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Evgeny Musicantov and Alexander Yom Din

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Reciprocity laws and K-theory

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We associate to a full flag $\mathcal{F}$ in an n-dimensional variety X over a field k, a "symbol map" $\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k)$. Here, F_X is the field of rational functions on X, and $K(\cdot)$ is the K-theory spectrum. We prove a "reciprocity law" for these symbols: given a partial flag, the sum of all symbols of full flags refining it is 0. Examining this result on the level of K-groups, we derive the following known reciprocity laws: the degree of a principal divisor is zero, the Weil reciprocity law, the residue theorem, the Contou-Carrère reciprocity law (when X is a smooth complete curve), as well as the Parshin reciprocity law and the higher residue reciprocity law (when X is higher-dimensional).

1. Introduction

1A. ***Overview.*** Several statements in number theory and algebraic geometry are "reciprocity laws". Let us consider, as an example, the Weil reciprocity law. Let X be a complete smooth curve over an algebraically closed field k, and let us fix $f, g \in F_X^\times$, two nonzero rational functions on X. Given a point $p \in X$, one defines the tame symbol:

$$(f, g)_p := (-1)^{v_p(f)\cdot v_p(g)} \frac{f^{v_p(g)}}{g^{v_p(f)}}(p).$$

Here, v_p is the valuation at p (that is, the order of the zero). The Weil reciprocity law states that $(f, g)_p = 1$ for all but finitely many $p \in X$, and that $\prod_{p\in X}(f, g)_p = 1$.

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More generally, one can describe the pattern as follows. There is a global object, exhausted by local pieces. One then associates an invariant to each local piece, as well as to the global object itself. The desired claim is then twofold.

(i) Global is trivial: the global invariant is trivial.

(ii) Local to global: the product of the local invariants equals the global invariant (usually this is an infinite product, and one should figure out how to make sense of it).

In the above example, the global object is the curve X, which is exhausted by the local pieces — the points of the curve. The invariant associated to a local piece is the tame symbol, while the global invariant is quite implicit.

Let us recall that the Weil reciprocity law admits a higher-dimensional analog, known as the Parshin reciprocity law [Parshin 1976; Soprounov 2002, Appendix A]; see page 34.

In this paper we define symbol maps and prove a reciprocity law using the machinery of algebraic K-theory. We then see how various reciprocity laws, such as the Parshin reciprocity law (generalizing the Weil reciprocity law), the higher residue reciprocity law (generalizing the residue theorem), and the Contou-Carrère reciprocity law, all follow from this one reciprocity law.

Let us describe our setup in more detail. Fix an n-dimensional irreducible variety X over a field k.[1] By a *full flag* $\mathcal{F}$ in X we mean a chain of closed irreducible subvarieties $X = X^0 \supset X^1 \supset \cdots \supset X^n$, where the codimension of X^i in X is i. Given a full flag $\mathcal{F}$, we shall define a morphism of spectra

$$\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k)$$

(we call it a symbol map). Here F_X denotes the field of rational functions on X, $K(\,\cdot\,)$ denotes the K-theory spectrum, and Σ^n denotes n-fold suspension. By a *partial flag* $\mathcal{G}$ in X, we mean a full flag with an element in some single codimension d omitted, for $0 < d \leq n$. Then, given a partial flag $\mathcal{G}$, we may consider the set $\mathrm{fl}(\mathcal{G})$ of full flags which refine it. The main result of this paper, Theorem 2.1, then states:

Theorem. *Let X be an n-dimensional irreducible variety over a field k. Let*

$$\mathcal{G} : X^0 \supset \cdots \supset X^{d-1} \supset X^{d+1} \supset \cdots \supset X^n$$

be a partial flag in X, with element in codimension $0 < d \leq n$ omitted. In the case $d = n$, assume additionally that the curve X^{n-1} is proper over k. Then

$$\sum_{\mathcal{F} \in \mathrm{fl}(\mathcal{G})} \mu_{\mathcal{F}} = 0.$$

[1]These assumptions on X and k are made here merely to simplify matters, and will be relaxed below.

Remark. The sum figuring in the theorem is infinite; however, in Appendix A we will make sense of it (inspired by [Clausen 2012]).

In fact, it is more "correct" to additionally define a symbol map

$$\mu_{\mathcal{G}} : K(F_X) \to \Sigma^n K(k)$$

associated to a partial flag $\mathcal{G}$. The theorem then divides into two parts: that $\mu_{\mathcal{G}}$ equals zero, and that the sum of all the morphisms $\mu_{\mathcal{F}}$ for $\mathcal{F} \in \mathrm{fl}(\mathcal{G})$ equals $\mu_{\mathcal{G}}$.

Notice how this setup instantiates the general pattern above. A fixed partial flag is the global object, exhausted by the local pieces which are the full flags refining the given partial flag. The symbol map is the associated invariant.

In order to derive the concrete reciprocity laws promised above from this abstract one, one considers its effect on K-groups.

Let us note that, in principle, the symbol map between spectra appears to contain more information than its "shadows" on K-groups. However, in this paper we have only recovered known reciprocity laws from it.

Let us also record here that relevant and independent work has been done in [Braunling et al. 2014a; 2014b; Osipov and Zhu 2014].

There are several further directions to consider. For example, one may consider the "curve" $\mathrm{Spec}(\mathbb{Z})$. Could our setup be altered so as to accommodate the Hilbert reciprocity law? For that to succeed, at least three phenomena should be addressed: the prime at infinity, ramification at the prime 2, and the sphere spectrum, which underlies all primes. A relevant treatment of the case of $\mathrm{Spec}(\mathbb{Z})$ is in [Clausen 2012].

1B. ***Relation to n-Tate vector spaces.*** There is a strong relation between our machinery and the theory of n-Tate vector spaces. In fact, n-Tate vector spaces could be seen as the actual "geometric" objects that the target of our symbol map $\mu_{\mathcal{F}}$ classifies, so that, in a sense, our approach "decategorifies" the actual picture.

The technical result underlying such a connection is the following. Let $\mathcal{C}$ be an exact category, and $\mathrm{Tate}(\mathcal{C})$ the exact category of "pro-ind" objects in $\mathcal{C}$, introduced by Beilinson [1987].

Theorem [Saito 2015]. $K(\mathrm{Tate}(\mathcal{C})) \approx \Sigma K(\mathcal{C})$.

Thus, we can say that the Tate construction acts as a delooping, when one passes to K-theory spectra.

In this paper we associate to a full flag $\mathcal{F}$ in an n-dimensional variety X a symbol map

$$\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k).$$

Taking the above theorem into account, one might interpret it as a map

$$\mu_{\mathcal{F}} : K(F_X) \to K(\mathrm{Tate}^n(k)),$$

where $\mathrm{Tate}^n(k)$ is the n-fold application of the $\mathrm{Tate}(\,\cdot\,)$ construction to the exact category $\mathrm{Vect}(k)$ of finite-dimensional vector spaces over k. At this point, one might wonder whether this map comes from a functor

$$\mathrm{Vect}(F_X) \to \mathrm{Tate}^n(k).$$

Indeed, such a functor can be constructed, and is essentially the adelic construction of [Beilinson 1980].

We will address and develop the above interesting ideas elsewhere.

Once again, we point out that relevant work has been done in [Braunling et al. 2014a; 2014b].

1C. *Organization.* This paper is organized as follows. Section 2 contains the formulation of the abstract reciprocity law (Section 2A) and the formulations of concrete reciprocity laws (Section 2B) which are obtained from the abstract reciprocity law by considering its effect on specific K-groups. Section 3 contains the construction of the abstract symbol map (Section 3A) and the proof of the abstract reciprocity law (Section 3B). Section 4 deals with the calculation of the symbol map on specific K-groups.

In Appendix A, we describe how to make sense of an infinite sum of morphisms of spectra. In Appendix B, we state some lemmas about K-theory which are used in calculations.

1D. *Notation.* We use [Thomason and Trobaugh 1990] as a reference for K-theory of schemes. Given a Noetherian scheme X, $K(X)$ denotes the K-theory spectrum of X. Given a closed subset $Z \subset X$, $K(X \text{ on } Z)$ denotes the K-theory spectrum of X with support in Z. By abuse of notation, given a commutative ring A and an ideal $I \subset A$, we also write $K(A) = K(X)$ and $K(A \text{ on } I) = K(X \text{ on } Z)$, where $X = \mathrm{Spec}(A)$ and $Z \subset X$ is the closed subset associated to the ideal I.

We use the following notation for the scheme X in this paper:

- $n = \dim(X)$ denotes the Krull dimension of X.
- $|X|$ denotes the underlying topological space of X. The usual partial order on $|X|$ (that of "containment in the closure of") is denoted by $\leq$, and $|X|^i$ denotes the subset of $|X|$ consisting of points of codimension i.
- γ denotes the generic point of $|X|$ (X will be assumed to be irreducible) — i.e., the only point in $|X|^0$ — and $F = F_X = \mathcal{O}_{X,\gamma}$ denotes the local ring at that point.
- For $p \in |X|$, we write $X_p := \mathrm{Spec}(\mathcal{O}_{X,p})$. There is a canonical map $X_p \to X$. As usual, we write $k(p)$ for the residue field of $\mathcal{O}_{X,p}$.
- If X is affine and $\mathfrak{p}$ is a prime ideal in $\mathcal{O}(X)$, then $p_{\mathfrak{p}} \in |X|$ denotes the corresponding point.

2. Statements

2A. ***The abstract reciprocity law.*** Let $X \to B$ be a morphism of schemes. We make the following assumptions:

(1) B is Noetherian, 0-dimensional (i.e., a finite disjoint union of Zariski spectra of local Artinian rings).

(2) X is Noetherian, of finite Krull dimension and irreducible.

(3) $X \to B$ is flat.

(4) For every $p \in |X|^n$ (recall $n = \dim(X)$), the composition $\mathrm{Spec}(k(p)) \to X \to B$ is a finite morphism.

We give two examples of morphisms that satisfy the above assumptions:

(1) $B = \mathrm{Spec}(k)$, where k is a field, and $X \to B$ is an irreducible scheme of finite type over B.

(2) $B = \mathrm{Spec}(k)$, where k is a field, and $X = \mathrm{Spec}(A)$, where $(A, \mathfrak{m})$ is a Noetherian local integral k-algebra, such that $A/\mathfrak{m}$ is finite over k. $X \to B$ is the corresponding structure map.

A convenient technical notion will be that of a collection C, by which we mean a family $C = (C^i)_{0 \le i \le n}$, where $C^i \subset |X|^i$. We only consider collections which satisfy $C^0 = \{\gamma\}$.

Given such a C, in Section 3A we construct a map of spectra ("symbol map")

$$\mu_C : K(F) \to \Sigma^n K(B).$$

We only consider and use collections attached to full and partial flags (to be now defined), for which we will state a reciprocity law. First, let

$$\mathcal{F} : x_n < x_{n-1} < \cdots < x_0 = \gamma$$

be a full flag of points in $|X|$ (thus, $\mathrm{codim}(x_i) = i$). We define a collection $C(\mathcal{F})$, by setting $C(\mathcal{F})^i = \{x_i\}$. Second, let

$$\mathcal{G} : x_n < x_{n-1} < \cdots < x_{d+1} < x_{d-1} < \cdots < x_0 = \gamma$$

be a partial flag, with the level d omitted, $0 < d \le n$. Here, we require $\mathrm{codim}(x_i) = i$. We define a collection $C(\mathcal{G})$ by setting $C(\mathcal{G})^i = \{x_i\}$ for $i \neq d$, and

$$C(\mathcal{G})^d = \{p \in |X|^d \mid x_{d+1} < p < x_{d-1}\}.$$

Note that we have the obvious notion of a full flag refining a partial one (meaning $C(\mathcal{F}) \subset C(\mathcal{G})$), which we denote by $\mathcal{F} > \mathcal{G}$. We sometimes write $\mu_{\mathcal{F}}$ instead of $\mu_{C(\mathcal{F})}$.

We prove the following "reciprocity" laws (for the meaning of the infinite sum in this statement, consult Appendix A).

Theorem 2.1. *Let $\mathcal{G}$ be a partial flag with level d omitted, where $0 < d \leq n$.*

(1) *Global is trivial*:
$$\mu_{C(\mathcal{G})} = 0,$$
where in the case $d = n$ we should assume that $\overline{x_{n-1}}$ is proper over B.

(2) *Local to global*:
$$\mu_{C(\mathcal{G})} = \sum_{\mathcal{F} > \mathcal{G}} \mu_{C(\mathcal{F})}.$$

2B. *Concrete reciprocity laws.* In the following, we give examples of concrete reciprocity laws, which one obtains by considering the effect of the abstract reciprocity law on various homotopy groups of the involved spectra.

***The case* $\mathbf{dim}(X) = 1$.** Let k be a field, $B = \mathrm{Spec}(k)$, and $X \to B$ a regular, connected, proper curve over B. We obtain, for every closed point $p \in |X|^1$, a map $\mu_p : K(F) \to \Sigma K(B)$. Here $\mu_p = \mu_{C(\mathcal{F})}$, where $\mathcal{F} : p < \gamma$. Applying the functor π_i, one has maps $\mu_p^i : K_i(F) \to K_{i-1}(k)$.

The degree law. We have the map $\mu_p^1 : F^\times \cong K_1(F) \to K_0(k) \cong \mathbb{Z}$.

Claim 2.2. *The integer $\mu_p^1(f)$ is equal to the valuation $v_p(f)$ of f at the point p, multiplied by* $[k(p) : k]$.

Applying the abstract reciprocity law, we recover the theorem about sum of degrees [Serre 1988, §II.3, Proposition 1]:

Corollary 2.3. *For $f \in F^\times$,*
$$\sum_{p \in |X|^1} [k(p) : k] \cdot v_p(f) = 0.$$

The Weil reciprocity law. Precomposing the map $\mu_p^2 : K_2(F) \to K_1(k)$ with the product in K-theory $K_1(F) \wedge K_1(F) \to K_2(F)$, we get a bilinear antisymmetric form $\mu_p^2 : F^\times \wedge F^\times \to k^\times$ (we also call it μ_p^2, by abuse of notation).

Claim 2.4.
$$\mu_p^2(f \wedge g) = N_{k(p)/k}\left((-1)^{v_p(f) \cdot v_p(g)} \frac{f^{v_p(g)}}{g^{v_p(f)}}(p) \right).$$

Applying the abstract reciprocity law, we recover the Weil reciprocity law [Serre 1988, §III.4]:

Corollary 2.5. *For $f, g \in F^\times$,*
$$\prod_{p \in |X|^1} N_{k(p)/k}\left((-1)^{v_p(f) \cdot v_p(g)} \frac{f^{v_p(g)}}{g^{v_p(f)}}(p) \right) = 1.$$

The residue law. Suppose that k is algebraically closed. Set $k_\epsilon := k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$, $B_\epsilon = \mathrm{Spec}(k_\epsilon)$, and $X_\epsilon = k_\epsilon \otimes_k X$. Then the local ring at the generic point of X_ϵ is just $F_\epsilon = k_\epsilon \otimes_k F$. By applying our construction to the morphism $X_\epsilon \to B_\epsilon$ we get a map $K(F_\epsilon) \to \Sigma K(k_\epsilon)$ for every closed point $p \in |X_\epsilon|^1 = |X|^1$. Applying the functor π_2 and using the product in K-theory as before, one gets a pairing $r_p : F_\epsilon^\times \wedge F_\epsilon^\times \to k_\epsilon^\times$.

Claim 2.6. *For* Res_p *the usual residue* [Serre 1988, §II.7],

$$r_p((1-\epsilon_1 f) \wedge (1-\epsilon_2 g)) = 1 - \epsilon_1\epsilon_2 \,\mathrm{Res}_p(f \cdot dg).$$

Applying the abstract reciprocity law, we recover the residue theorem [Serre 1988, §II.7, Proposition 6]:

Corollary 2.7. *For* $f, g \in F$,

$$\sum_{p \in |X|^1} \mathrm{Res}_p(f \cdot dg) = 0.$$

Remark 2.8. In fact, one can drop the assumption that k is algebraically closed. Then, one has

$$r_p((1-\epsilon_1 f) \wedge (1-\epsilon_2 g)) = 1 - \epsilon_1\epsilon_2 \,\mathrm{Tr}_{k(p)/k} \mathrm{Res}_p(f \cdot dg),$$

where $\mathrm{Res}_p(f \cdot dg)$ can be defined as follows: Choose an isomorphism $\widehat{\mathcal{O}_{X,p}} \simeq k'[\![z]\!]$, where $k' := k(p)$ is the residue field at p. Interpret $f \cdot dg$ as an element of $\Omega^1(k'((z))/k') = k'((z))\,dz$. Finally, define $\mathrm{Res}_p(f \cdot dg)$ as the coefficient of $z^{-1}\,dz$ in $f \cdot dg$. Note that in the case when k is algebraically closed, one recovers the usual definition.

The Contou-Carrère reciprocity law. More generally, let k be a local Artinian ring. Set $B = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(k[\![t]\!])$. Applying the functor π_2 to the symbol map $K(k((t))) \to \Sigma K(k)$, one gets a pairing $k((t))^\times \wedge k((t))^\times \to k^\times$. Although we do not spell out the details in this paper, one can check that it is the Contou-Carrère symbol [Contou-Carrère 1994]. Then the abstract reciprocity law implies the Contou-Carrère reciprocity law.

Let us note that [Osipov and Zhu 2014] also deals with the connection between K-theory and explicit formulas for Contou-Carrère symbols.

***The case* $\dim(X) > 1$.** Let k be a field, $B = \mathrm{Spec}(k)$, and $X \to B$ an irreducible scheme of finite type over B (recall $n = \dim(X)$). For every full flag $\mathcal{F}$ one has a map $\mu_{\mathcal{F}} : K(F) \to \Sigma^n K(B)$. Applying the functor π_i, one then gets maps $\mu^i_{\mathcal{F}} : K_i(F) \to K_{i-n}(k)$.

The Parshin reciprocity law. Let us assume that the flag $\mathcal{F} = x_n < x_{n-1} < \cdots < x_0 = \gamma$ is regular in the following sense: considering $X^i := \overline{x_i}$ as an integral closed subscheme of X, we demand $\mathcal{O}_{X^{i-1}, x_i}$ to be regular (here, $1 \le i \le n$).

Precomposing the map $\mu_{\mathcal{F}}^{n+1} : K_{n+1}(F) \to K_1(k)$ with the product in K-theory $\bigwedge^{n+1} K_1(F) \to K_{n+1}(F)$, one has a multilinear antisymmetric form

$$\mu_{\mathcal{F}}^{n+1} : \bigwedge^{n+1} F^\times \to k^\times$$

(we also denote it $\mu_{\mathcal{F}}^{n+1}$, by abuse of notation).

In order to write an explicit formula for the Parshin symbol, we introduce the following; see [Soprounov 2002, Appendix A]. For every $1 \le i \le n$, let us fix a uniformizer z_i in $\mathcal{O}_i := \mathcal{O}_{X^{i-1}, x_i}$. We attach, to any $f \in F^\times$, a sequence of integers $a_1, \ldots, a_n$ as follows. Note that the residue field of $\mathcal{O}_{i-1}$ can be identified with the fraction field of $\mathcal{O}_i$. We write $f = z_1^{a_1} u_1$, where u_1 is a unit in $\mathcal{O}_1$. Considering the residue class of u_1 as an element of the fraction field of $\mathcal{O}_2$, we proceed to write $u_1 = z_2^{a_2} u_2$, where u_2 is a unit in $\mathcal{O}_2$. We continue in this way to construct the sequence $a_1, \ldots, a_n$. Note that, generally speaking, this sequence depends on the choice of uniformizers $z_1, \ldots, z_n$.

Let $f_1, \ldots, f_{n+1} \in F^\times$. Write $a_{i1}, \ldots, a_{in}$ for the sequence of integers assigned to f_i as above. Construct the $(n+1) \times n$ matrix $A = (a_{ij})$. Set A_i to be the determinant of the $n \times n$ matrix that we get from A by omitting the i-th row. Set A_{ij}^k to be the determinant of the $(n-1) \times (n-1)$ matrix that we get from A by deleting the i-th and j-th rows and the k-th column. Set $B = \sum_k \sum_{i<j} a_{ik} a_{jk} A_{ij}^k$.

Claim 2.9. $\mu_{\mathcal{F}}^{n+1}(f_1, \ldots, f_{n+1}) = N_{k(x_n)/k}\left((-1)^B \left(\prod_{1 \le i \le n+1} f_i^{(-1)^{i+1} A_i}\right)(x_n)\right).$

By applying the abstract reciprocity law, we recover the Parshin reciprocity law; see [Soprounov 2002, Appendix A].

The Parshin higher residue reciprocity law. Considering

$$k_\epsilon := k[\epsilon_1, \ldots, \epsilon_{n+1}]/(\epsilon_1^2, \ldots, \epsilon_{n+1}^2)$$

and X_ϵ, B_ϵ, etc., as for the residue law on page 33, and considering the map $\mu^{n+1} : K_{n+1}(F_\epsilon) \to K_1(k_\epsilon)$, one can derive, in principle, the higher residue reciprocity law [Soprounov 2002, Appendix A], although we do not spell out the details in this paper.

3. Construction of μ_C and proof of the abstract reciprocity law

3A. ***Construction of μ_C.*** We recall the codimension filtration in K-theory [Thomason and Trobaugh 1990, (10.3.6)]. Write $S^d K(X)$ for the homotopy colimit of the spectra $K(X \text{ on } Z)$, where Z runs over closed subsets of X of codimension $\geq d$. Also, write

$$Q^d K(X) := \bigvee_{p \in |X|^d} K(X_p \text{ on } p).$$

Then we have the homotopy fiber sequence

$$S^{d+1}K(X) \longrightarrow S^d K(X) \xrightarrow{p_d} Q^d K(X) \xrightarrow{\partial_d} \Sigma S^{d+1} K(X).$$

Let us define Ψ^d to be the composition

$$\Psi^d : Q^d K(X) \xrightarrow{\partial_d} \Sigma S^{d+1} K(X) \xrightarrow{p_{d+1}} \Sigma Q^{d+1} K(X).$$

Also, given a collection $C = (C^i)_{0 \leq i \leq n}$ (for $C^i \subset |X^i|$), we define a map

$$\mathrm{sel}_{C^d} : Q^d K(X) \to Q^d K(X),$$

given by projecting on summands corresponding to $p \in C^d$.

We now define a map

$$I : Q^n K(X) \to K(B).$$

In order to do this, we first need to define maps $K(X_p \text{ on } p) \to K(B)$, which we do by pushing forward along $X_p \to B$. To justify the existence of the pushforward, let us fix convenient models for the K-spectra. As a model for $K(X_p \text{ on } p)$ we take strictly perfect complexes on X_p which are acyclic outside of the closed point p [Thomason and Trobaugh 1990, Lemma 3.8], and as a model for $K(B)$ we take perfect complexes on B [Thomason and Trobaugh 1990, Definition 3.1]. Pushing forward along $X_p \to B$ can be done termwise, since this morphism is affine. Thus, the result of pushing forward to B a strictly perfect complex on X_p, supported on p, is a strictly bounded complex, whose terms are flat (since $X_p \to B$ is assumed flat), and whose cohomologies are coherent (since $k(p) \to B$ is assumed finite). Thus, by criterion [Thomason and Trobaugh 1990, Proposition 2.2.12], the result is perfect.

Finally, we define μ_C as follows:[2,3]

$$\mu_C = I \circ \mathrm{sel}_{C^n} \circ \Psi^{n-1} \circ \cdots \circ \Psi^1 \circ \mathrm{sel}_{C^1} \circ \Psi^0.$$

3B. ***Proof of the reciprocity law.*** Let us show part (1) of Theorem 2.1.

First, consider the case $d \neq n$. Notice that the formula for $\mu_{C(\mathcal{G})}$ contains

$$\mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \mathrm{sel}_{C(\mathcal{G})^d} \circ \Psi^{d-1}.$$

[2] We assume that $C^0 = \{\gamma\}$.

[3] In this formula, as we compose, the target becomes more and more suspended; we do not write the obvious suspensions, by abuse of notation.

Since $C(\mathcal{G})^d$ contains all the points p such that $x_{d+1} < p < x_{d-1}$, one has

$$\mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \mathrm{sel}_{C(\mathcal{G})^d} = \mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d.$$

Thus, in fact,

$$\mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \mathrm{sel}_{C(\mathcal{G})^d} \circ \Psi^{d-1} = \mathrm{sel}_{C(\mathcal{G})^{d+1}} \circ \Psi^d \circ \Psi^{d-1},$$

which is zero since $\Psi^d \circ \Psi^{d-1} = 0$ (as it contains a composition of two consequent arrows in a long exact sequence).

Next, consider the case $d = n$. Write $Y = \overline{x_{n-1}}$. We will deal first with the case $X = Y$, to simplify matters.

Note that $\mu_{C(\mathcal{G})}$ equals the composition on the top horizontal line of the following commutative diagram:

$$\begin{tikzcd}
Q^0 K(X) \arrow[r, "\partial_0"] & \Sigma S^1 K(X) \arrow[r, "p_1"] \arrow[d, "i"] & \Sigma Q^1 K(X) \arrow[r, "I"] & \Sigma K(B) \\
 & \Sigma S^0 K(X) \arrow[urr, "\tilde{I}"'] & &
\end{tikzcd}$$

Here, i is the natural arrow, and $\tilde{I}$ is the arrow induced by pushforward. The crucial assumption here is that X is proper. Thus pushing forward preserves coherence, which in turn enables us to construct the map $\tilde{I}$ on K-spectra. Now, noticing that $i \circ \partial_0 = 0$ (as a composition of two consequent arrows in a long exact sequence) finishes the proof.

In general (not assuming $X = Y$), we want to do the same as in the case $X = Y$, but working with (X on Y) versions. To proceed, one considers the commutative diagram

$$\begin{tikzcd}
Q^{n-1} K(X) \arrow[r, "\mathrm{sel}_{C(\mathcal{G})^{n-1}}"] \arrow[dr] & Q^{n-1} K(X) \arrow[r, "\partial_0"] & \Sigma Q^n K(X) \arrow[r, "I"] & \Sigma K(B) \\
 & Q^{n-1} K(X \text{ on } Y) \arrow[u] \arrow[r, "\partial^Y_{n-1}"] & \Sigma Q^n K(X \text{ on } Y) \arrow[u] \arrow[ur, "I^Y"] &
\end{tikzcd}$$

and shows $I^Y \circ \partial^Y_{n-1} = 0$ as before.

Let us now show part (2) of Theorem 2.1. We note that the map $\mathrm{sel}_{C(\mathcal{G})^d}$ is the sum of the maps $\mathrm{sel}_{C(\mathcal{F})^d}$ (where $\mathcal{F} > \mathcal{G}$). Thus, the statement follows using Claims A.4 and A.5.

4. Calculation of local symbols

In this section, we calculate some symbol maps for local schemes. Using Lemma 4.7, these calculations imply the claims of Section 2B.

Let us fix the following notation and assumptions for this section. Let k be a field, and let $B = \mathrm{Spec}(k)$. Also, let A be a regular Noetherian local k-algebra, and set $X = \mathrm{Spec}(A)$. Denote by $\mathfrak{m}$ the maximal ideal of A, and $k' = A/\mathfrak{m}$. We assume that k' is finite over k. We denote by F the fraction field of A.

4A. ***The case*** $\mathbf{dim}(X) = 1$**.** In this subsection, we additionally assume that A is of Krull dimension 1. Let $v : F^\times \to \mathbb{Z}$ be the valuation, and let $[\,\cdot\,] : A \to k'$ be the quotient map. Finally, choose a uniformizer $z \in A$ (i.e., $v(z) = 1$).

Consider the unique full flag $\mathcal{F} : p_{\mathfrak{m}} < p_0$ in X. We have the corresponding symbol map

$$\mu = \mu_{\mathcal{F}} : K(F) \to \Sigma K(k).$$

We write μ^i for the induced map $K_i(F) \to K_{i-1}(k)$.

The degree.

Claim 4.1. *The morphism* $F^\times \cong K_1(F) \xrightarrow{\mu^1} K_0(k) \cong \mathbb{Z}$ *is equal to* $[k' : k] \cdot v$.

Proof. Since the composition $K_1(A) \to K_1(F) \to K_0(A \text{ on } \mathfrak{m})$ is zero (as part of a long exact sequence), it is enough to prove that

$$F^\times \cong K_1(F) \to K_0(A \text{ on } \mathfrak{m}) \to K_0(k) \cong \mathbb{Z}$$

maps z to $[k' : k]$. By Lemma B.3, the image of z under the above map is equal to the alternating sum of dimensions (over k) of cohomologies of the complex

$$\begin{array}{ccc} A & \xrightarrow{z} & A \\ -1 & & 0 \end{array}$$

which is $[k' : k]$. $\square$

The tame symbol.

Claim 4.2. *The morphism*

$$F^\times \wedge F^\times \cong K_1(F) \wedge K_1(F) \longrightarrow K_2(F) \xrightarrow{\mu^2} K_1(k) \cong k^\times$$

is given by

$$f \wedge g \mapsto N_{k'/k}\left((-1)^{v(f)\cdot v(g)} \left[\frac{f^{v(g)}}{g^{v(f)}}\right]\right).$$

Proof. We call the above morphism $F^\times \wedge F^\times \to k^\times$, by abuse of notation, μ^2. By bilinearity and antisymmetry of μ^2, it is enough to verify:

(i) $\mu^2(f \wedge g) = 0$ for $f, g \in A^\times$.

(ii) $\mu^2(f \wedge z) = N_{k'/k}([f])$ for $f \in A^\times$.

(iii) $\mu^2(z \wedge z) = N_{k'/k}(-1)$.

The first item follows since the following composition is zero (being a part of the localization long exact sequence):

$$K_2(A) \to K_2(F) \to K_1(A \text{ on } k').$$

For the second item, consider the commutative diagram

$$\begin{tikzcd}
K_1(A) \wedge K_1(F) \arrow[rr] \arrow[d] & & K_1(A) \wedge K_0(A \text{ on } k') \arrow[d] & \\
K_1(F) \wedge K_1(F) \arrow[r] & K_2(F) \arrow[r] & K_1(A \text{ on } k') \arrow[r] & K_1(k)
\end{tikzcd}$$

We have the element $f \wedge z$ in the upper-left group $K_1(A) \wedge K_1(F)$, and we should walk it through down, and then all the way right. Using commutativity of the diagram, we can chase the upper path instead, and using Lemma B.4, the result is represented by the automorphism of the following complex:

$$\begin{tikzcd}
A \arrow[r, "z"] \arrow[d, "f"'] & A \arrow[d, "f"'] \\
A \arrow[r, "z"] & A \\
-1 & 0
\end{tikzcd}$$

Taking the alternating determinant of cohomology, we see that the above automorphism represents the element $N_{k'/k}([f]) \in k^\times \cong K_1(k)$.

Let us handle the third item on our list. Denote the multiplication in K-theory by $\{\cdot, \cdot\} : K_1(F) \wedge K_1(F) \to K_2(F)$. Recall the Steinberg relation

$$\{x, 1-x\} = 0$$

for $x, 1-x \in F^\times \cong K_1(F)$. We then calculate

$$\{z, z\} = \{z, (1-z^{-1})^{-1}\}\{z, 1-z\}\{z, -1\} = \{z^{-1}, 1-z^{-1}\}\{z, 1-z\}\{z, -1\} = \{z, -1\}$$

(this calculation appears in [Snaith 1980, Theorem 2.6]). Hence, by (ii) above, $\mu^2(z \wedge z) = \mu^2(-1 \wedge z) = N_{k'/k}(-1)$. □

The residue. Consider a base change of our setup from k to $k_\epsilon := k[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$. Thus, we have $A_\epsilon := k_\epsilon \otimes_k A$, and similarly F_ϵ, X_ϵ, B_ϵ, etc. Hence, the basic morphism of schemes from which we build the symbol map is now $X_\epsilon \to B_\epsilon$.

Claim 4.3. *The morphism*

$$F_\epsilon^\times \wedge F_\epsilon^\times \cong K_1(F_\epsilon) \wedge K_1(F_\epsilon) \longrightarrow K_2(F_\epsilon) \xrightarrow{\mu_\epsilon^2} K_1(k_\epsilon) \cong k_\epsilon^\times$$

sends $(1-\epsilon_1 f)\wedge(1-\epsilon_2 g)$ *to* $1-\epsilon_1\epsilon_2 R(f,g)$ (*for* $f,g\in F$). *Here*, $R(f,g)$ *is defined as follows*: *Choose an isomorphism* $\hat{A}\simeq k'[\![z]\!]$. *Interpret* $f\cdot dg$ *as an element* $\sum_i a_i z^i\,dz\in\Omega^1\big(k'((z))/k'\big)=k'((z))\,dz$. *Finally*, *define* $R(f,g)=\mathrm{Tr}_{k'/k}(a_{-1})$.

Proof. In this proof let us denote by μ^2 the morphism $F_\epsilon^\times\wedge F_\epsilon^\times\to k_\epsilon^\times$ in the claim.

(a) We wish to reduce the computation to the case when $A=k[\![z]\!]$ and k is infinite. This is done by exploiting functoriality in a few steps; First, using Lemma 4.8, we may assume that A is complete. Hence, by Cohen's structure theorem, $A\simeq k'[\![z]\!]$. Second, since A is now a k'-algebra, $R(f,g)$ for A as a k-algebra is the trace $\mathrm{Tr}_{k'/k}$ of $R(f,g)$ for A as a k'-algebra. Hence, we may assume that $k=k'$. Finally, let l/k be a field extension. Consider the diagram

$$\begin{tikzcd} \mathrm{Spec}(l((z))) \arrow[r] \arrow[d] & \mathrm{Spec}(l[\![z]\!]) \arrow[d] & \mathrm{Spec}(l) \arrow[l] \arrow[d] \\ \mathrm{Spec}(k((z))) \arrow[r] & \mathrm{Spec}(k[\![z]\!]) & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

Note that the squares in the above diagram are pullback squares. Hence, the following diagram commutes:

$$\begin{tikzcd} k((x))_\epsilon^\times\wedge k((x))_\epsilon^\times \arrow[r, "\mu^2"] \arrow[d] & k_\epsilon^\times \arrow[d, hook] \\ l((x))_\epsilon^\times\wedge l((x))_\epsilon^\times \arrow[r, "\mu^2"] & l_\epsilon^\times \end{tikzcd}$$

Thus, we can replace the k-algebra $A=k[\![z]\!]$ by the l-algebra $l[\![z]\!]$, where l/k is any field extension. Hence, we may assume that k is infinite.

(b) Next, we show that $\mu^2(1-\epsilon_1 f, 1-\epsilon_2 g)$ is of the form $1-\epsilon_1\epsilon_2 R(f,g)$, where $R(f,g)\in k$. In other words, the "constant term" is 1, and there are no "linear terms". Towards this end, we perform "base change", sending $\epsilon_2\mapsto 0$. The operation μ^2 commutes with such a base change. We depict it as follows:

$$\begin{tikzcd} \mu^2(1-\epsilon_1 f, 1-\epsilon_2 g) \arrow[r, equal] \arrow[d, mapsto] & a+b\epsilon_1+c\epsilon_2+d\epsilon_1\epsilon_2 \arrow[d, mapsto] \\ \mu^2(1-\epsilon_1 f, 1) \arrow[r, equal] & a+b\epsilon_1 \end{tikzcd}$$

Here, the vertical assignment is base change, from k_ϵ to $k_\epsilon/(\epsilon_2)$. Note that the lower-left element is 1 (by bimultiplicativity of μ^2), so that we get $a=1$ and $b=0$. Similarly, one gets $c=0$.

(c) We notice that $R(f,g)$ is bilinear. The biadditivity follows immediately from the bimultiplicativity of μ^2 and (b). Next, let us show that $R(\alpha f,g)=\alpha R(f,g)$

for every $\alpha \in k$ (the homogeneity in the second variable is shown analogously). In case $\alpha = 0$, it is clear. Otherwise, we get the equality by performing “base change”, sending $\epsilon_1 \mapsto \alpha^{-1}\epsilon_1$.

(d) We now show the following properties, from which the statement follows by decomposing elements of F into Laurent expansions:

(1) $R(z^n, z^m) = 0$ for $n, m \in \mathbb{Z}$, provided $n + m \neq 0$.

(2) $R(z^{-n}, z^n) = n$ for $n \in \mathbb{Z}$.

(3) $R(z^{-n}, f) = 0$ for $n \in \mathbb{Z}_{\geq 0}$, provided that $v(f) \gg n$.

Consider the automorphism $z \mapsto \alpha z$, where $\alpha \in k^\times$. We notice that it does not alter the symbol μ^2, since it commutes with passing to the quotient $A \mapsto A/\mathfrak{m}$. Thus, we have $R(z^n, z^m) = R((\alpha z)^n, (\alpha z)^m)$. By bilinearity (see (c) above), we get $R(z^n, z^m) = \alpha^{n+m} R(z^n, z^m)$. Choosing α so that $\alpha^{n+m} \neq 1$, we conclude $R(z^n, z^m) = 0$. Such a choice of α is possible since k is infinite and $n + m \neq 0$.

To show the second item, note that

$$\mu^2(1 - \epsilon_1 z^{-n}, 1 - \epsilon_2 z^n) = \frac{\mu^2(z^n - \epsilon_1, 1 - \epsilon_2 z^n)}{\mu^2(z^n, 1 - \epsilon_2 z^n)},$$

and hence it is enough to calculate $\mu^2(z^n - \alpha\epsilon_1, 1 - \epsilon_2 z^n)$ (where $\alpha \in k$). By Lemmas B.3 and B.4, we should calculate the determinant of multiplication by $1 - \epsilon_2 z^n$ on the cohomology of

$$\begin{array}{ccc} A_\epsilon & \xrightarrow{z^n - \alpha\epsilon_1} & A_\epsilon \\ -1 & & 0 \end{array}$$

The only nonzero cohomology is the 0-th one. It is a free k_ϵ-module (with basis $1, z, \dots, z^{n-1}$). Multiplication by $1 - \epsilon_2 z^n$ is just multiplication by $1 - \alpha\epsilon_1\epsilon_2$. Thus, the determinant equals $(1 - \alpha\epsilon_1\epsilon_2)^n = 1 - n\alpha\epsilon_1\epsilon_2$, and consequently $R(z^{-n}, z^n) = n$.

The third item is verified similarly to the second one (when $v(f) \gg n$, the operator whose determinant we should consider is just the identity).

(e) By breaking f and g into sums of monomials in z and a reminder of large enough valuation, the proposition follows from (b), (c), and (d) . ☐

Remark 4.4. One could also obtain the residue symbol differently, by considering $k_\epsilon := k[\epsilon]/(\epsilon^3)$. Then $\mu^2(1 - \epsilon f, 1 - \epsilon g) = 1 - \epsilon^2 \operatorname{Res}(f dg)$.

4B. *The case* $\dim(X) > 1$. In this subsection, we drop the assumption that A is 1-dimensional. We denote the Krull dimension of A by n.

The Parshin symbol. Fix a full flag

$$\mathcal{F} : x_n < \cdots < x_0$$

in X, corresponding to a chain of prime ideals

$$0 = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{m}.$$

Consider $X^i := \overline{x_i}$ as an integral closed subscheme of X. We obtain a symbol

$$\mu = \mu_{\mathcal{F}} : K(F) \to \Sigma^n K(k).$$

As with the Parshin reciprocity law (see page 34), we consider the resulting map $\mu_{\mathcal{F}}^{n+1} : \bigwedge^{n+1} F^\times \to k^\times$. There, we essentially wrote a formula for this map (which we now want to verify) under the assumption that our flag is regular. In order to compute this map "recursively", we will use Quillen's dévissage (Lemma B.5) — application of which will be possible due to regularity of $\mathcal{F}$.

Claim 4.5. *The symbol $\mu_{\mathcal{F}} : K(F_X) \to \Sigma^n K(k)$ equals the composition*

$$\begin{tikzcd}
K(F_X) \arrow[r] & \Sigma K(X_{x_1} \text{ on } x_1) & \Sigma K(F_{X^1}) \arrow[l, "\sim"'] \arrow[r] & \Sigma^2 K(X^1_{x_2} \text{ on } x_2) \\
K(F_{X^2}) \arrow[urrr, bend left, "\sim"] \arrow[r] & \cdots \arrow[r] & \Sigma^n K(F_{X^n}) \arrow[r] & \Sigma^n K(k)
\end{tikzcd}$$

where the arrows $\xleftarrow{\sim}$ stand for Quillen's dévissage.

In view of this claim, $\mu_{\mathcal{F}}^{n+1}$ equals the composition

$$\bigwedge^{n+1} F_X^\times \longrightarrow K_{n+1}(F_X) \xrightarrow{\partial_0} K_n(F_{X_1}) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} K_1(F_{X_n}) \to K_1(k),$$

where ∂_i is the composition of the boundary map and the inverse of the dévissage.

The following lemma will allow us, in principle, to calculate $\mu_{\mathcal{F}}^{n+1}(f_1, \dots, f_{n+1})$ for any $f_1, \dots, f_{n+1} \in F^\times$.

Lemma 4.6. *Let R be a 1-dimensional regular local Noetherian ring with maximal ideal $\mathfrak{n}$, residue field ℓ, and fraction field L. Let $z \in R$ be a uniformizer. Consider the composition of the boundary map with the dévissage map*

$$K(L) \longrightarrow \Sigma K(R \text{ on } \mathfrak{n}) \xleftarrow{\sim} \Sigma K(\ell).$$

We use it to construct a map

$$\nu^m : \bigwedge^m L^\times \to K_m(L) \to K_{m-1}(\ell).$$

The following hold:

(i) $\nu^m(f_1, \dots, f_m) = 0$ *for* $f_1, \dots, f_m \in R^\times$.

(ii) $\nu^m(f_1, \dots, f_{m-2}, z, z) = \nu^m(f_1, \dots, f_{m-2}, -1, z)$ *for* $f_1, \dots, f_{m-2} \in R^\times$.

(iii) $\nu^m(f_1, \dots, f_{m-1}, z) = [f_1] \wedge \cdots \wedge [f_{m-1}]$ *for* $f_1, \dots, f_{m-1} \in R^\times$ (recall that $[f]$ denotes the residue in $\ell^\times$ of $f \in R^\times$, considered as an element of $K_1(\ell)$ in the case at hand).

Proof. The first item is clear, since $\nu^m(f_1, \dots, f_m)$ is the value of the composition $K_m(R) \to K_m(L) \to K_{m-1}(R \text{ on } \mathfrak{n})$ on $f_1 \wedge \cdots \wedge f_m \in K_m(R)$, and the composition is zero as part of a long exact sequence.

The second item follows from the Steinberg relation (as in the proof of Claim 4.2).

The third item follows from the commutativity of the following diagram:

$$
\begin{tikzcd}
K_{m-1}(R) \wedge K_1(L) \arrow[r] \arrow[dd] & K_{m-1}(R) \wedge K_0(R \text{ on } \mathfrak{n}) \arrow[dd] & K_{m-1}(R) \wedge K_0(\ell) \arrow[l, "\sim"'] \arrow[d] \\
 & & K_{m-1}(\ell) \wedge K_0(\ell) \arrow[d] \\
K_m(L) \arrow[r] & K_{m-1}(R \text{ on } \mathfrak{n}) & K_{m-1}(\ell) \arrow[l, "\sim"']
\end{tikzcd}
$$

Here the left square commutes since the boundary morphism is a morphism of $K(A)$-modules, while the right square commutes as Quillen's dévissage morphism is a morphism of $K(A)$-modules.

Note that the element $\nu^m(f_1, \dots, f_{m-1}, z)$ is the result of going right on the lower line, applied to $f_1 \wedge \cdots \wedge f_{m-1} \wedge z$. However, this element comes from an element at the upper-left corner, which we can chase through the right on the upper line, and then to the lower-left corner through the right line. □

4C. *Auxiliary lemmas.* We state two lemmas which are used above, and whose proofs are straightforward.

Lemma 4.7. *Let $X \to B$ be as in Section 2A. Let*

$$\mathcal{F} : x_n < x_{n-1} < \cdots < x_0 = \gamma$$

be a full flag of points in $|X|$. Writing $p := x_n$, we consider also the setting $X_p \to B$ and the obvious flag $\mathcal{F}_p$ on X_p induced by $\mathcal{F}$. We have two symbol maps:

$$\mu_{\mathcal{F}} : K(F) \to \Sigma^n K(k) \quad \textit{and} \quad \mu_{\mathcal{F}_p} : K(F) \to \Sigma^n K(k)$$

(note that the function field of X_p is identified with F). *Then these two symbol maps are equal.*

Lemma 4.8. *Let A be a* 1*-dimensional regular local Noetherian k-algebra whose residue field is finite over k, and let $\hat{A}$ be its completion. We write, as usual, $X = \operatorname{Spec}(A)$ and $B = \operatorname{Spec}(k)$, and also $\hat{X} = \operatorname{Spec}(\hat{A})$. Also, denote by F and $\hat{F}$ the fraction fields of A and $\hat{A}$, respectively. Associated to the unique full flags in*

X and $\hat{X}$ we have the symbols $K(F) \to \Sigma K(k)$ and $K(\hat{F}) \to \Sigma K(k)$. Then the diagram

$$\begin{tikzcd} K(F) \arrow[rr] \arrow[dr] & & K(\hat{F}) \arrow[dl] \\ & \Sigma K(k) & \end{tikzcd}$$

commutes.

Appendix A: Infinite sums of maps of spectra

In this paper, we consider spectra as a triangulated category $\mathcal{S}p$. We recall that a spectrum is called compact if maps from it commute with small direct sums. An example of a compact spectrum is $\Sigma^k \mathbb{S}$, a suspension of the sphere spectrum. The following definitions are inspired by [Clausen 2012, Appendix A].

Definition A.1. Let $f_i : \mathcal{S} \to \mathcal{T}$ $(i \in I)$ be a family of maps of spectra, and $f : \mathcal{S} \to \mathcal{T}$ an additional map. We say that *f is the sum of the f_i* (written $f = \sum_{i \in I} f_i$) if for every compact spectrum $\mathcal{C}$, and every element $e \in \mathrm{Hom}_{\mathcal{S}p}(\mathcal{C}, \mathcal{S})$, almost all (i.e., all but finitely many) of the maps $f_i \circ e$ are equal to zero, and the sum of all these $f_i \circ e$ is equal to $f \circ e$.

We note that we do not claim uniqueness of the sum (in whatever sense). In reality, this notion of "summability and summation on compact probes" is derived from a more holistic notion:

Definition A.2. Let $f_i : \mathcal{S} \to \mathcal{T}$ $(i \in I)$ be a family of maps of spectra, and $f : \mathcal{S} \to \mathcal{T}$ an additional map. An *evidence* for f being the sum of the f_i is a map

$$g : \mathcal{S} \to \bigvee_{i \in I} \mathcal{T}$$

such that when we compose g with the projection to the i-th summand we get f_i, while when we compose g with the fold map, we get f.

The following is evident:

Claim A.3. *Existence of an evidence for f being the sum of the f_i implies that f is the sum of the f_i.*

Let us also note the following two auxiliary claims (whose proofs are straightforward):

Claim A.4. *Let $h : \mathcal{U} \to \mathcal{S}$ and $g : \mathcal{T} \to \mathcal{V}$. If f is the sum of the f_i (we have evidence for f being the sum of the f_i), then $g \circ f \circ h$ is the sum of the $g \circ f_i \circ h$ (we have evidence for $g \circ f \circ h$ being the sum of the $g \circ f_i \circ h$).*

Claim A.5. *Let $\mathcal{S}_i$ ($i \in I$) be a collection of spectra, and write $\mathcal{S} = \bigvee_{i \in I} \mathcal{S}_i$. Then we have evidence for* id *being the sum of* pr_i ($i \in I$), *where* id *is the identity morphism of $\mathcal{S}$, while* pr_i *is the morphism of projection on the i-th summand. In particular*, $\mathrm{id} = \sum_{i \in I} \mathrm{pr}_i$.

Appendix B: *K*-theory calculation lemmas

We state some lemmas which are of use when calculating the concrete symbols. In what follows, X is a Noetherian scheme, $U \subset X$ an open subscheme, and Z the closed complement.

We denote by $\mathrm{SPerf}(X)$ the category of (strictly) bounded complexes of $\mathcal{O}_X$-modules, whose terms are locally free of finite rank. By $\mathrm{SPerf}(X \text{ on } Z)$ we denote the full subcategory of $\mathrm{SPerf}(X)$ consisting of complexes whose cohomologies are supported on Z.

Fact B.1. *There is a natural map from* (*the geometric realization of*) *the core groupoid of* $\mathrm{SPerf}(X)$ *to* $K(X)$. *In particular, every object in* $\mathrm{SPerf}(X)$ *defines a point in* $K(X)$. *In addition, the automorphism group of any object of* $\mathrm{SPerf}(X)$ *maps into* $K_1(X)$. *Since* $\mathcal{O}(X)^\times$ *maps into the automorphism group of the object* $\mathcal{O}_X \in \mathrm{SPerf}(X)$, *one then has a map* $\mathcal{O}(X)^\times \to K_1(X)$. *Thus, given an object or an automorphism in* $\mathrm{SPerf}(X)$, *one can view it as an element of an appropriate K-group* $K_i(X)$. *We will abuse this without further notice.*

Claim B.2. *Let X be local* (*i.e., the spectrum of a local ring*). *Then the above map* $\mathcal{O}(X)^\times \to K_1(X)$ *is an isomorphism.*

Lemma B.3. *Let $f \in \mathcal{O}(X)$ be such that $f|_U$ is invertible. Then the image of $f|_U \in \mathcal{O}(U)^\times$ under the map $K_1(U) \to K_0(X \text{ on } Z)$ which is obtained from the localization sequence*

$$K(X \text{ on } Z) \to K(X) \to K(U)$$

(see [Thomason and Trobaugh 1990, Theorem 7.4]) *is given by the complex*

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{f} & \mathcal{O}_X \\ -1 & & 0 \end{array}$$

Lemma B.4. *Let $f \in \mathcal{O}(X)^\times$, and $C \in \mathrm{SPerf}(X \text{ on } Z)$. Then the image of $f \wedge C$ under the product map $K_1(X) \wedge K_0(X \text{ on } Z) \to K_1(X \text{ on } Z)$ is given by the automorphism*

$$C \otimes \mathcal{O}_X \xrightarrow{1 \otimes f} C \otimes \mathcal{O}_X.$$

Lemma B.5 (Quillen's dévissage). *Suppose that X and Z are regular. Then the morphism $K(Z) \to K(X \text{ on } Z)$* (*induced by pushforward*) *is an equivalence of spectra.*

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EVGENY MUSICANTOV: evgenym@post.tau.ac.il
School of Mathematical Sciences, Tel Aviv University, 6997801 Tel Aviv, Israel

ALEXANDER YOM DIN: yomdina@post.tau.ac.il
School of Mathematical Sciences, Tel Aviv University, 6997801 Tel Aviv, Israel

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