to E(\mathcal{A}[1/t]) \xrightarrow{\partial} \Sigma E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}). \tag{8.8}$$

Now, recall from [Orlov 1992, §2] that we have two fully faithful dg functors

$$\begin{aligned} \iota_{-1} &: \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \to \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1), \quad M \mapsto \mathbb{L}p^*(M) \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1}(-1), \\ \iota_0 &: \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \to \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1), \quad M \mapsto \mathbb{L}p^*(M), \end{aligned}$$

where $p \colon \mathbb{P}^1 \to \mathrm{Spec}(k)$ stands for the projection morphism. The dg functor ι_{-1} induces a Morita equivalence between $\mathrm{perf}_{\mathrm{dg}}(\mathrm{pt})$ and Drinfeld's dg quotient $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)/\iota_0 \, \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt})$. Following [Tabuada 2008, §13], we obtain a *split* short exact sequence of dg categories (see also [Orlov 1992, Theorem 2.6])

$$0 \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \underset{\iota_0}{\overset{r}{\leftrightarrows}} \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) \underset{s}{\overset{\iota_{-1}}{\leftrightarrows}} \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \longrightarrow 0, \tag{8.9}$$

where r is the right adjoint of ι_0, $r \circ \iota_0 = \mathrm{id}$, ι_{-1} is right adjoint of s, and $\iota_{-1} \circ s = \mathrm{id}$. By first applying the functor $\mathcal{A} \otimes^{\mathbb{L}} -$ to (8.9), and then the functor E to the resulting split short exact sequence of dg categories, we obtain the isomorphism

$$[E(\mathrm{id} \otimes \iota_0), E(\mathrm{id} \otimes \iota_{-1})] : E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1). \tag{8.10}$$

The proof of the following general lemma is clear:

Lemma 8.11. *If $(f, g) : A \oplus A \xrightarrow{\sim} B$ is an isomorphism in an additive category, then $(f, f - g) : A \oplus A \xrightarrow{\sim} B$ is also an isomorphism.*

By applying Lemma 8.11 to (8.10), we obtain the isomorphism

$$[E(\mathrm{id} \otimes \iota_0), E(\mathrm{id} \otimes \iota_0) - E(\mathrm{id} \otimes \iota_{-1})] : E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1). \quad (8.12)$$

Proposition 8.13. *The composition*

$$E(\mathcal{A}) \xrightarrow{(8.3)} E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \longrightarrow E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1)$$

agrees with $E(\mathrm{id} \otimes \iota_0) - E(\mathrm{id} \otimes \iota_{-1})$.

Proof. As proved in [Tabuada 2005, Corollary 5.10], there is a bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B})$ and the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. Under this bijection, the composition law of $\mathrm{Hmo}(k)$ corresponds to the bifunctor

$$\mathrm{rep}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}(\mathcal{B}, \mathcal{C}) \to \mathrm{rep}(\mathcal{A}, \mathcal{C}), \quad (\mathrm{B}, \mathrm{B}') \mapsto \mathrm{B} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathrm{B}'. \quad (8.14)$$

Since the $\mathcal{A}$-$\mathcal{B}$-bimodules (4.1) belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, we have the $\otimes$-functor

$$\mathrm{dgcat}(k) \to \mathrm{Hmo}(k), \quad \mathcal{A} \mapsto \mathcal{A}, \; F \mapsto {}_F\mathrm{B}. \quad (8.15)$$

The *additivization* $\mathrm{Hmo}_0(k)$ *of* $\mathrm{Hmo}(k)$ is the additive category with the same objects and abelian groups of morphisms given by $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) := K_0 \,\mathrm{rep}(\mathcal{A}, \mathcal{B})$, where $K_0 \,\mathrm{rep}(\mathcal{A}, \mathcal{B})$ stands for the Grothendieck group of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. The composition law is induced by the above bitriangulated functor (8.14) and the symmetric monoidal structure by bilinearity from $\mathrm{Hmo}(k)$. Note that we also have the symmetric monoidal functor

$$\mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k), \quad \mathcal{A} \mapsto \mathcal{A}, \; \mathrm{B} \mapsto [\mathrm{B}]. \quad (8.16)$$

Let us denote by $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ the composition of (8.15) with (8.16).

Now, consider the composition of dg functors

$$\iota : \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \simeq \mathrm{P}(k)_{\mathbf{dg}} \xrightarrow{(8.2)} \mathrm{Nil}(k)_{\mathbf{dg}} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1).$$

Thanks to Proposition 8.17, below, and to the fact that U is a $\otimes$-functor, it suffices now to show that $U(\iota)$ agrees with $U(\iota_0) - U(\iota_{-1})$. As explained in [Grayson 1976, page 237], we have a short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \iota(\mathrm{pt}) \to 0$. Consequently, we obtain a short exact of dg functors

$$0 \to \iota_{-1} \to \iota_0 \to \iota \to 0, \quad \iota_{-1}, \iota_0, \iota : \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \to \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1).$$

By the construction of the additive category $\mathrm{Hmo}_0(k)$, we conclude that $[{}_\iota\mathrm{B}] = [{}_{\iota_0}\mathrm{B}] - [{}_{\iota_1}\mathrm{B}]$, i.e., that $U(\iota) = U(\iota_0) - U(\iota_{-1})$. This achieves the proof. □

Proposition 8.17. *Given a localizing invariant* $E : \mathrm{dgcat}(k) \to \mathrm{Spt}$, *there is an* (*unique*) *additive functor* $\overline{E} : \mathrm{Hmo}_0(k) \to \mathrm{Spt}$ *such that* $\overline{E} \circ U \simeq E$.

Proof. Recall from [Tabuada 2005] that a functor $E : \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category, is called an *additive invariant* if it inverts Morita equivalences and sends split short exact sequences of dg categories to direct sums. As proved in [Tabuada 2005, Theorems 5.3 and 6.3], the functor $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ is the *universal additive invariant*, i.e., given any additive category D there is an equivalence of categories

$$U^* : \mathrm{Fun}_{\mathrm{additive}}(\mathrm{Hmo}_0(k), \mathrm{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}),$$

where the left-hand side denotes the category of additive functors and the right-hand side the category of additive invariants. The proof follows now from the fact that every localizing invariant is in particular an additive invariant. $\square$

The distinguished triangle (8.8) gives rise to the long exact sequence

$$\cdots \to E_{q+1}(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1) \to E_{q+1}(\mathcal{A}[1/t]) \to E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \to E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1) \to \cdots$$

Note that the two compositions

$$\mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \underset{\iota_{-1}}{\overset{\iota_0}{\rightrightarrows}} \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) \xrightarrow{\mathbb{L} j^*} \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[1/t])) \tag{8.18}$$

agree with the inverse image dg functor induced by $\mathrm{Spec}(k[1/t]) \to \mathrm{pt}$. Making use of the isomorphism (8.12), we conclude that the above long exact sequence breaks up into shorter exact sequences

$$0 \to E_{q+1}(\mathcal{A}) \to E_{q+1}(\mathcal{A}[1/t]) \to E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \to E_q(\mathcal{A}) \to 0. \tag{8.19}$$

Moreover, making use of Proposition 8.13, we observe that the last morphism in (8.19) corresponds to the projection $\mathrm{Nil}\, E_q(\mathcal{A}) \oplus E_q(\mathcal{A}) \to E_q(\mathcal{A})$. Consequently, (8.19) can be further reduced to a short exact sequence

$$0 \longrightarrow E_{q+1}(\mathcal{A}) \longrightarrow E_{q+1}(\mathcal{A}[1/t]) \longrightarrow \mathrm{Nil}\, E_q(\mathcal{A}) \longrightarrow 0.$$

From this short exact sequence we obtain, finally, the sought-for isomorphism

$$N E_{q+1}(\mathcal{A}) \simeq \mathrm{cokernel}\big(E_{q+1}(\mathcal{A}) \to E_{q+1}(\mathcal{A}[1/t])\big) \simeq \mathrm{Nil}\, E_q(\mathcal{A}).$$

This concludes the proof of Theorem 8.4.

9. Step IV: module structure over the big Witt ring

Recall from [Bloch 1977, page 192] the construction of the big Witt ring $W(R)$ of a commutative ring R. As an additive group, $W(R)$ is equal to $(1 + tR[\![t]\!], \times)$. The multiplication $*$ is uniquely determined by naturality, formal factorization of the elements of $W(R)$ as $h(t)=\prod_{n\geq 1}(1-a_n t^n)$, and the equality $(1-at)*h(t)=h(at)$. The zero element is $1+0t+\cdots$ and the unit element is $(1-t)$.

Theorem 9.1. *Given a dg category $\mathcal{A}$, the abelian groups* $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, $q\in\mathbb{Z}$, *carry a $W(k)$-module structure.*

The remainder of this section is devoted to the proof of Theorem 9.1. Recall from [Grayson 1976] that for every positive integer $n\geq 1$ we have a *Frobenius* functor

$$F_n:\mathrm{End}(k)\to\mathrm{End}(k),\quad (M,f)\mapsto (M,f^n),$$

as well as a *Verschiebung* functor

$$V_n:\mathrm{End}(k)\to\mathrm{End}(k),\quad (M,f)\mapsto\left(M^{\oplus n},\begin{bmatrix}0&\cdots&\cdots&0&f\\ 1&\ddots&&&0\\ 0&\ddots&\ddots&&\vdots\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ 0&\cdots&0&1&0\end{bmatrix}_{n\times n}\right).$$

Both these functors are exact and preserve the full subcategory of nilpotent endomorphisms $\mathrm{Nil}(k)$. Moreover, the following diagrams are commutative:

$$\begin{tikzcd}\mathrm{End}(k)\arrow[r,"F_n"]\arrow[d,"(8.1)"'] & \mathrm{End}(k)\arrow[d,"(8.1)"]\\ \mathrm{P}(k)\arrow[r,equal] & \mathrm{P}(k)\end{tikzcd}\qquad\begin{tikzcd}\mathrm{End}(k)\arrow[r,"V_n"]\arrow[d,"(8.1)"'] & \mathrm{End}(k)\arrow[d,"(8.1)"]\\ \mathrm{P}(k)\arrow[r,"M\mapsto M^{\oplus n}"'] & \mathrm{P}(k)\end{tikzcd}\tag{9.2}$$

Following [Grayson 1976], let $\mathrm{End}_0(k)$ be the kernel of $K_0\,\mathrm{End}(k)\xrightarrow{(8.1)}K_0\mathrm{P}(k)$. Note that since $(8.1)\circ(8.2)=\mathrm{id}$, the homomorphism $K_0\mathrm{P}(k)\xrightarrow{(8.2)}K_0\,\mathrm{End}(k)$ gives rise to a splitting $K_0\,\mathrm{End}(k)\simeq\mathrm{End}_0(k)\oplus K_0\mathrm{P}(k)$. Note also that the image in $\mathrm{End}_0(k)$ of an endomorphism (M,f) is given by $[(M,f)]-[(M,0)]$. Under these notations, we have induced Frobenius and Verschiebung homomorphisms $F_n, V_n:\mathrm{End}_0(k)\to\mathrm{End}_0(k)$. Consider also the biexact functor

$$\mathrm{End}(k)\times\mathrm{Nil}(k)\to\mathrm{Nil}(k),\quad ((M,f),(M',f'))\mapsto(M\otimes M',f\otimes f'),\tag{9.3}$$

and the associated commutative diagram

$$\begin{tikzcd}\mathrm{End}(k)\times\mathrm{Nil}(k)\arrow[r,"(9.3)"]\arrow[d,"(8.1)\times(8.1)"'] & \mathrm{Nil}(k)\arrow[d,"(8.1)"]\\ \mathrm{P}(k)\times\mathrm{P}(k)\arrow[r,"(M{,}M')\mapsto M\otimes M'"'] & \mathrm{P}(k)\end{tikzcd}\tag{9.4}$$

Given a dg category $\mathcal{A}$, (9.2) and (9.4) give rise to the commutative diagrams

$$\begin{tikzcd} \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \arrow[r, "\mathrm{id}\otimes F_n"] \arrow[d] & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \arrow[d] \\ \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \arrow[r, equal] & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \arrow[r, "\mathrm{id}\otimes V_n"] \arrow[d] & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \arrow[d] \\ \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \arrow[r] & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{End}(k)_{\mathbf{dg}}\otimes^{\mathbb{L}}\mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \arrow[r] \arrow[d] & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \arrow[d] \\ \mathrm{P}(k)_{\mathbf{dg}}\otimes^{\mathbb{L}}\mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \arrow[r] & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \end{tikzcd} \tag{9.5}$$

In what follows, we will still denote by $F_n, V_n : \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$ the induced Frobenius and Verschiebung homomorphisms. Thanks to the work of Waldhausen [1985, page 342], a pairing of dg categories gives rise to a pairing on algebraic K-theory groups; see [Tabuada 2013, §4.2]. Therefore, since $\mathrm{End}_0(k)$ is the kernel of the homomorphism $K_0(\mathrm{End}(k)_{\mathbf{dg}}) \xrightarrow{(8.1)} K_0(\mathrm{P}(k)_{\mathbf{dg}})$, we obtain from (9.5) the bilinear pairings

$$-\cdot- : \mathrm{End}_0(k)\times \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}), \quad q\in\mathbb{Z}. \tag{9.6}$$

Remark 9.7 ($\mathrm{End}_0(k)$-module structure). The tensor product of k-modules gives rise naturally to a symmetric monoidal structure on the exact categories $\mathrm{P}(k)$ and $\mathrm{End}(k)$, making the forgetful functor (8.1) symmetric monoidal. Therefore, the abelian group $\mathrm{End}_0(k)$ comes equipped with an induced ring structure. Moreover, by construction, the bilinear pairings (9.6) endow the abelian groups $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, $q\in\mathbb{Z}$, with an $\mathrm{End}_0(k)$-module structure.

Proposition 9.8. *We have $V_n(\alpha\cdot F_n(\beta)) = V_n(\alpha)\cdot\beta$ for every $\alpha\in\mathrm{End}_0(k)$ and $\beta\in\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$.*

Proof. Let S be the multiplicatively closed subset of $\mathbb{Z}[x,y][s]$ generated by s and $s^n - x^n y$. In what follows, we denote by $\mathrm{End}(\mathbb{Z}[x,y];S)$ the full exact subcategory of $\mathrm{End}(\mathbb{Z}[x,y])$ consisting of those endomorphisms (N,g) for which there exists a polynomial $p(s)\in S$, depending on (N,g), such that $p(g)=0$. The endomorphisms

$$\epsilon_1 := \left(\mathbb{Z}[x,y]^{\oplus n}, \begin{bmatrix} 0 & \cdots & \cdots & 0 & x^n y \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n\times n} \right), \tag{9.9}$$

$$\epsilon_2 := \left(\mathbb{Z}[x, y]^{\oplus n}, \begin{bmatrix} 0 & \cdots & \cdots & 0 & xy \\ x & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x & 0 \end{bmatrix}_{n\times n} \right), \tag{9.10}$$

belong to $\mathrm{End}(\mathbb{Z}[x, y]; S)$ since they satisfy the equation $s^n - x^n y = 0$. Following [1982, §§5–6], consider the multiexact functor

$$\theta(-, -, -) : \mathrm{End}(\mathbb{Z}[x, y]; S) \times \mathrm{End}(k) \times \mathrm{Nil}(k) \longrightarrow \mathrm{Nil}(k)$$

which sends the triple $((N, g), (M, f), (M', f'))$ to the nilpotent endomorphism $(N \otimes_{\mathbb{Z}[x,y]} M \otimes M', g \otimes \mathrm{id} \otimes \mathrm{id})$, where the left $\mathbb{Z}[x, y]$-module structure on $M \otimes M'$ is given by $x \mapsto f'$ and $y \mapsto f$. Note that the following diagram commutes:

$$\begin{tikzcd} \mathrm{End}(\mathbb{Z}[x, y]; S) \times \mathrm{End}(k) \times \mathrm{Nil}(k) \arrow[r, "\theta(-,-,-)"] \arrow[d, "(8.1)\times(8.1)\times(8.1)"'] & \mathrm{Nil}(k) \arrow[d, "(8.1)"] \\ P(\mathbb{Z}[x, y]) \times \mathrm{P}(k) \times \mathrm{P}(k) \arrow[r, "(N,M,M')\mapsto N\otimes_{\mathbb{Z}[x,y]}M\otimes M'"'] & \mathrm{P}(k) \end{tikzcd} \tag{9.11}$$

Given a dg category $\mathcal{A}$, (9.11) leads to the commutative square

$$\begin{tikzcd} \mathrm{End}(\mathbb{Z}[x, y]; S)_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathrm{End}(k)_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}} \arrow[r] \arrow[d] & \mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}} \arrow[d] \\ \mathrm{P}(\mathbb{Z}[x, y])_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}} \arrow[r] & \mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}} \end{tikzcd} \tag{9.12}$$

In the same way that the diagram (9.5) gives rise to the bilinear pairings (9.6), the diagram (9.12) gives rise to the multilinear homomorphisms

$$\mathrm{End}_0(\mathbb{Z}[x, y]; S) \times \mathrm{End}_0(k) \times \mathrm{Nil}\, \mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\, \mathbb{K}_q(\mathcal{A}), \quad q \in \mathbb{Z}. \tag{9.13}$$

Thanks to Lemma 9.16, below, the evaluation of the homomorphism (9.13) at the class $[\epsilon_1] - [(\mathbb{Z}[x, y]^{\oplus n}, 0)] \in \mathrm{End}_0(\mathbb{Z}[x, y]; S)$ reduces to the bilinear pairing

$$\mathrm{End}_0(k) \times \mathrm{Nil}\, \mathbb{K}_n(\mathcal{A}) \longrightarrow \mathrm{Nil}\, \mathbb{K}_n(\mathcal{A}), \quad (\alpha, \beta) \mapsto V_n(\alpha \cdot F_n(\beta)). \tag{9.14}$$

Similarly, the evaluation of (9.13) at $[\epsilon_2] - [(\mathbb{Z}[x, y]^{\oplus n}, 0)]$ reduces to the pairing

$$\mathrm{End}_0(k) \times \mathrm{Nil}\, \mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\, \mathbb{K}_q(\mathcal{A}), \quad (\alpha, \beta) \mapsto V_n(\alpha) \cdot \beta. \tag{9.15}$$

Now, recall from [Almkvist 1974] (see also [Grayson 1978]) that the characteristic polynomial gives rise to an *injective* ring homomorphism

$$\mathrm{End}_0(\mathbb{Z}[x, y]; S) \to W(\mathbb{Z}[x, y]), \quad [(N, g)] - [(N, 0)] \mapsto \det(\mathrm{id} - gt).$$

Since the matrices (9.9)–(9.10) have the same characteristic polynomial, namely $1+(x^n y)t^n$, we conclude that $[\epsilon_1]-[(\mathbb{Z}[x,y]^{\oplus n},0)]=[\epsilon_2]-[(\mathbb{Z}[x,y]^{\oplus n},0)]$. This implies that the above pairings (9.14)–(9.15) agree and consequently that $V_n(\alpha\cdot F_n(\beta))=V_n(\alpha)\cdot\beta$ for every $\alpha\in\mathrm{End}_0(k)$ and $\beta\in\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$. $\square$

Lemma 9.16. *We have the commutative diagrams*

$$\begin{tikzcd} \mathrm{End}(k)\times\mathrm{Nil}(k) \arrow[r, "\theta(\epsilon_1,-,-)"] \arrow[d, "\mathrm{id}\times F_n"'] & \mathrm{Nil}(k) \\ \mathrm{End}(k)\times\mathrm{Nil}(k) \arrow[r, "(9.3)"'] & \mathrm{Nil}(k) \arrow[u, "V_n"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{End}(k)\times\mathrm{Nil}(k) \arrow[r, "\theta(\epsilon_2,-,-)"] \arrow[d, "V_n\times\mathrm{id}"'] & \mathrm{Nil}(k) \arrow[d, equal] \\ \mathrm{End}(k)\times\mathrm{Nil}(k) \arrow[r, "(9.3)"'] & \mathrm{Nil}(k) \end{tikzcd}$$

Proof. Let $(M,f)\in\mathrm{End}(k)$ and $(M',f')\in\mathrm{Nil}(k)$. By definition of ϵ_1 and ϵ_2, we observe that $\theta(\epsilon_1,(M,f),(M',f'))$ is naturally isomorphic to the endomorphism

$$\left((M\otimes M')^{\oplus n},\ \begin{bmatrix} 0 & \cdots & \cdots & 0 & f\otimes f'^n \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n\times n}\right)$$

and that $\theta(\epsilon_2,(M,f),(M',f'))$ is naturally isomorphic to the endomorphism

$$\left(M^{\oplus n}\otimes M',\ \begin{bmatrix} 0 & \cdots & \cdots & 0 & f \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n\times n}\otimes f'\right).$$

This achieves the proof. $\square$

Given an integer $m\geq 0$, let $\mathrm{Nil}(k)^m$ be the full exact subcategory of $\mathrm{Nil}(k)$ consisting of those nilpotent endomorphisms (M,f) with $f^m=0$. By construction, we have an exhaustive increasing filtration $\mathrm{Nil}(k)^m\subset\mathrm{Nil}(k)^{m+1}\subset\cdots\subset\mathrm{Nil}(k)$.

Given a dg category $\mathcal{A}$ and an integer $q\in\mathbb{Z}$, let us denote by $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$ the image of the induced homomorphism

$$\mathrm{kernel}\big(\mathbb{K}_q(\mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)^m_{\mathbf{dg}})\xrightarrow{\mathrm{id}\otimes(8.1)}\mathbb{K}_q(\mathcal{A}\otimes\mathrm{P}(k)_{\mathbf{dg}})\big)\longrightarrow\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}).$$

Note that $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})=\bigcup_m\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$ and that the Frobenius homomorphism $F_n:\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})\to\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$ vanishes on $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$ whenever $n\geq m$.

Given elements $a\in k$ and $\beta\in\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, consider the definition

$$(1-at^n)\odot\beta:=V_n\big([(k,a)]-[(k,0)]\big)\cdot\beta, \tag{9.17}$$

where (k, a) stands for the endomorphism of k given by multiplication by a. Thanks to Proposition 9.8, (9.17) agrees with $V_n\big(([(k,a)] - [(k,0)]) \cdot F_n(\beta)\big)$. Consequently, whenever $\beta \in \mathrm{Nil}\,\mathbb{K}_n(\mathcal{A})^m$ with $n \geq m$, we have $(1 - at^n) \odot \beta = 0$. Since $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) = \bigcup_m \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$, we obtain the bilinear pairings — the sum is finite! —

$$\begin{gathered} W(k) \times \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}), \\ \left(\prod_{n\geq 1}(1 - a_n t^n), \beta\right) \mapsto \sum_{n\geq 1}((1 - a_n t^n) \odot \beta). \end{gathered} \tag{9.18}$$

Now, recall from [Almkvist 1974] that the injective ring homomorphism

$$\mathrm{End}_0(k) \to W(k), \quad \big([(M,f)] - [(M,0)]\big) \mapsto \det(\mathrm{id} - ft),$$

sends $V_n\big([(k,a)] - [(k,0)]\big)$ to $1 - at^n$. Since every element of $W(k)$ can be written uniquely as $\prod_{n\geq 1}(1 - a_n t^n)$, we conclude that (9.18) extends (9.6). Moreover, thanks to Remark 9.7, the bilinear pairings (9.18) endow the abelian groups $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, $q \in \mathbb{Z}$, with a $W(k)$-module structure. This concludes the proof of Theorem 9.1.

10. Conclusion of the proof of Theorem 1.2

(i) As explained by Weibel [1981, Proposition 1.2], we have a ring homomorphism $\mathbb{Z}[1/l] \to W(\mathbb{Z}[1/l])$, $\lambda \mapsto (1-t)^\lambda$. Consequently, using the functoriality of $W(-)$ and the assumption $1/l \in k$, we observe that $W(k)$ is a $\mathbb{Z}[1/l]$-module. By combining Theorem 9.1 with Theorem 8.4 (with $E = \mathbb{K}$), we conclude that the groups $N\mathbb{K}_q(\mathcal{A})$, $q \in \mathbb{Z}$, carry a $\mathbb{Z}[1/l]$-module structure. The recursive formula (4.5) (with $H = \mathbb{K}_q$) implies that the groups $N^p\mathbb{K}_q(\mathcal{A})$, $p \geq 1$, are also $\mathbb{Z}[1/l]$-modules. Therefore, making use of the short exact sequences (see Step II)

$$0 \to N^p\mathbb{K}_q(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \to N^p\mathbb{K}_q(\mathcal{A}; \mathbb{Z}/l^\nu) \to \{l^\nu\text{-torsion in } N^p\mathbb{K}_{q-1}(\mathcal{A})\} \to 0,$$

we conclude that the groups $N^p\mathbb{K}_q(\mathcal{A}; \mathbb{Z}/l^\nu)$ are trivial. The convergent right half-plane spectral sequence (6.2) then implies that the edge morphisms

$$\mathbb{K}_q(\mathcal{A}; \mathbb{Z}/l^\nu) \to \mathbb{K}_q^h(\mathcal{A}; \mathbb{Z}/l^\nu)$$

are isomorphisms. The proof follows now from the fact that the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to a homotopy equivalence of spectra

$$\mathbb{K}^h(\mathcal{A}; \mathbb{Z}/l^\nu) \to \mathbb{K}^h(\mathcal{A}[t]; \mathbb{Z}/l^\nu);$$

see Step I.

(ii) We start with the following (arithmetic) result:

Lemma 10.1. *When l is nilpotent in k, the abelian groups* $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$ *are l-groups.*

Proof. Recall that the unit of $W(k)$ is $1-t$. Let $m \geq 0$ be a fixed integer. As explained by Weibel [1981, §1.5], whenever l is nilpotent in k there exists an integer $r \gg 0$ (which depends on m) such that $(1-t)^{l^r} \in 1+t^m k[\![t]\!]$. This implies that the formal factorization of $(1-t)^{l^r}$ only contains factors $(1-a_n t^n)$ with $n \geq m$. As in Step IV, we observe that every element β of $\operatorname{Nil}\mathbb{K}_q(\mathcal{A})^m$ is l^r-torsion. Finally, since $\operatorname{Nil}\mathbb{K}_q(\mathcal{A}) \simeq \bigcup_m \operatorname{Nil}\mathbb{K}_q(\mathcal{A})^m$, we conclude that $\operatorname{Nil}\mathbb{K}_q(\mathcal{A})$ is a l-group. $\square$

By combining Lemma 10.1 with Theorem 8.4 (with $E = \mathbb{K}$), we conclude that the abelian groups $N\mathbb{K}_q(\mathcal{A})$, $q \in \mathbb{Z}$, are l-groups. The recursive formula (4.5) (with $H = \mathbb{K}$) implies that the abelian groups $N^p\mathbb{K}_q(\mathcal{A})$, $p \geq 1$, are also l-groups. Therefore, $N^p\mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]} = 0$. Making use of the convergent right half-plane spectral sequence (6.3), we see that the edge morphisms $\mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]} \to \mathbb{K}_q^h(\mathcal{A})_{\mathbb{Z}[1/l]}$ are isomorphisms. The proof follows now from the fact that the dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra

$$\mathbb{K}^h(\mathcal{A}) \otimes \mathbb{Z}[1/l] \to \mathbb{K}^h(\mathcal{A}[t]) \otimes \mathbb{Z}[1/l];$$

see Step I.

11. Proof of Theorem 3.1

Thanks to Corollary 2.6, it suffices to compute the kernel and the cokernel of the (matrix) homomorphism (2.7) in the case where $m = 0$ and $Q = A_n$. The kernel is the solution of the system of linear equations with $\mathbb{Z}/l^\nu$-coefficients

$$\left\{\begin{aligned} -2x_1 + x_2 &= 0 \\ -x_1 - x_2 + x_3 &= 0 \\ &\vdots \\ -x_1 - x_{j-1} + x_j &= 0 \\ &\vdots \\ -x_1 - x_{n_1} + x_n &= 0 \\ -x_1 - x_n &= 0 \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} x_2 &= 2x_1 \\ x_3 &= 3x_1 \\ &\vdots \\ x_j &= jx_1 \\ &\vdots \\ x_n &= nx_1 \\ x_n &= -x_1 \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} (n+1)x_1 &= 0 \\ x_2 &= 2x_1 \\ &\vdots \\ x_j &= jx_1 \\ &\vdots \\ x_n &= nx_1 \end{aligned}\right\}.$$

From the above resolution of the system, we observe that the kernel is isomorphic to the $(n+1)$-torsion in $\mathbb{Z}/l^\nu$ or equivalently to the cyclic group $\mathbb{Z}/\gcd(n+1, l^\nu)$. Let us now compute the cokernel. Consider the (matrix) homomorphism

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ -1 & -1 & \ddots & \ddots & \vdots \\ -1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & 0 & \cdots & 0 & -1 \end{bmatrix} : \bigoplus_{r=1}^{n} \mathbb{Z} \to \bigoplus_{r=1}^{n} \mathbb{Z}. \tag{11.1}$$

Note that the cokernel of (11.1) is isomorphic to $\mathbb{Z}/(n+1)$. A canonical generator is given by the image of the vector $(0,\dots,0,-1) \in \bigoplus_{r=1}^{n} \mathbb{Z}$. Using the fact that the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu$ is right exact, we conclude that the cokernel of (2.7) is isomorphic to $\mathbb{Z}/(n+1) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \simeq \mathbb{Z}/\gcd(n+1, l^\nu)$. This concludes the proof.

Remark 11.2. Thanks to [Tabuada 2015a, Corollary 2.11], the Grothendieck group of $\mathcal{C}^{(0)}_{A_n}$ identifies with the cokernel of (11.1). We observe that $K_0(\mathcal{C}^{(0)}_{A_n}) \simeq \mathbb{Z}/(n+1)$.

12. Proof of Proposition 3.3

Similarly to the proof of Theorem 3.1, it suffices to compute the kernel and cokernel of the (matrix) homomorphism (2.7) in the case where $m = 1$ and Q is the generalized Kronecker quiver $1 \rightrightarrows 2$. The kernel is given by the solution of the system of linear equations with $\mathbb{Z}/l^\nu$-coefficients

$$\begin{cases} -9x_1 + 3x_2 = 0, \\ -3x_1 = 0. \end{cases} \tag{12.1}$$

Clearly, the solution of (12.1) is (3-torsion in $\mathbb{Z}/l^\nu$) $\times$ (3-torsion in $\mathbb{Z}/l^\nu$) or, equivalently, the cyclic group $\mathbb{Z}/\gcd(3, l^\nu) \times \mathbb{Z}/\gcd(3, l^\nu)$. Note that the latter group is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ when $l = 3$ and is zero otherwise. Let us now compute the cokernel. Consider the (matrix) homomorphism

$$\begin{bmatrix} -9 & 3 \\ -3 & 0 \end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}. \tag{12.2}$$

The cokernel of (12.2) is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$. Canonical generators are given by the image of the vectors $(1, 0)$ and $(-3, 1)$. Since the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu$ is right exact, we conclude that the cokernel of (2.7) is isomorphic to

$$(\mathbb{Z}/3 \times \mathbb{Z}/3) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \simeq \mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \times \mathbb{Z}/3 \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \simeq \mathbb{Z}/\gcd(3, l^\nu) \times \mathbb{Z}/\gcd(3, l^\nu).$$

Once again, the right-hand side abelian group is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ when $l = 3$ and is zero otherwise. This concludes the proof.

Remark 12.3. As in Remark 11.2, the Grothendieck group of $\mathcal{C}^{(1)}_{Q}$ is identified with the cokernel of (12.2). We observe that $K_0(\mathcal{C}^{(1)}_{Q}) \simeq \mathbb{Z}/3 \times \mathbb{Z}/3$.

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Reciprocity laws and K-theory

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We associate to a full flag $\mathcal{F}$ in an n-dimensional variety X over a field k, a "symbol map" $\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k)$. Here, F_X is the field of rational functions on X, and $K(\,\cdot\,)$ is the K-theory spectrum. We prove a "reciprocity law" for these symbols: given a partial flag, the sum of all symbols of full flags refining it is 0. Examining this result on the level of K-groups, we derive the following known reciprocity laws: the degree of a principal divisor is zero, the Weil reciprocity law, the residue theorem, the Contou-Carrère reciprocity law (when X is a smooth complete curve), as well as the Parshin reciprocity law and the higher residue reciprocity law (when X is higher-dimensional).

1. Introduction

1A. ***Overview.*** Several statements in number theory and algebraic geometry are "reciprocity laws". Let us consider, as an example, the Weil reciprocity law. Let X be a complete smooth curve over an algebraically closed field k, and let us fix $f, g \in F_X^\times$, two nonzero rational functions on X. Given a point $p \in X$, one defines the tame symbol:

$$(f, g)_p := (-1)^{v_p(f)\cdot v_p(g)} \frac{f^{v_p(g)}}{g^{v_p(f)}}(p).$$

Here, v_p is the valuation at p (that is, the order of the zero). The Weil reciprocity law states that $(f, g)_p = 1$ for all but finitely many $p \in X$, and that $\prod_{p \in X} (f, g)_p = 1$.

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More generally, one can describe the pattern as follows. There is a global object, exhausted by local pieces. One then associates an invariant to each local piece, as well as to the global object itself. The desired claim is then twofold.

(i) Global is trivial: the global invariant is trivial.

(ii) Local to global: the product of the local invariants equals the global invariant (usually this is an infinite product, and one should figure out how to make sense of it).

In the above example, the global object is the curve X, which is exhausted by the local pieces — the points of the curve. The invariant associated to a local piece is the tame symbol, while the global invariant is quite implicit.

Let us recall that the Weil reciprocity law admits a higher-dimensional analog, known as the Parshin reciprocity law [Parshin 1976; Soprounov 2002, Appendix A]; see page 34.

In this paper we define symbol maps and prove a reciprocity law using the machinery of algebraic K-theory. We then see how various reciprocity laws, such as the Parshin reciprocity law (generalizing the Weil reciprocity law), the higher residue reciprocity law (generalizing the residue theorem), and the Contou-Carrère reciprocity law, all follow from this one reciprocity law.

Let us describe our setup in more detail. Fix an n-dimensional irreducible variety X over a field k.[1] By a *full flag* $\mathcal{F}$ in X we mean a chain of closed irreducible subvarieties $X = X^0 \supset X^1 \supset \cdots \supset X^n$, where the codimension of X^i in X is i. Given a full flag $\mathcal{F}$, we shall define a morphism of spectra

$$\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k)$$

(we call it a symbol map). Here F_X denotes the field of rational functions on X, $K(\cdot)$ denotes the K-theory spectrum, and Σ^n denotes n-fold suspension. By a *partial flag* $\mathcal{G}$ in X, we mean a full flag with an element in some single codimension d omitted, for $0 < d \le n$. Then, given a partial flag $\mathcal{G}$, we may consider the set $\mathrm{fl}(\mathcal{G})$ of full flags which refine it. The main result of this paper, Theorem 2.1, then states:

Theorem. *Let X be an n-dimensional irreducible variety over a field k. Let*

$$\mathcal{G} : X^0 \supset \cdots \supset X^{d-1} \supset X^{d+1} \supset \cdots \supset X^n$$

be a partial flag in X, with element in codimension $0 < d \le n$ omitted. In the case $d = n$, assume additionally that the curve X^{n-1} is proper over k. Then

$$\sum_{\mathcal{F} \in \mathrm{fl}(\mathcal{G})} \mu_{\mathcal{F}} = 0.$$

[1] These assumptions on X and k are made here merely to simplify matters, and will be relaxed below.

Remark. The sum figuring in the theorem is infinite; however, in Appendix A we will make sense of it (inspired by [Clausen 2012]).

In fact, it is more "correct" to additionally define a symbol map

$$\mu_{\mathcal{G}} : K(F_X) \to \Sigma^n K(k)$$

associated to a partial flag $\mathcal{G}$. The theorem then divides into two parts: that $\mu_{\mathcal{G}}$ equals zero, and that the sum of all the morphisms $\mu_{\mathcal{F}}$ for $\mathcal{F} \in \mathrm{fl}(\mathcal{G})$ equals $\mu_{\mathcal{G}}$.

Notice how this setup instantiates the general pattern above. A fixed partial flag is the global object, exhausted by the local pieces which are the full flags refining the given partial flag. The symbol map is the associated invariant.

In order to derive the concrete reciprocity laws promised above from this abstract one, one considers its effect on K-groups.

Let us note that, in principle, the symbol map between spectra appears to contain more information than its "shadows" on K-groups. However, in this paper we have only recovered known reciprocity laws from it.

Let us also record here that relevant and independent work has been done in [Braunling et al. 2014a; 2014b; Osipov and Zhu 2014].

There are several further directions to consider. For example, one may consider the "curve" $\mathrm{Spec}(\mathbb{Z})$. Could our setup be altered so as to accommodate the Hilbert reciprocity law? For that to succeed, at least three phenomena should be addressed: the prime at infinity, ramification at the prime 2, and the sphere spectrum, which underlies all primes. A relevant treatment of the case of $\mathrm{Spec}(\mathbb{Z})$ is in [Clausen 2012].

1B. ***Relation to n-Tate vector spaces.*** There is a strong relation between our machinery and the theory of n-Tate vector spaces. In fact, n-Tate vector spaces could be seen as the actual "geometric" objects that the target of our symbol map $\mu_{\mathcal{F}}$ classifies, so that, in a sense, our approach "decategorifies" the actual picture.

The technical result underlying such a connection is the following. Let $\mathcal{C}$ be an exact category, and $\mathrm{Tate}(\mathcal{C})$ the exact category of "pro-ind" objects in $\mathcal{C}$, introduced by Beilinson [1987].

Theorem [Saito 2015]. $K(\mathrm{Tate}(\mathcal{C})) \approx \Sigma K(\mathcal{C})$.

Thus, we can say that the Tate construction acts as a delooping, when one passes to K-theory spectra.

In this paper we associate to a full flag $\mathcal{F}$ in an n-dimensional variety X a symbol map

$$\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k).$$

Taking the above theorem into account, one might interpret it as a map

$$\mu_{\mathcal{F}} : K(F_X) \to K(\mathrm{Tate}^n(k)),$$

where $\mathrm{Tate}^n(k)$ is the n-fold application of the $\mathrm{Tate}(\,\cdot\,)$ construction to the exact category $\mathrm{Vect}(k)$ of finite-dimensional vector spaces over k. At this point, one might wonder whether this map comes from a functor

$$\mathrm{Vect}(F_X) \to \mathrm{Tate}^n(k).$$

Indeed, such a functor can be constructed, and is essentially the adelic construction of [Beilinson 1980].

We will address and develop the above interesting ideas elsewhere.

Once again, we point out that relevant work has been done in [Braunling et al. 2014a; 2014b].

1C. ***Organization.*** This paper is organized as follows. Section 2 contains the formulation of the abstract reciprocity law (Section 2A) and the formulations of concrete reciprocity laws (Section 2B) which are obtained from the abstract reciprocity law by considering its effect on specific K-groups. Section 3 contains the construction of the abstract symbol map (Section 3A) and the proof of the abstract reciprocity law (Section 3B). Section 4 deals with the calculation of the symbol map on specific K-groups.

In Appendix A, we describe how to make sense of an infinite sum of morphisms of spectra. In Appendix B, we state some lemmas about K-theory which are used in calculations.

1D. ***Notation.*** We use [Thomason and Trobaugh 1990] as a reference for K-theory of schemes. Given a Noetherian scheme X, $K(X)$ denotes the K-theory spectrum of X. Given a closed subset $Z \subset X$, $K(X \text{ on } Z)$ denotes the K-theory spectrum of X with support in Z. By abuse of notation, given a commutative ring A and an ideal $I \subset A$, we also write $K(A) = K(X)$ and $K(A \text{ on } I) = K(X \text{ on } Z)$, where $X = \mathrm{Spec}(A)$ and $Z \subset X$ is the closed subset associated to the ideal I.

We use the following notation for the scheme X in this paper:

- $n = \dim(X)$ denotes the Krull dimension of X.
- $|X|$ denotes the underlying topological space of X. The usual partial order on $|X|$ (that of "containment in the closure of") is denoted by $\leq$, and $|X|^i$ denotes the subset of $|X|$ consisting of points of codimension i.
- γ denotes the generic point of $|X|$ (X will be assumed to be irreducible) — i.e., the only point in $|X|^0$ — and $F = F_X = \mathcal{O}_{X,\gamma}$ denotes the local ring at that point.
- For $p \in |X|$, we write $X_p := \mathrm{Spec}(\mathcal{O}_{X,p})$. There is a canonical map $X_p \to X$. As usual, we write $k(p)$ for the residue field of $\mathcal{O}_{X,p}$.
- If X is affine and $\mathfrak{p}$ is a prime ideal in $\mathcal{O}(X)$, then $p_{\mathfrak{p}} \in |X|$ denotes the corresponding point.

2. Statements

2A. ***The abstract reciprocity law.*** Let $X \to B$ be a morphism of schemes. We make the following assumptions:

(1) B is Noetherian, 0-dimensional (i.e., a finite disjoint union of Zariski spectra of local Artinian rings).

(2) X is Noetherian, of finite Krull dimension and irreducible.

(3) $X \to B$ is flat.

(4) For every $p \in |X|^n$ (recall $n = \dim(X)$), the composition $\operatorname{Spec}(k(p)) \to X \to B$ is a finite morphism.

We give two examples of morphisms that satisfy the above assumptions:

(1) $B = \operatorname{Spec}(k)$, where k is a field, and $X \to B$ is an irreducible scheme of finite type over B.

(2) $B = \operatorname{Spec}(k)$, where k is a field, and $X = \operatorname{Spec}(A)$, where $(A, \mathfrak{m})$ is a Noetherian local integral k-algebra, such that $A/\mathfrak{m}$ is finite over k. $X \to B$ is the corresponding structure map.

A convenient technical notion will be that of a collection C, by which we mean a family $C = (C^i)_{0 \le i \le n}$, where $C^i \subset |X|^i$. We only consider collections which satisfy $C^0 = \{\gamma\}$.

Given such a C, in Section 3A we construct a map of spectra ("symbol map")

$$\mu_C : K(F) \to \Sigma^n K(B).$$

We only consider and use collections attached to full and partial flags (to be now defined), for which we will state a reciprocity law. First, let

$$\mathcal{F} : x_n < x_{n-1} < \cdots < x_0 = \gamma$$

be a full flag of points in $|X|$ (thus, $\operatorname{codim}(x_i) = i$). We define a collection $C(\mathcal{F})$, by setting $C(\mathcal{F})^i = \{x_i\}$. Second, let

$$\mathcal{G} : x_n < x_{n-1} < \cdots < x_{d+1} < x_{d-1} < \cdots < x_0 = \gamma$$

be a partial flag, with the level d omitted, $0 < d \le n$. Here, we require $\operatorname{codim}(x_i) = i$. We define a collection $C(\mathcal{G})$ by setting $C(\mathcal{G})^i = \{x_i\}$ for $i \ne d$, and

$$C(\mathcal{G})^d = \{p \in |X|^d \mid x_{d+1} < p < x_{d-1}\}.$$

Note that we have the obvious notion of a full flag refining a partial one (meaning $C(\mathcal{F}) \subset C(\mathcal{G})$), which we denote by $\mathcal{F} > \mathcal{G}$. We sometimes write $\mu_{\mathcal{F}}$ instead of $\mu_{C(\mathcal{F})}$.

We prove the following "reciprocity" laws (for the meaning of the infinite sum in this statement, consult Appendix A).

Theorem 2.1. *Let $\mathcal{G}$ be a partial flag with level d omitted, where $0 < d \leq n$.*

(1) *Global is trivial*:
$$\mu_{C(\mathcal{G})} = 0,$$
where in the case $d = n$ we should assume that $\overline{x_{n-1}}$ is proper over B.

(2) *Local to global*:
$$\mu_{C(\mathcal{G})} = \sum_{\mathcal{F} > \mathcal{G}} \mu_{C(\mathcal{F})}.$$

2B. ***Concrete reciprocity laws.*** In the following, we give examples of concrete reciprocity laws, which one obtains by considering the effect of the abstract reciprocity law on various homotopy groups of the involved spectra.

***The case* $\mathbf{dim}(X) = 1$.** Let k be a field, $B = \mathrm{Spec}(k)$, and $X \to B$ a regular, connected, proper curve over B. We obtain, for every closed point $p \in |X|^1$, a map $\mu_p : K(F) \to \Sigma K(B)$. Here $\mu_p = \mu_{C(\mathcal{F})}$, where $\mathcal{F} : p < \gamma$. Applying the functor π_i, one has maps $\mu_p^i : K_i(F) \to K_{i-1}(k)$.

The degree law. We have the map $\mu_p^1 : F^\times \cong K_1(F) \to K_0(k) \cong \mathbb{Z}$.

Claim 2.2. *The integer $\mu_p^1(f)$ is equal to the valuation $v_p(f)$ of f at the point p, multiplied by $[k(p) : k]$.*

Applying the abstract reciprocity law, we recover the theorem about sum of degrees [Serre 1988, §II.3, Proposition 1]:

Corollary 2.3. *For $f \in F^\times$,*
$$\sum_{p \in |X|^1} [k(p) : k] \cdot v_p(f) = 0.$$

The Weil reciprocity law. Precomposing the map $\mu_p^2 : K_2(F) \to K_1(k)$ with the product in K-theory $K_1(F) \wedge K_1(F) \to K_2(F)$, we get a bilinear antisymmetric form $\mu_p^2 : F^\times \wedge F^\times \to k^\times$ (we also call it μ_p^2, by abuse of notation).

Claim 2.4.
$$\mu_p^2(f \wedge g) = N_{k(p)/k}\left((-1)^{v_p(f) \cdot v_p(g)} \frac{f^{v_p(g)}}{g^{v_p(f)}}(p) \right).$$

Applying the abstract reciprocity law, we recover the Weil reciprocity law [Serre 1988, §III.4]:

Corollary 2.5. *For $f, g \in F^\times$,*
$$\prod_{p \in |X|^1} N_{k(p)/k}\left((-1)^{v_p(f) \cdot v_p(g)} \frac{f^{v_p(g)}}{g^{v_p(f)}}(p) \right) = 1.$$

The residue law. Suppose that k is algebraically closed. Set $k_\epsilon := k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$, $B_\epsilon = \mathrm{Spec}(k_\epsilon)$, and $X_\epsilon = k_\epsilon \otimes_k X$. Then the local ring at the generic point of X_ϵ is just $F_\epsilon = k_\epsilon \otimes_k F$. By applying our construction to the morphism $X_\epsilon \to B_\epsilon$ we get a map $K(F_\epsilon) \to \Sigma K(k_\epsilon)$ for every closed point $p \in |X_\epsilon|^1 = |X|^1$. Applying the functor π_2 and using the product in K-theory as before, one gets a pairing $r_p : F_\epsilon^\times \wedge F_\epsilon^\times \to k_\epsilon^\times$.

Claim 2.6. *For* Res_p *the usual residue* [Serre 1988, §II.7],

$$r_p((1-\epsilon_1 f) \wedge (1-\epsilon_2 g)) = 1 - \epsilon_1\epsilon_2 \,\mathrm{Res}_p(f \cdot dg).$$

Applying the abstract reciprocity law, we recover the residue theorem [Serre 1988, §II.7, Proposition 6]:

Corollary 2.7. *For* $f, g \in F$,

$$\sum_{p \in |X|^1} \mathrm{Res}_p(f \cdot dg) = 0.$$

Remark 2.8. In fact, one can drop the assumption that k is algebraically closed. Then, one has

$$r_p((1-\epsilon_1 f) \wedge (1-\epsilon_2 g)) = 1 - \epsilon_1\epsilon_2 \,\mathrm{Tr}_{k(p)/k} \mathrm{Res}_p(f \cdot dg),$$

where $\mathrm{Res}_p(f \cdot dg)$ can be defined as follows: Choose an isomorphism $\widehat{\mathcal{O}_{X,p}} \simeq k'[[z]]$, where $k' := k(p)$ is the residue field at p. Interpret $f \cdot dg$ as an element of $\Omega^1(k'((z))/k') = k'((z))\,dz$. Finally, define $\mathrm{Res}_p(f \cdot dg)$ as the coefficient of $z^{-1}\,dz$ in $f \cdot dg$. Note that in the case when k is algebraically closed, one recovers the usual definition.

The Contou-Carrère reciprocity law. More generally, let k be a local Artinian ring. Set $B = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(k[[t]])$. Applying the functor π_2 to the symbol map $K(k((t))) \to \Sigma K(k)$, one gets a pairing $k((t))^\times \wedge k((t))^\times \to k^\times$. Although we do not spell out the details in this paper, one can check that it is the Contou-Carrère symbol [Contou-Carrère 1994]. Then the abstract reciprocity law implies the Contou-Carrère reciprocity law.

Let us note that [Osipov and Zhu 2014] also deals with the connection between K-theory and explicit formulas for Contou-Carrère symbols.

The case $\mathbf{dim(X) > 1}$. Let k be a field, $B = \mathrm{Spec}(k)$, and $X \to B$ an irreducible scheme of finite type over B (recall $n = \dim(X)$). For every full flag $\mathcal{F}$ one has a map $\mu_{\mathcal{F}} : K(F) \to \Sigma^n K(B)$. Applying the functor π_i, one then gets maps $\mu^i_{\mathcal{F}} : K_i(F) \to K_{i-n}(k)$.

The Parshin reciprocity law. Let us assume that the flag $\mathcal{F} = x_n < x_{n-1} < \cdots < x_0 = \gamma$ is regular in the following sense: considering $X^i := \overline{x_i}$ as an integral closed subscheme of X, we demand $\mathcal{O}_{X^{i-1}, x_i}$ to be regular (here, $1 \le i \le n$).

Precomposing the map $\mu_{\mathcal{F}}^{n+1} : K_{n+1}(F) \to K_1(k)$ with the product in K-theory $\bigwedge^{n+1} K_1(F) \to K_{n+1}(F)$, one has a multilinear antisymmetric form

$$\mu_{\mathcal{F}}^{n+1} : \bigwedge^{n+1} F^\times \to k^\times$$

(we also denote it $\mu_{\mathcal{F}}^{n+1}$, by abuse of notation).

In order to write an explicit formula for the Parshin symbol, we introduce the following; see [Soprounov 2002, Appendix A]. For every $1 \le i \le n$, let us fix a uniformizer z_i in $\mathcal{O}_i := \mathcal{O}_{X^{i-1}, x_i}$. We attach, to any $f \in F^\times$, a sequence of integers $a_1, \dots, a_n$ as follows. Note that the residue field of $\mathcal{O}_{i-1}$ can be identified with the fraction field of $\mathcal{O}_i$. We write $f = z_1^{a_1} u_1$, where u_1 is a unit in $\mathcal{O}_1$. Considering the residue class of u_1 as an element of the fraction field of $\mathcal{O}_2$, we proceed to write $u_1 = z_2^{a_2} u_2$, where u_2 is a unit in $\mathcal{O}_2$. We continue in this way to construct the sequence $a_1, \dots, a_n$. Note that, generally speaking, this sequence depends on the choice of uniformizers $z_1, \dots, z_n$.

Let $f_1, \dots, f_{n+1} \in F^\times$. Write $a_{i1}, \dots, a_{in}$ for the sequence of integers assigned to f_i as above. Construct the $(n+1) \times n$ matrix $A = (a_{ij})$. Set A_i to be the determinant of the $n \times n$ matrix that we get from A by omitting the i-th row. Set A_{ij}^k to be the determinant of the $(n-1) \times (n-1)$ matrix that we get from A by deleting the i-th and j-th rows and the k-th column. Set $B = \sum_k \sum_{i<j} a_{ik} a_{jk} A_{ij}^k$.

Claim 2.9. $\mu_{\mathcal{F}}^{n+1}(f_1, \dots, f_{n+1}) = N_{k(x_n)/k}\left((-1)^B \left(\prod_{1 \le i \le n+1} f_i^{(-1)^{i+1} A_i}\right)(x_n)\right).$

By applying the abstract reciprocity law, we recover the Parshin reciprocity law; see [Soprounov 2002, Appendix A].

The Parshin higher residue reciprocity law. Considering

$$k_\epsilon := k[\epsilon_1, \dots, \epsilon_{n+1}]/(\epsilon_1^2, \dots, \epsilon_{n+1}^2)$$

and X_ϵ, B_ϵ, etc., as for the residue law on page 33, and considering the map $\mu^{n+1} : K_{n+1}(F_\epsilon) \to K_1(k_\epsilon)$, one can derive, in principle, the higher residue reciprocity law [Soprounov 2002, Appendix A], although we do not spell out the details in this paper.

3. Construction of μ_C and proof of the abstract reciprocity law

3A. ***Construction of μ_C.*** We recall the codimension filtration in K-theory [Thomason and Trobaugh 1990, (10.3.6)]. Write $S^d K(X)$ for the homotopy colimit of the spectra $K(X \text{ on } Z)$, where Z runs over closed subsets of X of codimension $\geq d$. Also, write

$$Q^d K(X) := \bigvee_{p \in |X|^d} K(X_p \text{ on } p).$$

Then we have the homotopy fiber sequence

$$S^{d+1} K(X) \longrightarrow S^d K(X) \xrightarrow{p_d} Q^d K(X) \xrightarrow{\partial_d} \Sigma S^{d+1} K(X).$$

Let us define Ψ^d to be the composition

$$\Psi^d : Q^d K(X) \xrightarrow{\partial_d} \Sigma S^{d+1} K(X) \xrightarrow{p_{d+1}} \Sigma Q^{d+1} K(X).$$

Also, given a collection $C = (C^i)_{0 \leq i \leq n}$ (for $C^i \subset |X^i|$), we define a map

$$\mathrm{sel}_{C^d} : Q^d K(X) \to Q^d K(X),$$

given by projecting on summands corresponding to $p \in C^d$.

We now define a map

$$I : Q^n K(X) \to K(B).$$

In order to do this, we first need to define maps $K(X_p \text{ on } p) \to K(B)$, which we do by pushing forward along $X_p \to B$. To justify the existence of the pushforward, let us fix convenient models for the K-spectra. As a model for $K(X_p \text{ on } p)$ we take strictly perfect complexes on X_p which are acyclic outside of the closed point p [Thomason and Trobaugh 1990, Lemma 3.8], and as a model for $K(B)$ we take perfect complexes on B [Thomason and Trobaugh 1990, Definition 3.1]. Pushing forward along $X_p \to B$ can be done termwise, since this morphism is affine. Thus, the result of pushing forward to B a strictly perfect complex on X_p, supported on p, is a strictly bounded complex, whose terms are flat (since $X_p \to B$ is assumed flat), and whose cohomologies are coherent (since $k(p) \to B$ is assumed finite). Thus, by criterion [Thomason and Trobaugh 1990, Proposition 2.2.12], the result is perfect.

Finally, we define μ_C as follows:[2][3]

$$\mu_C = I \circ \mathrm{sel}_{C^n} \circ \Psi^{n-1} \circ \cdots \circ \Psi^1 \circ \mathrm{sel}_{C^1} \circ \Psi^0.$$

3B. ***Proof of the reciprocity law.*** Let us show part (1) of Theorem 2.1.

First, consider the case $d \neq n$. Notice that the formula for $\mu_{C(\mathcal{G})}$ contains

$$\mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \mathrm{sel}_{C(\mathcal{G})^d} \circ \Psi^{d-1}.$$

---

[2] We assume that $C^0 = \{\gamma\}$.

[3] In this formula, as we compose, the target becomes more and more suspended; we do not write the obvious suspensions, by abuse of notation.

Since $C(\mathcal{G})^d$ contains all the points p such that $x_{d+1} < p < x_{d-1}$, one has

$$\mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \mathrm{sel}_{C(\mathcal{G})^d} = \mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d.$$

Thus, in fact,

$$\mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \mathrm{sel}_{C(\mathcal{G})^d} \circ \Psi^{d-1} = \mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \Psi^{d-1},$$

which is zero since $\Psi^d \circ \Psi^{d-1} = 0$ (as it contains a composition of two consequent arrows in a long exact sequence).

Next, consider the case $d = n$. Write $Y = \overline{x_{n-1}}$. We will deal first with the case $X = Y$, to simplify matters.

Note that $\mu_{C(\mathcal{G})}$ equals the composition on the top horizontal line of the following commutative diagram:

$$\begin{tikzcd}
Q^0 K(X) \arrow[r, "\partial_0"] & \Sigma S^1 K(X) \arrow[r, "p_1"] \arrow[d, "i"] & \Sigma Q^1 K(X) \arrow[r, "I"] & \Sigma K(B) \\
& \Sigma S^0 K(X) \arrow[urr, "\tilde{I}"'] & &
\end{tikzcd}$$

Here, i is the natural arrow, and $\tilde{I}$ is the arrow induced by pushforward. The crucial assumption here is that X is proper. Thus pushing forward preserves coherence, which in turn enables us to construct the map $\tilde{I}$ on K-spectra. Now, noticing that $i \circ \partial_0 = 0$ (as a composition of two consequent arrows in a long exact sequence) finishes the proof.

In general (not assuming $X = Y$), we want to do the same as in the case $X = Y$, but working with (X on Y) versions. To proceed, one considers the commutative diagram

$$\begin{tikzcd}
Q^{n-1} K(X) \arrow[r, "\mathrm{sel}_{C(\mathcal{G})^{n-1}}"] \arrow[dr] & Q^{n-1} K(X) \arrow[r, "\partial_0"] & \Sigma Q^n K(X) \arrow[r, "I"] & \Sigma K(B) \\
& Q^{n-1} K(X \text{ on } Y) \arrow[u] \arrow[r, "\partial^Y_{n-1}"] & \Sigma Q^n K(X \text{ on } Y) \arrow[u] \arrow[ur, "I^Y"] &
\end{tikzcd}$$

and shows $I^Y \circ \partial^Y_{n-1} = 0$ as before.

Let us now show part (2) of Theorem 2.1. We note that the map $\mathrm{sel}_{C(\mathcal{G})^d}$ is the sum of the maps $\mathrm{sel}_{C(\mathcal{F})^d}$ (where $\mathcal{F} > \mathcal{G}$). Thus, the statement follows using Claims A.4 and A.5.

4. Calculation of local symbols

In this section, we calculate some symbol maps for local schemes. Using Lemma 4.7, these calculations imply the claims of Section 2B.

Let us fix the following notation and assumptions for this section. Let k be a field, and let $B = \mathrm{Spec}(k)$. Also, let A be a regular Noetherian local k-algebra, and set $X = \mathrm{Spec}(A)$. Denote by $\mathfrak{m}$ the maximal ideal of A, and $k' = A/\mathfrak{m}$. We assume that k' is finite over k. We denote by F the fraction field of A.

4A. *The case* $\dim(X) = 1$. In this subsection, we additionally assume that A is of Krull dimension 1. Let $v : F^\times \to \mathbb{Z}$ be the valuation, and let $[\,\cdot\,] : A \to k'$ be the quotient map. Finally, choose a uniformizer $z \in A$ (i.e., $v(z) = 1$).

Consider the unique full flag $\mathcal{F} : p_\mathfrak{m} < p_0$ in X. We have the corresponding symbol map

$$\mu = \mu_\mathcal{F} : K(F) \to \Sigma K(k).$$

We write μ^i for the induced map $K_i(F) \to K_{i-1}(k)$.

The degree.

Claim 4.1. *The morphism* $F^\times \cong K_1(F) \xrightarrow{\mu^1} K_0(k) \cong \mathbb{Z}$ *is equal to* $[k' : k] \cdot v$.

Proof. Since the composition $K_1(A) \to K_1(F) \to K_0(A \text{ on } \mathfrak{m})$ is zero (as part of a long exact sequence), it is enough to prove that

$$F^\times \cong K_1(F) \to K_0(A \text{ on } \mathfrak{m}) \to K_0(k) \cong \mathbb{Z}$$

maps z to $[k' : k]$. By Lemma B.3, the image of z under the above map is equal to the alternating sum of dimensions (over k) of cohomologies of the complex

$$\begin{array}{ccc} A & \xrightarrow{z} & A \\ -1 & & 0 \end{array}$$

which is $[k' : k]$. $\square$

The tame symbol.

Claim 4.2. *The morphism*

$$F^\times \wedge F^\times \cong K_1(F) \wedge K_1(F) \longrightarrow K_2(F) \xrightarrow{\mu^2} K_1(k) \cong k^\times$$

is given by

$$f \wedge g \mapsto N_{k'/k}\left((-1)^{v(f)\cdot v(g)} \left[\frac{f^{v(g)}}{g^{v(f)}}\right]\right).$$

Proof. We call the above morphism $F^\times \wedge F^\times \to k^\times$, by abuse of notation, μ^2. By bilinearity and antisymmetry of μ^2, it is enough to verify:

(i) $\mu^2(f \wedge g) = 0$ for $f, g \in A^\times$.

(ii) $\mu^2(f \wedge z) = N_{k'/k}([f])$ for $f \in A^\times$.

(iii) $\mu^2(z \wedge z) = N_{k'/k}(-1)$.

The first item follows since the following composition is zero (being a part of the localization long exact sequence):

$$K_2(A) \to K_2(F) \to K_1(A \text{ on } k').$$

For the second item, consider the commutative diagram

$$\begin{tikzcd} K_1(A) \wedge K_1(F) \arrow[rr] \arrow[d] & & K_1(A) \wedge K_0(A \text{ on } k') \arrow[d] & \\ K_1(F) \wedge K_1(F) \arrow[r] & K_2(F) \arrow[r] & K_1(A \text{ on } k') \arrow[r] & K_1(k) \end{tikzcd}$$

We have the element $f \wedge z$ in the upper-left group $K_1(A) \wedge K_1(F)$, and we should walk it through down, and then all the way right. Using commutativity of the diagram, we can chase the upper path instead, and using Lemma B.4, the result is represented by the automorphism of the following complex:

$$\begin{tikzcd} A \arrow[r, "z"] \arrow[d, "f"'] & A \arrow[d, "f"'] \\ A \arrow[r, "z"] & A \\ -1 & 0 \end{tikzcd}$$

Taking the alternating determinant of cohomology, we see that the above automorphism represents the element $N_{k'/k}([f]) \in k^\times \cong K_1(k)$.

Let us handle the third item on our list. Denote the multiplication in K-theory by $\{\cdot,\cdot\} : K_1(F) \wedge K_1(F) \to K_2(F)$. Recall the Steinberg relation

$$\{x, 1-x\} = 0$$

for $x, 1-x \in F^\times \cong K_1(F)$. We then calculate

$$\{z,z\} = \{z, (1-z^{-1})^{-1}\}\{z, 1-z\}\{z,-1\} = \{z^{-1}, 1-z^{-1}\}\{z, 1-z\}\{z,-1\} = \{z,-1\}$$

(this calculation appears in [Snaith 1980, Theorem 2.6]). Hence, by (ii) above, $\mu^2(z \wedge z) = \mu^2(-1 \wedge z) = N_{k'/k}(-1)$. □

The residue. Consider a base change of our setup from k to $k_\epsilon := k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$. Thus, we have $A_\epsilon := k_\epsilon \otimes_k A$, and similarly F_ϵ, X_ϵ, B_ϵ, etc. Hence, the basic morphism of schemes from which we build the symbol map is now $X_\epsilon \to B_\epsilon$.

Claim 4.3. *The morphism*

$$F_\epsilon^\times \wedge F_\epsilon^\times \cong K_1(F_\epsilon) \wedge K_1(F_\epsilon) \longrightarrow K_2(F_\epsilon) \xrightarrow{\mu_\epsilon^2} K_1(k_\epsilon) \cong k_\epsilon^\times$$

sends $(1-\epsilon_1 f)\wedge(1-\epsilon_2 g)$ *to* $1-\epsilon_1\epsilon_2 R(f,g)$ *(for* $f,g\in F$*). Here,* $R(f,g)$ *is defined as follows: Choose an isomorphism* $\hat{A}\simeq k'[\![z]\!]$*. Interpret* $f\cdot dg$ *as an element* $\sum_i a_i z^i\,dz\in\Omega^1\big(k'((z))/k'\big)=k'((z))\,dz$*. Finally, define* $R(f,g)=\mathrm{Tr}_{k'/k}(a_{-1})$*.*

Proof. In this proof let us denote by μ^2 the morphism $F_\epsilon^\times\wedge F_\epsilon^\times\to k_\epsilon^\times$ in the claim.

(a) We wish to reduce the computation to the case when $A=k[\![z]\!]$ and k is infinite. This is done by exploiting functoriality in a few steps; First, using Lemma 4.8, we may assume that A is complete. Hence, by Cohen's structure theorem, $A\simeq k'[\![z]\!]$. Second, since A is now a k'-algebra, $R(f,g)$ for A as a k-algebra is the trace $\mathrm{Tr}_{k'/k}$ of $R(f,g)$ for A as a k'-algebra. Hence, we may assume that $k=k'$. Finally, let l/k be a field extension. Consider the diagram

$$\begin{tikzcd}
\mathrm{Spec}(l((z))) \arrow[r] \arrow[d] & \mathrm{Spec}(l[\![z]\!]) \arrow[d] & \mathrm{Spec}(l) \arrow[l] \arrow[d] \\
\mathrm{Spec}(k((z))) \arrow[r] & \mathrm{Spec}(k[\![z]\!]) & \mathrm{Spec}(k) \arrow[l]
\end{tikzcd}$$

Note that the squares in the above diagram are pullback squares. Hence, the following diagram commutes:

$$\begin{tikzcd}
k((x))_\epsilon^\times\wedge k((x))_\epsilon^\times \arrow[r, "\mu^2"] \arrow[d] & k_\epsilon^\times \arrow[d, hook] \\
l((x))_\epsilon^\times\wedge l((x))_\epsilon^\times \arrow[r, "\mu^2"] & l_\epsilon^\times
\end{tikzcd}$$

Thus, we can replace the k-algebra $A=k[\![z]\!]$ by the l-algebra $l[\![z]\!]$, where l/k is any field extension. Hence, we may assume that k is infinite.

(b) Next, we show that $\mu^2(1-\epsilon_1 f,1-\epsilon_2 g)$ is of the form $1-\epsilon_1\epsilon_2 R(f,g)$, where $R(f,g)\in k$. In other words, the "constant term" is 1, and there are no "linear terms". Towards this end, we perform "base change", sending $\epsilon_2\mapsto 0$. The operation μ^2 commutes with such a base change. We depict it as follows:

$$\begin{tikzcd}
\mu^2(1-\epsilon_1 f,1-\epsilon_2 g) \arrow[r, equal] \arrow[d, mapsto] & a+b\epsilon_1+c\epsilon_2+d\epsilon_1\epsilon_2 \arrow[d, mapsto] \\
\mu^2(1-\epsilon_1 f,1) \arrow[r, equal] & a+b\epsilon_1
\end{tikzcd}$$

Here, the vertical assignment is base change, from k_ϵ to $k_\epsilon/(\epsilon_2)$. Note that the lower-left element is 1 (by bimultiplicativity of μ^2), so that we get $a=1$ and $b=0$. Similarly, one gets $c=0$.

(c) We notice that $R(f,g)$ is bilinear. The biadditivity follows immediately from the bimultiplicativity of μ^2 and (b). Next, let us show that $R(\alpha f,g)=\alpha R(f,g)$

for every $\alpha \in k$ (the homogeneity in the second variable is shown analogously). In case $\alpha = 0$, it is clear. Otherwise, we get the equality by performing "base change", sending $\epsilon_1 \mapsto \alpha^{-1}\epsilon_1$.

(d) We now show the following properties, from which the statement follows by decomposing elements of F into Laurent expansions:

(1) $R(z^n, z^m) = 0$ for $n, m \in \mathbb{Z}$, provided $n + m \neq 0$.

(2) $R(z^{-n}, z^n) = n$ for $n \in \mathbb{Z}$.

(3) $R(z^{-n}, f) = 0$ for $n \in \mathbb{Z}_{\geq 0}$, provided that $v(f) \gg n$.

Consider the automorphism $z \mapsto \alpha z$, where $\alpha \in k^\times$. We notice that it does not alter the symbol μ^2, since it commutes with passing to the quotient $A \mapsto A/\mathfrak{m}$. Thus, we have $R(z^n, z^m) = R((\alpha z)^n, (\alpha z)^m)$. By bilinearity (see (c) above), we get $R(z^n, z^m) = \alpha^{n+m} R(z^n, z^m)$. Choosing α so that $\alpha^{n+m} \neq 1$, we conclude $R(z^n, z^m) = 0$. Such a choice of α is possible since k is infinite and $n + m \neq 0$.

To show the second item, note that

$$\mu^2(1 - \epsilon_1 z^{-n}, 1 - \epsilon_2 z^n) = \frac{\mu^2(z^n - \epsilon_1, 1 - \epsilon_2 z^n)}{\mu^2(z^n, 1 - \epsilon_2 z^n)},$$

and hence it is enough to calculate $\mu^2(z^n - \alpha\epsilon_1, 1 - \epsilon_2 z^n)$ (where $\alpha \in k$). By Lemmas B.3 and B.4, we should calculate the determinant of multiplication by $1 - \epsilon_2 z^n$ on the cohomology of

$$\begin{array}{ccc} A_\epsilon & \xrightarrow{z^n - \alpha\epsilon_1} & A_\epsilon \\ -1 & & 0 \end{array}$$

The only nonzero cohomology is the 0-th one. It is a free k_ϵ-module (with basis $1, z, \ldots, z^{n-1}$). Multiplication by $1 - \epsilon_2 z^n$ is just multiplication by $1 - \alpha\epsilon_1\epsilon_2$. Thus, the determinant equals $(1 - \alpha\epsilon_1\epsilon_2)^n = 1 - n\alpha\epsilon_1\epsilon_2$, and consequently $R(z^{-n}, z^n) = n$.

The third item is verified similarly to the second one (when $v(f) \gg n$, the operator whose determinant we should consider is just the identity).

(e) By breaking f and g into sums of monomials in z and a reminder of large enough valuation, the proposition follows from (b), (c), and (d) . $\square$

Remark 4.4. One could also obtain the residue symbol differently, by considering $k_\epsilon := k[\epsilon]/(\epsilon^3)$. Then $\mu^2(1 - \epsilon f, 1 - \epsilon g) = 1 - \epsilon^2 \operatorname{Res}(f\,dg)$.

4B. ***The case* $\dim(X) > 1$.** In this subsection, we drop the assumption that A is 1-dimensional. We denote the Krull dimension of A by n.

The Parshin symbol. Fix a full flag

$$\mathcal{F} : x_n < \cdots < x_0$$

in X, corresponding to a chain of prime ideals

$$0 = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m}.$$

Consider $X^i := \overline{x_i}$ as an integral closed subscheme of X. We obtain a symbol

$$\mu = \mu_{\mathcal{F}} : K(F) \to \Sigma^n K(k).$$

As with the Parshin reciprocity law (see page 34), we consider the resulting map $\mu^{n+1}_{\mathcal{F}} : \bigwedge^{n+1} F^\times \to k^\times$. There, we essentially wrote a formula for this map (which we now want to verify) under the assumption that our flag is regular. In order to compute this map "recursively", we will use Quillen's dévissage (Lemma B.5) — application of which will be possible due to regularity of $\mathcal{F}$.

Claim 4.5. *The symbol $\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k)$ equals the composition*

$$\begin{tikzcd}
K(F_X) \arrow[r] & \Sigma K(X_{x_1} \textit{ on } x_1) & \Sigma K(F_{X^1}) \arrow[l, "\sim"'] \arrow[r] & \Sigma^2 K(X^1_{x_2} \textit{ on } x_2) \\
K(F_{X^2}) \arrow[urrr, bend left, "\sim"] \arrow[r] & \cdots \arrow[r] & \Sigma^n K(F_{X^n}) \arrow[r] & \Sigma^n K(k)
\end{tikzcd}$$

where the arrows $\xleftarrow{\sim}$ stand for Quillen's dévissage.

In view of this claim, $\mu^{n+1}_{\mathcal{F}}$ equals the composition

$$\textstyle\bigwedge^{n+1} F^\times_X \longrightarrow K_{n+1}(F_X) \xrightarrow{\partial_0} K_n(F_{X_1}) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} K_1(F_{X_n}) \to K_1(k),$$

where ∂_i is the composition of the boundary map and the inverse of the dévissage.

The following lemma will allow us, in principle, to calculate $\mu^{n+1}_{\mathcal{F}}(f_1, \ldots, f_{n+1})$ for any $f_1, \ldots, f_{n+1} \in F^\times$.

Lemma 4.6. *Let R be a 1-dimensional regular local Noetherian ring with maximal ideal $\mathfrak{n}$, residue field ℓ, and fraction field L. Let $z \in R$ be a uniformizer. Consider the composition of the boundary map with the dévissage map*

$$K(L) \longrightarrow \Sigma K(R \textit{ on } \mathfrak{n}) \xleftarrow{\sim} \Sigma K(\ell).$$

We use it to construct a map

$$\nu^m : \textstyle\bigwedge^m L^\times \to K_m(L) \to K_{m-1}(\ell).$$

The following hold:

(i) $\nu^m(f_1, \ldots, f_m) = 0$ *for* $f_1, \ldots, f_m \in R^\times$.

(ii) $\nu^m(f_1, \ldots, f_{m-2}, z, z) = \nu^m(f_1, \ldots, f_{m-2}, -1, z)$ *for* $f_1, \ldots, f_{m-2} \in R^\times$.

(iii) $\nu^m(f_1, \ldots, f_{m-1}, z) = [f_1] \wedge \cdots \wedge [f_{m-1}]$ *for* $f_1, \ldots, f_{m-1} \in R^\times$ (recall that $[f]$ denotes the residue in $\ell^\times$ of $f \in R^\times$, considered as an element of $K_1(\ell)$ in the case at hand).

Proof. The first item is clear, since $\nu^m(f_1, \ldots, f_m)$ is the value of the composition $K_m(R) \to K_m(L) \to K_{m-1}(R \text{ on } \mathfrak{n})$ on $f_1 \wedge \cdots \wedge f_m \in K_m(R)$, and the composition is zero as part of a long exact sequence.

The second item follows from the Steinberg relation (as in the proof of Claim 4.2).

The third item follows from the commutativity of the following diagram:

$$\begin{tikzcd}
K_{m-1}(R) \wedge K_1(L) \arrow[r] \arrow[dd] & K_{m-1}(R) \wedge K_0(R \text{ on } \mathfrak{n}) \arrow[dd] & K_{m-1}(R) \wedge K_0(\ell) \arrow[l, "\sim"'] \arrow[d] \\
 & & K_{m-1}(\ell) \wedge K_0(\ell) \arrow[d] \\
K_m(L) \arrow[r] & K_{m-1}(R \text{ on } \mathfrak{n}) & K_{m-1}(\ell) \arrow[l, "\sim"']
\end{tikzcd}$$

Here the left square commutes since the boundary morphism is a morphism of $K(A)$-modules, while the right square commutes as Quillen's dévissage morphism is a morphism of $K(A)$-modules.

Note that the element $\nu^m(f_1, \ldots, f_{m-1}, z)$ is the result of going right on the lower line, applied to $f_1 \wedge \cdots \wedge f_{m-1} \wedge z$. However, this element comes from an element at the upper-left corner, which we can chase through the right on the upper line, and then to the lower-left corner through the right line. $\square$

4C. *Auxiliary lemmas.* We state two lemmas which are used above, and whose proofs are straightforward.

Lemma 4.7. *Let $X \to B$ be as in Section 2A. Let*

$$\mathcal{F} : x_n < x_{n-1} < \cdots < x_0 = \gamma$$

be a full flag of points in $|X|$. Writing $p := x_n$, we consider also the setting $X_p \to B$ and the obvious flag $\mathcal{F}_p$ on X_p induced by $\mathcal{F}$. We have two symbol maps:

$$\mu_{\mathcal{F}} : K(F) \to \Sigma^n K(k) \quad \textit{and} \quad \mu_{\mathcal{F}_p} : K(F) \to \Sigma^n K(k)$$

(note that the function field of X_p is identified with F). *Then these two symbol maps are equal.*

Lemma 4.8. *Let A be a 1-dimensional regular local Noetherian k-algebra whose residue field is finite over k, and let $\hat{A}$ be its completion. We write, as usual, $X = \operatorname{Spec}(A)$ and $B = \operatorname{Spec}(k)$, and also $\hat{X} = \operatorname{Spec}(\hat{A})$. Also, denote by F and $\hat{F}$ the fraction fields of A and $\hat{A}$, respectively. Associated to the unique full flags in*

X and $\hat{X}$ we have the symbols $K(F) \to \Sigma K(k)$ and $K(\hat{F}) \to \Sigma K(k)$. Then the diagram

$$\begin{tikzcd} K(F) \arrow[rr] \arrow[dr] & & K(\hat{F}) \arrow[dl] \\ & \Sigma K(k) & \end{tikzcd}$$

commutes.

Appendix A: Infinite sums of maps of spectra

In this paper, we consider spectra as a triangulated category $\mathcal{S}p$. We recall that a spectrum is called compact if maps from it commute with small direct sums. An example of a compact spectrum is $\Sigma^k \mathbb{S}$, a suspension of the sphere spectrum. The following definitions are inspired by [Clausen 2012, Appendix A].

Definition A.1. Let $f_i : \mathcal{S} \to \mathcal{T}$ $(i \in I)$ be a family of maps of spectra, and $f : \mathcal{S} \to \mathcal{T}$ an additional map. We say that f *is the sum of the* f_i (written $f = \sum_{i \in I} f_i$) if for every compact spectrum $\mathcal{C}$, and every element $e \in \mathrm{Hom}_{\mathcal{S}p}(\mathcal{C}, \mathcal{S})$, almost all (i.e., all but finitely many) of the maps $f_i \circ e$ are equal to zero, and the sum of all these $f_i \circ e$ is equal to $f \circ e$.

We note that we do not claim uniqueness of the sum (in whatever sense). In reality, this notion of "summability and summation on compact probes" is derived from a more holistic notion:

Definition A.2. Let $f_i : \mathcal{S} \to \mathcal{T}$ $(i \in I)$ be a family of maps of spectra, and $f : \mathcal{S} \to \mathcal{T}$ an additional map. An *evidence* for f being the sum of the f_i is a map

$$g : \mathcal{S} \to \bigvee_{i \in I} \mathcal{T}$$

such that when we compose g with the projection to the i-th summand we get f_i, while when we compose g with the fold map, we get f.

The following is evident:

Claim A.3. *Existence of an evidence for f being the sum of the f_i implies that f is the sum of the f_i.*

Let us also note the following two auxiliary claims (whose proofs are straightforward):

Claim A.4. *Let $h : \mathcal{U} \to \mathcal{S}$ and $g : \mathcal{T} \to \mathcal{V}$. If f is the sum of the f_i (we have evidence for f being the sum of the f_i), then $g \circ f \circ h$ is the sum of the $g \circ f_i \circ h$ (we have evidence for $g \circ f \circ h$ being the sum of the $g \circ f_i \circ h$).*

Claim A.5. *Let $\mathcal{S}_i$ $(i \in I)$ be a collection of spectra, and write $\mathcal{S} = \bigvee_{i \in I} \mathcal{S}_i$. Then we have evidence for* id *being the sum of* pr_i $(i \in I)$, *where* id *is the identity morphism of $\mathcal{S}$, while* pr_i *is the morphism of projection on the i-th summand. In particular,* $\mathrm{id} = \sum_{i \in I} \mathrm{pr}_i$.

Appendix B: *K*-theory calculation lemmas

We state some lemmas which are of use when calculating the concrete symbols. In what follows, X is a Noetherian scheme, $U \subset X$ an open subscheme, and Z the closed complement.

We denote by $\mathrm{SPerf}(X)$ the category of (strictly) bounded complexes of $\mathcal{O}_X$-modules, whose terms are locally free of finite rank. By $\mathrm{SPerf}(X \text{ on } Z)$ we denote the full subcategory of $\mathrm{SPerf}(X)$ consisting of complexes whose cohomologies are supported on Z.

Fact B.1. *There is a natural map from (the geometric realization of) the core groupoid of* $\mathrm{SPerf}(X)$ *to $K(X)$. In particular, every object in* $\mathrm{SPerf}(X)$ *defines a point in $K(X)$. In addition, the automorphism group of any object of* $\mathrm{SPerf}(X)$ *maps into $K_1(X)$. Since $\mathcal{O}(X)^\times$ maps into the automorphism group of the object* $\mathcal{O}_X \in \mathrm{SPerf}(X)$, *one then has a map $\mathcal{O}(X)^\times \to K_1(X)$. Thus, given an object or an automorphism in* $\mathrm{SPerf}(X)$, *one can view it as an element of an appropriate K-group $K_i(X)$. We will abuse this without further notice.*

Claim B.2. *Let X be local (i.e., the spectrum of a local ring). Then the above map $\mathcal{O}(X)^\times \to K_1(X)$ is an isomorphism.*

Lemma B.3. *Let $f \in \mathcal{O}(X)$ be such that $f|_U$ is invertible. Then the image of $f|_U \in \mathcal{O}(U)^\times$ under the map $K_1(U) \to K_0(X \text{ on } Z)$ which is obtained from the localization sequence*

$$K(X \text{ on } Z) \to K(X) \to K(U)$$

(see [Thomason and Trobaugh 1990, Theorem 7.4]) *is given by the complex*

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{f} & \mathcal{O}_X \\ -1 & & 0 \end{array}$$

Lemma B.4. *Let $f \in \mathcal{O}(X)^\times$, and $C \in \mathrm{SPerf}(X \text{ on } Z)$. Then the image of $f \wedge C$ under the product map $K_1(X) \wedge K_0(X \text{ on } Z) \to K_1(X \text{ on } Z)$ is given by the automorphism*

$$C \otimes \mathcal{O}_X \xrightarrow{1 \otimes f} C \otimes \mathcal{O}_X.$$

Lemma B.5 (Quillen's dévissage). *Suppose that X and Z are regular. Then the morphism $K(Z) \to K(X \text{ on } Z)$ (induced by pushforward) is an equivalence of spectra.*

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On the cycle map of a finite group

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Let p be an odd prime number. We show that there exists a finite group of order p^{p+3} for which the mod p cycle map from the mod p Chow ring of its classifying space to its ordinary mod p cohomology is not injective.

1. Introduction

The Chow group $\mathrm{CH}^i X$ of a smooth algebraic variety X is the group of finite $\mathbb{Z}$-linear combinations of closed subvarieties of X of codimension i modulo rational equivalence and $\bigoplus_{i\geq 0}\mathrm{CH}^i X$, called the Chow ring of X, is a ring under intersection product. It is an important object of study in algebraic geometry. For a smooth complex algebraic variety, the cycle map is a homomorphism from the Chow ring to the ordinary integral cohomology of the underlying topological space. Thus, the cycle map relates algebraic geometry to algebraic topology. Totaro [1999] considered the Chow ring of the classifying space BG of an algebraic group G. In his recently published book, for each prime number p Totaro [2014] gave an example of a finite group K of order p^{2p+1} such that the mod p cycle map

$$\mathrm{cl}:\mathrm{CH}^2 BK/p \to H^4(BK)$$

is not injective, where $H^*(-)$ is the ordinary mod p cohomology and the finite group K is regarded as a complex algebraic group. Totaro wrote "…but there are probably smaller examples" in his book.

In this paper, we find a smaller example, possibly the smallest one. To be precise, we construct a finite group H of order p^{p+3} to prove the following result:

Theorem 1.1. *For each prime number p, there exists a finite group H of order p^{p+3} such that the mod p cycle map $\mathrm{cl}:\mathrm{CH}^2 BH/p \to H^4(BH)$ is not injective, where the finite group H is regarded as a complex algebraic group.*

For a complex algebraic group G, the following results were obtained by Totaro [1999, Corollary 3.5] using Merkurjev's theorem:

(1) $\mathrm{CH}^2 BG$ is generated by Chern classes.

(2) $\mathrm{CH}^2 BG \to H^4(BG;\mathbb{Z})$ is injective.

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Thus, we may use the ordinary integral cohomology and Chern classes to study the Chow group $\mathrm{CH}^2 BG$. A problem concerning the Chow group $\mathrm{CH}^2 BG$ in algebraic geometry could be viewed as a problem on the Chern subgroup of the ordinary integral cohomology $H^4(BG;\mathbb{Z})$, that is, the subgroup of $H^4(BG;\mathbb{Z})$ generated by Chern classes of complex representations of G, in classical algebraic topology. In what follows, we consider $\mathrm{CH}^2 BG$ as the Chern subgroup of the integral cohomology $H^4(BG;\mathbb{Z})$ and the mod p cycle map $\mathrm{CH}^2 BG/p \to H^4(BG)$ as the homomorphism induced by the mod p reduction $\rho : H^4(BG;\mathbb{Z}) \to H^4(BG)$. Since we consider the ordinary integral and mod p cohomology only, the group G could be a topological group and it need not be a complex algebraic group.

Throughout the rest of this paper, we assume that p is an odd prime number unless otherwise stated explicitly. Let p_+^{1+2} be the extraspecial p-group of order p^3 with exponent p. We consider it as a subgroup of the special unitary group $\mathrm{SU}(p)$. We will define a subgroup H_2 of $\mathrm{SU}(p)$ in Section 2. The group H in Theorem 1.1 is given in terms of p_+^{1+2} and H_2, that is,

$$H = p_+^{1+2} \times H_2/\langle \Delta(\xi)\rangle,$$

where $\langle \Delta(\xi)\rangle$ is a cyclic group in the center of $\mathrm{SU}(p) \times \mathrm{SU}(p)$. We define the group G as

$$G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi)\rangle.$$

We will give the detail of G, H and H_2 in Section 2. What we prove in this paper is the following theorem:

Theorem 1.2. *Let p be an odd prime number. Let K be a subgroup of*

$$G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi)\rangle$$

containing

$$H = p_+^{1+2} \times H_2/\langle \Delta(\xi)\rangle.$$

Then the mod p cycle map $\mathrm{cl} : \mathrm{CH}^2 BK/p \to H^4(BK)$ *is not injective.*

The order of the group $p_+^{1+2} \times H_2/\langle \Delta(\xi)\rangle$ is p^{p+3} and it is the group H in Theorem 1.1. Applying Theorem 1.2 to

$$K = p_+^{1+2} \times ((\mathbb{Z}/p^2)^{p-1} \rtimes \mathbb{Z}/p)/\langle \Delta(\xi)\rangle,$$

we obtain the example in [Totaro 2014, Section 15]. Thus our result not only gives a smaller group whose mod p cycle map is not injective but it extends Totaro's result. For $p = 2$, Theorem 1.1 was proved by Totaro [2014, Theorem 15.13]. For $p = 2$, the finite group H is the extraspecial 2-group 2_+^{1+4} of order 2^5. It is not difficult to see that we cannot replace H_2 by the extraspecial p-group p_+^{1+2} in Theorem 1.2. See Remark 6.3. This observation leads us to the following conjecture:

Conjecture 1.3. *Let p be a prime number. For a finite p-group K of order less than p^{p+3}, the mod p cycle map* $\mathrm{cl} : \mathrm{CH}^2 BK/p \to H^4(BK)$ *is injective.*

This paper is organized as follows: In Section 2, we define groups that we use in this paper, including G and H above. In Section 3, we recall the cohomology of the classifying space of the projective unitary group $\mathrm{PU}(p)$ up to degree 5. In Section 3, we prove that the mod p cycle map $\mathrm{CH}^2 BG/p \to H^4(BG)$ is not injective and describe its kernel. In Section 4, we collect some properties of the mod p cohomology of $B\tilde{\pi}(H_2)$, where $\tilde{\pi}$ is the restriction of the projection from $\mathrm{SU}(p)$ to $\mathrm{PU}(p)$. We use the mod p cohomology of $B\tilde{\pi}(H_2)$ in Section 5, where we study the mod p cycle map $\mathrm{CH}^2 BH/p \to H^4(BH)$ to complete the proof of Theorem 1.2.

Throughout the rest of this paper, by abuse of notation, we denote the map between classifying spaces induced by a group homomorphism $f : G \to G'$ by $f : BG \to BG'$.

2. Subgroups and quotient groups

In this section, we define subgroups of the unitary group $U(p)$ and of the product $\mathrm{SU}(p) \times \mathrm{SU}(p)$ of special unitary groups $\mathrm{SU}(p)$. We also define their quotient groups. For a finite subset $\{x_1, \dots, x_r\}$ of a group, we denote by $\langle x_1, \dots, x_r\rangle$ the subgroup generated by $\{x_1, \dots, x_r\}$. As we already mentioned, we assume that p is an odd prime number.

We start with subgroups of the special unitary group $\mathrm{SU}(p)$. Let $\xi = \exp(2\pi i/p)$, $\omega = \exp(2\pi i/p^2)$ and $\delta_{ij} = 1$ if $i \equiv j \bmod p$, $\delta_{ij} = 0$ if $i \not\equiv j \bmod p$. We consider the following matrices in $\mathrm{SU}(p)$:

$$\begin{aligned}
\xi &= (\xi\delta_{ij}) = \mathrm{diag}(\xi, \dots, \xi),\\
\alpha &= (\xi^{i-1}\delta_{ij}) = \mathrm{diag}(1, \xi, \dots, \xi^{p-1}),\\
\beta &= (\delta_{i,j-1}),\\
\sigma_1 &= \mathrm{diag}(\omega\xi^{p-1}, \omega, \dots, \omega).
\end{aligned}$$

Moreover, let σ_k be the diagonal matrix whose (i, i)-entry is $\omega\xi^{p-1}$ for $i = k$ and ω for $i \neq k$. Let us consider the following subgroups of $\mathrm{SU}(p)$:

$$\begin{aligned}
p_+^{1+2} &= \langle \alpha, \beta, \xi\rangle,\\
H_2 &= \langle \beta, \sigma_1, \dots, \sigma_p\rangle.
\end{aligned}$$

The group p_+^{1+2} is the extraspecial p-group of order p^3 with exponent p. Since $\sigma_1^p = \dots = \sigma_p^p = \xi$ and

$$\sigma_2\sigma_3^2 \cdots \sigma_p^{p-1} = \xi^{(p-1)/2}\alpha^{-1},$$

the group H_2 contains p_+^{1+2} as a subgroup. An element in the subgroup of H_2 generated by $\sigma_1, \dots, \sigma_p$ could be described as

$$\omega^j \operatorname{diag}(\xi^{i_1}, \dots, \xi^{i_p}),$$

where $0 \le j \le p-1,\ 0 \le i_1 \le p-1,\ \dots,\ 0 \le i_p \le p-1$ and $i_1 + \dots + i_p$ is divisible by p. So, the order of this subgroup is p^p. Since β acts on the subgroup of diagonal matrices as a cyclic permutation, the order of H_2 is p^{p+1}.

We write A_2 for the quotient group $p_+^{1+2}/\langle \xi \rangle$. The group A_2 is an elementary abelian p-group of rank 2. We denote by $\tilde{\pi}$ the obvious projection $\mathrm{SU}(p) \to \mathrm{PU}(p)$ and projections induced by this projection, e.g, $\tilde{\pi} : p_+^{1+2} \to \tilde{\pi}(p_+^{1+2}) = A_2$. We denote the obvious inclusions among p_+^{1+2}, H_2 and $\mathrm{SU}(p)$ and among A_2, $\tilde{\pi}(H_2)$ and $\mathrm{PU}(p)$ by ι.

Let us consider the following maps:

$$\Delta : \mathrm{SU}(p) \to \mathrm{SU}(p) \times \mathrm{SU}(p), \quad m \mapsto \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}.$$

$$\Gamma_1 : \mathrm{SU}(p) \to \mathrm{SU}(p) \times \mathrm{SU}(p), \quad m \mapsto \begin{pmatrix} m & 0 \\ 0 & I \end{pmatrix}.$$

$$\Gamma_2 : \mathrm{SU}(p) \to \mathrm{SU}(p) \times \mathrm{SU}(p), \quad m \mapsto \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix}.$$

Using these maps and matrices in $\mathrm{SU}(p)$ above, we consider the following groups:

$$\begin{aligned} G &= \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi) \rangle, \\ H &= \big\langle \Delta(\alpha), \Delta(\beta), \Delta(\xi), \Gamma_2(\beta), \Gamma_2(\sigma_1), \dots, \Gamma_2(\sigma_p) \big\rangle / \langle \Delta(\xi) \rangle, \\ A_3 &= \big\langle \Delta(\alpha), \Delta(\beta), \Delta(\xi), \Gamma_2(\xi) \big\rangle / \langle \Delta(\xi) \rangle, \\ A_3' &= \big\langle \Gamma_1(\alpha), \Gamma_2(\beta), \Delta(\xi), \Gamma_2(\xi) \big\rangle / \langle \Delta(\xi) \rangle. \end{aligned}$$

Since α and β are in H_2, the subgroup

$$\big\langle \Delta(\alpha), \Delta(\beta), \Delta(\xi), \Gamma_2(\beta), \Gamma_2(\sigma_1), \dots, \Gamma_2(\sigma_p) \big\rangle$$

contains

$$\Gamma_1(\alpha) = \Delta(\alpha)\Gamma_2(\alpha^{-1}), \quad \Gamma_1(\beta) = \Delta(\beta)\Gamma_2(\beta^{-1}), \quad \Gamma_1(\xi) = \Delta(\xi)\Gamma_2(\xi^{-1}).$$

Therefore, it is equal to the subgroup

$$p_+^{1+2} \times H_2 = \big\langle \Gamma_1(\alpha), \Gamma_1(\beta), \Gamma_1(\xi), \Gamma_2(\beta), \Gamma_2(\sigma_1), \dots, \Gamma_2(\sigma_p) \big\rangle.$$

Hence, we have

$$H = p_+^{1+2} \times H_2 / \langle \Delta(\xi) \rangle.$$

We denote the obvious inclusion of H by $f : H \to G$. It is also clear that A_3 and A_3' are elementary abelian p-subgroups of rank 3. We use the elementary abelian p-subgroup A_3' only in the proof of Proposition 6.4. In the above groups, $\Gamma_1(\xi) = \Gamma_2(\xi)$. We denote by π the obvious projections induced by $\pi : G \to \mathrm{PU}(p) \times \mathrm{PU}(p)$. It is clear that

$$\pi(H) = H/\langle\Gamma_2(\xi)\rangle = A_2 \times \tilde{\pi}(H_2)$$

and

$$\mathrm{PU}(p) \times \mathrm{PU}(p) = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle\Delta(\xi), \Gamma_2(\xi)\rangle.$$

Moreover, we have the following commutative diagram:

$$\begin{tikzcd}
A_3 \arrow[r, "g"] \arrow[d, "\varphi"'] & H \arrow[d, "\pi"] & A_3' \arrow[l, "g'"'] \arrow[d, "\varphi'"] \\
A_2 \arrow[r, "g"] & A_2 \times \tilde{\pi}(H_2) & A_2 \arrow[l, "g'"']
\end{tikzcd}$$

where the upper g and g' are the obvious inclusions, $A_2 = \langle\tilde{\pi}(\alpha), \tilde{\pi}(\beta)\rangle$,

$$\begin{aligned}
\varphi(\Delta(\alpha)) &= \tilde{\pi}(\alpha), & \varphi(\Delta(\beta)) &= \tilde{\pi}(\beta), \\
\varphi'(\Gamma_1(\alpha)) &= \tilde{\pi}(\alpha), & \varphi'(\Gamma_2(\beta)) &= \tilde{\pi}(\beta), \\
g(\tilde{\pi}(\alpha)) &= (\tilde{\pi}(\alpha), \tilde{\pi}(\alpha)), & g(\tilde{\pi}(\beta)) &= (\tilde{\pi}(\beta), \tilde{\pi}(\beta)), \\
g'(\tilde{\pi}(\alpha)) &= (\tilde{\pi}(\alpha), 1), & g'(\tilde{\pi}(\beta)) &= (1, \tilde{\pi}(\beta)).
\end{aligned}$$

We end this section by considering another subgroup H_2' of the unitary group $U(p)$ and its quotient group $\tilde{\pi}(H_2')$, which is a subgroup of $\mathrm{PU}(p)$. We use H_2' and $\tilde{\pi}(H_2')$ only in the proof of Proposition 5.2. Let T^p be the set of all diagonal matrices in $U(p)$, which is a maximal torus of $U(p)$. We define $H_2' = T^p \rtimes \mathbb{Z}/p$ as the subgroup generated by T^p and β. It is clear that $\tilde{\pi}(H_2)$ is a subgroup of $\tilde{\pi}'(H_2') \subset \mathrm{PU}(p)$, where we denote by $\tilde{\pi}'$ the obvious projection $U(p) \to \mathrm{PU}(p)$.

3. The cohomology of $B\,\mathrm{PU}(p)$

In this section, we recall the integral and mod p cohomology of $B\,\mathrm{PU}(p)$. Throughout the rest of this paper, we denote the integral cohomology of a space X by $H^*(X;\mathbb{Z})$ and its mod p cohomology by $H^*(X)$. Also, we denote the mod p reduction by

$$\rho : H^*(X;\mathbb{Z}) \to H^*(X).$$

We also define generators $u_2 \in H^2(B\,\mathrm{PU}(p))$ and $z_1 \in H^1(B\langle\xi\rangle)$ with $d_2(z_1) = x_1y_1$, $d_2(z_1) = u_2$ and $\iota^*(u_2) = x_1y_1$, where $x_1,\ y_1 \in H^1(BA_2)$ are generators corresponding to α and β in $\pi_1(BA_2) = \langle\tilde{\pi}(\alpha), \tilde{\pi}(\beta)\rangle$, and the d_2 are differentials in the Leray–Serre spectral sequence associated with the vertical fibrations $\tilde{\pi}$ in

$$\begin{tikzcd} Bp_+^{1+2} \arrow[r, "\iota"] \arrow[d, "\tilde{\pi}"'] & B\,\mathrm{SU}(p) \arrow[d, "\tilde{\pi}"] \\ BA_2 \arrow[r, "\iota"] & B\,\mathrm{PU}(p) \end{tikzcd} \tag{3.1}$$

where vertical maps are induced by the obvious projections and horizontal maps are induced by the obvious inclusions.

First, we set up notations related to the spectral sequence. Let

$$\pi : X \to B$$

be a fibration. Since the base space B is usually clear from the context, we write $E_r^{s,t}(X)$ for the Leray–Serre spectral sequence associated with the above fibration converging to the mod p cohomology $H^*(X)$. If it is clear from the context, we write $E_r^{s,t}$ for the Leray–Serre spectral sequence. We denote by

$$H^{s+t}(X) = F^0H^{s+t}(X) \supseteq F^1H^{s+t}(X) \supseteq \cdots \supseteq F^{s+t+1}H^{s+t}(X) = \{0\}$$

the filtration on $H^{s+t}(X)$ associated with the spectral sequence. Unless otherwise stated explicitly, by abuse of notation, we denote the cohomology class and the element it represents in the spectral sequence by the same symbol. Usually, it is clear from the context whether we mean the cohomology class or the element in the spectral sequence. Let R be an algebra or a graded algebra. Let $\{x_1, \dots, x_r\}$ be a finite set. We denote by $R\{x_1, \dots, x_r\}$ the free R-module spanned by $\{x_1, \dots, x_r\}$. For a graded module M, we say M is a free R-module up to degree m if the R-module homomorphism

$$f : (R\{x_1, \dots, x_r\})^i \to M^i$$

is an isomorphism for $i \leq m$ for some finite subset $\{x_1, \dots, x_r\}$ of M. We say a spectral sequence collapses at the E_r-level up to degree m if $E_r^{s,t} = E_\infty^{s,t}$ for $s + t \leq m$.

Next, we recall the integral and mod p cohomology of $B\,\mathrm{PU}(p)$. The mod 3 cohomology of $B\,\mathrm{PU}(3)$ was computed by Kono, Mimura and Shimada [Kono et al. 1975]. The integral and mod p cohomology of $B\,\mathrm{PU}(p)$ was computed by Vistoli [2007]. The mod p cohomology was computed by Kameko and Yagita [2008] independently. The computation up to degree 5 was also done by Antieau and Williams [2014]. Although the direct computation is not difficult, we prove the following proposition by direct computation because it is slightly different from the one in [Antieau and Williams 2014].

Proposition 3.2. *Up to degree* 5, *the integral cohomology of* $B\,\mathrm{PU}(p)$ *is given by*

$$H^i(B\,\mathrm{PU}(p);\mathbb{Z}) = \{0\} \quad \textit{for } i = 1, 2, 5,$$
$$H^i(B\,\mathrm{PU}(p);\mathbb{Z}) = \mathbb{Z}/p \quad \textit{for } i = 3,$$
$$H^i(B\,\mathrm{PU}(p);\mathbb{Z}) = \mathbb{Z} \quad \textit{for } i = 0, 4.$$

Up to degree 5, *the mod p cohomology of* $B\,\mathrm{PU}(p)$ *is given by*

$$H^i(B\,\mathrm{PU}(p)) = \{0\} \quad \textit{for } i = 1, 5,$$
$$H^i(B\,\mathrm{PU}(p)) = \mathbb{Z}/p \quad \textit{for } i = 0, 2, 3, 4.$$

Proof. Consider the Leray–Serre spectral sequence associated with

$$BU(p) \to B\,\mathrm{PU}(p) \to K(\mathbb{Z}, 3)$$

converging to $H^*(B\,\mathrm{PU}(p);\mathbb{Z})$. The integral cohomology of $BU(p)$ is a polynomial algebra generated by Chern classes, that is, $H^*(BU(p);\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_p]$, where $\deg c_i = 2i$. The integral cohomology $H^i(K(\mathbb{Z}, 3);\mathbb{Z})$ of the Eilenberg–Mac Lane space $K(\mathbb{Z}, 3)$ is $\mathbb{Z}$ for $i = 0, 3$ and $\{0\}$ for $i = 1, 2, 4, 5$. We fix a generator u_3 of $H^3(K(\mathbb{Z}, 3);\mathbb{Z})$. Up to degree 5, the only nontrivial E_2-terms are

$$E_2^{0,0} = E_2^{0,2} = \mathbb{Z}, \quad E_2^{0,4} = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad E_2^{3,0} = E_2^{3,2} = \mathbb{Z}.$$

Hence, up to degree 5, the only nontrivial differential is $d_3 : E_3^{0,t} \to E_3^{3,t-2}$, which is given by

$$d_3(c_1) = \alpha_1 u_3, \quad d_3(c_2) = \alpha_2 c_1 u_3,$$

where $\alpha_1, \alpha_2 \in \mathbb{Z}$. Since $B\,\mathrm{PU}(p)$ is simply connected and $\pi_2(B\,\mathrm{PU}(p)) = \mathbb{Z}/p$, by the Hurewicz theorem we have $H_1(B\,\mathrm{PU}(p);\mathbb{Z}) = \{0\}$ and $H_2(B\,\mathrm{PU}(p);\mathbb{Z}) = \mathbb{Z}/p$. By the universal coefficient theorem, we have $H^2(B\,\mathrm{PU}(p);\mathbb{Z}) = \{0\}$ and that $H^3(B\,\mathrm{PU}(p);\mathbb{Z})$ has $\mathbb{Z}/p$ as a direct summand. Therefore, α_1 must be $\pm p$ and $E_3^{3,0} = \mathbb{Z}/p$. The cohomology suspension $\sigma : H^4(BU(p)) \to H^3(U(p))$ maps $\rho(c_2)$ to a nontrivial primitive element in $H^3(U(p))$, but there exists no primitive element in $H^3(\mathrm{PU}(p))$ by the computation due to Baum and Browder [1965]. Hence, in the Leray–Serre spectral sequence $E_r^{s,t}(B\,\mathrm{SU}(p))$, the element $\rho(c_2)$ in $E_2^{0,4}(B\,\mathrm{SU}(p))$ must support a nontrivial differential. Therefore, α_2 is not divisible by p and, up to degree 5, the nontrivial E_3-terms are

$$E_3^{0,0} = E_3^{0,4} = \mathbb{Z}, \quad E_3^{3,0} = \mathbb{Z}/p.$$

As for $E_r^{s,t}(B\,\mathrm{PU}(p))$, we have

$$E_2^{0,0}(B\,\mathrm{PU}(p)) = E_2^{0,2}(B\,\mathrm{PU}(p)) = \mathbb{Z}/p, \quad E_2^{0,4}(B\,\mathrm{PU}(p)) = \mathbb{Z}/p \oplus \mathbb{Z}/p,$$
$$E_2^{3,0}(B\,\mathrm{PU}(p)) = E_2^{3,2}(B\,\mathrm{PU}(p)) = \mathbb{Z}/p,$$

and

$$d_3(\rho(c_1)) = 0, \quad d_3(\rho(c_2)) = \rho(\alpha_2 c_1 u_3) \neq 0.$$

So, we have the desired result. $\square$

With the following proposition, we choose generators

$$z_1 \in H^1(B\langle\xi\rangle), \quad u_2 \in H^2(B\,\mathrm{PU}(p))$$

such that

$$d_2(z_1) = u_2, \quad d_2(z_1) = x_1 y_1$$

in the spectral sequences associated with vertical fiber bundles in (3.1).

Proposition 3.3. *We may choose $u_2 \in H^2(B\,\mathrm{PU}(p))$ such that the induced homomorphism $\iota^* : H^2(B\,\mathrm{PU}(p)) \to H^2(BA_2)$ maps u_2 to $x_1 y_1$.*

Proof. From the commutative diagram (3.1), there exists the induced homomorphism between the Leray–Serre spectral sequences

$$\iota^* : E_r^{s,t}(B\,\mathrm{SU}(p)) \to E_r^{s,t}(Bp_+^{1+2}).$$

Since the group extension

$$\mathbb{Z}/p \to p_+^{1+2} \to A_2$$

corresponds to $x_1 y_1$ in $H^2(BA_2)$, the differential $d_2 : E_2^{0,1}(Bp_+^{1+2}) \to E_2^{2,0}(Bp_+^{1+2})$ is given by

$$d_2(z_1) = x_1 y_1$$

for some $z_1 \in H^1(B\langle\xi\rangle) = \mathbb{Z}/p[z_2] \otimes \Lambda(z_1)$. Hence,

$$d_2 : E_2^{0,1}(B\,\mathrm{SU}(p)) \to E_2^{2,0}(B\,\mathrm{SU}(p))$$

is nontrivial and we may define u_2 by $d_2(z_1)$. Hence, we have the desired result. $\square$

We end this section by computing $H^4(BG; \mathbb{Z})$ for $G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle\Delta(\xi)\rangle$. The following computation was done in the proof of [Totaro 2014, Theorem 15.4].

Proposition 3.4. *Consider a homomorphism*

$$\psi : H^4(BG; \mathbb{Z}) \to H^4(B\,\mathrm{PU}(p); \mathbb{Z}) \oplus H^4(B\,\mathrm{SU}(p); \mathbb{Z})$$

sending x to $(\Delta^*(x), \Gamma_2^*(x))$. It is an isomorphism.

Proof. Let $p_1 : \mathrm{PU}(p) \times \mathrm{PU}(p) \to \mathrm{PU}(p)$ be the projection onto the first factor. Then, the fiber of $p_1 \circ \pi$ is $\mathrm{SU}(p)$. Consider the spectral sequence associated with

$$B\,\mathrm{SU}(p) \xrightarrow{\Gamma_2} BG \xrightarrow{p_1 \circ \pi} B\,\mathrm{PU}(p).$$

The E_2-term is $H^s(B\,\mathrm{PU}(p); H^t(B\,\mathrm{SU}(p); \mathbb{Z}))$. By Proposition 3.2, $E_2^{s,t} = \{0\}$ unless $s = 0, 3, 4$ and $t = 0, 4$ up to degree 5. In particular, $E_2^{s,t} = \{0\}$ for $s + t = 5$. The nonzero E_2-terms of total degree 4 are given by

$$E_2^{4,0} = \mathbb{Z}, \quad E_2^{0,4} = \mathbb{Z}.$$

The nonzero E_2-term of total degree 3 is given by

$$E_2^{3,0} = \mathbb{Z}/p.$$

So, for dimensional reasons, we have $E_\infty^{s,t} = E_2^{s,t}$ for $s + t = 4$. Hence, we have $H^4(BG; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ and a short exact sequence

$$0 \to H^4(B\,\mathrm{PU}(p); \mathbb{Z}) \xrightarrow{(p_1\circ\pi)^*} H^4(BG; \mathbb{Z}) \xrightarrow{\Gamma_2^*} H^4(B\,\mathrm{SU}(p); \mathbb{Z}) \to 0.$$

Since the composition $p_1 \circ \pi \circ \Delta$ is the identity map, this short exact sequence splits and the homomorphism ψ is an isomorphism. □

4. The mod p cycle map for G

Let $G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi)\rangle$, as in Section 2. In this section, we define a virtual complex representation λ'' of G. Using the Chern class $c_2(\lambda'')$, we prove Theorem 1.2 for $K = G$. To be precise, we show that $c_2(\lambda'')$ is nonzero in $\mathrm{CH}^2 BG/p$ and the mod p reduction maps $c_2(\lambda'')$ to 0 in $H^4(BG)$. Theorem 1.2 for $K = G$ was obtained by Totaro [2014] and by the author in [Kameko 2015] independently. From now on, we denote the Bockstein operation of degree 1 by Q_0 and the Milnor operation of degree $2p - 1$ by Q_1. These are cohomology operations on the mod p cohomology.

Let $\lambda_1 : \mathrm{SU}(p) \to U(p)$ be the tautological representation, so that $\lambda_1(g)(v) = gv$ for $v \in \mathbb{C}^p$. Let

$$\lambda_1^* \otimes \lambda_1 : \mathrm{SU}(p) \times \mathrm{SU}(p) \to U(p^2)$$

be the complex representation defined by

$$(\lambda_1^* \otimes \lambda_1)(g_1, g_2)(v_1^* \otimes v_2) = (v_1^* g_1^{-1}) \otimes (g_2 v_2),$$

where $\mathbb{C}^{p^2} = (\mathbb{C}^p)^* \otimes \mathbb{C}^p$ and $(\mathbb{C}^p)^* = \mathrm{Hom}(\mathbb{C}^p, \mathbb{C})$. The complex representation $\lambda_1^* \otimes \lambda_1$ induces a complex representation $\lambda : G \to U(p^2)$. We define a complex representation λ' by $\lambda \circ \Delta \circ p_1 \circ \pi$. Using the complex representations λ and λ', we define a virtual complex representation λ'' by $\lambda'' = \lambda - \lambda'$. An element in the complex representation ring of G corresponds to an element in the topological K-theory $K^0(BG) = [BG, \mathbb{Z} \times BU]$. By abuse of notation, we denote by $\lambda'' : BG \to \mathbb{Z} \times BU$ a map in the homotopy class corresponding to λ''. It is clear that

$$\Delta^*(\lambda'') = 0 \quad \text{and} \quad \Gamma_2^*(\lambda'') = p\lambda_1$$

in the complex representation ring of G.

We denote by x_4 the cohomology class in $H^4(BG;\mathbb{Z})$ such that

(1) $\Gamma_2^*(x_4)=c_2(\lambda_1)$,

(2) $\Delta^*(x_4)=0$.

Then $c_2(\lambda'')=px_4$. Hence, $\rho(c_2(\lambda''))=0$ in $H^4(BG)$. It is clear from the definition that $c_2(\lambda'')\neq 0$ in $H^4(BG;\mathbb{Z})$. Thus, if we show that the Chern class $c_2(\lambda'')$ is not divisible by p in CH^2BG, then $c_2(\lambda'')$ represents a nonzero element in CH^2BG/p and the mod p cycle map is not injective for BG. We prove it by contradiction: Suppose that the Chern class $c_2(\lambda'')$ is divisible by p, that is, we suppose that there exists a virtual complex representation $\mu : BG\to\mathbb{Z}\times BU$ of G such that $x_4\in\operatorname{Im}\mu^*\subset H^4(BG;\mathbb{Z})$. Then $Q_1\rho(x_4)$ must be zero since $H^{\mathrm{odd}}(\mathbb{Z}\times BU)=\{0\}$. We prove the nonexistence of the above virtual complex representation by showing that $Q_1\rho(x_4)\neq 0$. To show that $Q_1\rho(x_4)\neq 0$, we show that $Q_1(f\circ g)^*(\rho(x_4))\neq 0$ in $H^*(BA_3)$, where f, g and A_3 are as defined in Section 2. The following Proposition 4.1 completes the proof of Theorem 1.2 for $K=G$.

We proved $(f\circ g)^*(\rho(x_4))=Q_0(x_1y_1z_1)$ in [Kameko 2015]. Because we use a similar but slightly different argument in the proof of Theorem 1.2 for $K=H$, we prove the following weaker form in this paper:

Proposition 4.1. *We have $Q_1(f\circ g)^*(\rho(x_4))\neq 0$ in $H^{2p+3}(BA_3)$.*

To prove Proposition 4.1, we compute the Leray–Serre spectral sequences and the homomorphism $(f\circ g)^*$ induced by the following commutative diagram:

$$\begin{tikzcd}
BA_3 \arrow[r, "f\circ g"] \arrow[d, "\varphi"'] & BG \arrow[d, "\pi"] & B\,\mathrm{SU}(p) \arrow[l, "\Gamma_2"'] \arrow[d, "\tilde{\pi}"] \\
BA_2 \arrow[r, "f\circ g"] & B\,\mathrm{PU}(p)\times B\,\mathrm{PU}(p) & B\,\mathrm{PU}(p) \arrow[l, "\Gamma_2"']
\end{tikzcd}$$

We denote by x_1 and y_1 the generators of the mod p cohomology of BA_3 corresponding to the generators $\Delta(\alpha)$ and $\Delta(\beta)$ of A_3, so that we have $\varphi^*(x_1)=x_1$ and $\varphi^*(y_1)=y_1$. Let z_1 be the element in $H^1(B\langle\Gamma_2(\xi)\rangle)$ such that $\Gamma_2^*(z_1)=-z_1\in E_2^{0,1}(B\,\mathrm{SU}(p))$. The element z_1 in $E_2^{0,1}(B\,\mathrm{SU}(p))$ and $u_2\in E_2^{2,0}(B\,\mathrm{SU}(p))$ are defined in Section 3, so that $d_2(z_1)=u_2$ in $E_2^{2,0}(B\,\mathrm{SU}(p))$. We define the generator u_3 of $H^3(B\,\mathrm{PU}(p))$ by $u_3=Q_0u_2$. Let us consider the E_2-term of the spectral sequence $E_r^{s,t}(BG)$. The E_2-term is as follows:

$$E_2^{*,*}=H^*(B\,\mathrm{PU}(p))\otimes H^*(B\,\mathrm{PU}(p))\otimes\mathbb{Z}/p[z_2]\otimes\Lambda(z_1).$$

Since $f\circ g=\Delta\circ\iota$, we have $(f\circ g)^*(1\otimes u)=(f\circ g)^*(u\otimes 1)=\iota^*(u)$. Moreover, we have $\Gamma_2^*(1\otimes u)=u$ and $\Gamma_2^*(u\otimes 1)=0$ for $\deg u>0$.

Let $a_i=u_i\otimes 1-1\otimes u_i$, $b_i=u_i\otimes 1$. Then, up to degree 6, the E_2-term is a free $\mathbb{Z}/p[a_2,z_2]\otimes\Lambda(z_1)$-module with basis $\{1,b_2,a_3,b_3,b_2^2,a_3b_3,b_2^3\}$. Since

$(f \circ g)^* d_2(z_1) = 0$ and $\Gamma_2^*(d_2(z_1)) = -u_2$, the first nontrivial differential is given by

$$d_2(z_1) = a_2.$$

So, up to degree 5, the E_3-term is a free $\mathbb{Z}/p[z_2]$-module with basis $\{1, b_2, a_3, b_3, b_2^2\}$. In particular, $a_3 b_2 = 0$ in $E_3^{5,0}$. Since $(f \circ g)^*(d_3(z_2)) = 0$ and $\Gamma_2^*(d_3(z_2)) = -u_3$, the second nontrivial differential is given by

$$d_3(z_2) = a_3.$$

Up to degree 4, the E_4-term is a free $\mathbb{Z}/p$-module with basis $\{1, b_2, b_3, b_2^2, b_2 z_2\}$ and the spectral sequence collapses at the E_4-level. Thus, the E_∞-terms of total degree 4 are as follows:

$$E_\infty^{0,4} = \{0\}, \quad E_\infty^{1,3} = \{0\}, \quad E_\infty^{2,2} = \mathbb{Z}/p\{b_2 z_2\}, \quad E_\infty^{3,1} = \{0\}, \quad E_\infty^{4,0} = \mathbb{Z}/p\{b_2^2\}.$$

The element b_2 is a permanent cocycle. By abuse of notation, we denote by b_2 the cohomology class in $F^2 H^2(BG)$ representing b_2. Since $H^2(B\,\mathrm{SU}(p)) = \{0\}$, we have

$$\Gamma_2^*(\pi^*(b_2)) = 0.$$

Moreover, $\pi^*\big(H^4(B\,\mathrm{PU}(p) \times B\,\mathrm{PU}(p))\big) = \mathbb{Z}/p\{b_2^2\}$. Hence, we have

$$\Gamma_2^*\big(\pi^*\big(H^4(B\,\mathrm{PU}(p) \times B\,\mathrm{PU}(p))\big)\big) = \{0\}.$$

On the other hand, $\Gamma_2^* \rho(x_4) = \rho(c_2(\lambda_1)) \neq 0$ in $H^4(B\,\mathrm{SU}(p))$. Therefore, $\rho(x_4)$ is not in the image of

$$\pi^* : H^4(B\,\mathrm{PU}(p) \times B\,\mathrm{PU}(p)) \to H^4(BG).$$

Hence, we have the following result:

Proposition 4.2. *The cohomology class $\rho(x_4)$ represents $\alpha b_2 z_2$ in $E_\infty^{2,2}$ for some $\alpha \neq 0$ in $\mathbb{Z}/p$.*

Now, we complete the proof of Proposition 4.1 using Proposition 4.2.

Proof of Proposition 4.1. Since $(f \circ g)^*(b_2) = x_1 y_1$, we have

$$(f \circ g)^*(b_2 z_2) = x_1 y_1 z_2$$

in the spectral sequence, where $z_2 = Q_0 z_1$ in $H^2(B\langle \Gamma_2(\xi)\rangle)$. Let $x_2 = Q_0 x_1$ and $y_2 = Q_0 y_1$. Then $H^*(BA_3) = \mathbb{Z}/p[x_2, y_2, z_2] \otimes \Lambda(x_1, y_1, z_1)$ and $\varphi^*(H^*(BA_2))$ is the subalgebra generated by x_1, y_1, x_2, y_2. Therefore, we have

$$(f \circ g)^*(\rho(x_4)) = \alpha x_1 y_1 z_2 + u' z_1 + u''$$

for some $u', u'' \in \varphi^*(H^*(BA_2))$. Let M be the $\varphi^*(H^*(BA_2))$-module generated by

$$1, \quad z_1, \quad z_1z_2, \quad z_2^i \quad \text{and} \quad z_1z_2^i \quad (i \geq 2),$$

so that

$$H^*(BA_3)/M = \varphi^*(H^*(BA_2))\{z_2\}.$$

Since $Q_1z_1 = z_2^p$, $Q_1z_2 = 0$ and Q_1 is a derivation, M is closed under the action of the Milnor operation Q_1. We have

$$(f \circ g)^*(\rho(x_4)) \equiv \alpha x_2^p y_1 z_2 - \alpha x_1 y_2^p z_2 \not\equiv 0 \mod M.$$

This completes the proof of Proposition 4.1. □

5. The mod p cohomology of $B\tilde{\pi}(H_2)$

In this section, we collect some facts on the mod p cohomology of $B\tilde{\pi}(H_2)$ as Propositions 5.1 and 5.2. We use these facts in the proof of Proposition 6.1.

We begin by defining generators of $H^1(B\tilde{\pi}(H_2))$. Since the commutator subgroup $[\tilde{\pi}(H_2), \tilde{\pi}(H_2)]$ is generated by $\tilde{\pi}(\mathrm{diag}(\xi^{a_1}, \dots, \xi^{a_p}))$ for $0 \leq a_i \leq p-1$, $1 \leq i \leq p$, with $a_1 + \cdots + a_p \equiv 0 \mod p$,

$$\tilde{\pi}(H_2)/[\tilde{\pi}(H_2), \tilde{\pi}(H_2)] = \mathbb{Z}/p \oplus \mathbb{Z}/p.$$

This elementary abelian p-group is generated by $\tilde{\pi}(\sigma_1)$ and $\tilde{\pi}(\beta)$. We denote by v_1 and w_1 the generators of $H^1\big(B\langle\tilde{\pi}(\sigma_1)\rangle\big)$ and $H^1\big(B\langle\tilde{\pi}(\beta)\rangle\big)$ corresponding to $\tilde{\pi}(\sigma_1)$ and $\tilde{\pi}(\beta)$, respectively. By abuse of notation, we denote the corresponding generators in $H^1(B\tilde{\pi}(H_2))$ by the same symbol, so that, for the inclusions

$$\iota_\beta : \langle\tilde{\pi}(\beta)\rangle \to \tilde{\pi}(H_2), \quad \iota_\sigma : \langle\tilde{\pi}(\sigma_1)\rangle \to \tilde{\pi}(H_2),$$

we have $\iota_\beta^*(w_1) = w_1$, $\iota_\beta^*(v_1) = 0$, $\iota_\sigma^*(w_1) = 0$ and $\iota_\sigma^*(v_1) = v_1$. Indeed, we have $H^*\big(B\langle\tilde{\pi}(\sigma_1)\rangle\big) = \mathbb{Z}/p[v_2] \otimes \Lambda(v_1)$ and $H^*(B\langle\tilde{\pi}(\beta)\rangle) = \mathbb{Z}/p[w_2] \otimes \Lambda(w_1)$, where $v_2 = Q_0v_1$ and $w_2 = Q_0w_1$. We denote the inclusion of $\tilde{\pi}(H_2)$ to $\mathrm{PU}(p)$ by

$$\iota : \tilde{\pi}(H_2) \to \mathrm{PU}(p)$$

and we recall that we defined the generator u_2 of $H^2(B\,\mathrm{PU}(p))$ in Proposition 3.3.

Proposition 5.1. *In $H^*(B\tilde{\pi}(H_2))$, we have $\iota^*(u_2)v_1 \neq 0$ and $\iota^*(u_2^2) \neq 0$.*

Proof. We consider the Leray–Serre spectral sequences associated with the vertical fibrations in the following commutative diagram:

$$\begin{tikzcd}
B\langle\sigma_1\rangle \arrow[r, "\iota_\sigma"] \arrow[d, "\tilde{\pi}"'] & BH_2 \arrow[r, "\iota"] \arrow[d, "\tilde{\pi}"] & B\,\mathrm{SU}(p) \arrow[d, "\pi"] \\
B\langle\tilde{\pi}(\sigma_1)\rangle \arrow[r, "\iota_\sigma"] & B\tilde{\pi}(H_2) \arrow[r, "\iota"] & B\,\mathrm{PU}(p)
\end{tikzcd}$$

Let $z_1 \in E_2^{0,1}(B\,\mathrm{SU}(p))$ and $u_2 \in E_2^{2,0}(B\,\mathrm{SU}(p))$ be elements defined in Section 3. By abuse of notation, we denote elements $\iota^*(z_1)$ in $E_2^{0,1}(BH_2)$ and $\iota_\sigma^*(\iota^*(z_1))$ in $E_2^{0,1}(B\langle\sigma_1\rangle)$ by z_1. Since $\langle\sigma_1\rangle = \mathbb{Z}/p^2$,

$$d_2(z_1) = \alpha v_2$$

for some $\alpha \neq 0$ in $\mathbb{Z}/p$ in the Leray–Serre spectral sequence $E_2^{2,0}(B\langle\sigma_1\rangle)$. Since $u_2 = d_2(z_1)$ in the Leray–Serre spectral sequence $E_2^{2,0}(B\,\mathrm{SU}(p))$, we have

$$\iota_\sigma^*(\iota^*(u_2)) = d_2(z_1) = \alpha v_2$$

in $H^*\big(B\langle\tilde{\pi}(\sigma_1)\rangle\big) = \mathbb{Z}/p[v_2] \otimes \Lambda(v_1)$. Hence, we have $\iota_\sigma^*(\iota^*(u_2)v_1) = \alpha v_1 v_2 \neq 0$ and $\iota_\sigma^*(\iota^*(u_2^2)) = \alpha^2 v_2^2 \neq 0$. Therefore, we obtain the desired result: $\iota^*(u_2)v_1 \neq 0$ and $\iota^*(u_2^2) \neq 0$ in $H^*(B\tilde{\pi}(H_2))$. □

Proposition 5.2. *In $H^*(B\tilde{\pi}(H_2))$, we have $\iota^*(u_2)w_1 = 0$.*

To prove Proposition 5.2, at the end of Section 2 we defined the subgroup $H_2' = T^p \rtimes \mathbb{Z}/p$ of the unitary group $U(p)$ generated by diagonal matrices and β. The quotient group $\tilde{\pi}'(H_2')$ contains $\tilde{\pi}(H_2)$ as a subgroup and they are subgroups of the projective unitary group $\mathrm{PU}(p)$. We denote by

$$\iota'' : \tilde{\pi}(H_2) \to \tilde{\pi}'(H_2'), \quad \iota' : \tilde{\pi}'(H_2') \to \mathrm{PU}(p)$$

the inclusions, so that $\iota = \iota' \circ \iota''$. We use the following lemma in the proof of Proposition 5.2:

Lemma 5.3. *In $H^*(B\tilde{\pi}'(H_2'))$, there exists an element $t_2 \in H^2(B\tilde{\pi}'(H_2'))$ such that $H^1(B\tilde{\pi}'(H_2')) = \mathbb{Z}/p\{w_1\}$ and $H^2(B\tilde{\pi}'(H_2')) = \mathbb{Z}/p\{t_2, w_2\}$, where $w_2 = Q_0 w_1$, $(\iota''\circ\iota_\sigma)^*(t_2) = v_2$ and $(\iota''\circ\iota_\beta)^*(t_2) = 0$. Moreover, we have $t_2 w_1 = 0$ in $H^*(B\tilde{\pi}'(H_2'))$.*

Now, we prove Proposition 5.2 assuming Lemma 5.3.

Proof of Proposition 5.2. We consider the Leray–Serre spectral sequences associated with the vertical fibrations in the commutative diagram

$$\begin{tikzcd}
B\langle\beta,\xi\rangle \arrow[r, "\iota_\beta"] \arrow[d, "\tilde{\pi}"'] & BH_2 \arrow[r, "\iota"] \arrow[d, "\tilde{\pi}"] & B\,\mathrm{SU}(p) \arrow[d, "\tilde{\pi}"] \\
B\langle\tilde{\pi}(\beta)\rangle \arrow[r, "\iota_\beta"] & B\tilde{\pi}(H_2) \arrow[r, "\iota"] & B\,\mathrm{PU}(p)
\end{tikzcd}$$

Suppose that $\iota'^*(u_2) = \alpha_1 t_2 + \alpha_2 w_2$, where $\alpha_1, \alpha_2 \in \mathbb{Z}/p$. Then, by Lemma 5.3, we have

$$\iota'^*(u_2)w_1 = \alpha_1 t_2 w_1 + \alpha_2 w_1 w_2 = \alpha_2 w_1 w_2.$$

Hence, we have $(\iota \circ \iota_\beta)^*(u_2)w_1 = \alpha_2 w_1 w_2$. On the other hand, since the group extension

$$\langle\xi\rangle \to \langle\beta,\xi\rangle \to \langle\tilde{\pi}(\beta)\rangle$$

is trivial, $d_2 : H^1(B\langle\xi\rangle) \to H^2\big(B\langle\tilde{\pi}(\beta)\rangle\big)$ in $E_2^{s,t}(B\langle\beta,\xi\rangle)$ is zero and

$$(\iota\circ\iota_\beta)^*(u_2) = d_2((\iota\circ\iota_\beta)^*(z_1)) = 0$$

in $H^*\big(B\langle\tilde{\pi}(\beta)\rangle\big) = E_2^{2,0}(B\langle\beta,\xi\rangle)$. Therefore, we have $\alpha_2 = 0$ and $w_1\iota'^*(u_2) = 0$ in $H^*(B\tilde{\pi}'(H_2'))$. Therefore, we have

$$\iota^*(u_2)w_1 = \iota''^*(\iota'^*(u_2)w_1) = 0$$

in $H^*(B\tilde{\pi}(H_2))$. □

We end this section by proving Lemma 5.3.

Proof of Lemma 5.3. We need to study the mod p cohomology only up to degree 3. We define t_2 by $\iota'^*(u_2)$, where u_2 is the generator of $H^2(B\,\mathrm{PU}(p))$.

We consider the Leray–Serre spectral sequence associated with the following commutative diagram:

$$\begin{array}{ccc} BT^p & \xrightarrow{\tilde{\pi}'} & BT^{p-1} \\ \downarrow & & \downarrow \\ BH_2' & \xrightarrow{\tilde{\pi}'} & B\tilde{\pi}'(H_2') \\ \downarrow & & \downarrow \\ B\langle\beta\rangle & \xrightarrow{\tilde{\pi}} & B\langle\tilde{\pi}(\beta)\rangle \end{array}$$

We choose a generator $t_2^{(i)} \in H^2(BT^p)$ corresponding to the i-th diagonal entry of T^p, so that $H^2(BT^p) = \mathbb{Z}/p\{t_2^{(1)},\dots,t_2^{(p)}\}$. The matrix β acts on T^p as the cyclic permutation of diagonal entries, so that it acts on $H^2(BT^p)$ as the cyclic permutation on $t_2^{(1)},\dots,t_2^{(p)}$. The induced homomorphism $\tilde{\pi}'^* : H^2(BT^{p-1}) \to H^2(BT^p)$ is injective and we may take a basis $\{u_2^{(1)},\dots,u_2^{(p-1)}\}$ for $H^2(BT^{p-1})$ such that $\tilde{\pi}'^*(u_2^{(i)}) = t_2^{(i)} - t_2^{(i+1)}$ for $i = 1,\dots,p-1$. Hence, $\langle\beta\rangle$ acts on $H^2(BT^{p-1})$ by

$$gu_2^{(i)} = u_2^{(i+1)}$$

for $i = 1,\dots,p-2$ and

$$gu_2^{(p-1)} = -(u_2^{(1)} + \cdots + u_2^{(p-1)})$$

for some generator g of $\langle\beta\rangle$. We consider the Leray–Serre spectral sequence converging to the mod p cohomology of $B\tilde{\pi}'(H_2')$. The E_1-term is additively given as follows:

$$E_1 = \mathbb{Z}/p[u_2^{(1)},\dots,u_2^{(p-1)}]\{w_2^i, w_1w_2^i \mid i \geq 0\}.$$

The first nontrivial differential is given by

$$d_1(uw_2^i) = ((1-g)u)w_1w_2^i, \quad d_1(uw_1w_2^i) = ((1-g)^{p-1}u)w_2^{i+1},$$

where $u \in \mathbb{Z}/p[u_2^{(1)}, \dots, u_2^{(p-1)}] = E_1^{0,*}$. The kernel of

$$(1-g) : \mathbb{Z}/p\{u_2^{(1)}, \dots, u_2^{(p-1)}\} \to \mathbb{Z}/p\{u_2^{(1)}, \dots, u_2^{(p-1)}\}$$

is spanned by a single element,

$$u_2^{(1)} + 2u_2^{(2)} + \cdots + (p-1)u_2^{(p-1)},$$

and the image of $(1-g)$ is spanned by the $p-2$ elements

$$u_2^{(1)} - u_2^{(2)}, \dots, \; u_2^{(p-2)} - u_2^{(p-1)}.$$

We denote the generator of the kernel of $(1-g)$ by $\tilde{u}$, that is,

$$\tilde{u} = u_2^{(1)} + 2u_2^{(2)} + \cdots + (p-1)u_2^{(p-1)}.$$

It is easy to see that

$$\tilde{u} \equiv (1 + \cdots + (p-1))u_2^{(p-1)} \equiv \tfrac{1}{2}p(p-1)u_2^{(p-1)} \equiv 0$$

modulo the image of $(1-g)$. By direct calculation, we have $(1-g)^{p-1}(u_2^{(1)}) = 0$ and $\mathrm{Ker}(1-g)^{p-1} = \mathbb{Z}/p\{u_2^{(1)}, \dots, u_2^{(p-1)}\}$. Hence, we have

$$\begin{aligned} E_2^{0,2} &= \mathrm{Ker}(1-g) = \mathbb{Z}/p\{\tilde{u}\}, \\ E_2^{1,2} &= \big(\mathrm{Ker}(1-g)^{p-1} / \operatorname{Im}(1-g)\big)\{w_1\} = \mathbb{Z}/p\{u_2^{(1)} w_1\}, \end{aligned}$$

respectively. Moreover, we have $E_r^{*,\mathrm{odd}} = \{0\}$ and $E_r^{*,0} = \mathbb{Z}/p[w_2] \otimes \Lambda(w_1)$ for $* \geq 0$ and $r \geq 1$. Since the elements in $E_r^{*,0}$ are permanent cocycles, the spectral sequence collapses at the E_2-level up to degree 3. Choose a cohomology class t_2' in $H^2(B\tilde{\pi}'(H_2'))$ representing the generator $\tilde{u}$ of $E_\infty^{0,2} = \mathbb{Z}/p$. Then, $H^2(B\tilde{\pi}'(H_2'))$ is generated by t_2' and w_2. Suppose that

$$\iota'^*(u_2) = \alpha_1 w_2 + \alpha_2 t_2',$$

where $\alpha_1, \alpha_2 \in \mathbb{Z}/p$. Since $(\iota' \circ \iota'' \circ \iota_\sigma)^*(u_2) = v_2$ and $(\iota'' \circ \iota_\sigma)^*(w_2) = 0$,

$$(\iota'' \circ \iota_\sigma)^*(\alpha_2 t_2') = v_2$$

and so $\alpha_2 \neq 0$. Hence, t_2 and w_2 generate $H^2(B\tilde{\pi}'(H_2'))$.

Next, we prove that $t_2 w_1 = 0$. The E_∞-terms of total degree 3 are given by

$$E_\infty^{0,3} = \{0\}, \quad E_\infty^{1,2} = \mathbb{Z}/p\{u_2^{(1)} w_1\}, \quad E_\infty^{2,1} = \{0\} \quad \text{and} \quad E_\infty^{3,0} = \mathbb{Z}/p\{w_1 w_2\}.$$

Therefore, we have

$$F^2 H^3(B\tilde{\pi}'(H_2')) = F^3 H^3(B\tilde{\pi}'(H_2')) = \mathbb{Z}/p\{w_1 w_2\}.$$

Since $\alpha_2 t'_2 w_1$ represents $\alpha_2 \tilde{u} w_1$ and $\tilde{u} \in \operatorname{Ker}(1-g)$ is congruent to zero modulo the image of $(1-g)$, we have $\tilde{u} w_1 = 0$ in $E_\infty^{1,2}$. So, we have

$$t_2 w_1 \in F^3 H^3(B\tilde{\pi}'(H'_2)) = \mathbb{Z}/p\{w_1 w_2\}.$$

Therefore, $t_2 w_1 = \alpha_3 w_1 w_2$ for some $\alpha_3 \in \mathbb{Z}/p$. We proved that $(\iota'' \circ \iota_\beta)^*(t_2) = (\iota' \circ \iota'' \circ \iota_\beta)^*(u_2) = 0$ in the proof of Proposition 5.2. Thus, we have $(\iota'' \circ \iota_\beta)^*(t_2 w_1) = 0$. On the other hand, we have $(\iota'' \circ \iota_\beta)^*(w_1 w_2) = w_1 w_2 \neq 0$ in $H^*\big(B\langle\tilde{\pi}(\beta)\rangle\big)$. Hence, we obtain $\alpha_3 = 0$. □

6. The mod p cycle map for H

In this section, we prove Theorem 1.2. Let G be $\mathrm{SU}(p) \times \mathrm{SU}(p)/\langle\Delta(\xi)\rangle$ and let $H = p_+^{1+2} \times H_2/\Delta(\xi)$, as in Section 3. Let K be a subgroup of G containing H, that is, $H \subset K \subset G$. We proved in Section 4 that the mod p cycle map $\mathrm{CH}^2 BG/p \to H^4(BG)$ is not injective. To be more precise, we defined the virtual complex representation $\lambda'' : BG \to \mathbb{Z} \times BU$ such that the Chern class $c_2(\lambda'') \in \mathrm{CH}^2 BG$ is nontrivial in $\mathrm{CH}^2 BG/p$, that is, $c_2(\lambda'')$ is not divisible by p, and the mod p cycle map maps $c_2(\lambda'')$ to $\rho(c_2(\lambda'')) = 0$. We denote the inclusions by $f' : K \to G$, $f'' : H \to K$ and $f : H \to G$, so that $f = f' \circ f'' : H \to G$. It is clear that $\rho(c_2(\lambda'' \circ f'))$ is zero in $H^4(BK)$. So, in order to prove Theorem 1.2, we need to show that $c_2(\lambda'' \circ f')$ remains nonzero in $\mathrm{CH}^2 BK \subset H^4(BK;\mathbb{Z})$ and that $c_2(\lambda'' \circ f')$ remains not divisible by p in $\mathrm{CH}^2 BK$. These follow immediately from:

(1) $c_2(\lambda'' \circ f) = f''^*(c_2(\lambda'' \circ f'))$ is not zero in $\mathrm{CH}^2 BH \subset H^4(BH;\mathbb{Z})$.

(2) $c_2(\lambda'' \circ f) = f''^*(c_2(\lambda'' \circ f'))$ is not divisible by p in $\mathrm{CH}^2 BH$.

To prove (1) and (2), we consider the spectral sequences associated with the vertical fibrations below and the induced homomorphism between them:

$$\begin{tikzcd} BH \arrow[r, "f"] \arrow[d, "\pi"'] & BG \arrow[d, "\pi"] \\ BA_2 \times B\tilde{\pi}(H_2) \arrow[r, "f"] & B\,\mathrm{PU}(p) \times B\,\mathrm{PU}(p) \end{tikzcd}$$

Let $g : BA_2 \to BA_2 \times B\tilde{\pi}(H_2)$ be the map defined in Section 2 by $g(\tilde{\pi}(\alpha)) = (\tilde{\pi}(\alpha), \tilde{\pi}(\alpha))$ and $g(\tilde{\pi}(\beta)) = (\tilde{\pi}(\beta), \tilde{\pi}(\beta))$. Let v_1 and w_1 be the generators of $H^1(B\tilde{\pi}(H_2))$ defined in the previous section; let x_1 and y_1 be those of $H^1(BA_2)$, as defined in Section 3. We denote by x_1, y_1, v_1 and w_1 the corresponding generators of $H^1(BA_2 \times B\tilde{\pi}(H_2))$, so that $g^*(x_1) = x_1$, $g^*(v_1) = 0$ and $g^*(y_1) = g^*(w_1) = y_1$. We denote by z_1 a generator of $H^1(B\langle\Gamma_2(\xi)\rangle) = E_2^{0,1}$ as in Section 4. Let $x_2 = Q_0 x_1$, $y_2 = Q_0 y_1$ and $z_2 = Q_0 z_1$, as usual, so that $H^*(BA_2) = \mathbb{Z}/p[x_2, y_2] \otimes \Lambda(x_1, y_1)$.

Also, let u_2 be the generator of $H^2(B\,\mathrm{PU}(p))$ defined in Section 3, and let $u_3 = Q_0u_2$, as in Section 4. Let ι be the map induced by the inclusion of $\tilde{\pi}(H_2)$ into $\mathrm{PU}(p)$. We need to compute the spectral sequence up to degree 4. Differentials d_2 and d_3 in the spectral sequence $E_r^{s,t}(BH)$ are given by

$$d_2(z_1) = x_1y_1 - \iota^*(u_2),$$
$$d_3(z_2) = x_2y_1 - x_1y_2 - \iota^*(u_3),$$

since

$$f^*(u_2 \otimes 1 - 1 \otimes u_2) = x_1y_1 - \iota^*(u_2),$$
$$f^*(u_3 \otimes 1 - 1 \otimes u_3) = x_2y_1 - x_1y_2 - \iota^*(u_3),$$

and the differentials d_2 and d_3 in the spectral sequence $E_r^{s,t}(BG)$ are given by $d_2(z_1) = u_2 \otimes 1 - 1 \otimes u_2$ and $d_3(z_2) = u_3 \otimes 1 - 1 \otimes u_3$, as we saw in Section 4.

Proposition 6.1. *The E_∞-terms $E_\infty^{s,t}$ (for $s = 0, 1, 2$ and $s + t = 3, 4$) for the spectral sequence $E_r^{s,t}(BH)$ are given as follows*: $E_\infty^{0,3} = E_\infty^{1,2} = E_\infty^{0,4} = E_\infty^{1,3} = \{0\}$,

$$E_\infty^{2,1} = \mathbb{Z}/p\{w_1x_1z_1, w_1y_1z_1\},$$
$$E_\infty^{2,2} = \mathbb{Z}/p\{x_1y_1z_2, w_1x_1z_2, w_1y_1z_2\}.$$

Proof. For the sake of notational simplicity, let

$$R = \mathbb{Z}/p[x_2, y_2] \otimes H^*(B\tilde{\pi}(H_2)),$$

so that

$$H^*(BA_2) \otimes H^*(B\tilde{\pi}(H_2)) = R\{1, x_1, y_1, x_1y_1\}.$$

The set $\{v_1, w_1\}$ is a basis for $H^1(B\tilde{\pi}(H_2))$. We consider a basis for $H^2(B\tilde{\pi}(H_2))$. By Proposition 5.1, we have $\iota^*(u_2)^2 \neq 0$. We choose a basis $\{m^{(i)}, \iota^*(u_2)\}$ for $H^2(B\tilde{\pi}(H_2))$, where $1 \leq i < \dim H^2(B\tilde{\pi}(H_2))$. Here, we do not exclude the possibility that $\{m^{(i)}\}$ could be the empty set. Then, the set $\{m^{(i)}, \iota^*(u_2), x_2, y_2\}$ is a basis for the subspace of R spanned by elements of degree 2 and $\{m^{(i)}, x_2, y_2\}$ is a basis for the subspace of $R/(\iota^*(u_2))$ spanned by elements of degree 2. The set

$$\{v_1, w_1, x_1, y_1\}$$

is a basis for $E_2^{1,0} = H^1(BA_2 \times B\tilde{\pi}(H_2))$ and

$$\{m^{(i)}, \iota^*(u_2), x_2, y_2, v_1x_1, v_1y_1, w_1x_1, w_1y_1, x_1y_1\}$$

is a basis for $E_2^{2,0} = H^2(BA_2 \times B\tilde{\pi}(H_2))$.

First, we compute E_3-terms $E_3^{0,3}$, $E_3^{2,1}$ and $E_3^{1,3}$. Let us consider R-module homomorphisms

$$\mathrm{pr}_2^{(k)} : E_2^{*,2k} = R\{z_2^k, x_1z_2^k, y_1z_2^k, x_1y_1z_2^k\} \to R\{x_1z_2^k, y_1z_2^k, x_1y_1z_2^k\}$$

sending z_2^k, $x_1z_2^k$, $y_1z_2^k$ and $x_1y_1z_2^k$ to 0, $x_1z_2^k$, $y_1z_2^k$ and $x_1y_1z_2^k$, respectively. Recall that

$$d_2(z_1) = x_1y_1 - \iota^*(u_2).$$

The E_2-term $E_2^{0,3}$ is spanned by z_1z_2. It is clear from $d_2(z_2) = 0$ that

$$d_2(z_1z_2) = d_2(z_1)z_2 = (x_1y_1 - \iota^*(u_2))z_2 \neq 0.$$

Hence the homomorphism $d_2 : E_2^{0,3} \to E_2^{2,2}$ is injective and we have $E_3^{0,3} = \{0\}$.

The E_2-term $E_2^{2,1}$ is spanned by

$$m^{(i)}z_1,\quad \iota^*(u_2)z_1,\quad x_2z_1,\quad y_2z_1,\quad v_1x_1z_1,\quad v_1y_1z_1,\quad w_1x_1z_1,\quad w_1y_1z_1,\quad x_1y_1z_1$$

and

$$d_2(\alpha_2z_1) = \alpha_2d_2(z_1) = \alpha_2x_1y_1 - \alpha_2\iota^*(u_2).$$

for any degree 2 element α_2 in $E_2^{2,0} = H^2(BA_2 \times B\tilde{\pi}(H_2))$ since $d_2(\alpha_2) = 0$. If α_2 is one of $m^{(i)}$, $\iota^*(u_2)$, x_2 or y_2, then $\alpha_2\iota^*(u_2) \in R\{1\}$ and so $\mathrm{pr}_2^{(0)}(\alpha_2\iota^*(u_2)) = 0$, by definition. Hence, for $\alpha_2 = m^{(i)}$, $\iota^*(u_2)$, x_2 and y_2, we have

$$\mathrm{pr}_2^{(0)}(d_2(\alpha_2z_1)) = \alpha_2x_1y_1.$$

So, we have

$$\begin{aligned}
\mathrm{pr}_2^{(0)}(d_2(m^{(i)}z_1)) &= m^{(i)}x_1y_1,\\
\mathrm{pr}_2^{(0)}(d_2(\iota^*(u_2)z_1)) &= \iota^*(u_2)x_1y_1,\\
\mathrm{pr}_2^{(0)}(d_2(x_2z_1)) &= x_2x_1y_1,\\
\mathrm{pr}_2^{(0)}(d_2(y_2z_1)) &= y_2x_1y_1.
\end{aligned}$$

If α_2 is one of v_1x_1, v_1y_1, w_1x_1, w_1y_1 or x_1y_1, then $\alpha_2x_1y_1 = 0$. So, we have

$$d_2(\alpha_2z_1) = -\alpha_2\iota^*(u_2) = -\iota^*(u_2)\alpha_2.$$

By Proposition 5.2, $\iota^*(u_2)w_1 = 0$ in $H^*(B\tilde{\pi}(H_2))$. Using this, we have

$$\begin{aligned}
d_2(w_1x_1z_1) &= -\iota^*(u_2)w_1x_1 = 0,\\
d_2(w_1y_1z_1) &= -\iota^*(u_2)w_1y_1 = 0.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mathrm{pr}_2^{(0)}(d_2(v_1x_1z_1)) &= -\iota^*(u_2)v_1x_1,\\
\mathrm{pr}_2^{(0)}(d_2(v_1y_1z_1)) &= -\iota^*(u_2)v_1y_1,\\
\mathrm{pr}_2^{(0)}(d_2(x_1y_1z_1)) &= -\iota^*(u_2)x_1y_1.
\end{aligned}$$

By Proposition 5.1, $\iota^*(u_2)v_1 \neq 0$. So, the kernel of $\mathrm{pr}_2^{(0)} \circ d_2$ is spanned by

$$x_1y_1z_1 + \iota^*(u_2)z_1,\quad w_1x_1z_1,\quad w_1y_1z_1.$$

On the other hand, we have

$$d_2(x_1y_1z_1+\iota^*(u_2)z_1)=x_1y_1(x_1y_1-\iota^*(u_2))+\iota^*(u_2)(x_1y_1-\iota^*(u_2))=-\iota^*(u_2)^2,$$

and, since $\iota^*(u_2)^2\neq 0$ by Proposition 5.1, $x_1y_1z_1+\iota^*(u_2)z_1$ is not in the kernel of d_2. Hence, the kernel of d_2 is spanned by $w_1x_1z_1$ and $w_1y_1z_1$, and the image of $d_2: E_2^{0,2}\to E_2^{2,1}$ is trivial since $E_2^{0,2}$ is spanned by z_2 and $d_2(z_2)=0$. Thus, we have $E_3^{2,1}=\mathbb{Z}/p\{w_1x_1z_1, w_1y_1z_1\}$.

As for the E_2-term $E_2^{1,3}$, we have a basis

$$\{x_1z_1z_2,\ y_1z_1z_2,\ v_1z_1z_2,\ w_1z_1z_2\}$$

and

$$d_2(\alpha_1z_1z_2)=-\alpha_1d_2(z_1)z_2=-\alpha_1x_1y_1z_2+\alpha_1\iota^*(u_2)z_2$$

for $\alpha_1=x_1,\ y_1,\ v_1,\ w_1$, since $d_2(\alpha_1)=d_2(z_2)=0$. For $\alpha_1=x_1,\ y_1$, since $\alpha_1x_1y_1=0$ we have

$$d_2(\alpha_1z_1z_2)=\alpha_1\iota^*(u_2)z_2=\iota^*(u_2)\alpha_1z_2.$$

For $\alpha_1=v_1,\ w_1$, since $\alpha_1\iota^*(u_2)z_2\in R\{z_2\}$, we have $\mathrm{pr}_2^{(1)}(\alpha_1\iota^*(u_2)z_2)=0$ by definition. Hence, we have

$$\mathrm{pr}_2^{(1)}(d_2(\alpha_1z_1z_2))=-\alpha_1x_1y_1z_2.$$

Thus, we obtain

$$\begin{aligned}
\mathrm{pr}_2^{(1)}(d_2(x_1z_1z_2))&=\qquad\qquad\iota^*(u_2)x_1z_2,\\
\mathrm{pr}_2^{(1)}(d_2(y_1z_1z_2))&=\qquad\qquad\qquad\qquad\iota^*(u_2)y_1z_2,\\
\mathrm{pr}_2^{(1)}(d_2(v_1z_1z_2))&=-v_1x_1y_1z_2,\\
\mathrm{pr}_2^{(1)}(d_2(w_1z_1z_2))&=-w_1x_1y_1z_2.
\end{aligned}$$

Hence, it is clear that the composition

$$\mathrm{pr}_2^{(1)}\circ d_2: E_2^{1,3}\to E_2^{3,2}\to R\{x_1z_2,\ y_1z_2,\ x_1y_1z_2\}$$

is injective and so is $d_2: E_2^{1,3}\to E_2^{3,2}$. Therefore, we have $E_3^{1,3}=\{0\}$.

Next we compute the E_4-terms $E_4^{0,4}$, $E_4^{1,2}$ and $E_4^{2,2}$. In the E_3-term, the relations are given by $x_1y_1=\iota^*(u_2)$, $\iota^*(u_2)x_1=0$ and $\iota^*(u_2)y_1=0$. In particular, $\iota^*(u_2)^2=0$. For simplicity, we write R' and R'' for $R/(\iota^*(u_2))$ and $R/(\iota^*(u_2)^2)$, respectively. We have

$$E_3^{*,2k}=R'\{x_1z_2^k,\ y_1z_2^k\}\oplus R''\{z_2^k\}$$

as a graded $\mathbb{Z}/p$-module. Let N be the subspace of $R'\{x_1\}$ spanned by elements of the form xx_1, where x ranges over a basis for $H^*(B\tilde{\pi}(H_2))/(\iota^*(u_2))\subset R'$. Here, we emphasize that N is not an R-submodule and that $\tilde{x}m^{(i)}x_1$, $\tilde{x}x_1$, $\tilde{x}v_1x_1$

and $\tilde{x}w_1x_1$ are linearly independent in $R'\{x_1\}/N$, where $\tilde{x}$ ranges over positive-degree monomials in x_2 and y_2. We consider a $\mathbb{Z}/p$-module homomorphism

$$\mathrm{pr}_3 : E_3^{*,0} = R'\{x_1, y_1\} \oplus R''\{1\} \to R'\{x_1\}/N \oplus R''\{1\},$$

sending $r'x_1$, $r'y_1$ and r'' to $r'x_1$, 0 and r'', respectively, where $r' \in R'$ and $r'' \in R''$. Recall that

$$d_3(z_2) = x_2y_1 - x_1y_2 - \iota^*(u_3).$$

The E_3-term $E_3^{0,4}$ is spanned by z_2^2 and, since $y_2x_1z_2$ is nontrivial in $R'\{x_1z_2\}$,

$$\begin{aligned} d_3(z_2^2) &= 2d_3(z_2)z_2 = 2x_2y_1z_2 - 2x_1y_2z_2 - 2\iota^*(u_3)z_2 \\ &= -2y_2x_1z_2 + 2x_2y_1z_2 - 2\iota^*(u_3)z_2 \end{aligned}$$

is nontrivial in $E_3^{*,2} = R'\{x_1z_2, y_1z_2\} \oplus R''\{z_2\}$. Hence, $d_3 : E_3^{0,4} \to E_3^{3,2}$ is injective and $E_4^{0,4} = \{0\}$.

The E_3-term $E_3^{1,2}$ is spanned by

$$v_1z_2, \quad w_1z_2, \quad x_1z_2, \quad y_1z_2,$$

since the subspace of R'' spanned by degree 1 elements is equal to $H^1(B\tilde{\pi}(H_2))$ and $H^1(B\tilde{\pi}(H_2))$ is spanned by v_1 and w_1. For $\alpha_1 = v_1, w_1, x_1, y_1$, since $d_3(\alpha_1) = 0$ we have

$$d_3(\alpha_1z_2) = -\alpha_1d_3(z_2) = -\alpha_1x_2y_1 + \alpha_1x_1y_2 + \alpha_1\iota^*(u_3).$$

Hence, for $\alpha_1 = v_1, w_1$, since $\mathrm{pr}_3(\alpha_1x_2y_1) = 0$ by definition, we have

$$\mathrm{pr}_3(d_3(\alpha_1z_2)) = \alpha_1x_1y_2 + \alpha_1\iota^*(u_3) = y_2\alpha_1x_1 + \alpha_1\iota^*(u_3).$$

For $\alpha_1 = x_1, y_1$, since $x_1^2 = y_1^2 = 0$, $x_1y_1 = \iota^*(u_2)$ and $y_1x_1 = -\iota^*(u_2)$, we have

$$\begin{aligned} d_3(x_1z_2) &= -x_1x_2y_1 + x_1\iota^*(u_3) = -\iota^*(u_3)x_1 - x_2\iota^*(u_2) \\ d_3(y_1z_2) &= y_1x_1y_2 + y_1\iota^*(u_3) = -\iota^*(u_3)y_1 - y_2\iota^*(u_2). \end{aligned}$$

Since $\iota^*(u_3)x_1$ is in N, $\mathrm{pr}_3(\iota^*(u_3)x_1) = 0$. By definition, $\mathrm{pr}_3(\iota^*(u_3)y_1) = 0$. Therefore, we have

$$\begin{aligned} \mathrm{pr}_3(d_3(v_1z_2)) &= v_1y_2x_1 + v_1\iota^*(u_3), \\ \mathrm{pr}_3(d_3(w_1z_2)) &= w_1y_2x_1 + w_1\iota^*(u_3), \\ \mathrm{pr}_3(d_3(x_1z_2)) &= - x_2\iota^*(u_2), \\ \mathrm{pr}_3(d_3(y_1z_2)) &= - y_2\iota^*(u_2). \end{aligned}$$

Since $v_1y_2x_1$ and $w_1y_2x_1$ are linearly independent in $R'\{x_1\}/N$, and $\iota^*(u_2)x_2$ and $\iota^*(u_2)y_2$ are linearly independent in $\mathbb{Z}/p\{x_2, y_2\} \otimes H^2(B\tilde{\pi}(H_2)) \subset R''\{1\}$, the four elements

$$d_3(v_1z_2),\quad d_3(w_1z_2),\quad d_3(x_1z_2),\quad d_3(y_1z_2)$$

are linearly independent in $E_3^{*,0} = R'\{x_1, y_1\} \oplus R''\{1\}$. Hence, the homomorphism $d_3 : E_3^{1,2} \to E_3^{4,0}$ is injective. Therefore, we have $E_4^{1,2} = \{0\}$.

The E_3-term $E_3^{2,2}$ is spanned by

$$m^{(i)}z_2,\quad \iota^*(u_2)z_2,\quad x_2z_2,\quad y_2z_2,\quad v_1x_1z_2,\quad v_1y_1z_2,\quad w_1x_1z_2,\quad w_1y_1z_2.$$

For $\alpha_2 = m^{(i)}$, $\iota^*(u_2)$, x_2, y_2, v_1x_1, v_1y_1, w_1x_1, $w_1y_1 \in E_3^{2,0}$, since $d_3(\alpha_2)$ is in $E_3^{5,-2} = \{0\}$ we have

$$d_3(\alpha_2z_2) = \alpha_2d_3(z_2) = \alpha_2x_2y_1 - \alpha_2x_1y_2 - \alpha_2\iota^*(u_3).$$

For $\alpha_2 = m^{(i)}$, $\iota^*(u_2)$, x_2, y_2, since $\mathrm{pr}_3(\alpha_2x_2y_1) = 0$ by definition, we have

$$\mathrm{pr}_3(d_3(\alpha_2z_2)) = -\alpha_2y_2x_1 - \alpha_2\iota^*(u_3).$$

Thus, we have

$$\begin{aligned}
\mathrm{pr}_3(d_3(m^{(i)}z_2)) &= -y_2m^{(i)}x_1 - m^{(i)}\iota^*(u_3),\\
\mathrm{pr}_3(d_3(x_2z_2)) &= -x_2y_2x_1 - x_2\iota^*(u_3),\\
\mathrm{pr}_3(d_3(y_2z_2)) &= -y_2^2x_1 - y_2\iota^*(u_3).
\end{aligned}$$

Moreover, since $\iota^*(u_2)\iota^*(u_3) = \iota^*(u_2u_3) = 0$ in $H^*(B\tilde{\pi}(H_2))$ by Proposition 3.2, and since $\iota^*(u_2)x_1 = \iota^*(u_2)y_1 = 0$ in $R'\{x_1, y_1\}$, we have

$$d_3(\iota^*(u_2)z_2) = 0.$$

For $\alpha_1 = v_1$, w_1, using the relations $x_1^2 = y_1^2 = 0$, $x_1y_1 = \iota^*(u_2)$ and $y_1x_1 = -\iota^*(u_2)$ in E_3, we have

$$\begin{aligned}
d_3(\alpha_1x_1z_2) &= \alpha_1x_1x_2y_1 - \alpha_1x_1x_1y_2 - \alpha_1x_1\iota^*(u_3) = \alpha_1\iota^*(u_3)x_1 + x_2\alpha_1\iota^*(u_2)\\
d_3(\alpha_1y_1z_2) &= \alpha_1y_1x_2y_1 - \alpha_1y_1x_1y_2 - \alpha_1y_1\iota^*(u_3) = \alpha_1\iota^*(u_3)y_1 + y_2\alpha_1\iota^*(u_2).
\end{aligned}$$

Since $\alpha_1\iota^*(u_3) \in H^*(B\tilde{\pi}(H_2))/(\iota^*(u_2))$, we obtain $\alpha_1\iota^*(u_3)x_1 \equiv 0$ in $R'\{x_1\}/N$, hence $\mathrm{pr}_3(\alpha_1\iota^*(u_3)x_1) = 0$. Moreover, $\mathrm{pr}_3(\alpha_1\iota^*(u_3)y_1) = 0$ by definition. So, we have

$$\begin{aligned}
\mathrm{pr}_3(d_3(\alpha_1x_1z_2)) &= \alpha_1x_2\iota^*(u_2) = x_2\alpha_1\iota^*(u_2)\\
\mathrm{pr}_3(d_3(\alpha_1y_1z_2)) &= \alpha_1y_2\iota^*(u_2) = y_2\alpha_1\iota^*(u_2).
\end{aligned}$$

By Proposition 5.2, $w_1\iota^*(u_2) = 0$. Hence, we have

$$\begin{aligned}
d_3(w_1x_1z_2) &= w_1\iota^*(u_3)x_1\\
d_3(w_1y_1z_2) &= w_1\iota^*(u_3)y_1.
\end{aligned}$$

Furthermore, by Proposition 5.2, $Q_0(w_1\iota^*(u_2)) = Q_0w_1 \cdot \iota^*(u_2) - w_1\iota^*(u_3) = 0$ in $H^*(B\tilde{\pi}(H_2))$, hence $w_1\iota^*(u_3)x_1 = (Q_0w_1)\iota^*(u_2)x_1 = 0$ in $R'\{x_1, y_1\} \subset E_3^{*,0}$. Thus, we obtain $d_3(w_1x_1z_2) = 0$. Similarly, we also have $d_3(w_1y_1z_2) = 0$. Thus, we have

$$\mathrm{pr}_3(d_3(v_1x_1z_2)) = x_2v_1\iota^*(u_2),$$
$$\mathrm{pr}_3(d_3(v_1y_1z_2)) = y_2v_1\iota^*(u_2),$$

and

$$d_3(w_1x_1z_2) = 0,$$
$$d_3(w_1y_1z_2) = 0.$$

Since $y_2m^{(i)}x_1$, $x_2y_2x_1$ and $y_2^2x_1$ are linearly independent in $R'\{x_1\}/N$ and, by Proposition 5.1, $x_2v_1\iota^*(u_2)$ and $y_2v_1\iota^*(u_2)$ are linearly independent in

$$\mathbb{Z}/p\{x_2, y_2\} \otimes H^3(B\tilde{\pi}(H_2)) \subset R''\{1\},$$

the kernel of $\mathrm{pr}_3 \circ d_3$ is spanned by $\iota^*(u_2)z_2$, $w_1x_1z_2$ and $w_1y_1z_2$, and, since these are in the kernel of d_3, the kernel of d_3 is spanned by these elements. Moreover, the image $d_3 : E_3^{-1,4} \to E_3^{2,2}$ is trivial. Therefore, we obtain

$$E_4^{2,2} = \mathbb{Z}/p\{\iota^*(u_2)z_2, w_1x_1z_2, w_1y_1z_2\} = \mathbb{Z}/p\{x_1y_1z_2, w_1x_1z_2, w_1y_1z_2\},$$

where $\iota^*(u_2)z_2 = x_1y_1z_2$.

Finally, we compute the E_∞-terms $E_\infty^{0,3}$, $E_\infty^{1,2}$, $E_\infty^{2,1}$ and $E_\infty^{0,4}$, $E_\infty^{1,3}$, $E_\infty^{2,2}$. Since $E_3^{0,3} = E_4^{1,2} = \{0\}$, we have $E_\infty^{0,3} = E_\infty^{1,2} = \{0\}$. Similarly, since $E_4^{0,4} = E_3^{1,3} = \{0\}$, we have $E_\infty^{0,4} = E_\infty^{1,3} = \{0\}$. Since the Leray–Serre spectral sequence is the first quadrant spectral sequence, for $s \le r-1$ and $t \le r-2$,

$$E_r^{s-r,t+r-1} = E_r^{s+r,t-r+1} = \{0\},$$

and the differentials

$$d_r : E_r^{s-r,t+r-1} \to E_r^{s,t}, \quad d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$$

are trivial. Hence, we have $E_r^{s,t} = E_\infty^{s,t}$ for $s \le r-1$ and $t \le r-2$. In particular, $E_3^{s,t} = E_\infty^{s,t}$ for $s \le 2$ and $t \le 1$, and $E_4^{s,t} = E_\infty^{s,t}$ for $s \le 3$ and $t \le 2$. Hence, we have $E_\infty^{2,1} = E_3^{2,1}$ and $E_\infty^{2,2} = E_4^{2,2}$. □

In Section 4, we defined $x_4 \in H^4(BG; \mathbb{Z})$, so that $c_2(\lambda'') = px_4$ in $H^4(BG; \mathbb{Z})$. Therefore, to show that $c_2(\lambda'' \circ f) \neq 0$ in $H^4(BH; \mathbb{Z})$ it is equivalent to show that $pf^*(x_4) \neq 0$ in $H^4(BH; \mathbb{Z})$. Hence, in order to prove (1), it suffices to show that the mod p reduction $\rho(f^*(x_4)) \in H^4(BH)$ of $f^*(x_4) \in H^4(BH; \mathbb{Z})$ is not in the image of the Bockstein homomorphism. So, we prove the following proposition:

Proposition 6.2. *The cohomology class $f^*(\rho(x_4))$ is not in the image of the Bockstein homomorphism*

$$Q_0 : H^3(BH) \to H^4(BH).$$

Proof. Since $E_\infty^{0,4} = E_\infty^{1,3} = \{0\}$, we have $F^2H^4(BH) = H^4(BH)$. Similarly, since $E_\infty^{0,3} = E_\infty^{1,2} = \{0\}$, we have $F^2H^3(BH) = H^3(BH)$. Hence, we have

$$Q_0H^3(BH) \subset F^2H^4(BH)$$

and each cohomology class in $Q_0H^3(BH)$ represents an element in

$$E_\infty^{2,2} = F^2H^4(BH)/F^3H^4(BH).$$

Since $E_\infty^{2,1}$ is spanned by $w_1x_1z_1$ and $w_1y_1z_1$, using the properties of the vertical operation $\beta\wp^0$ constructed by Araki [1957, Corollary 4.1] in the spectral sequence of a fibration, we have that if x is in $Q_0H^3(BH)$ then x represents a linear combination of $w_1x_1z_2$ and $w_1y_1z_2$ in $E_\infty^{2,2}$.

On the other hand, by Proposition 4.2, $\rho(x_4) \in H^4(BG)$ represents αb_2z_2 in $E_\infty^{2,2}(BG)$, where $\alpha \neq 0$ is in $\mathbb{Z}/p$. Using Proposition 3.3 and the definition of b_2 in Section 4, we have $f^*(b_2) = x_1y_1$. Therefore, $f^*(\rho(x_4))$ represents $\alpha x_1y_1z_2$ in $E_\infty^{2,2}$. Hence, $f^*(\rho(x_4))$ is not in the image of the Bockstein homomorphism Q_0. □

Remark 6.3. If we replace H_2 by the extraspecial p-group p_+^{1+2}, then (1) does not hold. To be more precise, $f^*(\rho(x_4))$ is in the image of the Bockstein homomorphism $Q_0 : H^3(Bp_+^{1+4}) \to H^4(Bp_+^{1+4})$ and $c_2(\lambda'' \circ f) = pf^*(x_4) = 0$ in $H^4(Bp_+^{1+4}; \mathbb{Z})$.

Finally, we prove (2) by proving the following proposition:

Proposition 6.4. *There exists no virtual complex representation*

$$\mu : BH \to \mathbb{Z} \times BU$$

such that $c_2(\lambda'' \circ f) \in p \cdot \operatorname{Im} \mu^*$.

Proof. We prove this by contradiction. Suppose that there exists a virtual complex representation

$$\mu : BH \to \mathbb{Z} \times BU$$

such that $c_2(\lambda'' \circ f) \in p \cdot \operatorname{Im} \mu^*$. Then, $p(\mu^*(y_4) - f^*(x_4)) = 0$ for some y_4 in $H^4(\mathbb{Z} \times BU; \mathbb{Z})$. Since Q_1 acts trivially on $H^*(\mathbb{Z} \times BU)$, we have

$$Q_1\rho(\mu^*(y_4)) = 0.$$

In what follows, we show that

$$Q_1\rho(\mu^*(y_4)) \neq 0,$$

which proves the proposition.

Since $p(\mu^*(y_4) - f^*(x_4)) = 0$, $\rho(\mu^*(y_4) - f^*(x_4))$ is in the image of the Bockstein homomorphism, that is, as in the proof of Proposition 6.2, $\rho(\mu^*(y_4) - f^*(x_4))$ represents

$$\alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2$$

in $E_\infty^{2,2}$ for some $\alpha_1, \alpha_2 \in \mathbb{Z}/p$. We already verified that $f^*(\rho(x_4)) = \rho(f^*(x_4))$ represents $\alpha x_1 y_1 z_2 \in E_\infty^{2,2}$, where $\alpha \neq 0$, in the proof of Proposition 6.2. So, $\rho(\mu^*(y_4))$ represents

$$\alpha x_1 y_1 z_2 + \alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2$$

in $E_\infty^{2,2}$ and $\alpha \neq 0$.

We recall the structure of H_2 defined in Section 2. Also, we recall the diagram

$$\begin{tikzcd}
A_3 \arrow[r, "g"] \arrow[d, "\varphi"] & H \arrow[d, "\pi"] & A_3' \arrow[l, "g'"'] \arrow[d, "\varphi'"] \\
A_2 \arrow[r, "g"] & A_2 \times \tilde{\pi}(H_2) & A_2 \arrow[l, "g'"']
\end{tikzcd}$$

where the upper g and g' are the obvious inclusions, $A_2 = \langle \tilde{\pi}(\alpha), \tilde{\pi}(\beta) \rangle$,

$$g(\tilde{\pi}(\alpha)) = (\tilde{\pi}(\alpha), \tilde{\pi}(\alpha)), \qquad g(\tilde{\pi}(\beta)) = (\tilde{\pi}(\beta), \tilde{\pi}(\beta)),$$
$$g'(\tilde{\pi}(\alpha)) = (\tilde{\pi}(\alpha), 1), \qquad g'(\tilde{\pi}(\beta)) = (1, \tilde{\pi}(\beta)).$$

In Section 5, we defined $w_1 \in H^1(B\tilde{\pi}(H_2))$, so that the induced homomorphism $H^1(B\tilde{\pi}(H_2)) \to H^1(B\langle \tilde{\pi}(\beta) \rangle)$ maps w_1 to the element corresponding to the generator $\tilde{\pi}(\beta)$. So, we see that the induced homomorphisms g^* and g'^* satisfy

$$g^*(x_1) = x_1, \quad g^*(y_1) = y_1, \quad g^*(w_1) = y_1,$$
$$g'^*(x_1) = x_1, \quad g'^*(y_1) = 0, \quad g'^*(w_1) = y_1.$$

Therefore, $g^*(\rho(\mu^*(y_4))) \in H^4(BA_3)$ represents

$$g^*(\alpha x_1 y_1 z_2 + \alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2) = \alpha x_1 y_1 z_2 + \alpha_1 y_1 x_1 z_2 = (\alpha - \alpha_1) x_1 y_1 z_2$$

in the spectral sequence for $H^*(BA_3)$ and $g'^*(\rho(\mu^*(y_4))) \in H^4(BA_3')$ represents

$$g'^*(\alpha x_1 y_1 z_2 + \alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2) = \alpha_1 y_1 x_1 z_2 = -\alpha_1 x_1 y_1 z_2$$

in the spectral sequence for $H^*(BA_3')$.

As in the proof of Proposition 4.1, let M be the $\varphi^*(H^*(BA_2))$-submodule of $H^*(BA_3)$ and M' the $\varphi'^*(H^*(BA_2))$-submodule of $H^*(BA_3')$ generated by

$$1, \quad z_1, \quad z_1 z_2, \quad z_2^i, \quad z_1 z_2^i \quad (i \geq 2),$$

where $\varphi : BA_3 \to BA_1$ and $\varphi' : BA_3' \to BA_2$ are the maps defined in Section 2, so that

$$H^*(BA_3)/M = \varphi^*(H^*(BA_2))\{z_2\} = \mathbb{Z}/p[x_2, y_2] \otimes \Lambda(x_1, y_1)\{z_2\},$$
$$H^*(BA_3')/M' = \varphi'^*(H^*(BA_2))\{z_2\} = \mathbb{Z}/p[x_2, y_2] \otimes \Lambda(x_1, y_1)\{z_2\},$$

respectively. Since $Q_1 z_1 = z_2^p$, $Q_1 z_2 = 0$, and Q_1 is a derivation, M and M' are closed under the action of Milnor operation Q_1. We have

$$g^*(\rho(\mu^*(y_4))) \equiv (\alpha - \alpha_1)x_1 y_1 z_2 \mod M,$$
$$g'^*(\rho(\mu^*(y_4))) \equiv -\alpha_1 x_1 y_1 z_2 \mod M'.$$

and so

$$Q_1 g^*(\rho(\mu^*(y_4))) \equiv (\alpha - \alpha_1)(x_2^p y_1 - x_1 y_2^p) z_2 \mod M,$$
$$Q_1 g'^*(\rho(\mu^*(y_4))) \equiv -\alpha_1 (x_2^p y_1 - x_1 y_2^p) z_2 \mod M'.$$

Since $\alpha \neq 0$, at least one of $\alpha - \alpha_1$ and $-\alpha_1$ is nonzero. Therefore, we have

$$Q_1 \rho(\mu^*(y_4)) \neq 0.$$

This completes the proof. □

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Chern classes and compatible power operations in inertial K-theory

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Let $\mathscr{X} = [X/G]$ be a smooth Deligne–Mumford quotient stack. In a previous paper we constructed a class of exotic products called *inertial products* on $K(I\mathscr{X})$, the Grothendieck group of vector bundles on the inertia stack $I\mathscr{X}$. In this paper we develop a theory of Chern classes and compatible power operations for inertial products. When G is diagonalizable these give rise to an augmented λ-ring structure on inertial K-theory.

One well-known inertial product is the *virtual product*. Our results show that for toric Deligne–Mumford stacks there is a λ-ring structure on inertial K-theory. As an example, we compute the λ-ring structure on the virtual K-theory of the weighted projective lines $\mathbb{P}(1, 2)$ and $\mathbb{P}(1, 3)$. We prove that, after tensoring with $\mathbb{C}$, the augmentation completion of this λ-ring is isomorphic as a λ-ring to the classical K-theory of the crepant resolutions of singularities of the coarse moduli spaces of the cotangent bundles $\mathbb{T}^*\mathbb{P}(1, 2)$ and $\mathbb{T}^*\mathbb{P}(1, 3)$, respectively. We interpret this as a manifestation of mirror symmetry in the spirit of the hyper-Kähler resolution conjecture.

1. Introduction

The work of Chen and Ruan [2002], Fantechi and Göttsche [2003], and Abramovich, Graber, and Vistoli [Abramovich et al. 2002; 2008] defined orbifold products for the cohomology, Chow groups and K-theory of the inertia stack $I\mathscr{X}$ of a smooth Deligne–Mumford stack $\mathscr{X}$. Moreover, there is an orbifold Chern character $\mathscr{C}h : K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}$ which respects these products [Jarvis et al. 2007]. In [Edidin et al. 2016] we showed that the orbifold product and Chern character fit into a more general formalism of *inertial products*, which are discussed below.

In this paper, we are motivated by mirror symmetry to find examples of elements in orbifold and inertial algebraic K-theory that play a role analogous to classes

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of vector bundles in the ordinary algebraic K-theory. Each such element should possess *orbifold Euler classes* analogous to the classically defined classes $\lambda_{-1}(\mathcal{E}^*)$ and $c_r(\mathcal{E})$ for vector bundles of rank r. This leads us to introduce the notions of an *orbifold* λ*-ring* and associated *Adams* (or *power*) *operations* which are suitably compatible with *orbifold Chern classes*, as we now explain.

Let $K(\mathscr{X})$ be the Grothendieck group of locally free sheaves on $\mathscr{X}$ with multiplication given by the ordinary tensor product. By definition, $K(\mathscr{X})$ is generated by classes of vector bundles and each such class possesses an Euler class. In the context of mirror symmetry we may be given a ring K which is conjectured to be the ordinary K-theory of some unknown variety. From the ring structure alone there is no way to solve the problem of identifying the elements of K which correspond to Chern classes of vector bundles on this unknown variety. However, a partial solution arises from observing that ordinary K-theory has the additional structure of a λ*-ring*. Every λ-ring has an associated invariant — the semigroup of λ*-positive elements* (Definition 6.11), which share many of the properties of classes of vector bundles in ordinary K-theory. In particular, λ-positive elements have Euler classes defined in terms of the λ-ring structure. In the case of ordinary K-theory of a scheme or stack, classes of vector bundles are always λ-positive, but there are other λ-positive classes as well.

Endowing the orbifold K-theory ring with the structure of a λ-ring with respect to its orbifold product allows one to identify its semigroup of λ-positive elements. Furthermore, defining suitably compatible orbifold Chern classes, should give these λ-positive elements orbifold Euler classes in orbifold K-theory, orbifold Chow theory, and orbifold cohomology theory. These λ-positive elements can be regarded as building blocks of orbifold K-theory.

We prove the following results about smooth quotient stacks $\mathscr{X} = [X/G]$ where G is a linear algebraic group acting with finite stabilizer on a smooth variety X.

Main results. (a) *Suppose $\mathscr{X}$ is Gorenstein, then there is an orbifold Chern class homomorphism* $c_t : K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}[[t]]$ (see Definition 5.1 and Theorem 5.18).

(b) *Suppose $\mathscr{X}$ is <u>strongly Gorenstein</u>* (see Definition 2.29); *then there are Adams ψ-operations and λ-operations defined on $K(I\mathscr{X})$ and $K(I\mathscr{X})_{\mathbb{Q}}$ compatible with the Chern class homomorphism* (see Definitions 5.4 and 5.7 and Theorem 5.18).

(c) *Suppose G is diagonalizable and $\mathscr{X}$ is strongly Gorenstein*; *then the Adams and λ-operations make $K(I\mathscr{X})_{\mathbb{Q}} := K(I\mathscr{X}) \otimes \mathbb{Q}$ with its orbifold product into a rationally augmented λ-ring* (see Theorem 5.23).

(d) *Suppose the orbifold $\mathscr{X}$ is strongly Gorenstein*; *then there is an <u>inertial dual</u> operation $\mathscr{F} \to \mathscr{F}^\dagger$ on $K(\mathscr{X})$ which is an involution and a ring homomorphism and which commutes with the orbifold Adams operations and the orbifold augmentation* (see Theorem 6.4).

Our method of proof is based on developing properties of *inertial pairs* defined in [Edidin et al. 2016]. An inertial pair $(\mathscr{R}, \mathscr{S})$ consists of a vector bundle $\mathscr{R}$ on the double inertia stack $I^2\mathscr{X}$ together with a class $\mathscr{S} \in K(\mathscr{X})_{\mathbb{Q}}$, where $\mathscr{R}$ and $\mathscr{S}$ satisfy certain compatibility conditions. The bundle $\mathscr{R}$ determines associative *inertial products* on $K(I\mathscr{X})$ and $A^*(I\mathscr{X})$, and the class $\mathscr{S}$ determines a Chern character homomorphism of inertial rings $\mathscr{C}h : K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}$.

The basic example of an inertial pair $(\mathscr{R}, \mathscr{S})$ is the orbifold obstruction bundle $\mathscr{R}$ and the class $\mathscr{S}$ defined in [Jarvis et al. 2007]. This pair corresponds to the usual orbifold product. However, this is far from being the only example. Each vector bundle V on $\mathscr{X}$ determines two inertial pairs, $(\mathscr{R}^+V, \mathscr{S}^+V)$ and $(\mathscr{R}^-V, \mathscr{S}^-V)$. For example, if we denote the tangent bundle of $\mathscr{X}$ by $\mathbb{T}$, then the inertial pair $(\mathscr{R}^-\mathbb{T}, \mathscr{S}^-\mathbb{T})$ produces the virtual orbifold product of [González et al. 2007].

We prove that the main results listed above hold for many inertial pairs. As a corollary, we obtain the following:

Corollary. (a) *The virtual orbifold product on $K(I\mathscr{X})$ admits a Chern series homomorphism $\tilde{c}_t : K(I\mathscr{X}) \to A^*(I\mathscr{X})_{\mathbb{Q}}[\![t]\!]$ as well as compatible Adams ψ-operations and λ-operations on $K(I\mathscr{X})_{\mathbb{Q}}$.*

(b) *If $\mathscr{X} = [X/G]$ with G diagonalizable, then the virtual orbifold λ-operations make $K(I\mathscr{X})_{\mathbb{Q}}$ with its orbifold product into a rationally augmented λ-ring with a compatible inertial dual.*

Whenever an inertial K-theory ring has a λ-ring structure compatible with its inertial Chern classes and inertial Chern character, then its semigroup of λ-positive elements will have an *inertial Euler class* in K, Chow, and cohomology theory (see (6.23)), but where all products, rank, Chern classes, and the Chern character are the inertial ones. Furthermore, in many cases, the semigroup of λ-positive elements in inertial K-theory can be used to give a nice presentation of both the inertial K-theory ring and inertial Chow ring.

A major motivation for the work in this paper is mirror symmetry. Beginning with the work of Ruan, a series of conjectures have been made that relate the orbifold quantum cohomology and Gromov–Witten theory of a Gorenstein orbifold to the corresponding quantum cohomology and Gromov–Witten theory of a crepant resolution of singularities of the orbifold [Coates and Ruan 2013]. When the orbifold also has a holomorphic symplectic structure, these conjectures predict that the orbifold cohomology ring should be isomorphic to the usual cohomology of a crepant resolution. In the literature this conjecture is often referred to as Ruan's hyper-Kähler resolution conjecture (HKRC), because in many examples the holomorphic symplectic structure is in fact hyper-Kähler.

In view of Ruan's HKRC conjecture, it is natural to investigate whether there is an orbifold λ-ring structure on orbifold K-theory that is isomorphic to the usual

λ-ring structure on $K(Z)$. One place to look is on the cotangent bundles of complex manifolds and orbifolds. These naturally carry a holomorphic symplectic structure, and in many cases these are hyper-Kähler. In [Edidin et al. 2016] we prove that, if $\mathscr{X} = [X/G]$, then the virtual orbifold Chow ring of $I\mathscr{X}$ (as defined in [González et al. 2007]) is isomorphic to the orbifold Chow ring of $T^*I\mathscr{X}$. Since the inertial pair defining the virtual orbifold product is strongly Gorenstein, we expect that the λ-ring structure on $K(I\mathscr{X})$ should be related to the usual λ-ring structure on $K(Z)$.

When $\mathscr{X}$ is an orbifold, $K(I\mathscr{X})$ typically has larger rank as an abelian group than the corresponding Chow group $A^*(I\mathscr{X})$, while $K(Z)$ and $A^*(Z)$ have the same rank by the Riemann–Roch theorem for varieties. Thus, it is not reasonable to expect an isomorphism of λ-rings between $K(I\mathscr{X})$ with the virtual product and $K(Z)$ with the tensor product.

But the Riemann–Roch theorem for Deligne–Mumford stacks implies that a summand $\widehat{K}(I\mathscr{X})_{\mathbb{Q}}$, corresponding to the completion at the classical augmentation ideal in $K(I\mathscr{X})_{\mathbb{Q}}$, is isomorphic as an abelian group to $A^*(I\mathscr{X})_{\mathbb{Q}}$. We prove the remarkable result (Theorem 4.3) that, if $(\mathscr{R}, \mathscr{S})$ is any inertial pair, then the classical augmentation ideal in $K(I\mathscr{X})_{\mathbb{Q}}$ and inertial augmentation ideal generate the same topology on the abelian group $K(I\mathscr{X})$. It follows that the summand $\widehat{K}(I\mathscr{X})$ inherits any inertial λ-ring structure from $K(I\mathscr{X})$.

This allows us to formulate a λ-ring variant of the HKRC for orbifolds $\mathscr{X} = [X/G]$ with G diagonalizable. Precisely, we expect there to be an isomorphism of λ-rings (after tensoring with $\mathbb{C}$) between $\widehat{K}(I\mathscr{X})$ with its virtual orbifold product and $K(Z)$, where Z is a hyper-Kähler resolution of the cotangent bundle $\mathbb{T}^*\mathscr{X}$.

We conclude by proving this conjecture for the weighted projective line $\mathbb{P}(1, n)$ for $n = 2, 3$. We also obtain an isomorphism $(A^*(I\mathbb{P}(1, n))_{\mathbb{C}}, \star_{\mathrm{virt}}) \cong A^*(Z)_{\mathbb{C}}$ of Chow rings commuting with the corresponding Chern characters. Furthermore, we show that the semigroup of inertial λ-positive elements induces an exotic integral lattice structure on $(K(I\mathbb{P}(1, n))_{\mathbb{C}}, \star_{\mathrm{virt}})$ and $(A^*(I\mathbb{P}(1, n))_{\mathbb{C}}, \star_{\mathrm{virt}})$ which corresponds to the ordinary integral lattice in $K(Z)_{\mathbb{C}}$ and $A^*(Z)_{\mathbb{C}}$, respectively.

Finally, our analysis suggests the following interesting question:

Question 1.1. *Is there a category associated to the crepant resolution Z whose Grothendieck group* (*with $\mathbb{C}$-coefficients*) *is isomorphic as a λ-ring to the virtual orbifold K-theory* $(K(I\mathscr{X})_{\mathbb{C}}, \star_{\mathrm{virt}})$ *before completion at the augmentation ideal*?

Remark 1.2. It has subsequently been shown [Kimura and Sweet 2013] that the results (namely Propositions 7.59 and 7.64 and Theorem 7.69) in this paper for the virtual K-theory of $\mathbb{P}(1, n)$ for $n = 2, 3$ generalize to all n. This verifies the conjectured relationship between the virtual K-theory ring and the K-theory of the crepant resolution Z_n of $T^*\mathbb{P}(1, n)$ for all n.

Outline of the paper. We begin by briefly reviewing the results of [Edidin et al. 2010; 2016] on inertial pairs, inertial products, and inertial Chern characters.

We then briefly recall the classical λ-ring and ψ-ring structures in ordinary equivariant K-theory, including the Adams (power) operations, Bott classes, Grothendieck's γ-classes, and some relations among these and the Chern classes.

For Gorenstein inertial pairs we define a theory of Chern classes and, for strongly Gorenstein inertial pairs, power (Adams) operations on inertial K-theory. Since the inertial pair associated to the virtual product of [González et al. 2007] is always strongly Gorenstein, this produces Chern classes and power operations in that theory.

We show that, for strongly Gorenstein inertial pairs, the inertial Chern classes satisfy a relation like that for usual Chern classes, expressing the Chern classes in terms of the orbifold ψ-operations and λ-operations. Finally we prove that, if G is diagonalizable, the orbifold Adams operations are homomorphisms relative to the inertial product. This shows that the virtual K-theory of a toric Deligne–Mumford stack has ψ-ring and λ-ring structures. We also give an example to show that the diagonalizability condition is necessary for obtaining a λ-ring structure.

We then develop the theory of λ-positive elements for a λ-ring and show that λ-positive elements of degree d share many of the same properties as classes of rank-d vector bundles; for example, they have a top Chern class in Chow theory and an Euler class in K-theory. We also introduce the notion of an inertial dual, which is needed to define the Euler class in inertial K-theory.

We conclude by working through some examples, including that of $B\mu_2$ and the virtual K-theory of the weighted projective lines $\mathbb{P}(1, 2)$ and $\mathbb{P}(1, 3)$.

The λ-positive elements, and especially the λ-line elements in the virtual theory, allow us to give a simple presentation of the K-theory ring with the virtual product and a simple description of the virtual first Chern classes. This allows us to prove that the completion of this ring with respect to the augmentation ideal is isomorphic as a λ-ring to the usual K-theory of the resolution of singularities of the cotangent orbifolds $T^*\mathbb{P}(1, 2)$ and $T^*\mathbb{P}(1, 3)$, respectively.

2. Background material

To make this paper self-contained, we recall some background material from [Edidin et al. 2010; 2016], but first we establish some notation and conventions.

Notation. We work entirely in the complex algebraic category. We will work exclusively with a smooth Deligne–Mumford stack $\mathscr{X}$ with *finite stabilizer*, by which we mean the inertia map $I\mathscr{X} \to \mathscr{X}$ is finite (see Definition 2.1 for the formal definition and more detail). We will also assume that every stack $\mathscr{X}$ has the *resolution property*. This means that every coherent sheaf is the quotient of a locally free sheaf. This assumption has two consequences. The first is that the natural map $K(\mathscr{X}) \to G(\mathscr{X})$ is an isomorphism, where $K(\mathscr{X})$ is the Grothendieck ring of vector bundles and $G(\mathscr{X})$ is the Grothendieck group of coherent sheaves. The second consequence is that $\mathscr{X}$ is a *quotient stack* [Totaro 2004]. This means that $\mathscr{X} = [X/G]$, where G is a linear algebraic group acting on an affine scheme X.

If $\mathscr{X}$ is a smooth Deligne–Mumford stack, we will explicitly choose a presentation $\mathscr{X} = [X/G]$. This allows us to identify the Grothendieck ring $K(\mathscr{X})$ with the equivariant Grothendieck ring $K_G(X)$, and the Chow ring $A^*(\mathscr{X})$ with the equivariant Chow ring $A^*_G(X)$. We will use the notation $K(\mathscr{X})$ and $K_G(X)$ (respectively $A^*(\mathscr{X})$ and $A^*_G(X)$) interchangeably.

Definition 2.1. Let G be an algebraic group acting on a scheme X. We define the *inertia scheme*

$$I_G X := \{(g, x) \mid gx = x\} \subseteq G \times X.$$

There is an induced action of G on $I_G X$ given by $g \cdot (m, x) = (gmg^{-1}, gx)$. The quotient stack $I\mathscr{X} = [I_G X/G]$ is the *inertia stack* of the quotient $\mathscr{X} := [X/G]$.

More generally, we define the higher inertia spaces to be the k-fold fiber products

$$I^k_G X = I_G X \times_X \cdots \times_X I_G X.$$

The quotient stack $I^k\mathscr{X} := [I^k_G X/G]$ is the corresponding higher inertia stack.

The composition $\mu : G \times G \to G$ induces a composition $\mu : I^2_G X \to I_G X$. This composition makes $I_G X$ into an X-group with identity section $X \to I_G X$ given by $x \mapsto (1, x)$. Furthermore, for $i = 1, 2$, the projection map $e_i : I^2_G X \to I_G X$ is called the *i-th evaluation map*, since it corresponds to the evaluation morphism in Gromov–Witten theory.

Definition 2.2. Let $\Psi \subset G$ be a conjugacy class. We define

$$I(\Psi) = \{(g, x) \mid gx = x,\ g \in \Psi\} \subset G \times X.$$

More generally, let $\Phi \subset G^l$ be a diagonal conjugacy class. We define $I^l(\Phi) = \{(m_1, \dots, m_l, x) \mid (m_1, \dots, m_l) \in \Phi \text{ and } m_i x = x \text{ for all } i = 1, \dots, l\}$.

By definition, $I(\Psi)$ and $I^l(\Phi)$ are G-invariant subsets of $I_G X$ and $I^l_G(X)$, respectively. Since G acts with finite stabilizer on X, the conjugacy class $I(\Psi)$ is empty unless Ψ consists of elements of finite order. Likewise, $I^l(\Phi)$ is empty unless every l-tuple $(m_1, \dots, m_l) \in \Phi$ generates a finite group. Since conjugacy classes of elements of finite order are closed, $I(\Psi)$ and $I^l(\Phi)$ are closed.

Proposition 2.3 [Edidin et al. 2010, Propositions 2.11 and 2.17]. *The conjugacy class $I(\Psi)$ is empty for all but finitely many Ψ, and each $I(\Psi)$ is a union of connected components of $I_G X$. Likewise, $I^l(\Phi)$ is empty for all but finitely many* *diagonal conjugacy classes* $\Phi \subset G^l$, *and each* $I^l(\Phi)$ *is a union of connected components of* $I^l_G(X)$.

Definition 2.4. In the special case that $\Psi = (1)$ is the class of the identity element $1 \in G$, the locus $I((1)) = \{(1, x) \mid x \in X\} \subset I_G X$, often written X^1, is canonically identified with X. It is an open and closed subset of $I_G X$, but is not necessarily connected. We often call X^1 the *untwisted sector of* $I_G X$ and the other loci $I(\Psi)$ for $\Psi \neq (1)$ the *twisted sectors*.

Similarly, the groups $A^*_G(X^1)$ and $K_G(X^1)$ are summands of $A^*_G(I_G X)$ and $K_G(I_G X)$, respectively, and each is called the *untwisted sector* of $A^*_G(I_G X)$ or $K_G(I_G X)$, respectively. The summands of $A^*_G(I_G X)$ and $K_G(I_G X)$ corresponding to the twisted sectors of $I_G X$ are also called *twisted sectors*.

Definition 2.5. If E is a G equivariant vector bundle on X, the element $\lambda_{-1}(E^*) = \sum_{i=0}^{\infty}(-1)^i[\Lambda^i E^*] \in K_G(X)$ is called the *K-theoretic Euler class* of E. (Note that this sum is finite.)

Likewise, we define the *Chow-theoretic Euler class* of E to be the element $c_{\mathbf{top}}(E) \in A^*_G(X)$, corresponding to the sum of the top Chern classes of E on each connected component of $[X/G]$ (see [Edidin and Graham 1998] for the definition and properties of equivariant Chern classes). These definitions can be extended to any nonnegative element by multiplicativity. It will be convenient to use the symbol $\mathrm{eu}(\mathscr{F})$ to denote both of these Euler classes for a nonnegative element $\mathscr{F} \in K_G(X)$.

Rank and augmentation homomorphisms. If $[X/G]$ is connected, then the rank of a vector bundle defines an augmentation homomorphism $\epsilon : K_G(X) \to \mathbb{Z}$. If we denote by $\mathbf{1}$ the class of the trivial bundle on X, then the decomposition of an element $x = \epsilon(x)\mathbf{1} + (x - \epsilon(x)\mathbf{1})$ gives a decomposition of $K_G(X)$ into a sum of $K_G(X)$-modules $K_G(X) = \mathbb{Z} + I$, where $I = \ker(\epsilon)$ is the augmentation ideal. From this point of view, we can equivalently define the augmentation as the projection endomorphism $K_G(X) \to K_G(X)$ given by $x \mapsto \mathrm{rk}(x)\mathbf{1}$, where rk is the usual notion of rank for classes in equivariant K-theory.

Since we frequently work with a group G acting on a space X where the quotient stack $[X/G]$ is not connected, some care is required in the definition of the rank of a vector bundle. Note that, for any X, the group $A^0_G(X)$ satisfies $A^0_G(X) = \mathbb{Z}^l$, where l is the number of connected components of the quotient stack $\mathscr{X} = [X/G]$. Since $\mathscr{X}$ has finite type, l is finite.

Definition 2.6. Any $\alpha \in K_G(X)$ uniquely determines an element α_U of $K(U)$ on each connected component U of $[X/G]$. If we fix an ordering of the components, then we define the *rank* of α to be the l-tuple in $\mathbb{Z}^l = A^0_G(X)$ whose component in the factor corresponding to a connected component U is the usual rank of α_U.

This agrees with the degree-zero part of the Chern character:

$$\mathrm{rk}(\alpha) := \mathrm{Ch}^0(\alpha) \in A^0_G(X) = \mathbb{Z}^l.$$

In this paper, where we study exotic λ and ψ-ring structures on equivariant K-theory of $K_G(I_G X)$, we will need to define corresponding exotic augmentations. To facilitate their definitions we introduce the more general notion of an augmented ring.

Definition 2.7 (compare [Cartan and Eilenberg 1956, p. 143]). An *augmentation homomorphism* of a ring R is an endomorphism ϵ of R that is a projection, i.e., $\epsilon \circ \epsilon = \epsilon$. The kernel of ϵ is called the *augmentation ideal* of R. The ring R is said to be a *ring with augmentation*.

Remark 2.8. In the language of [loc. cit.], the image of ϵ is called the augmentation module. Our definition is more restrictive than that of [loc. cit.], since it requires that R split as $R = \epsilon(R) + I$, where $\epsilon(R)$ is the augmentation module and I is the augmentation ideal.

Note that all rings have two trivial augmentations coming from the identity and zero homomorphisms. However, in our applications, ϵ will preserve unity in R.

We illustrate the use of this terminology by defining an augmentation homomorphism on $K_G(Y)$ when $[Y/G]$ is not necessarily connected.

Definition 2.9. In equivariant K-theory we define the *augmentation homomorphism* $\epsilon : K_G(Y) \to K_G(Y)$ to be the map which, for each connected component $[U/G]$ of $[Y/G]$, sends each $\mathscr{F}$ in $K_G(Y)$ supported on U to the rank of $\mathscr{F}$ times the structure sheaf $\mathscr{O}_U$:

$$\epsilon(\mathscr{F}|_U) := \mathrm{Ch}^0(\mathscr{F}|_U)\mathscr{O}_U.$$

Thus, for equivariant K-theory, the image of ϵ is isomorphic as a ring to $\mathbb{Z}^{\oplus l}$, where l is the number of connected components of $[Y/G]$. However, we will see that this property need not hold for inertial K-theory.

Inertial products, Chern characters, and inertial pairs. We review here the results from [Edidin et al. 2016], defining a generalization of orbifold cohomology, obstruction bundles, age grading, and stringy Chern character, by defining *inertial products* on $K_G(I_G X)$ and $A^*_G(I_G X)$ using *inertial pairs* $(\mathscr{R}, \mathscr{S})$, where $\mathscr{R}$ is a G-equivariant vector bundle on $I^2_G X$ and $\mathscr{S} \in K_G(I_G X)_{\mathbb{Q}}$ is a nonnegative class satisfying certain compatibility properties.

For each such pair, there is also a rational grading on the total Chow group, and a Chern character ring homomorphism. There are many inertial pairs, and hence there are many associative inertial products on $K_G(I_G X)$ and $A^*_G(I_G X)$ with rational gradings and Chern character ring homomorphisms. The orbifold products on $K(I\mathscr{X})$ and $A^*(I\mathscr{X})$ and the Chern character homomorphism of [Jarvis et al. 2007] are a special case, as is the virtual product of [González et al. 2007].

Definition 2.10. If $\mathscr{R}$ is a vector bundle on $I^2_G X$, we define products on $A^*_G(I_G X)$ (resp. $K_G(I_G X)$) via the formula

$$x \star_{\mathscr{R}} y := \mu_*(e_1^* x \cdot e_2^* y \cdot \mathrm{eu}(\mathscr{R})) \tag{2.11}$$

for $x, y \in A^*_G(I_G X)$ (resp. $K_G(I_G X)$), where $\mu : I^2_G X \to I_G X$ is the composition map, and $e_1, e_2 : I^2_G X \to I_G X$ are the evaluation maps.

To define an inertial pair requires a little more notation from [Edidin et al. 2010], which we recall here. Consider $(m_1, m_2, m_3) \in G^3$ such that $m_1 m_2 m_3 = 1$, and let $\Phi_{1,2,3}$ be the conjugacy class of (m_1, m_2, m_3). Let $\Phi_{12,3}$ be the conjugacy class of $(m_1 m_2, m_3)$ and $\Phi_{1,23}$ the conjugacy class of $(m_1, m_2 m_3)$. Let $\Phi_{i,j}$ be the conjugacy class of the (m_i, m_j) with $i < j$. Finally, let Φ_{ij} be the conjugacy class of $m_i m_j$, and let Φ_i be the conjugacy class of m_i. There are composition maps $\mu_{12,3} : I^3(\Phi_{1,2,3}) \to I^2(\Phi_{12,3})$ and $\mu_{1,23} : I^3(\Phi_{1,2,3}) \to I^2(\Phi_{1,23})$. The various maps we have defined are related by the following Cartesian diagrams, where all maps are local complete intersection morphisms:

$$\begin{array}{ccc} I^3(\Phi_{1,2,3}) & \xrightarrow{e_{1,2}} & I^2(\Phi_{1,2}) \\ \downarrow{\scriptstyle \mu_{12,3}} & & \downarrow{\scriptstyle \mu} \\ I^2(\Phi_{12,3}) & \xrightarrow{e_1} & I(\Phi_{12}) \end{array} \qquad \begin{array}{ccc} I^3(\Phi_{1,2,3}) & \xrightarrow{e_{2,3}} & I^2(\Phi_{2,3}) \\ \downarrow{\scriptstyle \mu_{1,23}} & & \downarrow{\scriptstyle \mu} \\ I^2(\Phi_{1,23}) & \xrightarrow{e_1} & I(\Phi_{23}) \end{array} \tag{2.12}$$

Let $E_{1,2}$ and $E_{2,3}$ be the respective excess normal bundles of the two diagrams (2.12).

Definition 2.13. Given a nonnegative element $\mathscr{S} \in K_G(I_G X)_{\mathbb{Q}}$ and G-equivariant vector bundle $\mathscr{R}$ on $I^2_G X$ we say that $(\mathscr{R}, \mathscr{S})$ is an *inertial pair* if the following conditions hold:

(a) The identity
$$\mathscr{R} = e_1^* \mathscr{S} + e_2^* \mathscr{S} - \mu^* \mathscr{S} + T_\mu \tag{2.14}$$
holds in $K_G(I^2_G X)$, where $T_\mu = T I^2_G X - \mu^*(T I_G X)$ is the relative tangent bundle of μ.

(b) $\mathscr{R}|_{I^2(\Phi)} = 0$ for every conjugacy class $\Phi \subset G \times G$ such that $e_1(\Phi) = 1$ or $e_2(\Phi) = 1$.

(c) $i^* \mathscr{R} = \mathscr{R}$, where $i : I^2_G X \to I^2_G X$ is the isomorphism
$$i(m_1, m_2, x) = (m_1 m_2 m_1^{-1}, m_1, x).$$

(d) $e_{1,2}^* \mathscr{R} + \mu_{12,3}^* \mathscr{R} + E_{1,2} = e_{2,3}^* \mathscr{R} + \mu_{1,23}^* \mathscr{R} + E_{2,3}$ for each triple m_1, m_2, m_3 with $m_1 m_2 m_3 = 1$.

Proposition 2.15 [Edidin et al. 2010, §3]. *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then the $\star_{\mathscr{R}}$ product is commutative and associative with identity $\mathbf{1}_X$, where $\mathbf{1}_X$ is the identity class in the untwisted sector $A^*_G(X^1)$ (respectively $K_G(X^1)$).*

Proposition 2.16 [Edidin et al. 2016, Proposition 3.8]. *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then the map*

$$\widetilde{\mathscr{C}h} : K_G(I_G X)_{\mathbb{Q}} \to A^*_G(I_G X)_{\mathbb{Q}},$$

defined by $\widetilde{\mathscr{C}h}(V) = \mathrm{Ch}(V) \cdot \mathrm{Td}(-\mathscr{S})$, is a ring homomorphism with respect to the $\star_{\mathscr{R}}$-inertial products on $K_G(I_G X)$ and $A^*_G(I_G X)$.

It is shown in [Edidin et al. 2016] that there are two inertial pairs for every G-equivariant vector bundle on X. Most of our results in this paper apply to general inertial pairs, but we have a special interest in the inertial pair associated to the *orbifold product* of [Chen and Ruan 2004; Abramovich et al. 2002; Fantechi and Göttsche 2003; Jarvis et al. 2007; Edidin et al. 2010] and in the inertial pair associated to the *virtual product* of [González et al. 2007].

Definition 2.17. Let $p : X \to \mathscr{X}$ be the quotient map, $T_{\mathscr{X}}$ be the tangent bundle of $\mathscr{X}$, and $\mathbb{T} = p^* T_{\mathscr{X}}$ in $K_G(X)$. In [Edidin et al. 2010, Lemma 6.6] we proved that $\mathbb{T} = T_X - \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G and T_X is the tangent bundle on X.

Definition 2.18. The inertial pair associated to the *orbifold product* is given by the element $\mathscr{S} = \mathscr{S}(\mathbb{T}) \in K_G(I_G X)_{\mathbb{Q}}$, defined as follows. For any $m \in G$ of finite order r, the element $\mathscr{S}$, when restricted to $X^m = \{(x, m) \mid mx = x\} \subset I_G X$, is

$$\mathscr{S}_m := \sum_{k=1}^{r-1} \frac{k}{r} \mathbb{T}_{m,k}, \tag{2.19}$$

where $\mathbb{T}_{m,k}$ is the eigenbundle of $\mathbb{T}$ on which m acts as $e^{2\pi i k/r}$. The first property of inertial pairs (see Definition 2.13(a)) then gives an explicit formula for $\mathscr{R}$:

$$\mathscr{R} = e_1^* \mathscr{S} + e_2^* \mathscr{S} - \mu^* \mathscr{S} + T_\mu.$$

Definition 2.20. The inertial pair associated to the *virtual product* is given by $\mathscr{S} = N$, where N is the quotient $q^* T_X / T_{I_G X}$ and $q : I_G X \to X$ is the canonical morphism, and

$$\mathscr{R} = \mathbb{T}|_{I^2_G X} + \mathbb{T}_{I^2_G X} - e_1^* \mathbb{T}_{I_G X} - e_2^* \mathbb{T}_{I_G X}. \tag{2.21}$$

Here $\mathbb{T}|_{I^2_G X}$ is the pullback of the bundle $\mathbb{T}$ to $I^2_G X$ via the natural map $I^2_G X \to X$ and $\mathbb{T}_{I_G X}$ (resp. $\mathbb{T}_{I^2_G X}$) is the pullback to $I_G X$ (resp. $I^2_G X$) of the tangent bundle to $I\mathscr{X} = [I_G X/G]$ (resp. the stack $I^2 \mathscr{X} = [I^2_G X/G]$).

Remark 2.22. By abuse of notation we will refer to the bundle N defined above as the normal bundle to the morphism $I_G X \to I X$.

Remark 2.23. In [Edidin et al. 2016] we showed that the pairs for both the orbifold product and the virtual orbifold product are indeed inertial pairs.

Definition 2.24. Given any nonnegative element $\mathscr{S} \in K_G(I_G X)_{\mathbb{Q}}$, we define the *$\mathscr{S}$-age* on a component U of $I_G X$ corresponding to a connected component $[U/G]$ of $[I_G X/G]$ to be the rational rank of $\mathscr{S}$ on the component U:

$$\mathrm{age}_{\mathscr{S}}(U) = \mathrm{rk}(\mathscr{S})_U.$$

We define the *$\mathscr{S}$-degree* of an element $x \in A^*_G(I_G X)$ on such a component U of $I_G X$ to be

$$\deg_{\mathscr{S}} x|_U = \deg x|_U + \mathrm{age}_{\mathscr{S}}(U),$$

where $\deg x$ is the degree with respect to the usual grading by codimension on $A^*_G(I_G X)$. Similarly, if $\mathscr{F} \in K_G(I_G X)$ is supported on U, then its $\mathscr{S}$-degree is

$$\deg_{\mathscr{S}} \mathscr{F} = \mathrm{age}_{\mathscr{S}}(U) \bmod \mathbb{Z}.$$

This yields a $\mathbb{Q}/\mathbb{Z}$-grading of the group $K_G(I_G X)$.

Proposition 2.25 [Edidin et al. 2016, Proposition 3.11]. *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then the $\mathscr{R}$-inertial products on $A^*_G(I_G X)$ and $K_G(I_G X)$ respect the $\mathscr{S}$-degrees. Furthermore, the inertial Chern character homomorphism*

$$\widetilde{\mathscr{C}h} : K_G(I_G X) \to A^*_G(I_G X)$$

preserves the $\mathscr{S}$-degree modulo $\mathbb{Z}$.

Definition 2.26. Let $A^{\{q\}}_G(I_G X)$ be the subspace in $A^*_G(I_G X)$ of elements with an $\mathscr{S}$-degree of $q \in \mathbb{Q}^l$, where l is the number of connected components of $I\mathscr{X}$.

Definition 2.27. Given a nonnegative $\mathscr{S} \in K_G(I_G X)_{\mathbb{Q}}$, the homomorphism $\widetilde{\mathscr{C}h}^0 : K_G(I_G X) \to A^{\{0\}}_G(I_G X)$ is called the *inertial rank for $\mathscr{S}$* or just the *$\mathscr{S}$-rank*.

The *inertial augmentation homomorphism* $\tilde{\epsilon} : K_G(I_G X) \to K_G(I_G X)$ is the map which, for each connected component $[U/G]$ of $[(I_G X)/G]$, sends each $\mathscr{F}$ in $K_G(I_G X)$ supported on U to

$$\tilde{\epsilon}(\mathscr{F}|_U) = \widetilde{\mathscr{C}h}^0(\mathscr{F}|_U)\mathscr{O}_U.$$

Hence, if $\star$ is an inertial product associated to an inertial pair $(\mathscr{R}, \mathscr{S})$, then $(K_G(I_G X), \star, 1, \tilde{\epsilon})$ is a ring with augmentation.

Remark 2.28. Note that the restriction $\widetilde{\mathscr{C}h}^0(\mathscr{F})|_U$ of the inertial rank to a component is equal to the classical rank if the $\mathscr{S}$-age of that component is zero, and $\widetilde{\mathscr{C}h}^0(\mathscr{F})|_U$ vanishes if the age is nonzero. Hence the product $\widetilde{\mathscr{C}h}^0(\mathscr{F}|_U)\mathscr{O}_U$ makes sense.

Definition 2.29. An inertial pair $(\mathscr{R}, \mathscr{S})$ is called *Gorenstein* if $\mathscr{S}$ has integral rank and *strongly Gorenstein* if $\mathscr{S}$ is represented by a vector bundle.

The Deligne–Mumford stack $\mathscr{X} = [X/G]$ is *strongly Gorenstein* if the inertial pair associated to the orbifold product (as in Definition 2.18) is strongly Gorenstein.

Note that the inertial pair for the virtual product is always strongly Gorenstein.

3. Review of λ-ring and ψ-ring structures in equivariant K-theory

In this section, we review the λ-ring and ψ-ring structures in equivariant K-theory and describe the Bott cannibalistic classes θ^j, as well as the Grothendieck γ-classes. The main theorems about these classes are the Adams–Riemann–Roch theorem (Theorem 3.34) and Theorem 3.25, which describes relations among the Chern character, the ψ-classes, the Chern classes, and the γ-classes.

Recall that a λ-ring is a commutative ring R with unity 1 and a map $\lambda_t : R \to R[\![t]\!]$, where

$$\lambda_t(a) =: \sum_{i\geq 0} \lambda^i(a)t^i, \tag{3.1}$$

such that the following are satisfied for all x, $y \in R$ and for all integers m, $n \geq 0$:

$$\lambda^0(x) = 1, \quad \lambda_t(1) = 1+t, \quad \lambda^1(x) = x, \quad \lambda_t(x+y) = \lambda_t(x)\lambda_t(y),$$

$$\lambda^n(xy) = \boldsymbol{P}_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y)), \tag{3.2}$$

$$\lambda^m(\lambda^n(x)) = \boldsymbol{P}_{m,n}(\lambda^1(x), \dots, \lambda^{mn}(x)), \tag{3.3}$$

where $\boldsymbol{P}_n$, and $\boldsymbol{P}_{m,n}$ are certain universal polynomials, independent of x and y (see [Fulton and Lang 1985, §I.1]).

Definition 3.4. If a λ-ring R is a $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field of characteristic 0, then we call $(R, \cdot, 1, \lambda)$ a *λ-algebra over* $\mathbb{K}$ if, for all α in $\mathbb{K}$ and all a in R, we have

$$\lambda_t(\alpha a) = \lambda_t(a)^\alpha := \exp(\alpha \log \lambda_t(a)). \tag{3.5}$$

Note that $\log \lambda_t$ makes sense because any series for λ_t starts with 1.

Remark 3.6. The significance of the universal polynomials in the definition of a λ-ring is that one can calculate $\lambda^n(xy)$ and $\lambda^m(\lambda^n(x))$ in terms of $\lambda^i(x)$ and $\lambda^j(y)$ by applying a formal splitting principle.

For example, suppose we wish to express $\lambda_t(x \cdot y)$ in terms of $\lambda_t(x)$ and $\lambda_t(y)$. First, replace x by the formal sum $x \mapsto \sum_{i=1}^\infty x_i$, where we assume that $\lambda_t(x_i) = 1 + tx_i$ for all i, and similarly replace y by the formal sum $y \mapsto \sum_{i=1}^\infty y_i$ in $\lambda_t(x \cdot y)$, where we assume that $\lambda_t(y_i) = 1 + ty_i$ for all i. The fact that $\lambda_t(x_i) = 1 + tx_i$ and

$\lambda_t(y_j) = 1 + ty_j$ means that $\lambda_t(x_i y_j) = 1 + tx_i y_j$, and multiplicativity gives us

$$\lambda_t(x \cdot y) = \prod_{i,j=1}^{\infty} (1 + tx_i y_j).$$

Therefore, $\lambda^n(x \cdot y)$ corresponds to the n-th elementary symmetric function $e_n(xy)$ in the variables $\{x_i y_j\}_{i,j=1}^{\infty}$, but $e_n(xy)$ can be uniquely expressed as a polynomial $\boldsymbol{P}_n$ in the variables $\{e_1(x), \dots, e_n(x), e_1(y), \dots, e_n(y)\}$, where $e_q(x)$ denotes the q-th elementary symmetric function in the variables $\{x_i\}_{i=1}^{\infty}$ and $e_r(y)$ denotes the r-th elementary symmetric function in the variables $\{y_i\}_{i=1}^{\infty}$. Replacing $e_q(x)$ by $\lambda^q(x)$ and $e_r(y)$ by $\lambda^r(y)$ in $\boldsymbol{P}_n$ for all $q, r \in \{1, \dots, n\}$ yields the universal polynomial $\boldsymbol{P}_n(\lambda^1(x), \dots, \lambda^n(x), \lambda^1(y), \dots, \lambda^n(y))$ appearing in the definition of a λ-ring.

A similar analysis holds for $\boldsymbol{P}_{m,n}$.

A closely related structure is that of a ψ-ring.

Definition 3.7. A commutative ring R with unity 1 together with a collection of ring homomorphisms $\psi^n : R \to R$ for each $n \geq 1$ is called a ψ-ring if, for all $x\ y \in R$ and all integers $n \geq 1$, we have

$$\psi^1(x) = x \quad \text{and} \quad \psi^m(\psi^n(x)) = \psi^{mn}(x).$$

The map $\psi^i : R \to R$ is called the *i-th Adams operation* (or *power operation*).

If the ψ-ring $(R, \cdot, 1, \psi)$ is a $\mathbb{K}$-algebra, then $(R, \cdot, 1, \psi)$ is said to be a *ψ-algebra over $\mathbb{K}$* if, in addition, ψ^n is a $\mathbb{K}$-linear map.

Theorem 3.8 (cf. [Knutson 1973, p. 49]). *Let $(R, \cdot, 1, \lambda)$ be a commutative λ-ring and let $\psi_t : R \to R[[t]]$ be given by*

$$\psi_t = -t \frac{d \log \lambda_{-t}}{dt}. \tag{3.9}$$

Expanding ψ_t as $\psi_t := \sum_{n \geq 1} \psi^n t^n$ defines $\psi^n : R \to R$ for all $n \geq 1$, and the resulting ring $(R, \cdot, 1, \psi)$ is a ψ-ring.

Conversely, if $(R, \cdot, 1, \psi)$ is a ψ-ring and $\lambda_t : R_{\mathbb{Q}} \to R_{\mathbb{Q}}[[t]]$ is defined by

$$\lambda_t = \exp\left(\sum_{r \geq 1} (-1)^{r-1} \psi^r \frac{t^r}{r} \right), \tag{3.10}$$

then $(R_{\mathbb{Q}}, \cdot, 1, \lambda)$ is a λ-algebra over $\mathbb{Q}$.

It follows from the definition of the ψ-operations in terms of λ-operations, (3.9), and (3.3) that

$$\lambda^i \circ \psi^j = \psi^j \circ \lambda^i \tag{3.11}$$

for all $i \geq 0$ and $j \geq 1$ as maps from $R \to R$.

Remark 3.12. As in Remark 3.6, the k-th λ-operation λ^k corresponds to the k-th elementary symmetric function. Equation (3.10) implies that the k-th power operation, ψ^k, corresponds to the k-th power sum symmetric function, since this equation is nothing more than the well-known relationship between the elementary symmetric functions and the power sums.

Let G be an algebraic group acting on an algebraic space X. The Grothendieck ring $(K_G(X), \cdot, 1)$ of G-equivariant vector bundles on X is a unital commutative ring, where $\cdot$ is the tensor product and 1 is the structure sheaf $\mathscr{O}_X$ of X.

It is well known that (nonequivariant) K-theory with exterior powers is a λ-ring, and the associated ψ-ring satisfies $\psi^k(\mathscr{L}) = \mathscr{L}^{\otimes k}$ for all line bundles $\mathscr{L}$. A lengthy but straightforward argument shows that an equivariant version of the splitting principle holds. One can then use the splitting principle with the fact that exterior powers (and the associated ψ-operations) respect G-equivariance to prove the following proposition:

Proposition 3.13 (cf. [Köck 1998, Lemma 2.4]). *For any G-equivariant vector bundle V on X, define $\lambda^k([V])$ to be the class $[\Lambda^k(V)]$ of the k-th exterior power. This defines a λ-ring structure $(K_G(X), \cdot, 1, \lambda)$ on $K_G(X)$. For any line bundle $\mathscr{L}$ and any integer $k \geq 1$, the corresponding homomorphisms ψ on $(K_G(X), \cdot, 1)$ satisfy*

$$\psi^k(\mathscr{L}) = \mathscr{L}^{\otimes k}. \tag{3.14}$$

Remark 3.15. The λ-ring $K_G(X)$ has still more structure, since any element can be represented as a difference of vector bundles. The collection $\boldsymbol{E}$ of classes of vector bundles in $K_G(X)$ endows the λ-ring $K_G(X)$ with a *positive structure* [Fulton and Lang 1985]. Roughly speaking, this means that $\boldsymbol{E}$ is a subset of the λ-ring consisting of elements of nonnegative rank such that any element in the ring can be written as a difference of elements in $\boldsymbol{E}$, and, for any $\mathscr{F}$ of rank d in $\boldsymbol{E}$, $\lambda_t(\mathscr{F})$ is a degree-d polynomial in t and $\lambda^d(\mathscr{F})$ is invertible (i.e., $\lambda^d(\mathscr{F})$ is a line bundle). Furthermore, $\boldsymbol{E}$ is closed under addition (but not subtraction) and multiplication, $\boldsymbol{E}$ contains the nonnegative integers, and there are special rank-one elements in $\boldsymbol{E}$, namely the line bundles; various other properties also hold. A positive structure on a λ-ring, if it exists, need not be uniquely determined by the λ-ring structure, nor does a general λ-ring possess a positive structure.

For example, if $G = \mathrm{GL}_n$, then the representation ring $R(G)$ can be identified as a subring of Weyl-group-invariant elements in the representation ring $R(T)$, where T is a maximal torus and the λ-ring structure on $R(T)$ restricts to the usual λ-ring structure on $R(G)$. However, the natural set of positive elements in $R(T)$ is generated by the characters of T, and this restricts to the set of positive symmetric linear combinations of characters, which contains, but does not equal, the set of irreducible representations of G.

In Section 6 we will introduce a different but related notion, called a *λ-positive structure*, which is a natural invariant of a λ-ring. This notion will play a central role in our analysis of inertial K-theory.

The λ- and ψ-ring structures behave nicely with respect to the augmentation on equivariant K-theory (Definition 2.9).

Proposition 3.16. *For all $\mathscr{F}$ in $K_G(X)$ and integers $n \geq 1$, we have*

$$\epsilon(\psi^n(\mathscr{F})) = \psi^n(\epsilon(\mathscr{F})) = \epsilon(\mathscr{F}) \tag{3.17}$$

and

$$\epsilon(\lambda_t(\mathscr{F})) = \lambda_t(\epsilon(\mathscr{F})) = (1+t)^{\epsilon(\mathscr{F})}. \tag{3.18}$$

Proof. Assume that $[X/G]$ is connected. Equation (3.18) holds if $\mathscr{F}$ is a rank-d, G-equivariant vector bundle on X since $\lambda^i(\mathscr{F})$ has rank $\binom{d}{i}$. Since $K_G(X)$ is generated under addition by isomorphism classes of vector bundles, the same equation holds for all $\mathscr{F}$ in $K_G(X)$ by multiplicativity of λ_t.

If $[X/G]$ is not connected, we have the ring isomorphism $K_G(X) = \bigoplus_\alpha K_G(X_\alpha)$, where the sum is over α such that $[X_\alpha/G]$ is a connected component of $[X/G]$. Equation (3.18) follows from multiplicativity of λ_t. Equation (3.17) follows from (3.18) and (3.9). □

This motivates the following definition:

Definition 3.19. Let $(R, \cdot, 1, \epsilon)$ be a ring with augmentation. Then $(R, \cdot, 1, \psi, \epsilon)$ is said to be an *augmented ψ-ring* if $(R, \cdot, 1, \psi)$ is a ψ-ring and, for all integers $n > 0$, we have $\epsilon \circ \psi^n = \psi^n \circ \epsilon = \epsilon$ as endomorphisms of R. If R is an augmented ψ-ring, we define $\psi^0 := \epsilon$.

Remark 3.20. The definition $\psi^0 = \epsilon$ is consistent with all the conditions in the definition of a ψ-ring (Definition 3.7).

Definition 3.21. Let $(R, \cdot, 1, \lambda)$ be a λ-algebra (Definition 3.4) over $\mathbb{Q}$ (respectively $\mathbb{C}$). Let $\epsilon : R \to R$ be an augmentation which is also a $\mathbb{Q}$-algebra (respectively $\mathbb{C}$-algebra) homomorphism. We say that $(R, \cdot, 1, \lambda, \epsilon)$ is an *augmented λ-algebra over* $\mathbb{Q}$ (respectively $\mathbb{C}$) if $\epsilon(\lambda_t(\mathscr{F})) = \lambda_t(\epsilon(\mathscr{F})) = (1+t)^{\epsilon(\mathscr{F})}$ for every $\mathscr{F} \in R$. Here the expression $(1+t)^x$ for an element x of the $\mathbb{Q}$-algebra R means

$$(1+t)^x := \sum_{n=0}^{\infty} \binom{x}{n} t^n, \quad \text{where } \binom{x}{n} := \frac{\prod_{i=0}^{n-1}(x-i)}{n!}.$$

The previous proposition implies that ordinary equivariant K-theory is an augmented ψ-ring. In fact, the equivariant Chow ring is also an augmented ψ-ring.

Definition 3.22. For all $n \geq 1$, the map $\psi^n : A^*_G(X) \to A^*_G(X)$ defined by

$$\psi^n(v) = n^d v \tag{3.23}$$

for all v in $A^d_G(X)$ endows $A^*_G(X)$ with the structure of a ψ-ring and, therefore, $A^*_G(X)_{\mathbb{Q}}$ with the structure of a λ-ring. The *augmentation* $\epsilon : A^*_G(X) \to A^0_G(X)$ is the canonical projection.

Associated to any λ-ring there is another (pre-λ-ring) structure, usually denoted by γ. These are the *Grothendieck γ-classes* $\gamma_t : R \to R[\![t]\!]$, given by the formula

$$\gamma_t := \sum_{i=0}^{\infty} \gamma^i t^i := \lambda_{t/(1-t)}. \tag{3.24}$$

Theorem 3.25 (see [Fulton and Lang 1985]). *If Y is a connected algebraic space with a proper action of a linear algebraic group G, and if, for each nonnegative integer i, Ch^i is the degree-i part of the Chern character and c^i is the i-th Chern class, then the following equations hold for all integers $n \geq 1$ and $i \geq 0$ and all $\mathscr{F}$ in $K_G(Y)$:*

$$\mathrm{Ch}^i \circ \psi^n = n^i\, \mathrm{Ch}^i, \tag{3.26}$$

$$c_t(\mathscr{F}) = \exp\biggl(\sum_{n\geq 1} (-1)^{n-1}(n-1)!\, \mathrm{Ch}^n(\mathscr{F}) t^n\biggr), \tag{3.27}$$

$$c^i(\mathscr{F}) = \mathrm{Ch}^i\bigl(\gamma^i(\mathscr{F} - \epsilon(\mathscr{F}))\bigr). \tag{3.28}$$

Remark 3.29. Equation (3.26) is precisely the statement that the Chern character $\mathrm{Ch} : K_G(X)_{\mathbb{Q}} \to A^*_G(X)_{\mathbb{Q}}$ is a homomorphism of ψ-rings and therefore of λ-rings.

In order to define inertial Chern classes and the inertial λ-ring and ψ-ring structures, we will need the so-called *Bott cannibalistic classes*.

Definition 3.30. Let Y be an algebraic space with a proper action of a linear algebraic group G. Denote by $K^+_G(Y)$ the semigroup of classes of G-equivariant vector bundles on Y.

For each $j \geq 1$, the *j-th Bott (cannibalistic) class* $\theta^j : K^+_G(Y) \to K_G(Y)$ is the multiplicative class, defined for any line bundle $\mathscr{L}$ by

$$\theta^j(\mathscr{L}) = \frac{1-\mathscr{L}^j}{1-\mathscr{L}} = \sum_{i=0}^{j-1} \mathscr{L}^i. \tag{3.31}$$

By the splitting principle, we can extend the definition of $\theta^j(\mathscr{F})$ to all $\mathscr{F}$ in $K^+_G(Y)$.

Definition 3.32. Let $\mathfrak{a}_Y$ denote the kernel of the augmentation $\epsilon : K_G(Y) \to K_G(Y)$. It is an ideal in the ring $(K_G(Y), \cdot)$, where $\cdot$ denotes the usual tensor product, and $\mathfrak{a}$ defines a topology on $K_G(Y)$. We denote the completion of $K_G(Y)_{\mathbb{Q}}$ with respect to that topology by $\widehat{K}_G(Y)_{\mathbb{Q}}$.

Remark 3.33. We will need to define Bott classes on elements of integral rank in rational K-theory. This can be done in a straightforward manner, but the resulting class will live in the augmentation completion of rational K-theory. Stated precisely, if $\mathscr{L}$ is a line bundle, then we can expand the power sum for $\psi^j(\mathscr{L})$ as $\psi^j(\mathscr{L}) = j(1 + a_1(\mathscr{L}-1) + \cdots + a_{j-1}(\mathscr{L}-1)^{j-1})$ for some rational numbers $a_1, \dots, a_{j-1}$. Since $(\mathscr{L}-1)$ lies in the augmentation ideal, any fractional power of the expression $1 + a_1(\mathscr{L}-1) + \cdots + a_{j-1}(\mathscr{L}-1)^{j-1}$ can be expanded using the binomial formula as an element of $\widehat{K}_G(Y)_{\mathbb{Q}}$. It follows that, if $\alpha = \sum_i q_i \mathscr{L}_i$ with $\sum_i q_i \in \mathbb{Z}$, then the binomial expansion of the expression

$$j^{\sum_i q_i} \prod_i (1 + a_1(\mathscr{L}_i - 1) + \cdots + a_{j-1}(\mathscr{L}-1)^{j-1})^{q_i}$$

defines $\theta^j(\alpha)$ as an element of $\widehat{K}_G(Y)_{\mathbb{Q}}$.

We will also need the following result:

Theorem 3.34 (the Adams–Riemann–Roch theorem for equivariant regular embeddings [Köck 1991; 1998]). *Let $\iota : Y \hookrightarrow X$ be a G-equivariant closed regular embedding of smooth manifolds. The following commutes for all integers $n \geq 1$:*

$$\begin{array}{ccc} K_G(Y) & \xrightarrow{\theta^n(N_\iota^*)\psi^n} & K_G(Y) \\ \big\downarrow{\scriptstyle \iota_*} & & \big\downarrow{\scriptstyle \iota_*} \\ K_G(X) & \xrightarrow{\psi^n} & K_G(X) \end{array} \tag{3.35}$$

where N_ι^* is the conormal bundle of the embedding ι.

4. Augmentation ideals and completions of inertial K-theory

We will use the Bott classes of $\mathscr{S}$ to define inertial λ- and ψ-ring structures as well as inertial Chern classes. Since $\mathscr{S}$ is generally not integral, we will often need to work in the augmentation completion $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ of $K_G(I_G X)_{\mathbb{Q}}$. However, it is not a priori clear that the inertial product behaves well with respect to this completion, since the topology involved is constructed by taking classical powers of the classical augmentation ideal instead of inertial powers of the inertial augmentation ideal. The surprising result of this section is that, when G is diagonalizable, these two completions are the same.

Definition 4.1. Given any inertial pair $(\mathscr{R}, \mathscr{S})$, define $\mathfrak{a}_{\mathscr{S}}$ to be the kernel of the inertial augmentation $\tilde{\epsilon} : K_G(I_G X) \to K_G(I_G X)$. It is an ideal with respect to the inertial product $\star := \star_{\mathscr{R}}$. Define $\mathfrak{a}_{I\mathscr{X}}$ to be the kernel of the classical augmentation $\epsilon : K_G(I_G X) \to K_G(I_G X)$. It is an ideal of $K_G(I_G X)$ with respect to the usual tensor product instead of the inertial product.

Each of these two ideals induces a topology on $K_G(I_G X)$, and we also consider a third topology induced by the augmentation ideal $\mathfrak{a}_{BG}$ of $R(G)$. By [Edidin and

Graham 2000, Theorem 6.1(a)] the $\mathfrak{a}_{BG}$-adic and $\mathfrak{a}_{I\mathscr{X}}$-adic topologies on $K_G(I_GX)$ are the same. In this section we will show that the $\mathfrak{a}_{\mathscr{S}}$-adic topology agrees with the other two.

Lemma 4.2. *If $(\mathscr{R}, \mathscr{S})$ is an inertial pair, then $(K_G(I_GX), \star_{\mathscr{R}})$ is an $R(G)$-algebra. Moreover, for any $x \in R(G)$, if $\beta_\Psi \in K_G(I(\Psi))$, we have $x\beta_\Psi = x\mathbf{1} \star_{\mathscr{R}} \beta_\Psi$.*

Proof. By definition of an inertial pair, if $\alpha_1 \in K_G(X)$ is supported in the untwisted sector, then $\alpha_1 \star_{\mathscr{R}} \beta_\Psi = f_\Psi^*\alpha \cdot \beta$, where $f_\Psi : I(\Psi) \to X$ is the projection. The lemma now follows from the projection formula for equivariant K-theory. □

Theorem 4.3. *When G is diagonalizable the $\mathfrak{a}_{BG}$-adic, $\mathfrak{a}_{I_GX}$-adic, and $\mathfrak{a}_{\mathscr{S}}$-adic topologies on $K_G(I_GX)$ are all equivalent. In particular, the $\mathfrak{a}_{BG}$-adic, $\mathfrak{a}_{I_GX}$-adic, and $\mathfrak{a}_{\mathscr{S}}$-adic completions of $K_G(I_GX)_{\mathbb{Q}}$ are equal.*

Proof. To prove that the topologies are equivalent we must show the following:

(1) For each positive integer n there is a positive integer r such that

$$\mathfrak{a}_{BG}^{\otimes r} K_G(I_GX) \subseteq (\mathfrak{a}_{\mathscr{S}})^{\star n}.$$

(2) For each positive integer n there is a positive integer r such that

$$(\mathfrak{a}_{\mathscr{S}})^{\star r} \subseteq \mathfrak{a}_{BG}^{\otimes n} K_G(I_GX)_{\mathbb{Q}}.$$

Condition (1) follows from Lemma 4.2 and observing that $\mathfrak{a}_{BG}K_G(I_GX) \subset \mathfrak{a}_{\mathscr{S}}$. In particular, we may take $r = n$.

Condition (2) is more difficult to check. Given a G-space Y, we denote by $\mathfrak{a}_Y$ the subgroup of $K_G(Y)$ of elements of rank 0. This is an ideal with respect to the tensor product.

For each connected component $[U/G]$ of $[I_GX/G]$, the inertial augmentation satisfies $\widetilde{\mathscr{C}h}_0(\alpha)|_U = 0$ if $\operatorname{age}_{\mathscr{S}}(U) > 0$ and $\widetilde{\mathscr{C}h}_0(\alpha)|_U = \mathrm{Ch}_0(\alpha)|_U$ if $\operatorname{age}_{\mathscr{S}}(U) = 0$ [Edidin et al. 2016, Theorem 2.3.9]. So $\mathfrak{a}_{\mathscr{S}}$ has the following decomposition as an abelian group:

$$\mathfrak{a}_{\mathscr{S}} = \bigoplus_{\substack{U \\ \operatorname{age}_{\mathscr{S}}(U)=0}} \mathfrak{a}_U \oplus \bigoplus_{\substack{U \\ \operatorname{age}_{\mathscr{S}}(U)>0}} K_G(U).$$

Lemma 4.4. *If $m \in G$ with $\alpha \in K_G(X^m) \cap \mathfrak{a}_{\mathscr{S}}$ and $\beta \in K_G(X^{m^{-1}}) \cap \mathfrak{a}_{\mathscr{S}}$, then $\alpha \star \beta \in \mathfrak{a}_{I\mathscr{X}}$.*

Proof. Since $mm^{-1} = 1$, we have $\alpha \star \beta \in K_G(X^1) \subset K_G(I_GX)$, so we must show $\alpha \star \beta \in \mathfrak{a}_X$. If $\operatorname{age}_{\mathscr{S}}(X^m) = 0$, then $\alpha_m \in \mathfrak{a}_{X^m}$, so the inertial product

$$\mu_*(e_1^*\alpha \cdot e_2^*\beta \cdot \mathrm{eu}(\mathscr{R}))$$

would automatically be in $\mathfrak{a}_X$ because the finite pushforward μ_* preserves the classical augmentation ideal. Thus we may assume that $\operatorname{age}_{\mathscr{S}}(X^m)$ and $\operatorname{age}_{\mathscr{S}}(X^{m^{-1}})$

are both nonzero and that α and β have nonzero rank as elements of $K_G(X^m)$ and $K_G(X^{m^{-1}})$, respectively. If the fixed locus $X^{m,m^{-1}}$ has positive codimension, then $\mu_*(K_G(X^{m,m^{-1}})) \subset K_G(X^1)$ is also in the classical augmentation ideal, since it consists of classes supported on subspaces of positive codimension. On the other hand, if $X^{m,m^{-1}} = X$, then $T_\mu|_{X^{m,m^{-1}}} = 0$. By definition of an inertial pair, $\mathscr{S}|_{X^1} = 0$, so $\mathscr{R}|_{X^{m,m^{-1}}} = (e_1^*\mathscr{S} + e_2^*\mathscr{S})|_{X^{m,m^{-1}}}$ is a nonzero vector bundle. It follows that $\mathrm{eu}(\mathscr{R}|_{X^{m,m^{-1}}}) \in \mathfrak{a}_{X^{m,m^{-1}}}$, and once again $\alpha \star \beta \in \mathfrak{a}_X$. □

Since G is diagonalizable and acts with finite stabilizer on X, there is a finite abelian subgroup $H \subset G$ such that $X^g = \varnothing$ for all $g \notin H$. Let $s = \sum_{h \in H}(\mathrm{ord}(h) - 1)$.

Lemma 4.5. *The* $(s+1)$*-fold inertial product* $(\mathfrak{a}_{\mathscr{S}})^{\star(s+1)}$ *is contained in* $\mathfrak{a}_{I_G X}$.

Proof. By the definition of s, any list $m_1, \dots, m_{s+1}$ of nonidentity elements of H contains at least one h with multiplicity at least $\mathrm{ord}(h)$. It follows that such a list contains subsets $m_1, \dots, m_k$ and $m_{k+1}, \dots, m_l$ with $m_1 \cdots m_k = (m_{k+1} \cdots m_l)^{-1}$.

Since the inertial product is commutative, we may write any product of the form $\alpha_{m_1} \star \cdots \star \alpha_{m_{s+1}}$ with $\alpha_{m_i} \in K_G(X^{m_i})$ as $\tilde{\alpha}_m \star \tilde{\beta}_{m^{-1}} \star \tilde{\gamma}_{m'}$ for some $\tilde{\alpha}_m \in K_G(X^m)$, $\tilde{\beta}_m \in K_G(X^{m^{-1}})$, and $\tilde{\gamma}_{m'} \in K_G(X^{m'})$. Lemma 4.4 now gives the result. □

To complete the proof of Theorem 4.3, observe first that we may use the equivalence of the $\mathfrak{a}_{BG}$-adic and the $\mathfrak{a}_{I^2_{\mathscr{X}}}$-adic topologies in the ring $(K_G(I^2_G X), \otimes)$ to see that, for any n, there is an r such that $\mathfrak{a}^{\otimes r}_{I^2_G X} \subset \mathfrak{a}^{\otimes n}_{BG} K_G(I^2_G X)$. This implies that $\mu_*(\mathfrak{a}^{\otimes r}_{I^2_G X}) \subset \mathfrak{a}^{\otimes n}_{BG} K_G(I_G X)$. It follows that

$$\mathfrak{a}_{\mathscr{S}}^{\star(r(s+1))} \subset \mathfrak{a}^{\otimes n}_{BG} K_G(I_G X). \qquad \square$$

Since the three topologies are the same we will not distinguish between them from now on, and will use the term *augmentation completion* to denote the completion with respect any one of these augmentation ideals. The completion of $K_G(I_G X)_{\mathbb{Q}}$ will be denoted by $\widehat{K}_G(I_G X)_{\mathbb{Q}}$. Note that this completion is a summand in $K_G(I_G X)_{\mathbb{Q}}$ [Edidin and Graham 2005, Proposition 3.6].

5. Inertial Chern classes and power operations

In this section we show that for each Gorenstein inertial pair $(\mathscr{R}, \mathscr{S})$ and corresponding Chern character $\widetilde{\mathscr{C}h}$, we can define inertial Chern classes. When $(\mathscr{R}, \mathscr{S})$ is strongly Gorenstein, there are also ψ-operations, λ-operations, and γ-operations on the corresponding inertial K-theory $K_G(I_G X)$. These operations behave nicely with respect to the inertial Chern character and satisfy many relations, including an analog of Theorem 3.25. When G is diagonalizable these operations make the inertial K-theory ring $K_G(I_G X)$ into a ψ-ring and $K_G(I_G X) \otimes \mathbb{Q}$ into a λ-ring.

Inertial Adams (power) operations and inertial Chern classes. We begin by defining inertial Chern classes. We then define inertial Adams operations associated to a strongly Gorenstein pair $(\mathscr{R}, \mathscr{S})$ and show that, for a diagonalizable group G, the corresponding rings are ψ-rings with many other nice properties.

Definition 5.1. For any Gorenstein inertial pair $(\mathscr{R}, \mathscr{S})$ the *$\mathscr{S}$-inertial Chern series* $\tilde{c}_t : K_G(I_G X) \to A^*_G(I_G X)_{\mathbb{Q}}[\![t]\!]$ is defined, for all $\mathscr{F}$ in $K_G(I_G X)$, by

$$\tilde{c}_t(\mathscr{F}) = \widetilde{\exp}\bigg(\sum_{n\geq 1}(-1)^{n-1}(n-1)!\,\widetilde{\mathscr{C}h}^n(\mathscr{F})t^n\bigg), \tag{5.2}$$

where the power series $\widetilde{\exp}$ is defined with respect to the $\star_{\mathscr{R}}$ product, and $\widetilde{\mathscr{C}h}^n(\mathscr{F})$ is the component of $\widetilde{\mathscr{C}h}(\mathscr{F})$ in $A^*(I_G X)$ with $\mathscr{S}$-age equal to n. For all $i \geq 0$, the *i-th $\mathscr{S}$-inertial Chern class $\tilde{c}^i(\mathscr{F})$ of $\mathscr{F}$* is the coefficient of t^i in $\tilde{c}_t(\mathscr{F})$.

Remark 5.3. The definition of inertial Chern classes could be extended to the non-Gorenstein case by introducing fractionally graded $\mathscr{S}$-inertial Chern classes, but the latter do not behave nicely with respect to the inertial ψ-structures.

Definition 5.4. Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. We define the *j-th inertial Adams* (or *power*) *operation* $\tilde{\psi}^j : K_G(I_G X) \to K_G(I_G X)$ for each integer $j \geq 1$ by the formula

$$\tilde{\psi}^j(\mathscr{F}) := \psi^j(\mathscr{F}) \cdot \theta^j(\mathscr{S}^*) \tag{5.5}$$

for all $\mathscr{F}$ in $K_G(I_G X)$. (Here $\cdot$ is the ordinary tensor product on $K_G(I_G X)$.)

We show in Theorem 5.23 that, in many cases, these inertial Adams operations define a ψ-ring structure on $(K_G(I_G X), \star_{\mathscr{R}})$.

Remark 5.6. If $(\mathscr{R}, \mathscr{S})$ is Gorenstein, then $\mathscr{S}$ has integral rank and $\theta^j(\mathscr{S}^*)$ may be defined as an element of the completion $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ (see Remark 3.33). Thus we can still define inertial Adams operations as maps $\tilde{\psi}^j : K_G(I_G X) \to \widehat{K}_G(I_G X)_{\mathbb{Q}}$.

Definition 5.7. Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. We define $\tilde{\lambda}_t : K_G(I_G X) \to K_G(I_G X)_{\mathbb{Q}}[\![t]\!]$ by (3.10) after replacing ψ, λ, and exp by their respective inertial analogs $\tilde{\psi}$, $\tilde{\lambda}$, and $\widetilde{\exp}$:

$$\tilde{\lambda}_t = \widetilde{\exp}\bigg(\sum_{r\geq 1}(-1)^{r-1}\tilde{\psi}^r\frac{t^r}{r}\bigg). \tag{5.8}$$

Define $\tilde{\lambda}^i$ to be the coefficient of t^i in $\tilde{\lambda}_t$. We call $\tilde{\lambda}^i$ the *i-th inertial λ operation.*

We now prove a relation between inertial Chern classes, the inertial Chern character, and inertial Adams operations, but first we need two lemmas connecting the classical Chern character, Adams operations, Bott classes, and Todd classes.

Lemma 5.9. *Let $\mathscr{F} \in K_G(I_G X)$ be the class of a G-equivariant vector bundle on $I_G X$. For all integers $n \geq 1$, we have the equality, in $A^*_G(I_G X)$,*

$$\mathrm{Ch}(\theta^n(\mathscr{F}^*))\,\mathrm{Td}(-\mathscr{F}) = n^{\mathrm{Ch}^0(\mathscr{F})}\,\mathrm{Td}(-\psi^n(\mathscr{F})). \tag{5.10}$$

More generally, if $\mathscr{F} \in K_G(I_G X)_{\mathbb{Q}}$ is such that $\mathscr{F} = \sum_{i=1}^k \alpha_i \mathscr{V}_i$, where $\mathscr{V}_i$ is a vector bundle, $\alpha_i \in \mathbb{Q}$ with $\alpha_i > 0$ for all $i = 1, \ldots, k$, and $\mathrm{Ch}^0(\mathscr{F}) \in \mathbb{Z}^l \subset A^0_G(I_G X)_{\mathbb{Q}}$ (l is the number of connected components of $[I_G X/G]$), then (5.10) *still holds in $A^*_G(I_G X)_{\mathbb{Q}}$, where $\theta^n(\mathscr{F}^*)$ is an element in the completion $\widehat{K}_G(I_G X)_{\mathbb{Q}}$.*

Proof. Let $\mathscr{L}$ in $K_G(I_G X)$ be a line bundle with ordinary first Chern class $c := c^1(\mathscr{L})$. For all $n \geq 1$ we have

$$\begin{aligned}
\mathrm{Ch}(\theta^n(\mathscr{L}^*))\,\mathrm{Td}(-\mathscr{L}) &= \mathrm{Ch}\left(\frac{1-(\mathscr{L}^*)^n}{1-\mathscr{L}^*}\right)\left(\mathrm{Td}(\mathscr{L})\right)^{-1} \\
&= \left(\frac{1-e^{-nc}}{1-e^{-c}}\right)\left(\frac{c}{1-e^{-c}}\right)^{-1} \\
&= n\left(\frac{nc}{1-e^{-nc}}\right)^{-1} \\
&= n\,\mathrm{Td}(\mathscr{L}^n)^{-1},
\end{aligned}$$

and we conclude that $\mathrm{Ch}(\theta^n(\mathscr{L}^*))\,\mathrm{Td}(-\mathscr{L}) = n\,\mathrm{Td}(-\psi^n(\mathscr{L}))$. Equation (5.10) now follows from the splitting principle, the multiplicativity of θ^n and Td, and the fact that Ch is a ring homomorphism.

The more general statement follows from the fact that Ch and Td factor through $\widehat{K}_G(I_G X)_{\mathbb{Q}}$ together with the fact that $\mathrm{Ch}^0(\theta^j(\mathscr{F}) - j^{\epsilon(\mathscr{F})}) = 0$. □

This lemma yields the following useful theorem:

Theorem 5.11. *Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. For any $\alpha \in \mathbb{N}$ and integer $n \geq 1$, we have*

$$\widetilde{\mathscr{C}h}^{\alpha}(\tilde{\psi}^n(\mathscr{F})) = n^{\alpha}\,\widetilde{\mathscr{C}h}^{\alpha}(\mathscr{F}) \tag{5.12}$$

in $A^{\{\alpha\}}_G(I_G X)_{\mathbb{Q}}$, where the grading is the $\mathscr{S}$-age grading.

Proof. We have

$$\begin{aligned}
\widetilde{\mathscr{C}h}(\tilde{\psi}^n(\mathscr{F})) &= \mathrm{Ch}(\psi^n(\mathscr{F})\theta^n(\mathscr{S}^*))\,\mathrm{Td}(-\mathscr{S}) \\
&= \mathrm{Ch}(\psi^n(\mathscr{F}))\,\mathrm{Ch}(\theta^n(\mathscr{S}^*))\,\mathrm{Td}(-\mathscr{S}) \\
&= \mathrm{Ch}(\psi^n(\mathscr{F}))\,\mathrm{Td}(-\psi^n(\mathscr{S}))n^{\mathrm{age}} \\
&= \sum_{\alpha\in\mathbb{N}} n^{\alpha}\,\widetilde{\mathscr{C}h}^{\alpha}(\mathscr{F}),
\end{aligned}$$

where the third equality follows from (5.10), and the final equality follows from the definition of $\widetilde{\mathscr{C}h}^{\alpha}$, (3.26), and the fact that, for all $j \geq 0$ and $k \geq 1$,

$$\mathrm{Td}^j \circ \psi^k = k^j \, \mathrm{Td}^j, \tag{5.13}$$

where $\mathrm{Td} = \sum_{j \geq 0} \mathrm{Td}^j$ is such that Td^j belongs to $A^j_G(I_G X)_{\mathbb{Q}}$. Equation (5.13) is proved in the same fashion as (3.26). □

Remark 5.14. If $(\mathscr{R}, \mathscr{S})$ is a Gorenstein inertial pair, then (5.12) also holds in $A^{\{\alpha\}}_G(I_G X)_{\mathbb{Q}}$, where $\tilde{\psi}^n$ is interpreted as a map

$$\tilde{\psi}^n : K_G(I_G X) \to \widehat{K}_G(I_G X)_{\mathbb{Q}}$$

(see Remark 5.6). This follows as $\widetilde{\mathscr{C}h}$ factors through the completion $\widehat{K}_G(I_G X)_{\mathbb{Q}}$.

Definition 5.15. Let $(\mathscr{R}, \mathscr{S})$ be a strongly Gorenstein inertial pair. We define the inertial operations $\tilde{\gamma}_t$ on inertial K-theory as in (3.24), that is,

$$\tilde{\gamma}_t := \sum_{i=0}^{\infty} \tilde{\gamma}^i t^i := \tilde{\lambda}_{t/(1-t)}. \tag{5.16}$$

Remark 5.17. If $(\mathscr{R}, \mathscr{S})$ is only Gorenstein, then we may still define γ_t as a map $K_G(I_G X) \to \widehat{K}_G(I_G X)_{\mathbb{Q}}[\![t]\!]$.

Theorem 5.18. *Let $(\mathscr{R}, \mathscr{S})$ be a Gorenstein inertial pair. The $\mathscr{S}$-inertial Chern series $\tilde{c}_t : K_G(I_G X) \to A^*_G(I_G X)_{\mathbb{Q}}[\![t]\!]$ satisfies the following properties*:

<u>Consistency with $\tilde{\gamma}$</u>: For all integers $n \geq 1$ and all $\mathscr{F}$ in $K_G(I_G X)_{\mathbb{Q}}$, we have the following equality in $A^*_G(I_G X)_{\mathbb{Q}}$:

$$\tilde{c}^n(\mathscr{F}) = \widetilde{\mathscr{C}h}^n\big(\tilde{\gamma}^n(\mathscr{F} - \tilde{\epsilon}(\mathscr{F}))\big), \tag{5.19}$$

where $\tilde{\gamma}_t$ is interpreted as a map $K_G(I_G X)_{\mathbb{Q}} \to \widehat{K}_G(I_G X)_{\mathbb{Q}}[\![t]\!]$.

<u>Multiplicativity</u>: For all $\mathscr{V}$ and $\mathscr{W}$ in $K_G(I_G X)_{\mathbb{Q}}$,

$$\tilde{c}_t(\mathscr{V} + \mathscr{W}) = \tilde{c}_t(\mathscr{V}) \star_{\mathscr{R}} \tilde{c}_t(\mathscr{W}).$$

<u>Zeroth Chern class</u>: For all $\mathscr{V}$ in $K_G(I_G X)_{\mathbb{Q}}$, we have $\tilde{c}^0(\mathscr{V}) = \mathbf{1}$.

<u>Untwisted sector</u>: For all $\mathscr{F} \in K_G(X^1) \subseteq K_G(I_G X)$ (i.e., supported only on the untwisted sector), the inertial Chern classes agree with the ordinary Chern classes, i.e., $\tilde{c}_t(\mathscr{F}) = c_t(\mathscr{F})$.

<u>Classes of unity</u>: All the inertial Chern classes of unity vanish, except for $\tilde{c}^0(\mathbf{1})$, so we have $\tilde{c}_t(\mathbf{1}) = \mathbf{1}$.

Remark 5.20. The theorem shows that (5.19) yields an alternative, but equivalent, definition of inertial Chern classes.

Proof. Multiplicativity and $\tilde{c}^0(\mathscr{V}) = \mathbf{1}$ follow immediately from the exponential form of (5.2) and the fact that $\widetilde{\mathscr{C}h}$ is a homomorphism.

On the untwisted sector, inertial products reduce to the ordinary products, and the inertial Chern character reduces to the classical Chern character, and this shows that (5.2) agrees with (3.27), which implies that the untwisted sector agrees with ordinary Chern classes. The classes of unity condition will follow immediately from (5.19).

The hard part of this proof is the consistency (5.19) of the inertial Chern classes with $\tilde{\gamma}$. To prove this, it will be useful to first introduce the ring homomorphism $\widetilde{\mathscr{C}h}_t : K_G(I_G X) \to A^*_G(I_G X)_{\mathbb{Q}}[[t]]$ via $\widetilde{\mathscr{C}h}_t(\mathscr{F}) := \sum_{n\geq 0} \widetilde{\mathscr{C}h}^n(\mathscr{F})t^n$. For the remainder of the proof, all products are understood to be inertial products. We have the equality, in $A^*_G(I_G X)_{\mathbb{Q}}[[t]]$,

$$
\begin{aligned}
\widetilde{\mathscr{C}h}_t(\tilde{\lambda}_u(\mathscr{F})) &= \widetilde{\exp}\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \widetilde{\mathscr{C}h}_t(\tilde{\psi}^k(\mathscr{F}))u^k \right) \\
&= \widetilde{\exp}\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \widetilde{\mathscr{C}h}_{kt}(\mathscr{F})u^k \right) \\
&= \widetilde{\exp}\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{\alpha\geq 0} \widetilde{\mathscr{C}h}^\alpha(\mathscr{F})(kt)^\alpha u^k \right) \\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0} \widetilde{\mathscr{C}h}^\alpha(\mathscr{F})t^\alpha \sum_{k\geq 1} (-1)^{k-1}k^{\alpha-1}u^k \right),
\end{aligned}
$$

where the first equality follows from the definition of $\tilde{\lambda}$ and the fact that $\widetilde{\mathscr{C}h}_t$ is a ring homomorphism, and the second equality follows from (5.12). From the definition of $\tilde{\gamma}_t$, it follows that

$$
\begin{aligned}
\widetilde{\mathscr{C}h}_t(\tilde{\gamma}_u(\mathscr{F} - \tilde{\epsilon}(\mathscr{F}))) &= \widetilde{\exp}\left(\sum_{\alpha\geq 0} \widetilde{\mathscr{C}h}^\alpha(\mathscr{F} - \tilde{\epsilon}(\mathscr{F}))t^\alpha \sum_{k\geq 1} (-1)^{k-1}k^{\alpha-1}\left(\frac{u}{1-u}\right)^k \right) \\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0} \sum_{k\geq 1} (-1)^{k-1}k^{\alpha-1} \widetilde{\mathscr{C}h}^\alpha(\mathscr{F})t^\alpha \sum_{n\geq k} u^n \binom{n-1}{k-1} \right) \\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0} \widetilde{\mathscr{C}h}^\alpha(\mathscr{F})t^\alpha \sum_{n\geq 1} u^n \sum_{k=1}^{n} (-1)^{k-1}k^{\alpha-1} \binom{n-1}{k-1} \right) \\
&= \widetilde{\exp}\left(\sum_{\alpha\geq 0} \widetilde{\mathscr{C}h}^\alpha(\mathscr{F})t^\alpha \sum_{n\geq 1} u^n (-1)^{n-1}(n-1)!\, S(\alpha, n) \right),
\end{aligned}
$$

where

$$S(\alpha, n) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^\alpha$$

are the Stirling numbers of the second kind. Projecting out those terms which are not powers of $z := ut$ yields the equality

$$\sum_{l\geq 0} \widetilde{\mathscr{C}h}^l\big(\tilde{\gamma}^l(\mathscr{F}-\tilde{\epsilon}(\mathscr{F}))\big)z^l = \widetilde{\exp}\bigg(\sum_{s\geq 0} z^s\,\widetilde{\mathscr{C}h}^s(\mathscr{F})(-1)^{n-1}(n-1)!\,S(n,n)\bigg).$$

The identity $S(n,n)=1$ and (5.2) yield (5.19). □

Even when an inertial pair $(\mathscr{R},\mathscr{S})$ is not Gorenstein, there are natural subrings of $K_G(I_G X)$ and $A^*_G(I_G X)$ where things behave well (as if $(\mathscr{R},\mathscr{S})$ were Gorenstein).

Definition 5.21. Let $(\mathscr{R},\mathscr{S})$ be an inertial pair, and let l be the number of connected components of $I\mathscr{X} = [I_G X/G]$. The subring of $K_G(I_G X)$ consisting of all elements of $\mathscr{S}$-grading $0 \in (\mathbb{Q}/\mathbb{Z})^l$ is called the *Gorenstein subring* $\check{K}_G(I_G X)$ of $K_G(I_G X)$, and the subring of $A^*_G(I_G X)$ consisting of all elements of $\mathscr{S}$-degree in $\mathbb{Z}^l \subseteq \mathbb{Q}^l$ is called the *Gorenstein subring* $\check{A}_G(I_G X)$ of $A^*_G(I_G X)$.

Remark 5.22. The previous theorem holds for a general inertial pair of a G-space X provided that $K_G(I_G X)$ and $A^*_G(I_G X)$ are replaced by their Gorenstein subrings $\check{K}_G(I_G X)$ and $\check{A}^*_G(I_G X)$, respectively.

ψ-ring and λ-ring structures on inertial K-theory. The main result of this section is the following:

Theorem 5.23. *If G is a diagonalizable group and $(\mathscr{R},\mathscr{S})$ is a strongly Gorenstein inertial pair on $I_G X$, then $(K_G(I_G X), \star_{\mathscr{R}}, \mathbf{1}, \tilde{\epsilon}, \tilde{\psi})$ is an augmented ψ-ring.*

Moreover, for general (possibly nondiagonalizable) G and any inertial pair $(\mathscr{R},\mathscr{S})$, the augmentation completion of the Gorenstein subring $\check{K}_G(I_G X)_{\mathbb{Q}}$ of $K_G(I_G X)_{\mathbb{Q}}$ is an augmented ψ-ring.

Remark 5.24. The hypothesis that G is diagonalizable is necessary, as is demonstrated later in this section (see Example 5.37).

With a little work we get the following corollary:

Corollary 5.25. *Let $(\mathscr{R},\mathscr{S})$ be a strongly Gorenstein inertial pair with G diagonalizable. Then $(K_G(I_G X)_{\mathbb{Q}}, \star_{\mathscr{R}}, \mathbf{1}, \tilde{\lambda})$ is an augmented λ-algebra over $\mathbb{Q}$.*

Moreover, for general (possibly nondiagonalizable) G and any inertial pair $(\mathscr{R},\mathscr{S})$, the augmentation completion of the Gorenstein subring $\check{K}_G(I_G X)_{\mathbb{Q}}$ of $K_G(I_G X)_{\mathbb{Q}}$ is an augmented λ-algebra over $\mathbb{Q}$.

Proof of Corollary 5.25. Combining Theorem 5.23 with Theorem 3.8, all that we must prove is that

$$\tilde{\epsilon}(\tilde{\lambda}_t(\mathscr{F})) = \tilde{\lambda}_t(\tilde{\epsilon}(\mathscr{F})) = (1+t)^{\tilde{\epsilon}(\mathscr{F})}. \tag{5.26}$$

Here we have omitted the $\star$ from the notation, but all products are the inertial product $\star$, and exponentiation is also with respect to the product $\star$.

For all $\mathscr{F} \in K_G(I_G X)$, we have

$$\tilde{\epsilon}(\tilde{\lambda}_t(\mathscr{F})) = \sum_{i \geq 0} t^i \tilde{\epsilon}(\tilde{\lambda}^i(\mathscr{F})),$$

but

$$\begin{aligned}\tilde{\epsilon}(\tilde{\lambda}_t(\mathscr{F})) &= \tilde{\epsilon}\left(\widetilde{\exp}\left(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} t^n \tilde{\psi}^n(\mathscr{F})\right)\right) \\ &= \widetilde{\exp}\left(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} t^n \tilde{\epsilon}(\tilde{\psi}^n(\mathscr{F}))\right) \\ &= \widetilde{\exp}\left(\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} t^n \tilde{\epsilon}(\mathscr{F})\right) \\ &= (1+t)^{\tilde{\epsilon}(\mathscr{F})},\end{aligned}$$

where the third line follows from $\tilde{\epsilon} \circ \tilde{\psi}^n = \tilde{\epsilon}$ (by Theorem 5.23). Finally, we have that $\tilde{\lambda}_t(\tilde{\epsilon}(\mathscr{F})) = (1+t)^{\tilde{\epsilon}(\mathscr{F})}$, since $\tilde{\epsilon}$ commutes with $\tilde{\psi}$ by Theorem 5.23. □

Proof of Theorem 5.23. It is straightforward from the definition that $\tilde{\psi}^n(\mathscr{F}+\mathscr{G}) = \tilde{\psi}^n(\mathscr{F}) + \tilde{\psi}^n(\mathscr{G})$, and also $\tilde{\psi}^1(\mathscr{F}) = \mathscr{F}$, since $\theta^1(\mathscr{G}) = \mathbf{1}$ for any $\mathscr{G}$. We also have $\tilde{\psi}^n(\mathbf{1}) = \mathbf{1}$, since $\mathbf{1}$ is supported only on $K_G(X^1)$, and $\mathscr{S}_{X^1} = 0$ (because $(\mathscr{R}, \mathscr{S})$ is an inertial pair). Now, to show for all $\mathscr{F}$ in $K_G(I_G X)$ that

$$\tilde{\psi}^n(\tilde{\psi}^l(\mathscr{F})) = \tilde{\psi}^{nl}(\mathscr{F}),$$

we observe that

$$\tilde{\psi}^n(\tilde{\psi}^l(\mathscr{F})) = \tilde{\psi}^n(\psi^l(\mathscr{F})\theta^l(\mathscr{S}^*)) = \psi^{nl}(\mathscr{F})\psi^n(\theta^l(\mathscr{S}^*)\theta^n(\mathscr{S}^*)).$$

Hence, we need to show that

$$\psi^n(\theta^l(\mathscr{S}^*))\theta^n(\mathscr{S}^*) = \theta^{nl}(\mathscr{S}^*).$$

This follows from the splitting principle in ordinary K-theory, the fact that the Bott classes are multiplicative, and the fact that for any line bundle $\mathscr{L}$ we have

$$\begin{aligned}\psi^n(\theta^l(\mathscr{L}))\theta^n(\mathscr{L}) &= \psi^n\left(\frac{1-\mathscr{L}^l}{1-\mathscr{L}}\right)\frac{1-\mathscr{L}^n}{1-\mathscr{L}} \\ &= \frac{1-\mathscr{L}^{nl}}{1-\mathscr{L}^n}\frac{1-\mathscr{L}^n}{1-\mathscr{L}} = \theta^{nl}(\mathscr{L}).\end{aligned} \tag{5.27}$$

It remains to show that $\tilde{\psi}$ preserves the inertial product defined by $\mathscr{R}$, i.e.,

$$\tilde{\psi}^n(\mathscr{F} \star \mathscr{G}) = \tilde{\psi}^n(\mathscr{F}) \star \tilde{\psi}^n(\mathscr{G}), \tag{5.28}$$

where $\star$ is understood to refer to the $\star_{\mathscr{R}}$-product. It is at this point in the proof that we need to use the hypothesis that G is diagonalizable.

Lemma 5.29. *Let G be a diagonalizable group. For each $(m_1, m_2) \in G \times G$ let $X^{m_1,m_2} = \{(m_1, m_2, x) \mid m_1 x = m_2 x = x\} \subset I^2_G X$. Then X^{m_1,m_2} is open and closed (but possibly empty) and the restriction of μ to X^{m_1,m_2} is a regular embedding.*

Proof. There is a decomposition of $I^2_G X$ into closed and open components indexed by conjugacy classes of pairs in $G \times G$. However, since G is diagonalizable, each conjugacy class consists of a single pair. If $\Psi = \{(m_1, m_2)\}$, then $I^2(\Psi) = X^{m_1,m_2}$ and the multiplication map restricts to the closed embedding $\mu : X^{m_1,m_2} \to X^{m_1 m_2}$, where $X^{m_1 m_2} = \{(m_1 m_2, x) \mid m_1 m_2 x = x\} \subset I_G X$. Since X is smooth, the fixed loci X^{m_1,m_2} and $X^{m_1 m_2}$ are also smooth, so the map is a regular embedding. $\square$

Let us prove that $\tilde{\psi}$ is compatible with the inertial product. First,

$$\begin{aligned}\tilde{\psi}^n(\mathscr{V} \star \mathscr{W}) &= \theta^n(\mathscr{S}^*) \cdot \psi^n(\mathscr{V} \star \mathscr{W}) \\ &= \theta^n(\mathscr{S}^*) \cdot \psi^n\big(\mu_*(e_1^*\mathscr{V} \cdot e_2^*\mathscr{W} \cdot \lambda_{-1}(\mathscr{R}^*))\big). \qquad (5.30)\end{aligned}$$

By our lemma $I^2_G X$ decomposes as a disjoint sum $\coprod X^{m_1,m_2}$ with $\mu|_{X^{m_1,m_2}}$ a closed regular embedding. Since an element $\alpha \in K_G(I^2_G X)$ decomposes as a sum $\alpha = \sum_{(m_1,m_2)} \alpha_{m_1,m_2}$ with $\alpha_{m_1,m_2} \in K_G(X^{m_1,m_2})$, we may invoke the equivariant Adams–Riemann–Roch theorem for closed embeddings (Theorem 3.34) on each α_{m_1,m_2} to conclude that $\psi^n \mu_* \alpha = \mu_*(\theta^n(N^*_\mu)\psi^n \alpha)$, where N^*_μ is the conormal bundle of μ. Writing $N^*_\mu = -T^*_\mu$ (see Definition 2.13) we obtain the equalities

$$\begin{aligned}\tilde{\psi}^n(\mathscr{V} \star \mathscr{W}) &= \theta^n(\mathscr{S}^*) \cdot \mu_*\big(\theta^n(-T^*_\mu) \cdot \psi^n(e_1^*\mathscr{V} \cdot e_2^*\mathscr{W} \cdot \lambda_{-1}(\mathscr{R})^*)\big) \\ &= \theta^n(\mathscr{S}^*) \cdot \mu_*\big(\theta^n(-T^*_\mu) \cdot e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \psi^n(\lambda_{-1}(\mathscr{R}^*))\big) \\ &= \theta^n(\mathscr{S}^*) \cdot \mu_*\big(\theta^n(-T^*_\mu) \cdot e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \lambda_{-1}(\psi^n(\mathscr{R}^*))\big) \\ &= \theta^n(\mathscr{S}^*) \cdot \mu_*\big(\theta^n(-T^*_\mu) \cdot e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \lambda_{-1}(\mathscr{R}^*) \cdot \theta^n(\mathscr{R}^*)\big) \\ &= \theta^n(\mathscr{S}^*) \cdot \mu_*\big(e_1^*\psi^n(\mathscr{V}) \cdot e_2^*\psi^n(\mathscr{W}) \cdot \lambda_{-1}(\mathscr{R}^*) \cdot \theta^n(\mathscr{R}^* - T^*_\mu)\big), \qquad (5.31)\end{aligned}$$

where the second equality follows from the fact that ψ^n respects the ordinary multiplication $\cdot$, the third from the definition of the Euler class and the fact [Knutson 1973, p. 48] that, for all i and n,

$$\psi^n \circ \lambda^i = \lambda^i \circ \psi^n,$$

the fourth from the fact that for any nonnegative element $\mathscr{F}$ in $K_G(I_G X)$ we have

$$\theta^n(\mathscr{F})\lambda_{-1}(\mathscr{F}) = \lambda_{-1}(\psi^n(\mathscr{F})),$$

and the fifth from the multiplicativity of θ^n. Since $\tilde{\psi}^n(\mathscr{F}) = \psi^n(\mathscr{F})\theta^n(\mathscr{S}^*)$, we may express the last line of (5.31) as

$$\theta^n(\mathscr{S}^*)\mu_*\big(e_1^*\tilde{\psi}^n(\mathscr{V})\cdot e_2^*\tilde{\psi}^n(\mathscr{W})\cdot\lambda_{-1}(\mathscr{R}^*)\cdot\theta^n(\mathscr{R}^*-T_\mu^*-e_1^*\mathscr{S}^*-e_2^*\mathscr{S}^*)\big). \quad (5.32)$$

Applying the projection formula to (5.32) yields

$$\begin{aligned}\tilde{\psi}^n(\mathscr{V}\star\mathscr{W}) = \mu_*\big(e_1^*\tilde{\psi}^n(\mathscr{V})\cdot e_2^*\tilde{\psi}^n(\mathscr{W})\cdot\lambda_{-1}(\mathscr{R}^*)&\\ \cdot\,\theta^n(\mathscr{R}^*-T_\mu^*-e_1^*\mathscr{S}^*-e_2^*\mathscr{S}^*+\mu^*\mathscr{S}^*)\big).&\end{aligned}$$

Now because $(\mathscr{R},\mathscr{S})$ is an inertial pair, we have

$$\mathscr{R} = e_1^*\mathscr{S}+e_2^*\mathscr{S}-\mu^*\mathscr{S}+T\mu,$$

so

$$\tilde{\psi}^n(\mathscr{V}\star\mathscr{W}) = \mu_*\big(e_1^*\tilde{\psi}^n(\mathscr{V})\cdot e_2^*\tilde{\psi}^n(\mathscr{W})\cdot\lambda_{-1}(\mathscr{R}^*)\big) = \tilde{\psi}(\mathscr{V})\star\tilde{\psi}(\mathscr{W}),$$

as claimed.

Finally, from the definition of $\tilde{\psi}$ and the fact that the ordinary augmentation in ordinary equivariant K-theory is preserved by and commutes with the ordinary ψ-operations, we have

$$\tilde{\epsilon}(\tilde{\psi}^n(\mathscr{V})) = \tilde{\psi}^n(\tilde{\epsilon}(\mathscr{V})) = \tilde{\epsilon}(\mathscr{V}). \quad (5.33)$$

When G is not diagonalizable, μ is a finite local complete intersection morphism, but in general it does not restrict to a closed embedding on each component of I_G^2X. In this case the equivariant Adams–Riemann–Roch theorem holds [Köck 1998, Theorem 4.5] after completing $K_G(I_GX)_\mathbb{C}$ and $K_G(I_G^2X)_\mathbb{C}$ at the augmentation ideal. Restricting to the augmentation completion of the Gorenstein subring ensures that the Bott class $\theta^k(\mathscr{S}^*)$ takes values in that subring (which has $\mathbb{Q}$ coefficients), whereas the Bott class in general would take values in the augmentation completion of $K_G(I_GX)\otimes\overline{\mathbb{Q}}$. The rest of the above argument goes through verbatim. □

Remark 5.34. Suppose G is not abelian, but the fixed locus X^g is empty if g is not in the center of G. Then, since the conjugacy classes of central elements are singletons, the argument of Lemma 5.29 shows that I_G^2X is a disjoint sum of components such that the restriction of μ to each of them is a regular embedding. Arguing as in the proof of Theorem 5.23 shows that in this case the inertial product would also commute with the inertial Adams operations.

Remark 5.35. If G is finite then, for each conjugacy class $\Phi\subset G\times G$ and $\Psi\subset G$ such that $\mu(I^2(\Phi))\subset I(\Psi)$, the pushforward map $\mu_*: K_G(I^2(\Phi))\to K_G(\boldsymbol{I}(\Psi))$ can be identified as a combination of pushforward along a regular embedding with an induction functor. Precisely, if $(m_1,m_2)\in\Phi$ is any element, then $K_G(I^2(\Phi))$

can be identified with $K_{Z_{1,2}}(X^{m_1,m_2})$, where $Z_{1,2}$ is the centralizer of m_1 and m_2 in G. Likewise, $K_G(I(\Psi))$ can be identified with $K_{Z_{12}}(X^{m_1m_2})$, where Z_{12} is the centralizer of the element m_1m_2. Let $i : X^{m_1,m_2} \hookrightarrow X^{m_1m_2}$ be the inclusion. Via these identifications the pushforward μ_* is the composition of the pushforward $i_* : K_{Z_{1,2}}(X^{m_1,m_2}) \to K_{Z_{1,2}}(X^{m_1m_2})$ with the induction functor $\mathrm{Ind}_{Z_{1,2}}^{Z_{12}} : K_{Z_{12}}(X^{m_1m_2}) \to K_{Z_{1,2}}(X^{m_1m_2})$. In this case, determining whether the equality $\tilde{\psi}^j(\alpha \star \beta) = \tilde{\psi}^j(\alpha) \star \tilde{\psi}^j(\beta)$ holds in $K_G(I_GX)_{\mathbb{Q}}$ boils down to the question of whether the classical Adams operations ψ^j commute with induction. This question has been studied in Section 6 of [Köck 1998], where it is proved that Adams operations commute with induction after completion at the augmentation ideal.

Remark 5.36. Let $(\mathscr{R}, \mathscr{S})$ be a Gorenstein inertial pair on I_GX. For each integer $k \geq 1$, let $\tilde{\psi}^k : A_G^*(I_GX) \to A_G^*(I_GX)$ be defined by (3.23). If

$$\tilde{\epsilon} : A_G^*(I_GX) \to A_G^{\{0\}}(I_GX)$$

is the canonical projection, then the inertial Chow theory $(A_G^*(I_GX), \star, 1, \tilde{\psi}, \tilde{\epsilon})$ is an augmented ψ-ring.

Moreover, if G is a diagonalizable group and $(\mathscr{R}, \mathscr{S})$ is a strongly Gorenstein inertial pair on I_GX, then the summand $\widehat{K}_G(I_GX)_{\mathbb{Q}}$ inherits an augmented ψ-ring structure from $K_G(I_GX)_{\mathbb{Q}}$. In addition, (5.12) means that the inertial Chern character homomorphism $\widetilde{\mathscr{C}h} : K_G(I_GX)_{\mathbb{Q}} \to A_G^*(I_GX)_{\mathbb{Q}}$ preserves the augmented ψ-ring structures and factors through an isomorphism $\widehat{K}_G(I_GX)_{\mathbb{Q}} \to A_G^*(I_GX)_{\mathbb{Q}}$ of augmented ψ-rings. In particular, if G acts freely on X, then the inertial Chern character is an isomorphism of augmented ψ-rings.

Example 5.37. The hypothesis of Theorem 5.23 that G is diagonalizable is necessary, as demonstrated by the following example: Let $G = S_3$ be the symmetric group S_3 on three letters, and consider the classifying stack $BS_3 = [\mathrm{pt}/S_3]$. The inertia stack IBS_3 is the disjoint union of three components, corresponding to the conjugacy classes of (1), (12), and (123) in S_3. The component corresponding to class Ψ is the stack $[\Psi/S_3]$, which is isomorphic to the classifying stack BZ, where Z is the centralizer of any element of Ψ. So the components of the inertia stack are isomorphic to BS_3, $B\mu_2$, and $B\mu_3$.

The double inertia I^2BS_3 is the disjoint union of eleven components: three isomorphic to a point ($B\{e\}$), corresponding to the conjugacy classes of the pairs ((12), (13)), ((12), (123)), and ((123), (12)), respectively; three isomorphic to $B\mu_2$, corresponding to the conjugacy classes of the pairs ((1), (12)), ((12), (1)), and ((12), (12)); four isomorphic to $B\mu_3$, corresponding to the conjugacy classes of ((1), (123)), ((123), (1)), ((123), (123)), and ((123), (132)); and the identity component, isomorphic to BS_3. Consider the inertial product with $\mathscr{R} = 0$ and $\mathscr{S} = 0$. (This is just the usual orbifold product on BS_3.)

Let $\chi \in R(\mu_2) = K(B\mu_2)$ be the defining character. Denote by $\chi|_{B\mu_2} \in K(IBS_3)$ the class which is χ on the sector isomorphic to $B\mu_2$ (corresponding to the conjugacy class of a transposition in S_3) and 0 on all other sectors. Likewise, let $1|_{B\mu_2} \in K(IBS_3)$ be the class which is the trivial representation on the sector isomorphic to $B\mu_2$ and 0 on all other sectors. We will compare $\psi^2(\chi|_{B\mu_2} \star 1|_{B\mu_2})$ and $\psi^2\chi|_{B\mu_2} \star \psi^2\chi|_{B\mu_2}$ and show that they are not equal in $K(IBS_3)$.

Since $\mathscr{R} = 0$, the orbifold product is given by the formula

$$\alpha \star \beta = \mu_*(e_1^*\alpha \cdot e_2^*\beta).$$

To compute the product, we note that if α is supported on the sector corresponding to the conjugacy class of (12) then $e_1^*\alpha$ is supported on the components of I^2BS_3 corresponding to the conjugacy classes of pairs

$$((12),(1)), \quad ((12),(13)), \quad ((12),(12)), \quad ((12),(123)).$$

Similarly, $e_2^*\alpha$ is supported on the components corresponding to the conjugacy classes of the pairs

$$((1),(12)), \quad ((12),(13)), \quad ((12),(12)), \quad ((123),(12)).$$

So if α and β are both supported on the sector corresponding to (12), then the classical product $e_1^*\alpha \cdot e_2^*\alpha$ is supported on components of I^2BS_3, corresponding to the conjugacy classes of the pairs ((12), (13)) and ((12), (12)). The multiplication map μ takes the component corresponding to the conjugacy class of ((12), (13)) to the twisted sector isomorphic to $B\mu_3$ corresponding to the conjugacy class of 3-cycles. Likewise, μ maps the component corresponding to the conjugacy class of ((12), (12)) to the untwisted sector BS_3, which corresponds to the conjugacy class of the identity.

Identifying $K(BG) = R(G)$, we see that $K(IBS_3) = R(S_3) \oplus R(\mu_2) \oplus R(\mu_3)$, while $K(I^2BS_3) = R(S_3) \oplus R(\{e\})^3 \oplus R(\mu_2)^3 \oplus R(\mu_3)^4$. Under this identification the pullbacks $e_i^* : K(IBS_3) \to K(I^2BS_3)$ correspond to restriction functors between the various representation rings. Likewise, the pushforward $\mu_* : K(I^2BS_3) \to K(IBS_3)$ corresponds to the induced representation functor. Hence,

$$\chi|_{B\mu_2} \star 1|_{B\mu_2} = (\operatorname{Ind}_{\mu_2}^{S_3}\chi)|_{BS_3} + (\operatorname{Ind}_{\{e\}}^{\mu_3}\operatorname{Res}_{\{e\}}^{\mu_2}\chi)|_{B\mu_3} = (\mathrm{sgn} + V_2)|_{BS_3} + V_3|_{B\mu_3},$$

where the sign representation of S_3 is denoted by sgn, the 2-dimensional irreducible representation of S_3 is denoted by V_2, and the regular representation of μ_3 is denoted by V_3. The character of $\psi^2(\mathrm{sgn} + V_2)$ has value 3 at the identity and at the conjugacy class of a 2-cycle, and it has value 0 on 3-cycles. On the other hand, $\psi^2(\chi) = \psi^2(1) = 1$ in $R(\mu_2)$, so

$$\psi^2(\chi|_{B\mu_2}) \star \psi^2(1|_{B\mu_2}) = 1|_{B\mu_2} \star 1|_{B\mu_2} = (1 + V_2)|_{BS_3} + V_3|_{B\mu_3}.$$

The character of $1 + V_2$ has value 1 on 2-cycles, so $\psi^2(\mathrm{sgn} + V_2) \neq (1 + V_2)$. Therefore,

$$\psi^2(\chi|_{B\mu_2} \star 1_{B\mu_2}) \neq \psi^2\chi|_{B\mu_2} \star \psi^2 1|_{B\mu_2}.$$

6. λ-positive elements, the inertial dual, and inertial Euler classes

Every λ-ring contains the semigroup of *λ-positive elements*, which is an invariant of the λ-ring structure. In the case of ordinary equivariant K-theory, every class of a rank-d vector bundle is a λ-positive element, although the converse need not be true. Nevertheless, λ-positive elements of degree d share many of the same properties as classes of rank-d vector bundles; for example, they have a top Chern class in Chow theory and an Euler class in K-theory. This is because the ordinary Chern character and Chern classes are compatible with the λ-ring and ψ-ring structures.

In this section, we will introduce the framework to investigate the λ-positive elements of inertial K-theory for strongly Gorenstein inertial pairs. We will see that the λ-positive elements of degree d in inertial K-theory satisfy the inertial versions of these properties. We will also introduce a notion of duality for inertial K-theory, which is necessary to define the inertial Euler class in inertial K-theory.

For the examples $\mathbb{P}(1, 2)$ and $\mathbb{P}(1, 3)$, we will see that the set of λ-positive elements yield integral structures on inertial K-theory and inertial Chow theory, which will correspond, under a kind of mirror symmetry, to the usual integral structures on ordinary K-theory and Chow theory of an associated crepant resolution of the orbifold cotangent bundle.

Remark 6.1. All results in this section hold for possibly nondiagonalizable G, provided that $K_G(I_G X)$ is replaced by the augmentation completion of its Gorenstein subring $\check{K}_G(I_G X)$.

We begin by defining the appropriate notion of duality for inertial K-theory.

Definition 6.2. Consider the inertial K-theory $(K_G(I_G X), \star, 1, \tilde{\epsilon}, \tilde{\psi})$ of a strongly Gorenstein pair $(\mathscr{R}, \mathscr{S})$ associated to a proper action of a diagonalizable group G on X. The *inertial dual* is the map $\widetilde{\boldsymbol{D}} : K_G(I_G X) \to K_G(I_G X)$ defined by

$$\widetilde{\boldsymbol{D}}(\mathscr{V}) := \mathscr{V}^\dagger := \mathscr{V}^* \cdot \rho(\mathscr{S}^*),$$

where

$$\rho(\mathscr{F}) := (-1)^{\epsilon(\mathscr{F})} \det(\mathscr{F}^*) \tag{6.3}$$

for all classes of locally free sheaves $\mathscr{F}$ in $K_G(I_G X)$ and $\det(\mathscr{F}) = \lambda^{\epsilon(\mathscr{F})}\mathscr{F}$ is the class of the usual determinant line bundle of $\mathscr{F}$. Note that in this definition the dual *, as well as both ϵ and det, are the usual, noninertial forms.

Theorem 6.4. *Consider the inertial K-theory* $(K_G(I_GX), \star, 1, \tilde{\epsilon}, \tilde{\psi})$ *of a strongly Gorenstein pair* $(\mathscr{R}, \mathscr{S})$ *for a diagonalizable group* G *with a proper action on* X.

(1) $\widetilde{\boldsymbol{D}}^2$ *is the identity map, i.e.,* $\mathscr{F}^{\dagger\dagger} = \mathscr{F}$ *for all* $\mathscr{F} \in K_G(I_GX)$.

(2) *For all* $l \geq 1$, *the inertial dual satisfies*

$$\widetilde{\boldsymbol{D}} \circ \tilde{\epsilon} = \tilde{\epsilon} \circ \widetilde{\boldsymbol{D}} = \tilde{\epsilon} \quad \text{and} \quad \tilde{\psi}^l \circ \widetilde{\boldsymbol{D}} = \widetilde{\boldsymbol{D}} \circ \tilde{\psi}^l. \tag{6.5}$$

(3) *The inertial dual is a homomorphism of unital rings.*

Before we give the proof of the theorem, we need to recall one fact from [Fulton and Lang 1985] about the ordinary dual in K-theory, and we need to prove a Riemann–Roch-type result for the ordinary dual.

Lemma 6.6 [Fulton and Lang 1985, Lemma I.5.1]. *Let* $\mathscr{F}$ *be any locally free sheaf of rank* d. *Then for all* i *with* $0 \leq i \leq d$ *we have*

$$\lambda^i(\mathscr{F}) = \lambda^{d-i}(\mathscr{F}^*)\lambda^d(\mathscr{F}). \tag{6.7}$$

Theorem 6.8 (Riemann–Roch for the ordinary dual). *Using the hypotheses and notation from Theorem 3.34, and the definition of* ρ *given in* (6.3), *for all* $\mathscr{F}$ *in* $K_G(Y)$ *we have*

$$(\iota_*(\mathscr{F}))^* = \iota_*(\rho(N_\iota^*) \cdot \mathscr{F}^*). \tag{6.9}$$

Proof. We first observe, using Lemma 6.6, that for any locally free sheaf $\mathscr{F} \in K_G(Y)$ we have

$$\lambda_{-1}(\mathscr{F})^* = \lambda_{-1}(\mathscr{F})\rho(\mathscr{F}). \tag{6.10}$$

We also observe that ordinary dualization commutes with pullback and is a ring homomorphism. Because of these properties, the ordinary dual is a so-called *natural operation*, and the desired result follows immediately from Köck's "Riemann–Roch theorem without denominators" [1991, Satz 5.1]. □

Proof of Theorem 6.4. Part (1) follows from the identity $\rho(\mathscr{F}^*) = (\rho(\mathscr{F}))^{-1}$.

The first equation of (2) is follows from the definition of $\tilde{\epsilon}$. The second equation of (2) follows from the identity $\theta^n(\mathscr{S}) = \theta^n(\mathscr{S}^*)(\det(\mathscr{S}))^{\epsilon(\mathscr{S})-1}$, which follows from the splitting principle in ordinary K-theory.

The proof of (3) is identical to the proof that $\tilde{\psi}^n$ is a homomorphism for all $n \geq 1$, but where the Bott class θ^n is replaced by the class ρ and Theorem 3.34 is replaced by Theorem 6.8. □

Definition 6.11. Let $(K, \cdot, 1, \lambda)$ be a λ-ring. For any integer $d \geq 0$, an element $\mathscr{V} \in K$ is said to have *λ-degree d* if $\lambda_t(\mathscr{V})$ is a degree-d polynomial in t. The element $\mathscr{V}$ is said to be a *λ-positive element of degree d of K* if it has λ-degree d for $d \geq 1$ and $\lambda^d(\mathscr{V})$ is a unit of K. A λ-positive element of degree 1 is said to be a λ-*line element of* K. Let $\mathcal{P}_d := \mathcal{P}_d(K)$ be the set of λ-positive elements of degree d in K, and let $\mathcal{P} = \sum_d \mathcal{P}_d \subset K$ be the semigroup of positive elements.

Remark 6.12. If the λ-ring $(K, \cdot, 1, \lambda)$ has an involutive homomorphism $K \to K$ taking $\mathscr{F}$ to $\mathscr{F}^\nabla$ that commutes with λ^i for all $i \geq 0$, then it may be useful in the definition of a λ-positive element of degree 1 to assume, in addition, that $\mathscr{V}^{-1} = \mathscr{V}^\nabla$. However, we will later see that this condition automatically holds for the virtual K-theory of $B\mu_2$ (Proposition 7.2), $\mathbb{P}(1,2)$ (Proposition 7.45), and $\mathbb{P}(1,3)$ (Proposition 7.64).

Proposition 6.13. *Let* $(K, \cdot, 1, \lambda)$ *be a* λ*-ring.*

(1) *Addition in* K *induces a map* $\mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \to \mathcal{P}_{d_1+d_2}$ *for all integers* $d_1, d_2 \geq 1$.

(2) *Multiplication in* K *induces a map* $\mathcal{P}_{d_1} \times \mathcal{P}_{d_2} \to \mathcal{P}_{d_1 d_2}$ *for all integers* $d_1, d_2 \geq 1$. *In particular, the set* $\mathcal{P}_1$ *of* λ*-line elements of* K *forms a group.*

(3) *If* K *is torsion-free, then an element* $\mathscr{L}$ *in* K *has* λ*-degree* 1 *if and only if*
$$\psi^l(\mathscr{L}) = \mathscr{L}^l \tag{6.14}$$
for all integers $l \geq 1$.

(4) *For all* $\mathscr{V}$ *in* $\mathcal{P}_d$,
$$\gamma_t(\mathscr{V} - d) = \sum_{i=0}^{d} t^i (1-t)^{d-i} \lambda^i(\mathscr{V}). \tag{6.15}$$

(5) *For all integers* $i \geq 0$ *and* $d \geq 1$, *we have* $\lambda^i : \mathcal{P}_d \to \mathcal{P}_{\binom{d}{i}}$. *Furthermore, if* K *is an augmented* λ*-algebra over* $\mathbb{Q}$ *with augmentation* ϵ *and* $\mathscr{V}$ *belongs to* $\mathcal{P}_d$, *then, in* K,
$$\epsilon(\mathscr{V}) = d, \tag{6.16}$$
and thus
$$\epsilon(\lambda^i(\mathscr{V})) = \binom{d}{i}. \tag{6.17}$$

Proof. Part (1) follows from the fact that the product of invertible elements is invertible. Part (2) follows from properties of the universal polynomials $\boldsymbol{P}_n$ appearing in (3.2) of the definition of a λ-ring. Part (3) follows immediately from (3.9) and the fact that K is torsion-free.

Equation (6.15) holds since, for all $\mathscr{V}$ in $\mathcal{P}_d$, we have
$$\gamma_t(\mathscr{V} - d) = \frac{\lambda_{t/(1-t)}(\mathscr{V})}{(1-t)^{-d}} = (1-t)^d \sum_{i=0}^{d} \left(\frac{t}{1-t}\right)^i \lambda^i(\mathscr{V}) = \sum_{i=0}^{d} t^i (1-t)^{d-i} \lambda^i(\mathscr{V}).$$

To prove (5), the properties of the universal polynomials $\boldsymbol{P}_{m,n}$ (see Remark 3.6) imply that $\lambda^i : \mathcal{P}_d \to \mathcal{P}_{\binom{d}{i}}$ for all $i \geq 0$. Hence, if $\mathscr{V}$ has λ-degree d, where $d, i \geq 1$, then, since $\lambda^d \mathscr{V}$ is invertible, so is $\lambda^{\binom{d}{i}}(\lambda^i(\mathscr{V})) = (\lambda^d \mathscr{V})^{\binom{d-1}{i-1}}$.

To prove (6.16) let us first suppose that $\mathscr{F} := \mathscr{L}$ belongs to $\mathcal{P}_1$. Applying ϵ to (6.14) for $l = 2$, we obtain $\epsilon(\psi^2(\mathscr{L})) = \epsilon(\mathscr{L}^2) = \epsilon(\mathscr{L})^2$, but $\epsilon(\psi^2(\mathscr{L})) = \epsilon(\mathscr{L})$. Thus $\epsilon(\mathscr{L})^2 = \epsilon(\mathscr{L})$ but, since $\mathscr{L}$ is invertible and ϵ is a homomorphism of unital rings, $\epsilon(\mathscr{L})$ is invertible. Therefore, $\epsilon(\mathscr{L}) = 1$. More generally, if $\mathscr{F}$ belongs to $\mathcal{P}_d$ for some integer $d \geq 1$, then (3.18) implies that $\binom{\epsilon(\mathscr{F})}{d} = 1$ and

$$0 = \binom{\epsilon(\mathscr{F})}{d+1} = \binom{\epsilon(\mathscr{F})}{d}\frac{\epsilon(\mathscr{F})-d}{d+1} = \frac{\epsilon(\mathscr{F})-d}{d+1}.$$

Therefore, $\epsilon(\mathscr{F}) = d$.

Finally, (6.17) follows from equations (3.18) and (6.16). □

In ordinary equivariant K-theory $(K_G(X), \otimes, 1, \epsilon)$, it is often useful to assume that $[X/G]$ is connected. This is not an actual restriction, since $K_G(X)$ can be expressed as the direct sum of λ-rings or ψ-rings of the form $K_G(U)$, where $[U/G]$ is a connected component of $[X/G]$. The condition that $[X/G]$ is connected is equivalent to the condition that the image of the augmentation is $\mathbb{Z}$ times the unit element 1, i.e., one may interpret the augmentation as a map $\epsilon : K_G(X) \to \mathbb{Z}$.

For an inertial K-theory $(K_G(I_G X), \star, 1, \tilde{\epsilon})$, an additional condition must be imposed in order for the inertial augmentation to have image equal to $\mathbb{Z}$.

Definition 6.18. Let X be an algebraic space with an action of G and let $(\mathscr{R}, \mathscr{S})$ be an inertial pair. For each $m \in G$, the restriction of $\mathscr{S}$ to X^m is denoted by $\mathscr{S}_m$.

We say that the action of G on X is *reduced* with respect to the inertial pair $(\mathscr{R}, \mathscr{S})$ if $\mathscr{S}_m = 0$ implies $m = 1$.

The following proposition is immediate:

Proposition 6.19. *Consider the inertial K-theory* $(K_G(I_G X), \star, 1, \tilde{\epsilon})$ (*respectively the rational inertial K-theory* $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon})$) *for some inertial pair* $(\mathscr{R}, \mathscr{S})$. *The image of the inertial augmentation* $\tilde{\epsilon}$ *is equal to* $\mathbb{Z}$ (*respectively* $\mathbb{Q}$) *times the unit element* 1 *of* $K_G(I_G X)$ *if and only if* $[X/G]$ *is connected and the action of* G *on* X *is reduced with respect to* $(\mathscr{R}, \mathscr{S})$.

In ordinary equivariant K-theory any vector bundle of rank d has λ-degree d. Thus, if $[X/G]$ is connected then, by definition, $(K_G(X), \cdot, 1, \epsilon, \lambda)$ (respectively $(K_G(X)_{\mathbb{Q}}, \cdot, 1, \epsilon, \lambda)$) is generated as a group (respectively $\mathbb{Q}$-vector space) by the classes of vector bundles and hence by elements of $\mathcal{P}$.

In inertial K-theory $(K_G(I_G X)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$, the situation is more complicated. Equation (6.17) implies that if $\mathscr{V}$ is in $\mathcal{P}_d$ then, for any connected component U of $I_G X \smallsetminus X^1$ which has $\mathscr{S}$-age equal to 0, the restriction $\mathscr{V}|_U$ must have ordinary rank equal to 0 on U. Therefore, the $\mathbb{Q}$-linear span of $\mathcal{P}_d$ cannot be equal to $K_G(I_G X)_{\mathbb{Q}}$. Furthermore, even if $[X/G]$ is connected and the action of G on X is reduced with respect to the inertial pair $(\mathscr{R}, \mathscr{S})$, there is no a priori reason that

$(K_G(I_GX)_{\mathbb{K}}, \cdot, 1, \tilde{\epsilon}, \tilde{\lambda})$ is generated as a $\mathbb{K}$-vector space by its λ-positive elements for any field $\mathbb{K}$ containing $\mathbb{Q}$.

Corollary 6.20. *The Gorenstein subring $(\mathscr{K}_G(I_GX)_{\mathbb{Q}}, \star, 1, \tilde{\lambda})$ is a λ-subring of the inertial K-theory which is preserved by the inertial dual.*

Proof. The proof follows from Proposition 6.13(2) and (4) and the fact that the inertial dual maps $\mathcal{P}_d$ to $\mathcal{P}_d$ for all d. □

One thing that makes the elements $\mathcal{P}_d$ in $(K_G(I_GX)_{\mathbb{Q}}, \cdot, 1, \tilde{\epsilon}, \tilde{\lambda})$ interesting is that in many ways they behave as though they were rank-d vector bundles. In particular, they have inertial Euler classes in both K-theory and Chow rings.

Proposition 6.21. *Let $(K_G(I_GX)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$ be the inertial K-theory of a strongly Gorenstein pair $(\mathscr{R}, \mathscr{S})$ associated to a diagonalizable group G with a proper action on X.*

(1) *The inertial Chern class $\tilde{c}^1 : \mathcal{P}_1 \to A_G^{\{1\}}(I_GX)_{\mathbb{Q}}$ is a group homomorphism.*

(2) *For all $\mathscr{V}$ in $\mathcal{P}_d$ and $\mathscr{L}$ in $\mathcal{P}_1$,*

$$\widetilde{\mathscr{C}h}(\mathscr{L}) = \widetilde{\exp}(\tilde{c}^1(\mathscr{L})) \tag{6.22}$$

and

$$\tilde{c}_t(\mathscr{V}) = \sum_{i=0}^{d} \tilde{c}^i(\mathscr{V})t^i, \tag{6.23}$$

so $\tilde{c}^i(\mathscr{V}) = 0$ for all $i > d$.

Proof. Part (1) follows from the fact that $\widetilde{\mathscr{C}h}(\mathscr{L}_1 \star \mathscr{L}_2) = \widetilde{\mathscr{C}h}(\mathscr{L}_1) \star \widetilde{\mathscr{C}h}(\mathscr{L}_2)$ for all $\mathscr{L}_1$ and $\mathscr{L}_2$ in $\mathcal{P}_1$. Picking off terms in $A_G^{\{1\}}(I_GX)_{\mathbb{Q}}$ and using $\widetilde{\mathscr{C}h}^1 = \tilde{c}^1$ and (6.17) yields the desired result.

Equation (6.22) follows from (5.2) and (6.23), which yields

$$1 + t\tilde{c}^1(\mathscr{L}) = \widetilde{\exp}\Bigg(\sum_{n \geq 1} (-1)^{n-1}(n-1)!\, t^n\, \widetilde{\mathscr{C}h}^n(\mathscr{L})\Bigg),$$

which implies that $\widetilde{\mathscr{C}h}^n(\mathscr{L}) = \tilde{c}^1(\mathscr{L})^n/n!$, as desired. Equation (6.23) follows from (5.19) and (6.15). □

The inertial dual allows us to introduce a generalization of the Euler class.

Definition 6.24. Let $(K_G(I_GX)_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda})$ be the inertial K-theory associated to $(\mathscr{R}, \mathscr{S})$. Let $\mathscr{V}$ belong to $\mathcal{P}_d$. The *inertial Euler class in $K_G(I_GX)_{\mathbb{Q}}$* of $\mathscr{V}$ is

$$\tilde{\lambda}_{-1}(\mathscr{V}^\dagger) = \sum_{i=0}^{d} (-1)^i \tilde{\lambda}^i(\mathscr{V}^\dagger).$$

The *inertial Euler class of $\mathscr{V}$ in $A_G^{\{d\}}(I_GX)_{\mathbb{Q}}$* is defined to be $\tilde{c}^d(\mathscr{F})$.

The inertial Euler classes are multiplicative by Proposition 6.13(1) and the multiplicativity of $\tilde{c}_t$ and $\tilde{\lambda}_t$.

Finally, we observe that $\mathcal{P}_1$ is preserved by the action of certain groups. This will be useful in our analysis of the virtual K-theory of $\mathbb{P}(1, n)$.

Definition 6.25. Let $(K, \cdot, 1, \psi, \epsilon)$ be a torsion-free, augmented ψ-ring. A *translation group of* K is an additive subgroup J of K such that, for all $n \geq 1$, $j \in J$, and $x \in K$, the following identities hold:

(1) $\psi^n(j) = nj$,

(2) $x \cdot j = \epsilon(x) j$,

(3) $\epsilon(x) j \in J$.

Proposition 6.26. *Let* $(K, \cdot, 1, \psi, \epsilon)$ *be a torsion-free augmented* ψ*-ring. If* J *is a translation subgroup of* K*, then* $\epsilon(J) = 0$, $J^2 = 0$, *and* J *is an ideal of the ring* K. *Furthermore,* J *acts freely on* $\mathcal{P}_1$, *where* $J \times \mathcal{P}_1 \to \mathcal{P}_1$ *is* $(j, \mathscr{L}) \mapsto j + \mathscr{L}$.

Proof. For all j in J and integers $n \geq 1$, $\epsilon(\psi^n(j)) = \epsilon(j)$ by the definition of an augmented ψ-ring. On the other hand, $\epsilon(\psi^n(j)) = \epsilon(nj) = n\epsilon(j)$ for all integers $n \geq 1$ by condition (1) in the definition of a translation group. Therefore, $\epsilon(j) = 0$ since K is torsion-free. The fact that $J^2 = 0$ and J is an ideal of K follows from conditions (2) and (3) in the definition of a translation group.

Consider $\mathscr{L}$ in $\mathcal{P}_1$ and j in J. We have

$$\psi^n(\mathscr{L} + j) = \psi^n(\mathscr{L}) + \psi^n(j) = \mathscr{L}^n + nj = (\mathscr{L} + j)^n,$$

where the second equality is by (6.14) and condition (1) in the definition of a translation group, and the last is from the binomial theorem and the fact that $J^2 = 0$ since $\epsilon(\mathscr{L}) = 1$. Hence, by (6.14), $\mathscr{L} + j$ has λ-degree 1. Also, notice that $(\mathscr{L}^{-1} - j)(\mathscr{L} + j) = 1$, so $\mathscr{L} + j$ is invertible and thus an element of $\mathcal{P}_1$. □

7. Examples

In this section, we work out some examples of inertial ψ-rings and λ-rings.

The classifying stack of a finite abelian group. In this section we discuss the case where X is a point with a trivial action by a finite group G and the trivial inertial pair $\mathscr{R} = 0$ and $\mathscr{S} = 0$. Since G is zero-dimensional, its tangent bundle is 0, so the orbifold and virtual inertial pairs (Definitions 2.18 and 2.20) are both trivial. We begin with some general results and conclude with explicit computations for the special case of the cyclic group $G = \mu_2$ of order 2.

General results. Let X be a point with the trivial action of a finite abelian group G. The inertia scheme is $I_G X = G$, which also has a trivial G action. The orbifold K-theory of $BG := [X/G]$ is additively the Grothendieck group $K_G(I_G X) = K_G(G)$ of G-equivariant vector bundles over G; however, the orbifold product on $K_G(G)$ differs from the ordinary one, as we now describe.

The double inertia manifold is $I^2_G X = G \times G$ with the diagonal conjugation action of G (again, trivial); the evaluation maps $e_i : G \times G \to G$ are the projection maps onto the i-th factor for $i = 1, 2$; and $\mu : G \times G \to G$ is the multiplication map. Let $\mathscr{F}$ and $\mathscr{G}$ be G-equivariant vector bundles on G; then $\mathscr{F} \star \mathscr{G} := \mu_*(\mathscr{F} \boxtimes \mathscr{G})$ is the G-equivariant vector bundle over G whose fiber over the point m in G is

$$(\mathscr{F} \star \mathscr{G})_m = \bigoplus_{m_1 m_2 = m} \mathscr{F}_{m_1} \otimes \mathscr{G}_{m_2}, \tag{7.1}$$

where the sum is over all pairs $(m_1, m_2) \in G^2$ such that $m_1 m_2 = m$.

The orbifold K-theory $(K_G(G), \star, 1)$ of BG can naturally be identified with two better-known rings: first, the group ring $R(G)[G]$ of G with coefficients in the representation ring $R(G)$ of G, and second, the representation ring $\operatorname{Rep}(D(G))$ of the Drinfeld double $D(G)$ of the group G (see [Kaufmann and Pham 2009, Theorem 4.13]). The ring $\operatorname{Rep}(D(G))$ has been studied in some detail in [Dijkgraaf et al. 1990; Kaufmann and Pham 2009; Witherspoon 1996].

In this case the orbifold Chern classes are all trivial, i.e., $\tilde{c}_t(\mathscr{F}) = 1$ for all $\mathscr{F}$. This follows from two facts. First, $\mathscr{S} = 0$, so $\widetilde{\mathscr{C}h}_t(\mathscr{F}) = \operatorname{Ch}_t(\mathscr{F})$ is the classical Chern character. Second, $A^i(BG)_{\mathbb{Q}} = 0$ for $i > 0$ because BG is a zero-dimensional Deligne–Mumford stack. Thus, $\operatorname{Ch}_t(\mathscr{F}) = \operatorname{rk}(\mathscr{F})$ for every $\mathscr{F} \in K_G(I_G X)$.

Since $\mathscr{S} = 0$ on $I_G X$, the orbifold Adams operations in $K_G(G)$ agree with the ordinary ones, i.e., $\tilde{\psi}^i := \psi^i$ for all $i \geq 1$.

The classifying stack $B\mu_2$. We now consider the special case where $G = \mu_2$ is the cyclic group of order 2. For each $m \in G$ and each irreducible representation $\alpha \in \operatorname{Irrep}(\mu_2) = \{\pm 1\}$, let V^α_m denote the bundle on G which is 0 away from the one-point set $\{m\} \in I_G X = \mu_2$ and which is equal to α on $\{m\}$. In this case the free abelian group $K_{\mu_2}(\mu_2)$ decomposes as

$$K(IB\mu_2) = K_{\mu_2}(\mu_2) = K_{\mu_2}(\{1\}) \oplus K_{\mu_2}(\{-1\})$$

and has a basis consisting of the four elements V^1_1, V^{-1}_1, V^1_{-1}, and V^{-1}_{-1}.

Proposition 7.2. *The orbifold λ-ring $(K(IB\mu_2)_{\mathbb{Q}}, \star, 1, \tilde{\lambda})$ satisfies the following*:

$$\tilde{\lambda}_t(V^1_1) = 1 + t, \tag{7.3}$$

$$\tilde{\lambda}_t(V^{-1}_1) = 1 + tV^{-1}_1, \tag{7.4}$$

$$\tilde{\lambda}_t(V^1_{-1}) = 1 + tV^1_{-1} + \frac{t^2}{2(1+t)}(1 - V^1_{-1}), \tag{7.5}$$

$$\tilde{\lambda}_t(V^{-1}_{-1}) = 1 + V^{-1}_{-1}t + \frac{t^2}{2(1-t^2)}(1 - tV^{-1}_1 - V^1_{-1} + tV^{-1}_{-1}). \tag{7.6}$$

There are four elements in $\mathcal{P}_1$, namely $V^{\pm 1}_1$ and

$$\sigma_\pm := \tfrac{1}{2}(V^1_1 + V^{-1}_1 \pm (V^1_{-1} - V^{-1}_{-1})),$$

with multiplication given by

$$\sigma_\pm \star \sigma_\pm = V^1_1, \quad V^{-1}_1 \star \sigma_\pm = \sigma_\mp, \quad \textit{and} \quad \sigma_+ \star \sigma_- = V^{-1}_1.$$

Proof. Equations (7.3) and (7.4) hold since $\{V^1_1, V^{-1}_1\}$ generates a subring of $(K_{\mu_2}(\mu_2)_{\mathbb{Q}}, \star)$ isomorphic as a λ-ring to the ordinary representation ring $K(B\mu_2)$.

Let us introduce some notation. If $f(t)$ is a formal power series in t, let

$$f_\pm(t) := \tfrac{1}{2}(f(t) \pm f(-t)).$$

In order to prove (7.5), we observe that $\tilde{\psi}^k = \psi^k = \psi^{k+2}$ for all $k \geq 1$. This can be seen from (3.14) and the fact that any irreducible representation V of G is a line element satisfying $V^2 = 1$.

Let $\lambda_t := \exp\bigl(\sum_{k=1}^\infty ((-1)^{k-1}/k)t^k\psi^k\bigr)$. Since

$$\psi^k(V^1_{-1}) = V^1_{-1} \quad \text{for all } k \geq 1, \tag{7.7}$$

we obtain

$$\tilde{\lambda}_t(V^1_{-1}) = \exp\biggl(\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} t^k V^1_{-1}\biggr) = \exp(V^1_{-1}\log(1+t)).$$

Since we have

$$(V^1_{-1})^k = \begin{cases} V^1_{-1} & \text{if } k \text{ is odd,} \\ V^1_1 = 1 & \text{if } k \text{ is even,} \end{cases}$$

we obtain

$$\begin{aligned} \exp(V^1_{-1}\log(1+t)) &= \exp_+(V^1_{-1}\log(1+t)) + \exp_-(V^1_{-1}\log(1+t)) \\ &= \exp_+(\log(1+t)) + V^1_{-1}\exp_-(\log(1+t)) \\ &= \frac{1+t+(1+t)^{-1}}{2} + V^1_{-1}\frac{1+t-(1+t)^{-1}}{2} \\ &= \frac{1+t}{2}(1 + V^1_{-1}) + \frac{1}{2(1+t)}(1 - V^1_{-1}), \end{aligned}$$

which agrees with (7.5).

The proof of (7.6) is similar. Since, for all $k \geq 1$,

$$\psi^k(V^{-1}_{-1}) = \begin{cases} V^{-1}_{-1} & \text{if } k \text{ is odd,} \\ V^1_{-1} & \text{if } k \text{ is even,} \end{cases} \tag{7.8}$$

we obtain $\tilde{\lambda}_t(V_{-1}^{-1}) = \exp(\psi_t(V_{-1}^1)) = \exp(V_{-1}^{-1}\log_-(1+t) + V_{-1}^1 \log_+(1+t))$ and

$$\tilde{\lambda}_t(V_{-1}^{-1}) = \exp(V_{-1}^{-1}\log_-(1+t))\exp(V_{-1}^1\log_+(1+t)). \tag{7.9}$$

Since

$$\begin{aligned}
&\exp(V_{-1}^{-1}\log_-(1{+}t))\\
&\quad= \exp_+(V_{-1}^{-1}\log_-(1{+}t)) + \exp_-(V_{-1}^{-1}\log_-(1{+}t))\\
&\quad= \exp_+(\log_-(1{+}t)) + V_{-1}^{-1}\exp_-(\log_-(1{+}t))\\
&\quad= \tfrac{1}{2}\big(\exp\big(\tfrac{1}{2}(\log(1{+}t)-\log(1{-}t))\big) + \exp\big(-\tfrac{1}{2}(\log(1{+}t)-\log(1{-}t))\big)\big)\\
&\qquad\quad + \tfrac{1}{2}V_{-1}^{-1}\big(\exp\big(\tfrac{1}{2}(\log(1{+}t)-\log(1{-}t))\big) - \exp\big(-\tfrac{1}{2}(\log(1{+}t)-\log(1{-}t))\big)\big)\\
&\quad= \frac{1}{2}\bigg(\Big(\frac{1{+}t}{1{-}t}\Big)^{\frac{1}{2}} + \Big(\frac{1{-}t}{1{+}t}\Big)^{\frac{1}{2}}\bigg) + \frac{1}{2}V_{-1}^{-1}\bigg(\Big(\frac{1{+}t}{1{-}t}\Big)^{\frac{1}{2}} - \Big(\frac{1{-}t}{1{+}t}\Big)^{\frac{1}{2}}\bigg),
\end{aligned}$$

we obtain

$$\exp(V_{-1}^{-1}\log_-(1+t)) = \frac{1+tV_{-1}^{-1}}{(1-t^2)^{\frac{1}{2}}}. \tag{7.10}$$

Also, since

$$\begin{aligned}
&\exp(V_{-1}^{1}\log_+(1{+}t))\\
&\quad= \exp_+(V_{-1}^{1}\log_+(1{+}t)) + \exp_-(V_{-1}^{1}\log_+(1{+}t))\\
&\quad= \exp_+(\log_+(1{+}t)) + V_{-1}^{1}\exp_-(\log_+(1{+}t))\\
&\quad= \tfrac{1}{2}\big(\exp\big(\tfrac{1}{2}(\log(1{+}t)+\log(1{-}t))\big) + \exp\big(-\tfrac{1}{2}(\log(1{+}t)+\log(1{-}t))\big)\big)\\
&\qquad\quad + \tfrac{1}{2}V_{-1}^{1}\big(\exp\big(\tfrac{1}{2}(\log(1{+}t)+\log(1{-}t))\big) - \exp\big(-\tfrac{1}{2}(\log(1{+}t)+\log(1{-}t))\big)\big)\\
&\quad= \tfrac{1}{2}((1-t^2)^{\frac{1}{2}} + (1-t^2)^{-\frac{1}{2}}) + \tfrac{1}{2}V_{-1}^1((1-t^2)^{\frac{1}{2}} - (1-t^2)^{-\frac{1}{2}}),
\end{aligned}$$

we obtain

$$\exp(V_{-1}^1\log_-(1+t)) = \frac{2-t^2-V_{-1}^1t^2}{2(1-t^2)^{\frac{1}{2}}}. \tag{7.11}$$

Plugging equations (7.10) and (7.11) into (7.9) and then expanding using (7.1) yields (7.6).

The fact that $V_1^{\pm 1}$ is in $\mathcal{P}_1$ is immediate, since the orbifold λ-ring structure reduces to the ordinary λ-ring structure on the untwisted sector. The fact that $\sigma_\pm$ is in $\mathcal{P}_1$ follows from (6.14) as follows: Since $\tilde{\psi}^k = \psi^k = \psi^{k+2}$ for all $k \geq 1$, it suffices to check that $\psi^2(\sigma_\pm) = \sigma_\pm \star \sigma_\pm = V_1^1$. But this is immediate from equations (7.7) and (7.8):

$$\psi^2(\sigma_\pm) = \tfrac{1}{2}((V_1^1)^2 + (V_1^{-1})^2 \pm (V_{-1}^1 - V_{-1}^1)) = V_1^1. \qquad \square$$

The virtual K-theory and virtual Chow ring of $\mathbb{P}(\mathbf{1}, \boldsymbol{n})$**.** Let $X := \mathbb{C}^2 \smallsetminus \{0\}$ and $G := \mathbb{C}^\times$, with the action $\mathbb{C}^\times \times X \to X$ defined by taking $(t, (a, b))$ to $(ta, t^n b)$. In this section, we first develop some general results about the virtual K-theory and virtual Chow theory of the weighted projective line $\mathbb{P}(1, n) := [X/\mathbb{C}^\times]$. Recall (see Definition 2.20) that the inertial pair associated to the *virtual product* is given by $\mathscr{S} = \boldsymbol{N}$, where $\boldsymbol{N}$ is the normal bundle of the projection morphism $I_{\mathbb{C}^\times} X \to X$, and $\mathscr{R}$ is given by (2.21). We work out the full inertial K-theory and Chow theory for the weighted projective spaces $\mathbb{P}(1, 2)$ and $\mathbb{P}(1, 3)$, and we compare our results with the usual K-theory and Chow theory of the resolution of singularities of the coarse moduli spaces of the cotangent bundles to these orbifolds.

General results on the K-theory of $\mathbb{P}(1, n)$ *and its inertia.* Since the action of $\mathbb{C}^\times$ on $\mathbb{C} \smallsetminus \{0\}$ has weights $(1, n)$, the only elements of $\mathbb{C}^\times$ with nonempty fixed loci are the n-th roots of unity. For $m \in \{0, \dots, n-1\}$, let X^m denote the fixed locus of the element $e^{2\pi i m/n}$ in X.

With this notation, $X^0 = X$, so $[X^0/\mathbb{C}^\times] = \mathbb{P}(1, n)$. For $m > 0$,

$$X^m = \{(0, b) \mid b \neq 0\} = \mathbb{C}^\times.$$

For each $m > 0$, the action of $\mathbb{C}^\times$ on X^m has weight n, so the quotient $[X^m/\mathbb{C}^\times]$ is the classifying stack $B\mu_n$. The inertia variety is $I_{\mathbb{C}^\times} X = \coprod_{m=0}^{n-1} X^m$, so the inertia stack $I\mathbb{P}(1, n)$ decomposes as $\mathbb{P}(1, n) \sqcup \coprod_{m=1}^{n-1} B\mu_n$.

We now compute the classical equivariant Grothendieck and Chow rings of the inertia variety, or equivalently the Grothendieck and Chow rings of the inertia stack.

Notation 7.12. Let χ be the defining character of $\mathbb{C}^\times$. We can associate to χ a $\mathbb{C}^\times$-equivariant line bundle on X. It is the trivial bundle $X \times \mathbb{C}$ with $\mathbb{C}^\times$ action given by $\beta(a, b, v) = (\beta a, \beta^n b, \beta v)$. For each m, denote by χ_m the class in $K_{\mathbb{C}^\times}(X^m)$ corresponding to the pullback of this $\mathbb{C}^\times$-equivariant line bundle to X^m.

The character χ has a first Chern class $c_1(\chi) \in A^1_{\mathbb{C}^\times}(\mathrm{pt})$, and we denote by c_m the pullback of $c_1(\chi)$ to $A^1_{\mathbb{C}^\times}(X^m)$ under the projection $X^m \to \mathrm{pt}$. With this notation, $c_1(\chi_m) = c_m$.

Proposition 7.13. *We have the following isomorphisms for all* $m \in \{1, \dots, n-1\}$:

$$K_{\mathbb{C}^\times}(X^0) = K(\mathbb{P}(1, n)) \cong \frac{\mathbb{Z}[\chi_0]}{\langle (\chi_0 - 1)(\chi_0^n - 1) \rangle}, \tag{7.14}$$

$$K_{\mathbb{C}^\times}(X^m) = K(B\mu_n) \cong \frac{\mathbb{Z}[\chi_m]}{\langle \chi_m^n - 1 \rangle}, \tag{7.15}$$

$$A^*_{\mathbb{C}^\times}(X^0) = A^*(\mathbb{P}(1, n)) \cong \frac{\mathbb{Z}[c_0]}{\langle n c_0^2 \rangle}, \tag{7.16}$$

$$A^*_{\mathbb{C}^\times}(X^m) = A^*(B\mu_n) \cong \frac{\mathbb{Z}[c_m]}{\langle n c_m \rangle}. \tag{7.17}$$

Proof. Since $\mathbb{C}^2$ is smooth, Thomason's equivariant resolution theorem [1987a] identifies the equivariant K-theory of vector bundles with the equivariant K-theory of coherent sheaves. It follows that there is a four-term localization exact sequence for equivariant K-theory [Thomason 1987b]

$$K_{\mathbb{C}^\times}(\{0\}) \xrightarrow{i_*} K_{\mathbb{C}^\times}(\mathbb{C}^2) \xrightarrow{j^*} K_{\mathbb{C}^\times}(X^0) \longrightarrow 0, \tag{7.18}$$

where $i : \{0\} \hookrightarrow \mathbb{C}^2$ is a closed embedding and $j : X^0 \to \mathbb{C}^2$ is an open immersion. Equation (7.18) implies that $K_{\mathbb{C}^\times}(X^0)$ is the quotient of $K_{\mathbb{C}^\times}(\mathbb{C}^2)$ by the image of $K_{\mathbb{C}^\times}(\{0\})$ under the pushforward induced by the inclusion i. Since $\mathbb{C}^2$ is a representation of $\mathbb{C}^\times$, the homotopy property of equivariant K-theory implies that $K_{\mathbb{C}^\times}(\mathbb{C}^2) = \mathrm{Rep}(\mathbb{C}^\times) = \mathbb{Z}[\chi, \chi^{-1}]$. The projection formula implies that $i_* K_{\mathbb{C}^\times}(\{0\})$ is an ideal in $\mathbb{Z}[\chi, \chi^{-1}]$, and $K_{\mathbb{C}^\times}(X^0)$ is the quotient of $\mathbb{Z}[\chi, \chi^{-1}]$ by this ideal. By the self-intersection formula in equivariant K-theory [Köck 1998, Corollary 3.9], $i^* i_* K_{\mathbb{C}^\times}(\{0\}) = \mathrm{eu}(N_{\{0\}}) K_{\mathbb{C}^\times}(\{0\})$, where $N_{\{0\}}$ is the normal bundle to the origin in $\mathbb{C}^2$. Since $\mathbb{C}^\times$ acts with weights $(1, n)$, the class of the normal bundle is $\chi + \chi^n$ and $\mathrm{eu}(N_{\{0\}}) = (1 - \chi^{-1})(1 - \chi^{-n})$. Since the pullback $i^* : K_{\mathbb{C}^\times}(\mathbb{C}^2) \to K_{\mathbb{C}^\times}(\{0\})$ is an isomorphism, $i_*(K_{\mathbb{C}^\times}(\{0\}))$ is the ideal generated by $(1 - \chi^{-1})(1 - \chi^{-n})$. Thus, $K_{\mathbb{C}^\times}(X^0) = \mathbb{Z}[\chi, \chi^{-1}]/\langle (1 - \chi^{-1})(1 - \chi^{-n})\rangle$. Clearing denominators and observing that the relation already implies that χ is a unit, we have the presentation $K_{\mathbb{C}^\times}(X^0) = \mathbb{Z}[\chi]/\langle (\chi - 1)(\chi^n - 1)\rangle$. Since χ_0 is our notation for the pullback of χ to X^0, we obtain the presentation $\mathbb{Z}[\chi_0]/\langle (\chi_0 - 1)(\chi_0^n - 1)\rangle$.

For $m > 0$ observe that, if $\mathbb{C}^\times$ acts on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ by $\lambda \cdot v = \lambda^n v$, then the $\mathbb{C}^\times$-equivariant normal bundle to $\{0\}$ in $\mathbb{C}$ is χ^n. The same argument as above implies that $K_{\mathbb{C}^\times}(\mathbb{C}^\times) = \mathbb{Z}[\chi, \chi^{-1}]/\langle 1 - \chi^{-n}\rangle$. Clearing denominators and using the notation χ_m for χ on X^m gives the desired presentation.

The proof in Chow theory is similar. We again use the five-term localization sequence for equivariant Chow groups [Edidin and Graham 1998] to see that $A^*_{\mathbb{C}^\times}(X^m)$ is a quotient of $A^*_{\mathbb{C}^\times}(pt) = \mathbb{Z}[c_1(\chi)]$. We can again apply the self-intersection formula. In Chow theory, $\mathrm{eu}(\chi) = c_1(\chi)$, while $\mathrm{eu}(\chi + \chi^n) = c_2(\chi + \chi^n) = n(c_1(\chi))^2$, which gives the relations in (7.17) and (7.16). □

Remark 7.19. As a consequence of the relations in Proposition 7.13, an additive basis for $K(I\mathbb{P}(1, n))$ is given by $n^2 + 1$ classes of the form χ_m^k, where the subscript refers to the sector while the superscript is an exponent. Including the untwisted sector X^0 there are n sectors, so $0 \le m \le n - 1$. If $m > 0$, then the exponent k is in $[0, n - 1]$, while if $m = 0$ then the exponent k is in $[0, n]$.

Similarly, the classes $\{c_m^k\}_{k \in \mathbb{N}}$ for $0 \le m \le n - 1$ generate $A^*_{\mathbb{C}^\times}(I_{\mathbb{C}^\times}(X)) = A^*(I\mathbb{P}(1, n))$. Again, in the notation c_m^k the subscript refers to the sector and the superscript to the exponent. Note the relations in the presentation imply that only c_0 and the fundamental classes $c_m^0 = [X^m]$ are nontorsion.

Remark 7.20. If $f : X \to Y$ is any morphism of G-varieties, then the pullback $f^* : K_G(Y) \to K_G(X)$ is a homomorphism of λ-rings, since, for any G-equivariant vector bundle, $\Lambda^k(f^*V) = f^*(\Lambda^k V)$. Applying this observation to the pullbacks $K_{\mathbb{C}^\times}(\mathbb{C}^2) \to K_{\mathbb{C}^\times}(X^0)$ and $K_{\mathbb{C}^\times}(\mathbb{C}) \to K_{\mathbb{C}^\times}(X^m)$, this means that for all $m \geq 0$ the classical λ-ring structure on $K_{\mathbb{C}^\times}(X^m)$ is induced from the usual λ-ring structure on $\mathbb{Z}[\chi_m, \chi_m^{-1}]$ defined by setting $\lambda_t(\chi_m^k) = 1 + t\chi_m^k$.

Remark 7.21. For any $m > 0$ the map $X^m \to X$ is an embedding of codimension 1, so the $\mathscr{S}$-age of X^m is 1 and the age of X^0 is 0. Hence the virtual degree of c_0 is 1, as is the virtual degree of the fundamental class $c_m^0 = [X^m]$ for $m > 0$.

The virtual Chern character homomorphism is very simple: in $A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$, $c_0^k = 0$ for $k > 1$ and, if $m > 1$, then $c_m^l = 0$ for $l \neq 0$. Stated more precisely, the map $\widetilde{\mathscr{C}h} : K(I\mathbb{P}(1,n)) \to A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$ satisfies

$$\widetilde{\mathscr{C}h}(\chi_0^a) = c_0^0 + ac_0^1 \tag{7.22}$$

for all $a \in \mathbb{Z}$ and, for $m \in \{1, \dots, n-1\}$, we have $\widetilde{\mathscr{C}h}(\chi_m^a) = c_m^0$.

We now compute the virtual product.

Theorem 7.23. *The virtual product on $K(I\mathbb{P}(1,n))$ satisfies*

$$\chi_{m_1}^{a_1} \star \chi_{m_2}^{a_2} = \begin{cases} \chi_{m_1+m_2}^{a_1+a_2} & \text{if } m_1 = 0 \text{ or } m_2 = 0, \\ \chi_0^{a_1+a_2}(1 - 2\chi_0^{-1} + \chi_0^{-2}) & \text{if } m_1 + m_2 = n, \\ \chi_{m_1+m_2}^{a_1+a_2}(1 - \chi_{m_1+m_2}^{-1}) & \text{otherwise}, \end{cases}$$

and the virtual product in $A^*(I\mathbb{P}(1,n))$ satisfies

$$c_{m_1}^{a_1} \star c_{m_2}^{a_2} = \begin{cases} c_{m_1+m_2}^{a_1+a_2} & \text{if } m_1 = 0 \text{ or } m_2 = 0, \\ c_0^{a_1+a_2+2} & \text{if } m_1 + m_2 = n, \\ c_{m_1+m_2}^{a_1+a_2+1} & \text{otherwise}. \end{cases}$$

Here the sum $m_1 + m_2$ is understood to be reduced modulo n and all products on the right-hand side are the classical product in $K_{\mathbb{C}^\times}(X^{m_1+m_2})$ (or $A^*_{\mathbb{C}^\times}(X^{m_1+m_2})$). In particular, the classes χ_m^{-1} are defined via (7.14) *and* (7.15).

Remark 7.24. Since $c_0^2 = 0$ in $A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$, and since for all $m > 0$ we have $c_m = 0$ in $A^*(I\mathbb{P}(1,n))_{\mathbb{Q}}$, Theorem 7.23 implies that all products $c_{m_1}^{a_0} \star c_{m_2}^{a_1}$ are equal to 0 unless one of the classes is the identity c_0^0. It follows that the rational virtual Chow ring is isomorphic to the graded ring $\mathbb{Q}[t_0, t_1, \dots, t_{n-1}]/\langle t_0, \dots, t_{n-1}\rangle^2$, where t_0 corresponds to c_0^1 and t_m corresponds to c_m^0 for all $m \in \{1, \dots, n-1\}$.

Before proving Theorem 7.23, we first need some notation for $K_{\mathbb{C}^\times}(I^2_{\mathbb{C}^\times}X)$ and $A^*_{\mathbb{C}^\times}(I^2_{\mathbb{C}^\times}X)$.

Notation 7.25. Given a pair $(m_1, m_2) \in (\mathbb{Z}_n)^2$ let $X^{m_1,m_2} = X^{m_1} \cap X^{m_2}$. We have $X^{m_1,m_2} = \{(0,b) \mid b \neq 0\} \subset X$ unless $m_1 = m_2 = 0$, and $X^{0,0} = X$. The double inertia decomposes as $I^2_{\mathbb{C}^\times} X = \coprod_{(m_1,m_2)\in(\mathbb{Z}_n)^2} X^{m_1,m_2}$. For each pair (m_1, m_2), let $\chi_{m_1,m_2} \in K_{\mathbb{C}^\times}(X^{m_1,m_2})$ be the class corresponding to the character $\chi \in \mathrm{Rep}(\mathbb{C}^\times)$. With this notation, Proposition 7.13 implies that

$$K_{\mathbb{C}^\times}(X^{m_1,m_2}) = \begin{cases} \mathbb{Z}[\chi_{m_1,m_2}]/\langle \chi^n_{m_1,m_2} - 1\rangle & \text{if } (m_1, m_2) \neq (0, 0), \\ \mathbb{Z}[\chi_{0,0}]/\langle(\chi_{0,0} - 1)(\chi^n_{0,0} - 1)\rangle & \text{if } (m_1, m_2) = (0, 0). \end{cases}$$

Similarly, we let c_{m_1,m_2} be the class in $A^1_{\mathbb{C}^\times}(X^{m_1,m_2})$ corresponding to $c_1(\chi)$.

Proof of Theorem 7.23. We first use (2.14) with $\mathscr{S} = N$ and compute the restriction of $\mathscr{R}$ to X^{m_1,m_2}. With our additive notation, the multiplication map $\mu : I^2_{\mathbb{C}^\times} X \to I_{\mathbb{C}^\times} X$ maps $X^{m_1,m_2} \to X^{m_1+m_2}$, so in $K_{\mathbb{C}^\times}(X^{m_1,m_2})$ we have

$$\mathscr{R}|_{X^{m_1,m_2}} = (e_1^* N_{m_1} + e_2^* N_{m_2} - \mu^* N_{m_1+m_2} + T_\mu,)|_{X^{m_1,m_2}}, \tag{7.26}$$

where N_m denotes the normal bundle to X^m in X.

First suppose that $m_1 = 0$. Then $X^{m_1,m_2} = X^{m_2} = X^{m_1+m_2}$. It follows that $\mu : X^{m_1,m_2} \to X^{m_1+m_2}$ is the identity map, so $(T_\mu)|_{X^{m_1,m_2}} = 0$. Also, $N_{m_1} = 0$ and $N_{m_1+m_2} = N_{m_2}$, so plugging into (7.26) gives $\mathscr{R}|_{X^{m_1,m_2}} = 0$. In this case, $\chi^{\alpha_1}_{m_1} \star \chi^{\alpha_2}_{m_2}$ corresponds to the usual product $\chi^{\alpha_1}\chi^{\alpha_2} = \chi^{\alpha_1+\alpha_2}$, but viewed as an element of $K_{\mathbb{C}^\times}(X^{m_1+m_2})$. In our notation, this class is $\chi^{\alpha_1+\alpha_2}_{m_1+m_2}$.

Next suppose that m_1 and m_2 are nonzero, but $m_1 + m_2 = n$. In this case, $X^{m_1,m_2} = X^{m_1} = X^{m_2} = \{(0, b) \mid b \neq 0\}$, while $X^{m_1+m_2} = X^0 = \mathbb{C}^2 \smallsetminus \{0\}$. Since $\mathbb{C}^\times$ acts with weights $(1, n)$, the normal bundle to $\{(0, b) \mid b \neq 0\} \subset \mathbb{C}^2 \smallsetminus \{0\}$ is the bundle determined by the character χ, so in our notation $N_{m_1} = \chi_{m_1}$ and $N_{m_2} = \chi_{m_2}$, and $N_{m_1+m_2} = 0$. The map $\mu : X^{m_1,m_2} \to X^{m_1+m_2}$ is the inclusion and $(T_\mu)|_{X^{m_1,m_2}} = -(N_\mu|_{X^{m_1,m_2}})$ corresponds to the class $-\chi$, which on X^{m_1,m_2} we denote by $-\chi_{m_1,m_2}$. Since

$$\begin{aligned} \mathscr{R}|_{X^{m_1,m_2}} &= e_1^*\chi_{m_1}|_{X^{m_1,m_2}} + e_2^*\chi_{m_2}|_{X^{m_1,m_2}} - \chi_{m_1,m_2} \\ &= \chi_{m_1,m_2} + \chi_{m_1,m_2} - \chi_{m_1,m_2} = \chi_{m_1,m_2}, \end{aligned}$$

it follows that

$$\chi^{\alpha_1}_{m_1} \star \chi^{\alpha_2}_{m_2} = \mu_*(\chi^{\alpha_1}_{m_1,m_2} \cdot \chi^{\alpha_2}_{m_1,m_2} \cdot \mathrm{eu}(\chi_{m_1,m_2})) = \mu_*(\chi^{\alpha_1+\alpha_2}_{m_1,m_2}(1 - \chi^{-1}_{m_1,m_2})).$$

Since the class χ_{m_1,m_2} is pulled back from the character $\chi \in \mathrm{Rep}(\mathbb{C}^\times)$, the projection formula yields the further simplification $\chi^{\alpha_1}_{m_1} \star \chi^{\alpha_2}_{m_2} = \chi^{\alpha_1+\alpha_2}_{m_1+m_2}(1 - \chi^{-1}_{m_1})\mu_*(1)$. To compute $\mu_*(1)$ consider the diagram of inclusions

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "j"] & \mathbb{C}^2 \\ X^{m_1,m_2} = \mathbb{C} \smallsetminus \{0\} \arrow[u, hook] \arrow[r, hook, "\mu"] & \mathbb{C}^2 \smallsetminus \{0\} = X^{m_1+m_2}. \arrow[u, hook] \end{tikzcd}$$

Then $\mu_*(1)$ is the restriction to $K_{\mathbb{C}^\times}(X^{m_1+m_2})$ of the image of $j_*(1)$. By the self-intersection formula, $j^*j_*(1) = \mathrm{eu}(N_j) = (1-\chi^{-1})$ under the identification of $K_{\mathbb{C}^\times}(\mathbb{C}) = \mathrm{Rep}(\mathbb{C}^\times)$. Since j^* is an isomorphism, we conclude that $j_*(1) = (1-\chi^{-1})$ and then, restricting to $K_{\mathbb{C}^\times}(X^{m_1+m_2})$, we obtain $\mu_*(1) = (1-\chi_{m_1+m_2}^{-1})$. Hence

$$\chi_{m_1}^{\alpha_1} \star \chi_{m_2}^{\alpha_2} = \chi_{m_1+m_2}^{\alpha_1+\alpha_2}(1-\chi_{m_1+m_2}^{-1})^2.$$

If $m_1, m_2 \neq 0$ and $m_1+m_2 \neq 0$, then $X^{m_1,m_2} = X^{m_1} = X^{m_2} = X^{m_1+m_2}$, so e_1, e_2, and μ are all identity maps. In this case,

$$\mathscr{R}|_{X^{m_1,m_2}} = e_1^*\chi_{m_1}|_{X^{m_1,m_2}} + e_2^*\chi_{m_2}|_{X^{m_1,m_2}} - \mu^*\chi_{m_1+m_2}|_{X^{m_1,m_2}} = \chi_{m_1,m_2}$$

and

$$\chi_{m_1}^{\alpha_1} \star \chi_{m_2}^{\alpha_2} = \chi_{m_1+m_2}^{\alpha_1+\alpha_2}(1-\chi_{m_1+m_2}^{-1}).$$

The proof in Chow theory is similar. If $m_1, m_2 \neq 0$, then $\mathrm{eu}(\mathscr{R}) = c_{m_1,m_2}$ is in $A^1_{\mathbb{C}^\times}(X^{m_1,m_2})$ and, if $m_1+m_2 = n$, then $\mu_*(1) = c_{m_1+m_2}$, which gives the factors of $c^2_{m_1+m_2}$ and $c_{m_1+m_2}$ appearing above. □

In order to calculate the virtual ψ-operations, for all $m \in \{1, \ldots, n-1\}$ we need the l-th Bott class $\theta^l(\mathscr{S}_m^*)$ in $K_{\mathbb{C}^\times}(X^m)$, which satisfies

$$\theta^l(\mathscr{S}_m^*) = \theta^l(\chi_m^{-1}) = \sum_{i=0}^{l-1} \chi_m^{-i}.$$

Applying (5.5) gives the virtual ψ-operations $\tilde{\psi}^k : K(I\mathbb{P}(1,n)) \to K(I\mathbb{P}(1,n))$.

Definition 7.27. Let $\mathbb{K}$ be $\mathbb{Q}$ or $\mathbb{C}$. For all $m \in \{1, \ldots, n-1\}$, let $\Delta_m = \sum_{i=0}^{n-1} \chi_m^i$ in $K_{\mathbb{C}^\times}(X^m)$ (respectively $K_{\mathbb{C}^\times}(X^m)_{\mathbb{K}}$) and $\Delta_0 = -\chi_0^0 + \chi_0^n$ in $K_{\mathbb{C}^\times}(X^0)$ (respectively $K_{\mathbb{C}^\times}(X^0)_{\mathbb{K}}$). Let J (respectively $J_{\mathbb{K}}$) be the additive group (respectively $\mathbb{K}$-vector space) generated by $\{\Delta_i\}_{i=0}^n$. Let $\tilde{\psi}^0$ be the inertial augmentation $\tilde{\epsilon}$.

Lemma 7.28. *Let* $\big(K(I\mathbb{P}(1,n)), \star, \mathbf{1}, \tilde{\epsilon}, \tilde{\psi}\big)$ *be the virtual K-theory ring.*

(1) *For all* $m \in \{0, \ldots, n-1\}$ *and* $\mathscr{F}_m$ *in* $K_{\mathbb{C}^\times}(X^m)$, *we have the identity with respect to the ordinary product*

$$\Delta_m \cdot \mathscr{F}_m = \epsilon_m(\mathscr{F}_m)\Delta_m. \tag{7.29}$$

(2) *For all* j *in* J *and* $\mathscr{F}$ *in the virtual K-theory ring* $K(I\mathbb{P}(1,n))$,

$$\mathscr{F} \star j = \tilde{\epsilon}(\mathscr{F})j, \quad J \star J = 0, \quad \textit{and} \quad \tilde{\epsilon}(J) = 0. \tag{7.30}$$

(3) *For all* $l \geq 1$ *and* $j \in J$, *we have the identity*

$$\tilde{\psi}^l(j) = lj. \tag{7.31}$$

In particular, J *is a translation group of the virtual K-theory* $K(I\mathbb{P}(1,n))$.

Proof. Equation (7.29) follows from the identity $(\chi_0^n - 1)(\chi_0^1 - 1) = 0$ in $K_{\mathbb{C}^\times}(X^0)$, and $\chi_m^n - 1 = 0$ in $K_{\mathbb{C}^\times}(X^m)$ for all $m \neq 0$.

Equation (7.30) follows from Theorem 7.23 and (7.29). The fact that $J \star J = 0$ follows from (7.30) and the fact that $\tilde{\epsilon}(\Delta_m) = 0$ for all m.

To prove (7.31), we first consider

$$\begin{aligned}\tilde{\psi}^l(\Delta_0) &= \psi^l(-1 + \chi_0^n) = -1 + \chi_0^{nl} = -1 + (1 + (\chi_0^n - 1))^l \\ &= -1 + (1 + l(\chi_0^n - 1)) = l\Delta_0,\end{aligned}$$

where we have used the binomial series and the relation $(\chi_0^n - 1)(\chi_0^1 - 1) = 0$ in the fourth equality. Let $m \neq 0$, $\zeta_n := e^{2\pi i/n}$, and $x = \chi_m^1$, and assume in the following that all products are ordinary products. By definition,

$$\tilde{\psi}^l(\Delta_m) = \psi^l(\Delta_m) \cdot \theta^l(x^{-1}) = \psi^l\Big(\sum_{i=0}^{n-1} x^i\Big) \sum_{j=0}^{l-1} (x^{-j}) = \sum_{i=0}^{n-1} (x^l)^i \sum_{j=0}^{l-1} x^{-j}.$$

To prove (7.31), consider the algebra isomorphism

$$K_{\mathbb{C}^\times}(X^m)_{\mathbb{Q}} = \frac{\mathbb{Q}[x]}{\langle x^n - 1\rangle} \xrightarrow{\Upsilon} \mathbb{Q} \times \mathbb{Q}[t]/(1 + t + \cdots + t^{n-1})$$

defined by $\Upsilon(f) := (f(1), f(\zeta_n))$. Then $\Upsilon(\tilde{\psi}^l(\Delta_m)) = (nl, 0) = l\Upsilon(\Delta_m)$. □

Proposition 7.32. *Let $\varphi_0 : K(I\mathbb{P}(1, n)) \to \mathbb{Z}$ be the additive map that is supported on $K_{\mathbb{C}^\times}(X^0)$ such that $\varphi_0(\chi_0^s) = s$ for all $s \in \{0, \ldots, n\}$.*

For all $k \geq 0$ and $a \in \{0, \ldots, n-1\}$, we have the identity in virtual K-theory $\big(K(I\mathbb{P}(1, n)), \star, 1, \tilde{\epsilon}, \tilde{\psi}\big)$,

$$\tilde{\psi}^{nk+a} = \tilde{\psi}^a + k\Delta_0\varphi_0 + \sum_{m=1}^{n} k\Delta_m\epsilon_m, \tag{7.33}$$

where $\epsilon_m(\mathscr{F})$ denotes the ordinary augmentation of $\mathscr{F}_m$ in $K_{\mathbb{C}^\times}(X^m)$ of $\mathscr{F}$.

Proof. For all $k \geq 1$, let $\tilde{\psi}_m^k(\mathscr{F}) := \tilde{\psi}^k(\mathscr{F}_m)$ for all $\mathscr{F} = \sum_{m=0}^n \mathscr{F}_m$, where $\mathscr{F}_m$ belongs to $K_{\mathbb{C}^\times}(X^m)$.

If $a \in \{0, \ldots, n-1\}$, $k \geq 0$, $s \in \{0, \ldots, n\}$, and $x = \chi_0^1$, then

$$\tilde{\psi}_0^{nk+a}(x^s) = (x^n)^{ks} x^{as} = (1 + (x^n - 1))^{ks} x^{as} = (1 + ks(x^n - 1))x^{as} = x^{sa} + ks\Delta_0,$$

where we have used the relation $(x^n - 1)(x - 1) = 0$ in $K_{\mathbb{C}^\times}(X^0)$ in the third and fourth equalities. Therefore, for all $n, k \geq 0$ and $a \in \{0, \ldots, n-1\}$, we have

$$\tilde{\psi}_0^{nk+a} = \tilde{\psi}_0^a + k\Delta_0\varphi_0. \tag{7.34}$$

If $m \in \{1, \dots, n-1\}$, then, adopting the convention that $\theta^0(0) = 1$ and $\theta^0(\chi_m^s) = 0$ for all s, we obtain

$$\begin{aligned}\tilde{\psi}_m^{nk+a}(\chi_m^s) &= \psi_m^{nk+a}(\chi_m^s)\theta^{nk+a}(\mathscr{S}_m^*)\\ &= \psi_m^a(\chi_m^s)(k\Delta_m + \theta^a(\mathscr{S}_m^*)) = k\psi_m^a(\chi_m^s)\Delta_m + \psi_m^a(\chi_m^s)\theta^a(\mathscr{S}_m^*)\\ &= k\epsilon_m(\psi_m^a(\chi_m^s))\Delta_m + \tilde{\psi}_m^a(\chi_m^s) = k\Delta_m + \tilde{\psi}_m^a(\chi_m^s),\end{aligned}$$

where we have used periodicity of ψ, the fact that $\mathscr{S}_m = \chi_m^1$ for all $m \in \{1, \dots, n-1\}$, the relation $(\chi_m^1)^n - 1 = 0$ in $K_G(X^m)$ (with respect to the ordinary multiplication), (7.29), and the fact that $\epsilon_m\psi_m^a = \epsilon_m$. Consequently, we have

$$\tilde{\psi}_m^{nk+a} = \tilde{\psi}_m^a + k\Delta_m\epsilon_m \tag{7.35}$$

for all $n, k \geq 0$, $a \in \{0, \dots, n-1\}$, and $m \in \{1, \dots, n-1\}$.

Equations (7.34) and (7.35) yield (7.33). □

Proposition 7.36. *In the virtual K-theory $\big(K(I\mathbb{P}(1,n))_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\psi}\big)$, an invertible element $\mathscr{L}$ is a λ-line element with respect to its inertial λ-ring structure if and only if $\tilde{\epsilon}(\mathscr{L}) = \mathbf{1}$ and* (6.14) *holds for all $l \in \{1, \dots, n\}$.*

Proof. First, (6.14) holds for $l = 1$ by definition of a ψ-ring. Suppose that $\mathscr{L}$ in $K(I\mathbb{P}(1,n))_{\mathbb{Q}}$ satisfies (6.14) for all $l \in \{1, \dots, n\}$. We now prove that (6.14) holds for all l. We do this by induction on k in the expression $nk + a$, as follows: Suppose for each $a \in \{1, \dots, n\}$ there exists $k \geq 0$ such that (6.14) holds for all $l \in \{a, n+a, \dots, nk+a\}$. Equation (7.33) implies that

$$\tilde{\psi}^{n(k+1)+a}(\mathscr{L}) = \tilde{\psi}^a(\mathscr{L}) + (k+1)j(\mathscr{L}), \tag{7.37}$$

where $j(\mathscr{L}) := \varphi_0(\mathscr{L})\Delta_0 + \sum_{m=1}^n \Delta_m\epsilon_m(\mathscr{L})$ belongs to J. However,

$$\begin{aligned}\mathscr{L}^{n(k+1)+a} &= \mathscr{L}^{nk+a}\mathscr{L}^n = (\tilde{\psi}^a(\mathscr{L}) + kj(\mathscr{L}))(\tilde{\psi}^0(\mathscr{L}) + j(\mathscr{L}))\\ &= (\tilde{\psi}^a(\mathscr{L}) + kj(\mathscr{L}))(1 + j(\mathscr{L}))\\ &= \tilde{\psi}^a(\mathscr{L}) + kj(\mathscr{L}) + \tilde{\psi}^a(\mathscr{L})j(\mathscr{L}) + kj(\mathscr{L})^2\\ &= \tilde{\psi}^a(\mathscr{L}) + (k+1)j(\mathscr{L})\\ &= \tilde{\psi}^{n(k+1)+a}(\mathscr{L}),\end{aligned}$$

where we have used the induction hypothesis and (7.37) in the second equality, the definition $\tilde{\psi}^0 = \tilde{\epsilon}$ in the third equality, Lemma 7.28 in the fifth, the fact that $\tilde{\epsilon} \circ \tilde{\psi}^q = \tilde{\epsilon}$ in the fifth, and (7.37) in the sixth. □

Remark 7.38. Proposition 7.36 reduces the problem of finding λ-line elements of $K(I\mathbb{P}(1,n))_{\mathbb{Q}}$ to solving a finite number of equations for $n^2 + 1$ (the rank of $K(I\mathbb{P}(1,n))$) unknowns. Furthermore, since the action of the translation group J, which is of rank n, respects $\mathcal{P}_1$ by Proposition 6.26, it is enough to solve for only

$n^2 - n + 1$ variables satisfying (6.14) for all $l \in \{0, \dots, n-1\}$, as all other λ-line elements will be their J-translates.

Corollary 7.39. *Let $\mathcal{P}_1$ be the semigroup of λ-line elements of the virtual K-theory $\big(K(I\mathbb{P}(1,n))_{\mathbb{Q}}, \star, 1, \tilde{\epsilon}, \tilde{\lambda}\big)$. Each $J_{\mathbb{Q}}$-orbit in $\mathcal{P}_1$ contains a unique representative $\mathscr{L}$ such that $\mathscr{L}^{\star n} = 1$.*

Proof. Given $\mathscr{F}$ in $\mathcal{P}_1$, we have $\mathscr{F}^{\star n} = \tilde{\psi}^n(\mathscr{F}) = 1 + j$ for some j in $J_{\mathbb{Q}}$ by Proposition 7.32. If $\mathscr{L} = \mathscr{F} - j/n$, then by (7.30) we have $\mathscr{L}^{\star n} = (\mathscr{F} - j/n)^{\star n} = \mathscr{F}^{\star n} - j = 1 + j - j = 1$. □

The virtual K-theory and virtual Chow ring of $\mathbb{P}(1,2)$. We now study the virtual K-theory and virtual Chow theory (with either $\mathbb{Q}$ or $\mathbb{C}$ coefficients) of the weighted projective line $\mathbb{P}(1,2) := [X/\mathbb{C}^{\times}]$. By [Edidin et al. 2016, Theorem 4.2.2] they are isomorphic to the orbifold K-theory and orbifold Chow theory, respectively, of the cotangent bundle $T^*\mathbb{P}(1,2)$.

Remark 7.40. For the remainder of this section, unless otherwise specified, all products are the virtual products.

Let $\tilde{\lambda} : K(I\mathbb{P}(1,2))_{\mathbb{Q}} \to K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ denote the induced virtual λ-ring structure. In order to describe the group of λ-line elements $\mathcal{P}_1$ of $\big(K(I\mathbb{P}(1,2))_{\mathbb{Q}}, \cdot, 1, \tilde{\lambda}\big)$, it will be useful to introduce the injective map $f : \mathbb{Q}^2 \to K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ defined by

$$f(\alpha, \beta) := \alpha\Delta_0 + \beta\Delta_1, \tag{7.41}$$

whose image is the translation group $J_{\mathbb{Q}}$ of $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$.

Consider the following injective maps from $\mathbb{Q}^2$ to $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$:

$$\rho_0(\alpha, \beta) := \chi_0^0 + f(\alpha, \beta), \tag{7.42}$$

$$\rho_1(\alpha, \beta) := \chi_0^1 + f(\alpha, \beta), \tag{7.43}$$

$$\rho_{\pm}(\alpha, \beta) := \tfrac{1}{2}(\chi_0^0 + \chi_0^1 \pm \chi_1^0) + f(\alpha, \beta). \tag{7.44}$$

Proposition 7.45. *In the virtual K-theory $\big(K(I\mathbb{P}(1,2))_{\mathbb{Q}}, \star, 1, \tilde{\lambda}\big)$, the group of λ-line elements $\mathcal{P}_1$ is the disjoint union of the images of the four maps ρ_0, ρ_1, and $\rho_{\pm}$, and the restriction of the inertial dual $\mathcal{P}_1 \to \mathcal{P}_1$ agrees with the operation of taking the inverse. In particular, $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ is spanned as a $\mathbb{Q}$-vector space by $\mathcal{P}_1$. The multiplication in $\mathcal{P}_1$ is given by the equations*

$$\rho_0(\alpha, \beta)\rho_0(\alpha', \beta') = \rho_0(\alpha + \alpha', \beta + \beta'), \tag{7.46}$$

$$\rho_0(\alpha, \beta)\rho_1(\alpha', \beta') = \rho_1(\alpha + \alpha', \beta + \beta'), \tag{7.47}$$

$$\rho_0(\alpha, \beta)\rho_{\pm}(\alpha', \beta') = \rho_{\pm}(\alpha + \alpha', \beta + \beta'), \tag{7.48}$$

$$\rho_1(\alpha, \beta)\rho_1(\alpha', \beta') = \rho_0(\alpha + \alpha' + 1, \beta + \beta'), \tag{7.49}$$

$$\rho_1(\alpha, \beta)\rho_{\pm}(\alpha', \beta') = \rho_{\mp}\big(\alpha + \alpha' + \tfrac{1}{2}, \beta + \beta' \pm \tfrac{1}{2}\big), \tag{7.50}$$

$$\rho_\pm(\alpha,\beta)\rho_\pm(\alpha',\beta') = \rho_0\big(\alpha+\alpha'+\tfrac{1}{2}, \beta+\beta' \pm \tfrac{1}{2}\big), \tag{7.51}$$

$$\rho_+(\alpha,\beta)\rho_-(\alpha',\beta') = \rho_1(\alpha+\alpha', \beta+\beta'). \tag{7.52}$$

The inverses are given by the equations

$$\rho_0(\alpha,\beta)^{-1} = \rho_0(-\alpha,-\beta), \tag{7.53}$$

$$\rho_1(\alpha,\beta)^{-1} = \rho_1(-(1+\alpha),-\beta), \tag{7.54}$$

$$\rho_\pm(\alpha,\beta)^{-1} = \rho_\pm\big(-\big(\alpha+\tfrac{1}{2}\big), -\beta \mp \tfrac{1}{2}\big). \tag{7.55}$$

Proof. We first show that the set of line elements $\mathcal{P}_1$ in the virtual K-theory $K := K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ is the union of the images of the maps ρ_0, ρ_1, and $\rho_\pm$. Since $\{\chi_0^0, \chi_0^1, \chi_1^1, \Delta_0, \Delta_1\}$ is a $\mathbb{Q}$-basis for K, it follows from Proposition 6.26 that every element of $\mathcal{P}_1$ can be uniquely written as $L + f(\alpha,\beta)$, where L is an element in $\mathcal{P}_1$ of the form $L = c_0^0\chi_0^0 + c_0^1\chi_0^1 + c_1^1\chi_1^1$, for some $c_0^0, c_0^1, c_1^1, \alpha, \beta \in \mathbb{Q}$. We will now find all such elements L in $\mathcal{P}_1$. By Proposition 7.36, L belongs to $\mathcal{P}_1$ if and only if it is invertible with $\tilde{\epsilon}(L) = 1$ and $\tilde{\psi}^2(L) = L^2$. Using the definition of $\tilde{\psi}^2$, we obtain

$$\tilde{\psi}^2(L) = c_0^0\chi_0^0 + c_0^1\chi_0^2 + c_1^1(\chi_1^0 + \chi_1^1),$$

and the virtual multiplication yields

$$\begin{aligned} L^2 &= (c_0^0\chi_0^0 + c_0^1\chi_0^1 + c_1^1\chi_1^1)^2 \\ &= (c_0^0)^2\chi_0^0 + (c_0^1)^2\chi_0^2 + (c_1^1)^2(\chi_1^1)^2 + 2c_0^0c_0^1\chi_0^1 + 2c_0^0c_1^1\chi_1^1 + 2c_0^1c_1^1\chi_0^1\chi_1^1 \\ &= (c_0^0)^2\chi_0^0 + (c_0^1)^2\chi_0^2 + (c_1^1)^2(\chi_0^0 - 2\chi_0^1 + \chi_0^2) + 2c_0^0c_0^1\chi_0^1 + 2c_0^0c_1^1\chi_1^1 + 2c_0^1c_1^1\chi_1^0 \\ &= ((c_0^0)^2 + (c_1^1)^2)\chi_0^0 + 2(c_0^0c_0^1 - (c_1^1)^2)\chi_0^1 + ((c_0^1)^2 + (c_1^1)^2)\chi_0^2 + 2c_0^1c_1^1\chi_1^0 + 2c_0^0c_1^1\chi_1^1, \end{aligned}$$

so $\tilde{\psi}^2(L) - L^2 = 0$ is equivalent to the simultaneous equations

$$0 = c_0^0(1-c_0^0) - (c_1^1)^2 = -c_0^0c_0^1 + (c_1^1)^2 = c_0^1(1-c_0^1) - (c_1^1)^2 = c_1^1(1-2c_0^1) = c_1^1(1-2c_0^0).$$

It follows that $\tilde{\psi}^2(L) = L^2$ if and only if $L = 0$, $\rho_0(0,0)$, $\rho_1(0,0)$, $\rho_\pm(0,0)$. However, the virtual augmentation satisfies $\tilde{\epsilon}(0) = 0$, while $\tilde{\epsilon}(\rho_0(0,0)) = \tilde{\epsilon}(\rho_1(0,0)) = \tilde{\epsilon}(\rho_\pm(0,0)) = 1$. Finally, $\rho_0(0,0)$, $\rho_1(0,0)$ are invertible, being classes of ordinary line bundles on the untwisted sector $\mathbb{P}(1,2)$, while a calculation shows that $\rho_\pm(0,0)^{-1} = \rho_\pm\big(-\tfrac{1}{2}, \mp\tfrac{1}{2}\big)$.

Therefore, by Proposition 6.26, $\mathcal{P}_1$ is the union of images of the maps ρ_0, ρ_1, and $\rho_\pm$. It is easy to see that these images are disjoint. Furthermore, K is spanned by $\mathcal{P}_1$, since $\{\rho_0(0,0), \rho_0(1,0), \rho_1(0,0), \rho_\pm(0,1)\}$ is a $\mathbb{Q}$-basis. Also, equations (7.53)–(7.55) follow from (7.46)–(7.52).

We will now write out a detailed proof of (7.51) to give the reader a feel for the calculation, noting that the proofs for (7.46)–(7.52) are similar. We first show that

(7.51) holds when $\alpha=\alpha'=\beta=\beta'=0$, since

$$\begin{aligned}(\rho_\pm(0,0))^2 &= \bigl(\tfrac{1}{2}(\chi_0^0+\chi_0^1\pm\chi_1^0)\bigr)^2\\ &= \tfrac{1}{4}\bigl((\chi_0^0)^2+(\chi_0^1)^2+(\chi_1^0)^2+2\chi_0^0\chi_0^1\pm 2\chi_0^0\chi_1^0\pm 2\chi_0^1\chi_1^0\bigr)\\ &= \tfrac{1}{4}\bigl(\chi_0^0+\chi_0^2+(\chi_0^0-2\chi_0^{-1}+\chi_0^{-2})+2\chi_0^1\pm 2\chi_1^0\pm 2\chi_1^1\bigr)\\ &= \tfrac{1}{4}\bigl(\chi_0^0+\chi_0^2+(\chi_0^0+\chi_0^2-2\chi_0^1)+2\chi_0^1\pm 2\chi_1^0\pm 2\chi_1^1\bigr)\\ &= \tfrac{1}{2}(\chi_0^0+\chi_0^2\pm(\chi_1^0+\chi_1^1))\\ &= \chi_0^0+\tfrac{1}{2}\Delta_0\pm\tfrac{1}{2}\Delta_1\\ &= \rho_0\bigl(\tfrac{1}{2},\pm\tfrac{1}{2}\bigr),\end{aligned}$$

where the third equality follows from Theorem 7.23 while the fourth is from the relations

$$\chi_0^{-1}=\chi_0^0+\chi_0^1-\chi_0^2 \quad\text{and}\quad \chi_0^{-2}=2\chi_0^0-\chi_0^2. \tag{7.56}$$

Now, (7.51) follows for all α, β, α', and β', since

$$\begin{aligned}&\rho_\pm(\alpha,\beta)\rho_\pm(\alpha',\beta')\\ &\quad= (\rho_\pm(0,0)+f(\alpha,\beta))(\rho_\pm(0,0)+f(\alpha',\beta'))\\ &\quad= \rho_\pm(0,0)\rho_\pm(0,0)+(f(\alpha,\beta)+f(\alpha',\beta'))\rho_\pm(0,0)+f(\alpha,\beta)f(\alpha',\beta')\\ &\quad= \rho_\pm\bigl(\tfrac{1}{2},\pm\tfrac{1}{2}\bigr)+f(\alpha+\alpha',\beta+\beta')\rho_\pm(0,0)\\ &\quad= \rho_\pm\bigl(\tfrac{1}{2},\pm\tfrac{1}{2}\bigr)+f(\alpha+\alpha',\beta+\beta')\tilde{\epsilon}(\rho_\pm(0,0))\\ &\quad= \rho_\pm\bigl(\tfrac{1}{2},\pm\tfrac{1}{2}\bigr)+f(\alpha+\alpha',\beta+\beta')=\rho_\pm\bigl(\alpha+\alpha'+\tfrac{1}{2},\beta+\beta'\pm\tfrac{1}{2}\bigr).\end{aligned}$$

Here, the third equality follows from the fact that $J^2=0$ in Lemma 7.28(2), from (7.51) when $\alpha=\beta=\alpha'=\beta'=0$, and from the definition of f. The fourth equality is from (7.30), the fifth is from Proposition 7.36, and the sixth is from the definition of $\rho_\pm$. This finishes the proof of (7.51).

Finally, we write details of the proof that $\rho_0(\alpha,\beta)^\dagger=\rho_0^{-1}(\alpha,\beta)$. The proof of the analogous statements for $\rho_1(\alpha,\beta)$, $\rho_\pm(\alpha,\beta)$, and, hence, for all elements in $\mathcal{P}_1$ is similar. The definition of the inertial dual, together with the fact that $\mathscr{S}_0=0$ and $\mathscr{S}_1=\chi_1^1$, yields the following identities for all $a,b\in\mathbb{Z}$:

$$(\chi_0^a)^\dagger=\chi_0^{-a} \quad\text{and}\quad (\chi_1^b)^\dagger=-\chi_1^{-b-1}. \tag{7.57}$$

It follows that

$$\begin{aligned}\rho_0(\alpha,\beta)^\dagger &= (\chi_0^0)^\dagger+\alpha(\Delta_0)^\dagger+\beta(\Delta_1)^\dagger\\ &= (\chi_0^0)^\dagger+\alpha((\chi_0^2)^\dagger-(\chi_0^0)^\dagger)+\beta((\chi_1^0)^\dagger+(\chi_1^1)^\dagger)\\ &= \chi_0^0+\alpha(\chi_0^{-2}-\chi_0^0)-\beta(\chi_1^{-1}+\chi_1^{-2})\end{aligned}$$

$$
\begin{aligned}
&= \chi_0^0 + \alpha((2\chi_0^0 - \chi_0^2) - \chi_0^0) - \beta(\chi_1^0 + \chi_1^1) \\
&= \chi_0^0 - \alpha\Delta_0 - \beta\Delta_1 \\
&= \rho_0(-\alpha, -\beta) = \rho_0(\alpha, \beta)^{-1},
\end{aligned}
$$

where the third equality follows from (7.57), the fourth from (7.56), and the last from (7.53). □

A direct calculation yields the following:

Proposition 7.58. *The inertial first Chern class for virtual K-theory is a homomorphism of groups* $\tilde{c}^1 : \mathcal{P}_1 \to A^{\{1\}}(I\mathbb{P}(1, 2))_{\mathbb{Q}}$, *where*

$$
\begin{aligned}
\tilde{c}^1(\rho_0(\alpha, \beta)) &= 2\alpha c_0^1 + 2\beta c_1^0, \\
\tilde{c}^1(\rho_1(\alpha, \beta)) &= (2\alpha + 1)c_0^1 + 2\beta c_1^0, \\
\tilde{c}^1(\rho_\pm(\alpha, \beta)) &= \big(2\alpha + \tfrac{1}{2}\big)c_0^1 + \big(2\beta \pm \tfrac{1}{2}\big)c_1^0.
\end{aligned}
$$

The virtual K-theory ring has a simple form in terms of these λ-line elements.

Proposition 7.59. *Let* $\big(K(I\mathbb{P}(1, 2))_{\mathbb{Q}}, \star, 1 := \chi_0^0\big)$ *be the virtual K-theory ring. We have two isomorphisms of* $\mathbb{Q}$*-algebras (and* ψ*-rings)*

$$
\Phi_\pm : \frac{\mathbb{Q}[\sigma, \tau]}{\langle(\tau-1)(\tau^2-1), (\sigma-1)(\sigma^2-1), (\sigma-\tau)(\tau-1)\rangle} \to K(I\mathbb{P}(1, 2))_{\mathbb{Q}}, \quad (7.60)
$$

where $\Phi_\pm(\sigma) := \rho_1(0, 0) = \chi_0^1$ *and* $\Phi_\pm(\tau) := \rho_\pm(0, 0) = \frac{1}{2}(\chi_0^0 + \chi_0^1 \pm \chi_1^0)$. *Here, the* ψ*-ring structure of the domain of* $\Phi_\pm$ *is given by* $\psi^l(\sigma^{\pm 1}) = \sigma^{\pm l}$ *and* $\psi^l(\tau^{\pm 1}) = \tau^{\pm l}$ *for all* $l \geq 1$. *Similarly, we have two isomorphisms of graded* $\mathbb{Q}$*-algebras*

$$
\Psi_\pm : \frac{\mathbb{Q}[\mu, \nu]}{\langle \mu, \nu\rangle^2} \to A^*(I\mathbb{P}(1, 2))_{\mathbb{Q}}, \quad (7.61)
$$

where $\mu, \nu \in A^{\{1\}}(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ *with* $\Psi_\pm(\nu) := \tilde{c}^1(\rho_\pm(0, 0)) = \frac{1}{2}(c_0^1 \pm c_1^0)$ *and* $\Psi_\pm(\mu) := \tilde{c}^1(\rho_1(0, 0)) = c_0^1$. *Under the identifications* $\Phi_\pm$ *and* $\Psi_\pm$, *the inertial Chern character* $\widetilde{\mathscr{C}h} : K(I\mathbb{P}(1, 2)) \to A^*(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ *corresponds to the map* $\sigma \mapsto \exp(\mu) = 1 + \mu$ *and* $\tau \mapsto \exp(\nu) = 1 + \nu$.

Proof. Since $(\chi_0^1)^2 = \chi_0^2$, $\chi_0^0 = 1$ and $\rho_\pm(0, 0)^2 = \frac{1}{2}((\chi_0^0 + \chi_0^2) \pm (\chi_1^0 + \chi_1^1))$, the set $\{\chi_0^0, \chi_0^1, \chi_0^2, \rho_+(0, 0), \rho_+(0, 0)^2\}$ is a basis for the $\mathbb{Q}$-vector space $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$. Thus, $K(I\mathbb{P}(1, 2))_{\mathbb{Q}}$ is generated as a $\mathbb{Q}$-algebra by χ_0^1 and $\rho_+(0, 0)$. A calculation shows that the following three polynomials are zero:

$$
\begin{aligned}
(\chi_0^1 - 1)((\chi_0^1)^2 - 1) &= (\rho_+(0, 0) - 1)(\rho_+(0, 0)^2 - 1) \\
&= (\chi_0^1 - \rho_+(0, 0))(\rho_+(0, 0) - 1) = 0.
\end{aligned}
$$

A dimension count shows that these are the only relations. Therefore, Φ_+ is an isomorphism of $\mathbb{Q}$-algebras. The previous analysis holds verbatim if $\rho_+(0, 0)$ is replaced by $\rho_-(0, 0)$ everywhere.

A similar analysis holds for the Chow theory. □

Remark 7.62. The presentation in the previous proposition yields an exotic integral structure in virtual K-theory and Chow theory, as we now explain.

Consider the subring $\boldsymbol{K}(I\mathbb{P}(1,2))$ (not sub-$\mathbb{Q}$-algebra) of $K(I\mathbb{P}(1,2))_{\mathbb{Q}}$ generated by $\{\rho_1(0,0), \rho_+(0,0)\}$. Under the isomorphism Φ_+ in Proposition 7.59, the ring $\boldsymbol{K}(I\mathbb{P}(1,2))$ is isomorphic to

$$\frac{\mathbb{Z}[\sigma,\tau]}{\langle(\tau-1)(\tau^2-1), (\sigma-1)(\sigma^2-1), (\sigma-\tau)(\tau-1)\rangle}$$

under the identification $\sigma=\rho_1(0,0)$ and $\tau=\rho_+(0,0)$.

We will now show that the group $\boldsymbol{P}_1$ of λ-line elements of $\boldsymbol{K}(I\mathbb{P}(1,2))$ is equal to $\mathcal{P}_1 \cap \boldsymbol{K}(I\mathbb{P}(1,2))$. To see this, notice that, since $\Delta_0=\sigma^2-1$ and $\Delta_1 = 2\tau^2-\sigma^2-1$,

$$f(\alpha,\beta) = 2\beta\tau^2 + (\alpha-\beta)\sigma^2 - (\alpha+\beta).$$

Hence, $\rho_s(\alpha,\beta)$ belongs to $\boldsymbol{K}(I\mathbb{P}(1,2))$ if and only if (α,β) belongs to

$$D := \{(p+\tfrac{1}{2}q, \tfrac{1}{2}q) \mid p,q\in\mathbb{Z}\}$$

for $s=0$, 1, and $\pm$, noting that $\rho_-(0,0)=\sigma\tau^{-1}$. Thus, by Proposition 7.59,

$$\mathcal{P}_1 \cap \boldsymbol{K}(I\mathbb{P}(1,2)) = \rho_0(D)\cup\rho_1(D)\cup\rho_+(D)\cup\rho_-(D),$$

but (7.53)–(7.55) imply that $\mathcal{P}_1 \cap \boldsymbol{K}(I\mathbb{P}(1,2))$ is closed under inversion. It follows that $\boldsymbol{P}_1 = \mathcal{P}_1 \cap \boldsymbol{K}(I\mathbb{P}(1,2))$.

We will now show that $\boldsymbol{P}_1$ is the subgroup generated by σ and τ. Notice that, since $\sigma^2=\rho_0(1,0)$ and $\tau^2=\rho_0(\frac{1}{2},\frac{1}{2})$, the element $\sigma^{2k}\tau^{2l} = \rho_0(k+\frac{1}{2}l, \frac{1}{2}l)$ belongs to $\langle\sigma,\tau\rangle$ for all $k,l\in\mathbb{Z}$, i.e., $\rho_0(D)\subseteq\langle\sigma,\tau\rangle$. Similarly, $\rho_1(0,0)\rho_0(D)=\rho_1(D)$, $\rho_+(0,0)\rho_0(D)=\rho_+(D)$, and $\rho_-(0,0)\rho_0(D)=\rho_-(D)$ are all subsets of $\langle\sigma,\tau\rangle$. It follows that $\langle\sigma,\tau\rangle = \boldsymbol{P}_1$.

Consider the subring $\boldsymbol{A}^*(I\mathbb{P}(1,2)) := \widetilde{\mathscr{C}h}(\boldsymbol{K}(I\mathbb{P}(1,2)))$ of the virtual Chow ring of $A^*(I\mathbb{P}(1,2))_{\mathbb{Q}}$. From this we obtain (see Proposition 7.58)

$$\boldsymbol{A}^{\{0\}}(I\mathbb{P}(1,2)) = \mathbb{Z}c_0^0 \quad\text{and}\quad \boldsymbol{A}^{\{1\}}(I\mathbb{P}(1,2)) = \{vc_0^1 + wc_1^0 \mid (v,w)\in D\}.$$

It follows that the first virtual Chern class $\tilde{c}^1 : \boldsymbol{P}_1 \to \boldsymbol{A}^{\{1\}}(I\mathbb{P}(1,2))$ is a group isomorphism by Proposition 7.58, since, for all p, q in $\mathbb{Z}$,

$$\tilde{c}^1(\sigma^p\tau^q) = p\tilde{c}^1(\sigma) + q\tilde{c}^1(\tau) = (p+\tfrac{1}{2}q)c_0^1 + \tfrac{1}{2}qc_1^0.$$

The virtual K-theory and virtual Chow ring of $\mathbb{P}(1,3)$. We now study the virtual K-theory and virtual Chow ring of $\mathbb{P}(1,3)$. Unlike the case of $\mathbb{P}(1,2)$, the formula of [Edidin et al. 2016, Theorem 4.2.2] implies that the rational virtual K-theory and rational virtual Chow rings of $\mathbb{P}(1,3)$ differ from the orbifold K-theory and the orbifold Chow rings of the cotangent bundle $T^*\mathbb{P}(1,3)$, respectively. Indeed the formula of [Edidin et al. 2016, Definition 4.0.11] shows that the class $\mathscr{S}^+T^*\mathbb{P}(1,3)$ is not integral, so the inertial pair from the orbifold theory of $T^*\mathbb{P}(1,3)$ is Gorenstein but not strongly Gorenstein. We will now describe the λ-positive elements of the virtual K-theory of $\mathbb{P}(1,3)$. Unlike the case of $\mathbb{P}(1,2)$, we need to work with coefficients in $\mathbb{C}$, so that the set of λ-line elements generate the entire virtual K-theory group.

Remark 7.63. For the remainder of this section, unless otherwise specified, all products are the virtual products.

Proposition 7.64. *Let $\big(K(I\mathbb{P}(1,3))_{\mathbb{C}}, \star, 1 := \chi_0^0, \tilde{\psi}\big)$ be the virtual K-theory ring with its virtual λ-ring structure. The set of its λ-line elements $\mathcal{P}_1$ spans the $\mathbb{C}$-vector space $K(I\mathbb{P}(1,3))_{\mathbb{C}}$. The restriction of the inertial dual $\mathcal{P}_1 \to \mathcal{P}_1$ agrees with the operation of taking the inverse. The space $\mathcal{P}_1$ consists of 27 orbits of the action of the translation group $J_{\mathbb{C}}$, where each orbit has a unique representative*[1] *in the set*

$$\{\Sigma_i\}_{i=1}^3 \sqcup \coprod_{\substack{i=1,2,3\\ j=1,2}} \mathcal{D}_{i,j} \sqcup \coprod_{\substack{i=1,\dots,6\\ k=0,1,2}} \mathcal{T}_{i,k}$$

given by the following, where $\zeta_3 = \exp\big(\tfrac{2}{3}\pi i\big)$, $j \in \{1,2\}$, and $k \in \{0,1,2\}$:

$$\Sigma_1 = \chi_0^0, \quad \Sigma_2 = \chi_0^1, \quad \Sigma_3 = \chi_0^2,$$

$$\begin{aligned}
\mathcal{D}_{1,j} &= \tfrac{1}{3}\chi_0^0 + \tfrac{1}{3}\chi_0^1 + \tfrac{1}{3}\chi_0^2 - \tfrac{1}{3}\zeta_3^j\chi_1^0 + \tfrac{1}{3}\chi_1^1 - \tfrac{1}{3}\zeta_3^{2j}\chi_2^0 + \tfrac{1}{3}\chi_2^1,\\
\mathcal{D}_{2,j} &= \tfrac{1}{3}\chi_0^0 + \tfrac{1}{3}\chi_0^1 + \tfrac{1}{3}\chi_0^2 - \tfrac{1}{3}\chi_1^0 + \tfrac{1}{3}\zeta_3^j\chi_1^1 - \tfrac{1}{3}\chi_2^0 + \tfrac{1}{3}\zeta_3^{2j}\chi_2^1,\\
\mathcal{D}_{3,j} &= \tfrac{1}{3}\chi_0^0 + \tfrac{1}{3}\chi_0^1 + \tfrac{1}{3}\chi_0^2 - \tfrac{1}{3}\zeta_3^{2j}\chi_1^0 + \tfrac{1}{3}\zeta_3^j\chi_1^1 - \tfrac{1}{3}\zeta_3^j\chi_2^0 + \tfrac{1}{3}\zeta_3^{2j}\chi_2^1,\\
\mathcal{T}_{1,k} &= \tfrac{1}{3}\chi_0^0 + \tfrac{2}{3}\chi_0^2 + \tfrac{1}{3}\zeta_3^k\chi_1^0 + \tfrac{1}{3}\zeta_3^{2k}\chi_2^0,\\
\mathcal{T}_{2,k} &= \tfrac{2}{3}\chi_0^0 + \tfrac{1}{3}\chi_0^2 - \tfrac{1}{3}\zeta_3^k\chi_1^0 - \tfrac{1}{3}\zeta_3^{2k}\chi_2^0,\\
\mathcal{T}_{3,k} &= \tfrac{2}{3}\chi_0^0 + \tfrac{1}{3}\chi_0^1 + \tfrac{1}{3}\zeta_3^k\chi_1^1 + \tfrac{1}{3}\zeta_3^{2k}\chi_2^1,\\
\mathcal{T}_{4,k} &= \tfrac{1}{3}\chi_0^0 + \tfrac{2}{3}\chi_0^1 - \tfrac{1}{3}\zeta_3^k\chi_1^1 - \tfrac{1}{3}\zeta_3^{2k}\chi_2^1,\\
\mathcal{T}_{5,k} &= \tfrac{1}{3}\chi_0^1 + \tfrac{2}{3}\chi_0^2 + \tfrac{1}{3}\zeta_3^k\chi_1^0 + \tfrac{1}{3}\zeta_3^k\chi_1^1 + \tfrac{1}{3}\zeta_3^{2k}\chi_2^0 + \tfrac{1}{3}\zeta_3^{2k}\chi_2^1,\\
\mathcal{T}_{6,k} &= \tfrac{2}{3}\chi_0^1 + \tfrac{1}{3}\chi_0^2 - \tfrac{1}{3}\zeta_3^k\chi_1^0 - \tfrac{1}{3}\zeta_3^k\chi_1^1 - \tfrac{1}{3}\zeta_3^{2k}\chi_2^0 - \tfrac{1}{3}\zeta_3^{2k}\chi_2^1.
\end{aligned}$$

Proof. The λ-line elements in $\mathcal{P}_1$ are calculated by applying the algorithm in Remark 7.38 and by showing that these λ-line elements are invertible. The fact

[1]This representative need not be the same as the one defined in Corollary 7.39.

that the elements of $\mathcal{P}_1$ span $K(\mathbb{P}(1,3))_{\mathbb{C}}$ is also a calculation. We omit the details to all of these calculations, which are straightforward but lengthy. □

Proposition 7.65. *Let $K(I\mathbb{P}(1,3))_{\mathbb{C}}$ be the virtual K-theory with its virtual λ-ring structure. We have an isomorphism $\Psi : \mathbb{C}[\sigma^{\pm 1}, \tau^{\pm 1}, \bar{\tau}^{\pm 1}]/\boldsymbol{I} \to K(I\mathbb{P}(1,3))_{\mathbb{C}}$ of $\mathbb{C}$-algebras with $\Psi(\sigma) = \Sigma_2$, $\Psi(\tau) = \mathcal{T}_{1,1}$, and $\Psi(\bar{\tau}) = \mathcal{T}_{1,2}$, where the ideal $\boldsymbol{I}$ is generated by the following ten relations*:

$$\begin{gathered}
\mathcal{R}_1 := \sigma^3 - 2\sigma^2 + \sigma - \tau^2 + \tau\bar{\tau} + \tau - \bar{\tau}^2 + \bar{\tau} - 1,\\
\mathcal{R}_2 := (\tau - 1)(\tau^2 - \sigma), \quad \overline{\mathcal{R}}_2 := (\bar{\tau} - 1)(\bar{\tau}^2 - \sigma),\\
\mathcal{R}_3 := (\tau - 1)(\sigma^2 - \tau), \quad \overline{\mathcal{R}}_3 := (\bar{\tau} - 1)(\sigma^2 - \bar{\tau}),\\
\mathcal{R}_4 := \sigma^2 - \sigma\tau - \sigma\bar{\tau} + \tau^2\bar{\tau} - \tau\bar{\tau} + \bar{\tau}^2 - \bar{\tau} + 1,\\
\overline{\mathcal{R}}_4 := \sigma^2 - \sigma\tau - \sigma\bar{\tau} + \tau^2 + \tau\bar{\tau}^2 - \tau\bar{\tau} - \tau + 1,\\
\mathcal{R}_5 := (\tau - 1)(\sigma\tau - 1), \quad \overline{\mathcal{R}}_5 := (\bar{\tau} - 1)(\sigma\bar{\tau} - 1),\\
\mathcal{R}_6 := -\sigma^2 + \sigma\tau\bar{\tau} + \sigma - \tau^2 + \tau\bar{\tau} - \bar{\tau}^2.
\end{gathered}$$

It follows that $(\sigma - 1)(\sigma^3 - 1)$ belongs to $\boldsymbol{I}$, which is the relation on the untwisted sector. Furthermore, every element $K(I\mathbb{P}(1,3))_{\mathbb{C}}$ can be uniquely presented as a polynomial $\{\sigma, \tau, \bar{\tau}\}$ of degree less than or equal to 2. *In particular, we have*

$$\begin{gathered}
\sigma^{-1} = -\sigma^2 + \sigma - \tau^2 + \tau\bar{\tau} + \tau - \bar{\tau}^2 + \bar{\tau},\\
\tau^{-1} = -\sigma\tau + \sigma + 1, \quad \text{and} \quad \bar{\tau}^{-1} = -\sigma\bar{\tau} + \sigma + 1.
\end{gathered}$$

Proof. $K(I\mathbb{P}(1,3))_{\mathbb{C}}$ is a 10-dimensional $\mathbb{C}$-vector space. A calculation shows that the set of all monomials in $\{\sigma, \tau, \bar{\tau}\}$ of degree less than or equal to 2 is a basis of this vector space. The ten relations correspond to the ten cubic monomials in $\{\sigma, \tau, \bar{\tau}\}$. The expression for the inverses can be verified by computation. We omit the details of these straightforward but lengthy calculations. □

Remark 7.66. Restricting Ψ to $\mathbb{Z}[\sigma^{\pm 1}, \tau^{\pm 1}, \bar{\tau}^{\pm 1}]/\boldsymbol{I}$ yields an exotic integral structure on the virtual K-theory $K(I\mathbb{P}(1,3))_{\mathbb{C}}$. The inertial Chern character homomorphism $\widetilde{\mathscr{C}h} : K(I\mathbb{P}(1,3))_{\mathbb{C}} \to A^*(I\mathbb{P}(1,3))_{\mathbb{C}}$ induces an exotic integral structure on virtual Chow theory.

The resolution of singularities of $\mathbb{T}^*\mathbb{P}(1,n)$ and the HKRC. We now connect the virtual λ-ring to the usual λ-ring structure on a crepant resolution of singularities of the coarse moduli space of the cotangent bundle stack $\mathbb{T}^*\mathbb{P}(1,n)$.

Proposition 7.67. *The cotangent bundle $\mathbb{T}^*\mathbb{P}(1,n)$ of $\mathbb{P}(1,n)$ is the quotient stack $[(X \times \mathbb{A}^1)/\mathbb{C}^\times]$, where $\mathbb{C}^\times$ acts with weights $(1, n, -(n+1))$.*

Proof. Since $\dim \mathbb{P}(1,n) = 1$ the cotangent bundle stack is a line bundle. Consider the quotient map $\pi : X^0 \to \mathbb{P}(1,n) = [X^0/\mathbb{C}^\times]$. We begin by determining $\pi^*\mathbb{T}^*\mathbb{P}(1,n)$ as an $\mathbb{C}^\times$-equivariant bundle L on X^0. Once we do this, we can identify $\mathbb{T}^*\mathbb{P}(1,n)$ with the quotient stack $[L/\mathbb{C}^\times]$.

The restriction map $\mathrm{Pic}_{\mathbb{C}^\times}(\mathbb{C}^2) \to \mathrm{Pic}_{\mathbb{C}^\times}(X^0) = \mathrm{Pic}(\mathbb{P}(1,n))$ is surjective, so any $\mathbb{C}^\times$-equivariant line bundle on X^0 is determined by a character ξ of $\mathbb{C}^\times$, so $L = X^0 \times \mathbb{A}^1$ and $\mathbb{C}^\times$ acts on L by $\lambda(a,b,v) = (\lambda a, \lambda^n b, \xi(\lambda)v)$.

To find the character ξ, note that, for any algebraic group G and any G-torsor $\pi : P \to X$, there is an exact sequence of G-equivariant vector bundles on P

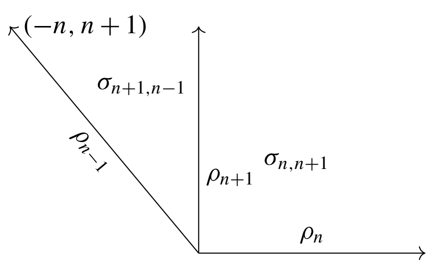
$$0 \to P \times \mathrm{Lie}(G) \to TP \to \pi^*TX \to 0,$$

where TP is the tangent bundle to P [Edidin and Graham 2005, Lemma A.1]. Applying this fact to the $\mathbb{C}^\times$-torsor $\pi : X^0 \to \mathbb{P}(1,n)$, we obtain an exact sequence of vector bundles

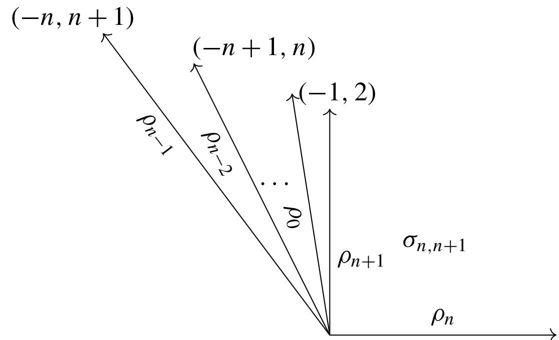
$$X^0 \times \mathbb{C} \to TX^0 \to \pi^*\mathbb{T}\mathbb{P}(1,n).$$

The action of $\mathbb{C}^\times$ is as follows: Since $\mathbb{C}^\times$ is abelian, the Lie algebra is the trivial representation, while $TX^0 = X^0 \times \mathbb{C}^2$, where $\mathbb{C}^\times$ acts on the $\mathbb{C}^2$ factor with weights $(1,n)$. Taking the determinant of this sequence shows $\pi^*\mathbb{T}\mathbb{P}(1,n)$ is the $\mathbb{C}^\times$-equivariant line bundle $X^0 \times \mathbb{C}$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight $n+1$. Hence, $\pi^*\mathbb{T}^*\mathbb{P}(1,n)$ is the $\mathbb{C}^\times$-equivariant bundle $X^0 \times \mathbb{C}$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight $-(n+1)$. $\square$

By Proposition 7.67, the coarse moduli space of $\mathbb{T}^*\mathbb{P}(1,n)$ is the geometric quotient $\big((\mathbb{C}^2 \smallsetminus \{0\}) \times \mathbb{C}\big)/\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts by $\lambda(a,b,v) = (\lambda a, \lambda^n b, \lambda^{-n-1}v)$. By the Cox construction [Cox et al. 2011, Section 5.1], this quotient is the toric surface associated to the simplicial fan Σ_n with two maximal cones $\sigma_{n+1,n-1}$ and $\sigma_{n,n+1}$. The cone $\sigma_{n+1,n-1}$ has rays ρ_{n-1} generated by $(-n, n+1)$ and ρ_{n+1} generated by $(0,1)$. The cone $\sigma_{n,n+1}$ has rays ρ_{n+1} and ρ_n spanned by $(1,0)$. The fan is as follows:



The cone $\sigma_{n+1,n-1}$ has multiplicity $n+1$ and, by the method of Hirzebruch–Jung continued fractions [Cox et al. 2011, Section 10.2], the nonsingular toric surface determined by the fan Σ'_n, where $\sigma_{n-1,n+1}$ is subdivided along the rays $\rho_0, \rho_1, \ldots, \rho_{n-2}$ with ρ_i generated by $(-(i+1), i+2)$, is a toric resolution of singularities of $X(\Sigma_n)$:



By [Cox et al. 2011, Exercise 8.2.13], $X(\Sigma_n)$ is Gorenstein, so by [Cox et al. 2011, Proposition 11.28] the resolution of singularities $X(\Sigma'_n) \to X(\Sigma_n)$ is crepant.

By the Cox construction, we can realize the smooth toric variety $X(\Sigma'_n)$ as the quotient of $\mathbb{A}^{n+2} \smallsetminus Z(\Sigma'_n)$ with coordinates $(x_0, \dots, x_{n+1})$ by the free action of $(\mathbb{C}^\times)^n$ with weights

$$(\chi_0, \dots, \chi_{n-1}, \chi_0\chi_1^2 \cdots \chi_{n-1}^n, \chi_0^{-2}\chi_1^{-3} \cdots \chi_{n-1}^{-(n+1)}),$$

where χ_i is the character of $(\mathbb{C}^\times)^n$ corresponding to the i-th standard basis vector of $\mathbb{Z}^n$ and

$$Z(\Sigma'_n) = V\big(x_2x_3 \cdots x_{n+1}, x_0x_3 \cdots x_{n+1}, x_0x_1x_4 \cdots x_{n+1}, \dots, \\ x_0 \cdots x_{n-3}x_nx_{n+1}, x_0 \cdots x_{n-1}, x_1x_2 \cdots x_n\big).$$

Proposition 7.68. *The following isomorphisms hold*, *where* $t_i = c_1(\chi_i)$:

$$K(X(\Sigma'_n)) = \frac{\mathbb{Z}[\chi_0, \chi_0^{-1}, \dots, \chi_{n-1}, \chi_{n-1}^{-1}]}{\langle \mathrm{eu}(\chi_0), \dots, \mathrm{eu}(\chi_{n-1})\rangle^2} \quad \textit{and} \quad A^*(X(\Sigma'_n)) = \frac{\mathbb{Z}[t_0, t_1, \dots, t_{n-1}]}{\langle t_0, t_1, \dots, t_{n-1}\rangle^2}.$$

Proof. The action of the torus is free, so $K(X(\Sigma'_n)) = K_{(\mathbb{C}^\times)^n}(\mathbb{C}^{n+2} \smallsetminus Z(\Sigma'_n))$ and $A^*(X(\Sigma'_n)) = A^*_{(\mathbb{C}^\times)^n}(\mathbb{C}^{n+2} \smallsetminus Z(\Sigma'_n))$. As in the proof of Proposition 7.13, the localization exact sequence in equivariant K-theory implies that $K_{(\mathbb{C}^\times)^n}(\mathbb{C}^{n+2} \smallsetminus Z(\Sigma'_n))$ is a quotient of $R((\mathbb{C}^\times)^n) = \mathbb{Z}[\chi_0, \chi_0^{-1}, \dots, \chi_{n-1}, \chi_{n-1}^{-1}]$. Because $Z(\Sigma'_n)$ is the union of intersecting linear subspaces, we use an inductive argument to establish the relations. The ideal

$$I = \big\langle x_2x_3 \cdots x_{n+1}, x_0x_3 \cdots x_{n+1}, x_0x_1x_4 \cdots x_{n+1}, \dots, \\ x_0 \cdots x_{n-3}x_nx_{n+1}, x_0 \cdots x_{n-1}, x_1x_2 \cdots x_n\big\rangle$$

has a primary decomposition as the intersection of the ideals of linear spaces $\langle x_i, x_j\rangle$ for $i \in \{0, \dots, n-1\}$ and $i+2 \le j \le n+1$. Thus $Z(\Sigma'_n)$ is the union of the linear subspaces $L_{i,j}$, where $L_{i,j} = Z(x_i, x_j)$. Order the pairs (i, j) lexicographically and set $U_{i,j} = \mathbb{C}^2 \smallsetminus \big(\bigcup_{(k,l)\le(i,j)} L_{k,l}\big)$, so that $\mathbb{C}^{n+2} \smallsetminus Z(\Sigma_n') = U_{n-1,n+1}$. If $j < n+1$, we have a localization sequence

$$K_{(\mathbb{C}^\times)^n}\Bigg(L_{i,j+1} \setminus \Bigg(\bigcup_{(j,k)<(i,j+1)} L_{j,k}\Bigg)\Bigg) \to K_{(\mathbb{C}^\times)^n}(U_{i,j}) \to K_{(\mathbb{C}^\times)^n}(U_{i,j+1}) \to 0.$$

The same self-intersection argument used in the proof of Proposition 7.13 shows that $K_{(\mathbb{C}^\times)^n}(U_{i,j+1}) = K_{(\mathbb{C}^\times)^n}(U_{i,j})/\langle \mathrm{eu}(N_{i,j+1})\rangle$, where $N_{i,j+1}$ is the normal bundle to $L_{i,j+1}$ in $\mathbb{C}^{n+2}$. Similarly, $K_{(\mathbb{C}^\times)^n}(U_{i+1,i+2}) = K_{(\mathbb{C}^\times)^n}(U_{i,i+2})/\langle \mathrm{eu}(L_{i+1,i+2})\rangle$. Hence, by induction we have that

$$K_{(\mathbb{C}^\times)^n}(U_{n-1,n+1}) = \mathbb{Z}[\chi_0, \chi_0^{-1}, \dots, \chi_{n-1}, \chi_{n-1}^{-1}]/\langle \{\mathrm{eu}(N_{i,j})\}_{i,j}\rangle.$$

The K-theoretic Euler class of the bundle $N_{i,j}$ can be read off from the weights of the $(\mathbb{C}^\times)^n$ action. We have

$$\mathrm{eu}(N_{i,j}) = \begin{cases} (1-\chi_i^{-1})(1-\chi_j^{-1}) & \text{if } j < n, \\ (1-\chi_i^{-1})(1-(\chi_0\chi_1^2\cdots\chi_{n-1}^n)^{-1}) & \text{if } j = n, \\ (1-\chi_i^{-1})(1-\chi_0^2\chi_1^3\cdots\chi_{n-1}^{n+1}) & \text{if } j = n+1. \end{cases}$$

We wish to show that the ideal $\mathfrak{b}$ generated by these Euler classes is the same as the ideal $\mathfrak{a} = \langle \mathrm{eu}(\chi_0), \dots, \mathrm{eu}(\chi_{n-1})\rangle^2$. If we set $e_i = \mathrm{eu}(\chi_i) = (1-\chi_i^{-1})$, then $\mathfrak{a} = \langle \{e_ie_j\}_{0\le i\le j\le n-1}\rangle$. Note that the ideal $\langle e_1, \dots, e_n\rangle$ is the ideal of Laurent polynomials in $\chi_0, \dots, \chi_n$ that vanish at $(1, 1, \dots, 1)$. If $j < n$, then $\mathrm{eu}(N_{i,j}) = e_ie_j \in \mathfrak{a}$. Also note that, since the expression $(1-(\chi_0\chi_1^2\cdots\chi_{n-1}^n)^{-1})$ vanishes when each χ_i is set to 1, it must be in the ideal generated by $e_1, \dots, e_n$, so $\mathrm{eu}(N_{i,n}) = (1-\chi_i^{-1})(1-(\chi_0\chi_1^2\cdots\chi_{n-1}^n)^{-1}) \in \langle e_0, \dots, e_n\rangle^2 = \mathfrak{a}$. Similarly, $\mathrm{eu}(N_{i,n+1}) \in \mathfrak{a}$.

If $i < n-1$ and $j \ge i+1$, then the generators e_ie_j are the Euler classes of the bundles $N_{i,j}$. The remaining generators of $\mathfrak{a}$ are of the form e_i^2 and e_ie_{i+1}. Since the χ_i are units, the fact that e_ie_j is in $\mathfrak{b}$ implies that, for all $k > 0$ and $|i-j| \ge 2$, all expressions of the form $e_i(1-\chi_j^{-k})$ and $(1-\chi_i^{-1})(\chi_j^k - 1)$ are in $\mathfrak{b}$. We can then perform repeated eliminations with the expression for $\mathrm{eu}(N_{i,n})$ to show that $e_i(1-\chi_i^{-(i+1)}\chi_i^{-(i+2)}) \in \mathfrak{b}$ for any i. A similar set of eliminations using the expression for $\mathrm{eu}(N_{i,n+1})$ shows that $e_i(1-\chi_i^{(i+2)}\chi_{i+1}^{-(i+3)}) \in \mathfrak{b}$. Since the χ_i are units, $e_i(1-\chi_i^{-(i+1)}\chi_{i+1}^{-(i+2)}) \in \mathfrak{b}$. Hence

$$e_i(-\chi_i^{-(i+2)(i+1)}\chi_{i+1}^{-(i+2)^2} + \chi_i^{-(i+1)(i+2)}\chi_{i+1}^{(i+1)(i+3)}) = \chi_i^{-(i+2)(i+1)}\chi_{i+1}^{-(i+1)(i+3)}e_ie_{i+1}.$$

A similar calculation shows that $e_i^2 \in \mathfrak{b}$.

The calculation for Chow groups is analogous, where the Chow-theoretic Euler class of the bundles $N_{i,j}$ are

$$\operatorname{eu}(N_{i,j}) = \begin{cases} t_i t_j & \text{if } j < n, \\ t_i(t_0 + 2t_1 + \cdots + nt_{n-1}) & \text{if } j = n, \\ t_i(-2t_0 - 3t_1 - \cdots - (n+1)t_{n-1}) & \text{if } j = n+1. \end{cases} \qquad \square$$

Theorem 7.69. *Let $X(\Sigma'_n)$ be the crepant resolution of singularities of the moduli space of $T^*\mathbb{P}(1, n)$ indicated by the toric diagram above. Then for $n = 2, 3$ there are isomorphisms of augmented λ-algebras over $\mathbb{C}$.*

$$\widehat{K}(I\mathbb{P}(1, n))_{\mathbb{C}} \to K(X(\Sigma'_n))_{\mathbb{C}},$$

where the augmentation completion $\widehat{K}(I\mathbb{P}(1, n))_{\mathbb{C}}$ has the inertial λ-ring structure described above.

Proof. We have calculated $K(I\mathbb{P}(1, 2))_{\mathbb{C}}$ and $K(I\mathbb{P}(1, 3))_{\mathbb{C}}$, and in both cases we obtain an Artin ring that is a quotient of a coordinate ring of a torus of rank 2 and 3, respectively. The inertial augmentation ideal corresponds to the identity in the corresponding torus. Thus for $n = 2, 3$ the ring $\widehat{K}(I\mathbb{P}(1, n))_{\mathbb{C}}$ is simply the localization of $K(I\mathbb{P}(1, n))_{\mathbb{C}}$ at the corresponding maximal ideal. A calculation, which we omit as it is straightforward but lengthy, shows that

$$\widehat{K}(I\mathbb{P}(1, 2))_{\mathbb{C}} = \mathbb{C}[\sigma, \sigma^{-1}, \tau, \tau^{-1}]/\langle \sigma - 1, \tau - 1\rangle^2$$
$$\widehat{K}(I\mathbb{P}(1, 3))_{\mathbb{C}} = \mathbb{C}[\sigma, \sigma^{-1}, \tau, \tau^{-1}, \bar{\tau}, \bar{\tau}^{-1}]/\langle \sigma - 1, \tau - 1, \bar{\tau} - 1\rangle^2,$$

which are readily seen to be isomorphic as λ-rings to $K(X(\Sigma'_2))_{\mathbb{C}}$ and $K(X(\Sigma'_3))_{\mathbb{C}}$, respectively. $\square$

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